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A NEW BASIS FOR THE REPRESENTATION RING OF A WEYL GROUP

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Abstract. Let W be a Weyl group. In this paper we define a new basis for the Grothendieck group of representations of W. This basis contains on the one hand the special representations of W and on the other hand the representations of W carried by the left cells of W. We show that the representations in the new basis have a certain bipositivity property.

Introduction and statement of results

0.1. Let W be an irreducible Weyl group. Let $\mathcal{R}_W$ be the (abelian) category of finite dimensional representations of W over $\mathbf{Q}$ and let $\mathcal{K}_W$ be the Grothendieck group of $\mathcal{R}_W$. Now $\mathcal{K}_W$ has a $\mathbf{Z}$-basis Irr_W consisting of the irreducible representations of W up to isomorphism. (We often identify a representation of W with its isomorphism class.)

Recall that Irr_W is partitioned into subsets called *families*, see [L2, §8], [L5, 4.2]; these are in 1-1 correspondence with the two-sided cells of W. For each family c of W we denote by $\mathcal{R}_c$ the (abelian) category of all $E \in \mathcal{R}_W$ which are direct sums of irreducible representations in c. Let $\mathcal{K}_c$ be the Grothendieck group of $\mathcal{R}_c$. It has a $\mathbf{Z}$-basis consisting of the irreducible representations in c. Thus we have $\mathcal{K}_W = \bigoplus_c \mathcal{K}_c$ where c runs over the families of W. We now fix a family c of W.

In [L1] we introduced a class of irreducible objects of $\mathcal{R}_W$ denoted by $\mathcal{S}_W$ (later called special representations); exactly one of these irreducible objects, denoted by E_c, is contained in c.

In [L4] we introduced a class of (not necessarily irreducible) objects of $\mathcal{R}_c$ called "cells" (later these objects were called the constructible representations). In [L6] we showed that the constructible representations in $\mathcal{R}_c$ are precisely the representations of W carried by the various left cells of W contained in c.

In this paper we introduce a class $\mathbf{B}_c$ of objects of $\mathcal{R}_c$ which includes both E_c and the constructible representations in $\mathcal{R}_c$ and which forms a $\mathbf{Z}$-basis of the group $\mathcal{K}_c$. The representations in $\mathbf{B}_c$ are called *new representations.* (Taking disjoint union over all families of W we obtain a new $\mathbf{Z}$-basis of $\mathcal{K}_W$.)

0.2. Let Γ be a finite group. As in [L2] we define $M(\Gamma)$ to be the set of all pairs (x,ρ) where $x \in \Gamma$ and $\rho \in \mathrm{Irr}(Z(x))$ where $Z(x)$ is the centralizer of x in Γ and $\mathrm{Irr}(Z(X))$ is the set of irreducible representations of $Z(x)$ over $\mathbf{C}$ up to isomorphism; these pairs are taken up to conjugacy by any element of Γ. Let $\mathbf{C}[M(\Gamma)]$ be the $\mathbf{C}$-vector space with basis $\{(x,\rho); (x,\rho) \in M(\Gamma)\}$.

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Let H be a subgroup of Γ. For $x \in \Gamma$ let $(\Gamma/H)^x$ be the fixed point set of the left translation action of x on Γ/H and let $\mathbf{C}[(\Gamma/H)^x]$ be the $\mathbf{C}$-vector space with basis $(\Gamma/H)^x$. Now $Z(x)$ acts by left translation on $(\Gamma/H)^x$ and this induces a linear action of $Z(x)$ on $\mathbf{C}[(\Gamma/H)^x]$. If $\rho \in \mathrm{Irr}(Z(x))$, let $N_{H,H,x,\rho}$ be the multiplicity of ρ in the $Z(x)$-module $\mathbf{C}[(\Gamma/H)^x]$. Let

(a) $$S_{H,H} = \bigoplus_{(x,\rho)\in M(\Gamma)} N_{H,H,x,\rho}(x,\rho) \in \mathbf{C}[M(\Gamma)].$$

More generally, let $H \subset H'$ be subgroups of Γ with H normal in H'. Then the obvious surjective map $\Gamma/H \to \Gamma/H'$ restricts to a map $(\Gamma/H)^x \to (\Gamma/H')^x$ and this induces a linear map $\mathbf{C}[(\Gamma/H)^x] \to \mathbf{C}[(\Gamma/H')^x]$ (compatible with $Z(x)$ actions) whose image is denoted by $\mathcal{I}$. Now $\mathcal{I}$ is a $Z(x)$-submodule of $\mathbf{C}[(\Gamma/H')^x]$. If $\rho \in \mathrm{Irr}(Z(x))$, let $N_{H,H',x,\rho}$ be the multiplicity of ρ in the $Z(x)$-module $\mathcal{I}$. Let

(b) $$S_{H,H'} = \bigoplus_{(x,\rho)\in M(\Gamma)} N_{H,H',x,\rho}(x,\rho) \in \mathbf{C}[M(\Gamma)].$$

For example,

$$S_{\{1\},\{1\}} = \sum_{\rho\in\mathrm{Irr}(\Gamma)} \dim\rho(1,\rho),$$

$$S_{\{1\},\Gamma} = (1,1),$$

$$S_{\Gamma,\Gamma} = \sum_{x\in\Gamma \text{ up to conjugacy}} (x,1).$$

0.3. As in [L5, §4] we attach to c a finite group $\mathcal{G}_c$ and an imbedding $c \to M(\mathcal{G}_c)$. Let $M_0(\mathcal{G}_c)$ be the image of this imbedding. For $(x,\rho) \in M_0(\mathcal{G}_c)$ let $E_{x,\rho}$ be the corresponding (irreducible) representation in c. For any $\mathcal{E} \in \mathcal{R}_c$ we define $\underline{\mathcal{E}} \in \mathbf{C}[M(\mathcal{G}_c)]$ by $\underline{\mathcal{E}} = \sum_{(x,\rho)\in M_0(\mathcal{G}_c)}(E_{x,\rho} : \mathcal{E})(x,\rho)$ where $(E_{x,\rho} : \mathcal{E}) \in \mathbf{N}$ is the multiplicity of $E_{x,\rho}$ in $\mathcal{E}$. Note that $\mathcal{E} \mapsto \underline{\mathcal{E}}$ defined an imbedding $\mathcal{K}_c \to \mathbf{C}[M(\mathcal{G}_c)]$.

As was pointed out in [L7], to any constructible representation E in $\mathcal{R}_c$ one can attach a subgroup H_E of $\mathcal{G}_c$, well defined up to conjugacy, such that $\underline{E} = S_{H_E,H_E}$; see 0.2(a). Moreover,

(a) $E \mapsto H_E$

is an injective map from the set of constructible representations in $\mathcal{R}_c$ to the set of subgroups of $\mathcal{G}_c$ (up to conjugacy). Let $\mathfrak{F}_c$ be the set of subgroups of $\mathcal{G}_c$ which are conjugate to a subgroup in the image of the map (a). We have $\mathcal{G}_c \in \mathfrak{F}_c$. We say that c is *anomalous* if $\{1\} \notin \mathfrak{F}_c$. If W is of classical-type, then c is not anomalous. If W is of exceptional-type, then c is anomalous in exactly the following cases:

(b) the unique c with $|c| = 2$ with W of type E_7;
(c) the two c with $|c| = 2$ with W of type E_8;
(d) the unique c with $|c| = 4$ with W of type G_2;
(e) the unique c with $|c| = 11$ with W of type F_4;
(f) the unique c with $|c| = 17$ with W of type E_8.

Let $\hat{\mathfrak{F}}_c$ be the set of subgroups of $\mathcal{G}_c$ which are either $\{1\}$ or are in $\mathfrak{F}_c$. Let $\tilde{\Theta}_c$ be the set of all pairs (H,H') where $H \in \hat{\mathfrak{F}}_c, H' \in \hat{\mathfrak{F}}_c$ and H is a normal subgroup of H'. Now $\mathcal{G}_c$ acts on $\tilde{\Theta}_c$ by simultaneous conjugation. We now state our main result.

Theorem 0.4. *There exists a $\mathcal{G}_c$-stable subset Θ_c of $\tilde{\Theta}_c$ such that the following hold:*

(i) *For any $H \in \mathfrak{F}_c$ we have $(H,H) \in \Theta_c$.*

(ii) *We have $(1, \mathcal{G}_c) \in \Theta_c$.*

(iii) *For any $(H,H') \in \Theta_c$ there is a unique object $E_{H,H'} \in \mathcal{R}_c$ such that $S_{H,H'} = \underline{E}_{H,H'}$, see 0.2(a). Let $\mathbf{B}_c$ be the set of isomorphism classes of objects of $\mathcal{R}_c$ of the form $E_{H,H'}$ for some $(H,H') \in \Theta_c$.*

(iv) *The map $(H,H') \mapsto E_{H,H'}$ defines a bijection from the set of $\mathcal{G}_c$-orbits on Θ_c to $\mathbf{B}_c$. Moreover $\mathbf{B}_c$ is a $\mathbf{Z}$-basis of $\mathcal{K}_c$.*

The representations in $\mathbf{B}_c$ are the new representations mentioned in 0.1. From (i) we see that any constructible representation of $\mathcal{R}_c$ is in $\mathbf{B}_c$. From (ii) we see that the special representation E_c is in $\mathbf{B}_c$.

In the case where W is of type A the theorem is trivial; we have $\mathcal{G}_c = \{1\}$ and $\mathbf{B}_c$ consists of the unique representation in c. The proof of the theorem for W of type B_n, C_n, D_n is given in §2. The proof of the theorem for W of exceptional-type is given in §3.

0.5. In this paper we also define a canonical bijection $c \xrightarrow{\sim} \mathbf{B}_c$, $E \mapsto \hat{E}$ which has the property that for any $E \in c$, E appears with multiplicity one in $\hat{E}$. For E, E' in c let $E' : \hat{E}$ be the multiplicity of E' in $\hat{E}$. Property (i) below will be proved in a sequel to this paper. (For W of exceptional-type (i) is easily deduced from the formulas in 3.2-3.8.)

(i) The matrix $(E' : \hat{E})$ indexed by $c \times c$ is upper triangular unipotent for a suitable partial order on c.

0.6. In the setup of 0.2 we define (following [L2, §4]) a pairing $\{,\} : M(\Gamma) \times M(\Gamma) \to \mathbf{C}$ by

$$\{(x,\rho),(x',\rho')\} = |Z(x)|^{-1}|Z(x')|^{-1} \sum_{g \in \Gamma; xgx'g^{-1} = gx'g^{-1}x} \overline{\mathrm{tr}(g^{-1}xg,\rho')}\,\mathrm{tr}(gx'g^{-1},\rho),$$

where $^-$ is complex conjugation. We define the non-abelian Fourier transform $A : \mathbf{C}[M(\Gamma)] \to \mathbf{C}[M(\Gamma)]$ as the $\mathbf{C}$-linear map such that

$$A(x,\rho) = \sum_{(x',\rho') \in M(\Gamma)} \{(x,\rho),(x',\rho')\}(x',\rho')$$

for any $(x,\rho) \in M(\Gamma)$. According to [L2], we have $A^2 = 1$. Let $M(\Gamma)_{\geq 0}$ be the set of elements

$$\sum_{(x,\rho) \in M(\Gamma)} c_{x,\rho}(x,\rho) \in \mathbf{C}[M(\Gamma)]$$

such that $c_{x,\rho} \in \mathbf{R}_{\geq 0}$ for any $(x,\rho) \in M(\Gamma)$.

An element $f \in \mathbf{C}[M(\Gamma)]$ is said to be *bipositive* if $f \in M(\Gamma)_{\geq 0}$ and $A(f) \in M(\Gamma)_{\geq 0}$. We have the following result.

Theorem 0.7. *Let $H \subset H'$ be subgroups of Γ with H normal in H'. Then $S_{H,H'} \in \mathbf{C}[M(\Gamma)]$ is bipositive. Hence (by 0.4), if $\Gamma = \mathcal{G}_c$ and $\mathcal{E}$ is a new representation in $\mathcal{R}_c$, then $\underline{\mathcal{E}} \in \mathbf{C}[M(\Gamma)]$ is bipositive.*

The proof is given in §4.

0.8. In a sequel to this paper we will extend the results of the paper by constructing a new basis for $\mathbf{C}[M(\mathcal{G}_c)]$ consisting of bipositive elements; this provides a new $\mathbf{Z}$-basis for the Grothendieck group of unipotent representations of a split Chevalley group over a finite field.

0.9. **Notation.** For $a \le b$ in $\mathbf{N}$ we write $[a,b] = \{z \in \mathbf{N}; a \le z \le b\}$. We set $[1,0] = \emptyset$. For a finite set Y we write $|Y|$ for the cardinal of Y. For a,b in $\mathbf{Z}$ we write $a =_2 b$ if $a = b \mod 2$ and $a \ne_2 b$ if $a \ne b \mod 2$. We write $\mathbf{Z}/2\mathbf{Z} = \mathbf{F}_2$.

1. The set S_D

1.1. Let $D \in \mathbf{N}$. A subset I of $[1,D]$ is said to be an *interval* if $I = [a,b]$ for some $a \le b$ in $[1,D]$. Let $\mathcal{I}_D$ be the set of intervals of $[1,D]$. For $I = [a,b], I' = [a',b']$ in $\mathcal{I}_D$ we write $I \prec I'$ whenever $a' < a \le b < b'$. We say that I, I' are non-touching (and we write $I \spadesuit I'$) if $a' - b \ge 2$ or $a - b' \ge 2$. Let $\mathcal{I}_D^1 = \{I \in \mathcal{I}_D; |I| = \text{ odd}\}$. Let R_D^1 be the set whose elements are the subsets of $\mathcal{I}_D^1$. Let $\emptyset \in R_D^1$ be the empty subset of $\mathcal{I}_D^1$.

When $D \ge 2$ and $i \in [1,D]$ we define an (injective) map $\xi_i : \mathcal{I}_{D-2} \to \mathcal{I}_D$ as follows:

$$\xi_i([a',b']) = [a'+2, b'+2] \text{ if } i \le a', \quad \xi_i([a',b']) = [a',b'] \text{ if } i \ge b'+2,$$

(a) $$\xi_i([a',b']) = [a', b'+2] \text{ if } a' < i < b'+2.$$

We have $\xi_i(\mathcal{I}_{D-2}^1) \subset \mathcal{I}_D^1$. We define $t_i : R_{D-2}^1 \to R_D^1$ by $B' \mapsto \{\xi_i(I'); I' \in B'\} \sqcup \{i\}$. We have $|t_i(B')| = |B'| + 1$.

1.2. We define a subset S_D of R_D^1 by induction on D as follows. When $D \in \{0,1\}$, S_D consists of a single element, namely $\emptyset \in R_D^1$. When $D \ge 2$ we say that $B \in R_D^1$ is in S_D if either $B = \emptyset$ or if

(i) there exists $i \in [1,D]$ (if D is even) or $i \in [1,D-1]$ (if D is odd) and $B' \in S_{D-2}$ such that $B = t_i(B')$.

If D is odd, we have $S_D = S_{D-1}$ (use induction on D).

Until the end of 1.8 we assume that D is even.

1.3. **The set S'_D.** Let $B \in R_D^1$. We consider the following properties $(P_0), (P_1)$ that B may or may not have.

(P_0) *If $I \in B$, $\tilde I \in B$, then either $I = \tilde I$, or $I \spadesuit \tilde I$, or $I \prec \tilde I$, or $\tilde I \prec I$.*

(P_1) *If $[a,b] \in B$ and $c \in \mathbf{N}$ satisfies $a < c < b$, $a - c =_2 1$ (hence $b - c =_2 1$), then there exists $[a_1, b_1] \in B$ such that $a < a_1 \le c \le b_1 < b$.*

From the definitions we see that if $D \ge 2$, $i \in [1,D]$, $B' \in R_{D-2}^1$ and $B = t_i(B') \in R_D^1$, the following holds:

(a) *B' satisfies (P_0) if and only if B satisfies (P_0); B' satisfies (P_1) if and only if B satisfies (P_1).*

Let S'_D be the set of all $B \in R_D^1$ which satisfy $(P_0), (P_1)$. In the setup of (a) we have the following consequence of (a):

(b) *We have $B' \in S'_{D-2}$ if and only if $B \in S'_D$.*

We show:

(c) $S_D = S'_D$.

We argue by induction on D. If $D = 0$, S'_D consists of the empty set hence (c) holds in this case. Assume now that $D \ge 2$. Let $B \in S_D$. We show that $B \in S'_D$. If $B = \emptyset$ this is clear. If $B \ne \emptyset$, then $B = t_i(B')$ for some $i, B' \in S_{D-2}$. By the induction hypothesis we have $B' \in S'_{D-2}$. By (b) we have $B \in S'_D$. We see that $B \in S_D \implies B \in S'_D$. Conversely, let $B \in S'_D$. We show that $B \in S_D$. If $B = \emptyset$ this is obvious. Thus we can assume that $B \neq \emptyset$. Let $[a, b] \in B$ be such that $b - a$ is minimum. If $a < z < b$, $z =_2 a+1$, then by (P_1) we have $z \in [a', b']$ with $[a', b'] \in B$, $b' - a' < b - a$, contradicting the minimality of $b - a$. We see that no z as above exists. Thus, $[a, b] = \{i\}$ for some $i \in [1, D]$. Using (P_0) and $\{i\} \in B$, we see that B does not contain any interval of the form $[a, i]$ with $a < i$, or $[i, b]$ with $i < b$, or $[a, i-1]$ with $a < i$ or $[i+1, b]$ with $i < b$; hence any interval of B other than $\{i\}$ is of the form $\xi_i[a', b']$ where $[a', b'] \in \mathcal{I}^1_{D-2}$. Thus we have $B = t_i(B')$ for some $B' \in S_{D-2}$. From (b) we see that $B' \in S'_{D-2}$. Using the induction hypothesis we deduce that $B' \in S_{D-2}$. By the definition of S_D, we have $B \in S_D$. This completes the proof of (c).

The following result has already been proved as a part of the proof of (c).

(d) *Assume that $D \geq 2$, $i \in [1, D]$. Let $B \in S_D$ be such that $\{i\} \in B$. Then there exists $B' \in S_{D-2}$ such that $B = t_i(B')$.*

1.4. For $B \in S_D$, $j \in [1, D]$ we set $B_j = \{I \in B; j \in I\}$. From the definitions we deduce:

(a) *Assume that $D \geq 2$, $i \in [1, D]$ and that $B' \in S_{D-2}, B = t_i(B') \in S_D$. Then for $r \in [1, D-2]$ we have:*

$|B'_r)| = |B_r|$ *if* $r \leq i-2$, $|B'_r| = |B_{r+2}|$ *if* $r \geq i$,

$|B_{i-1}| = |B_{i+1}| = |B'_{i-1}|$, $|B_i| = |B'_{i-1}| + 1$ *if* $1 < i < D$,

$|B_{i-1}| = 0$ *if* $i = D$, $|B_{i+1}| = 0$ *if* $i = 1$.

1.5. Let $B \in S_D, B \neq \emptyset$. In this case we must have $\{j\} \in B$ for some $j \in [1, D]$; we assume that j is as small as possible (then it is uniquely determined). As in the proof of 1.3(c) we have $B = t_j(B')$ where $B' \in S_{D-2}$. Let i be the smallest number in $\bigcup_{I \in B} I$. We have $i \leq j$. We show:

(a) *For any $h \in [i, j]$, we have $[h, \tilde{h}] \in B$ for a unique $\tilde{h} \in [h, D]$; moreover we have $j \leq \tilde{h}$.*

We argue by induction on D. When $D = 0$ the result is obvious. We now assume that $D \geq 2$. Assume first that $i = j$. By (P_0), $\{j\} \in B$ implies that we cannot have $[j, b] \in B$ with $j < b$; thus (a) holds in this case. In particular, (a) holds when $D = 2$ (in this case we have $i = j$). We now assume that $D \geq 4$. We can assume that $i < j$. We have $[i, b] \in B$ for some $b > i$ hence $|B| \geq 2$ so that $|B'| \geq 1$ and $B' \neq \emptyset$. Then i', j' are defined in terms of B' in the same way as i, j are defined in terms of B. From (P_1) we see that there exists j_1 such that $i < j_1 < b$ such that $\{j_1\} \in B$. By the minimality of j we must have $j \leq j_1$. Thus we have $i < j < b$. We have $[i, b] = \xi_j[i, b-2]]$ hence $[i, b-2] \in B'$. This implies that $i' \leq i$. We have $[i', c] \in B'$ for some $c \in [i', D-2]$, $c =_2 i'$; hence $[i', c'] \in B$ for some $c' \geq i'$ so that $i' \geq i$. Thus we have $i' = i$. By the induction hypothesis, the following holds:

(b) *For any $r \in [i, j']$, we have $[r, r_1] \in B'$ for a unique r_1; moreover $j' \leq r_1$.*

If $j' \leq j-2$, then $\{j'\} = \xi_j(\{j'\}) \in B$. Hence $j' \geq j$ by the minimality of j; this is a contradiction. Thus we have $j' \geq j-1$.

Let $r \in [i, j-1]$. Then we have also $r \in [i, j']$ hence r_1 is defined as in (b). We have $[r, r_1] \in B'$ hence $[r, r_1+2] \in B$ (we use that $r < j \leq j'+1 \leq r_1+1 < r_1+2$); we have $j < r_1+2$. Assume now that $[r, r_2] \in B$ with $r \leq r_2$. Then $r < r_2$ (by the minimality of j). If $j = r_2$ or $j = r_2+1$, then applying (P_0) to $\{j\}, [r, r_2]$ gives a contradiction. Thus we must have either $r < j < r_2$ or $j > r_2+1$. If $j > r_2+1$, then $[r, r_2] \in B'$ hence by (b), $r_2 = r_1$, hence $j > r_1 + 1$ contradicting $j < r_1 + 2$. Thus we have $r < j < r_2$, so that $[r, r_2 - 2] \in B'$ hence by (b), $r_2 - 2 = r_1$. Thus we have $r < j < r_2$ so that $[r, r_2 - 2] \in B'$ hence by (b), $r_2 - 2 = r_1$.

Next we assume that $r = j$. In this case we have $\{r\} \in B$. Moreover, if $[r, r'] \in B$ with $r \le r' \le D$, then we cannot have $r < r'$ (if $r < r'$, then applying (P_0) to $\{r\}, [r, r']$ gives a contradiction). This proves (a).

We show:

(c) *Assume that $j < D$ and that $i \le h < j$. Then $\tilde{h}$ in* (a) *satisfies $\tilde{h} > j$.*

Assume that $\tilde{h} = j$, so that $[h, j] \in B$. Since $h < j$, applying (P_0) to $\{j\}, [h, j]$ gives a contradiction. This proves (c).

(d) *Assume that $j < D$ and that $r \in [j + 1, D]$. We have $[j + 1, r] \notin B$.*

Assume that $[j + 1, r] \in B$. Applying (P_0) to $\{j\}, [j + 1, r]$ gives a contradiction. This proves (d).

We show:

(e) *For $h \in [i, j]$ we have $|B_h| = h - i + 1$. If $j < D$ we have $|B_{j+1}| = j - i$.*

Let $h \in [i, j]$. Then for any $h' \in [i, h]$, B_h contains $[h', \tilde{h}']$ (since $h \le \tilde{h}'$); see (a). Conversely, assume that $[a, b] \in B_h$. We have $a \le h$. By the definition of i we have $i \le a$. By the uniqueness statement in (a) we have $b = \tilde{a}$ so that $[a, b]$ is one of the $h - i + 1$ intervals $[h', \tilde{h}']$ above. This proves the first assertion of (e). Assume now that $j < D$. If $h' \in [i, j]$, $h' < j$, then $[h', \tilde{h}'] \in B_{j+1}$, by (c). Conversely, assume that $[a, b] \in B_{j+1}$. We have $a \le j + 1$ and by (d) we have $a \ne j + 1$ so that $a \le j$. If $a = j$, then by the uniqueness in (a) we have $b = j$ which contradicts $j + 1 \in [a, b]$. Thus we have $a \le j - 1$. We see that $[a, b]$ is one of the $j - i$ intervals $[h', \tilde{h}']$ with $h' \in [i, j]$, $h' < j$. This proves (e).

1.6. For $B \in S_D$, $j \in [1, D]$, we set

$$\epsilon_j(B) = |B_j|(|B_j| + 1)/2 \in \mathbf{F}_2.$$

We have $\epsilon_j(B) = 1$ if $|B_j| \in (4\mathbf{Z}+1) \cup (4\mathbf{Z}+2)$, $\epsilon_j(B) = 0$ if $|B_j| \in (4\mathbf{Z}+3) \cup (4\mathbf{Z})$.

Assume now that $B \ne \emptyset$. Let $i \le j$ in $[1, D]$ be as in 1.5. From 1.5(e) we deduce:

(a) *We have $(|B_i|, |B_{i+1}|, \dots, |B_j|) = (1, 2, 3, \dots, j - i, j - i + 1)$. If $j < D$, we have $|B_{j+1}| = j - i$.*

From (a) we deduce:

(b)

$$\begin{aligned}&(\epsilon_i(B), e_{i+1}(B), \dots, \epsilon_j(B))\\ &= (1 \times 2)/2, (2 \times 3)/2, (3 \times 4)/2, \dots, (j - i)(j - i + 1)/2, (j - i + 1)(j - i + 2)/2);\end{aligned}$$

(c) *if $j < D$, then $\epsilon_{j+1}(B) = (j - i)(j - i + 1)/2$.*

For future reference we note:

(d) *If $c \in \mathbf{Z}$, then $c(c + 1)/2 \ne_2 (c + 2)(c + 3)/2$.*

(e) *If $c \in 2\mathbf{Z}$, then $c(c + 1)/2 \ne_2 (c + 1)(c + 2)/2$.*

1.7. Let $B \in S_D$, $\tilde{B} \in S_D$ be such that $B \ne \emptyset, \tilde{B} \ne \emptyset$ and $\epsilon_h(B) = \epsilon_h(\tilde{B})$ for any $h \in [1, D]$. We show:

(a) *We can find $z \in [1, D]$ such that $\{z\} \in B$, $\{z\} \in \tilde{B}$.*

We associate $i \le j$ to B as in 1.5; let $\tilde{i} \le \tilde{j}$ be the analogous number for $\tilde{B}$. Assume first that $j < \tilde{j}$ (so that $j < D$) and $i < \tilde{i}$. From 1.6 for B we have $\epsilon_i(B) = (1 \times 2)/2 = 1$. Since $i < \tilde{i}$ we have $\epsilon_i(\tilde{B}) = 0$. Hence $1 =_2 0$, a contradiction. Thus we must have $i \ge \tilde{i}$.

Next we asssume that $j < \tilde{j}$ (so that $j < D$) and $\tilde{i} < i$. From 1.6 for $\tilde{B}$ we have $\epsilon_{\tilde{i}}(\tilde{B}) = (1 \times 2)/2$; moreover $\epsilon_{\tilde{i}}(B) = 0$. Hence $1 =_2 0$, a contradiction. Thus when $j < \tilde{j}$ we must have $i = \tilde{i}$. From 1.6(c) for B we have $e_{j+1}(B) = (j-i)(j-i+1)/2$ and from 1.6(b) for $\tilde{B}$ we have $e_{j+1}(\tilde{B}) = (j-i+2)(j-i+3)/2$. It follows that

$$(j-i)(j-i+1)/2) =_2 (j-i+2)(j-i+3)/2,$$

contradicting 1.6(d). We see that $j < \tilde{j}$ leads to a contradiction. Similarly, $\tilde{j} < j$ leads to a contradiction. Thus we must have $j = \tilde{j}$, so that (a) holds with $z = j = \tilde{j}$. This completes the proof of (a).

1.8. Let $B \in S_D$, $\tilde{B} \in S_D$.

(a) *Assume that $\tilde{B} = \emptyset$ and that $\epsilon_h(B) = \epsilon_h(\tilde{B})$ for any $h \in [1, D]$. Then $\tilde{B} = B$.*

The proof is similar to that of 1.7(a). Assume that $B \neq \emptyset$. Let $i \leq j$ be attached to B as in 1.5.

Using 1.6 we see that $e_i(B) = (1 \times 2)/2$. On the other hand we have $e_i(\tilde{B}) = 0$. We get $1 =_2 0$, a contradiction. This proves (a).

1.9. We no longer assume that D is even. Let V be the $\mathbf{F}_2$-vector space consisting of all functions $[1, D] \to \mathbf{F}_2$. For any subset I of $[1, D]$ let $e_I \in V$ be the function whose value at i is 1 if $i \in I$ and is 0 if $i \notin I$. For $i \in [1, D]$ we set $e_i = e_{\{i\}}$. Now $\{e_i; i \in [1, D]\}$ is a basis of V. We define a symplectic form $(,) : V \times V \to \mathbf{F}_2$ by $(e_i, e_j) = 1$ if $i - j = \pm 1$, $(e_i, e_j) = 0$ if $i - j \neq \pm 1$. This symplectic form is non-degenerate if D is even while if D is odd it has a one dimensional radical spanned by $e_1 + e_3 + e_5 + \cdots + e_D$.

For any subset Z of V we set $Z^\perp = \{x \in V; (x, z) = 0 \quad \forall z \in Z\}$.

When $D \geq 2$ we denote by V' the $\mathbf{F}_2$-vector space consisting of all functions $[1, D-2] \to \mathbf{F}_2$. For any $I' \subset [1, D-2]$ let $e'_{I'} \in V'$ be the function whose value at i is 1 if $i \in I'$ and is 0 if $i \notin I'$. For $i \in [1, D-2]$ we set $e'_i = e'_{\{i\}}$. Now $\{e'_i; i \in [1, D-2]\}$ is a basis of V'. We define a symplectic form $(,)' : V' \times V' \to \mathbf{F}_2$ by $(e'_i, e'_j) = 1$ if $i - j = \pm 1$, $(e'_i, e'_j) = 0$ if $i - j \neq \pm 1$.

When $D \geq 2$, for any $i \in [1, D]$ there is a unique linear map $T_i : V' \to V$ such that the sequence $T_i(e'_1), T_i(e'_2), \ldots, T_i(e'_{D-2})$ is:

$e_1, e_2, \ldots, e_{i-2}, e_{i-1} + e_i + e_{i+1}, e_{i+2}, e_{i+3}, \ldots, e_D$ (if $1 < i < D$),

$e_3, e_4, \ldots, e_D$ (if $i = 1$),

$e_1, e_2, \ldots, e_{D-2}$ (if $i = D$).

Note that T_i is injective and $(x, y)' = (T_i(x), T_i(y))$ for any x, y in V'. For any $I' \in \mathcal{I}^1_{D-2}$ we have $T_i(e'_{I'}) = e_{\xi_i(I')}$. Let V_i be the image of $T_i : V' \to V$. From the definitions we deduce:

(a) *We have $e_i^\perp = V_i \oplus \mathbf{F}_2 e_i$.*

We now assume that D is even. For $j \in [1, D-2]$ let $\epsilon'_j : S_{D-2} \to \mathbf{F}_2$ be the analogue of $\epsilon_i : S_D \to \mathbf{F}_2$ when D is replaced by $D - 2$.

For $B \in S_D$, we define $\epsilon(B) \in V$ by $i \mapsto \epsilon_i(B)$. For $B' \in S_{D-2}$ we define $\epsilon'(B') \in V'$ by $j \mapsto \epsilon'_j(B')$. We show:

(b) *Assume that $D \geq 2$, $i \in [1, D]$. Let $B' \in S_{D-2}, B = t_i(B') \in S_D$. Then $\epsilon(B) = T_i(\epsilon'(B')) + ce_i$ for some $c \in \mathbf{F}_2$.*

An equivalent statement is: for any $j \in [1, D] - \{i\}$ we have $\epsilon_j(B) = \epsilon'_{j'}(B')$ if $j' \in [1, D-2]$ is such that $j \in \xi_i(\{j'\})$; and $\epsilon_j(B) = 0$ if no such j' exists. It is enough to show:

$|B'_h| = |B_h|$ if $h \in [1, i-2]$,

$|B'_{h-2}| = |B_h|$ if $h \in [i+2, D]$,
$|B_{i-1}| = |B_{i+1}| = |B'_{i-1}|$ if $1 < i < D$,
$|B_{i-1}| \in \{0, -1\}$ (hence $\epsilon_{i-1}(B) = 0$) if $i = D$,
$|B_{i+1}| \in \{0, -1\}$ (hence $\epsilon_{i+1}(B) = 0$) if $i = 1$.
This follows from 1.4(a).

For $B \in S_D$ let $\langle B \rangle$ be the subspace of V generated by $\{e_I; I \in B\}$. For $B' \in S_{D-2}$ let $\langle B' \rangle$ be the subspace of V' generated by $\{e'_{I'}; I' \in B'\}$. We show:

(c) *Let $B \in S_D$. We have $\epsilon(B) \in \langle B \rangle$. If $D \geq 2, i \in [1, D]$, $B' \in S_{D-2}, B = t_i(B') \in S_D$, then $\langle B \rangle = T_i(\langle B' \rangle) \oplus \mathbf{F}_2 e_i$.*

To prove the first assertion of (c) we argue by induction on D. For $D = 0$ there is nothing to prove. Assume that $D \geq 2$. Let i, B' be as in (b). By the induction hypothesis we have $\epsilon'(B') \in \langle B' \rangle \subset V'$. Using (b) we see that it is enough to show that $T_i(\langle B' \rangle) \subset \langle B \rangle$. (Since $\{i\} \in B$, we have $e_i \in \langle B \rangle$.) Using the equality $T_i(e'_{I'}) = e_{\xi_i(I')}$ for any $I' \in B'$ it remains to note that $\xi_i(I') \in B$ for $I' \in B'$. This proves the first assertion of (c). The same proof shows the second assertion of (c).

1.10. Let $B \in S_D, \tilde{B} \in S_D$. We show:

(a) *If $\epsilon(B) = \epsilon(\tilde{B})$, then $B = \tilde{B}$.*

We argue by induction on D. If $D = 0$, there is nothing to prove. Assume that $D \geq 2$. If $\tilde{B} = \emptyset$, (a) follows from 1.8(a). Similarly, (a) holds if $B = \emptyset$. Thus, we can assume that $B \neq \emptyset$, $\tilde{B} \neq \emptyset$. By 1.7(a) we can find $i \in [1, D]$ such that $\{i\} \in B$, $\{i\} \in \tilde{B}$. By 1.3(d) we then have $B = t_i(B')$, $\tilde{B} = t_i(\tilde{B}')$ with $B' \in S_{D-2}$, $\tilde{B}' \in S_{D-2}$. Using our assumption and 1.9(b) we see that $T_i(\epsilon'(B')) = T_i(\epsilon'(\tilde{B}')) + ce_i$ for some $c \in \mathbf{F}_2$. Using 1.9(a) we see that $c = 0$ so that $T_i(\epsilon'(B')) = T_i(\epsilon'(\tilde{B}'))$. Since T_i is injective, we deduce $\epsilon'(B') = \epsilon'(\tilde{B}')$. By the induction hypothesis we have $B' = \tilde{B}'$ hence $B = \tilde{B}$. This proves (a).

1.11. Any $x \in V$ can be written uniquely in the form

$$x = e_{[a_1,b_1]} + e_{[a_2,b_2]} + \cdots + e_{[a_r,b_r]},$$

where $[a_r, b_r] \in \mathcal{I}_D$ are such that any two of them are non-touching and $r \geq 0$, $1 \leq a_1 \leq b_1 < a_1 \leq b_2 < \cdots < a_r \leq b_r \leq D$. Following [L3, 3.3] we set

(a) $$u(v) = |\{s \in [1, r]; a_s =_2 0, b_s =_2 1\}| - |\{s \in [1, r]; a_s =_2 1, b_s =_2 0\}| \in \mathbf{Z}.$$

This defines a function $u : V \to \mathbf{Z}$. When $D \geq 2$ we denote by $u' : V' \to \mathbf{Z}$ the analogous function with D replaced by $D - 2$. We show:

(b) *Assume that $D \geq 2, i \in [1, D]$. Let $v' \in V'$ and let $v = T_i(v') + ce_i \in V$ where $c \in \mathbf{F}_2$. We have $u(v) = u'(v')$.*

We write $v' = e'_{[a'_1,b'_1]} + e'_{[a'_2,b'_2]} + \cdots + e'_{[a'_r,b'_r]}$ where $r \geq 0$, $[a'_s, b'_s] \in \mathcal{I}_{D-2}$ for all s and any two of $[a'_s, b'_s]$ are non-touching. For each s, we have $T_i(e'_{[a'_s,b'_s]}) = e_{[a_s,b_s]}$ where $[a_s, b_s] = \xi_i[a'_s, b'_s]$ so that $a_s =_2 a'_s$, $b_s =_2 b'_s$ and the various $[a_s, b_s]$ which appear are still non-touching with each other. Hence $u(T_i(v')) = u'(v')$. We have $v = T_i(v')$ or $v = T_i(v') + e_i$. If $v = T_i(v')$, we have $u(v) = u'(v')$, as desired. Assume now that $v = T_i(v') + e_i$. From the definition of ξ_i we see that either

(i) $[i, i]$ is non-touching with any $[a_s, b_s]$, or

(ii) $[i, i]$ is not non-touching with some $[a, b] = [a_s, b_s]$ which is uniquely determined and we have $a < i < b$.

If (i) holds, then e_i does not contribute to $u(v)$ and $u(v) = u(T_i(v')) = u'(v')$. We now assume that (ii) holds. Then $e_{[a,b]} + e_i = e_{[a,i-1]} + e_{[i+1,b]}$. We consider six cases.

(1) a is even b is odd, i is even; then $|[i+1,b]|$ is odd so that the contribution of $e_{[a,i-1]} + e_{[i+1,b]}$ to $u(v)$ is $1+0$; this equals the contribution of $e_{[a,b]}$ to $u(T_i(v'))$ which is 1.

(2) a is even, b is odd, i is odd; then $|[a,i-1]|$ is odd so that the contribution of $e_{[a,i-1]} + e_{[i+1,b]}$ to $u(v)$ is $0+1$; this equals the contribution of $e_{[a,b]}$ to $u(T_i(v'))$ which is 1.

(3) a is odd, b is even, i is even; then $|[i+1,b]|$ is odd so that the contribution of $e_{[a,i-1]} + e_{[i+1,b]}$ to $u(v)$ is $0-1$; this equals the contribution of $e_{[a,b]}$ to $u(T_i(v'))$ which is -1.

(4) a is odd, b is even, i is odd; then $|[a,i-1]|$ is odd so that the contribution of $e_{[a,i-1]} + e_{[i+1,b]}$ to $u(v)$ is $-1+0$; this equals the contribution of $e_{[a,b]}$ to $u(T_i(v'))$ which is -1.

(5) $a =_2 b =_2 i+1$; then $|[a,i-1]|$ is odd, $|[i+1,b]|$ is odd so that the contribution of $e_{[a,i-1]} + e_{[i+1,b]}$ to $u(v)$ is $0+0$; this equals the contribution of $e_{[a,b]}$ to $u(T_i(v'))$ which is 0.

(6) $a =_2 b =_2 i$; then the contribution of $e_{[a,i-1]} + e_{[i+1,b]}$ to $u(v)$ is $1-1$ or $-1+1$; this equals the contribution of $e_{[a,b]}$ to $u(T_i(v'))$ which is 0.

This proves (b).

1.12. We view V as the set of vertices of a graph in which x, x' in V are joined whenever there exists $i \in [1, D]$ such that $x + x' = e_i$, $(x, e_i) = (x', e_i) = 0$. Similarly if $D \geq 2$, we view V' as the set of vertices of a graph in which y, y' in V' are joined whenever there exists $i \in [1, D-2]$ such that $y + y' = e'_i$, $(y, e'_i)' = (y', e'_i)' = 0$. We show:

(a) *If y, y' in V' are joined in the graph V', then $T_i(y), T_i(y')$ are in the same connected component of the graph V.*

We can find $j \in [1, 2d-2]$ such that $(y, e'_j)' = (y', e'_j)' = 0$, $y + y' = e'_j$. Hence $(\tilde{y}, T_i(e'_j)) = (\tilde{y}', T_i(e'_j)) = 0$, $\tilde{y} + \tilde{y}' = T_i(e'_j)$ where $\tilde{y} = T_i(y), \tilde{y}' = T_i(y')$. If $T_i(e'_j) = e_h$ for some $h \in [1, 2d]$, then $\tilde{y}, \tilde{y}'$ are joined in V, as required. If this condition is not satisfied, then $1 < i < D$, $j = i-1$ and $T_i(e'_j) = e_j + e_{j+1} + e_{j+2}$. We have $(\tilde{y}, e_j + e_{j+1} + e_{j+2}) = 0$, $\tilde{y} + \tilde{y}' = e_j + e_{j+1} + e_{j+2}$. Since $\tilde{y} \in V_i$, we have $(\tilde{y}, e_i) = 0$ hence $(\tilde{y}, e_{j+1}) = 0$ so that $(\tilde{y}, e_j) = (\tilde{y}, e_{j+2})$. We are in one of the two cases below.

(1) We have $(\tilde{y}, e_j) = (\tilde{y}, e_{j+2}) = 0$.

(2) We have $(\tilde{y}, e_j) = (\tilde{y}, e_{j+2}) = 1$.

In case (1) we consider the four term sequence $\tilde{y}, \tilde{y} + e_j, \tilde{y} + e_j + e_{j+2}, \tilde{y} + e_j + e_{j+1} + e_{j+2} = \tilde{y}'$; any two consecutive terms of this sequence are joined in the graph V. In case (2) we consider the four term sequence $\tilde{y}, \tilde{y} + e_{j+1}, \tilde{y} + e_j + e_{j+1}, \tilde{y} + e_j + e_{j+1} + e_{j+2} = \tilde{y}'$; any two consecutive terms of this sequence are joined in the graph V. We see that in both cases $\tilde{y}, \tilde{y}'$ are in the same connected component of V; (a) is proved.

Let $V_0 = \{x \in V; u(x) = 0\}$. Note that $0 \in V_0$. We show:

(b) *If $x \in V_0$, then $x, 0$ are in the same component of the graph V.*

We argue by induction on D. If $D=0$ there is nothing to prove. Assume now that $D\ge 2$. If $(x,e_i)=1$ for all $i\in[1,D]$, then

$$x=e_{[2,3]}+e_{[6,7]}+e_{[10,11]}+\dots+e_{[D-2,D-1]} \text{ if } D/2 \text{ is even,}$$
$$x=e_{[1,2]}+e_{[5,6]}+e_{[9,10]}+\dots+e_{[D-1,D]} \text{ if } D/2 \text{ is odd.}$$

In both cases we have $u(x)\ne 0$ contradicting our assumption. Thus we have $(x,e_i)=0$ for some $i\in[1,D]$. By 1.9(a) we have $x=T_i(x')+ce_i$ for some $x'\in V'$ and some $c\in\mathbf{F}_2$. By 1.11(b) we have $u'(x')=0$. By the induction hypothesis $x',0$ are in the same connected component of V'. By (a), $T_i(x'),0$ are in the same connected component of V. Clearly $x,T_i(x')$ are joined in the graph V. Hence $x,0$ are joined in the graph V. We see that (b) holds.

We show:

(c) *V_0 is a connected component of the graph V.*

If x,x' in V are in the same connected component of V, then $u(x)=u(x')$. (We can assume that x,x' are joined in the graph V. Then for some $i\in[1,D]$ we have $x=T_i(y)+ce_i$, $x'=T_i(y)+c'e_i$ where $y\in V'$, $c\in\mathbf{F}_2, c'\in\mathbf{F}_2$. By 1.11(b) we have $u(x)=u'(y)$, $u(x')=u'(y)$, hence $u(x)=u(x')$, as required.) Thus V_0 is a union of connected components of V. On the other hand, by (b), V_0 is contained in a connected component of the graph V. This proves (c).

1.13. We show:

(a) *If $B\in S_D$, then $\langle B\rangle\subset V_0$.*

We argue by induction on D. If $D=0$ there is nothing to prove. Assume that $D\ge 2$. If $B=\emptyset$ there is nothing to prove. Assume that $B\ne\emptyset$. We can find $i\in[1,D]$ and $B'\in S_{D-2}$ such that $B=t_i(B')$. By 1.9(c) we have $\langle B\rangle=T_i(\langle B'\rangle)\oplus\mathbf{F}_2e_i$. Using 1.11(b), to prove that $u=0$ on $\langle B\rangle$ it is enough to prove that $u'=0$ on $\langle B'\rangle$ and this follows from the induction hypothesis. This proves (a).

We show:

(b) *If $x\in V_0$, then $x\in\langle B\rangle$ for some $B\in S_d$.*

We argue by induction on D. If $D=0$ there is nothing to prove. Assume that $D\ge 2$. As in the proof of 1.12(b), from the fact that $u(x)=0$ we can deduce that $(x,e_i)=0$ for some $i\in[1,D]$. By 1.9(a) we have $x=T_i(x')+ce_i$ for some $x'\in V'$ and some $c\in\mathbf{F}_2$. By 1.11(b) we have $u'(x')=0$. By the induction hypothesis we have $x'\in\langle B'\rangle$ for some $B'\in S_{D-2}$. Then $x\in T_i(\langle B'\rangle)\oplus\mathbf{F}_2e_1=\langle B\rangle$ (we use 1.9(c)). This proves (b).

From (a),(b) we deduce:

(c) *We have $\bigcup_{B\in S_D}\langle B\rangle=V_0$.*

A closely related result is proved in [L3, 3.4].

1.14. The function $\epsilon:S_D\to V$ has values in $\bigcup_{B\in S_D}\langle B\rangle$ (see 1.9(c)) hence by 1.13(c) it has values in V_0. Thus, it can be viewed as a function $\epsilon:S_D\to V_0$.

From 1.10(a) we see that:

(a) *$\epsilon:S_D\to V_0$ is injective.*

1.15. Let F_0 be the $\mathbf{Q}$-vector space consisting of functions $V_0\to\mathbf{Q}$. For $x\in V_0$ let $\psi_x\in F_0$ be the characteristic function of x. For $B\in S_D$ let $\Psi_B\in F_0$ be the characteristic function of $\langle B\rangle$. (We use that $\langle B\rangle\subset V_0$; see 1.13.) Let $\tilde F_0$ be the $\mathbf{Q}$-subspace of F_0 generated by $\{\Psi_B; B\in S_D\}$. When $D\ge 2$ we define $\psi'_{x'}$ for $x'\in V'$ and $\Psi'_{B'}$ for $B'\in S_{D-2}$, $F'_0,\tilde F'_0$, in terms of S_{D-2} in the same way as

$\psi_x, \Psi_B, F_0, \tilde{F}_0$ were defined in terms of S_D. For any $i \in [1, D]$ we define a linear map $\theta_i : F'_0 \to F_0$ by $f' \mapsto f$ where $f(T_i(x') + ce_i) = f'(x')$ for $x' \in V', c \in \mathbf{F}_2$, $f(x) = 0$ for $x \in V - e_i^\perp$. We have

$\theta_i(\psi'_{x'}) = \psi_{T_i(x')} + \psi_{T_i(x')+e_i}$ for any $x' \in V'$,

$\theta_i(\Psi'_{B'}) = \Psi_{t_i(B')}$ for any $B' \in S_{D-2}$.

We show:

(a) *For any $x \in V_0$, we have $\psi_x \in \tilde{F}_0$.*

We argue by induction on D. If $D = 0$ the result is obvious. We now assume that $D \geq 2$. We first show:

(b) *If $x, \tilde{x}$ in V_0 are joined in the graph V and if* (a) *holds for x, then* (a) *holds for $\tilde{x}$.*

We can find $j \in [1, 2d]$ such that $x + \tilde{x} = e_j$, $(x, e_j) = 0$. We have $x = T_j(x') + ce_j$, $\tilde{x} = T_j(x') + c'e_j$ where $x' \in V'$ and $c \in \mathbf{F}_2, c' \in \mathbf{F}_2$, $c + c' = 1$. By the induction hypothesis we have $\psi'_{x'} = \sum_{B' \in S_{D-2}} a_{B'} \Psi'_{B'}$ where $a_{B'} \in \mathbf{Q}$. Applying θ_j we obtain

$$\psi_x + \psi_{\tilde{x}} = \sum_{B' \in S_{D-2}} a_{B'} \Psi_{t_j(B')}.$$

We see that $\psi_x + \psi_{\tilde{x}} \in \tilde{F}_0$. Since $\psi_x \in \tilde{F}$, by assumption, we see that $\psi_{\tilde{x}} \in \tilde{F}$. This proves (b).

We now prove (a). Since V_0 is the connected component of V containing 0, to prove (a) it is enough (by (b)) to show that (a) holds when $x = 0$. This follows from the fact that $\psi_0 = \Psi_B$ where $B = \emptyset$. This proves (a).

Since $\tilde{F}_0 \subset F_0$, we see that (a) implies:

(c) $F_0 = \tilde{F}_0$.

We have the following result.

Theorem 1.16. (a) *$\{\Psi_B; B \in S_D\}$ is a $\mathbf{Q}$-basis of F_0.*

(b) *$\epsilon : S_D \to V_0$ is a bijection.*

Proof. From the definition of $\tilde{F}_0$ we have $\dim \tilde{F}_0 \leq |S_D|$. By 1.14(a) we have $|S_D| \leq |V_0| = \dim F_0$. Since $F_0 = \tilde{F}_0$ (see 1.15(c)) it follows that $\dim \tilde{F}_0 = |S_D| = |V_0| = \dim F_0$. Using again the definition of $\tilde{F}_0$ and the equality $F_0 = \tilde{F}_0$ we see that (a) holds. Since the map in (b) is injective (see 1.14(a)) and $|S_D| = |V_0|$ we see that it is a bijection so that (b) holds. ∎

1.17. In this subsection we describe the bijection in 1.16(b) assuming that D is 2, 4, or 6. In each case we give a table in which there is one row for each $B \in S_D$; the row corresponding to B is of the form $\langle B \rangle : (\dots)$ where B is represented by the list of intervals of B (we write an interval such as $[4, 6]$ as 456) and $(\dots)$ is a list of the vectors in $\langle B \rangle$ (we write 1235 instead of $e_1 + e_2 + e_3 + e_5$, etc.). In each list $(\dots)$ we single out the vector corresponding $\epsilon(B)$ in 1.16(b) by putting it in a box. Any non-boxed entry in $(\dots)$ appears as a boxed entry in some previous row. We see that in these cases, 0.5(i) holds.

The table for $D = 2$.

$\emptyset : (\boxed{0})$

$\langle 1 \rangle : (0, \boxed{1})$

$\langle 2 \rangle : (0, \boxed{2})$.

The table for $D = 4$.

$\emptyset : (\boxed{0})$

$\langle 1\rangle : (0, \boxed{1})$

$\langle 2\rangle : (0, \boxed{2})$

$\langle 3\rangle : (0, \boxed{3})$

$\langle 4\rangle : (0, \boxed{4})$

$\langle 1,3\rangle : (0,1,3, \boxed{13})$

$\langle 1,4\rangle : (0,1,4, \boxed{14})$

$\langle 2,4\rangle : (0,2,4, \boxed{24})$

$\langle 2,123\rangle : (0,2,13, \boxed{123})$

$\langle 3,234\rangle : (0,3,24, \boxed{234})$.

The table for $D=6$.

$\emptyset : (\boxed{0})$

$\langle 1\rangle : (0, \boxed{1})$

$\langle 2\rangle : (0, \boxed{2})$

$\langle 3\rangle : (0, \boxed{3})$

$4\langle\rangle : (0, \boxed{4})$

$5\langle\rangle : (0, \boxed{5})$

$\langle 6\rangle : (0, \boxed{6})$

$\langle 1,4\rangle : (0,1,4, \boxed{14})$

$\langle 1,6\rangle : (0,1,6, \boxed{16})$

$\langle 2,4\rangle : (0,2,4, \boxed{24})$

$\langle 2,5\rangle : (0,2,5, \boxed{25})$

$\langle 2,6\rangle : (0,2,6, \boxed{26})$

$\langle 3,6\rangle : (0,3,6, \boxed{36})$

$\langle 4,6\rangle : (0,4,6, \boxed{46})$

$\langle 1,3\rangle : (0,1,3, \boxed{13})$

$\langle 1,5\rangle : (0,1,5, \boxed{15})$

$\langle 3,5\rangle : (0,3,5, \boxed{35})$

$\langle 2,123\rangle : (0,2,13, \boxed{123})$

$\langle 3,234\rangle : (0,3,24, \boxed{234})$

$\langle 4,345\rangle : (0,4,35, \boxed{345})$

$\langle 5,456\rangle : (0,5,46, \boxed{456})$

$\langle 1,3,5\rangle : (0,1,3,5,13,15,35, \boxed{135})$

$\langle 1,3,6\rangle : (0,1,3,6,13,16,36, \boxed{136})$

$\langle 1,4,345\rangle : (0,1,4,345,14,35,135, \boxed{1345})$

$\langle 1,4,6\rangle : (0,1,4,6,14,16,46, \boxed{146})$

$\langle 2,4,6\rangle : (0,2,4,6,24,26,46, \boxed{246})$

$\langle 1,5,456\rangle : (0,1,5,456,15,46,146, \boxed{1456})$

$\langle 2,5,456\rangle : (0,2,5,456,25,46,246, \boxed{2456})$

$\langle 2,5,123\rangle : (0,2,5,123,25,13,135, \boxed{1235})$

$\langle 2,6,123\rangle : (0,2,6,123,26,13,136, \boxed{1236})$

$\langle 2,4,12345\rangle : (0,2,4,24,1345,1235,135, \boxed{12345})$

$\langle 3, 234, 12345\rangle : (0, 3, 234, 12345, 24, 15, 135, \boxed{1245})$
$\langle 3, 6, 234\rangle : (0, 3, 6, 234, 24, 36, 246, \boxed{2346})$
$\langle 3, 5, 23456\rangle : (0, 3, 5, 2456, 35, 2346, 246, \boxed{23456})$
$\langle 4, 345, 23456\rangle : (0, 4, 345, 23456, 35, 26, 246, \boxed{2356})$.

2. The sets $\mathcal{F}_*(V), \mathcal{F}(V)$

2.1. We no longer assume that D is even. We define a collection $\mathcal{F}_*(V)$ and a collection $\mathcal{F}(V)$ of subspaces of V by induction on D as follows. If $D \in \{0, 1\}$, $\mathcal{F}_*(V)$ and $\mathcal{F}(V)$ consist of $\{0\}$. If $D \geq 2$, a subspace X of V is said to be in $\mathcal{F}_*(V)$ if there exists $i \in [1, D]$ (if D is even) or $i \in [1, D-1]$ (if D is odd) and $X' \in \mathcal{F}_*(V')$ such that $X = T_i(X') \oplus \mathbf{F}_2 e_i$; a subspace X of V is said to be in $\mathcal{F}(V)$ if either $X = 0$ or if there exists $i \in [1, D]$ (if D is even) or $i \in [1, D-1]$ (if D is odd) and $X' \in \mathcal{F}(V')$ such that $X = T_i(X') \oplus \mathbf{F}_2 e_i$. By induction on D we see that for $X \in \mathcal{F}_*(V)$ we have $X \in \mathcal{F}(V)$ and $\dim(X) = D/2$ if D is even, $\dim(X) = (D-1)/2$ if D is odd. When D is odd, let $\underline{V}$ be the subspace of V with basis $\{e_1, e_2, \ldots, e_{D-1}\}$. This vector space with basis is of the same kind as V in 1.9 (but of even dimension) hence $\mathcal{F}(\underline{V}), \mathcal{F}_*(\underline{V})$ are defined. Using induction on D we see that for D odd we have $\mathcal{F}(V) = \mathcal{F}(\underline{V}), \mathcal{F}_*(V) = \mathcal{F}_*(\underline{V})$. Thus, the study of $\mathcal{F}(V), \mathcal{F}_*(V)$ when D is odd is reduced to the similar study when D is even.

We now assume that D is even. If $B \in S_D$, then $\langle B\rangle \in \mathcal{F}(V)$ (this follows from 1.9(c) by induction on D). Conversely, if $X \in \mathcal{F}(V)$, then there exists $B \in S_D$ such that $X = \langle B\rangle$ (this again follows from 1.9(c) by induction on D). Thus we have a surjective map $S_D \to \mathcal{F}(V)$, $B \mapsto \langle B\rangle$. We show:

(a) *This map is a bijection.*

Indeed, if $B, \tilde{B}$ in S_D satisfy $\langle B\rangle = \langle \tilde{B}\rangle$, then the functions $\Psi_B, \Psi_{\tilde{B}}$ in F_0 coincide hence $B = \tilde{B}$ by 1.16(a). This proves (a).

For $B \in S_D$ we show:

(b) $\{e_I; I \in B\}$ *is an* $\mathbf{F}_2$*-basis of* $\langle B\rangle$.

We argue by induction on D. If $D = 0$ there is nothing to prove. Assume that $D \geq 2$. If $B = \emptyset$, then (b) is obvious. We now assume that $B \neq \emptyset$. Assume that $\sum_{I \in B} c_I e_I = 0$ with $c_I \in \mathbf{F}_2$ not all zero. We can find $I = [a, b] \in B$ with $c_I \neq 0$ and $|I|$ maximal. If $I' \in B$ is such that $a \in I'$, $I' \neq I$, $c_{I'} \neq 0$, then by (P_0) we have $I \prec I'$ (contradicting the maximality of $|I|$) or $I' \prec I$ (contradicting $a \in I'$). Thus no I' as above exists. Thus when $\sum_{I_1 \in B} c_{I_1} e_{I_1}$ is written in the basis $\{e_j; j \in [1, D]\}$, the coefficient of e_a is c_{I_1} hence $c_{I_1} = 0$, contradicting $c_{I_1} \neq 0$. This proves (b).

We show:

(c) *If* $X \in \mathcal{F}(V)$*, then* X *is an isotropic subspace of* V.

We argue by induction on D. If $D = 0$ there is nothing to prove. Assume that $D \geq 2$. If $X = 0$, then (c) is obvious. We now assume that $X \neq 0$. Then there exists $i \in [1, D]$ and $X' \in \mathcal{F}(V')$ such that $X = T_i(X') \oplus \mathbf{F}_2 e_i$. By the induction hypothesis, X' is isotropic in V'. Since T_i is compatible with the symplectic forms it follows that $T_i(X')$ is an isotropic subspace of V. Since $T_i(X')$ is contained in $e_i^\perp$, $T_i(X') \oplus \mathbf{F}_2 e_i$ is also isotropic. This proves (c). Alternatively, (c) can be deduced from property (P_0).

2.2. For $\delta \in \{0,1\}$ let $[1,D]^\delta = \{i \in [1,D]; i =_2 \delta\}$. Let V^δ be the subspace of V with basis $\{e_i; i \in [1,D]^\delta\}$. We have $V = V^0 \oplus V^1$. Similarly, if $D \geq 2$, we have $V' = V'^0 \oplus V'^1$ where V'^δ has basis $\{e'_i; i \in [1,D-2]^\delta\}$.

For any $I \in \mathcal{I}^1_D$ and $\delta \in \{0,1\}$ we set $I^\delta = I \cap [1,D]^\delta$, so that $I = I^0 \sqcup I^1$; we define $\kappa(I) \in \{0,1\}$ by $a =_2 \kappa(I)$ or equivalently $b =_2 \kappa(I)$ where $I = [a,b]$. We show:

(a) *Let $B \in S_D$ and let $I \in B$. Let $\delta = \kappa(I)$. We have $e_{I^\delta} = \sum_{I' \in B; I' \subset I} e_{I'}$.*

We argue by induction on $|I|$. If $|I| = 1$ the result is obvious. Assume now that $|I| > 1$. By $(P_0), (P_1)$, we can find $[a_1,b_1],[a_2,b_2],\dots,[a_k,b_k]$ in B such that $a_1 \leq b_1 < a_2 \leq b_2 < a_3 \leq b_3 <,\dots,$ $a_1,b_1,a_2,b_2,\dots,$ are all in $1-\delta+2\mathbf{Z}$ and $[a,b] \cap (1-\delta+2\mathbf{Z}) \subset [a_1,b_1] \cup [a_2,b_2] \cup \dots \cup [a_k,b_k]$. From the definition we have $e_{I^\delta} = e_I + \sum_{j=1}^k e_{[a_j,b_j]^{1-\delta}}$. By the induction hypothesis, for $j \in [1,k]$ we have $e_{[a_j,b_j]^{1-\delta}} = \sum_{I' \in B; I' \subset [a_j,b_j]} e_{I'}$. Thus we have

$$e_{I^\delta} = e_I + \sum_{I' \in B; I' \subset \cup_j [a_j,b_j]} e_{I'} = \sum_{I' \in B; I' \subset I} e_{I'}.$$

This proves (a).

We show:

(b) *Let $B \in S_D$. Then $\{e_{I^{\kappa(I)}}; I \in B\}$ is a basis of the vector space $\langle B \rangle$.*

From (a) we see that the collection of vectors $\{e_{I^{\kappa(I)}}; I \in B\}$ is related to the collection of vectors $\{e_I; I \in B\}$ by an upper triangular matrix with 1 on the diagonal. Hence the result follows from 2.1(b).

We deduce that if $B \in S_D$ and $X = \langle B \rangle \in \mathcal{F}(V)$, then for $\delta \in \{0,1\}$,

(c) *$X^\delta = X \cap V^\delta$ has basis $\{e_{I^{\kappa(I)}}; I \in B, \kappa(I) = \delta\}$; in particular, $X = X^0 \oplus X^1$.*

2.3. Assume that $D \geq 2$. Let $i \in [1,D]$ and let $\delta \in \{0,1\}$. There is a unique linear map $T_i^\delta : V'^\delta \to V^\delta$ such that

$T_i^\delta(e'_k) = e_k$ if $k \leq i-2$, $k =_2 \delta$;

$T_i^\delta(e'_k) = e_{k+2}$ if $k \geq i$, $k =_2 \delta$;

$T_i^\delta(e'_{i-1}) = e_{i-1} + e_{i+1}$ if $i =_2 \delta + 1$, $1 < i < D$.

Note that T_i^δ is injective and $(x,y)' = (T_i^0(x), T_i^1(y))$ for any $x \in V'^0, y \in V'^1$. For any $I' \in \mathcal{I}^1_{D-2}$ such that $\kappa(I') = \delta$ we have $T_i^\delta(e'_{I'^\delta}) = e_{\xi_i(I')^\delta}$. (Here $\kappa(I'), I'^\delta$ are defined in terms of I' in the same way as $\kappa(I), I^\delta$ are defined in 2.2.) Let V_i^δ be the image of $T_i^\delta : V'^\delta \to V^\delta$. From the definitions we deduce:

(a) *We have $V_i \oplus \mathbf{F}_2 e_i = V_i^0 \oplus V_i^1 \oplus \mathbf{F}_2 e_i$.*

We define a collection $\mathcal{C}(V^\delta)$ of subspaces of V^δ by induction on D as follows. If $D = 0$, $\mathcal{C}(V^\delta)$ consists of $\{0\}$. If $D \geq 2$, a subspace $\mathcal{L}$ of V^δ is said to be in $\mathcal{C}(V^\delta)$ if either $\mathcal{L} = 0$ or if there exists $i \in [1,D]$ and $\mathcal{L}' \in \mathcal{C}(V'^\delta)$ such that $\mathcal{L} = T_i^\delta(\mathcal{L}') \oplus \mathbf{F}_2 e_i$ (if $i =_2 \delta$) or $\mathcal{L} = T_i^\delta(\mathcal{L}')$ (if $i =_2 \delta + 1$).

We show:

(b) *If $X \in \mathcal{F}(V)$, then $X^\delta \in \mathcal{C}(V^\delta)$.*

We argue by induction on D. If $D = 0$ the result is obvious. Assume now that $D \geq 2$. If $X = 0$ there is nothing to prove. Assume that $X \neq 0$. We can find $i \in [1,D]$ and $X' \in \mathcal{F}(V')$ such that $X = T_i(X') \oplus \mathbf{F}_2 e_i$. By the induction hypothesis we have $X'^\delta \in \mathcal{C}(V'^\delta)$. Hence $T_i^\delta(X'^\delta) \oplus \mathbf{F}_2 e_i \in \mathcal{C}(V^\delta)$ if $i =_2 \delta$, $T_i^\delta(X'^\delta) \in \mathcal{C}(V^\delta)$ if $i =_2 \delta + 1$. It is enough to prove that $T_i^\delta(X'^\delta) \oplus \mathbf{F}_2 e_i = X^\delta$ if $i =_2 \delta$, $T_i^\delta(X'^\delta) = X^\delta$ if $i =_2 \delta + 1$, or that $T_i^\delta(X'^\delta) \oplus \mathbf{F}_2 e_i = (T_i(X') \oplus \mathbf{F}_2 e_i) \cap V^\delta$

if $i =_2 \delta$, $T_i^\delta(X'^\delta) = (T_i(X') \oplus \mathbf{F}_2 e_i) \cap V^\delta$ if $i =_2 \delta + 1$. This follows by comparing the definition of T_i^δ with that of T_i.

2.4. Let $\delta \in \{0,1\}$. If Z is a subspace of V^δ we set $Z^! = \{x \in V^{1-\delta}; (x,z) = 0 \quad \forall z \in Z\}$. Similarly, if Z' is a subspace of V'^δ we set $Z'^! = \{x \in V'^{1-\delta}; (x,z)' = 0 \quad \forall z \in Z'\}$. Let $\mathcal{L} \in \mathcal{C}(V^\delta)$. We show:

(a) *We have $\mathcal{L}^! \in \mathcal{C}(V^{1-\delta})$ and $\mathcal{L} \oplus \mathcal{L}^! \subset V$ is in $\mathcal{F}(V)$.*

The first statement of (a) follows from the second statement, using 2.3(b). We prove the second statement of (a) by induction on D. If $D = 0$ the result is immediate. Assume now that $D \geq 2$. If $\mathcal{L} = 0$, then $\mathcal{L}^! = V^{1-\delta} = \langle B \rangle$ where $B = \{\{j\}; j \in [1,D]^{1-\delta}\} \in S_D$; thus we have $\mathcal{L}^! \in \mathcal{F}(V)$. Next we assume that $\mathcal{L} \neq 0$. We can find $i \in [1,D]$ and $\mathcal{L}' \in \mathcal{C}(V'^\delta)$ such that $\mathcal{L} = T_i^\delta(\mathcal{L}') \oplus \mathbf{F}_2 e_i$ (if $i =_2 \delta$) or $\mathcal{L} = T_i^\delta(\mathcal{L}')$ (if $i =_2 \delta + 1$). By the induction hypothesis we have $\mathcal{L}' \oplus \mathcal{L}'^! \in \mathcal{F}(V')$. Hence $T_i(\mathcal{L}' \oplus \mathcal{L}'^!) \oplus \mathbf{F}_2 e_i \in \mathcal{F}(V)$. From the definition we have $T_i(\mathcal{L}' \oplus \mathcal{L}'^!) \oplus \mathbf{F}_2 e_i = T_i^\delta(\mathcal{L}') \oplus T_i^{1-\delta}(\mathcal{L}'^!) \oplus \mathbf{F}_2 e_i$. Thus we have $T_i^\delta(\mathcal{L}') \oplus T_i^{1-\delta}(\mathcal{L}'^!) \oplus \mathbf{F}_2 e_i \in \mathcal{F}(V)$ or equivalently $\mathcal{L} \oplus T_i^{1-\delta}(\mathcal{L}'^!) \in \mathcal{F}(V)$ (if $i =_2 \delta$) and $\mathcal{L} \oplus T_i^{1-\delta}(\mathcal{L}'^!) \oplus \mathbf{F}_2 e_i \in \mathcal{F}(V)$ (if $i =_2 \delta + 1$). It is enough to show: $\mathcal{L}^! = T_i^{1-\delta}(\mathcal{L}'^!)$ if $i =_2 \delta$ and $\mathcal{L}^! = T_i^{1-\delta}(\mathcal{L}'^!) \oplus \mathbf{F}_2 e_i$ if $i =_2 \delta + 1$. If $y \in \mathcal{L}'^!$, $x \in \mathcal{L}'$, we have $(T_i^{1-\delta}(y), T_i^\delta(x)) = (y,x)' = 0$; if $i =_2 \delta$ we have $(T_i^{1-\delta}(y), e_i) = 0$. If $i =_2 \delta + 1$ we have $(e_i, T_i^\delta(x)) = 0$. We see that $T_i^{1-\delta}(\mathcal{L}'^!) \subset \mathcal{L}^!$ if $i =_2 \delta$ and $T_i^{1-\delta}(\mathcal{L}'^!) \oplus \mathbf{F}_2 e_i \subset \mathcal{L}^!$ if $i =_2 \delta + 1$. The last two inclusions are between vector spaces of the same dimension; hence they must be equalities. This completes the proof of (a).

Let $S_{D,*} = \{B \in S_D; |B| = D/2\}$. From 2.1(b) we see that the bijection $S_D \xrightarrow{\sim} \mathcal{F}(V)$, $B \mapsto \langle B \rangle$ (see 2.1(a)) restricts to a bijection

(b) $S_{D,*} \xrightarrow{\sim} \mathcal{F}_*(V)$.

We show:

(c) *We have a bijection $\iota : \mathcal{C}(V^\delta) \xrightarrow{\sim} \mathcal{F}_*(V)$ given by $\iota(\mathcal{L}) = \mathcal{L} \oplus \mathcal{L}^!$.*

The fact that ι is well defined follows from (a). (For $\mathcal{L} \in \mathcal{C}(V^\delta)$ we have $\dim(\mathcal{L} \oplus \mathcal{L}^!) = D/2$.) We define $\iota' : \mathcal{F}_*(V) \to \mathcal{C}(V^\delta)$ by $X \mapsto X^\delta$. This is well defined by 2.3(b). Clearly, $\iota'\iota = 1$. Let $X \in \mathcal{F}_*(V)$. Then $X^{1-\delta} \subset (X^\delta)^!$ since X is isotropic so that $X^\delta \oplus (X^\delta)^! \subset X$; this is an inclusion of vector spaces of the same dimension, hence is an equality. Thus $\iota\iota' = 1$. This proves that ι is a bijection.

2.5. Let $\delta \in \{0,1\}$. We define a subset S_D^δ of R_D^1 by induction on D as follows. When $D = 0$, S_D^δ consists of $\emptyset \in R_D^1$. When $D \geq 2$ we say that $\beta \in R_D^1$ is in S_D^δ if either $\beta = \emptyset$ or if

(i) there exists $i \in [1,D]$ and $\beta' \in S_{D-2}^\delta$ such that $\beta = \{\xi_i(I'); I' \in \beta'\} \sqcup \{i\}$ if $i =_2 \delta$ and $\beta = \{\xi_i(I'); I' \in \beta'\}$ if $i =_2 \delta + 1$.

From the definition we see by induction on D that if $\beta \in S_D^\delta$ and $I \in \beta$, then $\kappa(I) = \delta$.

Let ${S'}_D^\delta$ be the set of all $\beta \in R_D^1$ such that $\kappa(I) = \delta$ for any $I \in \beta$ and such that the following holds:

(P_0^δ) *If $I \in \beta$, $\tilde{I} \in \beta$, then either $I = \tilde{I}$, or $I \spadesuit \tilde{I}$, or $I \prec \tilde{I}$, or $\tilde{I} \prec I$.*

By arguments similar to those in 1.3 we see that

(a) *We have $S_D^\delta = {S'}_D^\delta$.*

We show:

(b) *If $B \in S_D$, then ${}^\delta B := \{I \in B; \kappa(I) = \delta\}$ is in S_D^δ.*

From 2.5(c) we see that ${}^\delta B \in {S'}_D^\delta$ hence (using (a)) ${}^\delta B \in S_D^\delta$.

Using the definitions we can verify:

(c) *Assume that $D \geq 2$, that $B' \in S_{D-2}$, and that $B = t_i(B') \in S_D$. Let $\beta' = {}^\delta B' \in S^\delta_{D-2}$, $\beta = {}^\delta B \in S^d_D$. Then β is obtained from β' as in* (i) *above.*

Let ${}'S^\delta_D$ be the set of all subsets of R^1_D of the form ${}^\delta B$ for some $B \in S_{D,*}$. We show:

(d) ${}'S^\delta_D = S^\delta_D$.

The inclusion ${}'S^\delta_D \subset S^\delta_D$ follows from (b). Conversely we show that if $\beta \in S^\delta_D$, then $\beta \in {}'S^\delta_D$. We argue by induction on D. When $D = 0$ there is nothing to prove. Assume that $D \geq 2$. If $\beta = \emptyset$ there is nothing to prove. Assume that $\beta \neq \emptyset$. We can find $i \in [1, D]$ and $\beta' \in S^\delta_{D-2}$ such that β is obtained from β' as in (i) above. By the induction hypothesis we have $\beta' = {}^\delta B'$ where $B' \in S_{D-2,*}$. Let $B = t_i(B')$. We have $B \in S_{D,*}$. Let $\tilde{\beta} = {}^\delta B \in {}'S^\delta_D$. By (c), $\tilde{\beta}$ is obtained from β' as in (i) above. Since β has the same property, we have $\tilde{\beta} = \beta$. Thus $\beta \in {}'S^d_D$, as required. This proves (d).

We show:

(e) *The map $S_{D,*} \to {}'S^\delta_D$, $B \mapsto {}^\delta B$ is a bijection.*

It is enough to show that this map is injective. Assume that $B \in S_{D,*}, \tilde{B} \in S_{D,*}$ satisfy ${}^\delta B = {}^\delta \tilde{B}$. We must show that $B = \tilde{B}$. By the proof of 2.4(c) we have a bijection $\iota' : \mathcal{F}_*(V) \to \mathcal{C}(V^\delta)$ given by $X \mapsto X^\delta$. Now $\iota'(\langle B \rangle)$ has basis $\{e_{I^{\kappa(I)}}; I \in B, \kappa(I) = \delta\}$ and $\iota'(\langle \tilde{B} \rangle)$ has basis $\{e_{I^{\kappa(I)}}; I \in \tilde{B}, \kappa(I) = \delta\}$. Since ${}^\delta B = {}^\delta \tilde{B}$, these two bases coincide hence $\iota'(\langle B \rangle) = \iota'(\langle \tilde{B} \rangle)$. Since ι' is a bijection we deduce that $\langle B \rangle = \langle \tilde{B} \rangle$. Using 2.1(a) we see that $B = \tilde{B}$. This proves (e).

Combining (d),(e) we obtain:

(f) *The map $S_{D*} \to S^\delta_D$, $B \mapsto {}^\delta B$ is a bijection.*

For any $\beta \in S^\delta_D$ let $\langle \beta \rangle$ be the $\mathbf{F}_2$-subspace of V^δ spanned by $\{e_{I^{\kappa(i)}}; I \in \beta\}$. By the proof of (e), we have $\langle \beta \rangle \in \mathcal{C}(V^\delta)$ and $\dim \langle \beta \rangle = |\beta|$. We show:

(g) *The map $\beta \mapsto \langle \beta \rangle$ is a bijection $\iota'' : S^\delta_D \xrightarrow{\sim} \mathcal{C}(V^\delta)$.*

We have a commutative diagram

$$\begin{array}{ccc} S_{D,*} & \longrightarrow & \mathcal{F}_*(V) \\ \Big\downarrow & & {\scriptstyle \iota'}\Big\downarrow \\ S^\delta_D & \xrightarrow{\iota''} & \mathcal{C}(V^\delta) \end{array}$$

where the top horizontal map is a bijection as in 2.4(b), the left vertical map is a bijection as in (e) (see also (d)), and ι' is a bijection as in the proof of (e). It follows that ι'' is a bijection. This proves (g).

2.6. Let $\delta \in \{0, 1\}$. We define a bijection $S^\delta_D \xrightarrow{\sim} S^{1-\delta}_D, \beta \to \beta^!$ as follows. Let $\beta \in S^\delta_D$. By 2.5(g), we have $\langle \beta \rangle \in \mathcal{C}(V^\delta)$ and by 2.4(a) we have $\langle \beta \rangle^! \in \mathcal{C}(V^{1-\delta})$. Then $\beta^!$ is the unique element of $S^{1-\delta}_D$ such that $\langle \beta \rangle^! = \langle \beta^! \rangle$; see 2.5(g). From the definition we have $(\beta^!)^! = \beta$ and $|\beta^!| = (D/2) - |\beta|$. Recall that $\langle \beta \rangle \oplus \langle \beta^! \rangle = \langle B \rangle$ where $B \in S_{D,*}$ satisfies ${}^\delta B = \beta, {}^{1-\delta} B = \beta^!$.

The order reversing involution $i \mapsto i^* = D + 1 - i$ of $[1, D]$ induces an involution $R^1_D \to R^1_D$, $I \mapsto I^* = \{i^*; i \in I\}$ and an involution $S_D \to S_D$, $B \mapsto B^* := \{I^*; I \in B\}$. It also induces a bijection $\gamma_\delta : S^{1-\delta}_D \xrightarrow{\sim} S^\delta_D$. Then $\beta \mapsto \gamma_\delta(\beta^!)$ is a bijection $S^\delta_D \to S^\delta_D$ which carries any subset with m elements ($m \in [0, D/2]$) to a subset with $(D/2) - m$ elements.

2.7. Let $\delta \in \{0,1\}$. Let $U^\delta = \{(\mathcal{L}, \mathcal{L}') \in \mathcal{C}(V^\delta) \times \mathcal{C}(V^\delta); \mathcal{L} \subset \mathcal{L}'\}$. We define a map

(a) $\mathcal{F}(V) \to U^\delta$ by $X \mapsto (X^\delta, (X^{1-\delta})^!)$.

(We have $X^\delta \subset (X^{1-\delta})^!$ since X is isotropic.) This map is injective since X can be reconstructed from $X^\delta, X^{1-\delta}$: we have $X = X^\delta \oplus X^{1-\delta}$.

We note that the map (a) is not surjective. For example, if $D = 2, \delta = 0$ and $\mathcal{L} = 0$, $\mathcal{L}' = \mathbf{F}_2 e_2$, then $(\mathcal{L}, \mathcal{L}') \in U^0$ is not in the image of the map (a). The following result is a reformulation of 2.4(c).

(b) *The map* (a) *restricts to a bijection* $\mathcal{F}_*(V) \xrightarrow{\sim} \{(\mathcal{L}, \mathcal{L}') \in U^\delta; \mathcal{L} = \mathcal{L}'\}$.

2.8. In the remainder of this section we prove Theorem 0.4 assuming that W is a Weyl group of type B_n, C_n, or D_n. If $|c| = 1$ the theorem is trivial; we have $\mathcal{G}_c = \{1\}$ and $\mathbf{B}_c$ consists of the unique representation in c. Assume now that $|c| \ge 2$. As in [L5, 4.5,4.6], [L4], [L6], we can find $D \in \{2,4,6,\dots\}$ and $\delta \in \{0,1\}$ such that if V is the $\mathbf{F}_2$-vector space with basis $\{e_i; i \in [1,D]\}$ as in 1.9, then (i)-(iii) below hold.

(i) The group $\mathcal{G}_c$ in 0.3 is V^δ; hence $M(\mathcal{G}_c) = V^\delta \oplus \mathrm{Hom}(V^\delta, \mathbf{C}^*)$ can be identified with $V = V^\delta \oplus V^{1-\delta}$ (an element $y \in V^{1-\delta}$ can be identified with the homomorphism $V^\delta \to \mathbf{C}^*$ given by $x \mapsto (-1)^{(x,y)}$).

(ii) c is naturally in bijection with V_0 (see 1.12); hence any object $\mathcal{E} \in \mathcal{R}_c$ can be viewed as the function $f_\mathcal{E} : V_0 \to \mathbf{N}$ such that for $E \in c$ the multiplicity of E in $\mathcal{E}$ is equal to the value of $f_\mathcal{E}$ at the point of V_0 corresponding to E.

(iii) The constructible representations in $\mathcal{R}_c$ viewed as functions $V_0 \to \mathbf{N}$ are exactly the characteristic functions of the subsets $X \subset V$ with $X \in \mathcal{F}_*(V)$.

(More accurately, the results in [L4]–[L6] for W of type D_n are formulated in terms of a V as in 1.9 with odd D, but they can be restated in terms of a V as in 1.9 with D even, by the argument in the first part of 2.1.)

If $\mathcal{L}$ is a subspace of V^δ, then $S_{\mathcal{L},\mathcal{L}} \in \mathbf{C}[M(\mathcal{G}_c)] = \mathbf{C}[V]$ (see (i) and 0.2) can be identified with the function $V \to \mathbf{C}$ whose value is 1 at any element of $\mathcal{L} \oplus \mathcal{L}^!$ and is 0 at any element of $V - (\mathcal{L} \oplus \mathcal{L}^!)$. If $\mathcal{L} \in \mathcal{C}(V^\delta)$ this is the characteristic function of some $X \in \mathcal{F}_*(V)$ namely, $X = \mathcal{L} \oplus \mathcal{L}^!$; the converse also holds. We see that $\mathfrak{F}_c$ (see 0.3) consists of the subspaces $\mathcal{L} \in \mathcal{C}(V^\delta)$. We have $0 \in \mathcal{C}(V^\delta)$ hence $\hat{\mathfrak{F}}_c = \mathfrak{F}_c$. Now $\tilde{\Theta}_c$ becomes the set of pairs $(\mathcal{L}, \mathcal{L}') \in \mathcal{C}(V^\delta) \times \mathcal{C}(V^\delta)$ such that $\mathcal{L} \subset \mathcal{L}'$. We define Θ_c to be the set of pairs $(\mathcal{L}, \mathcal{L}') \in \mathcal{C}(V^\delta) \times \mathcal{C}(V^\delta)$ such that $\mathcal{L} \oplus \mathcal{L}'^! \in \mathcal{F}(V)$. (We then automatically have $\mathcal{L} \subset \mathcal{L}'$ since the subspaces in $\mathcal{F}(V)$ are isotropic. Thus $\Theta_c \subset \tilde{\Theta}_c$.) If $(\mathcal{L}, \mathcal{L}') \in \tilde{\Theta}_c$, then $S_{\mathcal{L},\mathcal{L}'} \in \mathbf{C}[M(\mathcal{G}_c)] = \mathbf{C}[V]$ (see (i) and 0.2) can be identified with the function $V \to \mathbf{C}$ whose value is 1 at any element of $\mathcal{L} \oplus \mathcal{L}'^!$ and is 0 at any element of $V - (\mathcal{L} \oplus \mathcal{L}'^!)$. If $(\mathcal{L}, \mathcal{L}') \in \Theta_c$, this is the characteristic function of some $X \in \mathcal{F}(V)$, namely $X = \mathcal{L} \oplus \mathcal{L}'^!$; the converse also holds. We see that Θ_c can be identified with $\mathcal{F}(V)$. With these identifications Theorem 0.4 follows from the results in §1 and §2. The representations in $\mathbf{B}_c$ corespond as in (ii) to the functions $f^X : V_0 \to \mathbf{N}$ which equal 1 on X and equal 0 on $V_0 - X$ (where $X \in \mathcal{F}(V)$). The bijection $c \to \mathbf{B}_c$ mentioned in 0.5 is $x \mapsto \langle \epsilon^{-1}(x) \rangle$ where ϵ is as in 1.16(b).

3. Exceptional Weyl groups

3.1. In this section we will prove Theorem 0.4 assuming that W is of exceptional-type. In 3.2-3.8 we will give a table of new representations in $\mathcal{R}_c$ in the form of a matrix M_c indexed by $c \times c$. (The table will be justified in 3.10.) The columns of

M_c are indexed by the representations in c. The rows of M_c are also indexed by the representations in c (for any $k \in [1, |c|]$, the kth row from up to down is indexed by the same representation in c as the kth column from left to right). Each row of M_c corresponds to a new representation; the entries of that row give the multiplicities of the various representations in c in the new representation. The first row in M_c stands for the special representation in c.

3.2. If $|c| = 1$, M_c is the 1×1 matrix with entry 1.

3.3. If $|c| = 2$ (so that W is of type E_7 or E_8) we order c using its bijection with $\{(1,1),(1,\epsilon)\}$ in [L5, 4.12, 4.13] (ordered from left to right); then M_c is

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The second row stands for a constructible representation.

3.4. If $|c| = 3$ we order c using its bijection with $\{(1,1),(g_2,1),(1,\epsilon)\}$ in [L5, 4.10, 4.11, 4.12, 4.13] (ordered from left to right); then M_c is

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

The last two rows stand for constructible representations.

3.5. If $|c| = 4$ (so that W is of type G_2) we order c using its bijection with $\{(1,1),(1,r),(g_2,1),(g_3,1)\}$ in [L5, 4.8] (ordered from left to right); then M_c is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

The last two rows stand for constructible representations.

3.6. If $|c| = 5$ (so that W is of type E_6, E_7, or E_8) we order c using its bijection with $\{(1,1),(1,r),(g_2,1),(g_3,1),(1,\epsilon)\}$ in [L5, 4.11, 4.12, 4.13] (ordered from left to right); then M_c is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{pmatrix}.$$

The last three rows stand for constructible representations.

3.7. If $|c| = 11$ (so that W is of type F_4) we write the elements of c (notation of [L5, 4.10]) in the order

$$12_1, 9_3, 6_2, 1_3, 16_1, 9_2, 4_4, 6_1, 4_3, 4_1, 1_2$$

(from left to right); then M_c is:

$$\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0&0&0\\
1&2&1&1&0&0&0&0&0&0&0\\
1&1&1&0&1&0&0&0&0&0&0\\
1&0&1&0&1&1&0&0&0&0&0\\
1&2&1&1&1&0&1&0&0&0&0\\
1&1&0&0&1&0&1&1&0&0&0\\
1&0&0&0&1&1&0&1&1&0&0\\
1&1&1&0&2&1&0&0&0&1&0\\
1&0&1&0&1&2&0&0&1&0&1
\end{pmatrix}.$$

The last five rows stand for constructible representations.

3.8. If $|c| = 17$ (so that W is of type E_8) we write the elements of c (with notation of [L5, 4.13.2] with subscripts omitted) in the order

$$4480, 5670, 4536, 1680, 1400, 70, 7168, 5600, 3150, 4200, 2688, 2016,$$
$$448, 1134, 1344, 420, 168$$

(from left to right); then M_c is:

$$\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&2&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&2&2&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&3&3&3&2&1&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
1&2&2&1&1&0&1&1&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&1&1&1&0&0&0&0&0&0&0&0\\
1&1&1&0&1&0&1&1&0&1&0&0&0&0&0&0&0\\
1&2&2&1&1&0&2&2&0&1&1&0&0&0&0&0&0\\
1&1&1&0&0&0&2&1&1&1&1&1&0&0&0&0&0\\
1&3&3&3&2&1&1&2&0&0&0&0&1&0&0&0&0\\
1&2&1&1&0&0&1&2&1&0&0&0&1&1&0&0&0\\
1&1&0&0&0&0&1&1&1&1&0&0&0&1&1&0&0\\
1&0&0&0&0&0&1&0&1&1&0&1&0&0&1&1&0\\
1&1&1&0&1&0&1&1&0&2&0&0&0&0&1&0&1
\end{pmatrix}.$$

The last seven rows stand for constructible representations.

3.9. For $N \geq 1$ let S_N be the group of all permutations of $[1, N]$. If $a_1 \geq a_2 \geq \dots$ is a partition of N (written as $a_1a_2\dots$) we say that a subgroup H of S_N is in $\mathcal{S}_{a_1a_2\dots}$ if H is conjugate to the subgroup of all permutations of $[1, N]$ which keep stable each of the subsets $[1, a_1], [a_1 + 1, a_1 + a_2], [a_1 + a_2 + 1, a_1 + a_2 + a_3], \dots$. We say that a subgroup H of S_N (with $N \geq 4$) is in $\tilde{\mathcal{S}}_N$ if it is conjugate to the subgroup of all permutations of $[1, N]$ which act as an identity on $[1, N] - [1, 4]$ and whose restriction to $[1, 4]$ commutes with the permutation $1 \mapsto 4 \mapsto 1, 2 \mapsto 3 \mapsto 2$.

The following results come from [L7].

If $|c| = 1$ we have $\mathcal{G}_c = \{1\}$ and $\hat{\tilde{\mathfrak{F}}}_c$ consists of $\{1\}$.

In the setup of 3.3 or 3.4 we have $\mathcal{G}_c = S_2$ and $\hat{\mathfrak{F}}_c$ consists of $S_2, \{1\}$.

In the setup of 3.5 or 3.6 we have $\mathcal{G}_c = S_3$ and $\hat{\mathfrak{F}}_c$ consists of $S_3, \{1\}$ and the subgroups of S_3 in $\mathcal{S}_{21}$.

In the setup of 3.7 we have $\mathcal{G}_c = S_4$ and $\hat{\mathfrak{F}}_c$ consists of $S_4, \{1\}$ and the subgroups of S_4 in $\mathcal{S}_{31}, \mathcal{S}_{22}, \mathcal{S}_{211}, \tilde{\mathcal{S}}_4$.

In the setup of 3.8 we have $\mathcal{G}_c = S_5$ and $\hat{\mathfrak{F}}_c$ consists of $S_5, \{1\}$ and the subgroups of S_5 in $\mathcal{S}_{41}, \mathcal{S}_{32}, \mathcal{S}_{311}, \mathcal{S}_{221}, \mathcal{S}_{2111}, \tilde{\mathcal{S}}_5$.

3.10. We describe the set $\tilde{\Theta}_c$ in each of the cases 3.2-3.8.

If $|c| = 1$, $\tilde{\Theta}_c$ consists of $(1, 1)$. (We shall write 1 instead of $\{1\}$.)

In the setup of 3.3 or 3.4, $\tilde{\Theta}_c$ consists of $(1, S_2), (S_2, S_2), (1, 1)$.

In the setup of 3.5 or 3.6, $\tilde{\Theta}_c$ consists of $(1, S_3), (1, H_{21}), (H_{21}, H_{21}), (S_3, S_3), (1, 1)$ where H_{21} runs through $\mathcal{S}_{21}$.

In the setup of 3.7, $\tilde{\Theta}_c$ consists of

$$(1, S_4), (1, H_{31}), (1, H_{22}), (1, H_{211}), (\tilde{H}_{211}, H_{22}), (\tilde{H}_{22}, \tilde{H}),$$
$$(H_{211}, H_{211}), (H_{31}, H_{31}), (S_4, S_4), (H_{22}, H_{22}), (\tilde{H}, \tilde{H}), (1, 1), (1, \tilde{H}),$$

where H_{211} runs through $\mathcal{S}_{211}$, H_{31} runs through $\mathcal{S}_{31}$, H_{22} runs through $\mathcal{S}_{22}$, $\tilde{H}$ runs through $\tilde{\mathcal{S}}_4$; for $H_{22} \in \mathcal{S}_{22}$, $\tilde{H}_{211}$ denotes one of the two subgroups in $\mathcal{S}_{211}$ contained in H_{22}; for $\tilde{H} \in \tilde{\mathcal{S}}_4$, $\tilde{H}_{22}$ denotes the unique subgroup in $\mathcal{S}_{22}$ contained in $\tilde{H}$.

In the setup of 3.8, $\tilde{\Theta}_c$ consists of

$$(1, S_5), (1, H_{41}), (1, H_{32}), (1, H_{311}), (1, H_{221}), (1, H_{2111}), (\tilde{H}_{2111}, H_{32}),$$
$$(\tilde{H}_{2111}, H_{221}), (\tilde{H}_{311}, H_{32}), (\tilde{H}_{221}, \tilde{H}), (H_{221}, H_{221}), (H_{32}, H_{32}),$$
$$(H_{2111}, H_{2111}), (H_{311}, H_{311}), (H_{41}, H_{41}), (S_5, S_5), (\tilde{H}, \tilde{H}), (1, 1), (1, \tilde{H}),$$

where H_{2111} runs through $\mathcal{S}_{2111}$, H_{221} runs through $\mathcal{S}_{221}$, H_{32} runs through $\mathcal{S}_{32}$, H_{311} runs through $\mathcal{S}_{311}$, H_{41} runs through $\mathcal{S}_{41}$, $\tilde{H}$ runs through $\tilde{\mathcal{S}}_5$; for $H_{221} \in \mathcal{S}_{221}$, $\tilde{H}_{2111}$ denotes one of the two subgroups in $\mathcal{S}_{2111}$ contained in H_{221}; for $H_{32} \in \mathcal{S}_{32}$, $\tilde{H}_{2111}$ denotes the unique subgroup in $\mathcal{S}_{2111}$ which is a normal subgroup of H_{32} and $\tilde{H}_{311}$ denotes the unique subgroup in $\mathcal{S}_{311}$ which is a normal subgroup of H_{32}; for $\tilde{H} \in \tilde{\mathcal{S}}_5$, $\tilde{H}_{221}$ denotes the unique subgroup in $\mathcal{S}_{221}$ contained in $\tilde{H}$.

3.11. We define the set Θ_c in each of the cases 3.2-3.8 by removing from $\tilde{\Theta}_c$ the pair $(1, 1)$ whenever c is anomalous (see 0.3) and by removing the pairs $(1, \tilde{H})$ with $\tilde{H}$ in $\tilde{\mathcal{S}}_4$ or $\tilde{\mathcal{S}}_5$ whenever $\tilde{\mathcal{S}}_4$ or $\tilde{\mathcal{S}}_5$ is defined. This guarantees that for $(H, H') \in \Theta_c$, H'/H is isomorphic to a product of symmetric groups.

If $|c| = 1$, Θ_c consists of $(1, 1)$.

In the setup of 3.3, Θ_c consists of $(1, S_2), (S_2, S_2)$.

In the setup of 3.4, $\Theta_c = \tilde{\Theta}_c$ consists of $(1, S_2), (S_2, S_2), (1, 1)$.

In the setup of 3.5, Θ_c consists of $(1, S_3), (1, H_{21}), (H_{21}, H_{21}), (S_3, S_3)$ (notation of 3.10).

In the setup of 3.6, $\Theta_c = \tilde{\Theta}_c$ consists of $(1, S_3), (1, H_{21}), (H_{21}, H_{21}), (S_3, S_3), (1, 1)$ (notation of 3.10).

In the setup of 3.7, Θ_c consists of

$$(1, S_4), (1, H_{31}), (1, H_{22}), (1, H_{211}), (\tilde{H}_{211}, H_{22}), (\tilde{H}_{22}, \tilde{H}),$$
$$(H_{211}, H_{211}), (H_{31}, H_{31}), (S_4, S_4), (H_{22}, H_{22}), (\tilde{H}, \tilde{H}),$$

(notation of 3.10).

In the setup of 3.8, Θ_c consists of

$$(1, S_5), (1, H_{41}), (1, H_{32}), (1, H_{311}), (1, H_{221}), (1, H_{2111}), (\tilde{H}_{2111}, H_{32}),$$
$$(\tilde{H}_{2111}, H_{221}), (\tilde{H}_{311}, H_{32}), (\tilde{H}_{221}, \tilde{H}), (H_{221}, H_{221}), (H_{32}, H_{32}),$$
$$(H_{2111}, H_{2111}), (H_{311}, H_{311}), (H_{41}, H_{41}), (S_5, S_5), (\tilde{H}, \tilde{H}),$$

(notation of 3.10).

In each case, the number of $\mathcal{G}_c$-orbits on Θ_c is equal to $|c|$. By computation we see that $S_{H,H'}$ with (H, H') running through a set of representatives for the $\mathcal{G}_c$-orbits on Θ_c are of the form $\underline{E_{H,H'}}$ (see 0.3) where $E_{H,H'} \in \mathcal{R}_c$ runs through the objects of $\mathcal{R}_c$ described by the rows of the matrix M_c in 3.2-3.8 (in the same order as the one used in the description of Θ_c given above). These objects form a basis of $\mathcal{G}_c$, due to the form of the matrix M_c. Now Theorem 0.4 follows in our case.

4. Proof of Theorem 0.7

4.1. Let $H \subset H'$ be subgroups of the finite group Γ with H normal in H'. For any $x \in \Gamma$ we consider the set $S(x)$ of all μ in Γ/H' such that for some γ in Γ/H contained in μ we have $x\gamma = \gamma$. Now $Z(x)$ acts on $S(x)$ by $y : \mu \mapsto y\mu$. For any $(x, \sigma) \in M(\Gamma)$ let $N_{x,\sigma} \in \mathbf{N}$ be the multiplicity of σ in the permutation representation of $Z(x)$ on $S(x)$. We have

$$N_{x,\sigma} = |Z(x)|^{-1} \sum_{y \in Z(x)} \sharp(\mu \in S(x); y\mu = \mu) \mathrm{tr}(y, \sigma),$$

where

$$\begin{aligned} &\sharp(\mu \in S(x); y\mu = \mu) \\ &= \sharp(\mu \in \Gamma/H';\ \text{for some } u \in \Gamma \text{ we have } xuH = uH, \mu = uH', yuH' = uH'). \end{aligned}$$

If the previous three equations hold for some u, then they hold for uh' for any $h' \in H'$. (Indeed, $xuh'H = uh'H$ since $h'H = Hh'$, and $\mu = uh'H', yuh'H' = uh'H'$.) Thus,

$$\sharp(\mu \in S(x); y\mu = \mu) = \sharp(u \in \Gamma; xuH = uH, yuH' = uH')/|H'|$$

and

$$\begin{aligned} N_{x,\sigma} &= |Z(x)|^{-1}|H'|^{-1} \sum_{y \in Z(x)} \sharp(u \in \Gamma; xuH = uH, yuH' = uH') \mathrm{tr}(y, \sigma) \\ &= |Z(x)|^{-1}|H'|^{-1} \sum_{y \in Z(x), u \in \Gamma; xuH = uH, yuH' = uH'} \mathrm{tr}(y, \sigma). \end{aligned}$$

Let $f=\sum_{(x,\sigma)\in M(\Gamma)}N_{x,\sigma}(x,\sigma)\in\mathbf{C}[M(\Gamma)]$. We have $f=S_{H,H'}$. We write $A(f)=\sum_{(x',\sigma')\in M(\Gamma)}N'_{x',\sigma'}(x',\sigma')$ with $N'_{x',\sigma'}\in\mathbf{C}$. We have

$$N'_{x',\sigma'}=\sum_{(x,\sigma)\in M(\Gamma)}N_{x,\sigma}(x,\sigma),(x',\sigma')$$
$$=\sum_{(x,\sigma)\in M(\Gamma)}|Z(x)|^{-1}|H'|^{-1}|Z(x)|^{-1}|Z(x')|^{-1}\sum_{y\in Z(x),u\in\Gamma;xuH=uH,yuH'=uH'}$$
$$\sum_{z\in\Gamma;zxz^{-1}x'=x'zxz^{-1}}\overline{\mathrm{tr}(zxz^{-1},\sigma')}\mathrm{tr}(z^{-1}x'z,\sigma)\mathrm{tr}(y,\sigma)$$
$$=\sum_{x\in\Gamma}|\Gamma|^{-1}|H'|^{-1}|Z(x)|^{-1}|Z(x')|^{-1}\sum_{y\in Z(x),u\in\Gamma;xuH=uH,yuH'=uH'}$$
$$\sum_{z\in\Gamma;zxz^{-1}x'=x'zxz^{-1}}\overline{\mathrm{tr}(zxz^{-1},\sigma')}\sum_{\sigma\in Irr(Z(x)}\mathrm{tr}(z^{-1}x'z,\sigma)\mathrm{tr}(y,\sigma).$$

The last sum over σ equals $|Z(x)\cap Z(y)|$ if $z^{-1}x'z=ay^{-1}a^{-1}$ for some $a\in Z(x)$ and equals 0 otherwise. Hence

$$N'_{x',\sigma'}=\sum_{x\in\Gamma}|\Gamma|^{-1}|H'|^{-1}|Z(x)|^{-1}|Z(x')|^{-1}\sum_{y\in Z(x),u\in\Gamma;xuH=uH,yuH'=uH'}$$
$$\sum_{z\in\Gamma;zxz^{-1}x'=x'zxz^{-1},a^{-1}nZ(x),z^{-1}x'z=ay^{-1}a^{-1}}\overline{\mathrm{tr}(zxz^{-1},\sigma')}.$$

We substitute $z_1=za$. We get

$$N'_{x',\sigma'}=\sum_{x\in\Gamma}|\Gamma|^{-1}|H'|^{-1}|Z(x)|^{-1}|Z(x')|^{-1}\sum_{y\in Z(x),u\in\Gamma;xuH=uH,yuH'=uH'}$$
$$\sum_{z_1\in\Gamma;z_1xz_1^{-1}x'=x'z_1xz_1^{-1},a^{-1}nZ(x),z_1^{-1}x'z_1=y^{-1}}\overline{\mathrm{tr}(z_1xz_1^{-1},\sigma')}.$$

We can eliminate a and change z_1 to z. We get

$$N'_{x',\sigma'}=\sum_{x\in\Gamma}|\Gamma|^{-1}|H'|^{-1}|Z(x')|^{-1}\sum_{y\in Z(x),u\in\Gamma;xuH=uH,yuH'=uH'}$$
$$\sum_{z\in\Gamma;zxz^{-1}x'=x'zxz^{-1},z^{-1}x'z=y^{-1}}\overline{\mathrm{tr}(zxz^{-1},\sigma')}.$$

We substitute $x_1=u^{-1}xu,y_1=u^{-1}yu,z_1=zu$. We get

$$N'_{x',\sigma'}=\sum_{x_1\in\Gamma}|\Gamma|^{-1}|H'|^{-1}|Z(x')|^{-1}\sum_{y_1\in Z(x_1),u\in\Gamma;x_1H=H,y_1H'=H'}$$
$$\sum_{z_1\in\Gamma;z_1x_1z_1^{-1}x'=x'z_1x_1z_1^{-1},z_1^{-1}x'z_1=y_1^{-1}}\overline{\mathrm{tr}(z_1x_1z_1^{-1},\sigma')}.$$

We can eliminate u and change x_1,y_1,z_1 to x,y,z. We get

$$N'_{x',\sigma'}=|H'|^{-1}|Z(x')|^{-1}\sum_{x\in H,y\in Z(x)\cap H'}$$
$$\sum_{z\in\Gamma;zxz^{-1}x'=x'zxz^{-1},z^{-1}x'z=y^{-1}}\overline{\mathrm{tr}(zxz^{-1},\sigma')}.$$

Here the condition $zxz^{-1}x' = x'zxz^{-1}$ follows from $z^{-1}x'z = y^{-1}$, $yx = xy$. Hence

$$N'_{x',\sigma'} = |H'|^{-1}|Z(x')|^{-1} \sum_{x\in H, y\in Z(x)\cap H'} \sum_{z\in\Gamma; z^{-1}x'z=y^{-1}} \overline{\mathrm{tr}(zxz^{-1},\sigma')},$$

that is,

$$N'_{x',\sigma'} = |H'|^{-1}|Z(x')|^{-1} \sum_{x\in H} \sum_{z\in\Gamma; z^{-1}x'z\in Z(x)\cap H'} \overline{\mathrm{tr}(zxz^{-1},\sigma')}.$$

We substitute $zxz^{-1} = x_1$. We get

$$N'_{x',\sigma'} = |H'|^{-1}|Z(x')|^{-1} \sum_{x_1\in\Gamma, z\in\Gamma; x'\in Z(x_1)\cap zH'z^{-1}, x_1\in zHz^{-1}} \overline{\mathrm{tr}(x_1,\sigma')},$$

that is,

$$N'_{x',\sigma'} = |H'|^{-1}|Z(x')|^{-1} \sum_{z\in\Gamma; z^{-1}x'z\in H'} \sum_{x_1\in Z(x')\cap zHz^{-1}} \overline{\mathrm{tr}(x_1,\sigma')}$$

and

$$N'_{x',\sigma'} = |H'|^{-1}|Z(x')|^{-1} \sum_{z\in\Gamma; z^{-1}x'z\in H'} (1 : \sigma'|(Z(x')\cap zHz^{-1}))|Z(x')\cap zHz^{-1}|,$$

where : denotes multiplicity. Thus we have

$$N_{x',\sigma'} \in \mathbf{Q}_{\geq 0}$$

so that $A(f) \in M(\Gamma)_{\geq 0}$. Since $f \in M(\Gamma)_{\geq 0}$ is obvious we see that f is bipositive. This proves Theorem 0.7.

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