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ON TYPICAL REPRESENTATIONS FOR DEPTH-ZERO COMPONENTS OF SPLIT CLASSICAL GROUPS

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ABSTRACT. Let $\mathbf{G}$ be a split classical group over a non-Archimedean local field F with the cardinality of the residue field $q_F > 5$. Let M be the group of F-points of a Levi factor of a proper F-parabolic subgroup of $\mathbf{G}$. Let $[M, \sigma_M]_M$ be an inertial class such that σ_M contains a depth-zero Moy–Prasad type of the form (K_M, τ_M), where K_M is a hyperspecial maximal compact subgroup of M. Let K be a hyperspecial maximal compact subgroup of $\mathbf{G}(F)$ such that K contains K_M. In this article, we classify $\mathfrak{s}$-typical representations of K. In particular, we show that the $\mathfrak{s}$-typical representations of K are precisely the irreducible subrepresentations of $\mathrm{ind}_J^K \lambda$, where (J, λ) is a level-zero G-cover of $(K \cap M, \tau_M)$.

1. Introduction

Let F be a non-Archimedean local field with ring of integers $\mathfrak{o}_F$. Let $\mathfrak{p}_F$ be the maximal ideal of $\mathfrak{o}_F$. Let k_F be the residue field of $\mathfrak{o}_F$, and we assume that k_F has cardinality $q_F > 5$. Let $\mathbf{G}$ be any reductive algebraic group over F, and let G be the group of F-rational points of $\mathbf{G}$. Let K be any maximal compact subgroup of G. All representations in this article are defined over complex vector spaces.

Let (M, σ_M) be a pair consisting of a Levi factor M of an F-parabolic subgroup of G, and a cuspidal representation σ_M of M. Recall that two such pairs (M_1, σ_{M_1}) and (M_2, σ_{M_2}) are called *inertially equivalent* if there exists an element $g \in G$ such that

$$M_1 = gM_2g^{-1} \text{ and } \sigma_{M_1} \simeq \sigma_{M_2}^g \otimes \chi,$$

where χ is an unramified character of M_1. Equivalence classes for this relation are called *inertial classes*. The inertial class containing the pair (M, σ_M) is denoted by $[M, \sigma_M]_G$ (or by $[M, \sigma_M]$ if G is clear from the context). The set of inertial classes of G is denoted by $\mathcal{B}(G)$. An inertial class of the form $[G, \sigma]_G$ is called a *cuspidal inertial class of* G.

Let $\mathcal{R}(G)$ be the category of smooth representations of G. Let $\mathfrak{s} = [M, \sigma_M]_G$ be an inertial class of G, and let $\mathcal{R}_\mathfrak{s}(G)$ be the full subcategory of $\mathcal{R}(G)$ consisting of smooth G-representations whose irreducible subquotients occur as subquotients of $i_P^G(\sigma_M \otimes \chi)$, where P is an F-parabolic subgroup such that M is a Levi factor of P and χ is an unramified character of M. Here, the functor i_P^G denotes the

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normalised parabolic induction. Bernstein in the article [Ber84] showed that the category $\mathcal{R}(G)$ can be decomposed as

$$\mathcal{R}(G) = \prod_{\mathfrak{s}\in\mathcal{B}(G)} \mathcal{R}_{\mathfrak{s}}(G).$$

The category $\mathcal{R}_{\mathfrak{s}}(G)$ is indecomposable. In particular, every smooth representation of G can be written as a direct sum of subrepresentations which belong to $\mathcal{R}_{\mathfrak{s}}(G)$. The category $\mathcal{R}_{\mathfrak{s}}(G)$ is called the *Bernstein component* associated to $\mathfrak{s}$.

Based on extensive examples for GL_n, SL_n, it turns out that for a given indecomposable block $\mathcal{R}_{\mathfrak{s}}(G)$, there is a natural set of irreducible smooth representations of K called $\mathfrak{s}$-typical representations: if an $\mathfrak{s}$-typical representation of K occurs in an irreducible smooth representation π of G, then π belongs to $\mathcal{R}_{\mathfrak{s}}(G)$. In this article, when K is hyperspecial, we classify $\mathfrak{s}$-typical representations of K for depth-zero inertial classes $\mathfrak{s}$ of split classical groups. We refer to the articles [BM02], [Pas05], [Nad19], [Nad17], [Lat17], and [Lat18] for some earlier works. We will now try to make this notation precise and describe our main theorem.

The theory of types, developed by Bushnell–Kutzko, describes the category $\mathcal{R}_{\mathfrak{s}}(G)$ in terms of modules over Hecke algebras. We refer to [BK98] for a systematic treatment. In particular, the formalism aims to construct a pair $(J_{\mathfrak{s}}, \lambda_{\mathfrak{s}})$ consisting of a compact open subgroup $J_{\mathfrak{s}}$ of G and an irreducible smooth representation $\lambda_{\mathfrak{s}}$ of $J_{\mathfrak{s}}$ such that, for any irreducible smooth representation π of G,

$$\mathrm{Hom}_{J_{\mathfrak{s}}}(\lambda_{\mathfrak{s}}, \pi) \neq 0 \text{ if and only if } \pi \in \mathcal{R}_{\mathfrak{s}}(G). \tag{1}$$

Such a pair $(J_{\mathfrak{s}}, \lambda_{\mathfrak{s}})$ is called a *type for* $\mathfrak{s}$ or an $\mathfrak{s}$*-type*.

A type $(J_{\mathfrak{s}}, \lambda_{\mathfrak{s}})$, for an inertial class $\mathfrak{s} = [M, \sigma_M]_G$, is generally constructed in two steps. First, a type $(J_{\mathfrak{t}}, \lambda_{\mathfrak{t}})$ is constructed for the cuspidal inertial class $\mathfrak{t} = [M, \sigma_M]_M$. For the inertial class $[M, \sigma_M]_G$, a type $(J_{\mathfrak{s}}, \lambda_{\mathfrak{s}})$ is then constructed as a G-cover of $(J_{\mathfrak{t}}, \lambda_{\mathfrak{t}})$, in the sense of [BK98, Section 8]. In particular, for any F-parabolic subgroup P of G such that M is a Levi factor of P, a G-cover $(J_{\mathfrak{s}}, \lambda_{\mathfrak{s}})$ has Iwahori decomposition with respect to the pair (P, M), i.e., $J_{\mathfrak{s}} \cap M$ is equal to $J_{\mathfrak{t}}$,

$$\mathrm{res}_{J_{\mathfrak{s}}\cap M} \lambda_{\mathfrak{s}} = \lambda_{\mathfrak{t}},$$

and the groups $J_{\mathfrak{s}} \cap U$ and $J_{\mathfrak{s}} \cap \bar{U}$ are both contained in the kernel of $\lambda_{\mathfrak{s}}$. Here U is the unipotent radical of P, and $\bar{U}$ is the unipotent radical of the opposite parabolic subgroup of P with respect to M.

Types $(J_{\mathfrak{s}}, \lambda_{\mathfrak{s}})$ are now constructed for many classes of reductive groups G. There are several constructions leading to different pairs $(J_{\mathfrak{s}}, \lambda_{\mathfrak{s}})$ as types for $\mathfrak{s}$. These types contain important arithmetic information. For $\mathrm{GL}_n(F)$, Bushnell and Kutzko [BK93a] constructed a set of types, which they called *maximal types*, for any cuspidal component. Later in the article [BK99], they constructed explicit G-covers for these maximal types. For $\mathrm{SL}_n(F)$, similar constructions are due to Bushnell–Kutzko and Goldberg–Roche (see [BK93b], [BK94], [GR02] and [GR05]), for inner forms of GL_n by Sécherre and Stevens (see [SS08] and [SS12]), for $\mathrm{Sp}_4(F)$ by Blasco and Blondel in [BB99] and [BB02]. Types for inertial classes of the form $[T, \chi]$, where T is a maximal split torus, are constructed by Roche [Roc98]. For an arbitrary connected reductive group and depth-zero components, types are constructed by Morris, and Moy and Prasad in [Mor99] and [MP96]; respectively. For classical groups (with p odd), these construction are due to Stevens [Ste08], and by Miyauchi and Stevens [MS14].

Let K be a maximal compact subgroup of G, and let $\mathfrak{s}$ be an inertial class of G. An irreducible smooth representation τ of K is called *$\mathfrak{s}$-typical* if every irreducible smooth representation π of G such that $\mathrm{Hom}_K(\tau,\pi) \neq 0$ is in $\mathcal{R}_{\mathfrak{s}}(G)$. This notion weakens that of an $\mathfrak{s}$-type introduced by Bushnell and Kutzko: τ is an $\mathfrak{s}$-type if it is $\mathfrak{s}$-typical and $\mathrm{Hom}_K(\tau,\pi) \neq 0$, for all irreducible smooth representations π in $\mathcal{R}_{\mathfrak{s}}(G)$. An irreducible smooth representation τ of K is called *atypical* if τ is not an $\mathfrak{s}$-typical representation for any $\mathfrak{s} \in \mathcal{B}(G)$. Let $(J_{\mathfrak{s}}, \lambda_{\mathfrak{s}})$ be an $\mathfrak{s}$-type such that $J_{\mathfrak{s}} \subseteq K$. Then Frobenius reciprocity shows that any irreducible subrepresentation of

$$\mathrm{ind}_{J_{\mathfrak{s}}}^{K} \tau_{\mathfrak{s}} \tag{2}$$

is $\mathfrak{s}$-typical. In general, the representation (2) is not irreducible, and hence, the isomorphism classes of $\mathfrak{s}$-typical representations of K are not necessarily unique. In the interest of arithmetic applications, it is important to understand the existence and classification of $\mathfrak{s}$-typical representations of K.

The representation theory of maximal compact subgroups of p-adic groups is quite involved. For example, a parametrisation of all irreducible smooth representations for $K = \mathrm{GL}_n(\mathfrak{o}_F)$ is not yet known. In this regard, it is interesting to understand irreducible smooth representations of K in terms of the Bernstein decomposition of G. Precisely, for any finite set of inertial classes $\mathcal{S}$ of G, one wants to understand those irreducible smooth representations τ of K such that, for an irreducible smooth representation π of G,

$$\mathrm{Hom}_K(\tau,\pi) \neq 0 \ \Rightarrow \ \pi \in \mathcal{R}_{\mathfrak{s}}(G), \text{ for some } \mathfrak{s} \in \mathcal{S}.$$

This paper belongs to this theme.

We now state the main results of this paper. Let (W, q) be a pair consisting of an F-vector space W, and a nondegenerate alternating or symmetric F-bilinear form q on W. Let G be the group of F-points of $\mathbf{G}$—the connected component of the isometry group associated to the pair (W, q). *We assume that* $\mathbf{G}$ *is an F-split group.* For any parahoric subgroup $\mathcal{K}$ of G we denote by $\mathcal{K}^+$ the pro-p-unipotent radical of $\mathcal{K}$. Let $\mathfrak{t}$ be an inertial class $[M, \sigma_M]_M$ such that $\sigma_M^{K_M^+} \neq 0$, for some maximal parahoric subgroup K_M of M. The representation σ_M is called a *depth-zero* cuspidal representation of M and the inertial class $\mathfrak{t}$ is called a *depth-zero inertial class*. Any irreducible K_M-subrepresentation of $\sigma_M^{K_M^+}$ is the inflation of a cuspidal representation of the finite reductive group K_M/K_M^+. Let τ_M be an irreducible K_M-subrepresentation of $\sigma_M^{K_M^+}$. The pair (K_M, τ_M) is called an *unrefined minimal K-type* by Moy and Prasad (see [MP94, Definition 5.1]). When K_M is a hyperspecial maximal compact subgroup, the pair (K_M, τ_M) is also a $[M, \sigma_M]_M$-type in the sense of Bushnell and Kutzko; in this case, we simply call the pair (K_M, τ_M) a *depth-zero* type.

Assume that K_M is a hyperspecial maximal compact subgroup of M. Let K be a hyperspecial maximal compact subgroup of G such that $K_M \subset K$. Let P be a parabolic subgroup of G such that M is a Levi factor of P. Let $P(1)$ be the group $(P \cap K)K^+$. Note that the group $P(1)$ is a parahoric subgroup of G, and we have $P(1) \cap M = K_M$. The representation τ_M of K_M extends as a representation of $P(1)$ such that $P(1) \cap U$ and $P(1) \cap \bar{U}$ are contained in the kernel of this extension. Here, U is the unipotent radical of P and $\bar{U}$ is the unipotent radical of the opposite parabolic subgroup of P with respect to M. With this notation, our main result can be stated as follows.

Theorem 1.1. *Let $\mathfrak{s} = [M, \sigma_M]_G$ be an inertial class such that $M \neq G$. Let K_M be a hyperspecial maximal compact subgroup of M. Assume that $\sigma_M^{K_M^+} \neq 0$, and let τ_M be an irreducible K_M-subrepresentation of $\sigma_M^{K_M^+}$. Let K be a hyperspecial maximal compact subgroup of G such that $K_M \subseteq K$. Then $\mathfrak{s}$-typical representations of K are exactly the subrepresentations of $\mathrm{ind}_{P(1)}^{K} \tau_M$.*

Let G be the group of F-points of a reductive algebraic group defined over F. For the depth-zero inertial classes of the form $\mathfrak{s} = [G, \sigma]_G$, and K is any maximal compact subgroup, Latham [Lat17] showed that an $\mathfrak{s}$-typical representation of K, if it exists, is unique. We will apply this result for split classical groups. However, for the present purposes of this article, we only need to consider hyperspecial maximal compact subgroups (see Lemma 4.4).

Let $\mathbf{T}$ be a maximal split torus of $\mathbf{G}$ defined over F. Using a Witt basis, we identify $\mathbf{T}(F)$ with the following subtorus of the diagonal torus of $\mathrm{GL}(W)$:

$$\{\mathrm{diag}(t_1, \dots, t_1^{-1}) : t_i \in F^\times, 1 \leq i \leq n\}.$$

Let χ be a character of $\mathbf{T}(F)$, and let

$$\chi(\mathrm{diag}(t_1, \dots, t_1^{-1})) = \chi_1(t_1) \cdots \chi_n(t_n),$$

where χ_i is a character of $F^\times$, for $1 \leq i \leq n$. The inertial class $[\mathbf{T}(F), \chi]_G$ is called a *toral inertial class*. For any character η of $F^\times$, let $l(\eta)$ be the least positive integer k such that $1 + \mathfrak{p}_F^k$ is contained in the kernel of η. In this article, we assume that

$$l(\chi_i) \neq l(\chi_j), \text{ for } 1 \leq i \neq j \leq n. \tag{3}$$

Let K be a hyperspecial maximal compact subgroup of $\mathbf{G}$ such that $\mathbf{T}(F) \cap K$ is the maximal compact subgroup of $\mathbf{T}(F)$. The proof of Theorem 1.1 can also be extended to obtain a classification of $\mathfrak{s}$-typical representations of K. In Section 7, we describe Roche's construction of a G-cover (J_χ, χ) for the pair $(\mathbf{T}(F) \cap K, \mathrm{res}_{\mathbf{T}(F)\cap K}\, \chi)$ (see [Roc98, Section 2,3]). This construction depends on the choice of a pinning. It is possible to choose a pinning such that $J_\chi \subset K$. We prove the following theorem for the toral inertial class $[\mathbf{T}(F), \chi]$.

Theorem 1.2. *Let K be any hyperspecial maximal compact subgroup of G. Let $\mathbf{T}$ be any maximal split torus of $\mathbf{G}$ defined over F. Assume that $K \cap \mathbf{T}(F)$ is the maximal compact subgroup of $\mathbf{T}(F)$. Let χ be a character of $\mathbf{T}(F)$ which satisfies the condition* (3)*. Then $[\mathbf{T}(F), \chi]_G$-typical representations of K are exactly the subrepresentations of $\mathrm{ind}_{J_\chi}^{K} \chi$.*

2. Notation

Let F be a non-Archimedean local field with ring of integers $\mathfrak{o}_F$. Let $\mathfrak{p}_F$ be the maximal ideal of $\mathfrak{o}_F$ with residue field $k_F = \mathfrak{o}_F/\mathfrak{p}_F$. Let q_F be the cardinality of k_F. In this article, we assume that $q_F > 5$. Let ϖ_F be a uniformiser of F. For any F-algebraic group $\mathbf{H}$, we denote by H the group $\mathbf{H}(F)$. The group H is considered as a topological group whose topology is induced from F.

Let $\mathbf{G}$ be any reductive algebraic group over F. For any closed subgroup H of G and a smooth representation σ of H, we denote by $\mathrm{ind}_H^G \sigma$ the compactly induced representation from H to G. For any parabolic subgroup P of G and σ any smooth representation of a Levi factor M of P, we denote by $i_P^G\sigma$ the normalised parabolically induced representation of G. For any representations ρ_1 and ρ_2 of the groups G_1 and G_2 respectively, we denote by $\rho_1 \boxtimes \rho_2$ the tensor product representation of the group $G_1 \times G_2$.

Let (V, q) be any pair consisting of a vector space V over a field k, and a k-bilinear form q on V. We denote by $G(V, q)$ (or by $G(V)$ when q is clear from the context) the group of k-points of the connected component of the isometry group of the pair (V, q).

3. Preliminaries

Let $\epsilon \in \{\pm 1\}$, and let W be an F-vector space with a nondegenerate F-bilinear form q such that

$$q(w_1, w_2) = \epsilon q(w_2, w_1), \text{ for } w_1, w_2 \in W.$$

Let W^+ be any maximal totally isotropic subspace of W. Let

$$(w_1, w_2, \dots, w_n)$$

be a basis of W^+. There exists a maximal totally isotropic subspace W^- with basis

$$(w_{-1}, w_{-2}, \dots, w_{-n})$$

such that

$$q(w_i, w_j) = 0, \text{ for } -n \le i \ne -j \le n, \text{ and } q(w_i, w_{-i}) = 1, \text{ for } 1 \le i \le n. \tag{4}$$

The space $W^+ \oplus W^-$ is a hyperbolic subspace of W. Let $(W^+ \oplus W^-) \perp W_0$ be a Witt decomposition of W. Note that W_0 is an anisotropic subspace of W. **In this article, we assume that** $\dim_F W_0 \le 1$**.** Let w_0 be any nonzero vector in W_0, if $W_0 \ne \{0\}$. The tuple of vectors

$$B := \begin{cases} (w_n, w_{n-1}, \dots, w_1, w_{-1}, w_{-2}, \dots, w_{-n}) & \text{if } \dim(W) = 2n, \\ (w_n, w_{n-1}, \dots, w_1, w_0, w_{-1}, w_{-2}, \dots, w_{-n}) & \text{if } \dim(W) = 2n+1 \end{cases} \tag{5}$$

is a basis of the space W. Any tuple of vectors as in B is called a *standard basis* of W. Let N be the cardinality of the basis B. Let $\mathbf{G}/F$ be the connected component of the isometry group associated to the pair (W, q). The group $\mathbf{G}$ is an F-split semisimple group. Any standard basis B gives the following isomorphism:

$$\mathbf{G} \simeq \begin{cases} \mathbf{SO}_{2n}/F & \text{if } \epsilon = 1, \text{ and } N = 2n, \\ \mathbf{SO}_{2n+1}/F & \text{if } \epsilon = 1 \text{ and } N = 2n+1, \\ \mathbf{Sp}_{2n}/F & \text{if } \epsilon = -1. \end{cases} \tag{6}$$

Given any maximal split torus $\mathbf{T}$ (defined over F) of $\mathbf{G}$, there exists a standard basis $B = (w_i : -n \le i \le n)$ of W such that T is the G-stabilizer of the decomposition

$$W = Fw_n \oplus Fw_{n-1} \oplus \cdots \oplus Fw_{-n+1} \oplus Fw_{-n}.$$

Conversely, any standard basis B gives rise to a maximal split torus $\mathbf{T}$ in $\mathbf{G}$ such that T is the G-stabilizer of the decomposition as above. We say that the torus $\mathbf{T}$ is associated to the standard basis B.

A *lattice chain* Λ is a function from $\mathbb{Z}$ to the set of lattices in W which satisfies the following conditions:

(1) $\Lambda(j) \subsetneq \Lambda(i)$, for $i < j$, and
(2) there exists an integer $e(\Lambda)$ such that $\Lambda(i + e(\Lambda)) = \mathfrak{p}_F\Lambda(i)$, for all $i \in \mathbb{Z}$.

Given any lattice $\mathcal{L}$, let $\mathcal{L}^{\#}$ be the lattice

$$\mathcal{L}^{\#} := \{w \in W \mid q(v, \mathcal{L}) \subset \mathfrak{p}_F\}.$$

Let $\Lambda^{\#}$ be the lattice chain defined by setting

$$\Lambda^{\#}(i) = \Lambda(-i)^{\#}, \text{ for all } i \in \mathbb{Z}.$$

A lattice chain Λ is called *self-dual* if there exists $d \in \mathbb{Z}$ such that $\Lambda^{\#}(i) = \Lambda(i+d)$, for all $i \in \mathbb{Z}$. For any integer i, let $a_i(\Lambda)$ be the set defined by

$$a_i(\Lambda) := \{T \in \mathrm{End}_F(W) \mid T\Lambda(j) \subset \Lambda(j+i) \ \forall\, j \in \mathbb{Z}\}.$$

Let $U_0(\Lambda)$ be the set of units in $a_0(\Lambda)$. Let $U_i(\Lambda)$ be the group $\mathrm{id}_V + a_i(\Lambda)$, for any $i > 0$. Given any self-dual lattice chain $\mathcal{L}$, there exists a standard basis B, called *a splitting* of Λ, such that for any $i \in \mathbb{Z}$:

$$\Lambda(i) = \mathfrak{p}_F^{a_n+i} w_n \oplus \mathfrak{p}_F^{a_{(n-1)}+i} w_{n-1} \oplus \cdots \oplus \mathfrak{p}_F^{a_{(-n+1)}+i} w_{-n+1} \oplus \mathfrak{p}_F^{a_{(-n)}+i} w_{-n}. \tag{7}$$

Given any hyperspecial maximal compact subgroup K of G, there exists a self-dual lattice chain Λ such that K is equal to $G \cap U_0(\Lambda)$. Note that $e(\Lambda) = 1$. Let $K(m)$ be the group $U_m(\Lambda) \cap G$, for $m \geq 1$. The group $K(m)$ is the principal congruence subgroup of level m. The group $K(m)$ is a normal subgroup of K, for $m \geq 1$. Let B be a standard basis such that B is a splitting of Λ. Let $\mathbf{T}$ be the maximal split torus of $\mathbf{G}$ associated to the standard basis B. The group $K \cap T$ is the maximal compact subgroup of T. Let $\mathcal{L}$ be the lattice

$$\mathcal{L} := \Lambda(0) = \mathfrak{p}_F^{a_n} w_n \oplus \mathfrak{p}_F^{a_{n-1}} w_{n-1} \oplus \cdots \oplus \mathfrak{p}_F^{a_{-n+1}} w_{-n+1} \oplus \mathfrak{p}_F^{a_{-n}} w_{-n}. \tag{8}$$

The lattice $\mathcal{L}$ is determined by the set of integers $\{a_i : -n \leq i \leq n\}$. Let L_0 be the ideal generated by the set $\{q(w_1, w_2) : w_1, w_2 \in \mathcal{L}\}$ in $\mathfrak{o}_F$. Let $\bar{q}$ be the following bilinear form:

$$\bar{q} : \frac{\mathcal{L}}{\mathfrak{p}_F\mathcal{L}} \times \frac{\mathcal{L}}{\mathfrak{p}_F\mathcal{L}} \to \frac{L_0}{\mathfrak{p}_F L_0}, \quad q(w_1, w_2) \mapsto \overline{q(w_1, w_2)} \ \forall\, w_1, w_2 \in W,$$

where $\overline{q(w_1, w_2)}$ is the image of $q(w_1, w_2)$ in $L_0/\mathfrak{p}_F L_0$. Since K is hyperspecial, the form $\bar{q}$ is nondegenerate (see [Tit79, 3.8.1]). We refer to the article [Lem09, Section 1.6] for these results.

Let $\mathbf{T}$ be any maximal split torus of $\mathbf{G}$, defined over F, such that $K \cap T$ is the maximal compact subgroup of T. Let B be the standard basis of W associated to the torus $\mathbf{T}$. There exists a self-dual lattice chain Λ such that B is a splitting of Λ and K is equal to $U_0(\Lambda) \cap G$.

Until the end of Section 5, we fix a hyperspecial maximal compact subgroup K of G. We fix a self-dual lattice chain Λ defining K. We fix a standard basis

$$B = (w_i : -n \leq i \leq n) \tag{9}$$

such that B is a splitting of Λ. We fix the set of integers $\{a_i : -n \leq i \leq n\}$ as in (8). We have a canonical homomorphism

$$\pi_1 : K \to K/K(1) \simeq G(\mathcal{L} \otimes k_F, \bar{q}). \tag{10}$$

Let I be a sequence of positive integers

$$n \geq n_1 \geq n_2 \geq \cdots \geq n_r \geq 1. \tag{11}$$

Consider the sets

$$S_i^{\pm} := \{w_{\pm n}, w_{\pm(n-1)}, \ldots, w_{\pm(n_i)}\},$$

for $1 \leq i \leq r$. Let $W_i^{\pm}$ be the subspace of W spanned by the set $S_i^{\pm}$. We denote by $V_i^{\pm}$ the space spanned by the set $S_{i+1}^{\pm} \backslash S_i^{\pm}$, for $i \leq r$. Let V_{r+1} be the space $(W_r^+ \oplus W_r^-)^{\perp}$. Let $\mathcal{F}_I$ be the flag

$$W_1^+ \subset W_2^+ \subset \cdots \subset W_r^+. \tag{12}$$

Let P_I be the G-stabiliser of the flag $\mathcal{F}_I$. Let M_I be the G-stabiliser of the decomposition

$$V_1^+ \oplus \cdots \oplus V_r^+ \oplus V_{r+1} \oplus V_r^- \oplus \cdots \oplus V_1^-.$$

The group P_I is the group of F-points of an F-parabolic subgroup of $\mathbf{G}$. Let U_I be the unipotent radical of P_I. We have $P_I = M_I \ltimes U_I$. We denote by $\bar{U}_I$ the unipotent radical of the opposite parabolic subgroup of P_I with respect to the group M_I.

Assume that G is a symplectic or special orthogonal group of odd dimension. In this case, the group of F-points of any F-parabolic subgroup of $\mathbf{G}$ is G-conjugate to P_I, for some sequence I as in (11). The subgroups P_I are called *standard parabolic subgroups*. The group M_I will be called a *standard Levi subgroup* of P_I.

Assume that G is a special orthogonal group of even dimension. In this case, there are two orbits of maximal totally isotropic subspaces of W. The representatives for these orbits are given by the spaces

$$W^+ = \quad Fw_n \oplus Fw_{n-1} \oplus \cdots \oplus Fw_1, \tag{13}$$

$$(W^+)' = \quad Fw_n \oplus Fw_{n-1} \oplus \cdots \oplus Fw_2 \oplus Fw_{-1}. \tag{14}$$

Let $\mathcal{F}_I'$ be a flag defined as in (12), except replacing w_1 with w_{-1}. Let P_I' and M_I' be parabolic subgroups, and Levi subgroups, respectively, defined similarly as above for the flag $\mathcal{F}_I'$. The group of F-points of an F-parabolic subgroup of $\mathbf{G}$ is G conjugate to at least one of the groups P_I or P_I' for some sequence $(n_1, n_2, \ldots, n_r)$ as in (11). The parabolic subgroups P_I' and P_I are called the *standard parabolic subgroups*. The Levi factors M_I and M_I', for P_I and P_I', respectively, are called the *standard Levi subgroups*.

Remark 3.1. There exist sequences I such that P_I and P_I' are G-conjugate. Hence, for even special orthogonal groups these groups P_I and P_I' are not a parametrisation. Nevertheless, any parabolic subgroup of G is conjugate to at least one such group.

Let P be a standard parabolic subgroup, and let M be a standard Levi factor of P. Let U be the unipotent radical of P, and let $\bar{U}$ be the unipotent radical of the opposite parabolic subgroup, $\bar{P}$, of P with respect to M. Let $P(m)$ be the following compact open group of G:

$$P(m) = K(m)(P \cap K).$$

Note that the group $P(1)$ is a parahoric subgroup of G. The group $P(m)$ has an Iwahori decomposition with respect to the pair (P, M). The group $K/K(1)$ can be identified with k_F-points of the connected component of the isometry subgroup associated to the pair $(\mathcal{L} \otimes_{\mathfrak{o}_F} k_F, \bar{q})$; let π_1 be the homomorphism as in (10). Let $P(k_F)$ be the image of $P(1)$ under π_1. $P(k_F)$ is a parabolic subgroup of $K/K(1)$. The group $M(k_F) = \pi_1(K \cap M)$ is a Levi factor of $P(k_F)$.

We identify M with the group

$$G_1 \times G_2 \times \cdots \times G_r \times G_{r+1},$$

where $G_i = \mathrm{GL}(V_i)$, for $1 \leq i \leq r$, and G_{r+1} is the group of F-points of the connected component of the isometry group associated to a nonsingular subspace

(V_{r+1}, q) of (W, q). Any cuspidal representation σ_M of M is isomorphic to

$$\sigma_1 \boxtimes \cdots \boxtimes \sigma_r \boxtimes \sigma_{r+1},$$

where σ_i is a cuspidal representation of G_i, for $1 \leq i \leq r+1$. Any inertial class $\mathfrak{s}$ of G is equal to $[M, \sigma_M]$.

Let K_M be the group $M \cap K$. Note that K_M is a hyperspecial maximal compact subgroup of M. Let γ_M be a cuspidal representation of $M(k_F)$. Let τ_M be a representation of K_M, obtained as the inflation of γ_M via the map

$$\pi_1 : K_M = M \cap K \to M(k_F).$$

Note that τ_M extends as a representation of $P(1)$ via inflation from the map

$$\tilde{\pi}_1 : P(1) \xrightarrow{\pi_1} P(k_F) \to M(k_F).$$

Let σ_M be a cuspidal representation of M containing the pair (K_M, τ_M).

Lemma 3.2. *Let $\mathfrak{s}$ be the inertial class $[M, \sigma_M]_G$. The pair $(P(1), \tau_M)$ is an $\mathfrak{s}$-type in the sense of Bushnell and Kutzko.*

Proof. This is essentially proved in [Mor99, Theorem 4.9]. However, we have to show that the group $P(1)$ coincides with the full normaliser of the facet corresponding to the parahoric subgroup $P(1)$, which is denoted by $\hat{P}$ in [Mor99]. First, we have $P(1) \subseteq \hat{P}$. From the Iwahori decomposition of $\hat{P}$ with respect to (P, M), we get that

$$\hat{P} = (\hat{P} \cap U)(\hat{P} \cap M)(\hat{P} \cap \bar{U}).$$

Since the groups $\hat{P} \cap U$ and $\hat{P} \cap \bar{U}$ are pro-p groups, they are contained in $P(1)$. Since $K_M = P(1) \cap M$ is a hyperspecial maximal compact subgroup, the group $P(1) \cap M$ is equal to $\hat{P} \cap M$. This shows that $\hat{P} = P(1)$. □

In this article, we classify the $[M, \sigma_M]_G$-typical representation of K. In particular, we show that the $[M, \sigma_M]_G$-typical representations of K are exactly the subrepresentations of $\mathrm{ind}_{P(1)}^{K} \tau_M$.

4. The first reduction

We begin with a few preliminary results. We will make a mild modification to the uniqueness result of typical representations proved for depth-zero inertial classes of $\mathrm{GL}_n(F)$. The following lemmas are essentially proved by Paškūnas in [Pas05] but are not stated in the form we need.

Lemma 4.1. *Let G be the group of k_F-points of a connected reductive group over k_F. Let H be a subgroup of G. Assume that there exists a proper parabolic subgroup P of G, with unipotent radical U such that $H \cap U = \{\mathrm{id}\}$. Let τ be an irreducible representation of G. For any irreducible subrepresentation ξ of $\mathrm{res}_H \tau$, there exists an irreducible noncuspidal G-representation τ' such that ξ occurs as a subrepresentation of $\mathrm{res}_H \tau'$.*

Proof. Using Mackey decomposition, we observe that the space

$$\mathrm{Hom}_U(\mathrm{ind}_H^G \xi, \mathrm{id})$$

is nontrivial. Therefore, there exists an irreducible noncuspidal G-subrepresentation τ' of $\mathrm{ind}_H^G \xi$. Frobenius reciprocity implies that ξ occurs in the irreducible noncuspidal representation τ' of G. □

For simplicity until the end of Lemmas 4.2 and 4.3, we denote the group $\mathrm{GL}_n(F)$ by G_n and the group $\mathrm{GL}_n(\mathfrak{o}_F)$ by K_n.

Lemma 4.2. *Let $n > 1$, and let $\mathfrak{s} = [G_n, \sigma]_{G_n}$ be a depth-zero inertial class. The representation $\mathrm{res}_{K_n} \sigma$ admits a decomposition:*

$$\mathrm{res}_{K_n} \sigma = \tau \oplus \tau'$$

such that τ is an $\mathfrak{s}$-typical representation of K_n, and any irreducible K_n-subrepresentation ξ of τ' occurs in $\mathrm{res}_{K_n} \pi_\xi$ for some irreducible noncuspidal representation π_ξ of G.

Proof. The representation σ is an unramified twist of the representation $\mathrm{ind}_{F^\times K_n}^{G_n} \tau$, where τ is a representation of $F^\times K_n$ such that: $\mathrm{res}_{K_n} \tau$ is obtained by inflation of a cuspidal representation of $\mathrm{GL}_n(k_F)$, and ϖ_F acts trivially on τ. Using Cartan decomposition for the group G_n, the representatives for the double cosets $F^\times K_n \backslash G_n / K_n$ are given by the elements of the form $\mathrm{diag}(\varpi_F^{i_1}, \dots, \varpi_F^{i_n})$, where $i_1 \geq \dots \geq i_n \geq 0$. Now

$$\mathrm{res}_{K_n} \sigma \cong \bigoplus_{t \in K_n \backslash \mathrm{GL}_n(F)/K_n} \mathrm{ind}_{K_n \cap t K_n t^{-1}}^{K_n} \tau.$$

Assume $t \neq \mathrm{id}$. Let H be the image of the group $K_n \cap t K_n t^{-1}$ under the reduction map $\pi_1 : K_n \to \mathrm{GL}_n(k_F)$. The group H is contained in a proper parabolic subgroup Q of $\mathrm{GL}_n(k_F)$.

Let U be the unipotent radical of an opposite parabolic subgroup of Q. Note that $H \cap U$ is the trivial group. Let ξ be an irreducible H-subrepresentation of τ. Using Lemma 4.1, we get that ξ occurs as a subrepresentation of $\mathrm{res}_H \gamma$, where γ is a noncuspidal irreducible representation of $\mathrm{GL}_n(k_F)$. This implies that any irreducible subrepresentation of $\mathrm{res}_{K_n \cap t K_n t^{-1}} \tau$ occurs as a subrepresentation of $\mathrm{res}_{K_n \cap t K_n t^{-1}} \tau'$ where τ' is the inflation of γ. This shows that any K_n-irreducible subrepresentation of $\mathrm{ind}_{K_n \cap t K_n t^{-1}}^{K_n} \tau$ occurs in $\mathrm{ind}_{K_n \cap t K_n t^{-1}}^{K_n} \tau'$ for some τ' as above.

The representation $\mathrm{ind}_{K_n \cap t K_n t^{-1}}^{K_n} \tau'$ is a subrepresentation of $\mathrm{res}_{K_n} \mathrm{ind}_{K_n}^{G} \tau'$. Let $Q(1)$ be a subgroup of K_n, obtained as the inverse image of Q via the map $\pi_1 : K_n \to \mathrm{GL}_n(k_F)$. Let N be a Levi factor of Q. The representation γ is a subrepresentation of $i_Q^{\mathrm{GL}_n(k_F)} \gamma_N$, where γ_N is a cuspidal representation of N. Let τ_N be the representation of $Q(1)$ obtained by inflation of γ_N via the map $\pi_1 : Q(1) \to Q$. The representation $\mathrm{ind}_{K_n}^{G} \tau'$ is a subrepresentation of $\mathrm{ind}_{Q(1)}^{G} \tau_N$. Any irreducible G-subquotient of $\mathrm{ind}_{Q(1)}^{G} \tau_N$ is a noncuspidal representation (see [BK93a, chapter 8]). This shows that irreducible subrepresentations of $\mathrm{ind}_{K_n \cap t K_n t^{-1}}^{K_n} \tau'$ occur in the restriction to K_n of a noncuspidal representation of G. □

Lemma 4.3. *Let $\mathfrak{s} = [M, \sigma]_{G_n}$ be a depth-zero noncuspidal inertial class. Let P be a parabolic subgroup of G such that M is a Levi factor of P. The representation $\mathrm{res}_{K_n} i_P^{G_n} \sigma$ admits a decomposition*

$$\mathrm{res}_{K_n} i_P^{G_n} \sigma = \tau \oplus \tau'$$

such that any irreducible K_n-subrepresentation of τ is $\mathfrak{s}$-typical, and any irreducible K_n-subrepresentation of τ' is atypical. Moreover, any irreducible K_n-subrepresentation of τ' occurs as a subrepresentation of $\mathrm{res}_{K_n} i_R^{G_n} \sigma_1$ such that P and R are not associate parabolic subgroups.

Proof. The first part of the lemma is proved in [Nad17, Theorem 3.2]. The last assertion follows from the proof of the result [Nad17, Theorem 3.2]. Note that there are no assumptions on q_F in the proof of this lemma. □

Let K be any hyperspecial maximal compact subgroup of G. We need the uniqueness of $\mathfrak{s}$-typical representations of K for the inertial class $[G,\sigma]$, where σ contains a depth-zero type of the form (K,λ). We only give a sketch of the following standard lemma for the completeness of the exposition. This result is generalised by Latham for arbitrary maximal compact subgroups and depth-zero cuspidal Bernstein components of a wide class of reductive groups G (see [Lat17]).

Lemma 4.4. *The K-representation λ is the unique $[G,\sigma]_G$-typical representation contained in σ.*

Proof. The representation σ is isomorphic to $\operatorname{ind}_K^G \lambda$. Now

$$\operatorname{res}_K \operatorname{ind}_K^G \lambda \simeq \bigoplus_{g\in K\backslash G/K} \operatorname{ind}_{K^g\cap K}^K \lambda^g.$$

Assume that $g \notin K$. Observe that the Cartan decomposition for $K\backslash G/K$ gives a representative $t \in KgK$ such that $K^{t^{-1}} \cap K \subset P(1)$ for some proper standard parabolic subgroup P of G. Using Lemma 4.1, we get that any irreducible subrepresentation ξ of

$$\operatorname{res}_{K^{t^{-1}}\cap K} \lambda$$

occurs as a subrepresentation of $\operatorname{res}_{K^{t^{-1}}\cap K} \operatorname{ind}_{R(1)}^K \tau'$, where τ' is the inflation of a cuspidal representation γ of $L(k_F)$, the standard Levi factor of $R(k_F)$, via the map

$$R(1) \to R(k_F) \to L(k_F).$$

Hence, any irreducible representation of $\operatorname{ind}_{K^g\cap K}^K \lambda^g$ occurs as a subrepresentation of

$$\operatorname{res}_K \operatorname{ind}_{R(1)}^G \tau'.$$

The pair $(R(1),\tau')$ is a type for the Bernstein component $[L,\sigma_L]$, where σ_L is any cuspidal representation of L containing the type $(K\cap L,\tau')$. Now any irreducible G-subquotients of $\operatorname{ind}_{R(1)}^G \tau'$ are noncuspidal. Hence the irreducible subrepresentations of $\operatorname{ind}_{K^g\cap K}^K \lambda^g$ are atypical. □

Consider a standard parabolic subgroup P with the standard Levi factor M isomorphic to

$$G_1 \times G_2 \times \cdots \times G_{r+1},$$

where G_i is the group of F-points of a general linear group over F, for $i \leq r$, and G_{r+1} is the group of F-points of the connected component of the isometry subgroup of a nonsingular subspace (W',q) of (W,q). The factor G_{r+1} is assumed to be trivial if M is contained in a maximal parabolic subgroup fixing a maximal totally isotropic flag. Let $\mathfrak{t}_i = [M_i,\sigma_i]_{G_i}$ be an inertial class of G_i, for $i \leq r$, and let $\mathfrak{t}_{r+1} = [G_{r+1},\sigma_{r+1}]$ be a cuspidal inertial class of G_{r+1}.

We assume that $\mathfrak{t}_i$ is a depth-zero inertial class of G_i for $1 \leq i \leq r$. We assume that σ_{r+1} contains a depth-zero type $(K\cap G_{r+1},\lambda)$. Let P_i be an F-parabolic subgroup of G_i with M_i as a Levi factor, and let

$$\operatorname{res}_{K\cap G_i} i_{P_i}^{G_i}\sigma_i = \tau_i \oplus \tau_i' \tag{15}$$

such that: any $K \cap G_i$-irreducible subrepresentation of τ_i' is atypical, $\tau_i \neq 0$, and any $K \cap G_i$-subrepresentation of τ_i is $\mathfrak{t}_i$-typical. Such a decomposition is possible by Lemmas 4.2 and 4.3 for $i \leq r$, and for G_{r+1} from Lemma 4.4.

Let $\mathfrak{s}$ be the inertial class $[L, \sigma_L]_G$, where $L \subset M$ is a standard Levi factor of a standard parabolic subgroup such that

$$L \simeq M_1 \times \cdots \times M_r \times G_{r+1},$$

and σ_L is isomorphic to $\sigma_1 \boxtimes \cdots \boxtimes \sigma_r \boxtimes \sigma_{r+1}$. We denote by τ_M the $K \cap M$-representation

$$\tau_1 \boxtimes \tau_2 \boxtimes \cdots \boxtimes \tau_{r+1}.$$

Let R be a standard parabolic subgroup such that L is the standard Levi factor of R. Let τ_M' be the representation $\mathrm{ind}_{R\cap M}^{M} \sigma_L/\tau_M$. With this notation, we have the following preliminary classification of $\mathfrak{s}$-typical representations of K.

Lemma 4.5. *Let $\mathfrak{s}$ be the inertial class $[L, \sigma_L]_G$. Any $\mathfrak{s}$-typical representation τ of K occurs as a subrepresentation of* $\mathrm{ind}_{K\cap P}^{K} \tau_M$.

Proof. The representation $\mathrm{ind}_K^G \tau$ is finitely generated and hence has an irreducible quotient π. From Frobenius reciprocity, the representation π occurs as a subquotient of $i_R^G(\sigma_L \otimes \chi)$, where R is a standard parabolic subgroup G with Levi factor L, and χ is some unramified character of L.

Let $\tilde{\sigma}_M$ be the representation $i_{R\cap M}^{M}\sigma_L$. Then τ occurs as a subrepresentation of $\mathrm{res}_K\, i_R^G \sigma_L$, and we have the restriction

$$\mathrm{res}_K\, i_R^G \sigma_L = \mathrm{ind}_{P\cap K}^{K}(\mathrm{res}_{K\cap M}\, \tilde{\sigma}_M) = \mathrm{ind}_{P\cap K}^{K} \tau_M \oplus \mathrm{ind}_{P\cap K}^{K} \tau_M'.$$

The Levi subgroup M is isomorphic to $G_1 \times G_2 \times \cdots \times G_r \times G_{r+1}$. We identify $\tilde{\sigma}_M$ with the representation $\tilde{\sigma}_1 \boxtimes \tilde{\sigma}_2 \boxtimes \cdots \boxtimes \tilde{\sigma}_r \boxtimes \tilde{\sigma}_{r+1}$, where $\tilde{\sigma}_i$ is the representation $i_{P_i}^{G_i}(\sigma_i \otimes \chi_i)$. Here P_i is the parabolic subgroup $R \cap G_i$ of G_i containing M_i as a Levi factor, and $\chi_i = \mathrm{res}_{M_i} \chi$ is an unramified character of M_i for all $1 \leq i \leq r+1$.

Let

$$\mathrm{res}_{K\cap G_i}\, \tilde{\sigma}_i = \bigoplus_j \xi_i^j,$$

where $\xi_i^0 = \tau_i$ as defined in the decomposition of $\mathrm{res}_{K\cap G_i} \tilde{\sigma}_i$ in (15), and for $j > 0$ the representation ξ_i^j is an irreducible subrepresentation of τ_i' in (15). Now the representation τ_M is isomorphic to $\xi_1^0 \boxtimes \cdots \boxtimes \xi_r^0 \boxtimes \xi_{r+1}^0$. Similarly define the representation τ_M' as the representation

$$\bigoplus_{(i_1, i_2, \ldots, i_{r+1}) \neq 0} \xi_1^{i_1} \boxtimes \xi_2^{i_2} \boxtimes \cdots \boxtimes \xi_{r+1}^{i_{r+1}}.$$

We denote by ξ_I the summand corresponding to the tuple $I = (i_1, i_2, \ldots, i_{r+1})$. Let I be the nonzero tuple $(i_1, i_2, \ldots, i_{r+1})$, and fix $1 \leq j \leq r+1$ such that $i_j \neq 0$. Now $\xi_j^{i_j}$ is atypical and hence occurs in

$$\mathrm{res}_{K\cap G_j}\, i_{R_j'}^{G_j} \gamma_j,$$

where R_j' is a parabolic subgroup of G_j, with a Levi factor M_j', and γ_j is a cuspidal representation of M_j' such that $[M_j', \gamma_j]$ is not equal to $[M_j, \sigma_j]$.

Let L' be the Levi subgroup $M_1 \times M_2 \times \cdots \times M_{j-1} \times M'_j \times \cdots \times G_{r+1}$, and let $\sigma'_{L'}$ be the cuspidal representation $\sigma_1 \boxtimes \cdots \boxtimes \sigma_{j-1} \boxtimes \gamma_j \boxtimes \cdots \boxtimes \sigma_{r+1}$. Let R' be any parabolic subgroup such that L' is a Levi factor of R'. Note that

$$\mathrm{ind}_{K\cap P}^{K} \xi_I \subset \mathrm{res}_K \, i_{R'}^{G} \sigma'_{L'}.$$

Now the cuspidal support of $i_{R'}^{G}\sigma'_{L'}$ is given by $[L', \sigma'_{L'}]$. If $j < r+1$, then using Lemmas 4.2 and 4.3, we know that M_j and M'_j are not conjugate in G_j. This shows that L and L' are not conjugate in G. Hence the inertial class $[L', \sigma'_{L'}]$ is not equal to $[L, \sigma_L]$. Assume that $j = r+1$. In this case, Lemma 4.4 shows that L' is a proper Levi subgroup of L. Hence the pairs (L, σ_L) and $(L', \sigma'_{L'})$ represent two distinct inertial classes. This shows that any irreducible subrepresentation of $\mathrm{ind}_{K\cap P_I}^{K} \xi_I$ is atypical. □

5. Decomposition of an auxiliary representation

Let P be any standard parabolic subgroup of G. Let U be the unipotent radical of P. Let M be the standard Levi subgroup of P. Let $\bar{P}$ be the opposite parabolic subgroup of P with respect to M. Let $\bar{U}$ be the unipotent radical of $\bar{P}$. Let $\mathfrak{s} = [M, \sigma_M]$ be a depth-zero Bernstein component such that σ_M contains a type (K_M, τ_M), where τ_M is the inflation of a cuspidal representation γ_M of $M(k_F)$.

Let $m \geq 1$ be any positive integer. Recall that $P(m)$ is defined as the group $(P \cap K)K(m)$. The group $P(m)$ has Iwahori decomposition with respect to the pair (P, M). Moreover,

$$P(m) \cap M = K \cap M \text{ and } P(m) \cap U = U \cap K.$$

The representation τ_M extends as a representation of $P(m)$ via inflation from the map $\pi_1 : P(1) \to P(k_F)$ defined in (10). The groups $U \cap P(m)$ and $\bar{U} \cap P(m)$ are contained in the kernel of this inflation. Note that

$$\bigcap_{m\geq 1} P(m) = P \cap K.$$

We obtain

$$\mathrm{ind}_{K\cap P}^{K} \tau_M = \bigcup_{m\geq 1} \mathrm{ind}_{P(m)}^{K} \tau_M.$$

We will show that the irreducible subrepresentations of the quotient

$$\mathrm{ind}_{P(m+1)}^{K} \tau_M / (\mathrm{ind}_{P(m)}^{K} \tau_M)$$

are atypical.

Given any irreducible representation τ of $M(k_F)$, we consider τ first as a representation of $P(k_F)$ via inflation. Then τ is considered as a representation of $P(1)$ via inflation from the map $\pi_1 : P(1) \to P(k_F)$ in (10). There exists a standard parabolic subgroup $R \subset P$ in G, containing L as its standard Levi factor, such that: $L \subset M$, and τ is a subrepresentation of

$$\mathrm{ind}_{R(k_F)\cap M(k_F)}^{M(k_F)} \tau',$$

where τ' is a cuspidal representation of $L(k_F)$. If

$$\mathrm{Hom}_{P(1)}(\tau, \pi) \neq 0,$$

for some irreducible smooth representation π of G, then the representation τ' of $R(1)$ occurs in π. The cuspidal support of the representation π is $[L, \sigma_L]$, where

σ_L is a cuspidal representation of L containing the pair (K_L, τ'). We call the component $[L, \sigma_L]_G$ the *inertial class associated to the pair* $(P(1), \tau)$.

For the purpose of inductive arguments it is useful to introduce more classes of compact open subgroups and prove some basic properties of these groups. Let I be a sequence of integers

$$n \geq n_1 \geq \cdots \geq n_r \geq 1.$$

Let I_1 be the sequence of integers as above consisting of a single integer n_r. Let $\mathcal{F}_I$ be the flag $W_1^+ \subset \cdots \subset W_r^+$ of totally isotropic subspaces of W, as defined in (12), corresponding to I (or possibly the flag defined for (14), if G is isomorphic to special orthogonal subgroup $\mathrm{SO}_{2n}(F)$). Let P be the standard parabolic subgroup fixing the flag $\mathcal{F}_I$. Let $\mathcal{F}_{I_1}$ be the flag W_r^+ (or possibly the space $(W_r^+)'$ if G is isomorphic to $\mathrm{SO}_{2n}(F)$). The standard parabolic subgroup P_1 fixing the flag $\mathcal{F}_{I_1}$ is the maximal proper parabolic subgroup containing the parabolic subgroup P. Let M_1 be the standard Levi factor of P_1. Let U_1 be the unipotent radical of P. Let $\bar{P}_1$ be the opposite parabolic subgroup of P_1 with respect to M_1. Let $\bar{U}_1$ be the unipotent radical of P_1.

Let $1 \leq i \leq r$ be any positive integer. Let $\bar{V}_i^{\pm}$ be the subspace $\mathcal{L} \otimes k_F$ spanned by set of vectors $\{\varpi_F^{a_i} w_i \otimes 1 \mid w_i \in S_i^{\pm}\}$. Let $\bar{V}_{r+1}$ be the space $(\bar{W}_r^+ \oplus \bar{W}_r^-)^{\perp}$. Let $\bar{W}_i$ be the totally isotropic space

$$\bar{V}_1^+ \oplus \bar{V}_2^+ \oplus \cdots \oplus \bar{V}_i^+.$$

The parabolic subgroup $P(k_F)$ is the $G(\mathcal{L} \otimes k_F, \bar{q})$-stabilizer of the flag

$$\bar{W}_1^+ \subset \bar{W}_2^+ \subset \cdots \subset \bar{W}_r^+.$$

The group $M(k_F)$ is the $G(\mathcal{L} \otimes k_F, \bar{q})$-stabilizer of the decomposition

$$\bar{V}_1^+ \oplus \bar{V}_2^+ \oplus \cdots \oplus \bar{V}_r^+ \oplus \bar{V}_{r+1} \oplus \bar{V}_r^- \oplus \bar{V}_{r-1}^- \oplus \cdots \oplus \bar{V}_1^-.$$

Moreover, the group $P_1(k_F)$ is the $G(\mathcal{L} \otimes k_F, \bar{q})$-stabilizer of the space $\bar{W}_r^+$, and $M_1(k_F)$ is the $G(\mathcal{L} \otimes k_F, \bar{q})$-stabilizer of the decomposition

$$W_r^+ \oplus V_{r+1} \oplus W_r^-.$$

Let m be a positive integer. We introduce a compact open subgroup $P(1, m) \subseteq P(1)$, which helps in inductive arguments. We set

$$P(1, m) = K(m)(P(1) \cap P_1).$$

Using Iwahori decomposition of the group $K(m)$, we get that the group $P(1, m)$ admits an Iwahori decomposition with respect to the pair (P_1, M_1). Let U_1 be the unipotent radical of P_1, and let $\bar{U}_1$ be the unipotent radical of the opposite parabolic subgroup of P_1 with respect to M_1. Using the Iwahori decomposition of $P(1)$ with respect to the pair (P_1, M_1), we get that

$$P(1) = (P(1) \cap \bar{U}_1)(P(1) \cap P_1).$$

Now, the group $P(1) \cap \bar{U}$ is contained in $K(1)$. Hence, we have $P(1, 1) = P(1)$. One of the main ingredients in the classification of typical representations is the description of the induced representation

$$\mathrm{ind}_{P_1(1,m+1)}^{P_1(1,m)} \mathrm{id}.$$

Since the unipotent radical of P_1 is not necessarily abelian, it is useful to introduce another family of compact subgroups $R(m)$ such that

$$P(1, m+1) \subset R(m) \subset P(1, m).$$

With respect to the basis

$$(\varpi_F^{a_n} w_n, \varpi_F^{a_{n-1}} w_{n-1}, \dots, \varpi_F^{a_{-n+1}} w_{-n+1}, \varpi_F^{a_{-n}} w_{-n}), \tag{16}$$

we identify the group K as a subgroup of $\mathrm{GL}_N(\mathfrak{o}_F)$ and P as a subgroup of invertible upper block matrices. With this identification, let $R(m)$ be the compact open subgroup of $P(1,m)$ consisting of matrices of the form

$$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ Z & * & * & * & * \end{pmatrix},$$

where entries of the matrix Z belong to $M_{n_r\times n_r}(\mathfrak{p}_F^{m+1})$. Since $m \geq 1$, the group $R(m)$ is well defined. Let $\mathfrak{n}_1$ be the Lie algebra of $\bar{U}_1(k_F)$. Now, with respect to the basis

$$(\varpi_F^{a_n} w_n \otimes 1, \varpi_F^{a_{n-1}} w_{n-1} \otimes 1, \dots, \varpi_F^{a_{-n+1}} w_{-n+1} \otimes 1, \varpi_F^{a_{-n}} w_{-n} \otimes 1) \tag{17}$$

of $\mathcal{L} \otimes k_F$, let $\bar{\mathfrak{n}}_1^1$ and $\bar{\mathfrak{n}}_1^2$ be the space of matrices in $\mathfrak{n}_1$ of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 \\ Y & 0 & 0 & 0 & 0 \\ 0 & Y' & a' & X' & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ Z & 0 & 0 & 0 & 0 \end{pmatrix},$$

respectively, where $X, Y, (X')^{\mathrm{tr}}, (Y')^{\mathrm{tr}} \in M_{(n-n_r)\times n_r}(k_F)$, and $a, (a')^{\mathrm{tr}} \in M_{1\times n_r}(k_F)$. The space $\mathfrak{n}_1$ is equal to $\mathfrak{n}_1^1 \oplus \mathfrak{n}_1^2$. Note that for symplectic groups and even orthogonal groups, the $n+1$th rows and columns are assumed to be absent.

Now we want to decompose the representations

$$\mathrm{ind}_{R(m)}^{P(1,m)} \mathrm{id} \text{ and } \mathrm{ind}_{P(1,m+1)}^{R(m)} \mathrm{id}.$$

We first consider two normal subgroups K_1 and K_2 of $P(1,m)$ and $R(m)$, respectively, with the properties that

$$K_1 \cap R(m) \trianglelefteq K_1 \text{ and } K_2 \cap P(1,m) \trianglelefteq K_2.$$

The groups K_1 and K_2 are kernels of the quotient maps

$$P(1,m) \to M_1(k_F) \text{ and } R(m) \to M_1(k_F),$$

respectively. Since K_1 and K_2 differ from $P(1,m)$ and $R(m)$ only by their intersections with Levi group M_1, we get that

$$K_1 R(m) = P(1,m) \text{ and } K_2 P(1,m+1) = R(m).$$

Lemma 5.1. *The subgroup $K_1 \cap R(m)$ is a normal subgroup of K_1, and $K_2 \cap P(1,m+1)$ is a normal subgroup of K_2.*

Proof. The groups K_1 and K_2 satisfy Iwahori decomposition with respect to the pair (P_1, M_1). Observe that

$$K_1 \cap P_1 = (K_1 \cap R(m)) \cap P_1 \text{ and } K_2 \cap P_1 = (K_2 \cap P(1,m+1)) \cap P_1.$$

We need to check that $K_1 \cap \bar{U}_1$ normalizes $K_1 \cap R(m)$, and $K_2 \cap \bar{U}_1$ normalizes $K_2 \cap P_I(1, m+1)$. We have $M_1 \cap P(1,m)$-equivariant isomorphisms

$$\frac{K_1 \cap \bar{U}_1}{(K_1 \cap R(m)) \cap \bar{U}_1} \simeq \bar{\mathfrak{n}}_1^1$$

and

$$\frac{K_2 \cap \bar{U}_1}{(K_2 \cap P_I(1, m+1)) \cap \bar{U}_1} \simeq \bar{\mathfrak{n}}_1^2.$$

Since $K_1 \cap M_1$ (respectively, $K_2 \cap M_1$) acts trivially on $\bar{\mathfrak{n}}_1^1$ (respectively, on $\bar{\mathfrak{n}}_1^2$), we get that $u^- j (u^-)^{-1}$ belongs to $K_1 \cap R(m)$ (respectively, $K_2 \cap P(1,m)$) for all $u^- \in K_i \cap \bar{U}_1$ and $j \in K_i \cap M_1$ for $i \in \{1, 2\}$.

With this, we are left with showing that $u^- u^+ (u^-)^{-1}$ belongs to $K_1 \cap R(m)$ (respectively, $K_2 \cap P(1,m)$) for all u^- in $K_1 \cap \bar{U}_1$ (respectively, $K_2 \cap \bar{U}_1$) and u^+ in $K_1 \cap U_1$ (respectively, $K_2 \cap U_1$). **We break the verification into two cases when W_r is a maximal or nonmaximal totally isotropic subspace**. Because of dimension reasons, we consider the symplectic and even orthogonal cases first and then consider the odd orthogonal case.

For any block matrix A in $M_{m \times n}(\mathfrak{o}_F)$, let $\mathrm{val}(A)$ be the least positive integer k such that $A \in M_{m\times n}(\mathfrak{p}_F^k)$. Let t be the dimension of W_r. First, suppose W_r is a maximal totally isotropic space, i.e., $t = n$. Consider the case where G is either a symplectic or an even orthogonal group. In this case, we have $R(m) = P(1, m+1)$. Let

$$\begin{pmatrix} I_n & 0 \\ X & I_n \end{pmatrix} \in K_1 \cap \bar{U}_1 \text{ and } \begin{pmatrix} I_n & A \\ 0 & I_n \end{pmatrix} \in K_1 \cap U_1,$$

where $X \in \mathrm{M}_n(\mathfrak{p}_F^{m+1})$ and $A \in \mathrm{M}_n(\mathfrak{o}_F)$. We have

$$\begin{pmatrix} I_n & 0 \\ X & I_n \end{pmatrix} \begin{pmatrix} I_n & A \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ X & I_n \end{pmatrix}^{-1} = \begin{pmatrix} I_n - AX & A \\ -XAX & I_n + XA \end{pmatrix}.$$

The lemma in this situation follows from the observation that $XAX \in \mathrm{M}_n(\mathfrak{p}_F^{m+1})$. For odd orthogonal groups,

$$u^- = \begin{pmatrix} I_n & 0 & 0 \\ a & 1 & 0 \\ X & a' & I_n \end{pmatrix} \text{ and } u^+ = \begin{pmatrix} I_n & b & Y \\ 0 & 1 & b' \\ 0 & 0 & I_n \end{pmatrix},$$

where a' and b' are uniquely determined by a and b, respectively. Now, the matrix $u^- u^+ (u^-)^{-1}$ in its block matrix form as above is equal to

$$\begin{pmatrix} * & * & * \\ a_1 & * & * \\ X_1 & a_1' & * \end{pmatrix},$$

where

$$\begin{aligned} a_1 &= -aba - (ay + b')(X + a'a), \\ X_1 &= X - (Xb + a')a - (XY + a'b' + 1)(X + aa'), \\ a_1' &= Xb - (XY + a'b')a'. \end{aligned}$$

Clearly, $\mathrm{val}(a_1)$, $\mathrm{val}(a_1')$, and $\mathrm{val}(X_1)$ are greater than or equal to $m+1$. This shows that $u^- u^+ (u^-)^{-1} \in K_1 \cap R(m)$ for similar reasons.

Now assume that W_r is a nonmaximal totally isotropic subspace of W, i.e., $t < n$. We first consider the symplectic or even orthogonal case. Let

$$u^- = \begin{pmatrix} I_t & 0 & 0 & 0 \\ A & I_{n-t} & 0 & 0 \\ B & 0 & I_{n-t} & 0 \\ C & B' & A' & I_t \end{pmatrix} \in K_i \cap \bar{U}_1 \text{ and } u^+ = \begin{pmatrix} I_t & X & Y & Z \\ 0 & I_{n-t} & 0 & Y' \\ 0 & 0 & I_{n-t} & X' \\ 0 & 0 & 0 & I_t \end{pmatrix} \in K_i \cap U_1,$$

for $i = 1, 2$. Hence $\mathrm{val}_F\{A, B, C\} \geq m$. Here again, A', B', X', and Y' are uniquely determined by A, B, X, and Y, respectively. The matrix $u^- u^+ (u^-)^{-1}$ looks like

$$u^- u^+ (u^-)^{-1} = \begin{pmatrix} * & * & * & * \\ P & * & * & * \\ Q & * & * & * \\ R & Q' & P' & * \end{pmatrix},$$

where

$$\begin{aligned} P &= -AXA - AYB - AZC - Y'C, \\ Q &= -BXA - BYB - BZC - X'C, \\ R &= -CXA - B'A - CYB - A'B - CZC - B'Y'C - A'X'C. \end{aligned} \tag{18}$$

Since $\mathrm{val}_F(R) \geq m + 1$, it follows that $K_1 \cap R(m)$ is normal in K_1. The remaining case, i.e., $K_2 \cap P(m+1)$ is normal in K_2, is similar. Indeed, in this case $\mathrm{val}_F\{A, B\} \geq m$ and $\mathrm{val}_F(C) \geq m + 1$. Hence normality follows from the fact that $\mathrm{val}_F\{P, Q\} \geq m + 1$.

Now finally we consider the odd orthogonal case. We have

$$u^- = \begin{pmatrix} I_t & 0 & 0 & 0 & 0 \\ A & I_{n-t} & 0 & 0 & 0 \\ x & 0 & 1 & 0 & 0 \\ B & 0 & 0 & I_{n-t} & 0 \\ C & B' & x' & A' & I_t \end{pmatrix} \text{ and } u^+ = \begin{pmatrix} I_t & X & a & Y & Z \\ 0 & I_{n-t} & 0 & 0 & Y' \\ 0 & 0 & 1 & 0 & a' \\ 0 & 0 & 0 & I_{n-t} & X' \\ 0 & 0 & 0 & 0 & I_t \end{pmatrix},$$

where $x \in \mathrm{M}_{1,t}(\mathfrak{p}_F^{m+1})$. Let A_1 denote the matrix $\left(\begin{smallmatrix} A \\ x \end{smallmatrix}\right) \in M_{n-t+1,t}(\mathfrak{p}_F^{m+1})$. Similarly, we define the matrix X_1 to be $X_1 = (X\ a) \in M_{t,n-t+1}(\mathfrak{o}_F)$. After redefining B' and Y' appropriately, we get

$$u^- = \begin{pmatrix} I_t & 0 & 0 & 0 \\ A_1 & I_{n-t+1} & 0 & 0 \\ B & 0 & I_{n-t} & 0 \\ C & B' & A' & I_t \end{pmatrix} \text{ and } u^+ = \begin{pmatrix} I_t & X_1 & Y & Z \\ 0 & I_{n-t+1} & 0 & Y' \\ 0 & 0 & I_{n-t} & X' \\ 0 & 0 & 0 & I_t \end{pmatrix}.$$

Now the normality follows from calculations similar to (18). □

Using Mackey decomposition and the fact that the quotients

$$K_1/(K_1 \cap R(m)) \text{ and } K_2/(K_2 \cap P(1, m+1))$$

are abelian, we have

$$\mathrm{res}_{K_1} \mathrm{ind}_{R(m)}^{P(1,m)} \mathrm{id} = \bigoplus_{\Lambda_1} \eta \text{ and } \mathrm{res}_{K_2} \mathrm{ind}_{P(1,m+1)}^{R(m)} \mathrm{id} = \bigoplus_{\Lambda_2} \eta,$$

where Λ_1 and Λ_2 are characters on the quotients $K_1/(K_1 \cap R(m))$ and $K_2/(K_2 \cap P(1, m+1))$, respectively. The groups $P(1, m)$ and $R(m)$ act on Λ_1 and Λ_2, respectively. We denote by Λ_1' and Λ_2' for a set of representatives for the action of

$P(1,m)$ and $R(m)$, respectively. Now using Clifford theory, we obtain

$$\operatorname{ind}_{R(m)}^{P(1,m)} \operatorname{id} \simeq \bigoplus_{\eta\in\Lambda_1'} \operatorname{ind}_{Z_{P(1,m)}(\eta)}^{P(1,m)} U_\eta \tag{19}$$

and

$$\operatorname{ind}_{P(m+1)}^{R(m)} \operatorname{id} \simeq \bigoplus_{\eta\in\Lambda_2'} \operatorname{ind}_{Z_{R(m)}(\eta)}^{R_m} U'_\eta, \tag{20}$$

where U_η and U'_η are some irreducible representations of $Z_{P(1,m)}(\eta)$ and $Z_{R(m)}(\eta)$, respectively. The precise description of U_η is not used in any argument.

It is crucial to understand the images of the groups $Z_{P(1,m)}(\eta)$ and $Z_{R(m)}(\eta)$ in the quotient $K/K(1)$. This is achieved in Lemma 5.4, and we begin with some preparations. We first note that the Iwahori decomposition gives us

$$Z_{P(1,m)}(\eta) = Z_{P(1,m)\cap M_1}(\eta)K_1$$

and

$$Z_{R(m)}(\eta) = Z_{R(m)\cap M_1}(\eta)K_2.$$

We have the following isomorphisms:

$$K_1/(K_1\cap R(m)) \cong \bar{\mathfrak{n}}_1^1$$

and

$$K_2/(K_2\cap P(1,m+1)) \cong \bar{\mathfrak{n}}_1^2,$$

respectively. The k_F-dual of the space $\bar{\mathfrak{n}}_1^i$ is isomorphic to $\bar{\mathfrak{n}}_1^i$ for $i\in\{1,2\}$ in a $M_1(k_F)$-equivariant way. This is because the representation of $M_1(k_F)$ on $\bar{\mathfrak{n}}_1^i$ is a self-dual for $i\in\{1,2\}$. Note that $P(1,m)\cap M_1 = R(m)\cap M_1$. Observe that the action of the groups $P(1,m)\cap M_1$ and $R(m)\cap M_1$ on the characters in Λ_1 and Λ_2 factors through the quotient map

$$\pi_1 : K\cap M_1 \to M_1(k_F). \tag{21}$$

We identify the group $M_1(k_F)$ with

$$\mathrm{GL}(\bar{W}_r^+)\times G(\bar{V}_{r+1}^+\oplus\bar{V}_{r+1}^-), \tag{22}$$

where $G(\bar{V}_{r+1}^+\oplus\bar{V}_{r+1}^-)$ is the group of k_F-points of the connected component of the isometry group of the pair $(\bar{V}_{r+1}^+\oplus\bar{V}_{r+1}^-,\bar{q})$. The image of $P(1,m)\cap M_1$ under the map (21) is contained in a group of the form

$$Q\times G(\bar{V}_{r+1}^+\oplus\bar{V}_{r+1}^-), \tag{23}$$

where Q is the parabolic subgroup of $\mathrm{GL}(\bar{W}_r^+)$ fixing the flag $\bar{W}_1^+\subset\cdots\subset\bar{W}_r^+$.

With the above observation, it is useful to recall the stabilisers in the case of general linear groups (see [Nad17, Lemma 3.8]). Let $r>1$ be an integer, and let $I=(n_1,n_2,\ldots,n_r)$ be a partition of n. We denote by P_I the parabolic subgroup of upper block diagonal matrices of size $n_i\times n_j$. The partition $(n_1,n_2,\ldots,n_{r-1})$ is denoted by J. Let $\mathcal{O}_A$ be an orbit for the action of $P_J(k_F)\times\mathrm{GL}_{n_r}(k_F)$ on the set of matrices $M_{(n-n_r)\times n_r}(k_F)$ given by

$$(g_1,g_2)X = g_1Xg_2^{-1}\ \forall\ g_1\in P_J(k_F),\ g_2\in\mathrm{GL}_{n_r}(k_F),\ X\in M_{(n-n_r)\times n_r}(k_F).$$

Let p_j be the composition of the quotient map $P_J(k_F)\times\mathrm{GL}_{n_r}(k_F)\to M_I(k_F)$ and the projection onto the jth factor of $M_I(k_F)=\prod_{i=1}^r\mathrm{GL}_{n_i}(k_F)$, i.e.,

$$p_j : P_J(k_F)\times\mathrm{GL}_{n_r}(k_F)\to\mathrm{GL}_{n_j}(k_F).$$

Lemma 5.2. *Let $\mathcal{O}_A$ be an orbit consisting of nonzero matrices in $M_{(n-n_r)\times n_r}(k_F)$. We can choose a representative A such that the $P_J(k_F) \times \mathrm{GL}_{n_r}(k_F)$-stabiliser $Z_{P_J(k_F)\times \mathrm{GL}_{n_r}(k_F)}(A)$ of A satisfies one of the following conditions:*

(1) *There exists a positive integer j with $j \leq r$ such that the image of*

$$p_j : Z_{P_J(k_F)\times \mathrm{GL}_{n_r}(k_F)}(A) \to \mathrm{GL}_{n_j}(k_F)$$

is contained in a proper parabolic subgroup of $\mathrm{GL}_{n_j}(k_F)$.

(2) *There exists a positive integer i with $1 \leq i \leq r-1$ such that $p_i(g) = p_r(g)$, for all g in*

$$Z_{P_J(k_F)\times \mathrm{GL}_{n_r}(k_F)}(A).$$

Now let us note a small observation which will be useful in the proof of Lemma 5.4.

Lemma 5.3. *Let G be a split reductive group with an automorphism θ. There exists a parabolic subgroup of $G \times G$ with unipotent radical U such that $\{(g, \theta(g))|g \in G\}$ has trivial intersection with U.*

Proof. Let P be any proper parabolic subgroup of G, and let $\bar{P}$ be any opposite parabolic subgroup of P. The unipotent radical of $P \times \bar{P}$ has trivial intersection with the diagonal subgroup of $G \times G$. The group $\{(g, \theta(g))|g \in G\}$ is the image by the automorphism $\mathrm{id} \times \theta$ of the diagonal subgroup of $G \times G$, and hence the lemma follows. □

The following is the technical heart of this article. **Here we use the condition that** $q_F > 5$. Let $\tilde{H}$ be the image of $P(1, m) \cap M_1$ under the map π_1 in (21). This is contained in the group $Q \times G(\bar{V}^+_{r+1} \oplus \bar{V}^-_{r+1})$ as in (23). Hence the lemma is based on the $Q \times G(\bar{V}^+_{r+1} \oplus \bar{V}^-_{r+1})$-stabilisers (which contain $\tilde{H}$-stabilisers) of nontrivial elements in $\bar{\mathfrak{n}}^1_1$ and $\bar{\mathfrak{n}}^2_1$. There are several cases to consider, primarily depending on whether or not the subspace $\bar{W}^+_r$ of the flag $\bar{W}^+_1 \subset \cdots \subset \bar{W}^+_r$ is maximal. Let θ be the quotient map

$$\theta : Q \times G(\bar{V}^+_{r+1} \oplus \bar{V}^-_{r+1}) \to M(k_F).$$

Lemma 5.4. *Let u be any nontrivial element of $\bar{\mathfrak{n}}^1_1$ or $\bar{\mathfrak{n}}^2_1$, and let H be the image of $Z_{\tilde{H}}(u)$ under the map θ. Let τ be a cuspidal representation of $M(k_F)$, and let ξ be an irreducible subrepresentation of $\mathrm{res}_H\, \tau$. There exists an irreducible representation τ' of $M(k_F)$ such that ξ occurs in the restriction $\mathrm{res}_H\, \tau'$, and the inertial classes associated to the pairs $(P(1), \tau)$ and $(P(1), \tau')$ are distinct.*

Proof. We will show that there exists a parabolic subgroup S of $M(k_F)$ such that $\mathrm{Rad}(S) \cap H$ is trivial. Using Lemma 4.1 we get a noncuspidal irreducible $M(k_F)$-representation τ' such that ξ occurs in $\mathrm{res}_H\, \tau$. The inertial classes associated to the pairs $(P_I(1), \tau)$ and $(P_I(1), \tau')$ are clearly distinct.

We begin with the case where **the space W^+_r is a maximal isotropic subspace of** (W, q). In this case, P is contained in the maximal parabolic subgroup P_1 fixing the maximal isotropic subspace W^+_r of W. Recall that the standard Levi factor of P_1 is denoted by M_1. The adjoint action of $M_1(k_F) \simeq \mathrm{GL}(\bar{W}^+_r)$ on $\bar{\mathfrak{n}}_1$, the Lie algebra of the unipotent radical of $\bar{P}_1(k_F)$, is the representation of $\mathrm{GL}(\bar{W}^+_r)$ on the space of $-\epsilon$ forms on $\bar{W}^+_r$.

Let B be a $-\epsilon$ bilinear form on $\bar{W}^+_r$ corresponding to u. In this case $\tilde{H}$ is contained in Q. Let $g = (g_{kl})$ and $B = (B_{k'l'})$ be the block matrix representation of the elements g in Q and the $-\epsilon$ bilinear form B on $\bar{W}_r^+$ with respect to the decomposition $\bar{V}_1^+ \oplus \cdots \oplus \bar{V}_r^+$ of $\bar{W}_r^+$. Let p be the largest positive integer such that B_{pq} is nonzero for some $1 \leq q \leq r$. Let q be the largest positive integer such that $B_{pq} \neq 0$. For any $g \in Z_Q(B)$ we have

$$g_{pp} B_{pq} g_{qq}^T = B_{pq},$$

where B_{pq} is a bilinear form on $\bar{V}_p^+ \times \bar{V}_q^+$. Without loss of generality assume that

$$\dim \bar{V}_p^+ > \dim \bar{V}_q^+.$$

Let S be the stabiliser of the kernel of the map $\bar{V}_p^+ \to (\bar{V}_q^+)^\vee$ induced by B_{pq}. Then g_{pp} belongs to a proper parabolic subgroup $\bar{S}$ of $\mathrm{GL}(\bar{V}_p^+)$. Hence H is contained in a proper parabolic subgroup $\bar{S}$ of $M(k_F)$. The required parabolic subgroup S can be taken to be any opposite parabolic subgroup of $\bar{S}$.

Consider the case where $\dim \bar{V}_p^+$ **is equal to** $\dim \bar{V}_q^+ > 1$. If the map $\bar{V}_p^+ \to (\bar{V}_q^+)^\vee$ induced by B_{pq} has a nontrivial kernel, then g_{pp} belongs to the proper parabolic subgroup of $\mathrm{GL}(\bar{V}_p^+)$ fixing this kernel. Hence H is contained in a proper parabolic subgroup $\bar{S}$ of $M(k_F)$. Let S be an opposite parabolic subgroup of $\bar{S}$. We get that $\mathrm{Rad}(S) \cap H$ is a trivial group. We assume that the map $\bar{V}_p^+ \to (\bar{V}_q^+)^\vee$, induced by B_{pq}, is an isomorphism. Now using Lemma 5.3, we get a proper parabolic subgroup S of $M(k_F)$, with unipotent radical U, such that $H \cap U$ is trivial.

We **consider the case where** $\dim \bar{V}_p^+$ **is equal to** $\dim \bar{V}_q^+ = 1$. In this case, the group H consists of elements of the form

$$\mathrm{diag}(g_1, \ldots, g_p, \ldots, g_q, \ldots, g_r),$$

where $g_i \in \mathrm{GL}(\bar{V}_i^+)$ for $i \in \{p, q\}$ and $g_p g_q = 1$. We identify the representation τ with $\tau_1 \boxtimes \tau_2 \boxtimes \cdots \boxtimes \tau_r$, where τ_i is a cuspidal representation of $\mathrm{GL}(\bar{V}_i^+)$. Let η be a nontrivial character of $k_F^\times$, and let τ' be the representation

$$\tau_1 \boxtimes \cdots \boxtimes \tau_p \eta \boxtimes \cdots \boxtimes \tau_q \eta^{-1} \boxtimes \cdots \boxtimes \tau_r.$$

Now the Bernstein components associated to the pairs $(P_I(1), \tau)$ and $(P_I(1), \tau')$ are the same if and only if the set $\{\tau_p \eta, \tau_p^{-1} \eta^{-1}\}$ is either equal to $\{\tau_p, \tau_p^{-1}\}$ or to $\{\tau_q \eta^{-1}, \tau_p^{-1} \eta\}$. Hence, the character η belongs to the set $\{\tau_p^{-2}, \tau_p \tau_q, \tau_p \tau_q^{-1}\}$. Since $q_F > 5$, we can find a character η such that η does not belong to the set $\{\tau_p^{-2}, \tau_p \tau_q, \tau_p \tau_q^{-1}\}$. For such a choice of η the Bernstein components associated to the pairs $(P(1), \tau)$ and $(P(1), \tau')$ are distinct, and from construction $\mathrm{res}_H \tau$ is equal to $\mathrm{res}_H \tau'$.

We come to the case when $\bar{W}_r^+$ **is not a maximal isotropic subspace**. In this case, the space $\bar{V}_{r+1}$ is nonzero. The standard Levi factor M_1 of P_1 is isomorphic to

$$\mathrm{GL}(\bar{W}_r^+) \times G(\bar{V}_{r+1}).$$

Recall the notation $\bar{V}_{r+1}$ for the space $(\bar{W}_r^+ \oplus \bar{W}_r^-)^\perp$. The adjoint action of M_1 on $\mathfrak{n}_1^2$ factors through the map

$$\mathrm{GL}(\bar{W}_r) \times G(\bar{V}_{r+1}) \to \mathrm{GL}(\bar{W}_r).$$

In this case, the action of $\mathrm{GL}(\bar{W}_r)$ on $\mathfrak{n}_1^2$ is its representation on the space of $-\epsilon$ forms. This case is similar to the case where $\bar{W}_r^+$ is maximal, and the proof of the lemma, in this case, follows from the analysis in the previous case.

The action of $M_1(k_F)$ on $\mathfrak{n}_1^1 \simeq \mathrm{Hom}(\bar{W}_r^+, \bar{V}_{r+1})$ is given by

$$(g_1, g_2)X = g_1 X g_2^{-1} \ \forall \ g_1 \in \mathrm{GL}(\bar{W}_r^+), \ g_2 \in G(\bar{V}_{r+1}).$$

We have to consider the stabilisers of $Q \times G(\bar{V}_{r+1})$ on the space $\mathrm{Hom}(\bar{W}_r^+, \bar{V}_{r+1})$. Let X be a nonzero element of $\mathrm{Hom}(\bar{W}_r^+, \bar{V}_{r+1})$. We have the decomposition

$$\mathrm{Hom}(\bar{W}_r^+, \bar{V}_{r+1}) \simeq \bigoplus_{i=1}^{r} \mathrm{Hom}(\bar{V}_r^+, \bar{V}_{r+1}).$$

Now decompose X as the sum $\sum_{i=1}^r X_i$ such that X_i belongs to $\mathrm{Hom}(\bar{V}_r^+, \bar{V}_{r+1})$. Let $g = (g_{mn})$ be the block matrix form of any element in Q with respect to the decomposition

$$\bar{W}_r^+ = \bar{V}_1^+ \oplus \cdots \oplus \bar{V}_r^+.$$

Let t be the least positive integer such that X_t is nonzero. We then have

$$g_{tt} X_t g^{-1} = X_t \ \forall \ g_{tt} \in \mathrm{GL}(\bar{V}_t^+), \tilde{g} \in G(\bar{V}_{r+1}).$$

Now let R be the group $\mathrm{GL}(\bar{V}_t^+) \times G(\bar{V}_{r+1})$.

Consider the case when $\dim(\bar{V}_t^+) > \dim(\bar{V}_{r+1})$. In this case $Z_R(X_t)$ is contained in a subgroup of the form $P \times G(\bar{V}_{r+1})$, where P is a proper parabolic subgroup of $\mathrm{GL}(\bar{V}_t^+)$ (see Lemma 5.2). Hence the unipotent radical of $\bar{P} \times G(\bar{V}_{r+1})$, for any opposite parabolic subgroup $\bar{P}$ of P, has trivial intersection with $Z_R(X_t)$. This shows that there exists a unipotent radical of $M(k_F)$ which has trivial intersection with H, and hence we get the lemma.

Now assume that $\dim(\bar{V}_t^+)$ **is equal to** $\dim(\bar{V}_{r+1})$. In this case **if the rank of X_t is not equal to** $\dim(\bar{V}_t^+)$, then $Z_R(X_t)$ is contained in $P \times G(\bar{V}_{r+1}^+)$, where P is a proper parabolic subgroup of $\mathrm{GL}(\bar{V}_t^+)$. From similar arguments of the previous case we prove the lemma. **If the rank of X_t is equal to** $\dim(\bar{V}_t)$, then $Z_R(X_t)$ is contained in a group of the form

$$\{(X_t g X_t^{-1}, g); g \in G(\bar{V}_{r+1}^+)\}.$$

Consider any Borel subgroup B of $\mathrm{GL}(V_{r+1}^+)$ such that $B \cap G(\bar{V}_{r+1}^+)$ is the Borel subgroup of $G(\bar{V}_{r+1}^+)$. Let $\bar{B}$ be any opposite Borel subgroup of B. The group $\bar{B} \times B$ can be identified with a Borel subgroup of $\mathrm{GL}(\bar{V}_t^+) \times G(\bar{V}_{r+1})$. Now the unipotent radical of the Borel subgroup $X_t \bar{B} X_t^{-1} \times B$ has trivial intersection with $Z_R(X_t)$, which proves the lemma in this case.

Let (g_1, g_2) be an element of the group $Z_R(X_t)$ such that $g_1 \in \mathrm{GL}(\bar{V}_t^+)$ and $g_2 \in G(\bar{V}_{r+1})$. **We are left with the case when** $\dim(\bar{V}_t^+) < \dim(\bar{V}_{r+1})$. Let $X_t \in \mathrm{Hom}_{k_F}(\bar{V}_t^+, \bar{V}_{r+1})$ be an operator such that $\ker(X_t)$ is a nonzero subspace (since X_t is nonzero operator, $\ker(X_t)$ is not equal to $\bar{V}_r^+$). The group $Z_R(X_t)$ is contained in a group of the form $P \times G(\bar{V}_{r+1})$, where P is a parabolic subgroup of $\mathrm{GL}(\bar{V}_t^+)$ fixing $\ker(X_t)$. This shows that H is contained in a proper parabolic subgroup of $M(k_F)$. Now assume that X_t is surjective. If $\mathrm{Rad}(X_t \bar{V}_t^+)$ is a proper nonzero subspace of $(X_t \bar{V}_t^+, \bar{q})$, then for any (g_1, g_2) in $Z_R(X_t)$ the element g_2 stabilises the space $X_t \bar{V}_t^+$. This implies that g_2 stabilises the space $\mathrm{Rad}(X_t \bar{V}_t^+)$. This shows that g_2 stabilises a proper isotropic subspace and hence is contained in a proper parabolic subgroup of $G(\bar{V}_{r+1})$.

Finally, consider the case where **the space $X_t \bar{V}_t^+$ is either totally isotropic or nonsingular**. If the space $X_t \bar{V}_t^+$ is totally isotropic, then the element g_2 belongs to a proper parabolic subspace of $G(\bar{V}_{r+1})$. If $X_t \bar{V}_t^+$ is a nonsingular space, then the form $\bar{h}'$, obtained by pulling $\bar{h}$ restricted to $X_t\bar{V}_t^+$ to $\bar{V}_t^+$, is preserved by g_1. Hence g_1 belongs to $G((\bar{V}_t^+, h'))$. In both the cases we can find a proper parabolic subgroup P of $\mathrm{GL}_r(\bar{W}_r^+) \times G(\bar{V}_{r+1})$ such that $Z_R(X_t)$ has trivial intersection with $\mathrm{Rad}(P)$ and hence prove the lemma. □

6. Classification of K-typical representations

We need the following well-known lemma (see [Nad17, Lemma 2.6]). For the sake of the next lemma consider any parabolic subgroup P of a reductive group G with a Levi factor M. Let U be the unipotent radical of P. Let $\bar{U}$ be the unipotent radical of the opposite parabolic subgroup of P with respect to M. Let J_1 and J_2 be two compact open subgroups of G such that J_1 contains J_2. Suppose J_1 and J_2 both satisfy an Iwahori decomposition with respect to the pair (P, M). Assume

$$J_1 \cap U = J_2 \cap U \text{ and } J_1 \cap \bar{U} = J_2 \cap \bar{U}.$$

Let λ be an irreducible smooth representation of J_2 which admits an Iwahori decomposition, i.e., $J_2 \cap U$ and $J_2 \cap \bar{U}$ are contained in the kernel of λ.

Lemma 6.1. *The representation $\mathrm{ind}_{J_2}^{J_1}(\lambda)$ is the extension of the representation $\mathrm{ind}_{J_2\cap M}^{J_1\cap M}(\lambda)$ such that $J_1\cap U$ and $J_1\cap\bar{U}$ are contained in the kernel of the extension.*

Let us resume with the present case where G is a split classical group. Let $\mathfrak{s} = [M, \sigma_M]_G$ be an inertial class such that $M \neq G$. Let K_M be a hyperspecial maximal compact subgroup of M. Let σ_M be a cuspidal representation of M such that σ_M contains a depth-zero type of the form (K_M, τ_M). Let the hyperspecial vertex in the Bruhat–Tits building of M, corresponding to K_M, be contained in the apartment corresponding to a maximal split torus T (defined over F) of M. Such a torus T is characterised by the property that $K_M \cap T$ is the maximal compact subgroup of T (see [MP94, 2.6]).

Let K be a hyperspecial maximal compact subgroup of G such that K contains K_M. Let T be a torus defined as in the above paragraph. Now $K \cap T$ is the maximal compact subgroup of T. This shows that K is the parahoric subgroup of G associated to a hyperspecial vertex in the apartment corresponding to T. Let B be the standard basis of W associated to T. There exists a self-dual lattice chain Λ such that B is a splitting of Λ and $K = U_0(\Lambda) \cap G$.

Now the group M is K-conjugate to a standard Levi subgroup defined with respect to the basis B and a flag $\mathcal{F}_I$ as defined in (12), for some sequence of integers I as defined in (11). Hence, we may (and do) assume that M is a standard Levi subgroup corresponding to $\mathcal{F}_I$. Let P be the standard parabolic subgroup fixing the flag $\mathcal{F}_I$. The group M is a Levi factor of P. Let $P(1)$ be the group $K(1)(P\cap K)$. The representation τ_M extends as a representation of $P(1)$ such that $P(1) \cap U$ and $P(1) \cap \bar{U}$ are contained in the kernel of this extension. With this we have the following theorem.

Theorem 6.2. *Let $\mathfrak{s} = [M, \sigma_M]_G$ be an inertial class such that $M \neq G$. Assume that σ_M contains a depth-zero type of the form (K_M, τ_M), where K_M is a hyperspecial maximal compact subgroup of M. Let K be a hyperspecial maximal compact subgroup of G containing K_M. If τ is an $\mathfrak{s}$-typical representation of K, then τ is a subrepresentation of $\mathrm{ind}_{P(1)}^{K} \tau_M$.*

Proof. Let P be the G stabilizer of the flag

$$\mathcal{F}_I = W_1^+ \subset W_2^+ \subset \cdots \subset W_r^+.$$

Let P_1 be the G-stabiliser of the space W_r^+. Let $\mathcal{F}_J$ be the flag

$$W_1^+ \subset W_2^+ \subset \cdots \subset W_{r-1}^+.$$

Let P_J be the parabolic subgroup of $G(W_r^+)$ fixing the flag $\mathcal{F}_J$. Let M_J be the subgroup of $\mathrm{GL}(W_r^+)$ fixing the decomposition

$$V_1^+ \oplus V_2^+ \oplus \cdots \oplus V_r^+.$$

The group M_J is a Levi factor of the parabolic subgroup P_J. We recall that

$$M \simeq G_1 \times G_2 \times \cdots \times G_r \times G_{r+1},$$

where $G_i = \mathrm{GL}(V_i^+)$, for $1 \leq i \leq r$, and G_{r+1} is the F-point of the connected component of the isotropy subgroup of (V_{r+1}, q).

We then identify σ_M with $\sigma_1 \boxtimes \cdots \boxtimes \sigma_{r+1}$, where σ_i is a cuspidal representation of the group G_i, for all $1 \leq i \leq r+1$. Let τ_i be the unique $K \cap G_i$-typical representation occurring in the cuspidal representation σ_i, for $1 \leq i \leq r+1$. The K_M representation τ_M is isomorphic to the representation

$$\tau_1 \boxtimes \cdots \boxtimes \tau_r \boxtimes \tau_{r+1}.$$

From Lemma 4.5 we know that any irreducible K-subrepresentation of

$$i_P^G \sigma_M / \operatorname{ind}_{P\cap K}^K \tau_M$$

is atypical. Now the representation $\operatorname{ind}_{P\cap K}^K \tau_M$ is the union of the representations $\operatorname{ind}_{P(m)}^K \tau_M$ for $m \geq 1$.

Let K' be the compact open subgroup $\mathrm{GL}(W_r^+) \cap K$ of $\mathrm{GL}(W_r^+)$. Let $K'(m)$ be the principal congruence subgroup of level m contained in K. The compact group $K'(m) \cap (P_J \cap K')$ is denoted by $P_J(m)$. Let τ_J be the $K' \cap M_J$-representation

$$\tau_1 \boxtimes \tau_2 \boxtimes \cdots \boxtimes \tau_r.$$

The representation τ_J extends as a representation of $P_J(m)$ via inflation from the map

$$P_J(m) \to P_J(k_F) \to M_J(k_F).$$

From transitivity of induction and using Lemma 6.1, we see that

$$\operatorname{ind}_{P(m)}^K \tau_M \simeq \operatorname{ind}_{P_1(m)}^K \{(\operatorname{ind}_{P_J(m)}^{K'} \tau_J) \boxtimes \tau_{r+1}\}.$$

The irreducible K'-subrepresentations of $\operatorname{ind}_{P_J(m)}^{K'} \tau_J / \operatorname{ind}_{P_J(1)}^{K'} \tau_J$ are atypical from the result [Nad17, Theorem 1.1]. Hence $\mathfrak{s}$-typical representations of K can only occur as subrepresentations of

$$\operatorname{ind}_{P_1(m)}^K \{(\operatorname{ind}_{P_J(1)}^{K'} \tau_J) \boxtimes \tau'\} \simeq \operatorname{ind}_{P(1,m)}^K \tau_M.$$

Now from Lemmas 3.2 and 2.5 we get that

$$\operatorname{ind}_{P(1,m+1)}^{P(1,m)} \mathrm{id} = \mathrm{id} \oplus \bigoplus_{i=1}^{k} \operatorname{ind}_{H_i}^{P(1,m)} U_i$$

such that any irreducible subrepresentation χ of $\mathrm{res}_{H_i}\tau_I$ occurs in $\mathrm{res}_{H_i}\tau_I'$. Moreover, the Bernstein components associated to the pairs $(P_I(1),\tau_I)$ and $(P_I(1),\tau_I')$ are distinct. Note that

$$\begin{aligned}\mathrm{ind}_{P(1,m+1)}^{K}\tau_M &\simeq \mathrm{ind}_{P(1,m)}^{K}\{\mathrm{ind}_{P(1,m+1)}^{P(1,m)}\mathrm{id}\}\otimes\tau_M\\ &\simeq \mathrm{ind}_{P(1,m)}^{K}\tau_M\oplus\mathrm{ind}_{H_i}^{P(1,m)}(U_i\times\mathrm{res}_{H_i}\tau_M).\end{aligned}$$

Using induction on m, any $\mathfrak{s}$-typical representation occurs as a subrepresentation of $\mathrm{ind}_{P(1)}^{K}\tau_M$. Recall that the subgroup $P(1,1)$ is equal to $P(1)$. Since $(P(1),\tau_M)$ is a Bushnell–Kutzko type for $[M,\sigma_M]$, we complete the proof of the theorem. □

7. Principal series components

Let $\mathbf{G}$ be the split classical group defined as the connected component of the isometry group of (W,q), as in Section 3. Let K be a hyperspecial maximal compact subgroup of G. Let $\mathbf{T}$ be a maximal split torus of $\mathbf{G}$ defined over F such that $K\cap T$ is the maximal compact subgroup of T. Let

$$(w_i : -n\le i\le n) \tag{24}$$

be a standard basis associated to T. Now there exists a self-dual lattice chain Λ such that the basis (24) is a splitting of Λ and $K=U_0(\Lambda)\cap G$. Let

$$\Lambda(0)=\mathfrak{p}_F^{a_n}w_n\oplus\mathfrak{p}_F^{a_{n-1}}w_{n-1}\oplus\cdots\oplus\mathfrak{p}_F^{a_{-n+1}}w_{-n+1}\oplus\mathfrak{p}_F^{a_{-n}}w_{-n}.$$

We fix a basis

$$\{\varpi_F^{a_n}w_n,\varpi_F^{a_{n-1}}w_{n-1},\ldots,\varpi_F^{a_{-n+1}}w_{-n+1},\varpi_F^{a_{-n}}w_{-n}\}$$

of W. Now, using this basis, we get an embedding

$$\iota : G\to \mathrm{GL}_N(F) \tag{25}$$

of G in $\mathrm{GL}_N(F)$. The image of the maximal compact subgroup K can be identified with $\mathrm{GL}_N(\mathfrak{o}_F)\cap\iota(G)$. The torus T is the group of diagonal matrices of $\iota(G)$. Let $\mathbf{B}$ be the Borel subgroup of $\mathbf{G}$ such that B is a subgroup of upper triangular matrices in $\mathrm{GL}_N(F)$. We denote by $\bar{\mathbf{B}}$ the opposite Borel subgroup of $\mathbf{B}$ with respect to $\mathbf{T}$. Let $\mathbf{U}$ and $\bar{\mathbf{U}}$ be the unipotent radicals of $\mathbf{B}$ and $\bar{\mathbf{B}}$, respectively.

We identify the torus T with $(F^\times)^n$ by the map

$$\mathrm{diag}(t_1,t_2,\ldots,t_n,t_n^{-1},\ldots,t_2^{-1},t_1^{-1})\mapsto(t_1,\ldots,t_n),\ t_i\in F^\times.$$

We also identify a character χ of T with

$$\chi=\chi_1\boxtimes\cdots\boxtimes\chi_n,$$

where χ_i is a character of $F^\times$. The conductor of χ_i, denoted by $l(\chi_i)$, is the least positive integer n such that $1+\mathfrak{p}_F^n$ is contained in the kernel of χ. **In this section, we assume that**

$$l(\chi_i)\ne l(\chi_j)\ \textbf{for all}\ i\ne j.$$

Let $\mathfrak{s}$ be the inertial class $[T,\chi]$. Let τ be an $\mathfrak{s}$-typical representation of K. The representation τ occurs as a subrepresentation of an irreducible smooth representation π of G. By definition, the inertial support of the representation π is equal to $\mathfrak{s}$. Hence, τ is an irreducible subrepresentation $\mathrm{res}_K i_B^G\chi$. The G-representations $i_B^G\chi$ and $i_B^G\chi^w$ have the same Jordan–Holder factors for all $w\in N_G(T)$. This shows that, for the purpose of understanding $\mathfrak{s}$-typical representations of K, we may (and do) arrange the characters $\chi_1, \chi_2, \ldots, \chi_n$ (conjugating by an element in the Weyl group if necessary) such that

$$l(\chi_i) > l(\chi_j) \text{ for } i < j. \tag{26}$$

The types for any Bernstein component $[T, \chi]$ of a split reductive group $\mathbf{G}$ are constructed by Roche in [Roc98]. We recall his constructions from [Roc98, Section 2,3]. Let $\mathbf{B}$ be any Borel subgroup of $\mathbf{G}$ containing a maximal split torus $\mathbf{T}$. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$, and let $\bar{\mathbf{U}}$ be the unipotent radical of the opposite Borel subgroup $\bar{\mathbf{B}}$ of $\mathbf{B}$ with respect to $\mathbf{T}$. Let Φ be the set of roots of $\mathbf{G}$ with respect to $\mathbf{T}$. Let Φ^+ and Φ^- be the set of positive and negative roots with respect to the choice of the Borel subgroup $\mathbf{B}$, respectively. Let f_χ be the function on Φ defined by

$$f_\chi(\alpha) = \begin{cases} [l(\chi\alpha^\vee)]/2 & \text{if } \alpha \in \Phi^+, \\ [(l(\chi\alpha^\vee)+1)/2] & \text{if } \alpha \in \Phi^-. \end{cases} \tag{27}$$

Let $x_\alpha : \mathbb{G}_a \to \mathrm{U}_\alpha$ be the root group isomorphism, and let $U_{\alpha,t}$ be the group $x_\alpha(\mathfrak{p}_F^t)$. Let T_0 be the maximal compact subgroup of T. Let $U_\chi^\pm$ be the group generated by $U_{\alpha, f_\chi(\alpha)}$, for all $\alpha \in \Phi^\pm$. Let J_χ be the group generated by U_χ^+, T_0, and U_χ^-. The group J_χ has Iwahori decomposition with respect to the pair (B, T) such that

$$J_\chi \cap U = U_\chi^+,\ J_\chi \cap \bar{U} = U_\chi^-, \text{ and } J_\chi \cap T = T_0.$$

The representation χ of T_0 extends to a representation of J_χ such that U_χ^+ and U_χ^- are both contained in the kernel of this extension. We use the same notation χ for this extension. The pair (J_χ, χ) is a type for the Bernstein component $[T, \chi]$. We apply these results to a split classical group $\mathbf{G}$ with the diagonal torus T and the Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ whose F-points are upper triangular matrices, to get a type (J_χ, χ) for s. Let $\mathcal{I}$ be the group $K(1)(B \cap K)$. The group $\mathcal{I}$ is an Iwahori subgroup of G, contained in K. We may (and do) choose the set of root group isomorphisms $\{x_\alpha : \mathbb{G}_a \to \mathrm{U}_\alpha |\ \alpha \in \Phi\}$ such that J_{id} is equal to $\mathcal{I}$. Moreover, for such a choice, we get that J_χ is a subgroup of $\mathcal{I}$.

Before going any further, we need some notation. Consider the isotropic space W_1^+ spanned by w_1, and W_1^- the space spanned by w_{-1}. Let P_1 be a parabolic subgroup of G fixing the space W_1^+. Let M_1 be the standard Levi factor of P_1, i.e., the G-stabiliser of the decomposition

$$W_1^+ \oplus (W_1^+ \oplus W_1^-)^\perp \oplus W_1^-.$$

The group M_1 isomorphic to $F^\times \times G(W')$, where W' is equal to $(W_1^+ \oplus W_1^-)^\perp$. Let $\bar{U}_1$ be the unipotent radical of the opposite parabolic subgroup $\bar{P}_1$ of P_1 with respect to M_1. Let m be any positive integer such that $m \geq l(\chi_1)$. Define the compact open subgroups $P_1^0(m)$ and $R^0(m)$ by

$$P_1^0(m) = (U_1 \cap P_1(m))(M_1 \cap J_\chi)(\bar{U}_1 \cap P_1(m))$$

and

$$R^0(m) = (U_1 \cap R(m))(M_1 \cap J_\chi)(\bar{U}_1 \cap R(m)),$$

respectively. Here $R(m)$ is the group as defined in Section 5.

For inductive arguments we will use the decomposition of the following representations:

$$\mathrm{ind}_{R^0(m)}^{P_1^0(m)} \mathrm{id} \text{ and } \mathrm{ind}_{P_1^0(m+1)}^{R^0(m)} \mathrm{id}.$$

Let K_1 and K_2 be the kernels of the maps

$$P_1^0(m) \xrightarrow{\pi_1} P_1(k_F) \to M_1(k_F) \text{ and } R^0(m) \xrightarrow{\pi_1} P_1(k_F) \to M_1(k_F),$$

respectively. Recall that the map π_1 is a reduction mod $\mathfrak{p}_F$ map. Using the arguments similar to Lemma 5.1 we get that

$$K_1 \cap R^0(m) \trianglelefteq K_1 \text{ and } K_2 \cap P_1^0(m+1) \trianglelefteq K_2.$$

Now let Λ_1 and Λ_2 be the set of representatives for the orbits of the action of the groups $P_1^0(m)$ and $R^0(m)$ on the set of characters of the groups $K_1/(K_1 \cap R^0(m))$ and $K_2/(K_2 \cap P_1^0(m+1))$. We then have

$$\mathrm{ind}_{R^0(m)}^{P_1^0(m)} \mathrm{id} \simeq \bigoplus_{\eta \in \Lambda_1} \mathrm{ind}_{Z_{P_1^0(m)}(\eta)}^{P_1^0(m)} U_\eta$$

and

$$\mathrm{ind}_{P_1(m+1)}^{R^0(m)} \mathrm{id} \simeq \bigoplus_{\eta \in \Lambda_2} \mathrm{ind}_{Z_{R^0(m)}(\eta)}^{R^0(m)} U_\eta.$$

We note that

$$Z_{P_1^0(m)}(\eta) = Z_{P_1^0(m) \cap M_1}(\eta) K_1 \text{ and } Z_{R^0(m)}(\eta) = Z_{R^0(m) \cap M_1}(\eta) K_2.$$

The group of characters of $K_1/(K_1 \cap R^0(m))$ and $K_2/(K_2 \cap P_1^0(m+1))$ are isomorphic to the groups $\bar{\mathfrak{n}}_1^1$ and $\bar{\mathfrak{n}}_1^2$, respectively. The action of the group $P_1^0(m) \cap M_1 = R^0(m) \cap M_1$ factors through the quotient map

$$P_1^0(m) \cap M_1 \to M_1(k_F).$$

The image of this quotient map is contained in $B(k_F) \cap M_1(k_F)$.

Lemma 7.1. *Let u be any nontrivial element of $\bar{\mathfrak{n}}_1^i$ for $i \in \{1, 2\}$. Let H be the group $Z_{M_1(k_F) \cap B(k_F)}(u)$. There exists a character χ' of T such that*

$$\mathrm{res}_H \chi = \mathrm{res}_H \chi'$$

and the inertial classes $[T, \chi]$ and $[T, \chi']$ are distinct.

Proof. The group $M_1(k_F) \cap B(k_F)$ is isomorphic to $k_F^\times \times B'$, where B' is a Borel subgroup of $G(\bar{W}', \bar{q})$. The action of the group $k_F^\times \times B'$ on $\bar{\mathfrak{n}}_1^2$ factors through the projection

$$k_F^\times \times B' \to k_F^\times.$$

The action is given by the character $x \mapsto x^2$. Hence if (x, b) belongs to $Z_{k_F^\times \times B'}(u)$ where $u \in \bar{\mathfrak{n}}_1^1 \backslash \{0\}$, then $x^2 = 1$. In this case, consider a nontrivial character η of $k_F^\times$ which is trivial on the group $\{\pm 1\}$. We consider the character η as a character of $\mathfrak{o}_F^\times$ via inflation. Set χ' to be the character $\chi_1 \eta \boxtimes \chi_2 \boxtimes \cdots \boxtimes \chi_n$. From the above definition we get

$$\mathrm{res}_H \chi = \mathrm{res}_H \chi'.$$

If the Bernstein component $[T, \chi_1]$ is equivalent to $[T, \chi_2]$, then $\eta^{-1} = \chi_1^2$. This is not possible as $l(\chi_1) \neq 1$. Hence the character χ' is the character satisfying the lemma.

Now consider the case when u belongs to $\bar{\mathfrak{n}}_1^1$. The unipotent radical U of $k_F^\times \times B'$ is a p-group. Hence there exists a flag $\{V_i; V_i \subset V_{i+1}\}$ of $\bar{\mathfrak{n}}_1^1$ stabilised by $k_F^\times \times B'$ such that U acts trivially on V_i/V_{i+1}. Let i be the least positive integer such that $u \in V_i$. The group H is contained in the $k_F^\times \times B'$-stabiliser of $\bar{u}$ in V_i/V_{i-1}. The group U acts trivially on V_i/V_{i-1}. Hence the image of H under the natural map $k_F^\times \times B' \to T(k_F)$ is contained in a group of the form

$$\{\mathrm{diag}(t_1, t_2, \dots, t_n, 1, t_{-n}, \dots, t_1)|\ t_1 t_j^{-1} = 1\}.$$

Without loss of generality, assume that $j > 0$. Consider the character χ' given by

$$\chi' = \chi_1\eta \boxtimes \cdots \boxtimes \chi_j\eta^{-1} \boxtimes \cdots \boxtimes \chi_n.$$

If (T, χ) and (T, χ') are inertially equivalent, then the multiplicity of $\{\chi_1, \chi_1^{-1}\}$ in the multisets

$$\{\{\chi_1, \chi_1^{-1}\}, \dots, \{\chi_n, \chi_n^{-1}\}\}$$

and

$$\{\{\chi_1\eta, \chi_1^{-1}\eta^{-1}\}, \dots, \{\chi_j\eta^{-1}, \chi_j^{-1}\eta\}, \dots, \{\chi_n, \chi_n^{-1}\}\}$$

must be the same. This implies that η belongs to $\{\chi_1^{-2}, \chi_1\chi_j, \chi_1\chi_j^{-1}\}$. Since $k_F^\times$ has cardinality bigger than 5, there exists a character η such that $[T, \chi]$ and $[T, \chi']$ are not inertially equivalent. This completes the proof of the lemma. □

We need the following technical observation. Let χ and η be two characters of T. Recall that T is identified with $(F^\times)^n$ using the diagonal embedding using ι in (25). We identify χ with $\boxtimes_{i=1}^n \chi_i$ and η with $\boxtimes_{i=1}^n \eta_i$.

Lemma 7.2. *Let $n > 1$, and let $[T, \chi]_{M_1}$ and $[T, \eta]_{M_1}$ be two inertial classes such that $\mathrm{res}_{\mathfrak{o}_F^\times} \chi_1 = \mathrm{res}_{\mathfrak{o}_F^\times} \eta_1$. If $[T, \chi]_{M_1} \neq [T, \eta]_{M_1}$, then $[T, \chi]_G \neq [T, \eta]_G$.*

Proof. Since $[T, \chi]_{M_1} \neq [T, \eta]_{M_1}$, there exists an integer i with $2 \leq i \leq n$ such that the multiplicity of the multiset $\{\mathrm{res}_{\mathfrak{o}_F^\times} \chi_i, \mathrm{res}_{\mathfrak{o}_F^\times} \chi_i^{-1}\}$ has different multiplicities in the multisets

$$\{\{\mathrm{res}_{\mathfrak{o}_F^\times} \chi_2, \mathrm{res}_{\mathfrak{o}_F^\times} \chi_2^{-1}\}, \dots, \{\mathrm{res}_{\mathfrak{o}_F^\times} \chi_n, \mathrm{res}_{\mathfrak{o}_F^\times} \chi_n^{-1}\}\}$$

and

$$\{\{\mathrm{res}_{\mathfrak{o}_F^\times} \eta_2, \mathrm{res}_{\mathfrak{o}_F^\times} \eta_2^{-1}\}, \dots, \{\mathrm{res}_{\mathfrak{o}_F^\times} \eta_n, \mathrm{res}_{\mathfrak{o}_F^\times} \eta_n^{-1}\}\}.$$

Hence, the multiset $\{\mathrm{res}_{\mathfrak{o}_F^\times} \chi_i, \mathrm{res}_{\mathfrak{o}_F^\times} \chi_i^{-1}\}$ will have different multiplicities in

$$\{\{\mathrm{res}_{\mathfrak{o}_F^\times} \chi_1, \mathrm{res}_{\mathfrak{o}_F^\times} \chi_1^{-1}\}, \{\mathrm{res}_{\mathfrak{o}_F^\times} \chi_2, \mathrm{res}_{\mathfrak{o}_F^\times} \chi_2^{-1}\}, \dots, \{\mathrm{res}_{\mathfrak{o}_F^\times} \chi_n, \mathrm{res}_{\mathfrak{o}_F^\times} \chi_n^{-1}\}\}$$

and

$$\{\{\mathrm{res}_{\mathfrak{o}_F^\times} \eta_1, \mathrm{res}_{\mathfrak{o}_F^\times} \eta_1^{-1}\}, \{\mathrm{res}_{\mathfrak{o}_F^\times} \eta_2, \mathrm{res}_{\mathfrak{o}_F^\times} \eta_2^{-1}\}, \dots, \{\mathrm{res}_{\mathfrak{o}_F^\times} \eta_n, \mathrm{res}_{\mathfrak{o}_F^\times} \eta_n^{-1}\}\}.$$

This shows the lemma. □

We are now ready to classify $\mathfrak{s} = [T, \chi]$-typical representations of K.

Theorem 7.3. *Let K be the fixed hyperspecial maximal compact subgroup G. Let $\mathfrak{s} = [T, \boxtimes_{i=1}^n \chi_i]_G$ be a toral inertial class such that $l(\chi_i) > l(\chi_i)$ for all $i < j$. If τ is an $\mathfrak{s}$-typical representation of K, then τ is a subrepresentation of $\mathrm{ind}_{J_\chi}^K \chi$.*

Proof. Using induction on n we show that the representation $\mathrm{ind}_{J_\chi}^K \chi$ is a subrepresentation of $\mathrm{res}_K\, i_B^G \chi$, and any irreducible subrepresentation of

$$(\mathrm{res}_K\, i_B^G \chi)/\, \mathrm{ind}_{J_\chi}^K \chi$$

is atypical.

Assume this hypothesis to be true for all $n' < n$. From induction hypothesis, we get that

$$\operatorname{res}_K i_{B\cap M_1}^{M_1}\chi = \operatorname{ind}_{J_\chi\cap M_1}^{K\cap M_1}\chi \oplus \tau'$$

such that any irreducible $(K\cap M_1)$-subrepresentation of τ' is atypical. Let ξ be a $(K\cap M_1)$-irreducible subrepresentation of τ'. Since the $(K\cap M_1)$-representation ξ is atypical, it occurs as a subrepresentation of $\operatorname{res}_{K\cap M_1} i_S^{M_1}\kappa$, where S is a standard parabolic subgroup of M_1 with Levi factor L and κ is a cuspidal representation of L such that $[L,\kappa]_{M_1} \neq [T,\chi]_{M_1}$. Any irreducible K-subrepresentation of $\operatorname{ind}_{K\cap P_1}^{K_1}\xi$ occurs as a K-subrepresentation of

$$i_{P_1}^{G}(i_S^{M_1}\kappa). \tag{28}$$

If $L \neq T$, then the cuspidal support of the representation (28) is not equal to $[T,\chi]_G$. Assume that $L = T$. Since we have $[T,\kappa]_{M_1} \neq [T,\chi]_{M_1}$, using Lemma 7.2, we get that $[T,\kappa]_G \neq [T,\chi]_G$. Hence, the irreducible subrepresentations of $\operatorname{ind}_{K\cap P_1}^{K_1}\xi$ are atypical.

Let τ be any $\mathfrak{s}$-typical representation of K. From the above discussion, we get that τ is a subrepresentation of

$$\operatorname{ind}_{K\cap P_1}^{K}\gamma \text{ with } \gamma = \operatorname{ind}_{J_\chi\cap M_1}^{K\cap M_1}\chi. \tag{29}$$

Now let N be the integer $l(\chi_1)$, the largest among the set of integers $\{l(\chi_i) : 1 \leq i \leq n\}$. Now the representation (29) is the union of the representations $\operatorname{ind}_{P_1(m)}^{K}\gamma$ for $m \geq N$. Hence any $\mathfrak{s}$-typical representation of K occurs as a subrepresentation of $\operatorname{ind}_{P_1(m)}^{K}\gamma$ for some $m \geq N$. Note that the representation $\operatorname{ind}_{P_1(m)}^{K}\gamma$ is isomorphic to the representation $\operatorname{ind}_{P_1^0(m)}^{K}\chi$ (see Lemma 6.1).

We use induction on $m \geq N$ to show that irreducible subrepresentations of

$$\operatorname{ind}_{P_1^0(m+1)}^{K}\chi / \operatorname{ind}_{P_1^0(m)}^{K}\chi$$

are atypical for all $m \geq N$. Now we have the isomorphism

$$\begin{aligned}\operatorname{ind}_{P_1^0(m+1)}^{K}\chi &\simeq \operatorname{ind}_{P_1^0(m)}^{K}\{\chi\otimes(\operatorname{ind}_{P_1^0(m+1)}^{P_1^0(m)}\mathrm{id})\}\\ &\simeq \operatorname{ind}_{P_1^0(m)}^{K}\chi \oplus_{\eta\in\Lambda_1}\operatorname{ind}_{Z_{P_1^0(m)}(\eta)}^{K}(\chi\otimes U_\eta)\\ &\quad \oplus_{\eta\in\Lambda_2}\operatorname{ind}_{Z_{R^0(m)}(\eta)}^{K}(\chi\otimes U_\eta).\end{aligned}$$

Using Lemma 7.1, we obtain a character χ' such that $\operatorname{res}_H\chi'$ is equal to $\operatorname{res}_H\chi$, where H is either $Z_{P_1^0(m)}(\eta)$ or $Z_{R^0(m)}(\eta)$. Moreover, $[T,\chi]$ and $[T,\chi']$ are distinct inertial classes. Hence, τ is contained in the representation $\operatorname{ind}_{P_1^0(N)}^{K}\chi$.

Let $\mathcal{I}$ be the Iwahori subgroup $K(1)(B\cap K)$; we have $J_\chi \subseteq \mathcal{I}$. Using the support of the G-intertwining of the pair (J_χ,χ) in [Roc98, Theorem 4.15], we note that the representation $\operatorname{ind}_{J_\chi}^{\mathcal{I}}\chi$ is irreducible. Moreover, we have that

$$\operatorname{Hom}_{\mathcal{I}}(\operatorname{ind}_{J_\chi}^{\mathcal{I}}\chi, \operatorname{ind}_{P_1^0(N)}^{\mathcal{I}}\chi) \neq 0.$$

From the definition of J_χ, we note that the dimensions of the representations $\operatorname{ind}_{J_\chi}^{\mathcal{I}}\chi$ and $\operatorname{ind}_{P_1^0(N)}^{\mathcal{I}}\chi$ are the same. This shows that these representations are isomorphic. We conclude that, for any $\mathfrak{s}$-typical representation τ of K, we get that τ is a subrepresentation of $\operatorname{ind}_{J_\chi}^{K}\chi$. Moreover, the representation $\operatorname{ind}_{J_\chi}^{K}\chi$ is a subrepresentation of $\operatorname{res}_K i_B^G\chi$. □

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