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DESCRIPTION OF UNITARY REPRESENTATIONS OF THE GROUP OF INFINITE p-ADIC INTEGER MATRICES

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ABSTRACT. We classify irreducible unitary representations of the group of all infinite matrices over a p-adic field ($p \neq 2$) with integer elements equipped with a natural topology. Any irreducible representation passes through a group GL of infinite matrices over a residue ring modulo p^k. Irreducible representations of the latter group are induced from finite-dimensional representations of certain open subgroups.

1. INTRODUCTION

1.1. Notations and definitions.

(a) RINGS. Let p be a prime,

$$p > 2.$$

Let $\mathbb{Z}_{p^n} := \mathbb{Z}/p^n\mathbb{Z}$ be a residue ring, $\mathbb{F}_p := \mathbb{Z}_p$ be the field with p elements. The ring of p-adic integers $\mathbb{O}_p$ is the projective limit

$$\mathbb{O}_p = \lim_{\longleftarrow n} \mathbb{Z}_{p^n}$$

of the following chain (see, e.g., [32]):

$$\dots \longleftarrow \mathbb{Z}_{p^{n-1}} \longleftarrow \mathbb{Z}_{p^n} \longleftarrow \mathbb{Z}_{p^{n+1}} \longleftarrow \dots,$$

we have $\mathbb{Z}_{p^n} = \mathbb{O}_p/p^n\mathbb{O}_p$. Denote by $\mathbb{Q}_p$ the field of p-adic numbers.

(b) THE INFINITE SYMMETRIC GROUP AND OLIGOMORPHIC GROUPS. Let Ω be a countable set. Denote by $S(\Omega)$ the group of all permutations of Ω; denote $S_\infty := S(\mathbb{N})$. The topology on the *infinite symmetric group* $S(\Omega)$ is determined by the condition: stabilizers of finite subsets are open subgroups and these subgroups form a fundamental system of neighborhoods of the unit.[1] Equivalently, a sequence $g^{(\alpha)}$ converges to g if for each $\omega \in \Omega$ we have $\omega g^{(\alpha)} = \omega g$ for sufficiently large α.

A closed subgroup G of $S(\Omega)$ is called *oligomorphic* if for each k it has only a finite number of orbits on the product $\Omega \times \dots \times \Omega$ of k copies of Ω; see [5].

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[1]Thus we get a structure of a Polish group. Moreover this topology is a unique separable topology on the infinite symmetric group; see [13]. In particular, this means that a unitary representation of S_∞ in a separable Hilbert space is automatically continuous.

(c) MODULES $\mathfrak{l}(\mathbb{Z}_{p^n})$ AND GROUPS $\mathrm{GL}(\infty,\mathbb{Z}_{p^n})$. Define the module $\mathfrak{l}(\mathbb{Z}_{p^n})$ as the set of all sequences $v=(v_1,v_2,\dots)$, where $v_j\in\mathbb{Z}_{p^n}$ and $v_j=0$ for sufficiently large j. The set $\mathfrak{l}(\mathbb{Z}_{p^n})$ is countable; we equip it with a discrete topology. Denote by e_j the standard basis elements, i.e., e_j has a unit on j-th place, other elements are 0.

Define groups $\mathrm{GL}(\infty,\mathbb{Z}_{p^n})$ as groups of infinite invertible matrices g over $\mathbb{Z}_{p^n}$ such that:

- each row of g contains only a finite number of nonzero elements;
- each column contains only a finite number of nonzero elements;
- the inverse matrix g^{-1} satisfies the same conditions.

Notice that rows of a matrix g are precisely vectors $e_i g$, and columns are $e_j g^t$ (we denote by g^t a *transposed matrix*).

Actually, the topic of this paper is representations of $\mathrm{GL}(\infty,\mathbb{Z}_{p^n})$.

This group is continual and we must define a *topology* on $\mathrm{GL}(\infty,\mathbb{Z}_{p^n})$. A sequence $g^{(\alpha)}\in\mathrm{GL}(\infty,\mathbb{Z}_{p^n})$ converges to g if all sequences $e_i g^{(\alpha)}$ and $e_i(g^{(\alpha)})^t$ are eventually constant and their limits are $e_i g$ and $e_j g^t$ respectively. Thus we get a structure of a totally disconnected topological group.

The group $\mathrm{GL}(\infty,\mathbb{Z}_{p^n})$ acts on the countable set $\mathfrak{l}(\mathbb{Z}_{p^n})\oplus\mathfrak{l}(\mathbb{Z}_{p^n})$ by transformations

$$(v,w)\mapsto(vg,w(g^t)^{-1}).$$

In particular, this defines an embedding of $\mathrm{GL}(\infty,\mathbb{Z}_{p^n})$ to a symmetric group $S\big(\mathfrak{l}(\mathbb{Z}_{p^n})\oplus\mathfrak{l}(\mathbb{Z}_{p^n})\big)$. The image of the group $\mathrm{GL}(\infty,\mathbb{Z}_{p^n})$ is a closed subgroup of $S\big(\mathfrak{l}(\mathbb{Z}_{p^n})\oplus\mathfrak{l}(\mathbb{Z}_{p^n})\big)$ and the induced topology coincides with the natural topology on $\mathrm{GL}(\infty,\mathbb{Z}_{p^n})$. By [27, Lemma 3.7], *the group* $\mathrm{GL}(\infty,\mathbb{Z}_{p^n})$ *is oligomorphic.*

(d) MODULES $\mathfrak{l}(\mathbb{O}_p)$ AND GROUPS $\mathrm{GL}(\infty,\mathbb{O}_p)$. Denote by $\mathfrak{l}(\mathbb{O}_p)$ the set of all sequences $r=(r_1,r_2,\dots)$, where $r_j\in\mathbb{O}_p$ and $|r_j|\to 0$ as $j\to\infty$. The space $\mathfrak{l}(\mathbb{O}_p)$ is a projective limit,

$$\mathfrak{l}(\mathbb{O}_p)=\lim_{\longleftarrow n}\mathfrak{l}(\mathbb{Z}_{p^n}),$$

we equip it with the *topology of the projective limit.* In other words, a sequence $r^{(j)}\in\mathfrak{l}(\mathbb{O}_p)$ converges if for any p^n the reduction of $r^{(j)}$ modulo p^n is eventually constant in $\mathbb{Z}_{p^n}$.

We define $\mathrm{GL}(\infty,\mathbb{O}_p)$ as the group of all infinite matrices g over $\mathbb{O}_p$ such that:

- each row of g is an element of $\mathfrak{l}(\mathbb{O}_p)$;
- each column of g is an element of $\mathfrak{l}(\mathbb{O}_p)$;
- the matrix g has an inverse and g^{-1} satisfies the same conditions.

We say that a sequence $g^{(\alpha)}\in\mathrm{GL}(\infty,\mathbb{O}_p)$ converges to g if for any i the sequence $e_i g^{(\alpha)}$ converges to $e_i g$ and for any j the sequence $e_i(g^{(\alpha)})^t$ converges to $e_j g^t$. This determines a structure of a totally disconnected topological group on $\mathrm{GL}(\infty,\mathbb{O}_p)$.

We have obvious homomorphisms $\mathrm{GL}(\infty,\mathbb{Z}_{p^n})\to\mathrm{GL}(\infty,\mathbb{Z}_{p^{n-1}})$, the group $\mathrm{GL}(\infty,\mathbb{O}_p)$ is the projective limit

$$\mathrm{GL}(\infty,\mathbb{O}_p)=\lim_{\longleftarrow n}\mathrm{GL}(\infty,\mathbb{Z}_{p^n})$$

and its topology is the topology of projective limit.

1.2. **Preliminary remarks.** A priori we know the following statement:

Theorem 1.1.

(a) *The group* $\mathrm{GL}(\infty, \mathbb{O}_p)$ *is a type I group; it has a countable number of irreducible unitary representations. Any unitary representation* $\mathrm{GL}(\infty, \mathbb{O}_p)$ *is a sum of irreducible representations. Any irreducible unitary representation of* $\mathrm{GL}(\infty, \mathbb{O}_p)$ *is in fact a representation of some group* $\mathrm{GL}(\infty, \mathbb{Z}_{p^n})$.

(b) *Each irreducible representation of* $\mathrm{GL}(\infty, \mathbb{Z}_{p^n})$ *is induced from a finite-dimensional representation of an open subgroup. More precisely, for any irreducible unitary representation of* $\mathrm{GL}(\infty, \mathbb{Z}_{p^n})$ *there exists an open subgroup* $\widehat{Q} \subset \mathrm{GL}(\infty, \mathbb{Z}_{p^n})$, *a normal subgroup* $Q \subset \widehat{Q}$ *of finite index and an irreducible representation* ν *of* $\widehat{Q}$, *which is trivial on* Q, *such that* ρ *is induced from* ν.

This is a special case of a theorem of Tsankov about unitary representations of oligomorphic groups and projective limits of holomorphic groups; see [34, Theorem 1.3].[2] It seems that [34], [2] are not sufficient to give a precise answer in our case.

Let us give a definition of an *induced representation* (see, e.g., [33, Sect. 7] and [15, Sect. 13]) which is appropriate in our case. Let G be a totally disconnected separable group, Q its open subgroup. Let ν be a unitary representation of Q in a Hilbert space V. Consider the space H of V-valued functions f on a countable homogeneous space $Q \setminus G$ such that

$$\sum_{x \in Q\setminus G} \|f(x)\|^2 < \infty.$$

Equip H with the inner product

$$\langle f_1, f_2 \rangle_H := \sum_{x \in Q\setminus G} \langle f_1(x), f_2(x) \rangle_V.$$

Let U be a function on $G \times (Q \setminus G)$ taking values in the group of unitary operators in V such that:

- Formula
$$\rho(g) f(x) = U(g, x) f(xg)$$
determines a representation of G in H.
- Let x_0 be the initial point of $Q \setminus G$, i.e., $x_0 Q = x_0$. Then for $q \in Q$ we have $U(q, x_0) = \nu(q)$.

The first condition implies that the function $U(g, x)$ satisfies the functional equation

$$U(x, g_1 g_2) = U(x, g_1)\, U(x g_1, g_2).$$

It can be shown that $U(g, x)$ is uniquely defined up to a natural calibration

$$U(g, x) \sim A(gx)^{-1} U(g, x) A(x),$$

where A is a function on $Q \setminus G$ taking values in the unitary group of V (see, e.g., [15, Sect 13.1]). For this reason, an induced representation $\rho(g) = \mathrm{Ind}_Q^G(\nu)$ is canonically defined up to a unitary equivalence.

[2] A reduction of representations of $\mathrm{GL}(\infty, \mathbb{O}_p)$ to representations of quotients $\mathrm{GL}(\infty, \mathbb{Z}_{p^\mu})$ easily follows from [20, Proposition VII.1.3]; see [27, Corollary 3.5]. In our proof of Theorem 1.5 Tsankov's theorem is used in the proof of Proposition 2.1, which was done in [27].

We also can choose $U(g,x)$ in the following way. For any $x \in Q \setminus G$ we choose an element $s(x) \in G$ such that $x_0 s(x) = x$. Then $U(g,x) = \nu(q)$, where q is determined from the condition $s(x)g = q\, s(xg)$.

1.3. **The statement.** The result of the paper is Theorem 1.5, which claims that irreducible representations of $\mathbb{G}$ are induced from finite dimensional representations of certain family of subgroups $\mathbb{G}^\circ[L;M]$; these subgroups are described in Lemma 1.3.

Thus we fix a ring $\mathbb{Z}_{p^\mu}$ and examine the group

$$\mathbb{G} := \mathrm{GL}(\infty, \mathbb{Z}_{p^\mu}).$$

We consider two right actions of $\mathbb{G}$ on $\mathfrak{l}(\mathbb{Z}_{p^\mu}), g : v \mapsto vg,\ g : v \mapsto v(g^t)^{-1}$. Define a pairing

$$\mathfrak{l}(\mathbb{Z}_{p^\mu}) \times \mathfrak{l}(\mathbb{Z}_{p^\mu}) \to \mathbb{Z}_{p^\mu}$$

by

$$\{v, w\} := \sum v_j w_j = vw^t, \tag{1.1}$$

our action preserves this pairing, i.e.,

$$\{vg, v(g^t)^{-1}\} = \{v, w\}.$$

Let $L \subset \mathfrak{l}(\mathbb{Z}_{p^\mu})$, $M \subset \mathfrak{l}(\mathbb{Z}_{p^\mu})$ be finitely generated $\mathbb{Z}_{p^\mu}$-submodules. Denote by $\widehat{\mathbb{G}}[L;M]$ the subgroup of $\mathbb{G}$ consisting of g such that $Lg = L$ and $M(g^t)^{-1} = M$. By $\mathbb{G}^\circ[L;M] \subset \widehat{\mathbb{G}}[L;M]$ we denote group of matrices fixing L and M pointwise. Obviously, the quotient group $\widehat{\mathbb{G}}[L;M]/\mathbb{G}^\circ[L;M]$ is finite; it acts on the direct sum $L \oplus M$ preserving the pairing $\{f, g\}$. Any irreducible representation τ of $\widehat{\mathbb{G}}[L;M]/\mathbb{G}^\circ[L;M]$ can be regarded as a representation $\widehat{\tau}$ the group $\widehat{\mathbb{G}}[L;M]$, which is trivial on $\mathbb{G}^\circ[L;M]$. For given L, M, τ we consider the representation

$$\mathrm{Ind}^{\mathbb{G}}_{\widehat{\mathbb{G}}[L;M]}(\widehat{\tau})$$

of $\mathbb{G}$ induced from the representation $\widehat{\tau}$ of the group $\widehat{\mathbb{G}}[L;M]$. Ol'shanskĭ [30] obtained the following statement[3] for the group $\mathrm{GL}(\infty, \mathbb{F}_p) = \mathrm{GL}(\infty, \mathbb{Z}_p)$.

Theorem 1.2.

(a) *Any irreducible unitary representation of the group* $\mathrm{GL}(\infty, \mathbb{F}_p)$ *has this form.*

(b) *Two irreducible representations can be equivalent only for a trivial reason, i.e.,*

$$\mathrm{Ind}^{\mathbb{G}}_{\widehat{\mathbb{G}}[L_1;M_1]}(\tau_1) \sim \mathrm{Ind}^{\mathbb{G}}_{\widehat{\mathbb{G}}[L_2;M_2]}(\tau_2)$$

if and only if there exists $h \in \mathbb{G}$ *such that* $L_1 h = L_2 h$, $M_1(h^t)^{-1} = M_2$ *and* $\tau_2(q) = \tau_1(hqh^{-1})$.

For groups $\mathrm{GL}(\infty, \mathbb{Z}_{p^\mu})$ with $\mu > 1$ the situation is more delicate. Let L, M actually be contained in $(\mathbb{Z}_{p^\mu})^m \subset \mathfrak{l}(\mathbb{Z}_{p^\mu})$. Fix a matrix b such that[4] $\ker b = L$ and a matrix c such that $\ker c^t = M$.

[3] A proof in [30] is only sketched; other proofs were given by Dudko [8] and Tsankov [34].

[4] We assume that each row of b and each column of c contain only a finite number of nonzero elements.

Lemma 1.3. *The group $\mathbb{G}^\circ[L;M]$ consists of all invertible matrices admitting the following representation as a block matrix of size $m+\infty$:*

$$g = \begin{pmatrix} a & bv \\ wc & z \end{pmatrix}, \tag{1.2}$$

where the block 'a' can be written in both forms

$$a = 1 - bS, \qquad a = 1 - Tc.$$

Next, define a subgroup $\mathbb{G}^\bullet[L;M] \subset \mathbb{G}^\circ[L;M]$ consisting of matrices having the form

$$g = \begin{pmatrix} 1-buc & bv \\ wc & z \end{pmatrix}. \tag{1.3}$$

Proposition 1.4. *The group $\mathbb{G}^\bullet[L;M]$ is the minimal subgroup of finite index in $\widehat{\mathbb{G}}[L;M]$, i.e., it is contained in any subgroup of finite index in $\widehat{\mathbb{G}}[L;M]$.*

Theorem 1.5.

(a) *Any irreducible unitary representation of $\mathbb{G}$ is induced from a representation τ of some group $\widehat{\mathbb{G}}[L;M]$ that is trivial on the subgroup $\mathbb{G}^\bullet[L;M]$.*

(b) *Two irreducible representations of this kind can be equivalent only for the trivial reason as in Theorem* 1.2.

Remark. Recall that $p \neq 2$. In several places of our proof we divide elements of residue rings $\mathbb{Z}_{p^\mu}$ by 2. Usually, this division can be replaced by longer considerations. But in Lemma 6.8 this seems crucial. ⊠

Remark. Let $L,\ M \subset p \cdot \mathfrak{l}(\mathbb{Z}_{p^\mu})$. Then $\mathbb{G}[L;M]$ contains a congruence subgroup N consisting of elements of $\mathbb{G}$ that are equal 1 modulo $p^{\mu-1}$. Since N is a normal subgroup in $\mathbb{G}$, it is normal in $\widehat{\mathbb{G}}[L;M]$. Let τ be trivial on N. Then the induced representation $\mathrm{Ind}_{\mathbb{G}[L;M]}^{\mathbb{G}}(\widehat{\tau})$ is trivial on the congruence subgroup N and actually we get representations of $\mathrm{GL}(\infty, \mathbb{Z}_{p^{\mu-1}})$. ⊠

Remark. The statement (b) is a general fact for oligomorphic groups; see [34, Proposition 4.1(ii)]. So we omit a proof (in our case this can be easily established by examination of intertwining operators). ⊠

1.4. **Remarks: Infinite-dimensional p-adic groups.** Now there exists a well-developed representation theory of infinite symmetric groups and of infinite-dimensional real classical groups. Parallel development in the p-adic case meets some difficulties. However, infinite dimensional p-adic groups were a topic of sporadic attacks since late 1980s; see [19], [36], [18]. We indicate some works on p-adic groups and their parallels with nontrivial constructions for real and symmetric groups.

(a) An extension of the Weil representation of the infinite-dimensional symplectic group $\mathrm{Sp}(2\infty, \mathbb{C})$ to the semigroup of lattices (Nazarov [19], [18]; see a partial exposition in [22, Sect. 11.1-11.2]).

(b) A construction of projective limits of p-adic Grassmannians and quasiinvariant actions of p-adic $\mathrm{GL}(\infty)$ on these Grassmannians [24]. This is an analog of virtual permutations (or Chinese restaurant process, see, e.g., [1, 11.19]; they are a base of harmonic analysis related to infinite symmetric group, see [14]), and of projective limits of compact symmetric spaces (see [31], [21]); they are a standpoint for a harmonic analysis related to infinite-dimensional classical groups; see [3].

(c) An attempt to describe a multiplication of double cosets (see the next section) for p-adic classical groups in [25]. In any case this leads to a strange geometric construction, namely to simplicial maps of Bruhat–Tits buildings whose boundary values are rational maps of p-adic Grassmannians.

(d) The work [4] contains a p-adic construction in the spirit of exchangeability,[5] namely, descriptions of invariant ergodic measures on spaces of infinite p-adic matrices. By the Wigner–Mackey trick (see, e.g., [15, Sect. 13.3]), such kind of statements can be translated to a description of spherical functions on certain groups.

So during last years new elements of a nontrivial picture related to infinite-dimensional p-adic groups appeared. For this reason, understanding of representations $\mathrm{GL}(\infty, \mathbb{O}_p)$ becomes necessary.

1.5. **Another completion of a group of infinite matrices over $\mathbb{Z}_{p^n}$.** Define a group $\mathcal{G}$ consisting of infinite matrices g over $\mathbb{Z}_{p^n}$ such that:

- g contains only a finite number of elements in each column;
- g^{-1} exists and satisfies the same property.

A sequence $g^{(\alpha)}$ converges to g if for each j we have a convergence of $e_j g^{(\alpha)}$.

Clearly, $\mathcal{G} \supset \mathbb{G}$. Classification of irreducible unitary representations of $\mathcal{G}$ is the following. For each finitely generated submodule in $\mathfrak{l}(\mathbb{Z}_{p^n})$ we consider the subgroup $\widehat{\mathcal{G}}[L]$ consisting of transformations sending L to itself and the subgroup $\mathcal{G}^\circ[L]$ fixing L pointwise.

Proposition 1.6. *Any irreducible unitary representation of $\mathcal{G}$ is induced from a representation of some group $\widehat{\mathcal{G}}[L]$ trivial on $\mathcal{G}^\circ[L]$.*

This follows from Theorem 1.5; on the other hand this can be deduced in a straightforward way from Tsankov's result [34].

2. Preliminaries: The category of double cosets

2.1. **Multiplication of double cosets and the category $\mathcal{K}$.** Here we discuss a version of a general construction of multiplication of double cosets (see [29], [30], [20], [26], [27]).

Denote by $\mathbb{G}_{\mathrm{fin}} \subset \mathbb{G}$ the subgroup of *finitary* matrices, i.e., matrices g such that $g - 1$ has only a finite number of nonzero elements. For $\alpha = 0, 1, \dots$ denote by $\mathbb{G}(\alpha) \subset \mathbb{G}$ the subgroups consisting of matrices having the form $\begin{pmatrix} 1_\alpha & 0 \\ 0 & u \end{pmatrix}$, where 1_α denotes the unit matrix of size α and u is an arbitrary invertible matrix over $\mathbb{Z}_{p^\mu}$. Obviously, $\mathbb{G}(\alpha)$ is isomorphic to $\mathbb{G}$. Consider double coset spaces $\mathbb{G}(\alpha) \setminus \mathbb{G}/\mathbb{G}(\beta)$; their elements are matrices determined up to the equivalence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} 1_\alpha & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1_\beta & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} a & bv \\ uc & udv \end{pmatrix}, \tag{2.1}$$

where a matrix g is represented as a block matrix of size $(\alpha + \infty) \times (\beta + \infty)$. For a matrix g we write the corresponding double coset as

$$\left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right]_{\alpha\beta},$$

[5] i.e., of higher analogs of the de Finetti theorem; see [1]

we will omit subscript $\alpha\beta$ if it is not necessary to indicate a size. We wish to define a natural multiplication

$$\mathbb{G}(\alpha)\setminus\mathbb{G}/\mathbb{G}(\beta)\ \times\ \mathbb{G}(\beta)\setminus\mathbb{G}/\mathbb{G}(\gamma)\ \to\ \mathbb{G}(\alpha)\setminus\mathbb{G}/\mathbb{G}(\gamma).$$

Let $\mathfrak{g}_1\in\mathbb{G}(\alpha)\setminus\mathbb{G}/\mathbb{G}(\beta)$, $\mathfrak{g}_2\in\mathbb{G}(\beta)\setminus\mathbb{G}/\mathbb{G}(\gamma)$ be double cosets. By [27, Lemma 4.1], any double coset has a representative in $\mathbb{G}_{\mathrm{fin}}$. Choose such representatives g_1 and g_2 for $\mathfrak{g}_1$, $\mathfrak{g}_2$,

$$g_1=\left[\begin{array}{c|cc} a & b & \\ \hline c & d & \\ & & 1_\infty \end{array}\right]_{\alpha\beta},\quad g_1=\left[\begin{array}{c|cc} p & q & \\ \hline r & t & \\ & & 1_\infty \end{array}\right]_{\beta\gamma}. \tag{2.2}$$

Let sizes of submatrices $\begin{pmatrix} a & b\\ c & d\end{pmatrix}$, $\begin{pmatrix} p & q\\ r & t\end{pmatrix}$, be $N\times N$. Denote by $\theta^\beta(j)$ the following matrix

$$\theta^\beta(j):=\left(\begin{array}{c|ccc} 1_\beta & & & \\ \hline & 0 & 1_j & \\ & 1_j & 0 & \\ & & & 1_\infty \end{array}\right)\in\mathbb{G}(\beta).$$

Consider the sequence

$$\mathbb{G}(\alpha)\cdot g_1\theta^\beta(j)\,g_2\cdot\mathbb{G}(\gamma)\ \in\ \mathbb{G}(\alpha)\setminus\mathbb{G}/\mathbb{G}(\gamma).$$

It is more or less obvious that this sequence is eventually constant and its limit is (2.3)

$$\mathfrak{g}_1\circ\mathfrak{g}_2=$$
$$=\left[\left(\begin{array}{c|ccc} a & b & & \\ \hline c & d & & \\ & & 1_L & \\ & & & 1_\infty \end{array}\right)\left(\begin{array}{c|ccc} 1_\beta & & & \\ \hline & 0 & 1_L & \\ & 1_L & 0 & \\ & & & 1_\infty \end{array}\right)\left(\begin{array}{c|ccc} p & q & & \\ \hline r & t & & \\ & & 1_L & \\ & & & 1_\infty \end{array}\right)\right]_{\alpha\gamma},$$

where $L\geqslant N-\beta$. The final expression is

$$\mathfrak{g}_1\circ\mathfrak{g}_2=\left[\begin{array}{c|ccc} ap & aq & b & \\ \hline cp & cq & d & \\ r & t & 0 & \\ & & & 1_\infty \end{array}\right]_{\alpha\gamma}\sim\left[\begin{array}{c|ccc} ap & b & aq & \\ \hline cp & d & cq & \\ r & 0 & t & \\ & & & 1_\infty \end{array}\right]_{\alpha\gamma}. \tag{2.4}$$

In calculations below we use the last expression for $\circ$-product.

It is easy to verify that this multiplication is associative, i.e., for any

$$\mathfrak{g}_1\in\mathbb{G}(\alpha)\setminus\mathbb{G}/\mathbb{G}(\beta),\quad \mathfrak{g}_2\in\mathbb{G}(\beta)\setminus\mathbb{G}/\mathbb{G}(\gamma),\quad \mathfrak{g}_3\in\mathbb{G}(\gamma)\setminus\mathbb{G}/\mathbb{G}(\delta),$$

we have

$$(\mathfrak{g}_1\circ\mathfrak{g}_2)\circ\mathfrak{g}_3=\mathfrak{g}_1\circ(\mathfrak{g}_2\circ\mathfrak{g}_3).$$

In other words, we get a category. Objects of this category are numbers $\alpha=0,1,2,\dots$. Sets of morphisms are

$$\mathrm{Mor}(\beta,\alpha):=\mathbb{G}(\alpha)\setminus\mathbb{G}/\mathbb{G}(\beta).$$

The multiplication is given by formula (2.4). Denote this *category* by $\mathcal{K}$.

The *group of automorphisms* $\mathrm{Aut}_{\mathcal{K}}(\alpha)$ is $\mathrm{GL}(\alpha, \mathbb{Z}_{p^\mu})$; it consists of double cosets of the form $\left[\begin{array}{c|c} a & 0 \\ \hline 0 & 1_\infty \end{array}\right]$.

Next, the map $g \mapsto g^{-1}$ induces maps

$$\mathbb{G}(\alpha) \setminus \mathbb{G}/\mathbb{G}(\beta) \to \mathbb{G}(\beta) \setminus \mathbb{G}/\mathbb{G}(\alpha),$$

denote these maps by $\mathfrak{g} \mapsto \mathfrak{g}^*$. It is easy to see that we get an *involution in the category* $\mathcal{K}$, i.e.,

$$(\mathfrak{g}_1 \circ \mathfrak{g}_2)^* = \mathfrak{g}_2^* \circ \mathfrak{g}_1^*.$$

The map $g \mapsto (g^t)^{-1}$ determines an *automorphism of the category* $\mathcal{K}$; denote it by $\mathfrak{g} \mapsto \mathfrak{g}^\bigstar$. It sends objects to themselves and

$$(\mathfrak{g}_1 \circ \mathfrak{g}_2)^\bigstar = \mathfrak{g}_2^\bigstar \circ \mathfrak{g}_1^\bigstar.$$

Remarks on notation.

(1) In formulas (2.2), (2.3), (2.4), the last columns, the last rows, and the blocks 1_∞ contain no information and only enlarge sizes of matrices. For this reason, below we will omit them. Precisely, for a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of finite size we denote

$$\left[\begin{array}{c|c} a & b \\ \hline c & d_\star \end{array}\right] := \left[\begin{array}{c|cc} a & b & 0 \\ \hline c & d & 0 \\ 0 & 0 & 1_\infty \end{array}\right] \qquad \left(\begin{array}{c|c} a & b \\ \hline c & d_\star \end{array}\right) := \left(\begin{array}{c|cc} a & b & 0 \\ \hline c & d & 0 \\ 0 & 0 & 1_\infty \end{array}\right).$$

(2) We will denote a multiplication of $[g]$ by an automorphism A as $A \cdot [g]$,

$$A \cdot \left[\begin{array}{c|c} a & b \\ \hline c & d_\star \end{array}\right] := \left[\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right] \circ \left[\begin{array}{c|c} a & b \\ \hline c & d_\star \end{array}\right] = \left[\begin{array}{c|c} Aa & Ab \\ \hline c & d_\star \end{array}\right];$$

$$\left[\begin{array}{c|c} a & b \\ \hline c & d_\star \end{array}\right] \cdot A' := \left[\begin{array}{c|c} a & b \\ \hline c & d_\star \end{array}\right] \circ \left[\begin{array}{c|c} A' & 0 \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} aA' & b \\ \hline cA' & d_\star \end{array}\right].$$

2.2. **The multiplicativity theorem.** Consider a unitary representation ρ of the group $\mathbb{G}$ in a Hilbert space H. Denote by $H_\alpha \subset H$ the space of $\mathbb{G}(\alpha)$-fixed vectors. Denote by P_α the operator of orthogonal projection to H_α.

Proposition 2.1.

(a) *For any β the sequence $\rho\bigl(\theta^\beta(j)\bigr)$ converges to P_β in the weak operator topology.*

(b) *The space $\cup H_\alpha$ is dense in H.*

The first statement is Lemma 1.1 from [27]; the claim (b) is a special case of Proposition VII.1.3 from [20].

Let $g \in \mathbb{G}$, α, $\beta \in \mathbb{Z}_+$. Consider the operator

$$\widetilde{\rho}_{\alpha\beta}(g) : H_\beta \to H_\alpha$$

given by

$$\widetilde{\rho}_{\alpha\beta}(g) := P_\alpha \rho(g)\Big|_{H_\beta}.$$

It is easy to see that for $h_1 \in \mathbb{G}(\alpha)$, $h_2 \in \mathbb{G}(\beta)$ we have

$$\widetilde{\rho}_{\alpha\beta}(g) = \widetilde{\rho}_{\alpha\beta}(h_1 g h_2),$$

i.e., $\widetilde{\rho}_{\alpha\beta}(g)$ actually depends on the double coset $\mathfrak{g}$ containing g.

Theorem 2.2.

(a) *The map* $g \mapsto \widetilde{\rho}_{\alpha\beta}(\mathfrak{g})$ *is a representation of the category* $\mathcal{K}$, *i.e., for any* α, β, γ *for any* $\mathfrak{g}_1 \in \mathrm{Mor}(\beta, \alpha)$, $\mathfrak{g}_2 \in \mathrm{Mor}(\gamma, \beta)$ *we have*

$$\widetilde{\rho}_{\alpha\beta}(\mathfrak{g}_1)\, \widetilde{\rho}_{\beta\gamma}(\mathfrak{g}_2) = \widetilde{\rho}_{\alpha\gamma}(\mathfrak{g}_1 \circ \mathfrak{g}_2).$$

(b) $\widetilde{\rho}$ *is a* $*$*-representation, i.e.,*

$$\widetilde{\rho}_{\alpha\beta}(\mathfrak{g})^* = \widetilde{\rho}_{\beta\alpha}(\mathfrak{g}^*).$$

The statement (a) is an automatic corollary of Proposition 2.1; see [27, Theorem 2.1]. The statement (b) is obvious.

Remark. The considerations of Subsections 2.1, 2.2 are one-to-one repetitions of similar statements for real classical groups and symmetric groups; see [30], [28], [23], [26]. Further considerations drastically differ from these theories.

2.3. **Structure of the paper.** We derive the classification of unitary representations of $\mathbb{G}$ from the multiplicativity theorem and the following argumentation. The semigroups $\Gamma(m) := \mathrm{End}_{\mathcal{K}}(m)$ are finite. It is known that a finite semigroup with an involution has a faithful $*$-representation in a Hilbert space if and only if it is an inverse semigroup (see discussion below, Subsection 3.3). More generally, if a category having finite sets of morphisms acts faithfully in Hilbert spaces, then it must be an inverse category; see [12]. However, semigroups $\mathrm{End}_{\mathcal{K}}(\alpha)$ are not inverse,[6] and $*$-representations of $\mathcal{K}$ pass through a smaller category.

Section 3 contains preliminary remarks on inverse semigroup and construction of an inverse category $\mathcal{L}$, which is a quotient of $\mathcal{K}$. This provides us lower estimate of maximal inverse semigroup quotients of semigroups $\Gamma(m)$.

In Section 4 we examine idempotents in maximal inverse semigroup quotients $\mathrm{inv}(\Gamma(m))$ of $\Gamma(m)$. In Section 5 we show that some of idempotents of $\mathrm{inv}(\Gamma(m))$ act by the same operators in all representations of $\mathbb{G}$. Next, for any representation of $\mathbb{G}$ there is a minimal m such that $H_m \neq 0$. In Section 6 we examine the image of $\Gamma(m)$ in such representation.

In Section 7 we discuss properties of the groups $\mathbb{G}^\circ[L; M]$ and $\mathbb{G}^\bullet[L; M]$.

The final part of the proof is contained in Section 8.

3. The reduced category and inverse semigroups

3.1. **Notation.** Below we work only with the group $\mathbb{G} := \mathrm{GL}(\infty, \mathbb{Z}_{p^\mu})$. To simplify notation, we write

$$\mathrm{GL}(m) := \mathrm{GL}(m, \mathbb{Z}_{p^\mu}), \qquad \Gamma(m) := \mathrm{End}_{\mathcal{K}}(m), \qquad \mathfrak{l}^m := (\mathbb{Z}_{p^\mu})^m.$$

For a unitary representation ρ of a $\mathbb{G}$ we define the *height* $h(\rho)$ as the minimum of α such that $H_\alpha \neq 0$.

By $x(\mathrm{mod}\ p)$ we denote a reduction of an object (a scalar, a vector, a matrix) defined over $\mathbb{Z}_{p^\mu}$ modulo p, i.e. to the field $\mathbb{F}_p$. Notice that a square matrix A of finite size over $\mathbb{Z}_{p^\mu}$ is invertible if and only if $A(\mathrm{mod}\ p)$ is invertible. A matrix B is nilpotent (i.e., $B^N = 0$ for sufficiently large N) if and only if $B(\mathrm{mod}\ p)$ is nilpotent. Indeed, if $B^k = 0(\mathrm{mod}\ p)$, then B^k has the form pC for some matrix C. Hence, $(B^k)^{\mu+1} = p^{\mu+1}C^{\mu+1} = 0$.

[6]This was observed by Ol'shanskiĭ [30] for $\mathrm{GL}(\infty, \mathbb{F}_p)$.

We use several symbols for equivalences in $\mathrm{Mor}_{\mathcal{K}}(\beta,\alpha)$; the $\sim$ was defined by (2.1); the symbols

$$\equiv, \qquad \approx, \qquad \approx_m$$

are defined in the next two subsections.

3.2. **The reduced category** $\mathrm{red}(\mathcal{K})$**.** Let $\mathfrak{g}_1$, $\mathfrak{g}_2 \in \mathrm{Mor}(\beta,\alpha)$. We say that they are $\approx$-*equivalent* if for any unitary representation of $\mathbb{G}$ we have $\widetilde{\rho}_{\alpha\beta}(\mathfrak{g}_1) = \widetilde{\rho}_{\alpha\beta}(\mathfrak{g}_2)$. The *reduced category* $\mathrm{red}(\mathcal{K})$ is the category whose objects are nonnegative integers and morphisms $\beta \to \alpha$ are $\approx$-equivalence classes of $\mathrm{Mor}(\beta,\alpha)$. Denote by $\mathrm{red}(\Gamma(m))$ semigroups of endomorphisms of $\mathrm{red}(\mathcal{K})$.

Also we define a weaker equivalence, $\mathfrak{g}_1 \approx_m \mathfrak{g}_2$ if $\widetilde{\rho}_{\alpha\beta}(\mathfrak{g}_1) = \widetilde{\rho}_{\alpha\beta}(\mathfrak{g}_1)$ for all ρ of height $\geqslant m$. Denote by $\mathrm{red}_m(\mathcal{K})$ the corresponding *m-reduced category.*

Our proof of Theorem 1.5 is based on an examination of the categories $\mathrm{red}(\mathcal{K})$ and $\mathrm{red}_m(\mathcal{K})$. We obtain an information sufficient for a classification of representations of $\mathbb{G}$. However, the author does not know an answer to Question 3.1.

Question 3.1. Find a transparent description of the category $\mathrm{red}(\mathcal{K})$.

3.3. **Inverse semigroups.** Let $\mathcal{P}$ be a **finite** semigroup with an involution $x \mapsto x^*$. Then the following conditions are equivalent.

(A) $\mathcal{P}$ admits a faithful representation in a Hilbert space.

(B) $\mathcal{P}$ admits an embedding to a semigroup of partial bijections[7] of a finite set compatible with the involutions in $\mathcal{P}$ and in partial bijections.

(C) $\mathcal{P}$ is an *inverse semigroup* (see [6], [17], [16]), i.e., for any x we have

$$xx^*x = x, \qquad x^*xx^* = x^* \tag{3.1}$$

and any two idempotents in $\mathcal{P}$ commute.

Discuss briefly some properties of inverse semigroups. Any idempotent in $\mathcal{P}$ is self-adjoint, and for any x, the element x^*x is an idempotent. Since idempotents commute, a product of idempotents is an idempotent. The semigroup of idempotents has a natural *partial order*,

$$x \preceq y \qquad \text{if} \qquad xy = x.$$

We have $xy \preceq x$. If $x \preceq y$ and $u \preceq v$, then $xu \preceq yv$. Since our semigroup is finite, the product of all idempotents is a *minimal idempotent* $\mathbf{0}$; we have $\mathbf{0}x = x\mathbf{0} = \mathbf{0}$ for any x.

Let $\mathcal{R}$ be a finite semigroup with involution. Then there exists an inverse semigroup $\mathrm{inv}(\mathcal{R})$ and epimorphism $\pi : \mathcal{R} \to \mathrm{inv}(\mathcal{R})$ such that any homomorphism ψ from $\mathcal{R}$ to an inverse semigroup $\mathcal{Q}$ has the form $\psi = \varkappa\pi$ for some homomorphism $\varkappa : \mathrm{inv}(\mathcal{R}) \to \mathcal{Q}$. We say that $\mathrm{inv}(\mathcal{R})$ is the *maximal inverse semigroup quotient* of $\mathcal{R}$.

Lemma 3.1. *The semigroups* $\Gamma(m)$ *are finite.*

This is a corollary of the following statement; see [27, Lemma 4.1.a].

Lemma 3.2. *Any double coset in* $\mathbb{G}(m)\backslash\mathbb{G}/\mathbb{G}(m)$ *has a representative in* $\mathrm{GL}(3m)$.

[7]Recall that a *partial bijection* σ from a set A to a set B is a bijection from a subset S of A to a subset T of B; see e.g., [17] or [20, Sect. VIII.1]. The adjoint partial bijection $\sigma^* : B \to A$ is the inverse bijection T to S.

We consider the following quotients of $\Gamma(m)$:

(1) $\mathrm{inv}(\Gamma(m))$ is the maximal inverse semigroup quotient of $\Gamma(m)$;

(2) $\mathrm{red}(\Gamma(m)) := \mathrm{End}_{\mathrm{red}\,\mathcal{K}}(m)$;

(3) $\mathrm{red}_m(\Gamma(m)) := \mathrm{End}_{\mathrm{red}_m(\mathcal{K})}(m)$.

We have the following sequence of epimorphisms[8]:

$$\Gamma(m) \to \mathrm{inv}(\Gamma(m)) \to \mathrm{red}(\Gamma(m)) \to \mathrm{red}_m(\Gamma(m)).$$

For $g \in \mathbb{G}_{\mathrm{fin}}$ we denote by $[g]_{mm}$ the corresponding element of $\Gamma(m)$ and by $[[g]]_{mm}$ the corresponding element of $\mathrm{inv}\bigl(\Gamma(m)\bigr)$. The equality in $\Gamma(m)$ we denote by $\sim$, in $\mathrm{inv}\,\Gamma((m))$ by $\equiv$, in $\mathrm{red}(\Gamma(m))$ by $\approx$, in $\mathrm{red}_m(\Gamma(m))$ by $\approx_m$. Denote by $[[g_1]] \diamond [[g_2]]$ the product in $\mathrm{inv}(\Gamma(m))$.

Our next purpose is to present some (non-maximal) inverse semigroup quotients of $\Gamma(m)$.

3.4. **The category $\mathcal{L}$ of partial isomorphisms.** Let V, W be modules over $\mathbb{Z}_{p^\mu}$. A *partial isomorphism* $p : V \to W$ is an isomorphism of a submodule $A \subset V$ to a submodule $B \subset W$. We denote $\mathrm{dom}\, p := A$, $\mathrm{im}\, p := B$. By p^* we denote the inverse map $B \to A$. Let $p : V \to W$, $q : W \to Y$ be partial isomorphisms. Then the *product* pq is defined in the following way:

$$\mathrm{dom}\, pq := p^*(\mathrm{dom}\, q) \cap \mathrm{dom}\, p,$$

for $v \in \mathrm{dom}\, pq$ we define $v(pq) = (vp)q$.

A partial isomorphism p is an *idempotent* if $\mathrm{dom}\, p = \mathrm{im}\, p$ and p is an identical map.

Objects of the category $\mathcal{L}$ are modules

$$\mathfrak{l}_+^\alpha \oplus \mathfrak{l}_-^\alpha := (\mathbb{Z}_{p^\mu})^\alpha \oplus (\mathbb{Z}_{p^\mu})^\alpha$$

equipped with the following pairing

$$\{v_+; v_-\} := \sum_j v_+^j v_-^j = v_+(v_-)^t,$$

where $v_\pm \in \mathfrak{l}_\pm^\alpha$. We say that two partial isomorphisms

$$\xi_+ : \mathfrak{l}_+^\alpha \to \mathfrak{l}_+^\beta, \qquad \xi_- : \mathfrak{l}_-^\alpha \to \mathfrak{l}_-^\beta$$

are *compatible* if for any $y_+ \in \mathrm{dom}\,\xi_+$ and $y_- \in \mathrm{dom}\,\xi_-$, we have

$$\{\xi_+(y_+), \xi_-(y_-)\} = \{y_+, y_-\}.$$

Next, we define a *category* $\mathcal{L}$. Its objects are spaces $\mathfrak{l}_+^\alpha \oplus \mathfrak{l}_-^\alpha$ and morphisms are pairs of compatible partial isomorphisms $\xi_+ : \mathfrak{l}_+^\alpha \to \mathfrak{l}_+^\beta$, $\xi_- : \mathfrak{l}_-^\alpha \to \mathfrak{l}_-^\beta$.

The category $\mathcal{L}$ is equipped with an involution

$$(\xi_+, \xi_-)^* = (\xi_+^*, \xi_-^*)$$

and an automorphism

$$(\xi_+, \xi_-)^\bigstar = (\xi_-, \xi_+).$$

Lemma 3.3. *The semigroups* $\mathrm{End}_{\mathcal{L}}(m)$ *are inverse.*

Indeed, $\mathrm{End}_{\mathcal{L}}(m)$ is a semigroup of partial bijections of a finite set $\mathfrak{l}_+^m \oplus \mathfrak{l}_-^m$. The whole category $\mathcal{L}$ is inverse for the same reason.

[8]All these semigroups are different.

3.5. **The functor $\Pi : \mathcal{K} \to \mathcal{L}$.** Consider $g \in \mathbb{G}_{\mathrm{fin}}$. Let actually g be contained in $\mathrm{GL}(N)$. Represent g as a block $(\beta + (N-\beta)) \times (\alpha + (N-\alpha))$ matrix and g^{-1} as an $(\alpha + (N-\alpha)) \times (\beta + (N-\beta))$-matrix,

$$g = \begin{pmatrix} a & b \\ c & d_\star \end{pmatrix}, \qquad g^{-1} = \begin{pmatrix} A & B \\ C & D_\star \end{pmatrix}.$$

Define maps $\xi_\pm : \mathfrak{l}_\pm^\alpha \to \mathfrak{l}_\pm^\beta$ by:

- $\operatorname{dom} \xi_+ := \ker b$ and ξ_+ is the restriction of a to $\ker b$;
- $\operatorname{dom} \xi_- := \ker C^t$ and ξ_- is the restriction of A^t to $\ker C^t$.

Proposition 3.4.

(a) *The pair ξ_+, ξ_- depends only on the double coset containing $\mathfrak{g}$.*

(b) *Partial isomorphisms ξ_+, ξ_- are compatible.*

(c) *The map $\mathfrak{g} \mapsto (\xi_+, \xi_-)$ determines a functor from the category $\mathcal{K}$ to the category $\mathcal{L}$.*

Denote this functor by Π. By $\Pi(\mathfrak{g})$ we denote the morphism of $\mathcal{L}$ corresponding to $\mathfrak{g}$. We have

$$\Pi(\mathfrak{g}^*) = (\Pi(\mathfrak{g}))^*, \qquad \Pi(\mathfrak{g}^\bigstar) = (\Pi(\mathfrak{g}))^\bigstar. \tag{3.2}$$

Proof. For any invertible matrix v we have, $\ker b = \ker bv$. Therefore ξ_+ depends only on a double coset. For ξ_- we apply (3.2).

(b) Let $v \in \ker b$, $w \in \ker C^t$. Then

$$\{v, w\} = vw^t = v(aA + bC)w^t = va \cdot (wA^t)^t + vb \cdot (wC^t)^t = \{va, wA^t\} + 0.$$

(c) We look to formula (2.4) for a product in $\mathcal{K}$. The new ξ_+ is a restriction of ap to $\ker b \cap \ker aq$. This is the product of two ξ-es. □

Remark. According Ol'shanskĭ [30], for the case $\mathrm{GL}(\infty, \mathbb{F}_p)$ the functor $\Pi : \mathcal{K} \to \mathcal{L}$ determines an isomorphism of categories $\mathrm{red}(\mathcal{K}) \to \mathcal{L}$. However, for $\mu > 1$ the maps $\Pi : \mathrm{red}(\Gamma(m)) \to \mathrm{Mor}_{\mathcal{L}}(m)$ are neither surjective nor injective. However we will observe that Π induces isomorphisms of semigroups of idempotents; this provides us an important argument for the proof of Proposition 6.1.

4. Idempotents in $\mathrm{inv}(\Gamma(m))$

Here we examine idempotents in the semigroup $\mathrm{inv}(\Gamma(m))$. The main statement of the section is Proposition 4.10.

4.1. **Projectors[9]: P_α.** Consider an irreducible representation ρ of $\mathbb{G}$; let subspaces $H_m \subset H$ and orthogonal projectors $P_m : H \to H_m$ be as above.

Lemma 4.1.

(a) *The projector*

$$P_\alpha \Big|_{H_m} : H_m \to H_\alpha$$

[9]This subsection contains generalities; $\mathcal{K}$ is an ordered category in the sense of [20, Sect. III.4]; this implies all statements of the subsection.

is given by the operator $\widetilde{\rho}_{mm}\big(\Theta^{\alpha}_{[m]}\big)$, *where*

$$\Theta^{\alpha}_{[m]} := \left[\begin{array}{cc|cc} 1_\alpha & 0 & 0 & 0 \\ 0 & 0 & 1_{m-\alpha} & 0 \\ \hline 0 & 1_{m-\alpha} & 0 & 0 \\ 0 & 0 & 0 & 1_\infty \end{array}\right]_{mm} \in \Gamma(m). \tag{4.1}$$

(b) *The tautological embedding* $H_\alpha \to H_m$ *is defined by the operator* $\widetilde{\rho}_{m\alpha}\big(\Lambda^{\alpha}_{[m]}\big)$, *where*

$$\Lambda^{\alpha}_{[m]} := \left[\begin{array}{cc|c} 1_\alpha & 0 & 0 \\ \hline 0 & 1_{m\alpha} & 0 \\ 0 & 0 & 1_\infty \end{array}\right]_{\alpha m} \in \mathrm{Mor}_{\mathcal{K}}(\alpha, m).$$

(c) *The orthogonal projector* $H_m \to H_\alpha$ *is given by* $\widetilde{\rho}_{\alpha m}\big((\Lambda^{\alpha}_{[m]})^*\big)$

$$(\Lambda^{\alpha}_{m})^* := \left[\begin{array}{c|cc} 1_\alpha & 0 & 0 \\ 0 & 1_{m-\alpha} & 0 \\ \hline 0 & 0 & 1_\infty \end{array}\right]_{m\alpha} \in \mathrm{Mor}_{\mathcal{K}}(m, \alpha).$$

Proof. (a) We apply Proposition 2.1(a). For $j > m-\alpha$ we have $[\theta^\alpha(j)]_{mm} = \Theta^{\alpha}_{[m]}$. The same argument proves (b) and (c). □

Lemma 4.2.
(a) *The map*

$$\iota^{\alpha}_{m} : \left[\begin{array}{c|c} a & b \\ \hline c & d_\star \end{array}\right]_{\alpha\alpha} \mapsto \left[\begin{array}{cc|cc} a & 0 & b & 0 \\ 0 & 0 & 0 & 1_{m-\alpha} \\ \hline c & 0 & d & 0 \\ 0 & 1_{m-\alpha} & 0 & 0_\star \end{array}\right]_{mm}$$

is a homomorphism $\Gamma(\alpha) \to \Gamma(m)$.
(b) *We have*

$$\iota^{\alpha}_{m}(\mathfrak{g}) \sim \Lambda^{\alpha}_{n} \circ \mathfrak{g} \circ (\Lambda^{\alpha}_{n})^* .$$

This follows from a straightforward calculation.

Corollary 4.3. *The map* ι^{α}_{m} *is compatible with representations* $\widetilde{\rho}$ *of* $\Gamma(\alpha)$ *and* $\Gamma(m)$. *Namely, operators* $\widetilde{\rho}_{mm}(\iota^{\alpha}_{m}(\mathfrak{g}))$ *have the following block structure with respect to the decomposition* $H_m = H_\alpha \oplus (H_m \ominus H_\alpha)$:

$$\widetilde{\rho}_{mm}(\iota^{\alpha}_{m}(\mathfrak{g})) = \begin{pmatrix} \widetilde{\rho}_{\alpha\alpha}(\mathfrak{g}) & 0 \\ 0 & 0 \end{pmatrix}.$$

4.2. **Idempotents in** $\mathrm{inv}(\Gamma(m))$**.** Here we formulate several lemmas (their proofs occupy Subsections 4.3–4.7); as a corollary we get Proposition 4.10.

Lemma 4.4. *Let for*

$$[g] = \begin{bmatrix} a & b \\ c & d_\star \end{bmatrix}_{mm} \in \Gamma(m)$$

one of the blocks a, d *be degenerate. Then* $[[g]]_{mm} \in \mathrm{inv}(\Gamma(m))$ *has a representative* $[g']$, *for which both blocks* a, d *are degenerate.*

Denote by

$$\Gamma^\circ(m)$$

the subsemigroup in $\Gamma(m)$ consisting of all $[g]$, for which both blocks a, d are nondegenerate.

Lemma 4.5. *Any idempotent in* $\mathrm{inv}(\Gamma(m))$ *has a representative of the form* $q \cdot [[R]] \cdot q^{-1}$ *with* q *ranging* $\mathrm{GL}(m)$ *and* R *having the form*

$$[R] := \left[\begin{array}{cc|cc} 1_\alpha & 0 & \varphi & 0 \\ 0 & 0 & 0 & 1_{m-\alpha} \\ \hline \psi & 0 & \varkappa & 0 \\ 0 & 1_{m-\alpha} & 0 & 0_\star \end{array}\right]_{mm} \in \Gamma(m), \tag{4.2}$$

where

$$\left[\begin{array}{c|c} 1 & \varphi \\ \hline \psi & \varkappa_\star \end{array}\right]_{\alpha\alpha} \in \Gamma(\alpha)$$

represents an idempotent in $\mathrm{inv}(\Gamma^\circ(\alpha))$. *The parameter* α *ranges in the set* $0, 1, 2, \dots, m$.

Remark. Denote

$$R^\square := \left[\begin{array}{cc|c} 1_\alpha & 0 & \varphi \\ 0 & 1_{m-\alpha} & 0 \\ \hline \psi & 0 & \varkappa_\star \end{array}\right]_{mm}.$$

Then the following elements of $\Gamma(m)$ coincide:

$$R = R^\square \Theta_m^\alpha = \Theta_m^\alpha R^\square = \Theta_m^\alpha R^\square \Theta_m^\alpha. \tag{4.3}$$

Denote

$$X(b,c) := \left(\begin{array}{c|cc} 1_m & b & 0 \\ \hline 0 & 1 & 0 \\ c & 0 & 1_\star \end{array}\right) \in \mathbb{G}_{\mathrm{fin}}.$$

Lemma 4.6. *Elements of the form* $[X(b,c)]$ *are idempotents in* $\Gamma^\circ(m)$. *They depend only on* $\ker b$ *and* $\ker c^t \subset \ell^m$.

Let $L := \ker b$ and $M := \ker c^t$. Denote

$$\mathcal{X}[L, M] := [X(b,c)]. \tag{4.4}$$

Lemma 4.7. *We have*

$$\mathcal{X}[L_1, M_1]\ \mathcal{X}[L_2, M_2] = \mathcal{X}[L_1 \cap L_2, M_1 \cap M_2].$$

Lemma 4.8. *Any idempotent in* $\mathrm{inv}(\Gamma^\circ(m))$ *has the form* $\mathcal{X}[L, M]$.

Corollary 4.9. *Idempotents* $\mathcal{X}[L, M]$ *are pairwise distinct in* $\mathrm{inv}(\Gamma^\circ(m))$.

Proof. Indeed, $\mathrm{End}_{\mathcal{L}}(m)$ is an inverse semigroup; therefore we have a chain of maps

$$\Gamma^\circ(m) \to \mathrm{inv}(\Gamma^\circ(m)) \to \mathrm{inv}(\Gamma(m)) \to \mathrm{Mor}_{\mathcal{L}}(m).$$

The image of $X(b,c)$ in $\mathrm{Mor}_{\mathcal{L}}(m)$ is precisely the pair of identical partial isomorphisms $M \to M$, $L \to L$. Therefore for nonequivalent $X(b,c)$ we have different images. □

Proposition 4.10. *Any idempotent in* $\mathrm{inv}(\Gamma(m))$ *has a representative of the form*

$$q \cdot \left[\begin{array}{cc|ccc} 1_\alpha & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_{m-\alpha} \\ \hline 0 & 0 & 1 & 0 & 0 \\ c & 0 & 0 & 1 & 0 \\ 0 & 1_{m-\alpha} & 0 & 0 & 0_\star \end{array}\right]_{mm} \cdot q^{-1},$$

where $q \in \mathrm{GL}(m) = \mathrm{Aut}_{\mathcal{K}}(m)$.

Proof. Lemma 4.2 defines a canonical embedding $i_m^\alpha : \Gamma(\beta) \to \Gamma(m)$ for $\alpha < m$. By Lemma 4.5 any idempotent in $\mathrm{inv}(\Gamma(m))$ is equivalent to an idempotent lying in some $i_m^\alpha(\Gamma^\circ(\alpha))$. Lemma 4.8 gives us a canonical form of this idempotent. □

Now we start proofs of Lemmas 4.4–4.8,

4.3. **Proof of Lemma 4.4.** Clearly $\Gamma(m) \setminus \Gamma^\circ(m)$ is a two-sided ideal in $\Gamma(m)$. Since $[[g \circ (g^{-1} \circ g)]]_{mm} = [[g]]_{mm}$, it is sufficient to prove the statement for idempotents.

Let

$$g = \begin{pmatrix} a & b \\ c & d_\star \end{pmatrix}, \qquad g^{-1} =: \begin{pmatrix} A & B \\ C & D_\star \end{pmatrix}. \tag{4.5}$$

Then

$$[[g]] \diamond [[g]]^* \equiv [[g \circ g^{-1}]] \equiv \left[\left[\begin{array}{c|c} aA & * \\ \hline * & * \end{array} \right]\right].$$

If a is degenerate, then aA is degenerate. Now let a be non-degenerate, d degenerate. Since the matrices (4.5) are inverse one to another, we have

$$aA = 1 - bC, \qquad Dd = 1 - Cb.$$

We see that $(1-Cb) (\mathrm{mod}\ p)$ is degenerate, $(1-bC) (\mathrm{mod}\ p)$ also is degenerate, and therefore aA is degenerate.

4.4. **Proof of Lemma 4.5.**

Step 1.

Lemma 4.11. *Let x be an idempotent in* $\mathrm{inv}(\Gamma(m))$. *Then it can be represented as* $[[u]]$, *where* $u = u^{-1}$.

Proof. Let $x = [[g]]$. Then

$$x = [[g]] \diamond [[g]]^* = [[g \circ g^{-1}]] = [[g\theta^m(j)g^{-1}]]$$

for sufficiently large j. We set $u := g\theta^m(j)g^{-1}$. □

Lemma 4.12. *Let $g = g^{-1} \in \mathbb{G}_{\mathrm{fin}}$. For any $N > 0$ there exists a representative $r \in \mathbb{G}_{\mathrm{fin}}$ of $[g]^{\circ 2N}$ such that $r = r^{-1}$.*

Proof. Let actually $g \in \mathrm{GL}(m+l)$. Then we choose the following representative of $[g]^{\circ 8}$:

$$r = g\ \theta^m(l)\ g\ \theta^m(2l)\ g\ \theta^m(4l)\ g\ \theta^m(8l)\ g\theta^m(4l)\ g\ \theta^m(2l)\ g\ \theta^m(l)\ g.$$

Step 2.

Lemma 4.13. *Let* $g = g^{-1} = \begin{pmatrix} a & b \\ c & d_\star \end{pmatrix} \in \mathbb{G}$. *Then there exists a matrix*

$$Z = \left(\begin{array}{c|c} \zeta & 0 \\ \hline 0 & 1_\star \end{array} \right) \in \mathrm{Aut}_{\mathcal{K}}(m), \qquad \text{where } \zeta \in \mathrm{GL}(m),$$

and N such that

$$[[Z \cdot g \cdot Z^{-1}]]^{\diamond N} = \left[\left[\begin{pmatrix} \zeta & 0 \\ 0 & 1_\star \end{pmatrix} \begin{pmatrix} a & b \\ c & d_\star \end{pmatrix} \begin{pmatrix} \zeta & 0 \\ 0 & 1_\star \end{pmatrix}^{-1} \right]\right]^{\diamond N}$$

has a form

$$r = \left[\left[\begin{array}{cc|c} 0 & 0 & * \\ 0 & 1_k & * \\ \hline * & * & * \end{array}\right]\right],$$

where k is the rank of the reduced matrix $a^m \pmod p$.

Clearly our lemma is a corollary of the following statement:

Lemma 4.14. *For any $m \times m$ matrix a over $\mathbb{Z}_{p^\mu}$ there exists $\zeta \in \mathrm{GL}(m)$ and N such that*

$$(\zeta a \zeta^{-1})^N = \begin{pmatrix} 0 & 0 \\ 0 & 1_k \end{pmatrix}.$$

Proof. We split the operator $a \pmod p$ over the field $\mathbb{F}_p$ as a direct sum of a nilpotent part S and an invertible part T. For sufficiently large M the matrix $\begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}^M$ has the form $\begin{pmatrix} 0 & 0 \\ 0 & P \end{pmatrix}$ with a nondegenerate P. Since the group $\mathrm{GL}(k, \mathbb{F}_p)$ is finite, $P^L = 1_k$ for some L.

Thus without a loss of generality, we can assume that a has a form

$$a = \begin{pmatrix} p\alpha & p\beta \\ p\gamma & 1 + p\delta \end{pmatrix},$$

where α, β, γ, δ are matrices over $\mathbb{Z}_{p^\mu}$. We conjugate it as follows

$$\begin{pmatrix} 1 & pu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p\alpha & p\beta \\ p\gamma & 1 + p\delta \end{pmatrix} \begin{pmatrix} 1 & -pu \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} * & \boxed{-p^2(\alpha u + u\gamma u) + p\bigl(\beta + u(1 + p\delta)\bigr)} \\ * & * \end{pmatrix}.$$

We wish to choose u to make zero in the boxed block. It is sufficient to find a matrix u satisfying the following equation:

$$u = \bigl(-\beta + p(\alpha u + u\gamma u)\bigr)(1 + p\delta)^{-1} = -\beta + p(-\delta + \alpha u + u\gamma u)(1 + p\delta)^{-1}. \tag{4.6}$$

We look for a solution in the form

$$u = \sum_{k=0}^{\mu} p^k S_k.$$

First, we consider S_k as formal noncommutative variables. Then we get a system of equations of the form

$$S_0 = -\beta, \qquad S_k = F_k(\alpha, \beta, \gamma, \delta; S_0, S_1, \ldots, S_{k-1}),$$

where F_k are polynomial expressions with integer coefficients. These equations can be regarded as recurrence formulas for S_k. In this way we get a solution u.

Thus without a loss of generality we can assume that a has the form

$$a = \begin{pmatrix} p\alpha' & 0 \\ p\gamma' & 1 + p\delta' \end{pmatrix}.$$

Raising it to μ-th power, we come to a matrix of the form

$$a = \begin{pmatrix} 0 & 0 \\ p\gamma'' & 1 + p\delta'' \end{pmatrix}.$$

We conjugate it as

$$\begin{pmatrix}1&0\\pv&1\end{pmatrix}\begin{pmatrix}0&0\\p\gamma''&1+p\delta''\end{pmatrix}\begin{pmatrix}1&0\\-pv&1\end{pmatrix}=\begin{pmatrix}0&0\\p(\gamma''-(1+p\delta'')v)&(1+\delta''v)\end{pmatrix}.$$

Taking $v=(1+p\delta'')^{-1}\gamma''$ we kill the left lower block and come to a matrix of the form $\begin{pmatrix}0&0\\0&1+p\delta'''\end{pmatrix}$. Raising it in $p^{\mu-1}$-th power we come to $\begin{pmatrix}0&0\\0&1\end{pmatrix}$. □

Step 3. Thus the element $[[g]]^{\diamond 2N}$ from Lemma 4.13 has a representative of the following block $(m-k)+k+(m-k)+\infty$ form:

$$r=r^{-1}=\left(\begin{array}{cc|cc}0&0&\beta_{11}&\beta_{12}\\0&1_k&\beta_{21}&\beta_{22}\\\hline\gamma_{11}&\gamma_{12}&\delta_{11}&\delta_{12}\\\gamma_{21}&\gamma_{22}&\delta_{21}&\delta_{22\star}\end{array}\right).$$

Lemma 4.15. *There is a matrix* $U=\left(\begin{array}{c|c}1_m&0\\\hline 0&u_\star\end{array}\right)$ *such that* UrU^{-1} *has the form*

$$\widetilde{r}=\left(\begin{array}{cc|cc}0&0&1_{m-k}&0\\0&1_k&0&\varphi\\\hline 1_{m-k}&0&0&0\\0&\psi&0&\varkappa_\star\end{array}\right). \tag{4.7}$$

Recall that $[r]\sim[UrU^{-1}]$.

Proof. Since the matrix $\begin{pmatrix}\beta_{11}&\beta_{12}\end{pmatrix}$ is nondegenerate (otherwise r is degenerate), we can choose a conjugation of r by matrices $U=\begin{pmatrix}1_m&0\\0&*\end{pmatrix}$ reducing this block to the form $\begin{pmatrix}1&0\end{pmatrix}$. We have $r^2=1$; evaluating r^2 we get γ_{11} in the left upper block. Therefore $\gamma_{11}=1$. Thus we come to new r,

$$r^{\sim}=\left(\begin{array}{cc|cc}0&0&1&0\\0&1_k&\beta_{21}&\beta_{22}\\\hline 1&\gamma_{12}&\delta_{11}&\delta_{12}\\\gamma_{21}&\gamma_{22}&\delta_{21}&\delta_{22\star}\end{array}\right)$$

with new β, γ, δ. Next, we conjugate this matrix by

$$\left(\begin{array}{c|cc}1_m&0&0\\\hline 0&1_{m-k}&0\\0&-\gamma_{21}&1_\star\end{array}\right)$$

and kill γ_{21}. Thus we come to new r,

$$r^{\sim\sim}=\left(\begin{array}{cc|cc}0&0&1&0\\0&1_k&\beta_{21}&\beta_{22}\\\hline 1&\gamma_{12}&\delta_{11}&\delta_{12}\\0&\gamma_{22}&\delta_{21}&\delta_{22\star}\end{array}\right).$$

But $(r^{\sim\sim})^2=1$. Looking to third row and third column of $(r^{\sim\sim})^2$ we observe that

$$\beta_{21},\ \delta_{11},\ \delta_{21},\ \gamma_{12},\ \delta_{12}\ \text{are zero.}$$

Thus, $r^{\sim\sim}$ has the desired form. □

4.5. **Proof of Lemma 4.6.** Denote

$$[X_+(A)] := \begin{bmatrix} 1 & A \\ 0 & 1_\star \end{bmatrix}.$$

We can conjugate this matrix by $\begin{pmatrix} 1 & 0 \\ 0 & u_\star \end{pmatrix}$. Therefore a matrix A is defined up to multiplications $A \sim Au$, where u is an invertible matrix. The invariant of this action is $\ker A$ (this is more or less clear; formally we can refer to Lemma 7.3 proved below).

Next,

$$[X_+(A)] \circ [X_+(A)] = \left[\begin{array}{c|cc} 1 & A & A \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1_\star \end{array}\right].$$

We have $\ker \begin{pmatrix} A & A \end{pmatrix} = \ker A$ and therefore $[X_+(A)]$ is an idempotent. In the same way, $[X_-(B)] := \begin{bmatrix} 1 & 0 \\ B & 1_\star \end{bmatrix}$ is an idempotent. It remains to notice that

$$[X(A,B)] = [X_+(A)] \diamond [X_-(B)].$$

Thus $[X(A,B)]$ *is an idempotent.*

4.6. **Proof of Lemma 4.6.** In notation of the previous subsection

$$X_-(A_1) \circ X_-(A_2) \sim X\Big(\begin{pmatrix} A_1 & A_2 \end{pmatrix}\Big),$$

i.e.,

$$\mathcal{X}[\ker A_1, 0] \diamond \mathcal{X}[\ker A_2, 0] \equiv \mathcal{X}[\ker A_1 \cap \ker A_2, 0],$$

or

$$\mathcal{X}[L_1, 0] \diamond \mathcal{X}[L_2, 0] \equiv \mathcal{X}[L_1 \cap L_2, 0].$$

On the other hand, we have

$$\mathcal{X}[L, 0] \diamond \mathcal{X}[0, M] \equiv \mathcal{X}[L, M],$$

and now the statement becomes obvious.

Proof of Lemma 4.7. Indeed,

$$[X(b_1, c_1)] \circ [X(b_1, c_1)] \sim \left[X\left(\begin{pmatrix} b_1 & b_2 \end{pmatrix}, \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\right)\right]$$

and $\ker \begin{pmatrix} b_1 & b_2 \end{pmatrix} = \ker b_1 \cap \ker b_2$. □

4.7. **Proof of Lemma 4.8.**

Step 1. Any idempotent $[[g]] \in \mathrm{inv}(\Gamma^\circ(m))$ has a representative of the form $\begin{pmatrix} 1 & a \\ b & 1_\star \end{pmatrix}$, where $ab = 0$.

Let $[[g]] = \left[\left[\begin{matrix} \alpha & \beta \\ \gamma & \delta_\star \end{matrix}\right]\right]$ be an idempotent; let α, δ be nondegenerate. By Lemma 4.11 without loss of generality we can assume $g = g^{-1}$. Taking an appropriate power $[r] = [g]^{\circ 2N}$, we can achieve $\alpha = 1$. By Lemma 4.12, we can assume $r = r^{-1}$.

Set $r = \begin{pmatrix} 1 & -a \\ b & c_\star \end{pmatrix}$. Evaluating $r^2 = 1$ we get the following collection of conditions

$$\boxed{ab = 0}, \qquad ac = -a, \qquad cb = -b, \qquad c^2 - ba = 1.$$

We replace r by an equivalent matrix

$$r \sim \begin{pmatrix}1 & -a\\ b & c_\star\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & c_\star^{-1}\end{pmatrix} = \begin{pmatrix}1 & -ac^{-1}\\ b & 1_\star\end{pmatrix} = \begin{pmatrix}1 & a\\ b & 1_\star\end{pmatrix},$$

here we used the identity $-ac^{-1} = a$.

Step 2. We evaluate $[r]^{\circ 2}$,

$$[r]^{\circ 2} = \left[\left(\begin{array}{c|cc}1 & a & 0\\ \hline b & 1 & 0\\ 0 & 0 & 1_\star\end{array}\right)\left(\begin{array}{c|cc}1 & 0 & a\\ \hline 0 & 1 & 0\\ b & 0 & 1_\star\end{array}\right)\right] = \left[\begin{array}{c|cc}1 & a & a\\ \hline b & 1 & ba\\ b & 0 & 1_\star\end{array}\right]$$
$$\sim \left[\left(\begin{array}{c|cc}1 & a & a\\ \hline b & 1 & ba\\ b & 0 & 1_\star\end{array}\right)\left(\begin{array}{c|cc}1 & 0 & 0\\ \hline 0 & 1 & -ba\\ 0 & 0 & 1_\star\end{array}\right)\right] = \left[\begin{array}{c|cc}1 & a & a-aba\\ \hline b & 1 & 0\\ b & 0 & 1_\star\end{array}\right].$$

But $ab = 0$ and therefore $aba = 0$. Repeating the same reasoning, we get

$$[[r]]^{\diamond N} \equiv [[q]] \equiv \left[\left[\begin{array}{c|ccc}1_m & a & \dots & a\\ \hline b & 1 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ b & 0 & \dots & 1_\star\end{array}\right]\right]. \tag{4.8}$$

Step 3. Next, we set $N = p^\mu$ in formula (4.8). Consider the following block matrix u of size p^μ,

$$u := \begin{pmatrix}1 & & & & \\ -1 & 1 & & & \\ 0 & -1 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \dots & 1\end{pmatrix}, \qquad u^{-1} := \begin{pmatrix}1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & 1 & 1 & \dots & 1\end{pmatrix}.$$

We conjugate the matrix q defined by (4.8) as

$$\begin{pmatrix}1 & 0\\ 0 & u_\star\end{pmatrix} q \begin{pmatrix}1 & 0\\ 0 & u_\star^{-1}\end{pmatrix}.$$

We have

$$u\begin{pmatrix}b\\ b\\ b\\ \vdots\end{pmatrix} = \begin{pmatrix}b\\ 0\\ 0\\ \vdots\end{pmatrix}, \qquad \begin{pmatrix}a & a & a & a & \dots\end{pmatrix} u^{-1} = \begin{pmatrix}0 & -a & -2a & -3a & \dots\end{pmatrix},$$

and we get a matrix of the form $X(A, B)$.

5. Idempotents in red($\Gamma(m)$)

Here the main statement is Proposition 5.1, which shows that all idempotents in red($\Gamma(m)$) have representatives in red($\Gamma^\circ(m)$); therefore they have the form $\mathfrak{X}[L, M]$), where $L \subset \ell^m$, $M \subset \ell^m$. The second fact (Proposition 5.3), which is important for the proof below, is a coherence of elements $\mathfrak{X}[L, M]$ in different semigroups red($\Gamma(n)$).

5.1. **Coincidence of idempotents.**

Proposition 5.1. *The following idempotents in* $\mathrm{inv}(\Gamma(m))$ *coincide as elements of* $\mathrm{red}(\Gamma(m))$*:*

$$[[X_\alpha^{\bigcirc}(b,c)]] := \left[\left[X\left(\begin{pmatrix} b & 0\\ 0 & 1_{m-\alpha}\end{pmatrix}, \begin{pmatrix} c & 0\\ 0 & 1_{m-\alpha}\end{pmatrix}\right)\right]\right]$$

$$= \left[\left[\left[\begin{array}{cc|cccc} 1_\alpha & 0 & b & 0 & 0 & 0\\ 0 & 1_{m-\alpha} & 0 & 1_{m-\alpha} & 0 & 0\\ \hline 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ c & 0 & 0 & 0 & 1 & 0\\ 0 & 1_{m-\alpha} & 0 & 0 & 0 & 1_\star \end{array}\right]\right]\right]_{mm},$$

and

$$[[X_\alpha^{\square}(b,c)]] := \left[\left[\left[\begin{array}{cc|ccc} 1_\alpha & 0 & b & 0 & 0\\ 0 & 0 & 0 & 0 & 1_{m-\alpha}\\ \hline 0 & 0 & 1 & 0 & 0\\ c & 0 & 0 & 1 & 0\\ 0 & 1_{m-\alpha} & 0 & 0 & 0_\star \end{array}\right]\right]\right]_{mm}.$$

Corollary 5.2. *Any idempotent in* $\mathrm{red}(\Gamma(m))$ *has the form* $\mathfrak{X}[L,M]$*.*

Proof of corollary. The semigroup $\mathrm{red}\,\Gamma(m))$ is a quotient of $\mathrm{inv}(\Gamma(m))$; the semigroup of idempotents also is a quotient of the semigroup of idempotents. By Proposition 4.10 all idempotents in $\mathrm{inv}(\Gamma(m))$ have $[[X_\alpha^{\bigcirc}[b,c]]$. By Proposition 5.1, they also can be written as $[[X_\alpha^{\square}[b,c]]$. □

Proposition will be proved in Subsection 5.3.

Remarks.

(a) The idempotents $[[X_\alpha^{\bigcirc}(b,c)]]$ and $[[X_\alpha^{\square}(b,c)]]$ are different in $\mathrm{inv}(\Gamma(m))$. Indeed, we have the following homomorphism from $\Gamma(m)$ to the inverse semigroup $\mathrm{End}_{\mathcal{L}}(m)$. On $\Gamma^\circ(m)$ we define it as the map Π described in Subsection 3.5. On the other hand, we send $\Gamma(m)\setminus\Gamma^\circ(m)$ to $\mathbf{0}$, i.e., to a pair of partial bijections with empty domains of definiteness. This map separates our idempotents.

(b) Idempotents $\mathfrak{X}[L,M]$ are pairwise different in $\mathrm{red}(\Gamma(m))$. To verify this, consider the representation of $\mathbb{G}$ in $\ell^2(\mathbb{G}/\mathbb{G}[L;M])$. It is easy to show that $\mathfrak{X}(L,M)$ is the minimal idempotent of $\mathrm{red}(\Gamma(m))$ acting in this representation nontrivially.

5.2. **Coherence.**

Let L, $M\subset \mathfrak{l}^m$ be submodules. Formula (4.4) defines the idempotent $\mathfrak{X}[L,M]=X(b,c)$ as an element of $\Gamma(m)$; recall that $L=\ker b$, $M=\ker c^t$. However, for $n>m$ we can regard L, $M\subset\mathfrak{l}^m$ as submodules L in $\mathfrak{l}^n\supset\mathfrak{l}^m$. In the larger space we have

$$L=\ker\begin{pmatrix} b & 0\\ 0 & 1_{n-m}\end{pmatrix},\qquad M=\ker\begin{pmatrix} c & 0\\ 0 & 1_{n-m}\end{pmatrix}.$$

Consider a unitary representation ρ of $\mathbb{G}$ in a Hilbert space H. For any $n\geqslant m$ we have an operator

$$\widetilde{\rho}_{nn}\left(X\left(\begin{pmatrix} b & 0\\ 0 & 1_{n-m}\end{pmatrix}, \begin{pmatrix} c & 0\\ 0 & 1_{n-m}\end{pmatrix}\right)\right): H_n\to H_n. \tag{5.1}$$

We claim that these operators as operators $H\to H$ depend only on L, M and not on n. Precisely, we have the following statement.

Proposition 5.3.

(a) *Let $n \geqslant m$. Then a block matrix structure of the operator* (5.1) *with respect to the orthogonal decomposition $H_n = H_m \oplus (H_n \ominus H_m)$ is*

$$\widetilde{\rho}_{nn}\left(X\left(\begin{pmatrix} b & 0\\ 0 & 1_{n-m}\end{pmatrix}, \begin{pmatrix} c & 0\\ 0 & 1_{n-m}\end{pmatrix}\right)\right) = \begin{pmatrix} \widetilde{\rho}_{mm}(X(b,c)) & 0\\ 0 & 0\end{pmatrix}. \tag{5.2}$$

(b) *For any L, $M \subset \mathfrak{l}^m$ we have a well-defined operator $\widetilde{\rho}(\mathfrak{X}[L,M])$ in H, which sends H_m to H_m as $\widetilde{\rho}_{mm}(\mathfrak{X}[L,M])$ and is zero on the orthocomplement $H \ominus H_m$.*

Proof. According Corollary 4.3, the right hand side of (5.2) is $\widetilde{\rho}_{nn}(X^{\square}_m(b,c))$. By Proposition 5.1, this operator coincides with $\widetilde{\rho}_{nn}(X^{\bigcirc}_m(b,c))$. ☐

5.3. **Proof of Proposition 5.1.**

Lemma 5.4. *Let $\mathfrak{g} \in \mathrm{red}(\Gamma(m))$ be an idempotent. Let g be a representative of $\mathfrak{g}$ in $\mathbb{G}_{\mathrm{fin}}$. Then for any unitary representation ρ of $\mathbb{G}$ in a Hilbert space H the image of the orthogonal projector $\widetilde{\rho}_{mm}(\mathfrak{g})$ coincides with the space of fixed points of the subgroup in $\mathbb{G}$ generated by $\mathbb{G}(m)$ and g.*

Proof. Let $v \in \mathrm{im}\,\widetilde{\rho}_{mm}(\mathfrak{g})$, i.e.,

$$P_m \rho(g) P_m v = v.$$

This happens if and only if $P_m v = v$, $\rho(g)v = v$. The condition $P_m v = v$ means that $\rho(h)v = v$ for all $h \in \mathbb{G}(m)$. ☐

Therefore, it is sufficient to show that the group generated by $\mathbb{G}(m)$ and $X^{\bigcirc}(b,c)$ coincides with the group generated by $\mathbb{G}(m)$ and $X^{\square}(b,c)$.

Lemma 5.5. *The group generated by the subgroup $\mathbb{G}(\beta)$ and the matrix*

$$X(1,1) = \left(\begin{array}{c|cc} 1_\beta & 1_\beta & 0\\ \hline 0 & 1_\beta & 0\\ 1_\beta & 0 & 1_{\beta\star}\end{array}\right)$$

coincides with $\mathbb{G}$.

Proof. Denote by G the group generated by $X(1,1)$ and $\mathbb{G}(\beta)$. Conjugating $X(1,1)$ by block diagonal matrices we can get any matrix of the form $X(A,B)$ with non-degenerate A, B. Multiplying such matrices we observe that elements of the form $X(A_1 + A_2, B_1 + B_2)$ are contained in G. In particular, $X(0,2) \in G$. Since $p \neq 2$, conjugating $X(0,2)$ by a block scalar matrix we come to $X(0,1) \in G$. In the same way $X(1,0) \in G$. Now the statement became more-or-less obvious. ☐

Lemma 5.6. *The group generated by $\mathbb{G}(\beta)$ and the matrix*

$$\left(\begin{array}{c|c} 0 & 1_\beta\\ \hline 1_\beta & 0_\star\end{array}\right) \tag{5.3}$$

coincides with $\mathbb{G}$.

Proof. Denote this group by G. Denote $S_\infty(\beta) := S_\infty \cap \mathbb{G}(\beta)$. Multiplying the matrix (5.3) from the left and right by elements of $S_\infty(\beta)$ we can get an arbitrary matrix of the form $\left(\begin{array}{c|c} 0 & \sigma_1\\ \hline \sigma_2 & 0_\star\end{array}\right)$ with σ_1, $\sigma_2 \in S_\beta$. Multiplying two matrices of this type we can get any matrix $\left(\begin{array}{c|c} \sigma & 1\\ \hline 0 & 1_\star\end{array}\right)$, where $\sigma \in S_\beta$. Therefore our group contains the subgroup $S_\beta \times S_\infty(\beta)$, which is maximal in S_∞. Therefore $G \supset S_\infty$. But S_∞ and $\mathbb{G}(\beta)$ generate $\mathbb{G}$; see [27, Lemma 3.6]. □

Proof of Proposition 5.1. Denote by

— $G^{\bigcirc}$ the group generated by $\mathbb{G}(m)$ and $X^{\bigcirc}_\alpha(b,c)$;

— $G^{\square}$ the group generated by $\mathbb{G}(m)$ and $X^{\square}_\alpha(b,c)$;

— G the group generated by $\mathbb{G}(\alpha)$ and the matrix $X_\diamond(b,c)$ defined by

$$X_\diamond(b,c) := \left(\begin{array}{cc|cc} 1_\alpha & 0 & b & 0 \\ 0 & 1_{m-\alpha} & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ c & 0 & 0 & 1_\star \end{array}\right).$$

Obviously, $G \supset G^{\bigcirc}$, $G \supset G_{\square}$. Let us verify the opposite inclusions.

The inclusion $G^{\bigcirc} \supset G$. Clearly $X_\alpha(-b,-c) \in G^{\bigcirc}$. Therefore $G^{\bigcirc}$ contains

$$X_\alpha(b,c)X_\alpha(-b,-c) = X\left(\begin{pmatrix}0&0\\0&2\end{pmatrix}, \begin{pmatrix}0&0\\0&2\end{pmatrix}\right) \sim X\left(\begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right) =: Y.$$

By Lemma 5.5, the group generated by Y and $\mathbb{G}(m)$ is $\mathbb{G}(\alpha)$. On the other hand, $Y^{-1}X^{\bigcirc}(b,c) \sim X_\diamond(b,c)$.

5.3.1. *The inclusion $G^{\square} \supset G$.* We have

$$X^{\square}_\alpha(b,c)^2 \sim X_\diamond(2b,2c) \sim X_\diamond(b,c).$$

Next, $X_\diamond(b,c)^{-1}X^{\square}_\alpha(b,c) \sim X_\alpha(0,0)$ and we refer to Lemma 5.6.

Thus, $G^{\bigcirc} = G^{\square}$. By Lemma 5.4, for any unitary representation ρ of $\mathbb{G}$ we have

$$\widetilde{\rho}_{mm}\big(X^{\bigcirc}(b,c))\big) = \widetilde{\rho}_{mm}\big(X^{\square}(b,c))\big)$$

and this completes the proof of Proposition 5.1. □

6. The semigroup $\mathrm{red}_m(\Gamma(m))$

6.1. **Structure of the semigroup** $\mathrm{red}_m(\Gamma(m))$**.** Denote by $\mathbf{0}$ the minimal idempotent of the semigroup $\mathrm{red}_m(\Gamma(m))$.

Proposition 6.1. *Any element $\neq \mathbf{0}$ in $\mathrm{red}_m(\Gamma(m))$ has a representative of a form $aX(b,c)$, where $a \in \mathrm{GL}(m)$.*

The proof occupies the rest of the section. As a byproduct of Lemma 6.3 we will get the following statement.

Lemma 6.2. *Any idempotent $[X(b,c)]$ by a conjugation by $a \in \mathrm{GL}(m)$ can be reduced to a form*

$$\left[X\left(\begin{pmatrix}0&\beta\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\\gamma&0\end{pmatrix}\right)\right],$$

where $\gamma\beta = 0 (\mathrm{mod}\ p)$, $\beta\gamma = 0 (\mathrm{mod}\ p)$.

6.2. **Proof of Proposition 6.1.**

Step 1.

Lemma 6.3.

(a) *Let* B *be an* $m \times N$ *matrix over* $\mathbb{Z}_{p^\mu}$, C *an* $N \times m$ *matrix. Then transformations*

$$B \mapsto u^{-1}Bv, \qquad C \mapsto v^{-1}Cu$$

allow to reduce them to the form

$$\widetilde{B} = \begin{pmatrix} 0 & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \qquad \widetilde{C} = \begin{pmatrix} 0 & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \tag{6.1}$$

where b_{12}, c_{21} *are square nondegenerate matrices of the same size, products* $b_{21}c_{12}$, $c_{12}b_{21}$ *are nilpotent and* $b_{22} = 0 (\text{mod } p)$, $c_{22} = 0 (\text{mod } p)$.

(b) *The transformations*

$$B \mapsto u^{-1}Bv, \qquad C \mapsto w^{-1}Cu,$$

where u, v, w *are invertible, allow to reduce a pair* (B, C) *to the form*

$$\widetilde{B} = \begin{pmatrix} 0 & b_{12} \\ 1 & 0 \end{pmatrix}, \qquad \widetilde{C} = \begin{pmatrix} 0 & 1 \\ c_{21} & 0 \end{pmatrix},$$

where $c_{21}b_{12} = 0 (\text{mod } p)$, $b_{12}c_{21} = 0 (\text{mod } p)$.

Proof. (a) Reduce our matrices modulo p. A canonical form of a pair of counter operators $P : \mathbb{F}_p^m \to \mathbb{F}_p^N$ and $Q : \mathbb{F}_p^N \to \mathbb{F}_p^m$ is a standard problem of linear algebra; see, e.g., [7], [11]. In particular, such operators in some bases admit block decompositions $P = \begin{pmatrix} P_r & 0 \\ 0 & P_n \end{pmatrix}$, $Q = \begin{pmatrix} Q_r & 0 \\ 0 & Q_n \end{pmatrix}$, where P_rQ_r, Q_rP_r are nondegenerate and P_nQ_n, Q_nP_n are nilpotent.

Thus the matrices B, C can be reduced to the form

$$B' = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \qquad C' = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$

where

(1) b_{21}, c_{12} are invertible matrices of the same size;

(2) products $b_{12}c_{21}$, $c_{21}b_{12}$ are nilpotent;

(3) the matrices b_{11}, b_{22}, c_{11}, c_{22} reduced (mod p) are zero.

Set

$$u_1 := \begin{pmatrix} 1 & b_{11}b_{12}^{-1} \\ 0 & 1 \end{pmatrix},$$

notice that u_1 (mod p) is 1. We pass to new matrices

$$B'' = u_1^{-1}B', \qquad C'' = C''u_1.$$

For new B the block $b_{11} = 0$; other properties (1)–(3) of matrices B, C are preserved. Next, we take a unique matrix of the form $u_2 = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$ such that $C''u_2$ has zero block c_{12}. On the other hand the block b_{11} of $u_2^{-1}B''$ is zero. We come to a desired form.

(b) We apply statement (a) and reduce (B, C) to the form (6.1). Next, we multiply $\widetilde{B}$ from right by $\begin{pmatrix} b_{21} & \\ & 1 \end{pmatrix}^{-1}$ and get 1 on the place of b_{21}. After this, we multiply new B from right by $\begin{pmatrix}1 & -b_{22}\\ 0 & 1\end{pmatrix}$ and kill b_{22}. Finally, we repeat the same transformations with $\widetilde{C}$.

Now the problem is reduced to the same question for a pair b_{12}, c_{21}. If $c_{21}b_{12} \neq 0 (\mathrm{mod}\ p)$, then we choose an invertible matrix U such that $b_{12}Uc_{21}$ is not nilpotent and again repeat (a). Etc. □

Step 2.

Lemma 6.4. *Let* $[g] \in \Gamma(m)$ *have the form*

$$[g] = \left[\begin{array}{c|c} 1 & b \\ \hline c & 1_\star \end{array}\right]_{mm}$$

and $[[g]] \not\approx_m \mathbf{0}$. *Then* bc *and* cb *are nilpotent.*

Proof. We apply the previous lemma and represent $[g]$ as

$$[g] = \left[\begin{array}{cc|cc} 1_\alpha & 0 & 0 & b_{12} \\ 0 & 1_{m-\alpha} & b_{21} & b_{22} \\ \hline 0 & c_{12} & 1_{m-\alpha} & 0 \\ c_{21} & c_{22} & 0 & 1_\star \end{array}\right]_{mm}.$$

Set

$$[h_m^\alpha] := \left[\begin{array}{cc|cc} 1_\alpha & 0 & 0 & 0 \\ 0 & 1_{m-\alpha} & 1_{m-\alpha} & 0 \\ \hline 0 & 0 & 1_{m-\alpha} & 0 \\ 0 & 1_{m-\alpha} & 0 & 1_{m-\alpha\star} \end{array}\right].$$

Let us show that

$$[g] \circ [h_m^\alpha] \sim [g]. \tag{6.2}$$

Indeed,

$$[g] \circ [h_m^\alpha] = \left[\begin{array}{cc|cccc} 1_\alpha & 0 & 0 & b_{12} & 0 & 0 \\ 0 & 1_{m-\alpha} & b_{21} & b_{22} & 1 & 0 \\ \hline 0 & c_{12} & 1_{m-\alpha} & 0 & c_{12} & 0 \\ c_{21} & c_{22} & 0 & 1 & c_{22} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1_\star \end{array}\right]_{mm} \tag{6.3}$$

$$\sim \left[\begin{array}{cc|cccc} 1_\alpha & 0 & 0 & b_{12} & 0 & 0 \\ 0 & 1_{m-\alpha} & b_{21} & b_{22} & \boxed{1_{m-\alpha}} & 0 \\ \hline 0 & c_{12} & 1_{m-\alpha} & 0 & 0 & 0 \\ c_{21} & c_{22} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \boxed{1_{m-\alpha}} & 0 & 0 & 0 & 1_\star \end{array}\right]_{mm} =: r,$$

to establish the equivalence we multiply $[g] \circ [h_m^\alpha]$ from the left by

$$\left(\begin{array}{c|cccc} 1_m & & & & \\ \hline & 1 & 0 & -c_{12} & \\ & 0 & 1 & -c_{22} & \\ & 0 & 0 & 1 & \\ & & & & 1_\star \end{array}\right).$$

Next, denote

$$v_1 := \left(\begin{array}{c|cccc} 1_m & & & & \\ \hline & 1 & 0 & -b_{21}^{-1} & \\ & & 1 & 0 & \\ & & & 1 & \\ & & & & 1_\star \end{array}\right), \qquad v_2 := \left(\begin{array}{c|cccc} 1_m & & & & \\ \hline & 1 & & & \\ & & 1 & & \\ & & 0 & 1 & \\ & & -c_{12}^{-1} & 0 & 1_\star \end{array}\right).$$

We have

$$[r] \sim [v_2 v_1^{-1} r v_1 v_2^{-1}],$$

the latter matrix is obtained from r, see (6.3), by removing two boxed blocks $\boxed{1_{m-\alpha}}$; all other blocks are the same. Thus $[r] \sim [g]$, i.e., we established (6.2).

Suppose that $\alpha \neq m$. Then by Proposition 5.1,

$$[g] \sim [g] \circ [h_m^\alpha] \approx [g] \circ \Theta_{[m]}^\alpha.$$

But $\Theta_{[m]}^\alpha \approx_m \mathbf{0}$; therefore $[[g]] \cong_m \mathbf{0}$. □

Step 3. Thus it is sufficient to prove Proposition 6.1 for $[g]$ having the form

$$[g] = \left[\begin{array}{c|c} 1 & b \\ \hline c & 1_\star \end{array}\right]_{mm}, \qquad \text{where } bc,\ cb \text{ are nilpotent.}$$

Lemma 6.5. *Let* $[g] = \left[\begin{array}{c|c} 1 & b \\ \hline c & 1_\star \end{array}\right]_{mm}$ *be invertible.*[10] *Then*

$$[g^{-1}] \circ [g] \equiv X(b, c).$$

Proof. By (3.1),

$$[g] \circ ([g^{-1}] \circ [g]) \equiv [g], \qquad \text{and } [g^{-1}] \circ [g] \text{ is an idempotent.}$$

We have (see, e.g., [9, Sect. 2.5])

$$[g^{-1}] = \left[\begin{array}{c|c} (1-bc)^{-1} & -(1-bc)^{-1}b \\ \hline -c(1-bc)^{-1} & (1-cb)_\star^{-1} \end{array}\right]_{mm} \sim \left[\begin{array}{c|c} (1-bc)^{-1} & b \\ \hline c(1-bc)^{-1} & 1_\star \end{array}\right]_{mm}.$$

We also keep in mind the identity

$$c(1-bc)^{-1} = (1-cb)^{-1}c, \tag{6.4}$$

to establish it, we multiply both sides from the left by $(1 - cb)$ and from the right by $(1 - bc)$.

Next,

$$[g^{-1}] \circ [g] = \left[\begin{array}{c|cc} (1-bc)^{-1} & b & (1-bc)^{-1}b \\ \hline c(1-bc)^{-1} & 1 & c(1-bc)^{-1}b \\ c & 0 & 1_\star \end{array}\right]_{mm} \sim \left[\begin{array}{c|cc} (1-bc)^{-1} & b & b \\ \hline c & 1 & 0 \\ c(1-bc)^{-1} & 0 & 1_\star \end{array}\right]_{mm}.$$

This matrix defines an idempotent in $\mathrm{inv}(\Gamma^\circ(m))$. We must verify the following statement: □

Lemma 6.6. *Under our conditions,*

$$[g^{-1}] \circ [g] \equiv X(b, c).$$

[10] This is equivalent to invertibility of $(1 - bc)^{-1}$ or invertibility of $(1 - cb)^{-1}$. Here we do not need a nilpotency of bc.

Proof. By Corollary 4.9 we can identify an idempotent in $\mathrm{inv}(\Gamma^\circ)$ evaluating its image in $\mathrm{Mor}_{\mathcal{L}}(m)$. So we get

$$[g]\circ[g^{-1}] \equiv [[X(B,C)]],$$

where

$$B := \begin{pmatrix} b & b \end{pmatrix}, \qquad C := \begin{pmatrix} c \\ (1-cb)^{-1}c \end{pmatrix}.$$

We have $\ker B = \ker b$, $\ker C^t = \ker c^t$; therefore by Lemma 4.6 we have $[[X(B,C)]] \equiv [[X(b,c)]]$. □

Corollary 6.7. *Let*

$$[g] = \left[\begin{array}{c|c} 1 & b \\ \hline c & 1_\star \end{array}\right]_{mm}, \qquad [g'] = \left[\begin{array}{c|c} 1 & bu \\ \hline c & 1_\star \end{array}\right]_{mm}$$

be invertible and u also be invertible. Then

$$[g^{-1}]\circ[g] \equiv [(g')^{-1}]\circ[g'].$$

Proof. Indeed, $\ker bu = \ker b$. So both sides are $[[X(b,c)]]$. □

Step 4.

Lemma 6.8. *Let* $[g] = \left[\begin{array}{c|c} 1 & b \\ \hline c & 1_\star \end{array}\right]_{mm}$*; let bc and cb be nilpotent. Then there exists u having the form*

$$u = -\frac{1}{2} + \sum_{j>0} \frac{\sigma_j}{2^{n_j}}(cb)^j, \qquad \text{where } \sigma_j \in \mathbb{Z},\ n_j \in \mathbb{Z}_+, \tag{6.5}$$

such that

$$\left(\left[\left(\begin{array}{c|c} 1 & bu \\ \hline c & 1_\star \end{array}\right)^{-1}\right]_{mm} \circ \left[\begin{array}{c|c} 1 & bu \\ \hline c & 1_\star \end{array}\right]_{mm}\right) \circ [g^{-1}] \tag{6.6}$$

$$\equiv \left[\begin{array}{c|cc} (1-buc)^{-1}(1-bc)^{-1} & b & 0 \\ \hline 0 & 1 & 0 \\ c & 0 & 1_\star \end{array}\right]_{mm}.$$

Proof. The product is

$$\left[\begin{array}{c|ccc} (1-buc)^{-1}(1-bc)^{-1} & bu & bu & (1-buc)^{-1}b \\ \hline c(1-bc)^{-1} & 1 & 0 & cb \\ c(1-buc)^{-1} & 0 & 1 & c(1-buc)^{-1} \\ c(1-bc)^{-1} & 0 & 0 & 1_\star \end{array}\right]_{mm}$$

$$\sim \left[\begin{array}{c|ccc} (1-buc)^{-1}(1-bc)^{-1} & bu & bu & (1-buc)^{-1}b \\ \hline c & 1 & 0 & 0 \\ c(1-buc)^{-1} & 0 & 1 & 0 \\ c(1-bc)^{-1} & 0 & 0 & 1_\star \end{array}\right]_{mm} =: \left[\begin{array}{c|c} A & br \\ \hline qc & 1_\star \end{array}\right]_{mm},$$

here

$$r := \begin{pmatrix} u & u & (1-cbu)^{-1} \end{pmatrix}, \qquad q := \begin{pmatrix} 1 \\ (1-cbu)^{-1} \\ (1-cb)^{-1} \end{pmatrix}.$$

We claim that *there exists a unique u such that $rq = 0$*. A straightforward calcula-

tion shows that
$$rq = 2u - ucbu + (1-cb)^{-1}.$$
Since cb is nilpotent, we can write the equation $rq = 0$ as
$$2u + 1 = ucbu - \sum_{j>0} (cb)^j,$$
the sum actually is finite. Clearly we can find a solution in the form $u = -1/2 + \sum_{j>0} s_j (cb)^j$, where s_j are dyadic rationals; for coefficients s_j we have a system of recurrent equations. This u is invertible (since we can write a finite series for u^{-1}).

Next, we must show that the matrix $\left(\begin{array}{c|c} 1 & bu \\ \hline c & 1_\star \end{array}\right)$ is invertible. Indeed, this is equivalent to existence of $(1-cbu)^{-1}$ and this is clear since by (6.5) cbu is nilpotent.

Next we wish to simplify the matrix $\left(\begin{array}{c|c} A & br \\ \hline qc & 1_\star \end{array}\right)$ by conjugations by matrices of the form $\begin{pmatrix} 1 & 0 \\ 0 & D_\star \end{pmatrix}$. In fact, we have transformations
$$r \mapsto r' = pD^{-1}, \qquad q \mapsto q' = Dq.$$
For such transformations we have $r'q' = rq$. Set
$$D = \begin{pmatrix} 1 & 1 & u^{-1}(1-cbu)^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1_\star \end{pmatrix}.$$
Then $r' = \begin{pmatrix} u & 0 & 0 \end{pmatrix}$. But u is invertible and $r'q' = 0$. Therefore q' has the form $\begin{pmatrix} 0 \\ * \\ * \end{pmatrix}$; on the other hand multiplication $q \mapsto Dq$ does not change the second and third elements of the column q. Thus we came to the matrix
$$R := \left[\begin{array}{c|ccc} (1-buc)^{-1}(1-bc)^{-1} & b\boxed{u} & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \boxed{(1-cbu)^{-1}}c & 0 & 1 & 0 \\ \boxed{(1-cb)^{-1}}c & 0 & 0 & 1_\star \end{array}\right]_{mm}.$$
Consider the following matrices:
$$S := \left[\begin{array}{c|ccc} 1 & & & \\ \hline & u & & \\ & & 1-cbu & \\ & & & (1-cb)_\star \end{array}\right], \qquad T := \left[\begin{array}{c|ccc} 1 & & & \\ \hline & 1 & & \\ & & 1 & \\ & & -1 & 1_\star \end{array}\right].$$
The conjugation $R \mapsto TRT^{-1}$ kills boxed elements of R. The conjugation $R \mapsto STRT^{-1}S^{-1}$ reduces the matrix to the desired form. □

Proof. Proof of Proposition 6.1 Thus we have
$$\left[\left(\begin{array}{c|c} 1 & b \\ \hline c & 1_\star \end{array}\right)^{-1}\right]_{mm} \equiv (1-buc)^{-1}(1-bc)^{-1} \cdot [[X\big((1-bc)(1-buc)b, c\big)]].$$
□

The second factor is

$$[[X\big(b(1-cb)(1-cbu),c\big)]]\equiv[[X(b,c)]].$$

Passing to adjoint elements we get

$$\begin{aligned}\left[\begin{array}{c|c}1&b\\\hline c&1\end{array}\right]_{mm}&\equiv[[X(b,c)]]\cdot(1-bc)(1-buc)\\&\equiv(1-bc)(1-buc)\cdot[[X((1-buc)^{-1}(1-bc)^{-1}b,c(1-bc)(1-buc))]]\\&\equiv(1-bc)(1-buc)\cdot[[X(b,c)]].\end{aligned}$$

It remains to notice that

$$\left[\begin{array}{c|c}a&b\\\hline c&1\end{array}\right]=a\cdot\left[\begin{array}{c|c}1&a^{-1}b\\\hline c&1\end{array}\right].$$

6.3. **Proof of Lemma 6.2.** We refer to Lemma 6.3.

7. The groups $\mathbb{G}^\bullet[L;M]$

In this section we examine subgroups $\mathbb{G}^\circ[L;M]$, $\mathbb{G}^\bullet[L;M]\subset\mathbb{G}$ defined in Subsection 1.3. We prove that $\mathbb{G}^\bullet[L;M]$ is well-defined. Lemma 7.5 shows that it is generated by $\mathbb{G}(m)$ and the element $X(b,c)$. Also we prove that it is a minimal subgroup of finite index in $\mathbb{G}[L;M]$ (equivalently, $\mathbb{G}^\bullet[L;M]$ has no subgroups of finite index, Proposition 7.11).

7.1. Several remarks on submodules in $\mathfrak{l}^k$.

Lemma 7.1. *Let $L\subset\mathfrak{l}^k$ be a submodule. Then there exists a basis $e_j\in\mathfrak{l}^k$ such that $M:=\oplus p^{s_j}\mathbb{Z}_{p^\mu}e_j$. The collection s_1, s_2, …is a unique* $\mathrm{GL}(m)$*-invariant of a submodule L.*

This is equivalent to a classification of sublattices in $(\mathbb{O}_p)^k$ under the action of $\mathrm{GL}(k,\mathbb{O}_p)$ or equivalently to a classification of pairs of lattices in $\mathbb{Q}_p^k$ under $\mathrm{GL}(k,\mathbb{Q}_p)$; the latter question is standard; see, e.g., [35, Theorem I.2.2].

Corollary 7.2. *Any submodule $L\subset\mathfrak{l}^k$ is a kernel of some endomorphism $\mathfrak{l}^k\to\mathfrak{l}^k$.*

Indeed, we pass to a canonical basis e_j as in the lemma and consider the map sending e_j to $p^{\mu-s_j}e_j$.

Lemma 7.3.

(a) *Let L be a submodule in $\mathfrak{l}^m$. Let b, $b':\mathfrak{l}^m\to\mathfrak{l}^N$ be morphisms of modules such that $L=\ker b=\ker b'$. Then there is a transformation $u\in\mathrm{GL}(N)$ such that $b'=bu$.*

(b) *Let $\ker b=L$, $\ker b'=L'\supset L$. Then there is an endomorphism $u:\mathfrak{l}^N\to\mathfrak{l}^N$ such that $b'=bu$.*

Proof. (a) The modules $\mathrm{im}\,b\simeq\mathrm{im}\,b'\simeq\mathfrak{l}^m/L$ are isomorphic. By the previous lemma there is an automorphism of $\mathfrak{l}^N$ identifying these submodules.

(b) L is a submodule of L'; therefore $\mathrm{im}\,b'$ is a quotient module of $\mathrm{im}\,b$. Therefore there is a projection map $\pi:\mathrm{im}\,b\to\mathrm{im}\,b'$; orders of elements do not increase under this map. By Lemma 7.1 we have a basis $e_j\in\mathfrak{l}^N$ such that $p^{s_j}e_j$, where $j=1$, …, m, is the system of generators of $\mathrm{im}\,b$. Choose arbitrary vectors v_j such that $p^{s_j}v_j=\pi(p^{s_j}e_j)$ and consider the map sending e_j to v_j. □

7.2. **The group $\mathbb{G}^\bullet$.** Here we show that $\mathbb{G}^\bullet[L;M]$ is a group, and its definition does not depend on the choice of matrices b, c.

Lemma 7.4.

(a) *Fix a matrix B of size $l \times N$. Then the set of invertible matrices g of the form $1 - BS$, where S ranges in the set of $N \times l$ matrices, is a group.*

(b) *Fix matrices B, C of sizes $l \times N$ and $N \times l$ respectively. Then the set of invertible matrices g of the form $g = 1 - BuC$ is a group.*

Proof. Clearly, both sets are closed with respect to multiplication. We must show that g^{-1} satisfies the same property. In the first case,

$$1 - g^{-1} = 1 - (1 - BS)^{-1} = -BS(1 - BS)^{-1}.$$

In the second case,

$$1 - g^{-1} = 1 - (1 - BuC)^{-1} = -BuC(1 - BuC)^{-1} = -Bu(1 - CBu)^{-1}C.$$

□

Lemma 7.5. *Fix matrices b, c of sizes $m \times N$ and $N \times m$ respectively.*

(a) *The set of invertible matrices $g = \begin{pmatrix} a & bv \\ wc & z \end{pmatrix}$ such that the block 'a' admits representations $a = 1 - bS$, $a = 1 - Tc$ is a group.*

(b) *The set $\mathbb{G}^\bullet[L;M]$, i.e., the set of all invertible matrices of the form $g = \begin{pmatrix} 1 - buc & bv \\ wc & z \end{pmatrix}$, is a group.*

Proof. In the first case we write

$$g = \begin{pmatrix} 1 - bS & bv \\ wc & z \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} - \begin{pmatrix} bS & -bv \\ -wc & 1 - z \end{pmatrix}$$
$$= \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} - \begin{pmatrix} b & \\ & 1 \end{pmatrix} \begin{pmatrix} S & -v \\ -wc & 1 - z \end{pmatrix},$$

and reduce the statement to the previous lemma.

In the second case we write

$$g = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} - \begin{pmatrix} b & \\ & 1 \end{pmatrix} \begin{pmatrix} u & -v \\ -w & 1 - z \end{pmatrix} \begin{pmatrix} c & \\ & 1 \end{pmatrix},$$

and again we apply the previous lemma. □

7.3. **The group $\mathbb{G}^\circ[L;M]$.**

Proof of Lemma 1.3. Let $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta_\star \end{pmatrix} \in \mathbb{G}^\circ[L,M]$, i.e., g fix pointwise $L \subset \ell^m$ and g^t fix pointwise of $M \subset \ell^m$. Then $L \subset \ker \beta$ and by Lemma 7.3(b) we have $\beta = bv$ for some matrix v. Also $L \subset \ker(1 - \alpha)$ and therefore $\alpha = 1 - bS$ for some S. □

7.4. **Changes of coordinates.**

Lemma 7.6. *Let L, $M \subset \ell^m$. Let $a \in \mathrm{GL}(m)$. Then*

$$a\mathbb{G}^\circ[L;M]a^{-1} = \mathbb{G}^\circ[aL, (a^t)^{-1}M], \qquad a\mathbb{G}^\bullet[L;M]a^{-1} = \mathbb{G}^\bullet[aL, (a^t)^{-1}M].$$

The first statement is an immediate consequence of the definition; the second is straightforward.

7.5. **Generators of $\mathbb{G}^\bullet[L;M]$.** Let m, b, c be as in Subsection 1.3, i.e., $L = \ker b$, $M = \ker c^t \subset \mathfrak{l}^m$.

Proposition 7.7. *The group $\mathbb{G}^\bullet[L;M]$ is generated by $\mathbb{G}(m)$ and the matrix $X(b,c)$.*

Proof. Consider the group G generated by $\mathbb{G}(m)$ and $X(b,c)$. Clearly, $\mathbb{G}^\bullet[L,M] \supset G$. Let us prove the converse.

(1) Conjugating $X(b,c)$ by block diagonal matrices $\in \mathbb{G}(m)$ we get arbitrary matrices of the form $X(bv, wc)$, where v, w are invertible matrices. Consider products

$$X(bv,wc)\,X(b'v,wc') = X((b+b')v, w(c+c')). \tag{7.1}$$

We set $b = -b'$; for any matrix σ we can find invertible matrices c, c' such that[11] $c + c' = \sigma$. Thus G contains all matrices of the form

$$\begin{pmatrix} 1_m & bv \\ & 1_{m\star} \end{pmatrix}, \qquad \begin{pmatrix} 1_m & \\ wc & 1_{m\star} \end{pmatrix}, \tag{7.2}$$

where v, w are arbitrary matrices.

(2) In virtue of Lemma 6.2, conjugating the matrices (7.2) by elements of $\mathrm{GL}(m)$ and multiplying from the left and the right by elements of $\mathbb{G}(m)$ we can reduce the matrices (7.2) to the forms

$$Y[\beta] := \begin{pmatrix} 1_{m-\alpha} & 0 & 0 & \beta \\ 0 & 1_\alpha & \boxed{1_\alpha} & 0 \\ 0 & 0 & 1_\alpha & 0 \\ 0 & 0 & 0 & 1_{m-\alpha\star} \end{pmatrix}, Z[\gamma] := \begin{pmatrix} 1_{m-\alpha} & 0 & 0 & 0 \\ 0 & 1_\alpha & 0 & 0 \\ 0 & \boxed{1_\alpha} & 1_\alpha & 0 \\ \gamma & 0 & 0 & 1_{m-\alpha\star} \end{pmatrix}, \tag{7.3}$$

where $\gamma\beta = 0 (\mathrm{mod}\ p)$, $\beta\gamma = 0 (\mathrm{mod}\ p)$. Multiplying $Y[\beta]$ from right by elements of $\mathbb{G}(m+\alpha)$ we can get any matrix $Y[\beta r]$ with invertible r. The condition of invertibility of r can be removed, because

$$Y[\beta r_1]\,Y[\beta r_2]^{-1}\,Y[\beta r_3] = Y[\beta(r_1 - r_2 + r_3)],$$

and we can represent any matrix r as a sum of 3 invertible matrices.

In the same way we get that G contains all elements of the form $Z[q\gamma]$.

Take $r = 0$, $q = 0$. Then the matrices $Y[0] = Y[\beta \cdot 0]$, $Z[0] = Z[0 \cdot \gamma]$ together with $\mathbb{G}(m)$ generate the group $\mathbb{G}(m-\alpha)$.

Next, G contains matrices $Y[\beta]Y[0]^{-1}$ and $Z[\gamma]Z[0]^{-1}$. They are matrices of the form (7.3), where boxed blocks are replaced by zeroes.

Therefore our problem is reduced to a description of the subgroup generated by $\mathbb{G}(m-\alpha)$ and $X(\beta,\gamma)$.

Thus, without loss of generality, we can assume that $\alpha = 0$ and $cb = 0 (\mathrm{mod}\ p)$, $bc = 0 (\mathrm{mod}\ p)$.

(3) Multiplying the matrices (7.2), we get

$$\begin{pmatrix} 1 - bvwc & bv \\ wc & 1_\star \end{pmatrix} \in G \quad \text{for any } v,\ w.$$

[11]It is sufficient to verify this statement for matrices over $\mathbb{F}_p$. Without loss of generality we can assume that σ is diagonal. For $p \neq 2$ any element of $\mathbb{F}_p$ is a sum of two nonzero elements, where σ can be represented as a sum of two diagonal matrices.

Since $cb = 0 (\mathrm{mod}\ p)$, the $bvwc$ is nilpotent, and therefore $1 - bvwc$ is invertible. We represent our matrix as

$$\begin{pmatrix} 1 & 0 \\ wc(1-bvwc)^{-1} & 1_\star \end{pmatrix} \begin{pmatrix} 1-bvwc & 0 \\ 0 & 1_\star \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 - wc(1-bvwc)^{-1}bv_\star \end{pmatrix} \begin{pmatrix} 1 & (1-bvwc)^{-1}bv \\ 0 & 1_\star \end{pmatrix}.$$

Since the whole product and three factors are contained in G, the fourth factor also is contained in G,

$$\begin{pmatrix} 1-bvwc & 0 \\ 0 & 1_\star \end{pmatrix} \in G$$

for any v, w.

(4) Now consider an arbitrary element of $\mathbb{G}^\bullet[L; M]$,

$$\begin{pmatrix} 1-buc & bv \\ wc & z_\star \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ wc(1-buc)^{-1} & 1_\star \end{pmatrix} \begin{pmatrix} 1-buc & 0 \\ 0 & 1_\star \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & z - wc(1-buc)^{-1}bv_\star \end{pmatrix} \begin{pmatrix} 1 & (1-buc)^{-1}bv \\ 0 & 1_\star \end{pmatrix}.$$

All factors of the right hand side are contained in G, and therefore $\mathbb{G}^\bullet[L; M]$ is contained in G. □

Corollary 7.8. *The group $\mathbb{G}^\bullet[L; M]$ does not depend on a choice of m.*

Proof. Let L, $M \subset \mathfrak{l}^m$; let $L = \ker b$, $M = \ker c^t$. Let us regard L, M as submodules L', M' of $\mathfrak{l}^m \oplus \mathfrak{l}^k$. Then

$$L' = \ker b',\ M' = \ker(c')t, \text{ where } b' = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix},\ c' = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}.$$

Clearly the subgroup G_m generated by $\mathbb{G}(m)$ and $X(b, c)$ and the subgroup G_{m+k} generated by $\mathbb{G}(m+k)$ and $X(b', c')$ coincide. Formally, we must repeat the first two steps of the previous proof. □

7.6. **The quotient $\mathbb{G}^\circ/\mathbb{G}^\bullet$.**

Lemma 7.9. *A group $\mathbb{G}^\bullet[L; M]$ has finite index in $\mathbb{G}^\circ[L; M]$.*

Proof. Without loss of generality we can assume that $cb = 0 (\mathrm{mod}\ p)$, $bc = 0 (\mathrm{mod}\ p)$. Denote by $A^\circ \subset \mathrm{GL}(m)$ the subgroup consisting of matrices a admitting representations $a = 1 - bS$, $a = 1 - Tc$. Notice that $1 - a$ is a nilpotent, since $TcbS = 0 (\mathrm{mod}\ p)$. Therefore a is invertible. Denote by $A^\bullet$ the subgroup consisting of elements of the form $1 - buc$.

The subgroup $A^\bullet$ is normal in A°. Indeed, let $a \in A^\circ$, $a = 1 - bS$, $a^{-1} = 1 - Tc$. Then

$$a(1-buc)a^{-1} = 1 - abuca^{-1} = 1 - (1-bS)buc(1-Tc) = 1 - b(1-Sb)u(1-cT)c.$$

Let $g = \begin{pmatrix} a & bv \\ wc & z \end{pmatrix} \in \mathbb{G}^\circ[L; M]$. Let us show that the map $g \mapsto a$ induces a homomorphism from $\mathbb{G}^\circ[L; M] \to A^\circ/A^\bullet$. Indeed,

$$g_1 g_2 = \begin{pmatrix} a_1 & bv_1 \\ w_1 & z_1 \end{pmatrix} \begin{pmatrix} a_2 & bv_2 \\ w_2 c & z_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + bv_1 w_2 c & * \\ * & * \end{pmatrix}.$$

In the left upper block we have

$$a_1 a_2 (1 + a_2^{-1} a_1^{-1} b v_1 w_2 c).$$

We represent $a_1^{-1} = 1 - bS_1$, $a_2^{-1} = 1 - bS_2$ and get

$$a_1 a_2 \bigl(1 + (1 - bS_2)(1 - bS_1) b v_1 w_2 c\bigr) = a_1 a_2 \bigl\{1 + b(1 - S_2 b)(1 - S_1 b) v_1 w_2 c\bigr\}.$$

The expression in the curly brackets is contained in $A^\bullet$.

Clearly, the kernel of the homomorphism is $\mathbb{G}^\bullet[L, M]$. Thus we have an isomorphism of quotient groups,

$$\mathbb{G}^\circ[L; M]/\mathbb{G}^\bullet[L; M] \simeq A^\circ[L; M]/A^\bullet[L; M].$$

The group on the right-hand side is finite. □

7.7. **Absence of subgroups of finite index.**

Lemma 7.10. *The group $\mathbb{G}$ has not proper open subgroups of finite index.*

Proof. Let P be a proper open subgroup. Then it contains some group $\mathbb{G}(\nu)$. On the other hand $\mathbb{G}$ contains a complete infinite symmetric group S_∞, and S_∞ has no subgroups of finite index. Therefore P contains S_∞. But the subgroup in $\mathbb{G}$ generated by $\mathbb{G}(\nu)$ and S_∞ is the whole group $\mathbb{G}$; see [27, Lemma 3.6]. □

Proposition 7.11. *The subgroup $\mathbb{G}^\bullet[L; M]$ has no proper open subgroups of finite index.*

Proof. Let Q be such subgroup. By the previous lemma, $\mathbb{G}(m)$ has not open subgroups of finite index; we have $Q \supset \mathbb{G}(m)$. Hence Q contains a minimal normal subgroup R containing $\mathbb{G}(m)$. The quotient Q/R is generated by the image ξ of $X(b, c)$; therefore Q/R is a cyclic group. But

$$X(b,c)^2 = X(2b, 2c) = \begin{pmatrix} \begin{array}{c|cc} 1 & & \\ \hline & 1/2 & \\ & & 2_\star \end{array} \end{pmatrix} \begin{pmatrix} \begin{array}{c|cc} 1 & b & 0 \\ \hline 0 & 1 & 0 \\ c & 0 & 1_\star \end{array} \end{pmatrix} \begin{pmatrix} \begin{array}{c|cc} 1 & & \\ \hline & 2 & \\ & & 1/2_\star \end{array} \end{pmatrix}.$$

Since $p \neq 2$ the elements $X(b, c)^2$ and $X(b, c)$ have the same images in Q/R. Therefore the image of $X[b, c]$ is 1. □

Corollary 7.12. *Any subgroup of finite index in $\mathbb{G}[L, M]$ contains $\mathbb{G}^\bullet[L, M]$.*

8. End of the proof

This section contains the end of the proof of Theorem 1.5. We know that all idempotents in semigroups $\mathrm{red}(\Gamma(n))$ have the form $\mathcal{X}[L, M]$, see Corollary 5.2, for different n they can be identified in a natural way; see Proposition 5.3. We also know that any non-zero element of $\mathrm{red}_m(\Gamma(m))$ is a product of an invertible element and an idempotent $\mathcal{X}[L, M]$; see Proposition 6.1. This implies that all irreducible representations of $\mathbb{G}$ are induced from representations τ of groups $\mathbb{G}[L; M]$. Proposition 7.11 implies that such τ must be trivial on $\mathbb{G}^\bullet[L; M]$.

8.1. **A preliminary remark.**

Lemma 8.1. *Consider an irreducible $*$-representation σ of the category $\mathcal{K}$ in a sequence of Hilbert spaces H_j. Let $\xi \in H_m$ be a nonzero vector. Then the matrix element*

$$c(\mathfrak{g}) = \langle \sigma(\mathfrak{g})\xi, \xi\rangle_{H_m}, \qquad \text{where } \mathfrak{g} \text{ ranges in } \mathrm{End}_{\mathcal{K}}(m),$$

determines σ up to equivalence.

This is a general statement on $*$-representations of categories (and a copy of a similar statement for unitary representations of groups); we give a proof for completeness.

Proof. For each $\mathfrak{g} \in \mathrm{Mor}_{\mathcal{K}}(m, \alpha)$ we define a vector

$$\omega_{\mathfrak{g}}^{\alpha} = \sigma(\mathfrak{g})\xi \in H_\alpha.$$

Since σ is irreducible, vectors $\omega_{\mathfrak{g}}^{\alpha}$, where $\mathfrak{g}$ ranges in $\mathrm{Mor}_{\mathcal{K}}(m, \alpha)$, generate the space H_α. Their inner products are determined by the function c:

$$\langle \omega_{\mathfrak{g}_1}^{\alpha}, \omega_{\mathfrak{g}_2}^{\alpha}\rangle_{H_\alpha} = \langle \sigma(\mathfrak{g}_1)\xi, \sigma(\mathfrak{g}_2)\xi\rangle_{H_\alpha} = \langle \sigma(\mathfrak{g}_2^* \circ \mathfrak{g}_1)\xi, \xi\rangle_{H_m} = c(\mathfrak{g}_2^* \circ \mathfrak{g}_1).$$

Next, let $\mathfrak{h} \in \mathrm{Mor}_{\mathcal{K}}(\alpha, \beta)$. Let $\mathfrak{g}$, $\mathfrak{f}$ range respectively in $\mathrm{Mor}_{\mathcal{K}}(m, \alpha)$, $\mathrm{Mor}_{\mathcal{K}}(m, \beta)$. Then

$$\langle \sigma(\mathfrak{h})\omega_{\mathfrak{g}}, \omega_{\mathfrak{f}}\rangle_{H_\beta} = \langle \sigma(\mathfrak{h})\sigma(\mathfrak{g})\xi, \sigma(\mathfrak{f})\xi\rangle_{H_\beta} = \langle \sigma(\mathfrak{f}^* \circ \mathfrak{h} \circ \mathfrak{g})\xi, \xi\rangle_{H_m} = c(\mathfrak{f}^* \circ \mathfrak{h} \circ \mathfrak{g}).$$

Clearly an operator $\sigma(\mathfrak{h})$ is uniquely determined by such inner products. $\square$

8.2. **Representations of the semigroup** $\mathrm{red}_m(\Gamma(m))$**.** Consider an irreducible representation of $\mathcal{K}$ of height m and the corresponding representation λ of the semigroup $\mathrm{End}_{\mathcal{K}}(m)$ in H_m. Recall that τ passes through semigroup $\mathrm{red}_m(\Gamma(m))$. By Proposition 6.1, any nonzero element of the latter semigroup can be represented as $a \cdot \mathcal{X}[L, M]$, where $a \in \mathrm{GL}(m)$. Denote

$$\widehat{\mathbb{G}}_n[L, M] = \mathrm{GL}(n) \cap \widehat{\mathbb{G}}[L, M], \qquad \widehat{\mathbb{G}}_{\mathrm{fin}}[L, M] = \mathbb{G}_{\mathrm{fin}} \cap \widehat{\mathbb{G}}_m[L, M].$$

Lemma 8.2 is a special case of general description of representations of finite inverse semigroups; see, e.g., [10]. However, due to Proposition 6.1 our case is simpler than general inverse semigroups. We show that the representation of $\mathrm{GL}(m)$ in H_m is induced from an irreducible representation of some subgroup $\widehat{\mathbb{G}}_m[L, M]$ and idempotents $\mathcal{X}[N, K]$ act in the induced representation as multiplications by indicator functions of certain sets. Precisely,

Lemma 8.2. *Let $\mathcal{X}[L, M]$ be the minimal idempotent in $\mathrm{red}_m(\Gamma(m))$ such that $\lambda(\mathcal{X}[L, M]) \neq 0$. Then there is an irreducible representation τ_m of $\widehat{\mathbb{G}}_m[L, M]$ in a space V such that H_m can be identified with the space ℓ_2 of V-valued functions on the homogeneous space $\widehat{\mathbb{G}}_m[L, M] \setminus \mathrm{GL}(m)$ and*

(1) *The group $\mathrm{GL}(m)$ acts by transformations of the form*

$$\lambda(g)f(x) = R(g, x)f(xg),$$

and for $q \in \widehat{\mathbb{G}}_m[L, M]$ we have $R(p, x_0) = \tau_m(q)$ (where x_0 denotes the initial point of $\widehat{\mathbb{G}}_m[L, M] \setminus \mathrm{GL}(m)$).

(2) *The semigroup of idempotents acts by multiplications by indicator functions. Namely* $\mathfrak{X}[K,N]$ *acts by multiplication by the function*

$$I_{K,N}(x_0 a) = \begin{cases} 1, & \text{if } K \supset aL,\ N \supset (a^t)^{-1}M; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Consider the image V of the projector $\lambda\big(\mathfrak{X}[L,M]\big)$. The idempotent $\mathfrak{X}[L,M]$ commutes with $\widehat{\mathbb{G}}_m[L,M]$. Indeed, for $q \in \widehat{\mathbb{G}}_m[L,M]$ we have

$$q \cdot \mathfrak{X}[L,M] \cdot q^{-1} = \mathfrak{X}[Lq, M(q^t)^{-1}] = \mathfrak{X}[L,M].$$

Therefore the subspace V is $\widehat{\mathbb{G}}_m[L,M]$-invariant. Denote by τ_m the representation of the group $\widehat{\mathbb{G}}_m[L,M]$ in V. We need Lemma 8.3:

Lemma 8.3. *For any* $\mathfrak{g} \in \mathrm{red}_m(\Gamma(m))$ *we have* $\lambda(\mathfrak{g})V = V$ *or* $\lambda(\mathfrak{g})V \perp V$.

Proof of Lemma 8.3. Let us apply an arbitrary element of $\mathrm{red}_m(\Gamma(m))$ to $v \in V$,

$$\lambda\left(a \cdot \mathfrak{X}[K,N]\right) v = \lambda(a) \cdot \lambda\left(\mathfrak{X}[K,N]\,\mathfrak{X}[L,M]\right) v = \lambda(a) \cdot \lambda\left(\mathfrak{X}[K \cap L, N \cap M]\right) v.$$

We have the following cases:

(1) If $K \not\supset L$ or $N \not\supset M$, then by our choice of $\mathfrak{X}[L,M]$, we have

$$\lambda\left(\mathfrak{X}[K \cap L, N \cap M]\right) = 0.$$

(2) Otherwise we come to $\lambda(a)\lambda\left(\mathfrak{X}[L,M]\right) v = \lambda(a)v$.

(2.1) If $a \in \widehat{\mathbb{G}}_m[L,M]$, we get $\lambda(a)v \in V$.

(2.2) Let $a \notin \widehat{\mathbb{G}}_m[L,M]$. Then

(8.1)
$$\begin{aligned} \lambda\left(\mathfrak{X}[L,M]\right)\lambda(a)\lambda\left(\mathfrak{X}[L,M]\right) v &= \lambda(a)\Big\{\lambda(a^{-1})\lambda\left(\mathfrak{X}[L,M]\right)\lambda(a)\Big\}\lambda\left(\mathfrak{X}[L,M]\right) v \\ &= \lambda(a)\lambda\left(\mathfrak{X}[La, M(a^t)^{-1}]\right)\lambda\left(\mathfrak{X}[L;M]\right) v \\ &= \lambda(a)\lambda\left(\mathfrak{X}[La \cap L, M(a^t)^{-1} \cap M]\right) v = 0. \end{aligned}$$

Since an idempotent $\mathfrak{X}[a^{-1}L \cap L, a^t M \cap M]$ is strictly smaller than $\mathfrak{X}[L,M]$, the $\lambda(\mathfrak{X}[\dots]) = 0$. ◻

End of proof of Lemma 8.2. Thus H_m is an orthogonal direct sum of spaces V_x, where x ranges in the homogeneous space $\widehat{\mathbb{G}}_m[L,M] \setminus \mathrm{GL}(m)$, and $\lambda(a)$ sends each V_x to V_{xa}. This means that the representation λ of $\mathrm{GL}(m)$ is induced from the representation of $\widehat{\mathbb{G}}_m[L,M]$ in V; see, e.g., [33, Sect.7.1].

Operators

$$\lambda\big(\mathfrak{X}[La^{-1}, a^t M]\big) = \lambda(a)\lambda(\mathfrak{X}[L,M])\lambda(a^{-1})$$

act as orthogonal projectors to $V_{x_0 a}$. A projector $\lambda\big(\mathfrak{X}[K,N]\big)$ is identical on $V_{x_0 a}$ if and only if $\mathfrak{X}[K,N]\,\mathfrak{X}[La^{-1}, Ma^t] = \mathfrak{X}[La^{-1}, Ma^t]$ and this gives us the action of the semigroup of idempotents.

It remains to show the representation of $\widehat{\mathbb{G}}_m[L,M]$ in V is irreducible. Assume that it contains a $\widehat{\mathbb{G}}_m[L,M]$-invariant subspace W; then each V_x contains a copy W_x of W and $\oplus_x W_x$ is a $\mathrm{GL}(m)$-invariant subspace in the whole H_m. ◻

Corollary 8.4. *Let* $\lambda(\mathfrak{g})$ *be a nonzero operator leaving* V *invariant. Then there is* $b \in \widehat{\mathbb{G}}_m[L,M]$ *such that*

$$\lambda(\mathfrak{g})\Big|_V = \rho(b)\Big|_V.$$

Proof. This operator can be represented as $\lambda(a)\lambda(\mathcal{X}[N,K])$. An operator $\lambda(\mathcal{X}[N,K])$ restricted to V is 0 or 1. Let this operator be 1. Then $\lambda(a)$ preserves V only if $a \in \widehat{\mathbb{G}}_m[L,M]$. In this case we set $b = a$. □

Keeping in mind Lemma 8.1 we get the following statement:

Corollary 8.5. *An irreducible $*$-representation of the category $\mathcal{K}$ is determined by its height m, a minimal idempotent $\mathcal{X}[L,M]$ acting nontrivially in H_m and an irreducible representation τ of the group $\widehat{\mathbb{G}}_m[L,M]$.*

We do not claim an existence of representation corresponding to given data of this kind.

8.3. **End of proof.** Let ρ be an irreducible unitary representation of $\mathbb{G}$ of height m in a Hilbert space H. Then we have a chain of subspaces in H:

$$H_m \longrightarrow H_{m+1} \longrightarrow H_{m+2} \longrightarrow \dots .$$

Lemma 4.2 defines a chain of semigroups

$$\Gamma(m) \longrightarrow \Gamma(m+1) \longrightarrow \Gamma(m+2) \longrightarrow \dots .$$

Each semigroup $\Gamma(n)$ acts in H as follows: in H_n it acts by operators $\widetilde{\rho}_{nn}(\cdot)$; on $H_n^\perp$ these operators are zero (see Lemma 4.1).

On the other hand, we have a chain of groups

$$\mathrm{GL}(m) \longrightarrow \mathrm{GL}(m+1) \longrightarrow \mathrm{GL}(m+2) \longrightarrow \dots$$

acting by unitary operators; their inductive limit is the group $\mathbb{G}_{\mathrm{fin}}$. Each group $\mathrm{GL}(n)$ preserves the subspace H_n; on this subspace the action of $\mathrm{GL}(n)$ coincides with the action of the group $\mathrm{Aut}_{\mathcal{K}}(n) = \mathrm{GL}(n)$.

Consider the data listed in Corollary 8.5. We regard the subspace

$$V = \operatorname{im} \widetilde{\rho}_{mm}(\mathcal{X}[L,M]) \subset H_m$$

as a subspace in H. Denote the $\mathrm{GL}(n)$-cyclic of V by W_n; it is a subspace in H_n.

Lemma 8.6. *Let $g \in \mathrm{GL}(n)$. If $g \in \widetilde{\mathbb{G}}_n[L;M]$, then $\rho(g)V = V$. Otherwise, $\rho(g)V \perp V$.*

Proof. In the first case, we have

$$\widetilde{\rho}_{nn}(\mathcal{X}[L,M])\,(\widetilde{\rho}_{nn}(g))^{-1}\,\widetilde{\rho}_{nn}(\mathcal{X}[Lg, M(g^t)^{-1}]) = \widetilde{\rho}_{nn}(\mathcal{X}[L,M])$$

and therefore the image V of $\widetilde{\rho}_{nn}(\mathcal{X}[L,M])$ is invariant with respect to $\rho(g)$.

In the second case we repeat the line (8.1). □

Thus the representation of $\mathrm{GL}(n)$ in W_n is induced from the subgroup $\widehat{\mathbb{G}}_n[L,M]$. If $k > n$, then we have embeddings

$$\mathrm{GL}(n) \to \mathrm{GL}(k), \quad \widehat{\mathbb{G}}_n[L,M] \to \widehat{\mathbb{G}}_k[L,M]$$

and therefore the map of homogeneous spaces

$$\Xi_{n,k} : \widehat{\mathbb{G}}_n[L,M] \setminus \mathrm{GL}(n) \to \widehat{\mathbb{G}}_k[L,M] \setminus \mathrm{GL}(k).$$

On the other hand, we have an embedding $W_n \to W_k$ regarding the orthogonal decompositions of these spaces into copies of V; therefore the map $\Xi_{n,k}$ is an embedding.

Finally, we get a representation of $\mathbb{G}_{\mathrm{fin}}$ induced from the subgroup $\widehat{\mathbb{G}}_{\mathrm{fin}}[L,M]$. By continuity, $\mathbb{G}$ acts regarding the same orthogonal decomposition $\oplus V_{xa}$. Hence a representation of $\mathbb{G}$ is induced from closure[12] of $\widehat{\mathbb{G}}_{\mathrm{fin}}[L,M]$, i.e., $\widehat{\mathbb{G}}[L,M]$.

Lemma 8.7. *The image of* $\widehat{\mathbb{G}}_{\mathrm{fin}}[L,M]$ *in the group of operators in* V *coincides with the image of* $\widehat{\mathbb{G}}_m[L,M]$.

Proof. Let $u \in \widehat{\mathbb{G}}_n[L,M]$. Then $\rho(u)$ preserves $V \subset H_m$. Therefore

$$\rho(u)\Big|_V = P_m \rho(u) P_m\Big|_V = \widetilde{\rho}([u]_{mm})\Big|_V.$$

By Corollary 8.4, this operator has the form $\rho(u')\Big|_V$, where $u' \in \widehat{\mathbb{G}}_m[L,M]$. □

Thus the representation τ of $\widehat{\mathbb{G}}_{\mathrm{fin}}[L;M]$ in V has a finite image. Its continuous extension to $\mathbb{G}[L;M]$ has the same image. The kernel of the representation τ is a closed subgroup. Since it has a finite index, it is open. By Proposition 7.11, τ is trivial on the subgroup $\mathbb{G}^\bullet[L;M]$.

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