

## GRADINGS OF LIE ALGEBRAS, MAGICAL SPIN GEOMETRIES AND MATRIX FACTORIZATIONS

ROLAND ABUAF AND LAURENT MANIVEL

ABSTRACT. We describe a remarkable rank 14 matrix factorization of the octic  $\text{Spin}_{14}$ -invariant polynomial on either of its half-spin representations. We observe that this representation can be, in a suitable sense, identified with a tensor product of two octonion algebras. Moreover the matrix factorisation can be deduced from a particular  $\mathbb{Z}$ -grading of  $\mathfrak{e}_8$ . Intriguingly, the whole story can in fact be extended to the whole Freudenthal-Tits magic square and yields matrix factorizations on other spin representations, as well as for the degree seven invariant on the space of three-forms in several variables. As an application of our results on  $\text{Spin}_{14}$ , we construct a special rank seven vector bundle on a double-octic threefold, that we conjecture to be spherical.

### 1. INTRODUCTION

Recall that a *matrix factorization* of a polynomial  $W$  is a pair  $(P, Q)$  of square matrices of the same size, say  $N$ , with polynomial entries, such that

$$PQ = QP = W.Id_N.$$

Matrix factorizations have attracted a lot of attention since their introduction by Eisenbud [13] in connection with Cohen-Macaulay modules over hypersurfaces. Important examples of matrix factorizations, when  $W$  is a quadratic form, are provided by Clifford modules [7, 8]. They can be obtained as follows. Suppose our base field is the field of complex numbers, which will be the case in the whole paper. Consider the simple Lie algebras  $\mathfrak{so}_n$ ,  $n \geq 5$ , with their spin representations. When  $n$  is even, there are two half-spin representations  $\Delta_+$  and  $\Delta_-$ , of the same dimension  $N = 2^{\frac{n}{2}-1}$ . Their direct sum can be defined as a module over the Clifford algebra of the natural representation  $V_n$  of  $\mathfrak{so}_n$ , with its invariant quadratic form  $q$ . The Clifford multiplication yields equivariant morphisms

$$V_n \otimes \Delta_+ \longrightarrow \Delta_- \quad \text{and} \quad V_n \otimes \Delta_- \longrightarrow \Delta_+.$$

So for each  $v \in V_n$ , we get morphisms  $P(v) : \Delta_+ \rightarrow \Delta_-$  and  $Q(v) : \Delta_- \rightarrow \Delta_+$ , depending linearly on  $v$ , and the fact that the total spin representation is a Clifford module yields the identities

$$P(v) \circ Q(v) = q(v)Id_{\Delta_-} \quad \text{and} \quad Q(v) \circ P(v) = q(v)Id_{\Delta_+}.$$

In other words, we get a of rank  $N$  matrix factorization of the quadratic form  $q$ .

Surprisingly, this is a non trivial matrix factorization of minimal size of a non degenerate quadratic form in  $n$  variables. This illustrates the difficulty to find explicit ones in general. One of the goals of this paper is precisely to describe several remarkable matrix factorizations, again related to spin representations. Our

---

Received by the editors April 17, 2020, and, in revised form, April 2, 2021.  
2020 *Mathematics Subject Classification*. Primary 15A66, 13A50, 17B45.

main result will be the description of a rank 14 matrix factorization of a particular degree eight polynomial in 64 variables, a  $\text{Spin}_{14}$ -invariant polynomial on a half-spin representation  $\Delta_{14}$ . In the next section, we will give a direct construction of this invariant and prove that it admits a matrix factorization (Theorem 2.3.2). Our proof is remarkably simple, and relies on the fact that  $\Delta_{14}$  contains an open orbit of the action of  $\mathbb{C}^* \times \text{Spin}_{14}$ . We will observe in passing the intriguing fact that fixing a point in this open orbit determines a factorisation of  $\Delta_{14}$  as the tensor product of two octonion algebras (Proposition 2.2.1).

In the last section, we will relate those observations to gradings of Lie algebras and the Freudenthal magic square. The point is that  $\Delta_{14}$  appears in a particular  $\mathbb{Z}$ -grading of the largest exceptional algebra  $\mathfrak{e}_8$ , and that the octic invariant and its matrix factorization can be constructed directly from  $\mathfrak{e}_8$ . Moreover, this particular  $\mathbb{Z}$ -grading turns out to be related to the space of “points” in the Freudenthal geometry associated to  $E_8$ .

Astonishingly, the whole story extends to the full magic square. Recall that this square associates to a pair  $(\mathbb{A}, \mathbb{B})$  of normed algebras (either  $\mathbb{R}, \mathbb{C}$ , the algebra  $\mathbb{H}$  of quaternions or the Cayley algebra  $\mathbb{O}$  of octonions) a semisimple Lie algebra  $\mathfrak{g}(\mathbb{A}, \mathbb{B})$ . In particular  $\mathfrak{g}(\mathbb{O}, \mathbb{O}) = \mathfrak{e}_8$ . The space of “points” in the corresponding Freudenthal geometry induces a grading of  $\mathfrak{g}(\mathbb{A}, \mathbb{B})$  whose main component is always a spin representation, and this yields a matrix factorization (Theorem 3.2.1). Moreover, once one chooses a general point in that representation, it gets naturally identified with  $\mathbb{A} \otimes \mathbb{B}$ .

Finally, we discuss the sporadic case of the third exterior power of a vector space of dimension seven, which is related with a certain  $\mathbb{Z}$ -grading of  $\mathfrak{e}_7$ .

One motivation for this study of matrix factorizations has been the construction of a special rank seven vector bundle on a double octic threefold obtained as a double cover of  $\mathbb{P}^3$  branched over a linear section of the octic hypersurface defined by the  $\text{Spin}_{14}$ -invariant of  $\Delta_{14}$ . We conjecture that this bundle is spherical. Such a double cover is in fact a Calabi-Yau threefold and there is an astonishing series of relationships, far from being completely understood yet, between exceptional Lie algebras and certain families of manifolds of Calabi-Yau type [1, 19]. We hope to come back to this conjecture in a subsequent paper.

## 2. SPIN GEOMETRY IN DIMENSION FOURTEEN

Spin geometry in dimension twelve has several very remarkable features, two of which we would like to recall briefly. Let  $\Delta_{12}$  be one of the half-spin representations of  $\text{Spin}_{12}$  (see [27, Section 5] for more details).

- (1) The action of  $\text{Spin}_{12}$  on  $\mathbb{P}\Delta_{12}$  has only four orbits, whose closures are the whole space, a degree four hypersurface, its singular locus, and inside the latter, the spinor variety  $S_{12}$ , which parametrizes one of the families of maximal isotropic subspaces of a quadratic twelve dimensional vector space  $V_{12}$ .
- (2) The spinor variety  $S_{12} \subset \mathbb{P}\Delta_{12}$  is a variety *with one apparent double point*, which means that through a general point of  $\mathbb{P}\Delta_{12}$  passes a unique line which is bisecant to  $S_{12}$ . One can deduce that the open orbit  $\mathcal{O}_0 \simeq \text{Spin}_{12}/\text{SL}_6 \times \mathbb{Z}_2$ , where the  $\mathbb{Z}_2$  factor exchanges the two points in  $S_{12}$  of the afore mentioned bisecant.

An interesting consequence is that a double cover of the open orbit is naturally built in the spin geometry, which turns out to be intimately related with the family of double quartic fivefolds. Those varieties have attracted some interest from the early ages of mirror symmetry, being Fano manifolds of Calabi-Yau type that can be considered as mirror to certain rigid Calabi-Yau threefolds [9, 34].

The goal of this section is to describe the similar properties that can be observed for the spin geometry in dimension fourteen. We will start by briefly recalling the orbit structure, which has classically been considered by several authors [20, 24, 32]. The quartic invariant hypersurface in  $\mathbb{P}\Delta_{12}$  is in particular replaced by an octic invariant hypersurface in  $\mathbb{P}\Delta_{14}$  on which the next section will focus. Here we will highlight a kind of multiplicative version of the one apparent double point property, which we find very remarkable.

**2.1. Orbits.** Recall that the half-spin representations of  $\text{Spin}_{14}$  can be defined by choosing a splitting  $V_{14} = E \oplus F$ , where  $E$  and  $F$  are maximal isotropic subspaces. Note that the quadratic form  $q$  on  $V_{14}$  induces a perfect duality between  $E$  and  $F$ . Then  $E$  acts on the exterior algebra  $\wedge^* E$  by the wedge product, and  $F$  by twice the contraction by the quadratic form. The resulting action of  $V_{14}$  on  $\wedge^* E$  upgrades to an action of its Clifford algebra. By restriction, one gets an action of  $\text{Spin}_{14}$ , as well as of its Lie algebra  $\mathfrak{spin}_{14} \simeq \wedge^2 V_{14}$ . The half-spin representations are then given by the even and odd parts,  $\wedge^+ E$  and  $\wedge^- E$ , of the exterior algebra  $\wedge^* E$ . We will let  $\Delta_{14} = \wedge^+ E$ . (Of course the construction works to any  $\text{Spin}_{2n}$ , starting from a splitting of a  $2n$ -dimensional vector space  $V_{2n}$  endowed with a non degenerate quadratic form. In the odd case,  $V_{2n+1}$  can only be split as  $E \oplus F \oplus L$ , with  $E$  and  $F$  isotropic and  $L$  a line. The unique spin representation can then be identified with  $\wedge^* E$ . See e.g. [10] for more details.)

According to Sato and Kimura [24, page 132], the fact that a half-spin representation  $\Delta_{14}$  of  $\text{Spin}_{14}$  is prehomogeneous under the action of  $\mathbb{C}^* \times \text{Spin}_{14}$  was first observed by Shintani in 1970, and the orbit structure was obtained by Kimura and Ozeki in 1973. The fact that there are only finitely many orbits is actually an immediate consequence of Kac and Vinberg's theory of  $\theta$ -groups [17, 36]. Indeed, the half-spin representation  $\Delta_{14}$  is a component of the  $\mathbb{Z}$ -grading of  $\mathfrak{e}_8$  defined by its first simple root.

Up to our knowledge, the classification of the orbits of  $\text{Spin}_{14}$  was first published in 1977 by Popov [32], with explicit representatives of each orbit and the types of their stabilizers. It also appears in the paper by Kac and Vinberg in [20], along with the the orbits of  $\text{Spin}_{13}$  on the same representation. More details about the geometry of the orbit closures can be found in [26].

As we already mentioned there is an octic invariant  $J_8$  (unique up to scalar), and each level set  $J_8^{-1}(c)$  is a single orbit of  $\text{Spin}_{14}$  for  $c \neq 0$ . Inside the octic hypersurface ( $J_8 = 0$ ), there are eight non trivial orbits. Among those, the most important one is the (pointed) cone over the spinor variety  $S_{14}$ , which parametrizes the maximal isotropic subspaces of  $V_{14}$  in the same family as  $F$ . The other family  $S'_{14}$  of such spaces, to which  $E$  belongs, is naturally embedded inside the projectivization of the other half-spin representation, the dual  $\Delta_{14}^\vee$ . By the way, although we will not use this fact, it is interesting to note that the projective dual of  $S'_{14}$  is precisely the octic hypersurface ( $J_8 = 0$ ) inside  $\mathbb{P}(\Delta_{14})$ .

For future use let us mention that the following points  $z_0$  and  $z_1$  are respectively outside the octic hypersurface, and inside the open orbit of this hypersurface [20]:

$$(1) \quad z_0 = 1 + e_{1237} + e_{4567} + e_{123456},$$

$$(2) \quad z_1 = 1 + e_{1237} + e_{1587} + e_{2467} + e_{123456}.$$

Here we have fixed a basis  $e_1, \dots, e_7$  of  $E$ , and we have used the notation  $e_{ijkl}$  for  $e_i \wedge e_j \wedge e_k \wedge e_l$ , and so on. We will also denote by  $f_1, \dots, f_7$  the dual basis of  $F$ , defined by the condition that  $q(e_i, f_j) = \delta_{ij}$ .

The stabilizer of  $z_0$  was computed explicitly in [24, Proposition 40], or more precisely its Lie algebra  $\mathfrak{h} \simeq \mathfrak{g}_2 \times \mathfrak{g}_2$ . Sato and Kimura observed that  $\mathfrak{h}$  stabilizes a unique pair of seven-dimensional subspaces of  $V_{14}$ , each copy of  $\mathfrak{g}_2$  acting irreducibly on one of these spaces and trivially on the other. To be more specific, the two invariant seven-dimensional subspaces are

$$V_7 = \langle e_1, e_2, e_3, f_1, f_2, f_3, e_7 - f_7 \rangle \quad \text{and} \quad V'_7 = \langle e_4, e_5, e_6, f_4, f_5, f_6, e_7 + f_7 \rangle.$$

Observe that  $V_7$  and  $V'_7$  are mutually orthogonal, and in direct sum (equivalently, the restriction of the quadratic form to these spaces is non degenerate). Moreover  $E_3 := E \cap V_7 = \langle e_1, e_2, e_3 \rangle$  and  $E'_3 := E \cap V'_7 = \langle e_4, e_5, e_6 \rangle$  are maximal isotropic subspaces of  $V_7$  and  $V'_7$ . In particular we may identify the spin representations of  $\mathfrak{spin}(V_7)$  and  $\mathfrak{spin}(V'_7)$  with  $\wedge^* E_3$  and  $\wedge^* E'_3$ , respectively. Observe that their tensor product is

$$\wedge^* E_3 \otimes \wedge^* E'_3 \simeq \wedge^*(E_3 \oplus E'_3) \simeq \wedge^+ E = \Delta_{14}.$$

Under this identification,

$$z_0 = 1 \otimes 1 + e_{123} \otimes 1 + 1 \otimes e_{456} + e_{123} \otimes e_{456} = (1 + e_{123}) \otimes (1 + e_{456})$$

turns out to be decomposable. Moreover, the action of  $\mathbb{C}^* \times \text{Spin}_7$  on the spin representation is prehomogeneous; in fact there is a quadratic invariant  $Q$ , and the non trivial orbits are the pointed quadric and its complement; the vector  $1 + e_{123}$  belongs to the latter, and its stabilizer is isomorphic to  $G_2 = \text{Aut}(\mathbb{O})$  (by [24, Proposition 25], the generic stabilizer has Lie algebra  $\mathfrak{g}_2$ ; that it is really isomorphic to  $G_2$  follows from the triality principle [5, Section 2.4]). This already implies that the identity component of the stabilizer of  $[z_0] \in \mathbb{P}(\Delta_{14})$  is  $G_2 \times G_2$ . The computations of [14, Example 21.1] show that:

**Proposition 2.1.1.** *The open orbit in  $\mathbb{P}(\Delta_{14})$  is isomorphic with the homogeneous space  $\text{Spin}_{14}/(G_2 \times G_2) \rtimes \mu_8$ , where  $\mu_8$  denotes the group of 8-th roots of unity.*

**2.2. A multiplicative double point property.** Note that an orthogonal decomposition  $V_{14} = V_7 \oplus V'_7$  always determines a decomposition of  $\Delta_{14}$  as  $\Delta_7 \otimes \Delta'_7$ , as follows. By the Borel-Weil theorem, we can realize  $\Delta_{14}$  as  $H^0(S_{14}, \mathcal{L}_{14})$ , where  $\mathcal{L}_{14}$  denotes the positive generator of the Picard group of the spinor variety  $S_{14}$ . Similarly, we can realize  $\Delta_7$  and  $\Delta'_7$  as  $H^0(S_7, \mathcal{L}_7)$  and  $H^0(S_7, \mathcal{L}'_7)$ , with similar notations. Points in  $S_7$  and  $S'_7$  are three-dimensional isotropic subspaces  $E_3$  and  $E'_3$  of  $V_7$  and  $V'_7$ . Their direct sum is still isotropic, and being of dimension six, it is contained in exactly two maximal isotropic spaces of  $V_{14}$ , one of each family. In particular this defines a regular map  $\phi : S_7 \times S'_7 \rightarrow S_{14}$ , such that  $\phi^* \mathcal{L}_{14} = \mathcal{L}_7 \boxtimes \mathcal{L}'_7$ . By restriction this yields a map

$$\Delta_{14} \simeq H^0(S_{14}, \mathcal{L}_{14}) \longrightarrow H^0(S_7 \times S'_7, \phi^* \mathcal{L}_{14}) = \Delta_7 \otimes \Delta'_7.$$

This map is equivariant under  $\text{Spin}(V_7) \times \text{Spin}(V'_7)$ , and certainly non zero. Its target being irreducible, it has to be surjective, and then an isomorphism since the source and target have the same dimension.

We can summarize this discussion as follows.

**Proposition 2.2.1.** *Let  $[\psi]$  be a generic element of  $\mathbb{P}(\Delta_{14})$ .*

- (1) *There exists a unique orthogonal decomposition  $V_{14} = V_7 \oplus V'_7$  preserved by the stabilizer of  $[\psi]$  in  $\text{Spin}_{14}$ .*
- (2) *Under the induced isomorphism  $\Delta_{14} \simeq \Delta_7 \otimes \Delta'_7$ , we have  $[\psi] = [\chi \otimes \chi']$  for  $[\chi]$  and  $[\chi']$  generic inside  $\mathbb{P}(\Delta_7)$  and  $\mathbb{P}(\Delta'_7)$ .*

*Remarks.*

- (1) Note that  $V_7$  must belong to the open subset of  $G(7, V_{14})$  defined by the condition that the restriction of the quadratic form remains non degenerate. This open subset has dimension 49, and for each choice of  $V_7$  there are 7 parameters for the generic  $[\chi]$  and  $[\chi']$ . This yields the correct number  $49 + 2 \times 7 = 63$  of parameters for the open orbit in  $\mathbb{P}(\Delta_{14})$ . Moreover, we get a remarkable partition of the open orbit in  $\mathbb{P}(\Delta_{14})$  by a 49-dimensional family of open subsets of  $\mathbb{P}^7 \times \mathbb{P}^7$ .
- (2) The stabilizer of a generic point  $[\chi]$  in  $\mathbb{P}\Delta_7$  is a copy of  $G_2$ , whose action on  $\Delta_7$  can be identified with the action of the latter on the Cayley algebra  $\mathbb{O}$  (and  $\chi$  becomes the unit in this algebra). As a consequence, a generic point in  $\Delta_{14}$  allows to interpret it as the tensor product  $\mathbb{O} \otimes \mathbb{O}$  of two Cayley algebras. It would be interesting to relate this observation to the work of Rosenfeld on the algebra of “octooctonions” [33].

**2.3. A matrix factorization for the octic invariant.** We will construct later on a matrix factorization for the octic invariant  $J_8$ . A first explicit but cumbersome construction was obtained by Gyoja [16]. Let us present a more direct approach.

Our main observation is that, according to [28], the symmetric square of  $\Delta_{14}$  contains a copy of  $\wedge^3 V_{14}$ . This can be deduced from the Clifford action on the full spin representation, which decomposes into equivariant maps

$$V_{14} \otimes \Delta_{14} \longrightarrow \Delta_{14}^\vee \quad \text{and} \quad V_{14} \otimes \Delta_{14}^\vee \longrightarrow \Delta_{14}.$$

(Recall that  $\Delta_{14}^\vee \simeq \wedge^- E$ , on which  $E \subset V_{14}$  acts by wedge product and  $F \subset V_{14}$  by twice the contraction.) Composing those maps we get an equivariant morphism  $\wedge^3 V_{14} \otimes \Delta_{14} \hookrightarrow V_{14} \otimes V_{14} \otimes V_{14} \otimes \Delta_{14} \rightarrow V_{14} \otimes V_{14} \otimes \Delta_{14}^\vee \rightarrow V_{14} \otimes \Delta_{14} \rightarrow \Delta_{14}^\vee$ . Taking the transpose we get

$$(3) \quad \Omega : S^2 \Delta_{14} \longrightarrow \wedge^3 V_{14}.$$

To be more explicit, we can fix a basis  $(v_1, \dots, v_{14})$  of  $V_{14}$  (for example  $(e_1, \dots, e_7, f_1, \dots, f_7)$ ) and denote the dual basis by  $(w_1, \dots, w_{14})$  (which would be  $(f_1, \dots, f_7, e_1, \dots, e_7)$  for the same example). Then

$$(4) \quad \Omega_z = \sum_{i < j < k} \langle z, v_i v_j v_k z \rangle w_i w_j w_k \in \wedge^3 V_{14}.$$

Here the natural pairing between  $z \in \Delta_{14} = \wedge^+ E$  and  $z' \in \wedge^- E$  is defined as the component of  $z \wedge z'$  on  $\wedge^7 E$ .

**Lemma 2.3.1.** *The equivariant map  $\Omega$  is non zero.*

*Proof.* We consider  $\Omega$  as a quadratic form on  $\Delta_{14}$ , with values in  $\wedge^3 V_{14}$ , and evaluate it at a general point, that is, at the point  $z_0$  of the open orbit. We get

$$\frac{1}{2}\Omega_{z_0} = e_{123} - e_{456} - f_{123} - f_{456} - \sum_{i=1}^6 \epsilon_i e_{i7} f_i - \sum_{i=1}^6 e_i f_{i7},$$

where  $\epsilon_i = 1$  for  $i \leq 3$  and  $\epsilon_i = -1$  for  $i \geq 4$ . This is of course non zero. □

Observe moreover that  $\frac{1}{2}\Omega_{z_0}$  decomposes nicely as  $\Omega - \Omega'$ , where  $\Omega \in \wedge^3 V_7$  and  $\Omega' \in \wedge^3 V'_7$  are given by

$$\begin{aligned} \Omega &= e_{123} - f_{123} + \left(\sum_{i=1}^3 e_i \wedge f_i\right) \wedge (e_7 - f_7), \\ \Omega' &= e_{456} + f_{456} + \left(\sum_{i=4}^6 e_i \wedge f_i\right) \wedge (e_7 + f_7). \end{aligned}$$

Those forms  $\Omega$  and  $\Omega'$  are generic elements of  $\wedge^3 V_7$  and  $\wedge^3 V'_7$  (up to normalizations, they coincide with the generic three-form explicited in [29]). Recall that we recover  $G_2$  as the stabilizer of such a generic form.

Our second ingredient will be the equivariant map

$$\Theta : S^2(\wedge^3 V_{14}) \longrightarrow \text{End}(V_{14})$$

obtained as follows. First embed  $\wedge^3 V_{14}$  inside  $V_{14} \otimes \wedge^2 V_{14}$ , Then recall that the quadratic form  $q$  on  $V_{14}$  induces a quadratic form

$$q_2 : S^2(\wedge^2 V_{14}) \longrightarrow \mathbb{C},$$

whose polarization is given by the formula

$$q_2(u_1 \wedge u_2, v_1 \wedge v_2) = \det(q(u_i, v_j))_{1 \leq i, j \leq 2}.$$

Use this quadratic form to define the composition

$$\Theta : S^2(\wedge^3 V_{14}) \hookrightarrow S^2(V_{14} \otimes \wedge^2 V_{14}) \longrightarrow S^2 V_{14} \otimes S^2(\wedge^2 V_{14}) \longrightarrow S^2 V_{14} \hookrightarrow \text{End}(V_{14}).$$

Finally, for  $z \in \Delta$ , let  $M_z = \Theta(\Omega_z) \in \text{End}(V_{14})$ . Using equation (4), we can compute explictely

$$(5) \quad M_z = \sum_{k, \ell} \left( \sum_{\substack{i < j, \\ i, j \neq k, \ell}} \langle z, v_i v_j v_k z \rangle \langle z, w_i w_j w_\ell z \rangle \right) w_k v_\ell.$$

**Theorem 2.3.2.** *The pair  $(M, M)$  is a matrix factorization of the octic invariant  $J_8$  of  $\Delta_{14}$ .*

*Proof.* We just need to check that  $M_{z_0}^2$  is a non zero multiple of the identity. So let us compute  $M_{z_0}$ . We have seen that  $\frac{1}{2}\Omega_{z_0} = \Omega - \Omega'$ , where  $\Omega$  and  $\Omega'$  belong to  $\wedge^3 V_7$  and  $\wedge^3 V'_7$ , respectively.

**Lemma 2.3.3.**  $\Theta(\Omega - \Omega') = \Theta(\Omega) + \Theta(\Omega')$ .

*Proof.* Indeed, let  $v_1, \dots, v_7$  and  $v'_1, \dots, v'_7$  be basis of  $V_7$  and  $V'_7$ , respectively. The polarisations of  $\Omega$  and  $\Omega'$  in  $V_{14} \otimes \wedge^2 V_{14}$  will respectively be of the form  $\sum_{i=1}^7 v_i \otimes \omega_i$  and  $\sum_{i=1}^7 v'_i \otimes \omega'_i$ , for some tensors  $\omega_i \in \wedge^2 V_7$  and  $\omega'_i \in \wedge^2 V'_7$ . When we apply  $\Theta$  and take the image in  $S^2 V_{14}$ , the mixed terms are of the form  $q_2(\omega_i, \omega'_j) v_i v'_j$ . But  $q_2(\omega_i, \omega'_j)$  is always zero since  $V_7$  and  $V'_7$  are orthogonal. □

In order to compute  $\Theta(\Omega)$ , we first send  $\Omega$  to  $V_{14} \otimes \wedge^2 V_{14}$  by polarizing it. Let  $e_0 = e_7 - f_7$ . We get

$$\begin{aligned} \bar{\Omega} = & e_1 \otimes (e_{23} + f_1 e_0) + e_2 \otimes (e_{31} + f_2 e_0) + e_3 \otimes (e_{12} + f_3 e_0) + \\ & + f_1 \otimes (f_{23} - e_1 e_0) + f_2 \otimes (f_{31} - e_2 e_0) + f_3 \otimes (f_{12} - e_3 e_0) + \\ & + e_0 \otimes (e_1 f_1 + e_2 f_2 + e_3 f_3). \end{aligned}$$

Now recall that  $q(e_i, f_i) = 1$  for all  $i$ , while  $q(e_0) = -2$ ; moreover  $q$  evaluates to zero on any other pair of basis vectors. We deduce that  $q_2(e_1 f_1 + e_2 f_2 + e_3 f_3) = 3$ ,  $q_2(e_{23} + f_1 e_0, f_{23} - e_1 e_0) = q_2(e_{31} + f_2 e_0, f_{31} - e_2 e_0) = q_2(e_{12} + f_3 e_0, f_{12} - e_3 e_0) = 3$ , and that all the other scalar products are zero. This yields

$$\Theta(\Omega) = 3e_0^2 - 6e_1 f_1 - 6e_2 f_2 - 6e_3 f_3 \in S^2 V_{14}.$$

With respect to the quadratic form  $q$ , the dual basis of  $(e_0, e_1, e_2, e_3, f_1, f_2, f_3)$  is  $(-\frac{1}{2}e_0, f_1, f_2, f_3, e_1, e_2, e_3)$ . Considered as an element of  $End(V_{14})$ , the tensor  $\Theta(\Omega)$  is thus exactly  $-6\pi_{V_7}$ , where  $\pi_{V_7}$  denotes the orthogonal projection to  $V_7$ . A similar computation shows that  $\Theta(\Omega')$  is  $+6\pi_{V_7}$ . We finally get

$$M_{z_0} = 24(\pi_{V_7'} - \pi_{V_7}),$$

whose square is 576 times the identity. This concludes the proof. □

*Remarks.*

- (1) Once we have observed that  $\Theta(\Omega_{z_0}) = \Theta(\Omega) + \Theta(\Omega')$ , we can in fact conclude without any extra computation. Indeed,  $\Theta(\Omega)$  is an element of  $S^2 V_7$  that must be preserved by the stabilizer of  $\Omega$ , hence by a copy of  $G_2$ . But up to scalar there is a unique such element. Moreover we already know one: the restriction to  $V_7$  of the quadratic form on  $V_{14}$ . The same being true for  $\Omega'$ , we conclude that there exist scalars  $a$  and  $a'$  such that

$$M_{z_0} = a\pi_{V_7} + a'\pi_{V_7'}.$$

But the trace of  $M_{z_0}$  must be zero, otherwise we would get a non trivial quartic invariant on  $\Delta_{14}$ , and we know there is none. So  $a + a' = 0$ , and the square of  $M_{z_0}$  is a homothety.

- (2) Let us also compute  $M_{z_1}$ . We start by computing  $\Omega_{z_1}$ :

$$\begin{aligned} \frac{1}{2}\Omega_{z_1} = & e_{123} + e_{156} + e_{246} - f_{135} - f_{234} - f_{456} + \\ & + (e_2 f_5 - e_1 f_4 + e_3 f_6)e_7 - (\sum_{i=1}^6 e_i f_i) f_7. \end{aligned}$$

Polarizing, we get the following tensor  $\Theta_1$  in  $V_{14} \otimes \wedge^2 V_{14}$ :

$$\begin{aligned} & e_1 \otimes (e_{23} + e_{56} + f_4 e_7 - f_{17}) + e_2 \otimes (e_{46} - e_{13} - f_5 e_7 - f_{27}) \\ & + e_3 \otimes (e_{12} - f_6 e_7 - f_{37}) - e_4 \otimes (e_{26} + f_{47}) - e_5 \otimes (e_{16} + f_{57}) \\ & + e_6 \otimes (e_{15} + e_{24} - f_{67}) + e_7 \otimes (e_2 f_5 - e_1 f_4 + e_3 f_6) \\ & + f_1 \otimes (e_1 f_7 - f_{35}) + f_2 \otimes (e_2 f_7 - f_{34}) + f_3 \otimes (e_3 f_7 + f_{15} + f_{24}) \\ & + f_4 \otimes (e_{17} + e_4 f_7 - f_{23} - f_{56}) + -f_5 \otimes (e_{27} - e_5 f_7 + f_{13} - f_{46}) \\ & - f_6 \otimes (e_{37} - e_6 f_7 + f_{45}) - f_7 \otimes (\sum_{i=1}^6 e_i f_i). \end{aligned}$$

If we write  $\Theta_1 = \sum_{i=1}^7 (e_i \otimes a_i + f_i \otimes b_i)$ , it is straightforward to check that the only non zero scalar products between the two-forms  $a_i, b_j$  are the following:

$$q_2(a_1, b_4) = -2, \quad q_2(a_2, b_5) = 2, \quad q_2(a_6, b_3) = -2, \quad q_2(b_7) = 6.$$

We thus finally get  $M_{z_1}$  as the following element of  $S^2V_{14}$ :

$$M_{z_1} = 8(3f_7^2 - e_1f_4 + e_2f_5 + e_6f_3).$$

As an endomorphism of  $V_{14}$ ,  $M_{z_1}$  has for image and kernel the same vector space  $V_7 = \langle f_7, e_1, f_4, e_2, f_5, e_6, f_3 \rangle$ . In particular, the square of  $M_{z_1}$  is zero, in agreement with the fact that  $J_8(z_1) = 0$ . Note that  $V_7$  is isotropic; moreover, since it meets  $E$  in odd dimension, it belongs to the same family of maximal isotropic subspaces, which is embedded in the other projectivized half-spin representation  $\mathbb{P}(\Delta_{14}^\vee)$ . This is in agreement with the fact that the octic invariant hypersurface in  $\mathbb{P}(\Delta_{14})$  can be obtained as the projective dual variety of the closed orbit  $S'_{14} \subset \mathbb{P}(\Delta_{14}^\vee)$ . In particular, the open orbit inside the octic is naturally fibered over  $S'_{14}$ , and our  $z_1$  must be sent to  $V_7$  by this fibration.

Let us summarize what we have proved so far, which is amazingly similar to what happens for  $S_{12}$  and  $\wedge^3V_6$ , see [31, sections 3.3 and 3.4].

**Proposition 2.3.4.**

- (1) *Let  $[z]$  belong to the open orbit in  $\mathbb{P}(\Delta_{14})$ , and let  $(V_7, V'_7)$  be the associated pair of orthogonal non degenerate subspaces of  $V_{14}$ . Then the associated three-form  $\Omega_z$  is the sum of generic three-forms  $\Omega \in \wedge^3V_7$  and  $\Omega' \in \wedge^3V'_7$ . Moreover*

$$M_z = m_z(\pi_{V'_7} - \pi_{V_7}), \quad \text{with } J_8(z) = m_z^2.$$

- (2) *Let  $[z]$  belong to the open orbit in the invariant hypersurface  $(J_8 = 0)$  of  $\mathbb{P}(\Delta_{14})$ , and let  $V_7$  be the associated maximal isotropic subspace of  $V_{14}$ . Then  $\Omega_z$  belongs to  $\wedge^2V_7 \wedge V_{14}$ , and  $M_z$  is a square zero endomorphism whose image and kernel are both equal to  $V_7$ .*

Beware that in (1), the pair  $(V_7, V'_7)$  is not ordered. Permuting  $V_7$  and  $V'_7$  changes the sign of  $m_z$ , so only its square is well-defined. And  $J_8$  is not globally a square.

**2.4. Double octics.** Consider the double cover  $\pi : D \rightarrow \mathbb{P}(\Delta_{14})$ , branched over the octic hypersurface  $(J_8 = 0)$ . This double cover can be interpreted as the octic hypersurface  $J_8(z) - y^2 = 0$  in the weighted projective space  $\tilde{\mathbb{P}} = \mathbb{P}(1^{64}, 4)$ . Moreover it inherits an action of the spin group  $\text{Spin}_{14}$ .

**Proposition 2.4.1.**

- (1) *The double cover  $D$  is smooth in codimension 5.*
- (2) *Its smooth locus  $D_0$  is endowed with two rank seven equivariant vector bundles  $\mathcal{E}_+$  and  $\mathcal{E}_-$ , exchanged by the deck transformation  $\iota$  of the double cover, which are two orthogonal subbundles of the trivial bundle  $\mathcal{V}$  with fiber  $V_{14}$ . Moreover there are exact sequences of vector bundles*

$$0 \rightarrow \mathcal{E}_\pm \rightarrow \mathcal{V} \rightarrow \mathcal{E}_\mp^\vee \rightarrow 0.$$

The matrix factorization  $(M_z, M_z)$  of  $J_8(z)$  upgrades to a matrix factorization  $(M_z + yId, M_z - yId)$  of  $J_8(z) - y^2$ . The bundles  $\mathcal{E}_+$  and  $\mathcal{E}_-$  can be defined at the point  $[z, y]$  as the kernels of  $M_z + yId$  and  $M_z - yId$ , respectively. Moreover, denote by  $\tilde{\mathbb{P}}_0$  the complement in  $\tilde{\mathbb{P}}$  of the singular locus of the hypersurface  $(J_8 = 0)$ .

Denote by  $j_0$  the embedding of  $D_0$  inside  $\tilde{\mathbb{P}}_0$ . Then we have the following exact sequences of sheaves on  $\tilde{\mathbb{P}}_0$ :

$$0 \longrightarrow V_{14} \otimes \mathcal{O}_{\tilde{\mathbb{P}}_0}(-4) \xrightarrow{M_z \pm y Id} V_{14} \otimes \mathcal{O}_{\tilde{\mathbb{P}}_0} \longrightarrow j_{0*} \mathcal{E}'_{\mp} \longrightarrow 0.$$

If  $L \subset \mathbb{P}(\Delta_{14})$  is a general linear subspace of dimension at most four, then it is contained in  $D_0$ . Over  $L$  we then get a double cover  $X_L$  with two rank seven vector bundles  $\mathcal{E}_L$  and  $\mathcal{E}'_L$ .

**Conjecture.** *For a general three dimensional subspace  $L \subset \mathbb{P}(\Delta_{14})$ , the vector bundles  $\mathcal{E}_L$  and  $\mathcal{E}'_L$  over the double cover  $X_L$ , are spherical.*

Note that in this case,  $X_L$  is a Calabi-Yau threefold. By spherical, one means that the bundles of endomorphisms of  $\mathcal{E}_L$  and  $\mathcal{E}'_L$  has the same cohomology as a three-dimensional sphere:

$$H^q(X, End(\mathcal{E}_L)) = H^q(X, End(\mathcal{E}'_L)) = \delta_{q,0} \mathbb{C} \oplus \delta_{q,3} \mathbb{C}.$$

As already discussed in [19], one can expect that a general double octic threefold  $X$  can always be represented as a section  $X_L$  of the octic in  $\mathbb{P}(\Delta_{14})$ . Moreover there should exist only a finite number  $N$  of such representations (up to isomorphism). This is the exact analogue of the statement, due to Beauville and Schreyer, that a general quintic threefold can be represented as a Pfaffian, in only finitely many ways [6, Proposition 8.9].

Moreover, our construction of the supposedly spherical vector bundles  $\mathcal{E}_L$  and  $\mathcal{E}'_L$  would parallel those of spherical bundles of rank seven on the generic cubic sevenfold [19], and of rank six on the generic double quartic fivefold [1].

Recall that a spherical object in the derived category of coherent sheaves of an algebraic variety defines a non trivial auto-equivalence of this category, called a spherical twist [35]. If the previous conjecture is true, we would thus get, on a general octic threefold,  $N$  pairs of spherical vector bundles, generating a certain group of symmetries of the derived category. It would be interesting to determine  $N$ , and this symmetry group.

The reconstruction problem also looks very intriguing: starting from a general octic threefold  $X$ , its branch divisor  $D$ , and the vector bundle  $\mathcal{E}_L$ , how can we reconstruct the embedding of  $D$  as a linear section of the invariant octic in  $\mathbb{P}(\Delta_{14})$ ?

### 3. RELATIONS WITH $\mathbb{Z}$ -GRADINGS OF LIE ALGEBRAS

**3.1. Morphisms from gradings.** Clerc was the first to realize that  $\mathbb{Z}$ -gradings can be useful to determine certain invariants [11]. He starts with a simple Lie algebra  $\mathfrak{g}$  whose adjoint representation is fundamental. In other words (once we have fixed a Cartan and a Borel subalgebra), the highest root  $\psi$  is a fundamental weight. The corresponding simple root  $\alpha$  defines a  $\mathbb{Z}$ -grading on  $\mathfrak{g}$ , of length five:

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

In this grading,  $\mathfrak{g}_2 = \mathfrak{g}_\psi$  and  $\mathfrak{g}_{-2} = \mathfrak{g}_{-\psi}$  are one dimensional. Once we have chosen generators  $X_\psi$  and  $X_{-\psi}$ , we can define a  $\mathfrak{g}_0$ -equivariant polynomial function  $J_4$  on  $\mathfrak{g}_1$ , homogeneous of degree four, by the relation

$$ad(z)^4 X_{-\psi} = J_4(z) X_\psi, \quad z \in \mathfrak{g}_1.$$

For  $\mathfrak{g}$  exceptional,  $J_4$  generates the space of semi-invariants for the action of  $\mathfrak{g}_0$  on  $\mathfrak{g}_1$ . (Recall that  $\mathfrak{g}_0$  is not semisimple but only reductive, with a non trivial center.

That  $J_4$  is semi-invariant means that  $\mathfrak{g}_0$  acts on it only through multiplication by some character.)

There exist other gradings of length five of simple Lie algebras, notably of the exceptional ones, such that  $\mathfrak{g}_2$  has dimension bigger than one. In this case, the very same idea yields morphisms

$$\text{Sym}^4 \mathfrak{g}_1 \longrightarrow \text{Hom}(\mathfrak{g}_{-2}, \mathfrak{g}_2) \quad \text{and} \quad \text{Sym}^4 \mathfrak{g}_{-1} \longrightarrow \text{Hom}(\mathfrak{g}_2, \mathfrak{g}_{-2}).$$

The representations  $\mathfrak{g}_1$  and  $\mathfrak{g}_{-1}$ , as well as  $\mathfrak{g}_2$  and  $\mathfrak{g}_{-2}$ , are in perfect duality through the Cartan-Killing form on  $\mathfrak{g}$ . But it is often the case that  $\mathfrak{g}_2$  is in fact self-dual. The quite unexpected fact, discovered in [1], is that we can construct matrix factorizations from the resulting maps.

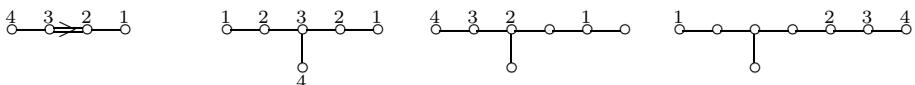
Note also that gradings of Lie algebras and their associated  $\theta$ -groups have been discussed in [15] in connection with certain types of polarized abelian varieties. This is another instance of the intriguing relationships that exist between the exceptional Lie algebras and certain families of algebraic varieties, often with trivial canonical class.

**3.2. A magic square of matrix factorizations.** A remarkable framework in which we will obtain matrix factorizations is that of the Tits-Freudenthal magic square (see [5, Section 4.3] and references therein). Either in its algebraic, or in its geometric versions, this magic square (and its enriched triangular version due to Deligne and Gross [12]) encodes all sorts of surprising phenomena related to the exceptional Lie algebras. We will describe another one in this section.

Remember that the magic square has its lines and columns indexed by the normed algebras  $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$  and is symmetric at the algebraic level, in the sense that there is a way to associate to a pair  $(\mathbb{A}, \mathbb{B})$  a semisimple Lie algebra  $\mathfrak{g}(\mathbb{A}, \mathbb{B})$ , with  $\mathfrak{g}(\mathbb{B}, \mathbb{A}) = \mathfrak{g}(\mathbb{A}, \mathbb{B})$ . Here is the magic square over the complex numbers:

$\mathfrak{so}_3$	$\mathfrak{sl}_3$	$\mathfrak{sp}_6$	$\mathfrak{f}_4$
$\mathfrak{sl}_3$	$\mathfrak{sl}_3 \times \mathfrak{sl}_3$	$\mathfrak{sl}_6$	$\mathfrak{e}_6$
$\mathfrak{sp}_6$	$\mathfrak{sl}_6$	$\mathfrak{so}_{12}$	$\mathfrak{e}_7$
$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

Freudenthal enhanced this construction by associating to each pair  $(\mathbb{A}, \mathbb{B})$  some special geometry, in a way which is not symmetric, but uniform along the lines (see [27, Section 2.1] and references therein). This means that for each  $\mathbb{A}$ , one has four special geometries associated to the pairs  $(\mathbb{A}, \mathbb{R}), (\mathbb{A}, \mathbb{C}), (\mathbb{A}, \mathbb{H}), (\mathbb{A}, \mathbb{O})$ , with completely similar formal properties. Each of these geometries is made of elements that we call F-points (for the first line of the magic square), plus F-lines (for the second line), plus F-planes (for the third line), plus F-symplecta (for the fourth line). Moreover, each type of elements involved in the special geometry of the pair  $(\mathbb{A}, \mathbb{B})$  is parametrized by a rational homogeneous space  $G/P$ , where  $G$  is an algebraic group whose Lie algebra is precisely  $\mathfrak{g}(\mathbb{A}, \mathbb{B})$ . The parabolic subgroup  $P$ , usually maximal, depends on the type of the element. All these data can be encoded in the following diagrams:



This must be interpreted as follows. Each of these diagrams corresponds to one column in the magic square. Recall that a Dynkin diagram encodes a semisimple Lie algebra (or Lie group, up to finite covers), and that a subset of vertices encodes a

conjugacy class of parabolic subgroups. The geometry associated to the box  $(i, j)$ , on the  $i$ -th line and the  $j$ -th column, is defined by considering the  $j$ -th Dynkin diagram above and suppressing the vertices numbered by integers bigger than  $i$ . This gives a Dynkin diagram  $D_{i,j}$  marked by integers from 1 to  $i$ . The elements of the corresponding geometry are then parametrized by the homogeneous spaces  $G/P(k)$ ,  $1 \leq k \leq i$ , where  $G$  is a semisimple Lie group with Dynkin diagram  $D_{i,j}$ , and  $P(k)$  is a parabolic subgroup defined, up to conjugacy, by the vertices of  $D_{i,j}$  marked by  $k$ . The spaces  $G/P(1)$  parametrize F-points, while the  $G/P(2)$  parametrize F-lines (for  $i \geq 2$ ), the  $G/P(3)$  parametrize F-planes (for  $i \geq 3$ ), and the  $G/P(4)$  parametrize F-symplecta (for  $i = 4$ ). For example, the Dynkin diagram  $D(3, 2)$  has type  $A_5$ , with the marks 1 at its extremities; so that the space of F-points for  $\mathbb{A} = \mathbb{H}$  and  $\mathbb{B} = \mathbb{C}$  is  $A_5/P_{1,5}$ , the flag variety of incident points and hyperplanes in  $\mathbb{P}^5$ .

Let us focus on the square of homogeneous spaces  $X(\mathbb{A}, \mathbb{B})$  parametrizing the F-points of the Freudenthal geometries, namely:

$A_1/P_1$	$A_2/P_{1,2}$	$C_3/P_2$	$F_4/P_1$
$A_2/P_1$	$A_1 \times A_1/P_{1,1}$	$A_5/P_2$	$E_6/P_1$
$C_3/P_1$	$A_5/P_{1,5}$	$D_6/P_2$	$E_7/P_1$
$F_4/P_1$	$E_6/P_{1,6}$	$E_7/P_6$	$E_8/P_1$

Now, we define a five-step grading of  $\mathfrak{g}(\mathbb{A}, \mathbb{B})$  as follows. As we have just explained, each  $X(\mathbb{A}, \mathbb{B})$  is  $G/P_I$ , where the standard parabolic subgroup  $P_I$  of  $G$  is defined by the subset  $I$  of the set  $\Delta$  of simple roots (which correspond bijectively with the vertices of the Dynkin diagram). Let  $\omega_I^\vee = \sum_{i \in I} \omega_i^\vee$  denote the sum of the corresponding fundamental coweights. If we express a root  $\alpha$  as a linear combination of simple roots,  $\alpha = \sum_{j \in \Delta} n_j \alpha_j$ , then  $\omega_I^\vee(\alpha) = \sum_{i \in I} n_i$ . The associated  $\mathbb{Z}$ -grading of  $\mathfrak{g} = \mathfrak{g}(\mathbb{A}, \mathbb{B})$  is

$$\mathfrak{g}_k = \delta_{k,0} \mathfrak{t} \oplus \left( \bigoplus_{\omega_I^\vee(\alpha)=k} \mathfrak{g}_\alpha \right).$$

It turns out that  $\mathfrak{g}_0$  is always made of orthogonal Lie algebras, that  $\mathfrak{g}_1$  is always a spin representation, while  $\mathfrak{g}_2$  is a natural representation, in particular self dual. Moreover  $\mathfrak{g}_k = 0$  for  $k > 2$ . It was argued in [27, Proposition 3.2] that  $\mathfrak{g}_1$  should be thought of as  $\mathbb{A} \otimes \mathbb{B}$  and  $\mathfrak{g}_2$  as  $\mathbb{A}_0 \oplus \mathbb{B}_0$ , where  $\mathbb{A}_0$  is the hyperplane of imaginary elements in  $\mathbb{A}$ . Here is the table giving the semisimple part of  $\mathfrak{g}_0$ :

0	0	$\mathfrak{sl}_2^2$	$\mathfrak{spin}_7$
$\mathfrak{sl}_2$	$\mathfrak{sl}_2^2$	$\mathfrak{sl}_2 \times \mathfrak{sl}_4$	$\mathfrak{spin}_{10}$
$\mathfrak{sp}_4$	$\mathfrak{sl}_4$	$\mathfrak{sl}_2^2 \times \mathfrak{sl}_3$	$\mathfrak{spin}_{12}$
$\mathfrak{spin}_7$	$\mathfrak{spin}_8$	$\mathfrak{sl}_2 \times \mathfrak{spin}_{10}$	$\mathfrak{spin}_{14}$

**Theorem 3.2.1.** *For each pair  $(\mathbb{A}, \mathbb{B})$  the representations  $\mathfrak{g}_1, \mathfrak{g}_2$  of the reductive Lie algebra  $\mathfrak{g}_0$  have the following properties:*

- (1)  $\mathfrak{g}_1$  is prehomogeneous under the action of  $\mathfrak{g}_0$ , and the generic stabilizer is  $\text{aut}(\mathbb{A}) \times \text{aut}(\mathbb{B})$ ,
- (2) there exist equivariant morphisms

$$P, Q : \text{Sym}^4 \mathfrak{g}_1 \longrightarrow \text{Sym}^2 \mathfrak{g}_2 \hookrightarrow \text{End}(\mathfrak{g}_2)$$

such that  $(P, Q)$  is a matrix factorization of a semi-invariant  $J_8$  on  $\mathfrak{g}_1$ .

This statement generalizes Theorem 2.3.2, which corresponds to the pair  $(\mathbb{O}, \mathbb{O})$ . All the other cases are somewhat degenerate, in the following ways.

- (1) For the pairs  $(\mathbb{O}, \mathbb{C})$  and  $(\mathbb{O}, \mathbb{H})$  of the last line, the fundamental semi-invariant has degree four, and we get a matrix factorization of its square.
- (2) For the pair  $(\mathbb{H}, \mathbb{O})$  of the third line,  $\mathfrak{g}_2$  is one dimensional, the fundamental invariant has degree four and admits a matrix factorization induced by the 5-grading, as in [1].
- (3) For the pair  $(\mathbb{C}, \mathbb{O})$  of the second line, there is no non trivial semi-invariant and  $\mathfrak{g}_2$  is actually zero.

The matrices  $P_z$  and  $Q_z$  will always be obtained as  $ad(z)^4$ , up to some homothety, but the proof of the Theorem is unfortunately a case by case check. In the next section we will discuss the first degenerate case, that of  $V_2 \otimes \Delta_{10}$  where  $V_2$  is two-dimensional. The other cases are similar and simpler.

*Remark.* The last line of our square is connected with the work of Allison and Faulkner on structurable algebras. Indded, Allison devised in [3] a construction of a 5-graded simple Lie algebra from a central simple structurable algebra. The structurable algebras  $\mathbb{O} \otimes \mathbb{A}$ , for  $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ , yield (after complexification) the simple Lie algebras  $\mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$ . [3, 8(c) p. 1869]. Moreover there is a natural notion of norm on a structurable algebra [4], and the norm is a semi-invariant for the so-called structure group. The norms of  $\mathbb{O} \otimes \mathbb{A}$  have degree 2, 4, 4, 8 respectively. In particular this provides a description of the basic invariants of these representations.

**3.3. The case of  $\mathbb{C}^2 \otimes \Delta_{10}$ .** Here the action of  $GL(V_2) \times Spin_{10}$  is prehomogeneous, and there is a quartic semi-invariant  $J_4$  [25]. Note that this is the representation used by Hitchin in order to defined (a substitute for)  $GL_2(\mathbb{O})$ , with  $J_4$  playing the rôle of the determinant [21].

An important observation is that the natural map, again induced by Clifford multiplication, yields an isomorphism

$$\wedge^2 \Delta_{10} \simeq \wedge^3 V_{10}.$$

We therefore get a morphism of  $SL(V_2) \times Spin_{10}$ -modules

$$(6) \quad \Omega : S^2(V_2 \otimes \Delta_{10}) \longrightarrow \wedge^2 V_2 \otimes \wedge^2 \Delta_{10} \simeq \wedge^3 V_{10}.$$

This in turn induces a morphism

$$(7) \quad M : S^4(V_2 \otimes \Delta_{10}) \longrightarrow S^2(\wedge^3 V_{10}) \longrightarrow S^2 V_{10} \hookrightarrow End(V_{10}),$$

where the first arrow is given by the square of  $\Omega$ , and the second arrow is induced by the quadratic form on  $V_{10}$ . Let us compute this morphism explicitly. According to [25], a generic element of  $V_2 \otimes \Delta_{10}$  is given by

$$z = v_1 \otimes (1 + e_{1234}) + v_2 \otimes (e_{1235} + e_{45}).$$

Letting  $e_0 = e_4 + f_4$  and  $f_0 = f_4 - e_4$ , the associated three-form is

$$\Omega_z = e_{123} + f_{123} + e_0(e_1 f_1 + e_2 f_2 + e_3 f_3) + f_0 e_5 f_5.$$

Observe that this decomposes as the sum of  $\Omega \in \wedge^3 V_7$  and  $\Omega' = f_0 e_5 f_5 \in \wedge^3 V_3$ , where  $V_7 = \langle e_0, e_1, e_2, e_3, f_1, f_2, f_3 \rangle$  and  $V_3 = \langle f_0, e_5, f_5 \rangle$  are orthogonal spaces, on which the quadratic form  $q$  is non degenerate. The associated element in  $S^2 V_{10}$  is  $6e_1 f_1 + 6e_2 f_2 + 6e_3 f_3 + 3e_0^2 + f_0^2 - 2e_5 f_5$ , and since  $q(e_0) = 2$  and  $q(f_0) = -2$ , the corresponding endomorphism is

$$M_z = 6\pi_{V_7} - 2\pi_{V_3} \in End(V_{10}),$$

where  $\pi_{V_7}$  and  $\pi_{V_3}$  are the two projections relative to the direct sum decomposition  $V_{10} = V_7 \oplus V_3$ . This endomorphism has non zero-trace, which allows to define a degree four invariant  $J_4(z) := \text{trace}(M_z)$ . We then get

$$(M_z - \frac{1}{18}J_4(z)Id_{V_{10}})^2 = \frac{1}{162}J_4(z)^2 Id_{V_{10}},$$

a matrix factorization of the octic  $J_8 = \frac{1}{162}J_4^2$ .

**3.4. Three-forms in seven variables.** A sporadic case to which the same circle of ideas can be applied is that of the degree seven invariant of  $\wedge^3 V_7$ . This is a very old story since it goes back at least to Engel's 1900 article (see [2] for historical details and references). The heptic invariant gives the equation of the projective dual of the Grassmannian  $G(3, V_7)$ , or of the complement of the open  $GL_7$ -orbit of  $\wedge^3 V_7$  consisting of forms with stabilizer bigger than  $G_2$ .

This case is associated with the grading of  $\mathfrak{e}_7$  defined by the simple root  $\alpha_2$ :

$$\mathfrak{e}_7 = V_7 \oplus \wedge^3 V_7^\vee \oplus \mathfrak{gl}(V_7) \oplus \wedge^3 V_7 \oplus V_7^\vee.$$

(For simplicity we wrote this grading as a decomposition into  $\mathfrak{sl}(V_7)$ -modules.) For any  $\omega \in \wedge^3 V_7$  and  $y \in V_7^\vee$ , we denote by  $i_y(\omega) \in \wedge^2 V_7$  the contraction of  $\omega$  with  $y$ . Then  $P(\omega, y) = \frac{1}{6}\omega \wedge i_y(\omega) \wedge i_y(\omega)$  belongs to  $\wedge^7 V_7 \simeq \mathbb{C}$ . This defines an equivariant morphism

$$(8) \quad P : S^3(\wedge^3 V_7) \longrightarrow S^2 V_7 \otimes \det V_7.$$

On the other hand, recall that  $\wedge^3 V_7 \simeq (\wedge^4 V_7)^\vee$  and that  $(\wedge^4 V_7)^\vee$  is a submodule of  $V_7^\vee \otimes (\wedge^3 V_7)^\vee$ . Moreover the natural map  $End(V_7) \longrightarrow End(\wedge^3 V_7)$  has for transpose an equivariant map  $End(\wedge^3 V_7) = (\wedge^3 V_7)^\vee \otimes (\wedge^3 V_7) \longrightarrow End(V_7)$ . This yields a morphism

$$(9) \quad \theta : S^2(\wedge^3 V_7) \longrightarrow V_7^\vee \otimes End(V_7) \otimes \det V_7.$$

Taking its symmetric square and composing with the trace map, we get

$$(10) \quad R : S^4(\wedge^3 V_7) \longrightarrow S^2 V_7^\vee \otimes S^2 End(V_7) \otimes (\det V_7)^2 \longrightarrow S^2 V_7^\vee \otimes (\det V_7)^2.$$

*Remark.* The morphism  $R$  is induced from the quartic  $SL_6$ -invariant of  $\wedge^3 U_6$ , in the following way. This quartic corresponds to a  $GL_6$ -equivariant linear map:

$$\psi_{U_6} : S^4(\wedge^3 U_6) \longrightarrow \det(U_6)^{\otimes 2}.$$

Now the Borel-Weil Theorem implies that  $\wedge^3 V_7 = H^0(\mathbb{P}(V_7), \wedge^3 Q)$ , where  $Q$  is the tautological quotient on  $\mathbb{P}(V_7)$ , the projective space of lines in  $V_7$ . We can then consider the composition

$$S^4 H^0(\mathbb{P}(V_7), \wedge^3 Q) \longrightarrow H^0(\mathbb{P}(V_7), S^4(\wedge^3 Q)) \longrightarrow H^0(\mathbb{P}(V_7), \det(Q)^{\otimes 2}),$$

where the first arrow is surjective because  $Q$  is globally generated, and the second arrow is defined by  $\psi_Q$ . Finally, the Borel-Weil Theorem implies that

$$H^0(\mathbb{P}(V_7), \det(Q)^{\otimes 2}) = S^2 V_7^\vee \otimes \det(V_7)^{\otimes 2},$$

and the resulting morphism is nothing else but  $R$ .

As was first done by Gyoja and Kimura [22], we can now define the heptic semi-invariant  $J_7$  on  $\wedge^3 V_7$  by contracting  $R$  with  $P$ :

$$J_7(z) = \langle P_z, R_z \rangle \in (\det V_7)^{\otimes 3}.$$

But in fact a much stronger statement is true. Observe that  $S^2V_7$  is a submodule of  $Hom(V_7^\vee, V_7)$ , as well as  $S^2V_7^\vee$  is a submodule of  $Hom(V_7, V_7^\vee)$ . In other words, we can consider  $R_z$  and  $P_z$  as symmetric morphisms from  $V_7$  to  $V_7^\vee$  and from  $V_7^\vee$  to  $V_7$ , respectively. The following result appears in [23, Example 2.6]. We give a short proof, without computation.

**Theorem 3.4.1.** *The pair of symmetric morphisms  $(P, R)$  is a matrix factorization of the heptic semi-invariant  $J_7$  of  $\wedge^3V_7$ .*

*Proof.* Because of the quasi-homogeneity, it is enough to check this assertion at a generic point  $z$  of  $\wedge^3V_7$ . The stabilizer of this form is then a copy of  $G_2$ . The quadratic forms  $R_z$  and  $P_z$  on  $V_7^\vee$  and  $V_7$  must be preserved by this stabilizer. But up to scalar, there is a unique quadratic form on  $V_7$  (or its dual) preserved by  $G_2$ . After identifying  $V_7$  with its dual through this non degenerate quadratic form (which depends on  $z$ ), we get  $R_z$  and  $P_z$  as (non zero) homotheties, and our statement immediately follows.  $\square$

*Remark.* In many respects, the case of  $(Spin_{14}, \Delta_{14})$  is a doubled version of the case of  $(SL_7, \wedge^3C^7)$ . Similarly,  $(Spin_{12}, \Delta_{12})$  is a doubled version of  $(SL_6, \wedge^3C^6)$  (they both appear on the same line of Freudenthal’s magic square, see [27]), and  $(Spin_{10}, \Delta_{10})$  is a doubled version of  $(SL_5, \wedge^2C^5)$  (see [30, Introduction] for more details). The second pair has a clear complex-quaternionic interpretation. It would be very nice to find a similar, or may be quaternion-octonionic interpretation of the two other pairs.

**3.5. Three-forms in eight variables.** For completeness let us briefly discuss the case of the degree sixteen  $SL(V_8)$ -invariant of  $\wedge^3V_8$ , the last prehomogeneous space of skew-symmetric three-forms. This invariant is given by the equation of the projective dual of the Grassmannian  $G(3, V_8)$ . This case is associated with the grading of length seven of  $\mathfrak{e}_8$  defined by the simple root  $\alpha_2$ :

$$\mathfrak{e}_8 = V_8^\vee \oplus \wedge^2V_8 \oplus \wedge^3V_8^\vee \oplus \mathfrak{gl}(V_8) \oplus \wedge^3V_8 \oplus \wedge^2V_8^\vee \oplus V_8.$$

(For simplicity we wrote this grading as a decomposition into  $\mathfrak{sl}(V_8)$ -modules.) The adjoint action induces maps  $Sym^{2k}\mathfrak{g}_1 \rightarrow Hom(\mathfrak{g}_{-k}, \mathfrak{g}_k)$ , which for  $k = 2$  and  $k = 3$  yield the following:

$$P : Sym^4(\wedge^3V_8) \longrightarrow Hom(\wedge^2V_8, \wedge^2V_8^\vee),$$

$$Q : Sym^6(\wedge^3V_8) \longrightarrow Hom(V_8^\vee, V_8).$$

Putting together  $P$  and  $Q$ , and using the contraction map  $\wedge^2V_8^\vee \otimes V_8 \rightarrow V_8^\vee$  and its dual, we get a morphism

$$R : Sym^{10}(\wedge^3V_8) \longrightarrow Hom(V_8, V_8^\vee).$$

The computations made by Kimura in [23, Example 2.7] imply the following result:

**Theorem 3.5.1.** *The pair of symmetric morphisms  $(Q, R)$  is a matrix factorization of the degree sixteen semi-invariant  $J_{16}$  of  $\wedge^3V_8$ .*

ACKNOWLEDGMENTS

We thank S. Alsaody, A. Elduque and S. Garibaldi for useful comments and references, and V. Benedetti for his kind help with LiE.

## REFERENCES

- [1] Roland Abuaf, *On quartic double fivefolds and the matrix factorizations of exceptional quaternionic representations*, Ann. Inst. Fourier (Grenoble) **70** (2020), no. 4, 1403–1430. MR4245576
- [2] Ilka Agricola, *Old and new on the exceptional group  $G_2$* , Notices Amer. Math. Soc. **55** (2008), no. 8, 922–929. MR2441524
- [3] B. N. Allison, *Models of isotropic simple Lie algebras*, Comm. Algebra **7** (1979), no. 17, 1835–1875, DOI 10.1080/00927877908822432. MR547712
- [4] B. N. Allison and J. R. Faulkner, *Norms on structurable algebras*, Comm. Algebra **20** (1992), no. 1, 155–188, DOI 10.1080/00927879208824337. MR1145331
- [5] John C. Baez, *The octonions*, Bull. Amer. Math. Soc. (N.S.) **39** (2002), no. 2, 145–205, DOI 10.1090/S0273-0979-01-00934-X. MR1886087
- [6] Arnaud Beauville, *Determinantal hypersurfaces*, Michigan Math. J. **48** (2000), 39–64, DOI 10.1307/mmj/1030132707. MR1786479
- [7] José Bertin, *Clifford algebras and matrix factorizations*, Adv. Appl. Clifford Algebr. **18** (2008), no. 3–4, 417–430, DOI 10.1007/s00006-008-0079-6. MR2490565
- [8] Ragnar-Olaf Buchweitz, David Eisenbud, and Jürgen Herzog, *Cohen-Macaulay modules on quadrics*, Singularities, representation of algebras, and vector bundles (Lambrech, 1985), Lecture Notes in Math., vol. 1273, Springer, Berlin, 1987, pp. 58–116, DOI 10.1007/BFb0078838. MR915169
- [9] P. Candelas, E. Derrick, and L. Parkes, *Generalized Calabi-Yau manifolds and the mirror of a rigid manifold*, Nuclear Phys. B **407** (1993), no. 1, 115–154, DOI 10.1016/0550-3213(93)90276-U. MR1242064
- [10] Claude Chevalley, *The algebraic theory of spinors and Clifford algebras*, Springer-Verlag, Berlin, 1997. Collected works. Vol. 2; Edited and with a foreword by Pierre Cartier and Catherine Chevalley; With a postface by J.-P. Bourguignon. MR1636473
- [11] Jean-Louis Clerc, *Special prehomogeneous vector spaces associated to  $F_4$ ,  $E_6$ ,  $E_7$ ,  $E_8$  and simple Jordan algebras of rank 3*, J. Algebra **264** (2003), no. 1, 98–128, DOI 10.1016/S0021-8693(03)00115-7. MR1980688
- [12] Pierre Deligne and Benedict H. Gross, *On the exceptional series, and its descendants* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **335** (2002), no. 11, 877–881, DOI 10.1016/S1631-073X(02)02590-6. MR1952563
- [13] David Eisenbud, *Homological algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc. **260** (1980), no. 1, 35–64, DOI 10.2307/1999875. MR570778
- [14] Skip Garibaldi, *Cohomological invariants: exceptional groups and spin groups*, Mem. Amer. Math. Soc. **200** (2009), no. 937, xii+81, DOI 10.1090/memo/0937. With an appendix by Detlev W. Hoffmann. MR2528487
- [15] Laurent Gruson, Steven V. Sam, and Jerzy Weyman, *Moduli of abelian varieties, Vinberg  $\theta$ -groups, and free resolutions*, Commutative algebra, Springer, New York, 2013, pp. 419–469, DOI 10.1007/978-1-4614-5292-8\_13. MR3051381
- [16] Akihiko Gyoja, *Construction of invariants*, Tsukuba J. Math. **14** (1990), no. 2, 437–457, DOI 10.21099/tkbjm/1496161465. MR1085211
- [17] V. G. Kac, *Some remarks on nilpotent orbits*, J. Algebra **64** (1980), no. 1, 190–213, DOI 10.1016/0021-8693(80)90141-6. MR575790
- [18] Jun-ichi Igusa, *A classification of spinors up to dimension twelve*, Amer. J. Math. **92** (1970), 997–1028, DOI 10.2307/2373406. MR277558
- [19] Atanas Iliev and Laurent Manivel, *Fano manifolds of Calabi-Yau Hodge type*, J. Pure Appl. Algebra **219** (2015), no. 6, 2225–2244, DOI 10.1016/j.jpaa.2014.07.033. MR3299729
- [20] V. Gatti and E. Viniberghi, *Spinors of 13-dimensional space*, Adv. in Math. **30** (1978), no. 2, 137–155, DOI 10.1016/0001-8708(78)90034-8. MR513846
- [21] Nigel Hitchin,  *$SL(2)$  over the octonions*, Math. Proc. R. Ir. Acad. **118A** (2018), no. 1, 21–38, DOI 10.3318/pria.2018.118.04. MR3811489
- [22] Tatsuo Kimura, *Remark on some combinatorial construction of relative invariants*, Tsukuba J. Math. **5** (1981), no. 1, 101–115, DOI 10.21099/tkbjm/1496159322. MR630524
- [23] Tatsuo Kimura, *Introduction to prehomogeneous vector spaces*, Translations of Mathematical Monographs, vol. 215, American Mathematical Society, Providence, RI, 2003. Translated

- from the 1998 Japanese original by Makoto Nagura and Tsuyoshi Niitani and revised by the author. MR1944442
- [24] M. Sato and T. Kimura, *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya Math. J. **65** (1977), 1–155. MR430336
- [25] W. Kraskiewicz, J. Weyman, *Geometry of orbit closures for the representations associated to gradings of Lie algebras of type  $E_7$* , arXiv:1301.0720.
- [26] W. Kraskiewicz, J. Weyman, *Geometry of orbit closures for the representations associated to gradings of Lie algebras of type  $E_8$* , preprint.
- [27] J. M. Landsberg and L. Manivel, *The projective geometry of Freudenthal’s magic square*, J. Algebra **239** (2001), no. 2, 477–512, DOI 10.1006/jabr.2000.8697. MR1832903
- [28] LiE, A computer algebra package for Lie group computations, available online at <http://www.mathlabo.univ-poitiers.fr/~maavl/LiE/>
- [29] Laurent Manivel, *The Cayley Grassmannian*, J. Algebra **503** (2018), 277–298, DOI 10.1016/j.jalgebra.2018.02.010. MR3779996
- [30] Laurent Manivel, *Double spinor Calabi-Yau varieties* (English, with English and French summaries), Épijournal Géom. Algébrique **3** (2019), Art. 2, 14, DOI 10.46298/epiga.2019.volume3.3965. MR3936623
- [31] Laurent Manivel, *Ulrich and aCM bundles from invariant theory*, Comm. Algebra **47** (2019), no. 2, 706–718, DOI 10.1080/00927872.2018.1495222. MR3935376
- [32] V. L. Popov, *Classification of the spinors of dimension fourteen* (Russian), Uspehi Mat. Nauk **32** (1977), no. 1(193), 199–200. MR0437459
- [33] Boris Rosenfeld, *Geometry of Lie groups*, Mathematics and its Applications, vol. 393, Kluwer Academic Publishers Group, Dordrecht, 1997. MR1443207
- [34] Rolf Schimmrigk, *Mirror symmetry and string vacua from a special class of Fano varieties*, Internat. J. Modern Phys. A **11** (1996), no. 17, 3049–3096, DOI 10.1142/S0217751X96001486. MR1394772
- [35] Paul Seidel and Richard Thomas, *Braid group actions on derived categories of coherent sheaves*, Duke Math. J. **108** (2001), no. 1, 37–108, DOI 10.1215/S0012-7094-01-10812-0. MR1831820
- [36] È. B. Vinberg, *Classification of homogeneous nilpotent elements of a semisimple graded Lie algebra* (Russian), Trudy Sem. Vektor. Tenzor. Anal. **19** (1979), 155–177. MR549013

PARIS, FRANCE

*Email address:* rabuaf@gmail.com

INSTITUT DE MATHÉMATIQUES DE TOULOUSE, UMR 5219, UNIVERSITÉ DE TOULOUSE, CNRS, UPS, F-31062 TOULOUSE CEDEX 9, FRANCE

*Email address:* manivel@math.cnrs.fr