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PARTIAL FLAG MANIFOLDS OVER A SEMIFIELD

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ABSTRACT. For any semifield K we define a K-form of a partial flag manifold of a semisimple group of simply laced type over the complex numbers.

INTRODUCTION

0.1. Let G be the group of simply connected-type associated in [MT], [Ma], [Ti], [PK], to a not necessarily positive definite symmetric Cartan matrix and to the field $\mathbf{C}$. We assume that a pinning of G is given. It consists of a "Borel subgroup" B^+, a "maximal torus" $T \subset B^+$ and one parameter subgroups $x_i : \mathbf{C} \to G, y_i : \mathbf{C} \to G \quad (i \in I)$ analogous to those in [Lus94]. We have $x_i(\mathbf{C}) \subset B^+$. We fix a subset $J \subset I$. Let Π^J be the subgroup of G generated by B^+ and by $\bigcup_{i \in J} y_i(\mathbf{C})$. Let $\mathcal{P}^J$ be the set of subgroups of G which are G-conjugate to Π^J (a partial flag manifold). As in [Lus94, 2.20] we consider the submonoid $G_{\geq 0}$ of G generated by $x_i(a), y_i(a)$ with $i \in I, a \in \mathbf{R}_{\geq 0}$ and by the "vector part" $T_{>0}$ of T. (T is a product of $T_{>0}$ and a compact torus.) Let K be a semifield. Let $\mathfrak{G}(K)$ be the monoid associated to G, K by generators and relations in [L18, 3.1(i)-(viii)]. When $K = \mathbf{R}_{>0}$ this can be identified with $G_{\geq 0}$ by an argument given in [L19a].

The main result of this paper is a definition of an analogue $\mathcal{P}^J(K)$ of the partial flag manifold $\mathcal{P}^J$ in the case where $\mathbf{C}$ is replaced by any semifield K. This is a set $\mathcal{P}^J(K)$ with an action of the monoid $\mathfrak{G}(K)$.

A part of our argument involves a construction of an analogue of the highest weight integrable representations of G when G is replaced by the monoid $\mathfrak{G}(K)$. The possibility of such a construction comes from the positivity properties of the canonical basis [Lus93]. A key role in our argument is played by a classical theorem of Kostant which describes any flag manifold by a system of quadratic equations.

0.2. In this subsection we assume that our Cartan matrix is of finite-type. If $K = \mathbf{R}_{>0}$, the set $\mathcal{P}^J(K)$ coincides with the subset $\mathcal{P}^J_{\geq 0}$ of $\mathcal{P}^J$ defined in [Lus98]. If K is the semifield $\mathbf{Z}$ and $J = \emptyset$, a definition of the flag manifold over $\mathbf{Z}$ was given in [L19b]; we expect that it agrees with the definition in this paper, but we have not proved that. In the case where $G = SL_n$, a form over $\mathbf{Z}$ of a Grassmannian was defined earlier in [SW].

1. THE SET $\mathcal{P}^J(K)$

1.1. Let $\mathcal{X} = \mathrm{Hom}(T, \mathbf{C}^*)$. This is a free abelian group with basis $\{\omega_i; i \in I\}$ consisting of fundamental weights. Let $\mathcal{X}^+ = \sum_{i \in I} \mathbf{N}\omega_i \subset \mathcal{X}$ be the set of dominant weights. For $\lambda \in \mathcal{X}$ let $\mathrm{supp}(\lambda)$ be the set of all $i \in I$ such that ω_i appears with $\neq 0$

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coefficient in λ. Let $\mathcal{X}_J^+ = \{\lambda \in \mathcal{X}^+; \mathrm{supp}(\lambda) = I - J\}$, $\mathcal{X}_{\bar{J}}^+ = \{\lambda \in \mathcal{X}^+; \mathrm{supp}(\lambda) \subset I - J\}$.

The irreducible highest weight integrable representations of G are indexed by their highest weight, an element of $\mathcal{X}^+$. For $\lambda \in \mathcal{X}^+$ let ${}^\lambda V$ be a $\mathbf{C}$-vector space which is an irreducible highest weight integrable representation of G indexed by λ. Let ${}^\lambda P$ be the set of lines in ${}^\lambda V$. Let ${}^\lambda\xi^+$ be a highest weight vector of ${}^\lambda V$. Let ${}^\lambda\beta$ be the canonical basis of ${}^\lambda V$ (see [Lus93, 11.10]) containing ${}^\lambda\xi^+$.

For a nonzero vector ξ in a vector space V we denote by $[\xi]$ the line in V that contains ξ. Note that Π^J (see 0.1) is the stabilizer of $[{}^\lambda\xi]$ in G where $\lambda \in \mathcal{X}_J^+$.

For λ, λ' in $\mathcal{X}^+$ we define a linear map

$$E : {}^\lambda V \times {}^{\lambda'} V \to {}^\lambda V \otimes {}^{\lambda'} V$$

by $(\xi, \xi') \mapsto \xi \otimes \xi'$ and a linear map

$$\Gamma : {}^{\lambda+\lambda'} V \to {}^\lambda V \otimes {}^{\lambda'} V$$

which is compatible with the G-actions and takes ${}^{\lambda+\lambda'}\xi^+$ to ${}^\lambda\xi^+ \otimes {}^{\lambda'}\xi^+$. Let ${}^{\lambda,\lambda'}P$ be the set of lines in ${}^\lambda\xi^+ \otimes {}^{\lambda'}\xi^+$. Now E induces a map $\bar{E} : {}^\lambda P \times {}^{\lambda'} P \to {}^{\lambda,\lambda'}P$ and Γ induces a map $\bar{\Gamma} : {}^{\lambda+\lambda'} P \to {}^{\lambda,\lambda'}P$.

Let $\mathcal{C}$ be the set of all collections $\{x_\lambda \in {}^\lambda V; \lambda \in \mathcal{X}_{\bar{J}}^+\}$ such that for any λ, λ' in $\mathcal{X}_{\bar{J}}^+$ we have $\Gamma(x_{\lambda+\lambda'}) = E(x_\lambda, x_{\lambda'})$. Let $\mathcal{C}^*$ be the set of all $(x_\lambda) \in \mathcal{C}$ such that $x_\lambda \neq 0$ for any $\lambda \in \mathcal{X}_{\bar{J}}^+$. Let H be the group consisting of all collections $\{z_\lambda \in \mathbf{C}^*; \lambda \in \mathcal{X}_{\bar{J}}^+\}$ such that for any λ, λ' in $\mathcal{X}_{\bar{J}}^+$ we have $z_{\lambda+\lambda'} = z_\lambda z_{\lambda'}$. Now H acts on $\mathcal{C}$ by $(z_\lambda), (x_\lambda) \mapsto (z_\lambda x_\lambda)$. This restricts to a free action of H on $\mathcal{C}^*$. Let ${}'\mathcal{P}^J$ be the set of orbits for this action. Note that G acts on $\mathcal{C}$ by $g(x_\lambda) = (g(x_\lambda))$. This induces a G-action on $\mathcal{C}^*$ and on ${}'\mathcal{P}^J$. We define a map $\theta : \mathcal{P}^J \to {}'\mathcal{P}^J$ by $g\Pi^J g^{-1} \mapsto H$-orbit of $(g({}^\lambda\xi))$ where $g \in G$. This is well defined since $({}^\lambda\xi) \in \mathcal{C}$ and since for $g \in \Pi^J$, $(g({}^\lambda\xi))$ is in the same H-orbit as $({}^\lambda\xi)$. We show the following.

Lemma 1.2. *$\theta : \mathcal{P}^J \to {}'\mathcal{P}^J$ is a bijection.*

For $\lambda \in \mathcal{X}_{\bar{J}}^+$ we denote by $\Pi(\lambda)$ the stabilizer of $[{}^\lambda\xi]$ in G. Now θ is injective since if $\lambda \in \mathcal{X}_J^+$, a subgroup $\Pi \in \mathcal{P}^J$ is uniquely determined by the Π-stable line in ${}^\lambda V$. Now let $(x_\lambda) \in \mathcal{C}^*$. We show that the H-orbit of (x_λ) is in $\theta(\mathcal{P}^J)$. Let $\lambda \in \mathcal{X}_{\bar{J}}^+$. We have $\Gamma(x_{2\lambda}) = E(x_\lambda, x_\lambda)$. Thus, E_{x_λ, x_λ} is contained in the irreducible summand of ${}^\lambda V \otimes {}^\lambda V$ which is isomorphic to ${}^{2\lambda}V$, hence by a theorem of Kostant (see [Gar82] for the finite-type case and [PK] for the general case), we must have $[x_\lambda] = g_\lambda[{}^\lambda\xi]$ for some $g_\lambda \in G$. Since $(x_\lambda) \in \mathcal{C}^*$, for λ, λ' in $\mathcal{X}_{\bar{J}}^+$ we have

$$\bar{E}([g_{\lambda+\lambda'}({}^\lambda\xi)], [g_{\lambda+\lambda'}({}^{\lambda'}\xi)]) = \bar{\Gamma}([g_{\lambda+\lambda'}({}^{\lambda+\lambda'}\xi)]) = \bar{E}([g_\lambda({}^\lambda\xi)], [g_{\lambda'}({}^{\lambda'}\xi)]).$$

Since $\bar{E}$ is injective, it follows that

$$[g_{\lambda+\lambda'}({}^\lambda\xi)] = [g_\lambda({}^\lambda\xi)], [g_{\lambda+\lambda'}({}^{\lambda'}\xi)] = [g_{\lambda'}({}^{\lambda'}\xi)],$$

so that

(a) $g_\lambda^{-1} g_{\lambda+\lambda'} \in \Pi(\lambda)$.

Assuming that $\lambda, \lambda' \in \mathcal{X}_J^+$, we see that $g_\lambda^{-1} g_{\lambda+\lambda'} \in \Pi^J$ and similarly $g_{\lambda'}^{-1} g_{\lambda+\lambda'} \in \Pi^J$, so that $g_{\lambda'}^{-1} g_\lambda \in \Pi^J$. Thus, there exists $g \in G$ such that for any $\lambda \in \mathcal{X}_J^+$ we have $g_\lambda = gp_\lambda$ with $p_\lambda \in \Pi^J$. Replacing g_λ by $g_\lambda p_\lambda^{-1}$, we see that we can assume that

(b) $g_\lambda = g$ for any $\lambda \in \mathcal{X}_J^+$.

If $\lambda \in \mathcal{X}_J^+$, $\lambda' \in \mathcal{X}_J^+$, we have $\lambda + \lambda' \in \mathcal{X}_J^+$ hence by (b), $g_{\lambda+\lambda'} = g$, so that (a) implies $g_\lambda^{-1} g \in \Pi(\lambda)$ and $[x_\lambda] = [g_\lambda({}^\lambda\xi)] = [g({}^\lambda\xi)]$. Thus for any $\lambda \in \mathcal{X}_J^+$ we have $x_\lambda = z_\lambda g({}^\lambda\xi)$ for some $z_\lambda \in \mathbf{C}^*$. Since $(x_\lambda) \in \mathcal{C}^*$ and $(g({}^\lambda\xi)) \in \mathcal{C}^*$, we necessarily have $(z_\lambda) \in H$. Thus the H-orbit of (x_λ) is in the image of θ. The lemma is proved.

1.3. Let $\mathcal{D}$ be the category whose objects are pairs (V, β) where V is a $\mathbf{C}$-vector space and β is a basis of V; a morphism from (V, β) to (V', β') is a $\mathbf{C}$-linear map $f : V \to V'$ such that for any $b \in \beta$ we have $f(b) = \sum_{b' \in \beta'} c_{b,b'} b'$ where $c_{b,b'} \in \mathbf{N}$ for all b, b' and $c_{b,b'} = 0$ for all but finitely many b'.

Let K be a semifield. As in [L19b] we define $K^! = K \sqcup \{\circ\}$ where $\circ$ is a symbol. We extend the sum and product on K to a sum and product on $K^!$ by defining $\circ + a = a$, $a + \circ = a$, $\circ \times a = \circ$, $a \times \circ = \circ$ for $a \in K$ and $\circ + \circ = \circ$, $\circ \times \circ = \circ$. Thus $K^!$ becomes a monoid under addition and a monoid under multiplication. Moreover, the distributivity law holds in $K^!$.

A K-semivector space is an abelian (additive) semigroup $\mathcal{V}$ with neutral element $\underline{\circ}$ in which a map $K^! \times \mathcal{V} \to \mathcal{V}$, $(k, v) \mapsto kv$ ("scalar multiplication") is given such that $(kk')v = k(k'(v))$, $(k + k')v = kv + k'v$ for k, k' in $K^!$, $v \in \mathcal{V}$ and $k(v + v') = kv + kv'$ for $k \in K^!$, v, v' in $\mathcal{V}$; moreover, we assume that $k\underline{\circ} = \underline{\circ}$ for $k \in K^!$.

Let $\mathcal{D}(K)$ be the category whose objects are K-semivector spaces $\mathcal{V}$; a morphism from $\mathcal{V}$ to $\mathcal{V}'$ is a map $f : \mathcal{V} \to \mathcal{V}'$ of semigroups preserving the neutral elements and commuting with scalar multiplication. For any $\mathcal{V} \in \mathcal{D}(K)$ let $\mathrm{End}(\mathcal{V}) = \mathrm{Hom}_{\mathcal{D}(K)}(\mathcal{V}, \mathcal{V})$; this is a monoid under composition of maps.

For $(V, \beta) \in \mathcal{D}$ let $V(K)$ be the set of formal sums $\xi = \sum_{b \in \beta} \xi_b b$ with $\xi_b \in K^!$ for all $b \in \beta$ and $\xi_b = \circ$ for all but finitely many b. We can define addition on $V(K)$ by $(\sum_{b \in \beta} \xi_b b) + (\sum_{b \in \beta} \xi'_b b) = \sum_{b \in \beta} (\xi_b + \xi'_b) b$. We can define scalar multiplication by elements in $K^!$ by $k(\sum_{b \in \beta} \xi_b) = \sum_{b \in \beta} (k\xi_b) b$. Then $V(K)$ becomes an object of $\mathcal{D}(K)$. The neutral element for addition is $\underline{\circ} = \sum_{b \in \beta} \circ b$. Let f be a morphism from (V, β) to (V', β') in $\mathcal{D}$. For $b \in \beta$ we have $f(b) = \sum_{b' \in \beta'} c_{b,b'} b'$ where $c_{b,b'} \in \mathbf{N}$. We define a map $f(K) : V(K) \to V'(K)$ by $f(K)(\sum_{b \in \beta} \xi_b b) = \sum_{b' \in \beta'} (\sum_{b \in \beta} c_{b,b'} \xi_b) b'$. Here for $c \in \mathbf{N}, k \in K^!$ we set $ck = k + k + \cdots + k$ (c terms) if $c > 0$ and $ck = \circ$ if $c = 0$. Note that $f(K)$ is a morphism in $\mathcal{D}(K)$. We have thus defined a functor $(V, \beta) \mapsto V(K)$ from $\mathcal{D}$ to $\mathcal{D}(K)$.

Let $\lambda \in \mathcal{X}^+$. We have $({}^\lambda V, {}^\lambda\beta) \in \mathcal{D}$ hence ${}^\lambda V(K) \in \mathcal{D}(K)$ is defined. For $i \in I, m \in \mathbf{Z}$, the linear maps $e_i^{(n)}, f_i^{(n)}$ from ${}^\lambda V$ to ${}^\lambda V$ (as in [L19b, 1.4]) are morphisms in $\mathcal{D}$ (we use the positivity property [Lus93, 22.1.7] of ${}^\lambda\beta$; in [Lus93]this property is stated assuming that the Cartan matrix is of simply laced-type, but the same proof applies in our case). Hence they define morphisms $e_i^{(n)}(K), f_i^{(n)}(K)$ from ${}^\lambda V(K)$ to ${}^\lambda V(K)$. For $i \in I, k \in K$ we define $i^k \in \mathrm{End}({}^\lambda V(K))$, $(-i)^k \in \mathrm{End}({}^\lambda V(K))$ by

$$i^k(b) = \sum_{n \in \mathbf{N}} k^n e_i^{(n)}(K) b, \quad (-i)^k(b) = \sum_{n \in \mathbf{N}} k^n f_i^{(n)}(K) b$$

for any $b \in {}^\lambda\beta$.

For any $i \in I$ there is a well defined function $l_i : {}^\lambda\beta \to \mathbf{Z}$ such that for $b \in {}^\lambda\beta$, $t \in \mathbf{C}^*$ we have $i(t)b = t^{l_i(b)} b$. (Here i is viewed as a simple coroot homomorphism $\mathbf{C} \to T$.) For $i \in I, k \in K$ we define $\underline{i}^k \in \mathrm{End}({}^\lambda V(K))$ by $\underline{i}^k(b) = k^{l_i(b)} b$ for any $b \in {}^\lambda\beta$. As in [L19b, 1.5], the elements $i^k, (-i)^k, \underline{i}^k$ (with $i \in I, k \in K$) in

End($^\lambda V(K)$) satisfy the relations in [L19a, 2.10(i)-(vii)] defining the monoid $\mathfrak{G}(K)$ hence they define a monoid homomorphism $\mathfrak{G}(K) \to \mathrm{End}(^\lambda V(K))$. It follows that $\mathfrak{G}(K)$ acts on $^\lambda V(K)$.

1.4. In the setup of 1.4 for λ, λ' in $\mathcal{X}^+$ we can view $^\lambda V \otimes {}^{\lambda'}V$ with its basis $\mathcal{S} = {}^\lambda\beta \otimes {}^{\lambda'}\beta$ as an object of $\mathcal{D}$. Hence $(^\lambda V \otimes {}^{\lambda'}V)(K) \in \mathcal{D}(K)$ is defined. We define $E(K) : {}^\lambda V(K) \times {}^{\lambda'}V(K) \to (^\lambda V \otimes {}^{\lambda'}V)(K)$ by

$$(\sum_{b\in{}^\lambda\beta} \xi_b b), (\sum_{b'\in{}^{\lambda'}\beta} \xi'_{b'} b') \mapsto \sum_{(b,b')\in\mathcal{S}} \xi_b\xi'_{b'}(b \otimes b').$$

(This is not a morphism in $\mathcal{D}(K)$.) We define a map

$$\mathrm{End}(^\lambda V(K)) \times \mathrm{End}(^{\lambda'}V(K)) \to \mathrm{End}((^\lambda V \otimes {}^{\lambda'}V)(K))$$

by $(\tau, \tau') \mapsto [b \otimes b') \mapsto E(K)(\tau(b), \tau'(b'))]$. Composing this map with the map

$$\mathfrak{G}(K) \to \mathrm{End}(^\lambda V(K)) \times \mathrm{End}(^{\lambda'}V(K))$$

whose components are the maps

$$\mathfrak{G}(K) \to \mathrm{End}(^\lambda V(K)), \quad \mathfrak{G}(K) \to \mathrm{End}(^{\lambda'}V(K))$$

in 1.4 we obtain a map $\mathfrak{G}(K) \to \mathrm{End}((^\lambda V \otimes {}^{\lambda'}V)(K))$ which is a monoid homomorphism. Thus $\mathfrak{G}(K)$ acts on $(^\lambda V \otimes {}^{\lambda'}V)(K)$; it also acts on $^\lambda V(K) \times {}^{\lambda'}V(K)$ (by 1.4) and the two actions are compatible with $E(K)$.

Let $\Gamma : {}^{\lambda+\lambda'}V \to {}^\lambda V \otimes {}^{\lambda'}V$ be as in 1.1. For $b \in {}^{\lambda+\lambda'}\beta$ we have

$$\Gamma(b) = \sum_{(b_1,b'_1)\in\mathcal{S}} e_{b,b_1,b'_1} b_1 \otimes b'_1$$

where $e_{b,b_1,b'_1} \in \mathbf{N}$. (This can be deduced from the positivity property [Lus93, 14.4.13(b)] of the homomorphism r in [Lus93, 1.2.12].) Thus Γ is a morphism in $\mathcal{D}$ hence $\Gamma(K) : {}^{\lambda+\lambda'}V(K) \to (^\lambda V \otimes {}^{\lambda'}V)(K)$ is a well defined morphism in $\mathcal{D}(K)$. Note that $\Gamma(K)$ is compatible with the action of $\mathfrak{G}(K)$ on the two sides.

1.5. In the setup of 1.4 let $\mathcal{C}(K)$ be the set of all collections $\{x_\lambda \in {}^\lambda V(K); \lambda \in \mathcal{X}^+_{\bar{J}}\}$ such that for any λ, λ' in $\mathcal{X}^+_{\bar{J}}$ we have $\Gamma(K)(x_{\lambda+\lambda'}) = E(K)(x_\lambda, x_{\lambda'})$. Let $\mathcal{C}^*(K)$ be the set of all $(x_\lambda) \in \mathcal{C}(K)$ such that $x_\lambda \neq \underline{o}$ for any $\lambda \in \mathcal{X}^+_{\bar{J}}$. Let $H(K)$ be the group (multiplication component by component) consisting of all collections $\{z_\lambda \in K; \lambda \in \mathcal{X}^+_{\bar{J}}\}$ such that for any λ, λ' in $\mathcal{X}^+_{\bar{J}}$ we have $z_{\lambda+\lambda'} = z_\lambda z_{\lambda'}$. Now $H(K)$ acts on $\mathcal{C}(K)$ by $(z_\lambda), (x_\lambda) \mapsto (z_\lambda x_\lambda)$. This restricts to a free action of $H(K)$ on $\mathcal{C}^*(K)$. Let $\mathcal{P}^J(K)$ be the set of orbits for this action. Note that $\mathfrak{G}(K)$ acts on $\mathcal{C}(K)$ by acting component by component (see 1.4); we use that $E(K), \Gamma(K)$ are compatible with the $\mathfrak{G}(K)$-actions (see 1.5). This induces a $\mathfrak{G}(K)$-action on $\mathcal{P}^J(K)$.

1.6. In this subsection we assume that $K = \mathbf{R}_{>0}$. If $(x_\lambda) \in \mathcal{C}^*(K)$, we can view (x_λ) as an element of $\mathcal{C}^*$ by viewing $^\lambda V(K)$ as a subset of $^\lambda V$ in an obvious way. The inclusion $\mathcal{C}^*(K) \subset \mathcal{C}^*$ is compatible with the actions of $H(K)$ and H (we have $H(K) \subset H$) hence it induces an (injective) map $\mathcal{P}^J(K) \to {}'\mathcal{P}^J$. Composing this with the inverse of the bijection $\mathcal{P}^J \to {}'\mathcal{P}^J$ (see 1.3) we obtain an injective map $\mathcal{P}^J(K) \to \mathcal{P}^J$. We define $\mathcal{P}^J_{\geq 0}$ to be the image of this map.

Assuming further that our Cartan matrix is of finite-type, we show that the last definition of $\mathcal{P}^J_{\geq 0}$ agrees with the definition in [Lus98]. Applying [Lus98, 3.4] to a

$\lambda \in \mathcal{X}_J^+$ with large enough coordinates we see that $\mathcal{P}^J_{\geq 0}$ (in the new definition) is contained in $\mathcal{P}^J_{\geq 0}$ (in the definition of [Lus98]). The reverse inclusion follows from [Lus98, 3.2].

1.7. Any homomorphism of semifields $K \to K'$ induces in an obvious way a map $\mathcal{P}^J(K) \to \mathcal{P}^J(K')$.

1.8. We expect that when K' is the semifield $\{1\}$ with one element, one can identify $\mathcal{P}^\emptyset(K')$ with the set of pairs (a, a') in the Weyl group W of G such that $a \leq a'$ for the standard partial order of W. If K is any semifield one can also expect that the fibre of the map $\mathcal{P}^\emptyset(K) \to \mathcal{P}^\emptyset(\{1\})$ induced by the obvious map $K \to \{1\}$ (see 1.8) at the element corresponding to (a, a') is in bijection with $K^{|a'|-|a|}$ where $a \mapsto |a|$ is the length function on W.

2. The semiring $M(K)$

2.1. In this section we assume that our Cartan matrix is of finite-type. Let K be a semifield. Let $M(K) = \bigoplus_{\lambda \in \mathcal{X}_J^+} {}^\lambda V(K)$ viewed as a monoid under addition and with scalar multiplication by elements of $K^!$.

We define a multiplication $\mu : M(K) \times M(K) \to M(K)$ which is "bilinear" with respect to addition and scalar multiplication and satisfies $\mu(b_1, b_1') = \sum_{b \in {}^{\lambda+\lambda'}\beta} e_{b,b_1,b_1'} b$ where $\lambda \in \mathcal{X}_J^+, \lambda' \in \mathcal{X}_J^+$, $b_1 \in {}^\lambda\beta, b_1' \in {}^{\lambda'}\beta$, and $e_{b,b_1,b_1'} \in \mathbf{N}$ (viewed as an element of $K^!$) is as in the definition of $\Gamma(K)$ in 1.5, so that it comes from the homomorphism r in [Lus93, 1.2.12]. This can be viewed as a direct sum of "transposes" of maps like $\Gamma(K)$. From the properties of r we see that the multiplication μ is associative and commutative; it is clearly distributive with respect to addition. This multiplication has a unit element, given by the unique element in β^λ with $\lambda = 0$. Note that $M(K)$ is a semiring. Now $M(K)$ can be viewed as a form over K of the coordinate ring of G/U^+ where U^+ is the unipotent radical of B^+. Let $M'(K)$ be the set of maps $M(K) \to K^!$ which are compatible with addition, multiplication, and with scalar multiplication by elements of $K^!$, take the unit element of $M(K)$ to the unit element of $K^!$, and take the element with all components equal to $\underline{\circ}$ to $\circ \in K^!$. It is easy to show that $M'(K)$ is in canonical bijection with $\mathcal{C}(K)$.

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