REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 26, Pages 1–16 (January 5, 2022)
https://doi.org/10.1090/ert/592

COORDINATE RINGS AND BIRATIONAL CHARTS

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Abstract. Let G be a semisimple simply connected complex algebraic group. Let U be the unipotent radical of a Borel subgroup in G. We describe the coordinate rings of U (resp., G/U, G) in terms of two (resp., four, eight) birational charts introduced by Lusztig [*Total positivity in reductive groups*, Birkhäuser Boston, Boston, MA, 1994; Bull. Inst. Math. Sin. (N.S.) 14 (2019), pp. 403–459] in connection with the study of total positivity.

Introduction

Let G be a simply connected, almost simple algebraic group over $\mathbb{C}$. Fix a maximal torus T of G and a pair B^+, B^- of opposite Borel subgroups containing T, with unipotent radicals U^+, U^-. Let $\nu = \dim(U^+)$ and $r = \dim(T)$. For an irreducible quasi-affine variety X over $\mathbb{C}$, we denote by $O(X)$ the algebra of regular functions $X \to \mathbb{C}$, and let $[O(X)]$ be the quotient field of $O(X)$.

In this paper, we show (see Theorems 0.3, 4.2 and 5.2) that the algebra $O(U^+)$ (resp., $O(G/U^-)$ and $O(G)$) can be completely described in terms of two (resp., four and eight) birational charts $\mathbb{C}^\nu \to U^+$ (resp., $\mathbb{C}^\nu \times (\mathbb{C}^*)^r \to G/U^-$ and $\mathbb{C}^\nu \times (\mathbb{C}^*)^r \times \mathbb{C}^\nu \to G$) which were introduced in [Lus94],[Lus19] in connection with the study of total positivity.

Theorem 0.3 provides a proof of a conjecture in [Lus19, 6.1(a)]. Theorem 4.2 (resp., Theorem 5.2) establishes a weak form of a conjecture in [Lus19, 6.3(a)] (resp., [Lus19, 6.2(a)]) in which only two birational charts, instead of four (resp., eight), were used. The proof of Theorem 0.3 given in Section 3 relies on the results in [BZ97] and [FZ99] that describe the inverse of the charts for U^+ in terms of "generalized minors." Theorems 4.2 and 5.2 are proved in Sections 4 and 5, respectively, using reduction to the case of U^+. In particular, our proof of Theorem 5.2 does not use the more complete results on generalized minors in [FZ99]. (The latter technique would have allowed to decrease the number of charts from eight to two, but then the two charts used would not be canonical, unlike the eight that we consider here.)

In order to state our main result (Theorem 0.3), we will need to introduce some notation.

Let U_i^+ $(i \in I)$ be the simple root subgroups of U^+, and let U_i^- $(i \in I)$ be the corresponding root subgroups of U^-; here I is a finite indexing set. We assume that for any $i \in I$ we are given isomorphisms of algebraic groups $x_i : \mathbb{C} \xrightarrow{\sim} U_i^+$ and $y_i : \mathbb{C} \xrightarrow{\sim} U_i^-$ such that $(T, B^+, B^-, x_i, y_i; i \in I)$ is a pinning for G.

Received by the editors February 20, 2021, and, in revised form, October 25, 2021.
2020 *Mathematics Subject Classification.* Primary 22E46; Secondary 20G20, 14M15.
Key words and phrases. Lie group, coordinate ring, birational chart, generalized minor.
The first author was supported by NSF grants DMS-1664722, DMS-2054231 and by a Simons Fellowship. The second author was supported by DMS-1855773.

Definition 0.1. Let I^* be the set of all pairs $(i,j) \in I \times I$ such that any element in U_i commutes with any element in U_j. There is a unique (up to a labeling convention) partition $I = I_0 \sqcup I_1$ into two disjoint subsets such that $I_0 \times I_0 \subset I^*$ and $I_1 \times I_1 \subset I^*$. Let $r_0 = \sharp(I_0)$ and $r_1 = \sharp(I_1)$ be the cardinalities of I_0 and I_1.

It is known that $h = 2\nu/r$ is an integer (the Coxeter number).

For $\varepsilon \in \mathbb{Z}$, we define $[\varepsilon] \in \{0,1\}$ by $\varepsilon \equiv [\varepsilon] \bmod 2$. With this notation, we have

$$\nu = \underbrace{r_{[\varepsilon]} + r_{[\varepsilon+1]} + \cdots + r_{[\varepsilon+h-1]}}_{h \text{ terms}}.$$

(If h is even, this follows from $r_0 + r_1 = r$; if h is odd, we use that $r_0 = r_1 = r/2$.)

For $\varepsilon \in \{0,1\}$, let us fix the ordering of the elements of I_ε:

$$I_\varepsilon = \{i_1^\varepsilon, i_2^\varepsilon, \ldots, i_{r_\varepsilon}^\varepsilon\}.$$

We then define the sequence $\mathbf{j}^\varepsilon \in I^\nu$ (a distinguished reduced expression) by

$$\begin{aligned}\mathbf{j}^\varepsilon &= (j_1^\varepsilon, j_2^\varepsilon, \ldots, j_\nu^\varepsilon)\\ &= (i_1^{[\varepsilon]}, i_2^{[\varepsilon]}, \ldots, i_{r_{[\varepsilon]}}^{[\varepsilon]}, i_1^{[\varepsilon+1]}, i_2^{[\varepsilon+1]}, \ldots, i_{r_{[\varepsilon+1]}}^{[\varepsilon+1]},\\ &\quad i_1^{[\varepsilon+2]}, i_2^{[\varepsilon+2]}, \ldots, i_{r_{[\varepsilon+2]}}^{[\varepsilon+2]}, \ldots, i_1^{[\varepsilon+h-1]}, i_2^{[\varepsilon+h-1]}, \ldots, i_{r_{[\varepsilon+h-1]}}^{[\varepsilon+h-1]}).\end{aligned} \tag{0.1.1}$$

(The upper indices are not exponents.) Thus, the first $r_{[\varepsilon]}$ terms of $\mathbf{j}^\varepsilon$ are the elements of $I_{[\varepsilon]}$ in their order, the next $r_{[\varepsilon+1]}$ terms are the elements of $I_{[\varepsilon+1]}$ in their order, and these patterns keep alternating until we accumulate ν entries.

For a sequence of indices $\mathbf{i} = (i_1, i_2, \ldots, i_n) \in I^n$ of length $n \geq 0$, we define the map $f_\mathbf{i} : \mathbb{C}^n \to U^+$ by

$$f_\mathbf{i}(a_1, a_2, \ldots, a_n) = x_{i_1}(a_1) x_{i_2}(a_2) \ldots x_{i_n}(a_n). \tag{0.1.2}$$

In particular, one can choose $\mathbf{i} = \mathbf{j}^\varepsilon$ for $\varepsilon \in \{0,1\}$, as in (0.1.1) above. The following fact is well known:

Proposition 0.2. *The maps $f_{\mathbf{j}^0}, f_{\mathbf{j}^1}$ are birational isomorphisms from $\mathbb{C}^\nu$ to U^+.*

Proposition 0.2 can be deduced from the proof of [Lus94, 2.7] using (1.3.1) below; it can also be deduced from [BZ97]. See also 3.12(d).

By Proposition 0.2, each map $f_{\mathbf{j}^\varepsilon}$ ($\varepsilon \in \{0,1\}$) induces an isomorphism of fields $f^*_{\mathbf{j}^\varepsilon} : [O(U^+)] \xrightarrow{\sim} [O(\mathbb{C}^\nu)]$.

Theorem 0.3. *An element $\phi \in [O(U^+)]$ belongs to $O(U^+)$ if and only if the rational function $f^*_{\mathbf{j}^\varepsilon}(\phi) \in [O(\mathbb{C}^\nu)]$ belongs to $O(\mathbb{C}^\nu)$ for $\varepsilon = 0$ and for $\varepsilon = 1$.*

The proof of Theorem 0.3 is given in Section 3.

The instances of Theorem 0.3 for G of types A_2 and A_3 have been verified in [Lus19, Section 6.1]. In the rest of this section, we work out the latter case in detail.

Example 0.4. Let $G = \mathrm{SL}_4(\mathbb{C})$, with T, B^+, and B^- its subgroups of diagonal, upper-triangular, and low-triangular matrices, respectively. Then

$$\begin{aligned}
U^+ &= \left\{ u = \begin{bmatrix} 1 & u_{12} & u_{13} & u_{14} \\ 0 & 1 & u_{23} & u_{24} \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| u_{12}, u_{13}, u_{14}, u_{23}, u_{24}, u_{34} \in \mathbb{C} \right\}, \\
O(U^+) &= \mathbb{C}[u_{12}, u_{13}, u_{14}, u_{23}, u_{24}, u_{34}], \\
r &= 3, \\
I &= \{1, 2, 3\}, \\
x_1(a) &= \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad x_2(a) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad x_3(a) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
\nu &= 6, \\
h &= 4.
\end{aligned}$$

We set $I_0 = \{2\}$ and $I_1 = \{1, 3\}$. Then $r_0 = 1$, $r_1 = 2$, and

$$\begin{aligned}
\mathbf{j}^0 &= (2, 1, 3, 2, 1, 3), \\
\mathbf{j}^1 &= (1, 3, 2, 1, 3, 2), \\
f_{\mathbf{j}^0}(a_1, a_2, a_3, a_4, a_5, a_6) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & a_1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & a_2 + a_5 & a_2 a_4 & a_2 a_4 a_6 \\ 0 & 1 & a_1 + a_4 & a_1 a_3 + a_1 a_6 + a_4 a_6 \\ 0 & 0 & 1 & a_3 + a_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad (0.4.1) \\
f_{\mathbf{j}^1}(b_1, b_2, b_3, b_4, b_5, b_6) &= \begin{bmatrix} 1 & b_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & b_6 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & b_1 + b_4 & b_1 b_3 + b_1 b_6 + b_4 b_6 & b_1 b_3 b_5 \\ 0 & 1 & b_3 + b_6 & b_3 b_5 \\ 0 & 0 & 1 & b_2 + b_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (0.4.2)
\end{aligned}$$

Proposition 0.2 asserts that each of the 6 parameters a_1, a_2, a_3, a_4, a_5, a_6 (resp., b_1, b_2, b_3, b_4, b_5, b_6) can be expressed as a rational function in the 6 matrix entries u_{ij} $(1 \le i < j \le 4)$ of the unipotent upper-triangular matrix

$$u = (u_{ij}) = f_{\mathbf{j}^0}(a_1, a_2, a_3, a_4, a_5, a_6)$$

(resp., $f_{\mathbf{j}^1}(b_1, b_2, b_3, b_4, b_5, b_6)$). For example,

$$a_1 = \frac{u_{13}u_{24} - u_{14}u_{23}}{u_{13}u_{34} - u_{14}}\,,\ a_2 = \frac{u_{13}u_{34} - u_{14}}{u_{23}u_{34} - u_{24}}\,,\ a_3 = \frac{u_{13}u_{34} - u_{14}}{u_{13}}\,,$$
$$a_4 = \frac{u_{13}(u_{23}u_{34} - u_{24})}{u_{13}u_{34} - u_{14}}\,,\ a_5 = u_{12} - \frac{u_{13}u_{34} - u_{14}}{u_{23}u_{34} - u_{24}}\,,\ a_6 = \frac{u_{14}}{u_{13}}\,. \tag{0.4.3}$$

(For explicit formulas for matrices of arbitrary size, see [BFZ96, Theorem 1.4].)

Any rational function

$$\phi(u) = \phi(u_{12}, u_{13}, u_{14}, u_{23}, u_{24}, u_{34}) \in [O(U^+)]$$

can be rewritten in terms of the parameters a_i (resp., b_i), by substituting the appropriate expressions for the u_{ij} from (0.4.1)–(0.4.2):

$$\begin{aligned} \phi(u) &= \phi(a_2 + a_5,\ a_2a_4,\ a_2a_4a_6,\ a_1 + a_4,\ a_1a_3 + a_1a_6 + a_4a_6,\ a_3 + a_6) \\ &= \phi(b_1 + b_4,\ b_1b_3 + b_1b_6 + b_4b_6,\ b_1b_3b_5,\ b_3 + b_6,\ b_3b_5,\ b_2 + b_5). \end{aligned} \tag{0.4.4}$$

Theorem 0.3 asserts that ϕ is a polynomial in the variables u_{ij} if and only if both functions in the parameters a_i (resp., b_i) appearing in (0.4.4) are polynomial.

Theorem 0.3 can also be restated entirely in terms of the parameters a_i and b_i. As observed in [Lus94] (in a more general setting of an arbitrary pair of reduced expressions), the birational map relating the ν-tuples (a_i) and (b_j) to each other can be obtained as a composition of simple birational transformations associated to individual braid moves. In our example, calculations based on those rules yield the following formulas expressing $a_1, a_2, a_3, a_4, a_5, a_6$ in terms of $b_1, b_2, b_3, b_4, b_5, b_6$:

$$a_1 = \frac{b_3b_4b_5b_6}{R}, a_2 = \frac{R}{Q}, a_3 = \frac{R}{P}, a_4 = \frac{PQ}{R}, a_5 = \frac{b_2b_3b_4}{Q}, a_6 = \frac{b_1b_3b_5}{P}, \tag{0.4.5}$$

where

$$\begin{aligned} P &= b_1b_3 + b_1b_6 + b_4b_6\,, \\ Q &= b_2b_3 + b_2b_6 + b_5b_6\,, \\ R &= b_1b_2b_3 + b_1b_2b_6 + b_1b_5b_6 + b_2b_4b_6 + b_4b_5b_6. \end{aligned} \tag{0.4.6}$$

Theorem 0.3 (in this example) says that a polynomial $\Phi(a_1, a_2, a_3, a_4, a_5, a_6)$ lies in the subring

$$\mathbb{C}[a_2 + a_5, a_2a_4, a_2a_4a_6, a_1 + a_4, a_1a_3 + a_1a_6 + a_4a_6, a_3 + a_6] \subset \mathbb{C}[a_1, a_2, \dots, a_6]$$

(cf. (0.4.1)) if and only if substituting (0.4.5)–(0.4.6) into $\Phi(a_1, a_2, a_3, a_4, a_5, a_6)$ produces a *polynomial* (rather that merely a rational function) in $b_1,\ b_2,\ \dots,\ b_6$. (An alternative criterion would be to substitute (0.4.3) into $\Phi(a_1, a_2, a_3, a_4, a_5, a_6)$ and verify that the result lies in $\mathbb{C}[u_1,\ u_2,\ \dots,\ u_6]$.)

1. Preliminaries on the Weyl group and weights

1.1. Let $\iota : G \to G$ be the unique automorphism of G such that $\iota(x_i(a)) = y_i(a)$, $\iota(y_i(a)) = x_i(a)$ for $i \in I, a \in \mathbb{C}$ and $\iota(t) = t^{-1}$ for $t \in T$. We have $\iota^2 = 1$.

Let $\mathcal{Y} = \mathrm{Hom}(\mathbb{C}^*, T)$ and $\mathcal{X} = \mathrm{Hom}(T, \mathbb{C}^*)$. We write the operation in each of these groups as addition. Let $\langle\,,\rangle : \mathcal{Y} \times \mathcal{X} \to \mathbb{Z}$ be the obvious perfect pairing. For $i \in I$, let $\alpha_i \in \mathcal{X}$ be the simple root corresponding to U_i and let $\check{\alpha}_i$ be the corresponding simple coroot. Let $\mathcal{X}^+ = \{\lambda \in \mathcal{X} \mid \langle\check{\alpha}_i, \lambda\rangle \geq 0\ \forall i \in I\}$. For $i \in I$, the fundamental weight $\omega_i \in \mathcal{X}$ is defined by the condition $\langle\check{\alpha}_j, \omega_i\rangle = \delta_{ij}$ for $j \in I$. We have $\omega_i \in \mathcal{X}^+$.

For $i \in I$, we denote by P_i the (parabolic) subgroup of G generated by B^+ together with $\bigcup_{j \in I - \{i\}} U_j^-$.

1.2. For $i \in I$ define $s_i : \mathcal{Y} \to \mathcal{Y}$ by $\chi \mapsto \chi - \langle \chi, \alpha_i \rangle \check{\alpha}_i$. Let W be the subgroup of $\mathrm{Aut}(\mathcal{Y})$ generated by $\{s_i; i \in I\}$. This is a Coxeter group with the simple reflections $\{s_i \mid i \in I\}$ and with the length function that we denote by $w \mapsto |w|$. Let $w_0 \in W$ be the unique element such that $|w_0| = \nu$, the maximal possible length. Now W acts on $\mathcal{X}$ by the rule $\langle \chi, w(\lambda) \rangle = \langle w^{-1}(\chi), \lambda \rangle$ for $\chi \in \mathcal{Y}, \lambda \in \mathcal{X}$. For $i \in I$, let $W\omega_i$ be the W-orbit of the weight ω_i in $\mathcal{X}$ and let $W_{I-\{i\}}$ be the subgroup of W generated by $\{s_j; j \in I - \{i\}\}$. This is exactly the stabilizer of ω_i with respect to the W-action on $\mathcal{X}$.

Let NT be the normalizer of T in G. Now NT/T acts in an obvious way on $\mathcal{Y}$. This gives an embedding of $NT/T \hookrightarrow \mathrm{Aut}(\mathcal{Y})$ that identifies NT/T with W.

For $i \in I$, set $\dot{s}_i = x_i(1)y_i(-1)x_i(1) \in NT$ and $\ddot{s}_i = y_i(1)x_i(-1)y_i(1) \in NT$. We extend this to define representatives $\dot{w} \in NT$ and $\ddot{w} \in NT$ for all $w \in W$ by requiring that for any $w, w', w'' \in W$ satisfying $w'' = ww'$ and $|w''| = |w| + |w'|$, we have $\dot{w}'' = \dot{w}\dot{w}'$ and $\ddot{w}'' = \ddot{w}\ddot{w}'$.

For $\varepsilon \in \{0, 1\}$, we set

$$z^{\varepsilon} = \prod_{i \in I_\varepsilon} s_i \in W.$$

(Here the factors commute, so the order does not matter.)

Lemma 1.3 (see [Bou59, Chapter V, §6, Ex. 2])**.** *We have*

$$|z^{[\varepsilon]} z^{[\varepsilon+1]} \dots z^{[\varepsilon+h-1]}| = |z^{[\varepsilon]}| + |z^{[\varepsilon+1]}| + \dots + |z^{[\varepsilon+h-1]}| = \nu. \tag{1.3.1}$$

It follows that, if $I' \subset I_{[\varepsilon]}, I'' \subset I_{[\varepsilon+l+1]}$, $w' = \prod_{i \in I'} s_i$, $w'' = \prod_{i \in I''} s_i$, *then*

$$|wz^{[\varepsilon+1]} \dots z^{[\varepsilon+l]} w'| = |w| + |z^{[\varepsilon+1]}| + |z^{[\varepsilon+2]}| + \dots + |z^{[\varepsilon+l]}| + |w'|, \tag{1.3.2}$$

provided either (a) $l \in \{0, 1, \dots, h-2\}$ *or* (b) $w = 1$ *and* $l \in \{0, 1, \dots, h-1\}$, *or* (c) $w' = 1$ *and* $l \in \{0, 1, \dots, h-1\}$.

1.4. We denote

$$Y' = \{\omega_i \mid i \in I\}, \tag{1.4.1}$$

$$Y'' = \{w_0 \omega_i \mid i \in I\}. \tag{1.4.2}$$

If $\gamma \in Y'$, then $\langle \check{\alpha}_j, \gamma \rangle \geq 0$ for all $j \in I$. If $\gamma \in Y''$, then $\langle \check{\alpha}_j, \gamma \rangle \leq 0$ for all $j \in I$.

1.5. Fix $\varepsilon \in \{0, 1\}$. Recall that $\mathbf{j}^\varepsilon = (j_1^\varepsilon, j_2^\varepsilon, \dots, j_\nu^\varepsilon)$ was defined in (0.1.1). For $k \in \{1, 2, \dots, \nu\}$, we set

$$\gamma_k^\varepsilon = s_{j_\nu^\varepsilon} \dots s_{j_{k+1}^\varepsilon} s_{j_k^\varepsilon} \omega_{j_k^\varepsilon},$$
$$\tilde{\gamma}_k^\varepsilon = s_{j_\nu^\varepsilon} \dots s_{j_{k+2}^\varepsilon} s_{j_{k+1}^\varepsilon} \omega_{j_k^\varepsilon}.$$

In order to represent γ_k^ε and $\tilde{\gamma}_k^\varepsilon$ more explicitly, we will need to introduce some additional notation. For $l \in \{1, 2, \dots, h-2\}$ and $i \in I_{[\varepsilon+h-l]}$, let

$$v_{l,i}^\varepsilon = z^{[\varepsilon+h-1]} z^{[\varepsilon+h-2]} \dots z^{[\varepsilon+h-l+1]} s_i \omega_i.$$

Let $\mathcal{X}_1^\varepsilon \sqcup \mathcal{X}_2^\varepsilon \cdots \sqcup \mathcal{X}_h^\varepsilon$ be the partition of $\{1, 2, \ldots, \nu\}$ given by

$$\begin{aligned}
&\mathcal{X}_1^\varepsilon = \{1, 2, \ldots, r_{[\varepsilon]}\},\\
&\mathcal{X}_2^\varepsilon = \{r_{[\varepsilon]}+1, r_{[\varepsilon]}+2, \ldots, r_{[\varepsilon]}+r_{[\varepsilon+1]}\},\\
&\mathcal{X}_3^\varepsilon = \{r_{[\varepsilon]}+r_{[\varepsilon+1]}+1, r_{[\varepsilon]}+r_{[\varepsilon+1]}+2, \ldots, r_{[\varepsilon]}+r_{[\varepsilon+1]}+r_{[\varepsilon+2]}\},\\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

Since $s_j s_{j'} = s_{j'} s_j$ for j, j' in the same I_ε and $s_j \omega_{j'} = \omega_{j'}$ if $j \neq j'$, we see that

$$\begin{aligned}
\gamma_k^\varepsilon &= v_{l,j_k^\varepsilon}^\varepsilon \quad \text{if } \ l \in \{1, 2, \ldots, h-2\},\ k \in \mathcal{X}_{l+2}^\varepsilon \subset \{r+1, r+2, \ldots, \nu\},\\
\tilde{\gamma}_k^\varepsilon &= v_{l,j_k^\varepsilon}^\varepsilon \quad \text{if } \ l \in \{1, 2, \ldots, h-2\},\ k \in \mathcal{X}_l^\varepsilon \subset \{1, 2, , \ldots, \nu-h\}\\
\tilde{\gamma}_k^\varepsilon &\in Y' \quad \text{if } \ k \in \mathcal{X}_{h-1}^\varepsilon \sqcup \mathcal{X}_h^\varepsilon = \{\nu-h+1, \nu-h+2, \ldots, \nu\},\\
\gamma_k^\varepsilon &\in Y'' \quad \text{if } \ k \in \mathcal{X}_1^\varepsilon \sqcup \mathcal{X}_2^\varepsilon = \{1, 2, \ldots, r\}.
\end{aligned}$$

For k, k' in $\{1, 2, \ldots, \nu\}$ such that $(j_k^\varepsilon, j_{k'}^\varepsilon) \notin I^*$ (see Definition 0.1), we set

$$\gamma_{k,k'}^\varepsilon = s_{j_\nu^\varepsilon} \ldots s_{j_{k+1}^\varepsilon} s_{j_k^\varepsilon} \omega_{j_{k'}^\varepsilon}.$$

From the definitions we see that under these assumptions,

(a) $\gamma_{k,k'}^\varepsilon$ is either equal to one of the elements $\gamma_{k''}^\varepsilon$ or lies in Y'.

Lemma 1.6. *Let $\gamma = v_{l,i}^\varepsilon$ where $\varepsilon \in \{0, 1\}$, $i \in I_{[\varepsilon+h-l]}$, $l \in \{1, 2, \ldots, h-2\}$.*

(a) *If $j \in I_{[\varepsilon+h]}$, then $\langle \alpha_j, \gamma \rangle \geq 0$.*

(b) *There exists $j \in I_{[\varepsilon+h+1]}$ such that $\langle \alpha_j, \gamma \rangle < 0$.*

Proof. Let us prove (a). We have

$$\langle \alpha_j, \gamma \rangle = \langle s_i z^{[\varepsilon+h-l+1]} \ldots z^{[\varepsilon+h-1]} \alpha_j, \omega_i \rangle.$$

To show that this is nonnegative, it suffices to prove that

(c) $s_i z^{[\varepsilon+h-l+1]} \ldots z^{[\varepsilon+h-1]} \alpha_j$ is a positive coroot.

We have

$$|s_i z^{[\varepsilon+h-l+1]} \cdots z^{[\varepsilon+h-1]}| = |s_i| + |z^{[\varepsilon+h-l+1]}| + \cdots + |z^{[\varepsilon+h-1]}|.$$

(Use $i \in I_{[\varepsilon+j-l]}$ and (1.3.2) which holds since $l-1 \leq h-1$.) Therefore, to prove (c), it is enough to show that

$$|s_i z^{[\varepsilon+h-l+1]} \cdots z^{[\varepsilon+h-1]} s_j| = |s_i| + |z^{[\varepsilon+h-l+1]}| + \cdots + |z^{[\varepsilon+h-1]}| + |s_j|.$$

The latter follows from (1.3.2) since $l-1 \leq h-2$. This proves (a).

Now suppose that (b) does not hold. Then by (a), we have $\langle \alpha_j, \gamma \rangle \geq 0$ for every $j \in I$. Therefore $\gamma \in \mathcal{X}^+$. Since $\gamma \in W\omega_i$, we have $\gamma = \omega_i$. Hence $z^{[\varepsilon+h-1]} \ldots z^{[\varepsilon+h-l+1]} s_i$ is in the stabilizer of ω_i, i.e., in $W_{I-\{i\}}$. This contradicts

$$|z^{[\varepsilon+h-1]} \cdots z^{[\varepsilon+h-l+1]} s_i| = |z^{[\varepsilon+h-1]}| + \cdots + |z^{[\varepsilon+h-l+1]}| + |s_i|$$

which holds by (1.3.2). □

Lemma 1.7. *Let $\varepsilon \in \{0, 1\}$, $i \in I_{[\varepsilon+h-l]}$, $l \in \{1, 2, \ldots, h-2\}$. Let $w \in W$ be the unique element of minimal length in $\{w_1 \in W \mid w_1 \omega_i = v_{l,i}^\varepsilon\}$.*

(a) *If $j \in I_{[\varepsilon+h]}$, then $|s_j w| > |w|$.*

(b) *There exists $j \in I_{[\varepsilon+h+1]}$ such that $|s_j w| < |w|$.*

Proof. Assume that $j \in I$ satisfies $|s_j w| < |w|$. Then $|w^{-1}s_j| < |w^{-1}|$, and using [BZ97, Proposition 2.6] we see that $\langle \alpha_j, v^\varepsilon_{l,i}\rangle < 0$. Now using Lemma 1.6(a), we deduce that $j \notin I_{[\varepsilon+h]}$, proving (a). Now suppose (b) does not hold. Then by (a), $|s_j w| > |w|$ for all $j \in I$. Hence $w = 1$ and $v^\varepsilon_{l,i} = \omega_i$ so that $\langle \alpha_j, v^\varepsilon_{\lambda,i}\rangle \geq 0$ for all $j \in I$. This contradicts Lemma 1.6(b). □

1.8. Let $\varepsilon \in \{0,1\}$. Denote

$$Y^\varepsilon = \{v^\varepsilon_{l,i} \mid i \in I_{[\varepsilon+h-l]}, l \in \{1, 2, \dots, h-2\}\}. \tag{1.8.1}$$

We are going to show that

(a) all the weights in Y^ε are distinct.

To prove this, suppose that $v^\varepsilon_{l,i} = v^\varepsilon_{l',i'}$ where $i \in I_{[\varepsilon+h-l]}$, $i' \in I_{[\varepsilon+h-l']}$, and l, $l' \in \{1, 2, \dots, h-2\}$. Then $W\omega_i = W\omega_{i'}$ and therefore $\omega_i = \omega_{i'}$ and so $i = i'$.

Suppose that $l \neq l'$. Without loss of generality, we may assume that $l > l'$. Setting $e = l - l' \geq 1$ we get:

$$z^{[\varepsilon+h-l+e]} z^{[\varepsilon+h-l+e-1]} \cdots z^{[\varepsilon+h-l+1]} s_i \omega_i = s_i \omega_i.$$

Hence

(b) $s_i z^{[\varepsilon+h-l+e]} z^{[\varepsilon+h-l+e-1]} \cdots z^{[\varepsilon+h-l+1]} s_i \in W_{I-\{i\}}$.

From (1.3.2) we see that

$$\begin{aligned} &|s_i z^{[\varepsilon+h-l+e]} z^{[\varepsilon+h-l+e-1]} \cdots z^{[\varepsilon+h-l+1]} s_i| \\ =&|s_i z^{[\varepsilon+h-l+e]}| + |z^{[\varepsilon+h-l+e-1]}| + \cdots + |z^{[\varepsilon+h-l+1]}| + |s_i| \end{aligned}$$

which contradicts (b). Hence $l = l'$ and (a) is proved.

We note that

$$\sharp(Y^\varepsilon) = \sum_{l=1}^{h-2} r_{[\varepsilon+h-l]} = \sum_{l=1}^{h} r_{[\varepsilon+h-l]} - r_{[\varepsilon+1]} - r_{[\varepsilon]} = \nu - r. \tag{1.8.2}$$

Lemma 1.9. *With the notation introduced in* (1.4.1), (1.4.2), (1.8.1), *we have* $Y^\varepsilon \cap Y' = \varnothing$ *and* $Y^\varepsilon \cap Y'' = \varnothing$ *(assuming G is not of type A_1).*

Proof. If $\gamma \in Y^\varepsilon$, then $\langle \alpha_j, \gamma\rangle < 0$ for some j by Lemma 1.6(b); thus $\gamma \notin Y'$ by 1.4.

Assume that $v^\varepsilon_{l,i} = w_0\omega_j$ for some $l \in \{1, 2, \dots, h-2\}$, $i \in I_{[\varepsilon+h-l]}$, $j \in I$. Then ω_i and ω_j are in the same W-orbit. Hence $i = j$ and we have

$$\begin{aligned} &z^{[\varepsilon+h-1]} z^{[\varepsilon+h-2]} \cdots z^{[\varepsilon+h-l+1]} s_i \omega_i \\ =&z^{[\varepsilon+h-1]} z^{[\varepsilon+h-2]} \cdots z^{[\varepsilon+h-l+1]} z^{[\varepsilon+h-l]} \cdots z^{[\varepsilon]} \omega_i\,. \end{aligned}$$

This implies that $s_i z^{[\varepsilon+h-l]} \cdots z^{[\varepsilon]} \omega_i = \omega_i$, i.e., $s_i z^{[\varepsilon+h-l]} \cdots z^{[\varepsilon]}$ lies is in the stabilizer of ω_i, that is, in $W_{I-\{i\}}$. So any reduced expression of it does not contain s_i. If $l \geq 2$, this contradicts

$$|s_i z^{[\varepsilon+h-l]} \cdots z^{[\varepsilon]}| = |s_i| + |z^{[\varepsilon+h-l]}| + \cdots + |z^{[\varepsilon]}|.$$

since $h - l + 1 + 1 \leq h$. Therefore $l = 1$ and moreover any reduced expression of $s_i w_0$ does not contain s_i. But this cannot happen if G is of type other than A_1. Indeed, for some $\varepsilon \in \{0,1\}$,

$$z^{[\varepsilon+h-1]} z^{[\varepsilon+h-2]} \cdots z^{[\varepsilon+h-l+1]} z^{[\varepsilon+h-l]} \cdots z^{[\varepsilon]}$$

gives a reduced expression of w_0 such that s_i appears in the first group $z^{[\varepsilon+h-1]}$. If s_i does not appear in any other group, then there are only two factors and $h=2$. But $h>2$ in any type other than A_1. $\square$

2. An irreducibility property

In this section, we prove the following result.

Proposition 2.1. *Let $w, w' \in W$. The set $U^+ \cap (B^- \dot{w}' B^+ \dot{w}^{-1})$ is empty if $w' \not\leq w$; it is smooth and irreducible, of dimension $\nu - |w'|$, if $w' \leq w$.*

2.2. For $y \in W$, let

$$U_y^+ = \{u \in U^+, \dot{y}^{-1} u \dot{y} \in U^-\},$$
$$U^{+y} = \{u \in U^+, \dot{y}^{-1} u \dot{y} \in U^+\}.$$

The multiplication map $U_y^+ \times U^{+y} \xrightarrow{\sim} U^+$ is an isomorphism of varieties.

2.3. For $x \in G$ and a subgroup C of G, we shall write ${}^xC$ instead of xCx^{-1}. For $w \in W$, we shall write ${}^wC$ instead of ${}^{\dot{w}}C$.

We denote by $\mathcal{B}$ the variety of Borel subgroups in G. For $B', B'' \in \mathcal{B}$, there is a unique $w \in W$ such that for some x', x'' in G we have $B' = {}^{x'}B^+, B'' = {}^{x''}B^+$, $x'^{-1}x'' \in B^+ \dot{w} B^+$; we then write $w = \mathrm{pos}(B', B'')$.

For $z, z' \in W$, we denote

$$\mathcal{R}_{z,z'} = \{B \in \mathcal{B} \mid \mathrm{pos}(B^-, B) = z',\ \mathrm{pos}(B, B^+) = z^{-1}w_0\}.$$

It is known [KL79] that $\mathcal{R}_{z,z'}$ is nonempty if and only if $z \leq z'$. We show:

Proposition 2.4. *If $z \leq z'$ then $\mathcal{R}_{z,z'}$ is smooth, irreducible of dimension $|z'|-|z|$.*

Proof. We shall adapt an argument in [Lus98, 1.4] by replacing $\mathbb{R}$ by $\mathbb{C}$. The set

$$\tilde{\mathcal{R}}_{z,z'} = \{B \in \mathcal{B} \mid \mathrm{pos}(B^-, B) = z', \mathrm{pos}(B, {}^{w_0 z}B^-) = w_0\}.$$

is an open nonempty subset in $\{B \mid \mathrm{pos}(B^-, B) = z'\} \cong \mathbb{C}^{|z'|}$. Hence it is smooth irreducible of dimension $|z'|$. Clearly, the map $(B, u) \mapsto {}^uB$ is an isomorphism $\mathcal{R}_{z,z'} \times (U^- \cap {}^{w_0 z}U^-) \xrightarrow{\sim} \tilde{\mathcal{R}}_{z,z'}$. Now the claim follows since $U^- \cap {}^{w_0 z}U^- \cong \mathbb{C}^{|z|}$. $\square$

2.5. A result related to Proposition 2.4 holds for the analogue of $\mathcal{R}_{z,z'}$ over a finite field $\mathbb{F}_q$. By [KL79], the number of $\mathbb{F}_q$-rational points in this analogue is given by the polynomial $R_{z,z'}$ in *loc.cit.* evaluated at q. By the inductive formula in *loc.cit.*, the latter is monic of degree $|z'|-|z|$.

Proof of Proposition 2.1. Setting $B = {}^xB^+$, we can reformulate Proposition 2.4 as the statement that

$$\begin{aligned}
&\{xB^+ \in G/B^+ \mid \mathrm{pos}(B^-, {}^xB^+) = z', \mathrm{pos}({}^xB^+, B^+) = z^{-1}w_0\} \\
&= ((U^+ w_0 \dot{z} B^+) \cap (B^- (w_0 z'^{-1})^{\cdot} B^+))/B^+ \\
&= (U^+_{w_0 z} (w_0 z)^{\cdot}) \cap (B^- (w_0 z'^{-1})^{\cdot} B^+)
\end{aligned}$$

is smooth, irreducible of dimension $|z'|-|z|$ if $z \leq z'$, and is empty if $z \not\leq z'$.

Replacing here $w_0 z, w_0 z'$ by w, w' we deduce that $(U_w^+ \dot{w}) \cap (B^- \dot{w}' B^+)$ is smooth, irreducible of dimension $|w|-|w'|$ if $w' \leq w$, and is empty if $w' \not\leq w$.

Using 2.2, we see that the map

$$(U_w^+ \dot{w}) \cap (B^- \dot{w}' B^+) \times U^{+w} \to (U^+ \dot{w}) \cap (B^- \dot{w}' B^+)$$

given by $(u'\dot{w}, u'') \mapsto u'u''\dot{w}$ with $u' \in U_w^+$ such that $u'\dot{w} \in B^-\dot{w}'B^+$ and $u'' \in U^{+w}$ is an isomorphism of varieties. Since $U^{+w} \cong \mathbb{C}^{\nu-|w|}$, we conclude that $(U^+\dot{w}) \cap (B^-\dot{w}'B^+)$ is smooth, irreducible of dimension $\nu - |w'|$ if $w' \leq w$, and is empty if $w' \not\leq w$. This completes the proof of Proposition 2.1. □

3. Proof of Theorem 0.3

When G is of type A_1, we have $\mathbf{j}^0 = \mathbf{j}^1$ and the theorem is trivial. For the rest of this section, we assume that G is not of type A_1.

3.1. Fix $i \in I$. Let

$$V_i = \{f \in O(G) \mid f(utg) = \omega_i(t)f(g) \ \forall u \in U^-, t \in T, g \in G\}.$$

The group G acts on V_i by $g_1 : f \mapsto g_1 f$ where $(g_1 f)(g) = f(gg_1)$. There is a unique $f \in V_i$ such that $f(gu) = f(g)$ for all $g \in G, u \in U^+$ and such that $f(1) = 1$. We denote it by Δ. (Note that Δ depends on the choice of i.)

We show that $\Delta(\dot{s}_i) = 0$. Setting $g_c = y_i(-c)\check{\alpha}_i(c^{-1})x_i(c)$ for $c \in \mathbb{C}^*$, we see that $\lim_{c\to\infty} g_c = \dot{s}_i$ in G. We have $\Delta(g_c) = \omega_i(\check{\alpha}_i(c^{-1})) = c^{-1}$, so $\Delta(\dot{s}_i) = \lim_{c\to\infty} c^{-1} = 0$. It follows that Δ vanishes on $U^-\dot{s}_iB^+$, hence also on the closure

(a) $Z = \overline{U^-\dot{s}_iB^+} = \cup_{w; s_i \leq w} U^-\dot{w}B^+ = \cup_{w \in W - W_{I-\{i\}}} U^-\dot{w}B^+ = G - (U^-P_i)$.

The function Δ is preserved (up to a nonzero scalar) by the action of P_i on V_i. Hence Δ takes only nonzero values on the open subset U^-P_i of G, implying that

(b) $Z = \{g \in G; \Delta(g) = 0\}$.

Definition 3.2. Let $i \in I$ and $\gamma \in W\omega_i$. Following [BZ97], we set $\Delta_\gamma = \ddot{w}\Delta$, where $w \in W$ is such that $w\omega_i = \gamma$. This does not depend on the choice of w. In particular, $\Delta_{\omega_i} = \Delta$.

Let Δ_γ^+ be the restriction of Δ_γ to U^+. For $u \in U^+$, we have $\Delta_\gamma^+(u) = \Delta(u\ddot{w})$, with w as above. (Note that Δ_γ^+ is not identically zero on U^+. Otherwise we would have $\Delta(U^-B^+\ddot{w}) = 0$; but $U^-B^+\ddot{w}$ is dense in G; hence $\Delta = 0$, a contradiction.)

We will also use the notation

$$\mathscr{Z}_\gamma = \{u \in U^+ \mid \Delta_\gamma^+(u) = 0\}. \tag{3.2.1}$$

Lemma 3.3. *Let $i \in I$, $w \in W$, and $\gamma = w\omega_i$. Then:*
(a) $\mathscr{Z}_\gamma = \bigcup_{y \in W - W_{I-\{i\}}} (U^+ \cap (U^-\dot{y}B^+\dot{w}^{-1}))$;
(b) *if $s_i \not\leq w$ then $\mathscr{Z}_\gamma$ is empty;*
(c) *if $s_i \leq w$, then $\mathscr{Z}_\gamma$ is the closure of $U^+ \cap (U^-\dot{s}_iB^+\dot{w}^{-1})$ (a smooth irreducible variety of dimension $\nu - 1$).*

Proof. Using 3.1(a),(b), we get

$$\begin{aligned}\mathscr{Z}_\gamma &= \{u \in U^+; \Delta(u\ddot{w}) = 0\} \\ &= \{u \in U^+; u\ddot{w} \in Z\} \\ &= \{u \in U^+ \mid u\ddot{w} \in \cup_{y \in W - W_{I-\{i\}}} U^-dyB^+\},\end{aligned}$$

and (a) follows. (We used that $B^+\ddot{w}^{-1} = B^+\dot{w}^{-1}$.)

By Proposition 2.1, $U^+ \cap (U^-\dot{s}_iB^+\dot{w}^{-1})$ is smooth irreducible of dimension $\nu - 1$ provided that $s_i \leq w$ and is empty if $s_i \not\leq w$. Moreover if y satisfies $s_i < y$, then the same Proposition shows that $U^+ \cap (U^-\dot{y}B^+\dot{w}^{-1})$ is either empty or irreducible of dimension $\nu - |y| \leq \nu - 2$. Since, by Krull's theorem, $\mathscr{Z}_\gamma$ is either empty or of pure dimension $\nu - 1$, the statements (b) and (c) follow. □

Lemma 3.4. *Let $\varepsilon \in \{0,1\}$, $\gamma \in Y^\varepsilon$. Then:*
(a) *$\mathcal{Z}_\gamma$ (see (3.2.1)) is an irreducible variety of dimension $\nu - 1$;*
(b) *for any $j \in I_{[\varepsilon+h]}$ and any $c \in \mathbb{C}$ we have $\mathcal{Z}_\gamma x_j(c) \subset \mathcal{Z}_\gamma$;*
(c) *there exists $j \in I_{[\varepsilon+h+1]}$ such that for some $c \in \mathbb{C}$ we have $\mathcal{Z}_\gamma x_j(c) \not\subset \mathcal{Z}_\gamma$.*

Proof of (a). We write $\gamma = w\omega_i$ with $i \in I, w \in W$. By Lemma 1.9, we have $\gamma \notin Y'$ hence $w \notin W_{I-\{i\}}$ so that $s_i \le w$. Now (a) follows from Lemma 3.3(c). □

Proof of (b). We write $\gamma = w\omega_i$ where $i \in I$ and $w \in W$ is the unique element of minimal length in $\{w_1 \in W \mid w_1\omega_i = \gamma\}$. Using Lemma 3.3(c), we see that it is enough to show that for j, c as in (b) we have

$$(U^+ \cap (U^-\dot{y}B^+\dot{w}^{-1}))x_j(c) \subset U^+ \cap (U^-\dot{y}B^+\dot{w}^{-1})$$

for any $y \in W - W_{I-\{i\}}$. This follows from $\dot{w}^{-1}x_j(c) \in U^+\dot{w}^{-1}$ which in turn follows from $|s_jw| > |w|$ (see Lemma 1.7(a)) or equivalently $|w^{-1}s_j| > |w^{-1}|$. □

Proof of (c). Suppose that (c) does not hold. Using (b), we see that for any $j \in I$ and any $c \in \mathbb{C}$ we have $\mathcal{Z}_\gamma x_j(c) \subset \mathcal{Z}_\gamma$. Since the elements $x_j(c)$ for various j, c generate the group U^+, it follows that $\mathcal{Z}_\gamma U^+ \subset \mathcal{Z}_\gamma$. Since $\mathcal{Z}_\gamma \ne \varnothing$, we conclude that $\mathcal{Z}_\gamma = U^+$. This contradicts Lemma 3.3(b),(c). □

Lemma 3.5. *Let $\gamma \in Y^0$ and $\gamma' \in Y^1$. Then every irreducible component of $\mathcal{Z}_\gamma \cap \mathcal{Z}_{\gamma'}$ has dimension $\le \nu - 2$.*

Proof. By Lemma 3.4(c) with $\varepsilon = 0$, there exist $j \in I_{[h+1]}$ and $c \in \mathbb{C}$ such that $\mathcal{Z}_\gamma x_j(c) \not\subset \mathcal{Z}_\gamma$. By Lemma 3.4(b) with $\varepsilon = 1$, we have $\mathcal{Z}_{\gamma'} x_j(c) \subset \mathcal{Z}_{\gamma'}$. Therefore $\mathcal{Z}_\gamma \ne \mathcal{Z}_{\gamma'}$. Since $\mathcal{Z}_\gamma, \mathcal{Z}_{\gamma'}$ are irreducible of dimension $\nu - 1$, the lemma follows. □

3.6. Consider the partition

$$U^+ = \bigsqcup_{z \in W} U^+(z) \tag{3.6.1}$$

where

$$U^+(z) = U^+ \cap B^-\dot{z}B^-$$

is smooth and irreducible of dimension $|z|$ (cf. Proposition 2.1 with (w, w') replaced by (w_0, zw_0)). Furthermore, the closure of $U^+(z)$ in U^+ is equal to $\bigsqcup_{z';z' \le z} U^+(z')$. It follows that $U^+(w_0)$ is open dense in U^+. For $z \in W$, we set

$$U^-(z) = U^- \cap B^+\dot{z}B^+ = \iota(U^+(z))$$

(see 1.1 for the definition of ι). Then

$$U^- = \bigsqcup_{z \in W} U^-(z)$$

and $U^-(w_0)$ is open dense in U^-.

Let

$$A : U^+(w_0) \xrightarrow{\sim} U^+(w_0)$$

be the composition

$$U^+(w_0) \xrightarrow{\sim} \{B \in \mathcal{B} \mid \mathrm{pos}(B^+, B) = \mathrm{pos}(B, B^-) = w_0\} \xrightarrow{\sim} U^-(w_0) \xrightarrow{\sim} U^+(w_0)$$

where the first isomorphism is $u \mapsto {}^uB^-$, the second isomorphism is the inverse of $u' \mapsto {}^{u'}B^+$, and the third isomorphism is the restriction of ι.

We will show that A is an involution. For $u \in U^+$, we have ${}^{u}B^- = {}^{\iota(A(u))}B^+$. Replacing u by $A(u)$ we obtain ${}^{A(u)}B^- = {}^{\iota(A^2(u))}B^+$. Applying ι, we obtain ${}^{\iota(A(u))}B^+ = {}^{A^2(u)}B^-$, i.e., ${}^{u}B^- = {}^{A^2(u)}B^-$. Hence $u = A^2(u)$ and $A^2 = 1$.

3.7. Let $\varepsilon \in \{0, 1\}$. We denote

$$\mathcal{V}^\varepsilon = \{u \in U^+ \mid \Delta^+_{\gamma^\varepsilon_k}(u) \neq 0, \Delta^+_{\gamma^\varepsilon_k}(u) \neq 0, k = \{1, 2, \dots, \nu\}\}.$$

This set is open in U^+. It is also nonempty, since each of $\Delta^+_{\gamma^\varepsilon_k}, \Delta^+_{\gamma^\varepsilon_k}$ is not identically zero on U^+. We denote

$$\mathcal{V}^\varepsilon_* = \{u \in U^+(w_0) \mid A(u) \in \mathcal{V}^\varepsilon\} = U^+(w_0) \cap A^{-1}(\mathcal{V}^\varepsilon).$$

This set is open in U^+. It is also nonempty, as it is the intersection of two open nonempty subsets of U^+. We shall need the following result from [BZ97], [FZ99].

Lemma 3.8. *The map $f_{\mathbf{j}^\varepsilon} : \mathbb{C}^\nu \to U^+$ restricts to an isomorphism $(\mathbb{C}^*)^\nu \xrightarrow{\sim} \mathcal{V}^\varepsilon_*$.*

3.9. Using the results in 1.5, we see that

$$\mathcal{V}^\varepsilon_* = \{u \in U^+(w_0) \mid \Delta^+_\gamma(Au) \neq 0 \text{ for all } \gamma \in Y^\varepsilon \cup Y' \cup Y''\}. \tag{3.9.1}$$

If $\gamma = \omega_i$ with $i \in I$, then Δ^+_γ is the function $u \mapsto \Delta_{\omega_i}(u) = \Delta_{\omega_i}(1) = 1$ (a constant function). If $\gamma = w_0\omega_i$ with $i \in I$, then $\Delta^+_\gamma(u) \neq 0$ for any $u \in U^+(w_0)$. (Indeed, writing $u = u'\dot{w}_0 b'$ with $u' \in U^-, b' \in B^-$, so that $u\dot{w}_0 = u'tu_1$ with $t \in T, u_1 \in U^+$, we have $\Delta^+_\gamma(u) = \Delta_{\omega_i}(u\dot{w}_0) = \Delta_{\omega_i}(u'tu_1) = \omega_i(t) \neq 0$.) It follows that Y' and Y'' can be eliminated from (3.9.1), and we conclude that

$$\mathcal{V}^\varepsilon_* = \{u \in U^+(w_0) \mid \Delta^+_\gamma(Au) \neq 0 \text{ for all } \gamma \in Y^\varepsilon\}. \tag{3.9.2}$$

Lemma 3.10. $\dim(U^+(w_0) - (\mathcal{V}^0_* \cup \mathcal{V}^1_*)) \leq \nu - 2$.

Proof. From (3.9.2), we obtain

$$U^+(w_0) - (\mathcal{V}^0_* \cup \mathcal{V}^1_*) = \bigcup_{(\gamma,\gamma') \in Y^0 \times Y^1} A(U^+(w_0) \cap \mathcal{Z}_\gamma \cap \mathcal{Z}_{\gamma'}).$$

It remains to use that $\dim(\mathcal{Z}_\gamma \cap \mathcal{Z}_{\gamma'}) \leq \nu - 2$ for $(\gamma, \gamma') \in Y^0 \times Y^1$ (see Lemma 3.5). ☐

Lemma 3.11. *Let $\mathbf{i} = (i_1, i_2, \dots, i_n) \in I^n$ be a reduced expression, that is, the element $w = s_{i_1} \dots s_{i_n} \in W$ has length n. Let*

$$'f_{\mathbf{i}} : (\mathbb{C}^*)^n \to U^+$$

be the restriction of the map $f_{\mathbf{i}}$ in (0.1.2). Then $'f_{\mathbf{i}}$ is an isomorphism of $(\mathbb{C}^*)^n$ onto an open subset $'U^+_{\mathbf{i}}$ of $U^+(w)$.

Proof. Induction on n. For $n = 0$, the result is obvious. Assume that $n \geq 1$. Let $\mathbf{i}' = (i_1, i_2, \dots, i_{n-1}) \in I^{n-1}$ and let $w' = s_{i_1} \dots s_{i_{n-1}}$. The map

$$U^+(w') \times \mathbb{C}^* \to U^+(w)$$
$$(u', c) \mapsto u' x_{i_n}(c)$$

is an isomorphism of $U^+(w') \times \mathbb{C}^*$ onto an open subset of $U^+(w)$. It restricts to an isomorphism of $'U^+_{\mathbf{i}'} \times \mathbb{C}^*$ onto an open subset $'U^+_{\mathbf{i}}$ of $U^+(w)$. ☐

3.12. Let $\varepsilon \in \{0,1\}$ and let $i \in I_{[\varepsilon+h+1]}$. Define $k \in \mathcal{X}_h^\epsilon$ (in the notation of 1.5) by $j_k^\varepsilon = i$. Let $\mathbb{C}_i^\nu$ (resp. $'\mathbb{C}_i^\nu$) be the subset of $\mathbb{C}^\nu$ consisting of all $(a_1, a_2, \ldots, a_\nu)$ such that $a_l \in \mathbb{C}^*$ for $l \neq k$ whereas $a_k \in \mathbb{C}$ (resp. $a_k = 0$). By restricting $f_{\mathbf{j}^\varepsilon} : \mathbb{C}^\nu \to U^+$ to $\mathbb{C}_i^\nu$ (resp. $'\mathbb{C}_i^\nu$), we obtain maps $f_{\mathbf{j}^\varepsilon;i} : \mathbb{C}_i^\nu \to U^+$ and $'f_{\mathbf{j}^\varepsilon;i} : {'\mathbb{C}_i^\nu} \to U^+$.

It follows from Lemma 3.11 that

(a) $'f_{\mathbf{j}^\varepsilon;i}$ is an isomorphism of $'\mathbb{C}_i^\nu$ onto an open subset $'U^+_{\mathbf{j}^\varepsilon;i}$ of $U^+(w_0 s_i)$.
We next prove that
(b) $f_{\mathbf{j}^\varepsilon;i}$ is an isomorphism of $\mathbb{C}_i^\nu$ onto an open subset $U^+_{\mathbf{j}^\varepsilon;i}$ of $U^+(w_0) \cup U^+(w_0 s_i)$ containing $'U^+_{\mathbf{j}^\varepsilon;i}$.

Proof. The map $U^+(w_0 s_i) \times \mathbb{C} \to U^+$, $(u', c) \mapsto u' x_{i_n}(c)$, is an isomorphism of $U^+(w_0 s_i) \times \mathbb{C}$ onto an open subset of $U^+(w_0) \cup U^+(w_0 s_i)$. It restricts to an isomorphism of $'U^+_{\mathbf{j}^\varepsilon;\mathbf{i}} \times \mathbb{C}$ onto an open subset $U^+_{\mathbf{j}^\varepsilon;i}$ of $U^+(w_0) \cup U^+(w_0 s_i)$. □

The following is a special case of Lemma 3.11:
(c) $'f_{\mathbf{j}^\varepsilon}$ is an isomorphism of $(\mathbb{C}^*)^\nu$ onto an open subset $'U^+_{\mathbf{j}^\varepsilon}$ of $U^+(w_0)$.
From (c), we deduce that
(d) $f_{\mathbf{j}^\varepsilon}$ is a birational isomorphism from $\mathbb{C}^\nu$ to U^+.

Lemma 3.13. *Let $\mathcal{U}$ be the open subset of U^+ defined by*

$$\mathcal{U} = \mathcal{V}_*^0 \cup \mathcal{V}_*^1 \cup \bigcup_{\varepsilon \in \{0,1\}, i \in I_{[\varepsilon+h+1]}} U^+_{\mathbf{j}^\varepsilon;i}. \tag{3.13.1}$$

Then $\dim(U^+ - \mathcal{U}) \leq \nu - 2$.

Proof. Using the partition (3.6.1), it is enough to show that

$$\dim(U^+(z) \cap (U^+ - \mathcal{U})) \leq \nu - 2 \tag{3.13.2}$$

for any $z \in W$.

Case 1. $z = w_0$. We have

$$U^+(w_0) \cap (U^+ - \mathcal{U}) \subset U^+(w_0) - (\mathcal{V}_*^0 \cup \mathcal{V}_*^1).$$

Therefore

$$\dim(U^+(w_0) \cap (U^+ - \mathcal{U})) \leq \dim(U^+(w_0) - (\mathcal{V}_*^0 \cup \mathcal{V}_*^1)) \leq \nu - 2$$

(see Lemma 3.10), and (3.13.2) follows.

Case 2. $z = w_0 s_i$ with $i \in I$. Define $\varepsilon \in \{0,1\}$ by $i \in I_{[\varepsilon+h+1]}$. Then

$$U^+(w_0 s_i) \cap (U^+ - \mathcal{U}) \subset U^+(w_0 s_i) \cap (U^+ - U^+_{\mathbf{j}^\varepsilon;i}) \subset U^+(w_0 s_i) - {'U^+_{\mathbf{j}^\varepsilon;i}}.$$

The last difference has dimension $\leq \nu - 2$ (as desired) since $U^+(w_0 s_i)$ is irreducible of dimension $\nu - 1$ and $'U^+_{\mathbf{j}^\varepsilon;i}$ is a nonempty open subset of $U^+(w_0 s_i)$.

Case 3. z is not of the form w_0 or $w_0 s_i$. Then $|z| \leq \nu - 2$. Therefore $\dim(U^+(z)) \leq \nu - 2$ which implies (3.13.2). □

3.14. *Proof of Theorem* 0.3. The "only if" part of Theorem 0.3 is obvious. Let us prove the "if" statement. Consider $\phi \in [O(U^+)]$ such that $f^*_{\mathbf{j}^\varepsilon}(\phi) \in [O(\mathbb{C}^\nu)]$ belongs to $O(\mathbb{C}^\nu)$ for $\varepsilon = 0$ and for $\varepsilon = 1$. From our assumption we see that $\phi|_{\mathcal{V}_*^\varepsilon}$ is regular for $\varepsilon \in \{0,1\}$ (see Lemma 3.8) and that $\phi|_{U^+_{\mathbf{j}^\varepsilon;i}}$ is regular for $\varepsilon \in \{0,1\}$ and $i \in I_{[\varepsilon+h+1]}$ (see 3.12(b)). Hence ϕ is regular on $\mathcal{U}$. Using this and Lemma 3.13, we conclude that ϕ is regular on U^+. Theorem 0.3 is proved. □

4. The study of $O(G/U^-)$

4.1. For $\mathbf{i} = (i_1, i_2, \ldots, i_\nu) \in I^\nu$, we define the maps

$$f_{\mathbf{i};+} : \mathbb{C}^\nu \times T \to G/U^-$$
$$f_{\mathbf{i};-} : \mathbb{C}^\nu \times T \to G/U^-$$

by

$$f_{\mathbf{i};+}(a_1, a_2, \ldots, a_\nu, t) = x_{i_1}(a_1)x_{i_2}(a_2)\ldots x_{i_\nu}(a_\nu)\, t\, U^-,$$
$$f_{\mathbf{i};-}(a_1, a_2, \ldots, a_\nu, t) = y_{i_1}(a_1)y_{i_2}(a_2)\ldots y_{i_\nu}(a_\nu)\, t\, \dot{w}_0\, U^-.$$

Of particular interest to us are the cases where $\mathbf{i} = \mathbf{j}^\varepsilon$, for $\varepsilon \in \{0, 1\}$, as in (0.1.1). Proposition 0.2 implies that both $f_{\mathbf{j}^\varepsilon;+}$ and $f_{\mathbf{j}^\varepsilon;-}$ are birational isomorphisms from $\mathbb{C}^\nu \times T$ to G/U^-. Consequently the maps $f^*_{\mathbf{j}^\varepsilon;+}$ and $f^*_{\mathbf{j}^\varepsilon;-}$ are well-defined isomorphisms $[O(G/U^-)] \xrightarrow{\sim} [O(\mathbb{C}^\nu \times T)]$.

Theorem 4.2. *An element $\phi \in [O(G/U^-)]$ belongs to $O(G/U^-)$ if and only if each of the four rational functions $f^*_{\mathbf{j}^0;+}(\phi), f^*_{\mathbf{j}^1;+}(\phi), f^*_{\mathbf{j}^0;-}(\phi), f^*_{\mathbf{j}^1;-}(\phi) \in [O(\mathbb{C}^\nu \times T)]$ belongs to $O(\mathbb{C}^\nu \times T)$.*

The proof of Theorem 4.2 will rely on the following statement.

Lemma 4.3. *We have*

$$\dim(G/U^- - ((U^+TU^- \cup U^-T\dot{w}_0U^-)/U^-)) \le \dim(G/U^-) - 2. \tag{4.3.1}$$

Proof. The inequality (4.3.1) is equivalent to

$$\dim(G/B^- - ((U^+B^- \cup (U^-\dot{w}_0B^-)/B^-)) \le \dim(G/B^-) - 2,$$

which is equivalent to the inequality

$$\dim(\mathcal{B} - (\{B \in \mathcal{B} \mid \mathrm{pos}(B, B^+) = w_0\} \cup \{B \in \mathcal{B}; \mathrm{pos}(B, B^-) = w_0\}) \le \dim \mathcal{B} - 2$$

and thus to the statement that, for any $z \in W - \{1\}$ and $z' \in W - \{w_0\}$, we have

$$\dim(\{B \in \mathcal{B} \mid \mathrm{pos}(B^-, B) = z',\ \mathrm{pos}(B, B^+) = z^{-1}w_0\}) \le \nu - 2.$$

The last claim follows from Proposition 2.4 since $|z'| - |z| \le \nu - 2$. □

4.4. *Proof of Theorem* 4.2. The "only if" statement in the theorem is obvious. Let us prove the "if" statement. Thus, let $\phi \in [O(G/U^-)]$ be such that the four conditions in the theorem are satisfied. We need to show that $\phi \in O(G/U^-)$.

Suppose G is of type A_1. Then ϕ is regular on $(U^+TU^- \cup U^-T\dot{w}_0U^-)/U^-$. Hence by (4.3.1), it is regular on G/U^-, and we are done.

In the rest of the proof, we assume that G is of type other than A_1.

We first show that ϕ regular on the open subset U^+TU^-/U^- of G/U^-. With the notation as in 3.7 and 3.12, we see as in the proof in 3.14 that ϕ is regular on each of the following open subsets of U^+TU^-/U^-:

- $\mathcal{V}^\varepsilon TU^-/U^-$, for $\varepsilon \in \{0, 1\}$;
- $U^+_{\mathbf{j}^\varepsilon;i}TU^-/U^-$, for $\varepsilon \in \{0, 1\}$ and $i \in I_{[\varepsilon+h+1]}$.

Hence ϕ is regular on the union of these subsets, i.e., on $\mathcal{U}TU^-/U^-$ (here $\mathcal{U} \subset U^+$ is as in (3.13.1)). By Lemma 3.13, we have

$$\dim((U^+TU^- - \mathcal{U}TU^-)/U^-) \le \nu + r - 2 = \dim(G/U^-) - 2.$$

Since ϕ is regular on $\mathcal{U}TU^-/U^-$, it follows that ϕ is regular on U^+TU^-/U^-.

We next show that ϕ is regular on the open subset $U^-T\dot{w}_0U^-/U^-$ of G/U^-.

We denote $\mathcal{V}^{\varepsilon-}_* = \iota(\mathcal{V}^\varepsilon_*) \subset U^-$ (cf. 3.7) and $U^-_{\mathbf{j}^\varepsilon;i} = \iota(U^+_{\mathbf{j}^\varepsilon;i}) \subset U^-$ (cf. 3.12).

As in the proof in 3.14 (with U^+ replaced by U^-), we see that ϕ is regular on each of the following open subsets of $U^-T\dot{w}_0U^-/U^-$:

- $\mathcal{V}^{\varepsilon-}T\dot{w}_0U^-/U^-$, for $\varepsilon \in \{0,1\}$;
- $U^-_{\mathbf{j}^\varepsilon;i}T\dot{w}_0U^-/U^-$, for $\varepsilon \in \{0,1\}$ and $i \in I_{[\varepsilon+h+1]}$.

Hence ϕ is regular on the union of these subsets, i.e., on $\mathcal{U}^-T\dot{w}_0U^-/U^-$ where $\mathcal{U}^- = \iota(\mathcal{U}) \subset U^-$. By Lemma 3.13 (with U^- instead of U^+), we have

$$\dim((U^-T\dot{w}_0U^- - \mathcal{U}^-T\dot{w}_0U^-)/U^-) \leq \nu + r - 2 = \dim(G/U^-) - 2.$$

Since ϕ is regular on $\mathcal{U}^-T\dot{w}_0U^-/U^-$, it follows that ϕ is regular on $U^-T\dot{w}_0U^-/U^-$. Thus ϕ is regular on the open subset $((U^+TU^-) \cup (U^-T\dot{w}_0U^+))/U^-$ of G/U^-. Using this and Lemma 4.3, we conclude that ϕ is regular on G/U^-, as desired. $\square$

5. The study of $O(G)$

5.1. For $\mathbf{i} = (i_1, i_2, \dots, i_\nu) \in I^\nu$ and $\mathbf{i}' = (i'_1, i'_2, \dots, i'_\nu) \in I^\nu$, we define the maps

$$f_{\mathbf{i},\mathbf{i}';\pm} : \mathbb{C}^\nu \times T \times \mathbb{C}^\nu \to G,$$
$$f_{\mathbf{i},\mathbf{i}';\mp} : \mathbb{C}^\nu \times T \times \mathbb{C}^\nu \to G$$

by

$$\begin{aligned}
&f_{\mathbf{i},\mathbf{i}';\pm}(a_1, a_2, \dots, a_\nu, t, b_1, b_2, \dots, b_\nu)\\
&= x_{i_1}(a_1)x_{i_2}(a_2)\dots x_{i_\nu}(a_\nu)\, t\, y_{i'_1}(b_1)y_{i'_2}(b_2)\dots y_{i'_\nu}(b_\nu),\\
&f_{\mathbf{i},\mathbf{i}';\mp}(a_1, a_2, \dots, a_\nu, t, b_1, b_2, \dots, b_\nu)\\
&= y_{i_1}(a_1)y_{i_2}(a_2)\dots y_{i_\nu}(a_\nu)\, t^{-1}\, x_{i'_1}(b_1)x_{i'_2}(b_2)\dots x_{i'_\nu}(b_\nu).
\end{aligned}$$

Thus, $f_{\mathbf{i},\mathbf{i}';\mp} = \iota f_{\mathbf{i},\mathbf{i}';\pm}$.

Let $\mathbf{j}^\varepsilon$, for $\varepsilon \in \{0,1\}$, be as in (0.1.1). From Proposition 0.2 one can deduce that for each of the four possible pairs $(\varepsilon, \varepsilon') \in \{0,1\} \times \{0,1\}$, both maps $f_{\mathbf{j}^\varepsilon,\mathbf{j}^{\varepsilon'};\pm}$ and $f_{\mathbf{j}^\varepsilon,\mathbf{j}^{\varepsilon'};\mp}$ are birational isomorphisms from $\mathbb{C}^\nu \times T \times \mathbb{C}^\nu$ to G. It follows that both $f^*_{\mathbf{j}^\varepsilon,\mathbf{j}^{\varepsilon'};\pm}$ and $f^*_{\mathbf{j}^\varepsilon,\mathbf{j}^{\varepsilon'};\mp}$ are well defined isomorphisms $[O(G)] \xrightarrow{\sim} [O(\mathbb{C}^\nu \times T \times \mathbb{C}^\nu)]$.

Theorem 5.2. *An element $\phi \in [O(G)]$ belongs to $O(G)$ if and only if for each of the four possible pairs $(\varepsilon, \varepsilon') \in \{0,1\} \times \{0,1\}$, both rational functions*

$$f^*_{\mathbf{j}^\varepsilon,\mathbf{j}^{\varepsilon'};\pm}(\phi), f^*_{\mathbf{j}^\varepsilon,\mathbf{j}^{\varepsilon'};\mp}(\phi) \in [O(\mathbb{C}^\nu \times T \times \mathbb{C}^\nu)]$$

belong to $O(\mathbb{C}^\nu \times T \times \mathbb{C}^\nu)$.

The proof of Theorem 5.2 will rely on the following statement.

Lemma 5.3. $\dim(G - ((U^+TU^-) \cup (U^-TU^+))) \leq \dim(G) - 2$.

Proof. Using the Bruhat decomposition, we obtain:

$$\begin{aligned}
G - ((U^+TU^-) \cup (U^-TU^+)) &= (G - (B^+U^-)) \cap (G - (B^-U^+))\\
&= \Big(\bigcup_{w \in W - \{1\}} B^+\dot{w}U^- \Big) \cap \Big(\bigcup_{w' \in W - \{1\}} B^-\dot{w}'U^+ \Big)\\
&= \bigcup_{w,w' \text{ in } W-\{1\}} (B^+\dot{w}U^-) \cap (B^-\dot{w}'U^+).
\end{aligned}$$

It is therefore enough to show that for any $w \neq 1$ and $w' \neq 1$, we have

$$\dim((B^+\dot{w}U^-) \cap (B^-\dot{w}'U^+)) \leq \dim(G) - 2. \tag{5.3.1}$$

This is clear if either $B^+\dot{w}U^-$ or $B^-\dot{w}'U^+$ has dimension $\leq \dim(G) - 2$. Thus we can assume that $\dim(B^+\dot{w}U^-) = \dim(B^-\dot{w}'U^+) = \dim(G) - 1$ or equivalently $|w| = |w'| = 1$. Then both $\overline{B^+\dot{w}U^-}$ and $\overline{B^-\dot{w}'U^+}$ (closures in G) are irreducible of dimension $\dim(G) - 1$. If $\overline{B^+\dot{w}U^-} \neq \overline{B^-\dot{w}'U^+}$, then

$$\dim(\overline{B^+\dot{w}U^-} \cap \overline{B^-\dot{w}'U^+}) \leq \dim(G) - 2,$$

implying (5.3.1). Thus we may assume that $\overline{B^+\dot{w}U^-} = \overline{B^-\dot{w}'U^+}$. By our assumption, $w = s_i$ for some $i \in I$. For any $c \in \mathbb{C}$ we have $y_i(c)B^-\dot{w}'U^+ \subset B^-\dot{w}'U^+$ hence $y_i(c)\overline{B^-\dot{w}'U^+} \subset \overline{B^-\dot{w}'U^+}$. Using our assumption, we also deduce that $y_i(c)\overline{B^+\dot{s}_iU^-} \subset \overline{B^+\dot{s}_iU^-}$ for any $c \in \mathbb{C}$. We have $B^+\dot{s}_iU^- = B^+(s_iw_0)^{\cdot}U^+\dot{w}_0^{-1}$. For $c \in \mathbb{C}^*$, we have

$$y_i(c)B^+\dot{s}_iU^- \subset B^+\dot{s}_iB^+B^+(s_iw_0)^{\cdot}U^+\dot{w}_0^{-1} \subset B^+\dot{w}_0B^+\dot{w}_0^{-1} = B^+U^-$$

and this is disjoint from $B^+\dot{s}_iU^-$. (We have used that $|s_i(s_iw_0)| = |s_i| + |s_iw_0|$.) This contradicts the inclusion $y_i(c)\overline{B^+\dot{s}_iU^-} \subset \overline{B^+\dot{s}_iU^-}$. ☐

5.4. *Proof of Theorem* 5.2. The "only if" statement in Theorem 5.2 is obvious. Let us prove the "if" statement. Consider $\phi \in [O(G)]$ such that the eight conditions in Theorem 5.2 are satisfied. We need to show that $\phi \in O(G)$.

Suppose that G is of type A_1. Then ϕ is regular on $U^+TU^- \cup U^-TU^+$. Hence by Lemma 5.3, it is regular on G, and we are done.

In the rest of the proof, we assume that G is of type other than A_1.

We will first show that ϕ is a regular function on the open set U^+TU^-.

From our assumptions we see—as in the proof in 3.14—that (using the same notation as 4.4) ϕ is regular on each of the following open subsets of U^+TU^-:

- $\mathcal{V}_*^{\varepsilon}T\mathcal{V}_*^{\varepsilon'-}$, for $\varepsilon, \varepsilon' \in \{0,1\}$;
- $\mathcal{V}_*^{\varepsilon}TU^-_{\mathbf{j}^{\varepsilon};i}$, for $\varepsilon \in \{0,1\}$ and $i \in I_{[\varepsilon+h+1]}$;
- $U^+_{\mathbf{j}^{\varepsilon};i}T\mathcal{V}_*^{\varepsilon'-}$, for $\varepsilon, \varepsilon' \in \{0,1\}$ and $i \in I_{[\varepsilon+h+1]}$;
- $U^+_{\mathbf{j}^{\varepsilon};i}TU^-_{\mathbf{j}^{\varepsilon'};i'}$, for $\varepsilon, \varepsilon' \in \{0,1\}$, $i \in I_{[\varepsilon+h+1]}$, and $i' \in I_{[\varepsilon'+h+1]}$.

Hence ϕ is regular on the union of these subsets, i.e., on $\mathcal{U}T\mathcal{U}^-$ (where $\mathcal{U} \subset U^+$ is given by (3.13.1) and $\mathcal{U}^- = \iota(\mathcal{U}) \subset U^-$). We have

$$U^+TU^- - \mathcal{U}T\mathcal{U}^- = ((U^+ - \mathcal{U})T\mathcal{U}^-) \cup (\mathcal{U}T(U^- - \mathcal{U}^-)).$$

By Lemma 3.13, we have

$$\dim((U^+ - \mathcal{U})T\mathcal{U}^-) \leq \nu - 2 + r + \nu = \dim(G) - 2$$

and similarly

$$\dim(\mathcal{U}T(U^- - \mathcal{U}^-)) \leq \dim(G) - 2.$$

It follows that

$$\dim(U^+TU^- - \mathcal{U}T\mathcal{U}^-) \leq \dim(G) - 2.$$

Since ϕ is regular on $\mathcal{U}T\mathcal{U}^-$, we conclude that ϕ is regular on U^+TU^-. An entirely similar argument shows that ϕ is regular on U^-TU^+. It follows that ϕ is regular on the open subset $(U^+TU^-) \cup (U^-TU^+)$ of G. Together with Lemma 5.3, this implies that ϕ is regular on G. ☐

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