A PROOF OF THE POLYNOMIAL CONJECTURE FOR RESTRICTIONS OF NILPOTENT LIE GROUPS REPRESENTATIONS

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This work is dedicated to the memory of Takaaki Nomura

Abstract. Let $G$ be a connected and simply connected nilpotent Lie group, $K$ an analytic subgroup of $G$ and $π$ an irreducible unitary representation of $G$ whose coadjoint orbit of $G$ is denoted by $Ω(π)$. Let $𝒰(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}_C$, $\mathfrak{g}$ designating the Lie algebra of $G$. We consider the algebra $D_π(G)^K ≃ (𝒰(\mathfrak{g})/\ker(π))^K$ of the $K$-invariant elements of $𝒰(\mathfrak{g})/\ker(π)$. It turns out that this algebra is commutative if and only if the restriction $π|_K$ of $π$ to $K$ has finite multiplicities (cf. Baklouti and Fujiwara [J. Math. Pures Appl. (9) 83 (2004), pp. 137-161]). In this article we suppose this eventuality and we provide a proof of the polynomial conjecture asserting that $D_π(G)^K$ is isomorphic to the algebra $C[Ω(π)^K]$ of $K$-invariant polynomial functions on $Ω(π)$. The conjecture was partially solved in our previous works (Baklouti, Fujiwara, and Ludwig [Bull. Sci. Math. 129 (2005), pp. 187-209]; J. Lie Theory 29 (2019), pp. 311-341).

1. Introduction

Let $G = \exp \mathfrak{g}$ be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and $K = \exp \mathfrak{k}$ an analytic subgroup of $G$. We denote by $\mathfrak{g}^*$ (resp. $\mathfrak{k}^*$) the dual vector space of $\mathfrak{g}$ (resp. $\mathfrak{k}$). Then, $G$ (resp. $K$) acts on $\mathfrak{g}^*$ (resp. $\mathfrak{k}^*$) by the coadjoint action whose orbit space realizes by the orbit method [8], [12], [21] the unitary dual $\hat{G}$ (resp. $\hat{K}$) of $G$ (resp. $K$). We denote by $θ_G : \mathfrak{g}^* → \hat{G}$ the Kirillov map and by $Ω(π) = Ω_G(π) = θ_G^{-1}(π)$ the coadjoint orbit of $G$ associated to $π ∈ \hat{G}$. Although we use the notation $≃$ for the unitary equivalence, we often identify an irreducible unitary representation with its equivalence class.

We know in the nilpotent case the branching laws of induced and restricted representations ([15], [16]). Let $p : \mathfrak{g}^* → \mathfrak{k}^*$ be the restriction mapping. For $π ∈ \hat{G}$, we consider a finite measure $μ_π$ on $\mathfrak{g}^*$ equivalent to the canonical measure on the orbit $Ω_G(π)$ which is regarded as a measure on $\mathfrak{g}^*$. Put $ν_π = (θ_K ∘ p)_*(μ_π)$. The restriction $π|_K$ of $π$ to $K$ is disintegrated as:

$$π|_K ≃ \int_K \oplus \sigma m_\sigma^\mathfrak{g} dν_π(\sigma),$$

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where the multiplicities $m_\sigma^\pi$ are obtained as the number of the $K$-orbits contained in $\Omega_G(\pi) \cap p^{-1}(\Omega_K(\sigma))$ (cf. [11] and [17]).

In other respects, it is well known ([2], [10], [11]) that in these situations the multiplicities are either uniformly bounded almost everywhere or equal to the infinity almost everywhere. According to these two eventualities, we say that the representation $\pi|_K$ has either finite or infinite multiplicities.

We denote by $\mathcal{U}(g)$ the enveloping algebra of $g_\mathbb{C}$ and let $\text{ker}(\pi)$ be the primitive ideal of $\mathcal{U}(g)$ associated to $\pi$. We introduce the algebra

\[ \mathcal{U}_\pi^\ast(g)^f = \{A \in \mathcal{U}(g); [A, t] \subset \text{ker}(\pi)\} \]

and its image

\[ D_\pi(G)^K \equiv \mathcal{U}_\pi^\ast(g)^f / \text{ker}(\pi) \equiv (\mathcal{U}(g) / \text{ker}(\pi))^K, \]

where the last member designates the quotient algebra of $K$-invariant elements. The algebra $D_\pi(G)^K$ was the object of our three previous works [4], [5] and [6]. In particular, we proved [5] that our algebra $D_\pi(G)^K$ is commutative if and only if the restricted representation $\pi|_K$ has finite multiplicities (cf. [19]). We then substantiated in [6] Conjecture 1.1 (cf. [17]):

**Conjecture 1.1** (cf. [17]). Let $G$ be a connected and simply connected nilpotent Lie group, $K$ an analytic subgroup of $G$. Let $\pi \in \hat{G}$ be a unitary and irreducible representation of $G$ such that $\pi|_K$ is of finite multiplicities. Then the algebra $D_\pi(G)^K$ is isomorphic to the algebra $\mathbb{C}[\Omega(\pi)]^K$ of the $K$-invariant polynomial functions on $\Omega(\pi)$.

We positively proved Conjecture 1.1 in many settings, especially when $K$ is a normal subgroup of $G$ or where the orbit $\Omega(\pi)$ is flat in [6] and further, the case where $K$ is abelian or where $\Omega(\pi)$ admits a normal polarizing subgroup [7]. The aim of the present paper is to provide a proof of Conjecture 1.1.

The outline of the paper is as follows: We introduce in the next section some backgrounds about the algebra $D_\pi(G)^K$ and some algebraic tools to describe its generators in term of the enveloping algebra of $g_\mathbb{C}$. This makes use of Pedersen’s construction of the kernel $\text{ker}(\pi)$, $\pi$ being the Kirillov’s model associated to $\Omega(\pi)$ (cf. [21]). Section 3 is devoted to prepare the ingredients to prove the main result, mainly an algorithm which allows to define a rational function $P_W$ on $\Omega(\pi)$, for a given $W \in \mathcal{U}_\pi(g)^f$. Sections 4 and 5 are devoted to prove Conjecture 1.1.

2. Backgrounds

2.1. Let $G$ be a connected and simply connected nilpotent Lie group. We consider a unipotent representation of $G$ on a real vector space $V$ of finite dimension. Let $v \in V$ be an invariant vector by the action of $G$, i.e. $g \cdot v = v$ for all $g \in G$. Put for $x \in V$ arbitrarily fixed, $L_x = \{x + tv; t \in \mathbb{R}\}$, the straight line passing through $x$ and having the direction of $v$. Then, there are two possibilities: either $L_x \cap G \cdot x = L_x$ or $L_x \cap G \cdot x = \{x\}$. According to these two possibilities, we shall say that the orbit $G \cdot x$ is either saturated or non-saturated in the direction $\mathbb{R}v$. We shall utilize in what follows this fact applied to the coadjoint representation of $G$ (or a subgroup $K$ of $G$), where the invariant vector $v$ will be a linear form which vanishes on an ideal $g'$ of codimension 1 of $g$. In this situation, we shall say that the orbit in question is either saturated or non-saturated with respect to $g'$.
2.2. Let
(1) \{0\} = g_0 \subset g_1 \subset \cdots \subset g_{n-1} \subset g_n = g
be a Jordan-Hölder sequence of g, i.e. an increasing sequence of ideals of g such that \text{dim}(g_j) = j, j = 0, \ldots, n. Let \{Y_1, \ldots, Y_n\} be a Jordan-Hölder basis of g, associated to this Jordan-Hölder sequence, and \{Y_1^*, \ldots, Y_n^*\} the basis of g* such that \text{dim}(g_j^*) = \delta_{i,j}, 1 \leq i, j \leq n. Let \pi_i : g^* \to g_i^* be the canonical projection which intertwines the actions of G on g* and g_i*. For \ell \in g^*, we put \pi_i(\ell) = \text{dim}G-p_i(\ell), e(\ell) = (e_1(\ell), \ldots, e_n(\ell)) and \mathcal{E} = \{e(\ell), \ell \in g^*\}. For e \in \mathcal{E}, we define the G-invariant layer \Upsilon_e = \{\ell \in g^* : e(\ell) = e\}. Putting \epsilon_0 = 0, we define also
\[ S(e) = \{i : e_i = 1 + e_{i-1}\}, \quad g_S^* = \mathbb{R} - \text{vect}\{Y_i^* : i \in S(e)\} \]
\[ T(e) = \{i : e_i = e_{i-1}\}, \quad g_T^* = \mathbb{R} - \text{vect}\{Y_i^* : i \in T(e)\}. \]
Then we have \( g^* = g_S^* \oplus g_T^* \). There exists an order among the elements of \mathcal{E} = \{e^{(1)} > \cdots > e^{(k)}\} such a manner that \Upsilon_{e^{(1)}} \cup \cup_{j \leq i} \Upsilon_{e^{(j)}} are Zariski open sets of g* for every i. In this way all the layers \Upsilon_e are semi-algebraic set, i.e. difference of two Zariski open sets of g*. Let \Upsilon_e be an arbitrary layer, we write \S(e) = \{j_1 < \cdots < j_r\} where r designates the dimension of the G-orbits in \Upsilon_e. Then there exist some functions \( R^e_j : \Upsilon_e \times \mathbb{R}^r \to \mathbb{R}, \quad j = 1, \ldots, n \) such that:
(a) For \( f \in \Upsilon_e \) fixed, \( x = (x_1, \ldots, x_r) \to R^e_j(f, x) : \mathbb{R}^r \to \mathbb{R} \) is a polynomial function in x and the coefficients are G-invariant functions on \Upsilon_e;
(b) \( R^e_j(f, x) = x_k \) for \( j = j_k \in S(e), \quad f \in \Upsilon_e \);
(c) If \( j_k \leq j < j_{k+1} \), then \( R^e_j(f, x) \) depends only on \( x_1, \ldots, x_k \);
(d) For any \( f \in \Upsilon_e \), the coadjoint orbit \( G \cdot f \) is given by:
\[ G \cdot f = \left\{ \sum_{j=1}^n R^e_j(f, x)Y^*_j; x \in \mathbb{R}^r \right\}, \]
(see [22]).

Let \( r^e_j(f) \) be the image of \( \mathcal{W}(g) \) by the symmetrization of the element
\[ R^e_j(f, -iY_j, \ldots, -iY_{j^*}) \]
in the symmetric algebra \( S(g) \) of \( g_C \), namely, we replace the variable \( x_k \) in \( R^e_j(f, x) \) by \(-iX_{j_k} \). Notice in particular that \( r^e_j(f) = -iY_{j_k} \). Let \( V_e \) be the subspace of \( S(g) \) spanned by the elements of the form \( Y_{j_1}^{\alpha_1} \cdots Y_{j^*_r}^{\alpha_r}, \alpha_1, \ldots, \alpha_r \in \mathbb{N} \), and let \( F_e \) be the image in \( \mathcal{W}(g) \) of \( V_e \) by the symmetrization. On the other hand, let \( E_e \) be the subspace of \( \mathcal{W}(g) \) spanned by the elements of the form \( Y_{j_1}^{\alpha_1} \cdots Y_{j^*_r}^{\alpha_r}, \alpha_1, \ldots, \alpha_r \in \mathbb{N} \). If \( S(e) = \emptyset \), we put \( V_e = F_e = E_e = \mathbb{C} \cdot 1 \). Pedersen proved that the primitive ideal \( \ker(\pi) \), where \( \pi \in \hat{G} \) such that \( f \in \Omega(\pi) \) is generated by the elements
\[ u^e_j(f) = Y_j - ir^e_j(f), \quad j \in T(e) \]
and that
\[ \mathcal{W}(g) = \ker(\pi) \oplus E_e = \ker(\pi) \oplus F_e \]
(see Theorem 2.1.1 and Theorem 2.2.1 in [22]). In the same way, the actions of \( \pi \) on \( E_e \) and \( F_e \) are faithful (see Lemma 2.2.12 and Lemma 2.2.13 in [22]). In this way, identifying \( E_e \) and \( F_e \) à \( \mathcal{W}(g)/\ker(\pi) \) and abusing notations, we have
\[ D_\pi(G)^K \simeq E_e^K \simeq F_e^K \simeq C[Y_{j_1}, \ldots, Y_{j_r}]^K. \]
These isomorphisms are simply isomorphisms of vector spaces.
2.3. In [13], Corwin and Greenleaf showed that Pedersen’s construction of the kernel \( \ker(\pi_\ell) \), where \( \pi_\ell \) designates the Kirillov’s model [21] which represents the class \( \theta_G(\ell) \), for \( \ell \in U_e \) leads to construct \( e \)-central elements (cf. Theorem 3.1 in [13]). These are elements \( A \) of the enveloping algebra \( \mathcal{U}(g) \) such that the operators \( \pi_\ell(A) \) are scalars for \( \ell \in U_e \). Then \( \pi_\ell(A) = \pi_\ell(A) \) for all \( \ell' \in G \cdot \ell \). More precisely, let \( U_e \subset g^* \) be one of the layers constructed above. Then there exists a Zariski open set \( Z \subset U_e \) such that \( Z \cap U_e \) is non-empty \( G \)-invariant and for all \( j \in T(e) \) there exists an \( e \)-central element \( A_j \in \mathcal{U}(g_j) \) on \( Z \cap U_e \), i.e. the operators \( \pi_\ell(A_j) \) are scalars for all \( \ell \in Z \cap U_e \) with the following properties:

1. \( A_j = P_j Y_j + Q_j \), where \( P_j, Q_j \) are in \( \mathcal{U}(g_{j-1}) \).
2. \( P_j \) is \( e \)-central on \( Z \cap U_e \) and does not belong to \( \ker(\pi_\ell) \).
3. \( \pi_\ell(A_j) = \phi_j(\ell) \text{Id} \) for \( \ell \in Z \cap U_e \), where \( \phi_j(\ell) = \tilde{p}_j(\ell) Y_j + \tilde{q}_j(\ell) \), \( \tilde{p}_j \) and \( \tilde{q}_j \) being non-singular rational functions on \( Z \cap U_e \) depending only on \( (Y_1, \ldots, Y_{j-1}) \).

While the rational function \( \tilde{p}_j(\ell) \) is \( G \)-invariant and never vanishes on \( Z \cap U_e \).

Moreover, we easily see that the system \( \{ A_j ; j \in T(e) \} \) of these \( e \)-central elements separates the orbits in \( Z \cap U_e \).

Having given the construction of \( A_j \), Corwin-Greenleaf [13] remarked the following: Dropping out the Zariski open set \( Z \cap U_e \) from \( U_e \), we notice that, \( U_e \backslash Z \) being \( G \)-invariant and semi-algebraic, the parametrization of the orbits in \( U_e \) is carried out and retains all its properties on this sub-layer in \( U_e \). We are able to repeat the whole process starting from \( U_e \backslash Z \). Since \( U_e \) is semi-algebraic, the ascendent chain condition for the ideals in \( \mathbb{C}[g^*] \) assures that the process terminates after a finite number of steps. So, patching the pieces together, we may suppose that \( Z \cap U_e = U_e \).

Let \( \rho \) be a unitary representation of \( G \). We denote by \( \mathcal{H}_\rho, \mathcal{H}_\rho^\infty \) and \( \mathcal{H}_\rho^{-\infty} \) respectively the space of \( \rho \), that of its differentiable vectors and the anti-dual of \( \mathcal{H}_\rho^\infty \) (cf. [9] and [23]). For \( a \in \mathcal{H}_\rho^\pm\infty \) and \( b \in \mathcal{H}_\rho^{-\infty} \), we denote by \( \langle a, b \rangle \) the image of \( b \) by \( a \), so that \( \langle a, b \rangle = \langle \bar{b}, a \rangle \). Being given a subgroup \( H \) of \( G \) and its unitary character \( \chi \), put

\[
(\mathcal{H}_\rho^{-\infty})^H,\chi = \{ a \in \mathcal{H}_\rho^{-\infty} ; \rho(h)a = \chi(h)a, \forall h \in H \}.
\]

3. First preparations to the proof of Conjecture [13].

3.1. Recall once again our situation. Let \( G = \exp g \) be a connected and simply connected nilpotent Lie group with Lie algebra \( g \), \( K = \exp \mathfrak{k} \) an analytic subgroup of \( G \) and \( \pi \) an irreducible unitary representation of \( G \) whose coadjoint orbit is denoted by \( \Omega(\pi) \). For \( \ell \in \Omega(\pi) \), we designate by \( b[\ell|_\mathfrak{k}] \) a polarization of \( \mathfrak{t} \) at \( \ell|_\mathfrak{k} \in \mathfrak{t}^\ast \). We know [5] that \( \pi|_K \) has finite multiplicities if and only if \( b[\ell|_\mathfrak{k}] + b(\ell) \) is a Lagrangian subspace for the bilinear form \( B_\ell : (X, Y) \mapsto \ell([X, Y]) \), at \( \mu_\pi \)-almost all \( \ell \) in \( \Omega(\pi) \).

At the flag of ideals [1] of \( g \), let \( \mathcal{I} = \{ i_1 < \cdots < i_d \} \) where \( d = \dim \mathfrak{t} \) be the set of indices \( 1 \leq i \leq n \) such that \( \mathfrak{t} \cap g_i \neq \mathfrak{t} \cap g_{i-1} \) and put

\[
\mathcal{I} = \{ j_1 < \cdots < j_q \} = \{ 1, 2, \ldots, n \} \setminus \mathcal{I}
\]

with \( q = \dim(g/\mathfrak{t}) \). Putting \( \mathfrak{t}_d = \mathfrak{t} \) and \( \mathfrak{t}_{d+r} = \mathfrak{t} + g_{j_r} \) for \( 1 \leq r \leq q \), we obtain a sequence of subalgebras of \( g \):

\[
(2) \quad \mathfrak{t} = \mathfrak{t}_d \subset \mathfrak{t}_{d+1} \subset \cdots \subset \mathfrak{t}_{n-1} \subset \mathfrak{t}_n = g, \quad \dim(\mathfrak{t}_r/\mathfrak{t}_{r-1}) = 1.
\]
Furthermore, considering \( \mathfrak{k}_s = \mathfrak{k} \cap \mathfrak{g}_s \), \( 1 \leq s \leq d \), we get a flag of ideals of \( \mathfrak{k} \):

\[
\{0\} = \mathfrak{k}_0 \subset \mathfrak{k}_1 \subset \cdots \subset \mathfrak{k}_{d-1} \subset \mathfrak{k}_{d} = \mathfrak{k}, \quad \dim \mathfrak{k}_s = s.
\]

3.2. Let \( \ell \in \Omega(\pi) \). Taking there a real polarization \( \mathfrak{b}[\ell] \) of \( \mathfrak{g} \), we realize \( \pi \) as \( \pi = \text{ind}_{B[\ell]}^G \mathcal{X}_\ell \) with \( B[\ell] = \exp(\mathfrak{b}[\ell]) \) and \( \mathcal{X}_\ell \) is the unitary character of \( B[\ell] \) whose differential is \( i\mathcal{X}_\ell[I] \).

On the other hand, by means of the flag \([3]\), we construct \([8]\) the Vergne polarization \( \mathfrak{b}[\ell]|_s \) of \( \ell \) at \( \ell|_s \in \mathfrak{k}^s \). Put \( B[\ell]|_s = \exp(\mathfrak{b}[\ell]|_s) \). It is easy to verify \([6]\) that the formula

\[
\langle a^K_\ell, \varphi \rangle = \int_{B[\ell]|_s/(B[\ell]|_s \cap B[\ell])} \frac{\varphi(b)}{\mathcal{X}_\ell(b)} db \quad (\forall \varphi \in \mathcal{H}_\pi^\infty),
\]

designating an invariant measure on the homogeneous space \( B[\ell]|_s/(B[\ell]|_s \cap B[\ell]) \), gives us a semi-invariant generalized vector \( a_\ell \) in \( (\mathcal{H}_\pi^\infty)^{B[\ell]|_s, \mathcal{X}_\ell} \).

Suppose that \( \pi|_K \) has finite multiplicities. This would say as in the case of the monomial representations, that \( \mathfrak{b}[\ell]|_s + \mathfrak{g}(\ell) \) is a Lagrangian subspace of \( \mathfrak{g} \) for \( \mathfrak{b}[\ell] \) at almost all \( \ell \in \Omega(\pi) \) with respect to the invariant measure. Then, it results \( \mu_{\pi}|_s \) almost everywhere in \( \Omega(\pi) \) that \( a_\ell \) is a vector for all the elements of \( D(\pi)(G)^K \) acting on \( \mathcal{H}_\pi^\infty \) by continuity. This also means that for every \( W \in \mathcal{H}_\pi \mathfrak{g}(\ell)^s \) we have

\[
W \cdot a_\ell := \pi(W) a_\ell = \lambda_\ell(W) a_\ell
\]

with a certain scalar \( \lambda_\ell(W) \) (cf. \([6]\)). Remark that this scalar \( \lambda_\ell(W) \) does not depend on the choice of the polarization \( \mathfrak{b}[\ell] \) and of the flag \([3]\) (cf. \([15]\), Proposition 3).

Further, we also have the

**Theorem 1** \([6]\), Theorem 3.4. Suppose that \( \pi|_K \) has finite multiplicities. The homomorphism \( \mathcal{Z}_\pi(\mathfrak{g})^s \ni W \mapsto P_W : \ell \mapsto \lambda_\ell(W) \) defines an imbedding of \( D(\pi)(G)^K \) into the field \( \mathbb{C}(\Omega(\pi))^K \) of rational \( K \)-invariant functions on \( \Omega(\pi) \).

We can say even more. Aligning the two sequences \([2]\) and \([3]\), we have a sequence of subalgebras of \( \mathfrak{g} \):

\[
\{0\} = \mathfrak{k}_0 \subset \mathfrak{k}_1 \subset \cdots \subset \mathfrak{k}_d = \mathfrak{k} \subset \cdots \subset \mathfrak{k}_{n-1} \subset \mathfrak{k}_n = \mathfrak{g}.
\]

Relatively to this sequence, let us extract again a vector \( X_k \in \mathfrak{k}_k \setminus \mathfrak{k}_{k-1} \) and put \( \ell_k = \mathfrak{k}(X_k) \) for \( 1 \leq k \leq n \). Consider the action of \( K \) on the sequence \([3]\) and define two sets \( S_k, T_k \) of jump and non-jump indices. Namely, we denote by \( e^K_\ell \) the dimension of the \( K \)-orbit of \( \ell|_{\mathfrak{g}(\ell)} \in \mathfrak{g}^s \) for every \( 1 \leq j \leq n \). Then we agree \( e^K_\ell(\ell) = 0 \).

For each index \( j \), the same possibility of the alternative \( e^K_j(\ell) = e^K_{j-1}(\ell) + 1 \) or \( e^K_j(\ell) = e^K_{j-1}(\ell) \) happens \( \mu_{\pi}-\text{almost everywhere on } \Omega(\pi) \). We denote by \( S_K \) the set of the indices \( 1 \leq j \leq n \) which verify the first eventuality and by \( T_K \) that of indices of the second eventuality. Put \( \mathcal{Z}_\pi(\mathfrak{g})^s = \mathcal{Z}_\pi(\mathfrak{g})^s \cap \mathcal{Z}(\mathfrak{g})^s \). Theorem 2 is proved in \([5]\).

**Theorem 2.** We keep the same notations and hypotheses. Then:

1. If \( j \in S_K \), then \( \mathcal{Z}_\pi(\mathfrak{g})^s = \mathcal{Z}_\pi(\mathfrak{g})^s \chi_j(\mathfrak{g}) + \mathcal{Z}_\pi(\mathfrak{g})^s \chi_j(\mathfrak{g}) \mathcal{Z}(\mathfrak{g})^s \chi_j(\mathfrak{g}) \).

2. If \( j \in T_K \), then \( \mathcal{Z}_\pi(\mathfrak{g})^s \neq \mathcal{Z}_\pi(\mathfrak{g})^s \chi_j(\mathfrak{g})^s + \mathcal{Z}_\pi(\mathfrak{g})^s \chi_j(\mathfrak{g}) \mathcal{Z}(\mathfrak{g})^s \chi_j(\mathfrak{g}) \) and there exists \( W_j \in \mathcal{Z}_\pi(\mathfrak{g})^s \) having the form \( W_j = aX_j + b(a, b \in \mathcal{Z}(\mathfrak{g})^s) \), \( a \in \mathcal{Z}_\pi(\mathfrak{g})^s \) with \( \pi(a) \neq 0 \).

For \( j \in T_K \) and \( \ell \in \Omega(\pi) \), \( P_{W_j}(\ell) = \varphi_j(\ell) \ell_j + \psi_j(\ell) \), where \( \varphi_j(\ell), \psi_j(\ell) \) are two rational functions of \( \ell_1, \ldots, \ell_{j-1} \).
As a direct consequence of this result, we obtain as in [6]:

**Proposition 1.**

1. Let $A$ be an element of $\mathcal{Z}_x(t_m)\ell$ for $1 \leq m \leq n$ satisfying $\pi(A) \neq 0$. Then there exists two non-zero polynomials $\beta_A$ and $\gamma_A$ of the elements $\{W_j; j \in T_K, j \leq m\}$ such that $\beta_A A \equiv \gamma_A$ modulo $\ker(\pi)$.

2. The functions $\{P_{W_j}(\ell); j \in T_K\}$ rationally generate the field $\mathbb{C}(\Omega(\pi))^K$.

### 3.3. On the coordinates of the coadjoint orbit.

As in Section 2, we start from the flag of ideals (1) of $\mathfrak{g}$ to parameterize the orbit $\Omega = \Omega(\pi)$ and denote there by $S_\Omega$ and $T_\Omega$ respectively the sets of jump and non-jump indices. Let $\{Y_1, \ldots, Y_n\}$ be a Malcev basis adapted to the flag (1), $\ell_j = \ell(Y_j)$ ($1 \leq j \leq n$) for $\ell \in \Omega$, $S_\Omega = \{s_1 < \cdots < s_r\}$, $r = \dim \Omega$ and $x_k = \ell_{s_k}$ for $1 \leq k \leq r$. Describe as in Section 2 the orbit $\Omega$ by the polynomial relations

$$\ell_j = F_j(x_1, \ldots, x_k), \quad s_k < j < s_{k+1},$$

where $x = (x_1, \ldots, x_r)$ runs through $\mathbb{R}^r$. In these circumstances the rational functions on $\Omega$ are nothing but the rational functions of the variables $(x_1, \ldots, x_r)$.

For $1 \leq k \leq r$, let $I^{(k)}$ be the set of the $K$-invariant polynomial functions on $\Omega$, which depend only on the variables $\{x_i; i \leq k\}$. The arguments developed in the pages 60–61 of [24] make us see that every $R$ in $\mathbb{C}(\Omega)^K$ verifying

$$\frac{\partial R}{\partial x_k} \neq 0 \text{ and } \frac{\partial R}{\partial x_i} = 0 (i > k)$$

is written in the form $P/Q$, where $P$ and $Q$ belong to $I^{(k)}$. Therefore, the existence of such an element $R$ means that $I^{(k-1)}$ is strictly contained in $I^{(k)}$. Next, let $Q = \sum_{i=0}^{m} Q_i x_k^i$ ($m > 0$) be an element of $I^{(k)} \setminus I^{(k-1)}$, where $Q_i$ ($0 \leq i \leq m$) designate polynomials of $(x_1, \ldots, x_{k-1})$ verifying $Q_m \neq 0$. We then confirm that $Q_m$ and $m Q_m x_k + Q_{m-1}$ are $K$-invariant polynomials.

### 4. Proof of Conjecture [1.1] First part

We keep all our notations. We first define the following:

**Definition 1.**

1. We say that $W \in \mathcal{Z}_x(\mathfrak{g})\ell$ is $K$-diagonal, if

$$\pi(W) a_\ell = P_W(\ell) a_\ell$$

for a certain scalar $P_W(\ell) \in \mathbb{C}$ independent of the polarizations chosen to describe the distribution $a_\ell$ and $\ell \mapsto P_W(\ell)$ extends to a rational function on $\Omega$.

2. Let $\mathcal{C}$ be the set of $K$-diagonal elements of $\mathcal{Z}_x(\mathfrak{g})\ell$. Let

$$\Theta : \mathcal{C} \ni W \mapsto P_W$$

**Remark 1.**

1. From ([1], Theorem 4.1), any $K$-diagonal element of $\mathcal{C}(\mathfrak{g})$ belongs to $\mathcal{Z}_x(\mathfrak{g})\ell$.

2. Definition 1 is posed independently from the fact that $\pi|_K$ has finite multiplicities or not. In the case of finiteness, any element of $\mathcal{Z}_x(\mathfrak{g})\ell$ is $K$-diagonal (cf. Theorem [1]).

Next, we can easily adapt the arguments of ([6], Lemma 3.2) to prove the following:
Lemma 1. Let $W \in \mathcal{U}(\mathfrak{g})$ be $K$-diagonal. Then $P_W$ is identically zero if and only if $W \in \ker(\pi)$.

Proof. If $W \in \ker(\pi)$, $P_W(\ell) \equiv 0$ because $a_\ell \in \mathcal{H}_0^{-\infty}$. Suppose that $P_W(\ell) = 0$ almost everywhere on $\Omega$ and let us prove that $W \in \ker(\pi)$ by induction on $\dim G$.

Let $p : \mathfrak{g}^\ast \to (\mathfrak{k}_{n-1})^\ast$ be the restriction mapping and $K_{n-1} = \exp(\mathfrak{k}_{n-1})$. If $\Omega$ is non-saturated with respect to $\mathfrak{k}_{n-1}$, there exists in $\ker(\pi)$ an element $A$ having the form $A = X_n + V$ with a certain $V \in \mathcal{U}(\mathfrak{k}_{n-1})$. Making use of $A$ to kill from $W$ the part which is found outside of $\mathcal{U}(\mathfrak{k}_{n-1})$, we can suppose that $W \in \mathcal{U}(\mathfrak{k}_{n-1})$. Since $p(\Omega)$ is a $K_{n-1}$-orbit, the induction hypothesis gives us immediately the desired result.

Suppose now that $\Omega$ is saturated with respect to $\mathfrak{k}_{n-1}$. This implies that $W$ belongs to $\mathcal{U}(\mathfrak{k}_{n-1})$. The restriction $\pi|_{K_{n-1}}$ is disintegrated as $\pi|_{K_{n-1}} \simeq \int_{\mathbb{R}} \pi_t dt$ into a one parameter family $\{\pi_t\}_{t \in \mathbb{R}}$ of irreducible unitary representations of $K_{n-1}$ and accordingly the restriction $p(\Omega) = \Omega|_{\mathfrak{k}_{n-1}}$ is decomposed as $p(\Omega) = \cup_{t \in \mathbb{R}} \omega_t$, where $\omega_t$ is the coadjoint orbit of $K_{n-1}$ associated to $\pi_t$. Then, the induction hypothesis says that $W$ belongs to $\ker(\pi_t)$ for almost all $t \in \mathbb{R}$ and hence $W \in \ker(\pi)$. \hfill $\square$

The first step to prove Conjecture \ref{eq:14} consists in proving Theorem \ref{thm:3}.

Theorem 3. Let $\pi \in \hat{G}$ and let $W \in \mathcal{U}(\mathfrak{g})$ be $K$-diagonal. The function $P_W$ extends to a $K$-invariant polynomial function on $\Omega$.

The proof of Theorem \ref{thm:3} will be achieved through different steps. Let us start with the following:

4.1. A preliminary inductive proof. In order to prove Theorem \ref{thm:3} we proceed by induction on $\delta(G, K) = \dim G + \dim G/K$. For $\delta(G, K)$, $G$ turns out to be abelian and the answer is immediate. Consider the flag of algebras $Y_k$ of $\mathfrak{g}$ and for the sake of simplicity of notation, denote $\mathfrak{g}' = \mathfrak{k}_{n-1}$ which contains $\mathfrak{k}$. Put $G' = \exp \mathfrak{g}'$ and suppose that Theorem \ref{thm:3} holds for $G'$.

4.1.1. Case where the ideal $\mathfrak{g}'$ is of non-saturation. Suppose that the orbit $\Omega$ is non-saturated with respect to $\mathfrak{g}'$, namely that $n \in T_\Omega$. Then the projection $pr : \mathfrak{g}^\ast \to \mathfrak{g}'^\ast$ turns out to be a $K$-equivariant homeomorphism between $\Omega$ and $\omega = pr(\Omega)$ which is a $G'$-orbit. Hence, $\mathbb{C}[\Omega]^{K} \cong \mathbb{C}[\omega]^{K}$. On the other hand, $\pi' = \pi|_{G'}$ is irreducible and there exists in $\ker(\pi)$ an element $W'$ having the form $W' = X_n + A$ with $A \in \mathcal{U}(\mathfrak{g}')$ which allows us to identify $D_\pi(G)^K$ with $D_{\pi'}(G')^K$. Since $\omega$ is the coadjoint orbit of $G'$ associated to $\pi'$ and since $a_\ell = a_\ell|_{\mathfrak{g}'}$, the induction hypothesis proves Theorem \ref{thm:3} in this case.

4.1.2. Case where the ideal $\mathfrak{g}'$ is of saturation. Suppose now that $\Omega$ is saturated with respect to $\mathfrak{g}'$, namely that $n \in S_\Omega$. We have Lemma \ref{lem:2}.

Lemma 2 (\cite{2}, [6] Lemma 4.1). There exists one and only one index $2 \leq j \leq n-1$ belonging to $S_\Omega$ and $b \in \mathcal{U}(\mathfrak{g}_{j-1})$ such that $Y_j + b \in \mathcal{U}(\mathfrak{g}_j)^{\mathfrak{g}'}$.

Likewise, if $j = s_i$ ($1 \leq i \leq r-1$), there exists a $G'$-invariant polynomial function

$$\alpha = x_i + \varphi(x_1, \ldots, x_{i-1})$$
Let \( \omega \), which separates the \( G' \)-orbits \( w_\alpha = \{ \ell \in \Omega : \alpha(\ell) = \alpha \} \) contained in \( pr(\Omega) \). This means that \( pr(\Omega) = \bigcap_{\alpha \in \mathbb{R}} \omega_\alpha \), the disjoint union of \( G' \)-orbits \( \omega_\alpha \). Accordingly,

\[
(9) \quad \pi|_{G'} \simeq \int_{\mathbb{R}} \pi_\alpha d\alpha
\]

with \( \pi_\alpha = \theta_{G'}(\omega_\alpha) \) for all \( \alpha \in \mathbb{R} \).

Since the orbit \( \Omega \) is saturated with respect to \( \mathfrak{g}' \), for any \( \ell \in \Omega \) there exists then a polarization \( \mathfrak{h}[\ell] \) at \( \ell \) contained in \( \mathfrak{g}' \), which is also a polarization at \( \ell|_{\mathfrak{g}'} \). Furthermore we can suppose that \( W \in \mathcal{U}(\mathfrak{g}') \), since \( \pi(W) a_\ell = P_W(\ell) a_\ell, \ell \in \Omega, \) and \( \pi_\ell = \text{ind}_{G'}^\pi_{\ell|_{\mathfrak{g}'}} \). It follows then from the definition of \( a_\ell, \ell \in \Omega, \) that

\[
P_W^G(\ell) = P_W^G(\ell|_{\mathfrak{g}'}, \ell \in \Omega,
\]

where the index \( G \) (resp. \( G' \)) indicates the action of \( W \) on \( a_\ell \) (resp. on \( a_\ell|_{\mathfrak{g}'} \)). We apply the induction hypothesis to \( W \) and \( G' \). Then it follows that the function \( P_W \), which is rational on \( \Omega \) restricts to the \( G' \)-orbits \( \omega_\alpha, \alpha \in \mathbb{R}, \) as a polynomial function. Let \( \ell \) be a point of \( \Omega \), for each real number \( t \), let \( \alpha \) be such that \( Ad^t(\exp(tX_n)) \ell \in \omega_\alpha \), then:

\[
P_W(Ad^t(\exp(tX_n)g')\ell) = P_W(\alpha, g') = \frac{A(\alpha, g')}{B(\alpha, g')}, g' \in G',
\]

for two polynomial functions \( A, B \). Since \( P_W|_{\omega_\alpha} \) is polynomial, we have that \( B \) is independent of the variable \( g' \) and so \( P_W \) is given by a polynomial function \( A \) devided by a polynomial function in \( \alpha \).

The following consequence is then immediate.

**Corollary 1.** Suppose that \( \mathfrak{k} \) is contained in an ideal of codimension 2. Then for every \( K \)-diagonal \( W \in \mathcal{U}_x(\mathfrak{g})^\ell \), the function \( \ell \mapsto P_W(\ell) \) is polynomial.

**Proof.** Let \( \mathfrak{h}_j, j = 1, 2 \) be two distinct ideals of codimension 1 containing \( \mathfrak{k} \). Accordingly to Subsection [1.1], we can assume that the orbit \( \Omega \) is saturated with respect to \( \mathfrak{h}_j, j = 1, 2 \). We fix the flag (1) such that \( \mathfrak{g}_n = \mathfrak{h}_1 \cap \mathfrak{h}_2 \), and \( \mathfrak{g}_{n-1} = \mathfrak{h}_1 \), thus, if \( w \) is the \( G_{n-2} \)-orbit of \( \ell|\mathfrak{g}_{n-2} \), the set of jump indices are \( S_\Omega = S_w \cup \{ n - 1, n \} \), or \( S_\Omega = S_w \cup \{ i, k, n - 1, n \} \). In the first case, we have \( n - 1 = s_{i, n} = s_{i+1} \), by (8), there is a \( H_1 \)-invariant polynomial function \( \alpha_1 = x_i + \varphi_1(x_1, x_{i-1}) \) separating the \( H_1 \)-orbits, and replacing \( \mathfrak{h}_1 \) by \( \mathfrak{h}_2 \) in the flag (1), there is a \( H_2 \)-invariant polynomial function \( \alpha_2 = x_{i+1} + \varphi_2(x_1, x_{i-1}) \) separating the \( H_2 \)-orbits. Moreover, for any complex numbers \( c_j \), there is no common divisor for \( \alpha_1 + c_1 \) and \( \alpha_2 + c_2 \). In the second case, suppose the jump indices for the \( H_1 \)-orbit \( w_1 \) of \( \ell|\mathfrak{h}_1 \) are \( S_w \cup \{ i, n - 1 \}, \) with \( i = s_{i, 1} \), and by (8), there is a \( G_{n-2} \)-invariant polynomial function \( \beta_1 = x_{i+1} + \varphi_1(x_1, x_{i-1}) \) separating the \( G_{n-2} \)-orbits in the \( H_1 \)-orbit \( w_1 \). Suppose \( X_n \) be in \( \mathfrak{h}_2 \setminus \mathfrak{h}_1 \), and \( k = s_{i, 2} \), by (8) there is \( \alpha_2 = x_{i+2} + \varphi_2(x_1, x_{i-1}) \) separating the \( H_2 \)-orbits in \( \Omega \). Fix \( X_n \) such that \( \alpha_2(\exp(tX_n))l = t \) for each \( l \) in \( \Omega \) such that \( \alpha_2(l) = 0 \). Finally put: \( \alpha_1(\exp(tX_n))l = \beta_1(l|\mathfrak{h}_1) \) or

\[
\alpha_1(x_i) = (e^{-\alpha_2(x_i)Ad^t(X_n)}(x_i)\beta_1(x_i) = \sum_m (-\alpha_2(x_i))m \frac{m!}{m} \beta_1((Ad^t(X_n))m(x_i))
\]
The function $\alpha_2$, polynomial on $\Omega$ is $H_1$-invariant and separates the $H_1$ orbits in $\Omega$. Moreover, since for any complex numbers $c_1$ and $c_2$,

$$\alpha_1 + c_1 = e^{2z\pi i}(z) + c_1 + \sum_{m>0} \frac{(-\alpha_2 - c_2)^m}{m!} e^{2z\pi i}(z)^m,$$

and $e^{2z\pi i}(z) = x_i + \psi(x_i, \ldots, x_{i-1})$, there is no common divisor for $\alpha_1 + c_1$ and $\alpha_2 + c_2$. In both cases, applying the induction hypothesis to $W$ and $H_j$, we can write $P_W$ as a quotient of a polynomial function $A_j$ by a function $B_j(\alpha_j)$, polynomial in $\alpha_j$. Thus:

$$P_W = \frac{A_1}{B_1(\alpha_1)} = \frac{A_2}{B_2(\alpha_2)}, \quad B_2(\alpha_2)A_1 = B_1(\alpha_1)A_2.$$

Since $\alpha_1 + c_1$ and $\alpha_2 + c_2$ have no common divisor, $P_W$ itself is a polynomial function.

On the other hand, let $W \in \mathcal{U}_\pi(\mathfrak{t}_v)$. If $v \leq d$, $W$ belongs to $\mathcal{U}_\pi(\mathfrak{t})^\ell$ and the operator $\sigma(W)$ is a scalar for almost all $\sigma \in \hat{K}$ with respect to the measure $\nu_\sigma$ used in the irreducible decomposition of $\pi|_K$. Then, we can apply Theorem 2.1.1 in [19] to get:

**Proposition 2.** For any $W \in \mathcal{U}_\pi(\mathfrak{t})^\ell$, the function $\ell \mapsto P_W(\ell)$ is polynomial on $\Omega$.

### 4.2. Proof of Theorem 3

As usual, we can assume that the center $\mathfrak{z}$ of $\mathfrak{g}$ has dimension 1, that $\pi(= \pi_\ell)$ is not 0 on $\mathfrak{z}$ and that $\mathfrak{z} \subset \mathfrak{t}$. Also according to [10], we can assume that for every subalgebra $\mathfrak{g}'$ of codimension one containing $\mathfrak{t}$, that $\Omega$ is saturated with respect to $\mathfrak{g}'$. In particular a polarization $b[\ell]$ with $B[\ell] = \exp(b[\ell])$ of $\ell$ can always be found in $\mathfrak{g}'$ and $W \in \mathcal{U}(\mathfrak{g}')$.

We make now a further induction on $j_0$, the smallest index $j \in \{1, \ldots, n\}$, such that $W \in \mathcal{U}(\mathfrak{t}_{j_0})$. If $j_0 \leq d$, then $W$ is an e-central element of Corwin-Greenleaf for the projection of $\Omega$ on $\mathfrak{t}^*$ and hence the function $P_W(\ell)$ is polynomial as in Proposition 2. We can therefore assume that $j_0 \geq d+1$.

Let now $\mathfrak{l} = \mathfrak{t}_{d+1}$ and $L = \exp \mathfrak{l}$. If the generic $L$-orbits in $\Omega_1$ are non-saturated with respect to $\mathfrak{t}$, then exists a $\nu = aX_{d+1} + b, a, b \in \mathcal{U}(\mathfrak{t})$ which is e-central for $\Omega_1$. Applying $W, \nu$ to the Penney distribution $a_\ell(\ell \in \Omega)$, we see that they commute modulo ker($\pi$) and so $W$ is also $L$-invariant. If we use the Penney distributions $a_\ell^L$ (as in formula (11)) and if $b[\ell]|_1 \cap b[\ell] = b[\ell]|_1 \cap b[\ell]$, we see that for some $S \in (l[\ell]|_1 \cap \ker(\ell)) \setminus \mathfrak{t}$, we have for any $\varphi \in \mathcal{H}^\infty$:

$$\langle W \cdot a_\ell^L, \varphi \rangle = \int_{B[\ell]|_1/(B[\ell]|_1 \cap B[\ell])} \frac{\pi(W^*)\varphi(b)\chi_\ell(b)db}{\pi(W^*)\varphi(\exp(sS)b)\chi_\ell(\exp(sS)b)dbds}$$

$$= \int_{R} \int_{B[\ell]|_1/(B[\ell]|_1 \cap B[\ell])} \frac{\pi(W^*)\varphi(\exp(-sS)b)\chi_\ell(\exp(-sS)b)dbds}{\pi(W^*)(\pi(\exp(-sS))\varphi(b)\chi_\ell(\exp b)dbds}$$

$$= \int_{R} P_W(\ell|_1) \int_{B[\ell]|_1/(B[\ell]|_1 \cap B[\ell])} \frac{\varphi(\exp(sS)b)\chi_\ell(\exp b)dbds}{P_W^K(\ell)(a_\ell^L, \varphi).}
Therefore $W$ is $L$-diagonal. Since $\delta(G, L) < \delta(G, K)$, the induction hypothesis implies that $P^K_W = P^L_W$ is polynomial.

Recall now that we are in the situation where the orbit $\Omega$ is saturated with respect to $\mathfrak{k}_{n-1}$. There exists then by Lemma 2 a unique index $2 \leq r_0 \leq n - 1$ belonging to $S_\Omega$ and $b \in \mathcal{U}(\mathfrak{g}_{r_0-1})$ such that

$$\kappa = Y_{r_0} + b \in \mathcal{U}_n(\mathfrak{g}_{r_0})^g$$

and $[X_n, \kappa] \neq 0 \text{ mod } \ker(\pi)$. The polynomial function $P_n$ on $\Omega|_{\mathfrak{k}_{n-1}}$ then separates the $K_{n-1}$-orbits $\omega_y, y \in \mathbb{R}$ and, as we have seen in 11.1.2, $W$ belongs to $\mathcal{U}(\mathfrak{k}_{n-1})$ and $P_W$ can be written as $\frac{\partial}{\partial \bar{\nu}}$ for a polynomial function $A$ on $\Omega$ divided by a polynomial $B$ in the variable $P_n$.

Let now $\bar{\mathfrak{g}}$ be another ideal of $\mathfrak{g}$ of codimension 1. If $\mathfrak{t} \subset \bar{\mathfrak{g}}$, then Theorem 3 holds by Corollary 1. Hence we assume that $\mathfrak{t} \not\subset \bar{\mathfrak{g}}$. Let us treat first the case where $\Omega$ is not saturated with respect to $\bar{\mathfrak{g}}$. Write $\mathfrak{g} = \mathfrak{R} \bar{\mathfrak{X}} + \bar{\mathfrak{g}}$ and $G = \exp\bar{\mathfrak{g}}$. We can again assume as in 11.1.1 that $W \in \mathcal{U}(\bar{\mathfrak{g}})$. Let $\mathfrak{L} = \mathfrak{t} \cap \bar{\mathfrak{g}}$ and $K = \exp\mathfrak{L}$. If $b[\mathfrak{l}|_{\mathfrak{t}}] \subset \bar{\mathfrak{g}}$ almost everywhere on $\Omega$, then $a_{\mathfrak{L}} = a_{\mathfrak{L}}|_{\mathfrak{t}}$ and the induction hypothesis tells us that $P_W(\mathfrak{l})$ is a polynomial function on the $\bar{G}$-orbit $\bar{\Omega} = \bar{p}(\Omega)$, where $\bar{p} : \mathfrak{g}^* \to (\bar{\mathfrak{g}})^*$ is the restriction map. Hence $P_W$ is also a polynomial function on $\Omega$. If $b[\mathfrak{l}|_{\mathfrak{t}}] \not\subset \bar{\mathfrak{g}}$ for almost all $\mathfrak{l} \in \Omega$, let us write $b[\mathfrak{l}|_{\mathfrak{t}}] = \mathfrak{R}X(\mathfrak{l}) + b[\bar{\mathfrak{L}}]$, where $\bar{\mathfrak{L}} = \bar{p}(\mathfrak{l})$. We remark that we can take $b[\bar{\mathfrak{L}}] = \mathfrak{L}$ to be the Vergne polarisation at $\bar{\mathfrak{L}}|_{\mathfrak{t}} \in (\mathfrak{t})^*$ built from a Jordan-Hölder sequence $\mathcal{S} \cap \bar{\mathfrak{g}}$ of $\bar{\mathfrak{g}}$, $\mathcal{S}$ denoting the flag (I) of $\mathfrak{g}$. As $W$ is $K$-invariant, we see that

$$\langle W \cdot a_{\mathfrak{L}}, \varphi \rangle = \int_{\mathbb{R}} \langle W \cdot a_{\bar{\mathfrak{L}}}, \varphi(\exp(tX(\mathfrak{l}))) \rangle dt \ (\mathfrak{l} \in \Omega)$$

for $\varphi \in \mathcal{H}^{-\infty}_\pi$. We identify $\mathcal{H}^{-\infty}_\pi$ with $\mathcal{H}^{-\infty}_\bar{\pi}$. Fixing a generic $\mathfrak{l} \in \Omega$ and taking a Malcev basis in $\mathfrak{g}$ relative to $b[\mathfrak{l}]$, which contains a Malcev basis in $b[\mathfrak{l}|_{\mathfrak{t}}]$ relative to $b[\mathfrak{l}|_{\mathfrak{t}}] \cap b[\mathfrak{L}]$, we identify the space $\mathcal{H}_\pi$ of $\pi$ with $\mathbb{R}^m$, $m = \dim(\mathfrak{g}/b[\mathfrak{l}])$. Since

$$B[\mathfrak{l}|_{\mathfrak{t}}]/(B[\mathfrak{l}|_{\mathfrak{t}}] \cap B[\mathfrak{L}]) \simeq B[\mathfrak{l}|_{\mathfrak{t}}]B[\mathfrak{L}]/B[\mathfrak{l}] = \exp(\mathfrak{R}X(\mathfrak{l}))B[\bar{\mathfrak{L}}]B[\mathfrak{l}]/B[\mathfrak{l}],$$

we finally get the following two eventualities: either

$$B[\mathfrak{l}|_{\mathfrak{t}}]/(B[\mathfrak{l}|_{\mathfrak{t}}] \cap B[\mathfrak{L}]) \simeq B[\bar{\mathfrak{L}}]/(B[\bar{\mathfrak{L}}] \cap B[\bar{\mathfrak{L}}])$$

or

$$B[\mathfrak{l}|_{\mathfrak{t}}]/(B[\mathfrak{l}|_{\mathfrak{t}}] \cap B[\mathfrak{L}]) \simeq \exp(\mathfrak{R}X(\mathfrak{l}))B[\bar{\mathfrak{L}}]B[\mathfrak{l}]/B[\mathfrak{l}]$$

$$\simeq \exp(\mathfrak{R}X(\mathfrak{l})) \times B[\bar{\mathfrak{L}}]/(B[\bar{\mathfrak{L}}] \cap B[\bar{\mathfrak{L}}]).$$

In the first case, the distribution $a_{\mathfrak{L}}$ associated to $\pi$ can be identified with the generalized vector $a_T$ of $\bar{\pi} = \pi|_{\bar{G}}$. In the second case, for $\varphi \in \mathcal{H}^{\infty}_\pi$ satisfying

$$\varphi(\exp(tX(\bar{\mathfrak{g}}))) = \varphi(t) \psi(\bar{\mathfrak{g}}), \ t \in \mathbb{R}, \bar{\mathfrak{g}} \in \bar{G},$$

with $\varphi \in C_c(\mathbb{R})$, $\psi \in \mathcal{H}^{\infty}_\pi$, the $\mathfrak{t}$-invariance of $W$ implies that

$$\langle W \cdot a_{\mathfrak{L}}, \varphi \rangle = \langle \left( \int_{\mathbb{R}} \varphi(t) e^{itX(\mathfrak{l})} dt \right) W \cdot a_{\bar{\mathfrak{L}}}, \psi \rangle = P_W(\mathfrak{l}) \left( \int_{\mathbb{R}} \varphi(t) e^{itX(\mathfrak{l})} dt \right) \langle a_{\bar{\mathfrak{L}}}, \psi \rangle.$$
In both cases we see that $W \cdot a_{\bar{t}} = P_W(\ell) a_{\bar{t}}$. According to the induction hypothesis $P_W(\ell) = P_W(\bar{\ell})$ is a polynomial function on $\Omega$ and hence also on $\Omega$.

We can now assume, as we have seen before, that $\Omega$ is saturated with respect to $\bar{g}$, that for generic $\ell \in \Omega$, the $L$-orbits of $\ell|_1$ are saturated with respect to $\mathfrak{g}$ and that $\mathfrak{g} \not\subset \bar{g}$.

Recall again $\bar{\mathfrak{g}} := \mathfrak{g} \cap \bar{g}$. If $W \in \mathcal{U}(\bar{g})$, then the last computation tells us that
\[
W \cdot a_{\bar{t}_{i\bar{g}}} = P_W(\ell|_{\bar{g}}) a_{\bar{t}_{i\bar{g}}}.
\]
Since $\delta(\tilde{G}, \tilde{K}) < \delta(G, K)$, by the induction hypothesis, $P_W(\ell|_{\bar{g}})$ is a polynomial on the $G$-orbit of $\bar{\ell}$.

Suppose that $P_\kappa(\ell) \neq 0$ and $ad^s(X_n)P_\kappa = 1$. Let $\tilde{\kappa}_1 = Y_{n-1} + \tilde{U}, \tilde{U} \in \mathcal{U}(\mathfrak{g}_{\tilde{r}^{-1}_{n-1}})$ be the $e$-central element of Corwin-Greenleaf in $\mathcal{U}(\bar{g})$ associated to $\mathfrak{g}_{n-1} \cap \bar{g}$ and $G$-orbit of $\bar{\ell}$ as in (10). Then as in the proof of Corollary 1 we conclude that the denominator of the rational function $P_W$ is a polynomial in $P_{\tilde{\kappa}_1}(\exp(-P_\kappa(\ell|X_n))\ell)$. Since the denominator is also a polynomial in $P_\kappa(\ell)$, it follows that $P_W$ is in fact a polynomial function.

Therefore we can finally assume that $W$ is not contained in $\mathcal{U}(\bar{g})$. This means that $\mathfrak{b}[\ell|] \not\subset \mathfrak{g}$ for generic $\ell \in \Omega$. This being assumed, we suppose that the denominator of the rational function $P_W(\ell)$ is not trivial. We are brought to the case where this denominator is equal to $P_{\kappa-c}(\ell)$ for some $c \in \mathbb{C}$. Take $\tilde{X}$ in $\mathfrak{g}$. In these circumstances, there exists in $\mathcal{U}(\mathfrak{g})$ an element
\[
(11) \quad \sigma = \tilde{a}\tilde{x} + \tilde{b}, \quad \tilde{a}, \tilde{b} \in \mathcal{U}(\mathfrak{g})
\]
which is $e$-central for $\Omega|_\mathfrak{g}$. If $W$ is of degree $m$ relatively to $\tilde{X}$ with the dominant term $w_m\tilde{x}^m, w_m \in \mathcal{U}(\bar{g})$, we saw in Subsection 3.3 that $w_m$ and $\tilde{a}$ are $\mathfrak{g}$-invariant. Then, applying $\tilde{a}$ and $w_m$ to $a_\ell(\ell \in \Omega)$, we see that they commute each other modulo ker($\pi$). Thus,
\[
(12) \quad W_1 = \tilde{a}^m W - w_m \sigma^m
\]
is of degree inferior to $m$ relatively to $\tilde{X}$. Repeating this process, we build an element $\tilde{W} \in \mathcal{U}(\bar{g})$ such that $P_{\tilde{W}}(\ell)$ is a polynomial function on $\Omega$. This means that $\alpha$ is a factor of $\tilde{a}$.

Recall that $j_0$ is the smallest index such that $W \in \mathcal{U}(\mathfrak{g}_{j_0})$ modulo ker($\pi$). We now prove the following:

**Lemma 3.** There exists a $K$-diagonal element
\[
\nu = \beta X_{j_0} + \gamma, \quad \beta, \gamma \in \mathcal{U}(\mathfrak{g}_{j_0-1}),
\]
in $\mathcal{U}_\pi(\mathfrak{g}_{j_0})$ such that $P_\nu(\ell)$ extends to a polynomial function on $\Omega$ and such that $\beta$ is not divisible by $\alpha$ modulo ker($\pi$).

**Proof.** We proceed by induction on dim $\mathfrak{g}$. Let first dim $\mathfrak{g} = 1$, namely $\mathfrak{g}$ is abelian. At each point $\ell \in \Omega$, the Penney’s distribution $a_\ell$ is nothing but the Dirac measure at the unit element of $G$. Put $\mathfrak{b} = \cap_{\ell \in \Omega} \mathfrak{b}[\ell]$, which is an ideal of $\mathfrak{g}$. Then, the existence of $W$ allows us to take $X_{j_0}$ in $\mathfrak{b}$. This being done, $\nu = X_{j_0}$ suits us. Suppose now that dim $\mathfrak{g} > 1$. Let us repeat the above construction of the element $\tilde{W} \in \mathcal{U}(\mathfrak{g})$ such that $P_{\tilde{W}}(\ell) = P_{\tilde{W}}(\ell)$ extends to a polynomial function on $\Omega$. Here, $\ell = \ell|_{\bar{g}}$ and $\tilde{P}$ designates the object obtained from the pair $(a_{\bar{t}}, \mathfrak{g})$. 

In the first step of construction, if $w_m \notin \mathcal{W}(\bar{\mathfrak{t}}_{j_0-1})$, then we put $W' = w_m$ which belongs to $\mathcal{W}_K(\bar{\mathfrak{g}})^{\mathbf{t}}$ but not in $\mathcal{W}(\bar{\mathfrak{t}}_{j_0-1})$, where $\bar{\mathfrak{t}}_{j_0-1} = \mathfrak{t}_{j_0-1} \cap \bar{\mathfrak{g}}$. Otherwise, the element $W_1$ defined in equation (12) does not belong to $\mathcal{W}(\bar{\mathfrak{t}}_{j_0-1})$, and we replace $W$ by $W_1$, and continue the construction of $\bar{W}$. At the end of this process, we get an element $W'$ in $\mathcal{W}_K(\bar{\mathfrak{g}})^{\mathbf{t}}$ but not in $\mathcal{W}(\bar{\mathfrak{t}}_{j_0-1})$.

Hence, by the induction hypothesis, there exists a $\bar{K}$-diagonal element

$$\tilde{\nu} = \tilde{a}X_{j_0} + \tilde{b}, \quad \tilde{a}, \tilde{b} \in \mathcal{W}(\bar{\mathfrak{t}}_{j_0-1}),$$

in $\mathcal{W}_K(\bar{\mathfrak{t}}_{j_0})^{\mathbf{t}}$ such that $\tilde{P}_\nu(\ell)$ extends to a polynomial function on $\Omega$ and that $\tilde{a}$ is not divisible by $\alpha$ modulo $\ker(\pi)$. Since $\sigma$ is $e$-central for $\Omega|_{\mathbf{t}}$, it gives us the polynomial function $P_{\sigma}(\ell)$ when it is applied to Penney’s distributions for $\mathfrak{t}$. It follows that $[\sigma, [\sigma, \nu]] \in \ker(\pi)$. Thus, $\tilde{\nu}$ turns out to be $\mathbf{t}$-invariant and $P_{\nu}(\ell) = \tilde{P}_\nu(\ell)$. \hfill \Box

We continue the proof of Theorem 3. Let us write

$$W = \sum_{j=0}^{r} w_j X_{j_0}^j, \quad w_j \in \mathcal{W}(\mathfrak{t}_{j_0-1})(0 \leq j \leq r).$$

We go now to engage a double induction on the index $j_0 > d$ and on the degree $r$ of $X_{j_0}$ in the expression of $W$. As $w_r$ is $\mathbf{t}$-invariant, it follows from the induction hypothesis that $w_r a_\ell = P_{w_r}(\ell) a_\ell$ for $\ell \in \Omega$ with a function $P_{w_r}(\ell)$ which extends into a polynomial function on $\Omega$. Next, in the expression (13), let us suppose our assertion established for the elements whose degree relative to $X_{j_0}$ is inferior or equal to $r - 1$. We see that

$$\tilde{W} = \beta^r W - w_r \nu^r$$

is of degree inferior to $r$ relative to $X_{j_0}$ and hence $P_{W}(\ell)$ is a polynomial function on $\Omega$. One deduces from this that $P_{W}(\ell)$ is polynomial because $\beta$ is not divisible by $\alpha$.

\hfill \Box

**Corollary 2.** Suppose that $\pi|_K$ has finite multiplicities. Then the rational function $\ell \mapsto P_{W}(\ell) = \Theta(W)(\ell)$ extends to a polynomial function on $\Omega$, where $\Theta$ is defined as in equation (7). 

5. **Proof of Conjecture**

Second part

Recall first the flag of subalgebras (2), where $\mathfrak{t} = \mathfrak{t}_d$, $j_0 \geq d + 1$ the smallest index such that $W \in \mathcal{W}(\mathfrak{t}_{j_0})$ and $\alpha$ as given in equation (8). Let us first prove the following result, which could be regarded as a substitute to Lemma 3. Repeating this process, we get the element $\tilde{\nu}$ in Lemma 3.

**Proposition 3.** Let $m \leq d$ such that the generic $K_m$-orbits in $\Omega|_{\mathfrak{t}_m}$ are saturated with respect to $\mathfrak{t}_{m-1}$. Write $\mathfrak{t}_m = \mathbb{R}X_m + \mathfrak{t}_{m-1}$ for some $X_m \in \mathfrak{t}_m \setminus \mathfrak{t}_{m-1}$ and let

$$\tau_m = d'_m X_{k_m} + b'_m, \quad d'_m, b'_m \in \mathcal{W}(\mathfrak{t}_{k_m-1})$$

be an $e$-central element for $\Omega|_{\mathfrak{t}_m-1}$ which is not $e$-central for $\Omega|_{\mathfrak{t}_m}$ with the index $k_m$ as small as possible. Then:

1. $\tau_m$ and $[X_m, \tau_m]$ can be choosen in a way that they are not divisible by $\alpha$ modulo $\ker(\pi)$. 

□
(2) Suppose that $h_m' = h_m + g_{j_0-1}$ is strictly included in $h_m = \mathfrak{h}_m + g_{j_0}$ and there exists $W_m \in \mathcal{U}(h_m)^{\mathfrak{f}_m}$ such that $W_m \notin \mathcal{U}(h_m')^{\mathfrak{f}_m}$, which gives us a rational function on $\Omega$ when it is applied to Penney’s distributions for $\mathfrak{f}_m$, then there exists an element

$$\nu_m = a_mX_{j_0} + b_m, \quad a_m, b_m \in \mathcal{U}(h_m'),$$

where $g_{j_0} = R^X_{j_0} + g_{j_0-1}$, which is $\mathfrak{f}_m$-invariant and gives us a polynomial function on $\Omega$ when it is applied to Penney’s distributions for $\mathfrak{f}_m$ and such that $a_m$ is not divisible by $\alpha$ modulo $\ker(\pi)$.

Proof. Let us proceed by induction on $\dim \mathfrak{f}$. The claim is trivial when $\dim \mathfrak{f} \leq 3$. We prove both the assertions at the same time in case of saturation. Let $4 \leq m \leq d$ and suppose that the generic orbits by $K_m = \exp(\mathfrak{f}_m)$ in $\Omega|_{\mathfrak{f}_m}$ are saturated with respect to $\mathfrak{f}_{m-1}$. Let

$$\tau_m = a'_mX_{k_m} + b'_m, a'_m, b'_m \in \mathcal{U}(\mathfrak{f}_{k_m-1})$$

be an $e$-central element for $\Omega|_{\mathfrak{f}_{m-1}}$ which is not $e$-central for $\Omega|_{\mathfrak{f}_m}$ and which is not divisible by $\alpha$. Choose the index $k_m$ as small as possible.

Replacing $\mathfrak{f}$ by $\mathfrak{f}_m$, Lemma 3 gives us the element $\tau_m = a'_mX_{k_m} + b'_m$, with $a'_m, b'_m \in \mathcal{U}(\mathfrak{f}_{k_m-1})$ and $a'_m$ is not divisible by $\alpha$ modulo $\ker(\pi)$. Now $[X_m, \tau_m]$ is by construction in $\mathcal{U}(\mathfrak{f}_{k_m-1})$, thus it is not divisible by $\alpha$ modulo $\ker(\pi)$.

This being done, suppose that there exists $W_m \in \mathcal{U}(h_m)^{\mathfrak{f}_m}\setminus \mathcal{U}(h_m')^{\mathfrak{f}_m}$, where $h_m = \mathfrak{h}_m + g_{j_0}$, which gives us a rational function on $\Omega$ when it is applied to Penney’s distributions for $\mathfrak{f}_m$ and let us build the element $\nu_m$ with the properties cited in the proposition.

By the saturation argument, we see that $W_m \in \mathcal{U}(h_m')^{\mathfrak{f}_m}$ and that the Penney’s distributions for $\mathfrak{f}_m$ are the same as those for $\mathfrak{f}_{m-1}$. Therefore, by the induction hypothesis, there exists a $K_{m-1}$-diagonal element

$$\nu_{m-1} = a_{m-1}X_{j_0} + b_{m-1}, \quad a_{m-1}, b_{m-1} \in \mathcal{U}(h_m')^{\mathfrak{f}_m}$$

in $\mathcal{U}(h_m')^{\mathfrak{f}_m}$ which gives us a polynomial function on $\Omega$ when it is applied to Penney’s distributions for $\mathfrak{f}_{m-1}$ and such that $a_{m-1}$ is not divisible by $\alpha$. If $\nu_{m-1}$ is $\mathfrak{f}_m$-invariant, it is qualified as our desired $\nu_m$. Suppose that $\nu_{m-1}$ is not $\mathfrak{f}_m$-invariant and retake the construction of our $\nu$ introduced in 7. For a sufficiently large integer $v \in \mathbb{N}$, we consider

$$\psi = \nu_{m-1} + F(\tau_m),$$

where $F(t)$ is a polynomial in one variable $t$ of degree $2v$. For $k \in \mathbb{N}$, put

$$\psi_0 = \psi, \quad \psi_k = (\text{ad}(X_m))^k(\psi).$$

Remark that $[X_m, [X_m, \tau_m]] \in \ker(\pi)$. Therefore, if $v$ is sufficiently large, then

$$\psi_{2v} \notin \ker(\pi), \quad \psi_{2v+1} \in \ker(\pi).$$

We now build an element of $\mathcal{U}(h_m')^{\mathfrak{f}_m}$ by the formula

$$\nu_m = (\psi_0\psi_{2v} + \psi_{2v}\psi_0) - (\psi_1\psi_{2v-1} + \psi_{2v-1}\psi_1) + \cdots$$

$$+ (-1)^{v-2}(\psi_{v-2}\psi_{v+2} + \psi_{v+2}\psi_{v-2})$$

$$+ (-1)^{v-1}(\psi_{v-1}\psi_{v+1} + \psi_{v+1}\psi_{v-1}) + (-1)^v\psi_v^2.$$
Remark once again the fact that $v$ is sufficiently large. This assures that $\nu_m$ is of degree 1 with respect to $X_{j_0}$. Moreover, $[X_m, \nu_{m-1}]$ applied to $a_\ell$ gives us a polynomial function on $\Omega$. Indeed, we see by definition that

$$P_{[X_m, \nu_{m-1}]}(\ell) = \frac{d}{dt} P_{\nu_{m-1}}(\exp(t X_m) \cdot \ell)|_{t=0}, \ \ell \in \Omega.$$  

It follows that $\nu_m a_\ell = P_{\nu_m}(\ell) a_\ell$ for generic $\ell \in \Omega$ with a polynomial function $P_{\nu_m}(\ell)$ on $\Omega$.

Finally, since, for any $k$, $(\text{ad}(X_m))^k F(\tau_m)$ belongs to $\mathcal{U}(\mathfrak{h}'_m)$, we can choose the polynomial $F$ such that $\nu_m = a_m X_{j_0} + b_m$, $a_m, b_m \in \mathcal{U}(\mathfrak{h}'_m)$ and $a_m$ not divisible by $\alpha$ modulo $\ker(\pi)$. Indeed, let

$$F(t) = \lambda_0 + \lambda_1 t + \cdots + \lambda_{2v-1} t^{2v-1} + \lambda_{2v} t^{2v}, \ \lambda_j \in \mathbb{C} \ (0 \leq j \leq 2v).$$

Suppose that $(\text{ad}(X_m))^k(a_{m-1})$ are not divisible by $\alpha$ modulo $\ker(\pi)$, but that $(\text{ad}(X_m))^{n_0+1}(a_{m-1})$ and hence all the elements $(\text{ad}(X_m))^k(a_{m-1})$, $k \geq n_0 + 1$ are divisible by $\alpha$ modulo $\ker(\pi)$.

Considering the $\lambda_j$ as variables, and supposing that for any choice of these variables, the coefficient $a_m$ of $X_{j_0}$ in $\nu_m$ is divisible by $\alpha$ modulo $\ker(\pi)$, thus for any $j$, the coefficient of $\lambda_j X_{j_0}$ of $\nu_m$ is divisible by $\alpha$ modulo $\ker(\pi)$. Remark now that the terms $\lambda_{2v-n_0} X_{j_0}$ in $\nu_m$ appear only in the sum:

$$\sum_{k \geq n_0} (-1)^k (\psi_k \psi_{2v-k} + \psi_{2v-k} \psi_k) \equiv 2 \sum_{k \geq n_0} (-1)^k \psi_{2v-k} \psi_k \ (\text{mod } \ker(\pi))$$

and they are modulo $\ker(\pi)$:

$$\left( \sum_{k \geq n_0} c_k(\text{ad}(X_m))^{2v-k}(\tau_m^{2v-n_0})(\text{ad}(X_m)^k a_{m-1}) \lambda_{2v-n_0} X_{j_0},$$

where $c_k$ is a numerical constant. Each term in this sum is divisible by $\alpha$ except the first one, by definition of $n_0$. This proves that there is a polynomial $F$ such that the conditions of the proposition hold for $\nu_m$.

Now, suppose that the generic $K_m$-orbits in $\Omega|_{\xi_m}$ are non-saturated with respect to $\xi_{m-1}$. Then, there exists an element

$$\sigma_m = c_m X_m + d_m, \ c_m, d_m \in \mathcal{U}(\xi_{m-1})$$

which is $e$-central for $\Omega|_{\xi_m}$. If

$$W_m = v_r X_m + v_{r-1} X_m^{-1} + \cdots + v_1 X_m + v_0, \ v_j \in \mathcal{U}(\mathfrak{h}_{m-1})(0 \leq j \leq r)$$

with $v_r \not\in \ker(\pi)(r > 0)$, $W_m = c_r W_m - c_r^* v_r$ is $\xi_{m}$-invariant and of degree smaller or equal to $r - 1$ relative to $X_m$ because $v_r, c_m$ are also $\xi_{m}$-invariant and commute each other modulo $\ker(\pi)$. Repeating these manipulations if necessary, we arrive to a $\xi_{m}$-invariant element $W_{m-1} \in \mathcal{U}(\mathfrak{h}_{m-1})$ which gives us a rational function on $\Omega$ when it is applied to Penney’s distributions. From the induction hypothesis there exists a $\xi_{m-1}$-invariant element $\nu_{m-1}$ which satisfies the required conditions as above. Applying $\nu_{m-1}, \sigma_m$ to Penney’s distributions for $\xi_{m-1}$, we confirm that they commute each other modulo $\ker(\pi)$. In this way, $\nu_{m-1}$ turns out to be $\xi_{m}$-invariant and is qualified as our desired $\nu_m$.

\[\Box\]

**Corollary 3.** Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{e}$ and $\mathfrak{h}'$ an ideal of codimension 1 in $\mathfrak{h}$ such that the generic orbits by $H = \exp \mathfrak{h}$ in $\Omega|_{\mathfrak{h}}$ are saturated with respect to $\mathfrak{h}'$. Let

$$\tau = a' X_{k'} + b', \ a', b' \in \mathcal{U}(\xi_{k'-1}), \ a' \not\in \ker(\pi),$$


be a $c$-central element for $\Omega|_{\mathfrak{h}'}$, which is not $c$-central for $\Omega|_{\mathfrak{h}}$ for which $k'$ is minimal. Then $\tau$ and $[X, \tau]$ can be chosen in a way that they are not divisible by $\alpha$, where $\mathfrak{h} = \mathbb{R}X + \mathfrak{h}'$.

We now look at the surjectivity of the homomorphism $\Theta$ defined by equation (7). We first record the following, which will be of use later

**Proposition 4** ([7] Proposition 4.4). Keep the same notations and hypotheses and let us denote by $y$ the variable corresponding to the polynomial function defined as in equation (8). Then for every polynomial $\zeta(x) \in \mathbb{C}[\Omega]^K$, there exists a polynomial $s(y')$ of $y'$ such that the product $s(y')\zeta(x)$ is in the image of $\Theta$.

Let $V$ be the set of $K$-diagonal elements $W \in \mathcal{U}_{\tau}(\mathfrak{g})^\mathfrak{f}$ such that $W \cdot a_\ell = P_W(\ell) a_\ell$ with a function $P_W(\ell)$ which extends to a polynomial function on $\Omega$. We consider the image $M$ of the mapping

$$\Theta_{\nu} : \nu \ni W \mapsto P_W \in \mathbb{C}[\Omega]^K.$$

We now prove the following:

**Proposition 5.** Let $q(\ell) \in \mathbb{C}[\Omega]^K$. If there exists $0 \neq u(\ell) \in M$ such that the product $u(\ell)q(\ell)$ belongs to $M$, then the function $q(\ell)$ itself belongs to $M$.

**Proof.** We proceed by induction on $\dim G + \dim(G/K)$. Let $u(\ell) = P_{W_1}(\ell)$ and $u(\ell)q(\ell) = P_{W_2}(\ell)$ with $W_1, W_2 \in V$. Examine first the case where $\mathfrak{f} = \{0\}$. Put $\mathfrak{b} = \cap_{\ell \in \Omega} \mathfrak{b}[\ell]$, which is an ideal of $\mathfrak{g}$. It is seen that $\mathcal{U}(\mathfrak{b})$ is identified modulo $\ker(\pi)$ to the symmetric algebra $S(\mathfrak{b})$ of $\mathfrak{b}$ because $[\mathfrak{b}, \mathfrak{b}] \subset \ker(\pi)$. Then, $W_1, W_2$ belong to $\mathcal{U}(\mathfrak{b}) \simeq S(\mathfrak{b})$ and $W_2$ is divisible by $W_1$, namely that there exists $W \in S(\mathfrak{b}) \simeq \mathcal{U}(\mathfrak{b})$ such that $W_2 = W_1 W$. It is clear that $W \in \mathcal{U}_{\tau}(\mathfrak{g})^\mathfrak{f}$ and $P_W(\ell) = q(\ell)$. In sum, $q(\ell) \in M$.

Suppose that $\dim \mathfrak{f} \geq 1$. Keep the notations introduced before. When $\Omega$ is non-saturated with respect to $\mathfrak{x}_{n-1}$, $W_1, W_2$ are taken in $\mathcal{U}_{\tau}(\mathfrak{x}_{n-1})^\mathfrak{f}$ and the result derives immediately from the induction hypothesis.

Suppose that $\Omega$ is saturated with respect to $\mathfrak{x}_{n-1}$. It follows that $W_1, W_2 \in \mathcal{U}(\mathfrak{x}_{n-1})$ and that $q(\ell)$ depends only on $\ell' = \ell|_{\mathfrak{x}_{n-1}}$. For almost all $\ell \in \Omega$, there exists by the induction hypothesis an element $W_1 \in \mathcal{U}_{\tau}(\mathfrak{x}_{n-1})^\mathfrak{f}$ verifying $P_{W_1}(\ell') = q(\ell')$ for almost all $\ell' \in \omega$. Here, $W_1$ depends rationally on $\ell \in \Omega$. By Proposition 4 there exists a polynomial $s(y')$ of $y' = P_{\kappa}(\ell)$ such that $s(y')q(\ell) \in M$.

Now take an ideal $\mathfrak{g} \neq \mathfrak{x}_{n-1}$ of codimension 1 in $\mathfrak{g}$. Suppose first that $\Omega$ is non-saturated with respect to $\mathfrak{g}$. Then $W_1, W_2$ are in $\mathcal{U}_{\tau}(\mathfrak{g})^\mathfrak{f}$ modulo $\ker(\pi)$. If $\mathfrak{f} \subseteq \mathfrak{g}$, the induction hypothesis provides us the result. If $\mathfrak{f} \not\subseteq \mathfrak{g}$, put $\tilde{\mathfrak{f}} = \mathfrak{f} \cap \mathfrak{g}$ and $\tilde{K} = \exp \tilde{\mathfrak{f}}$. The induction hypothesis assures that there exists a $\tilde{K}$-diagonal $\tilde{W} \in \mathcal{U}_{\tau}(\mathfrak{g})^\mathfrak{f}$ so that we have $\tilde{P}_{\tilde{W}}(\ell) = q(\ell)$. Since $q(\ell)$ is $\tilde{\mathfrak{f}}$-invariant, $\tilde{W}$ turns out to be $\tilde{\mathfrak{f}}$-invariant and hence $\tilde{P}_{\tilde{W}}(\ell) = q(\ell)$. In this way, $q(\ell) \in M$.

Recall now our previous notations: $\mathfrak{g}' = \mathfrak{x}_{n-1}$, $\kappa$ its corresponding $c$-central element and $y'$ as in equation (8). Suppose that $\Omega$ is saturated with respect to $\mathfrak{g}$. If $\mathfrak{f} \subseteq \mathfrak{g}$, $W_1, W_2$ belong to $\mathcal{U}(\mathfrak{g})$. As above, there exists a polynomial $\tilde{s}(\tilde{y})$ of $\tilde{y} = P_{\kappa}(\ell)$ such that $\tilde{s}(\tilde{y})q(\ell) \in M$. Let $s(y')q(\ell) = P_{W'}(\ell)$ and $\tilde{s}(\tilde{y})q(\ell) = P_{\tilde{W}}(\ell)$ for some $W', \tilde{W} \in V$. Then, $\tilde{s}(\kappa)W' \equiv s(\kappa)\tilde{W}$ modulo $\ker(\pi)$. Therefore, $W'$ must be divisible modulo $\ker(\pi)$ by $s(\kappa)$ and $W' \equiv s(\kappa)W$ modulo $\ker(\pi)$ with a certain $K$-diagonal $W \in \mathcal{U}_{\tau}(\mathfrak{g})^\mathfrak{f}$. Thus, $q(\ell) = P_W(\ell)$. 

Finally, suppose that $\mathfrak{t}$ is not found in $\tilde{\mathfrak{g}}$. We shall argue similarly as in the proof of Theorem 3. If $b[\ell]|_{\mathfrak{t}} \subset \mathfrak{t}$ almost everywhere on $\Omega$, $W_1, W_2$ belong to $\mathcal{W}(\mathfrak{g})$ and hence $q(\ell)$ depends only on $\ell|_{\tilde{\mathfrak{g}}}$. From the induction hypothesis applied to $\tilde{\mathfrak{t}}$, there exists a $\tilde{K}$-diagonal $W \in \mathcal{U}_{\tau}(\tilde{\mathfrak{g}})$ such that $q(\ell) = \tilde{P}_W(\ell)$. Since $q(\ell)$ is $\mathfrak{t}$-invariant, $\tilde{W}$ is $\mathfrak{t}$-invariant too and $\tilde{P}_W(\ell) = P_W(\ell)$. Therefore, $q(\ell) \in M$.

We place in the last possibility where $b[\ell]|_{\mathfrak{t}} \not\subset \tilde{\mathfrak{t}}$ almost everywhere on $\Omega$. It is sufficient for us to treat the case where $s(y') = \alpha$ which is a polynomial in $y'$ of degree 1.

Let $j_0$ be the smallest index such that $q(\ell)$ belongs to the symmetric algebra $S(\mathfrak{t}_{j_0}) = \mathbb{C}[\mathfrak{t}_{j_0}^*]$ of $\mathfrak{t}_{j_0}$ with respect to the sequence (5) of subalgebras. Aligning back to Subsection 3.3 let $\{Y_k\}_{k=1}^n$ be a Jordan-Hölder basis of $\mathfrak{g}$ adapted to the flag (11) and let
\begin{equation}
S = \{s_1 < \cdots < s_r\}
\end{equation}
be the set of jump indices for $\Omega$ with respect to the flag (11) which appear in $\mathfrak{t}_{j_0}$.

Set $x_i = \ell(Y_{s_i})$ for $1 \leq i \leq r$, where $Y_{s_r} = X_{j_0}$ changing the ordering. So, $q(\ell)$ depends on $\{x_1, \ldots, x_r\}$. Write
\begin{equation}
q(\ell) = \sum_{j=0}^v q_j(\ell)x_r^j,
\end{equation}
where $q_j(\ell)(0 \leq j \leq v)$ are polynomial functions of $x_1, \ldots, x_{r-1}$.

Everything as in the proof of Lemma 3, we now prove by induction on the dimension of $\mathfrak{t}$ that there exists in $M$ an element
\begin{equation}
\nu(\ell) = \beta(\ell)x_r + \gamma(\ell),
\end{equation}
where $\beta(\ell), \gamma(\ell)$ are polynomials of $\{x_1, \ldots, x_{r-1}\}$ and where $\beta(\ell) \in M$ is not divisible by $\alpha$. Indeed, assume first that $j_0 > d$. Making use of the $e$-central element $\sigma$ for $\Omega|_{\mathfrak{t}}$ as in equation (11), one finds in $S((\mathfrak{t}_{j_0} \cap \tilde{\mathfrak{g}})) \cap \mathbb{C}[\Omega]^{\tilde{K}}$, an element $\tilde{q}(\ell)$ outside $S(\mathfrak{t}_{j_0-1})$ such that $\alpha\tilde{q}(\ell) \in \tilde{M}$, the corresponding set for $\tilde{\mathfrak{t}}$. By the induction hypothesis, there exists in $\tilde{M}$ an element
\begin{equation}
\tilde{\nu}(\ell) = \tilde{\beta}(\ell)x_r + \tilde{\gamma}(\ell),
\end{equation}
where $\tilde{\beta}(\ell), \tilde{\gamma}(\ell)$ are polynomials of $\{x_1, \ldots, x_{r-1}\}$ and where $\tilde{\beta}(\ell) \in \tilde{M}$ is not divisible by $\alpha$. Now, using the element $\sigma$ as in (11), $\tilde{\nu}(\ell)$ turns out to be $K$-invariant and hence belongs to $M$ as is to be shown.

When $j_0 \leq d$, we first prove Lemma 4.

**Lemma 4.** We regard the symmetric algebra $S(\mathfrak{t})$ of $\mathfrak{t}$ as the algebra of polynomial functions on $\Omega|_{\mathfrak{t}}$ through the evaluation $\Omega|_{\mathfrak{t}} \ni \ell \mapsto \sqrt{-1}\ell(X)$ for $X \in \mathfrak{t}$. Let $\zeta : S(\mathfrak{t}) \to \mathcal{W}(\mathfrak{t})$ be the symmetrization map. Then, $\zeta(q)$ is $K$-diagonal and
\begin{equation}
\zeta(q) - a_\ell = q(\ell)a_\ell, \; \ell \in \Omega|_{\mathfrak{t}}.
\end{equation}

**Proof.** We proceed by induction on $\dim \mathfrak{t}$. When $\dim \mathfrak{t} = 1$, the claim is trivial. Let $\mathfrak{z}(\mathfrak{t})$ be the center of $\mathfrak{t}$. If $\dim \mathfrak{z}(\mathfrak{t}) = 1$, $\mathfrak{z}(\mathfrak{t})$ is nothing but the center $\mathfrak{z}$ of $\mathfrak{g}$. As $\mathfrak{p}|_{\mathfrak{z}} \neq 0$, $q \in S(\mathfrak{t}')$ where $\mathfrak{t}'$ denotes the centralizer of $\mathfrak{t}_2$ in $\tilde{\mathfrak{t}}$, where $\mathfrak{t}_2$ is as in the flag (13). Since $b[\ell]|_{\mathfrak{t}} \subset \mathfrak{t}'$ for $\ell \in \Omega$, we can apply the induction hypothesis to $\mathfrak{t}'$. Suppose $\dim \mathfrak{z}(\mathfrak{t}) \geq 2$. For $\ell \in \Omega|_{\mathfrak{t}}$, we put $\mathfrak{a} = \mathfrak{z}(\mathfrak{t}) \cap \ker(\ell), \mathfrak{f} = \mathfrak{t}/\mathfrak{a}$ and $\ell \in (\mathfrak{f})^*$ such that $\ell p = \ell$ with the canonical projection $p : \mathfrak{t} \to \mathfrak{f}$. Let $a_\ell$ be the Penney distribution of $\mathfrak{f}$ at $\ell$. Then, we have $\zeta(\tilde{q}) - a_\ell = \tilde{q}(\ell)a_\ell$ from the induction hypothesis.
applied to \( \tilde{k} \). Here, \( \zeta : S(\tilde{k}) \to \mathcal{U}(\tilde{k}) \) denotes the symmetrization map and \( \tilde{q} \in S(\tilde{k}) \) is such that \( \tilde{q} \circ p = q \). Thus, we get the claim. \( \square \)

Now if \( j_0 > d \), we use assertion 2 of Proposition 3 to argue similarly as in the previous case.

We now utilize a new induction on the degree \( v \) of \( q \) relatively to \( x_r \). If \( \kappa \) is such that \( \zeta(p) = q \), then \( \zeta(q) = \zeta(p) \circ \kappa = \kappa \cdot \zeta(p) = \kappa \cdot q \).

Thus, we get the claim.

□

Remark 2. It is worth noting here that by a result of M. Duflo (cf. [14]), for any \( \sigma \in \hat{K} \) and any \( q \in S(\mathfrak{k})^K \), \( \sigma(\zeta(q)) = q(\ell)id \) for any \( \ell \) in the orbit associated to \( \sigma \) in \( K^* \). It remains unclear to us whether this results provides directly a proof of Lemma 4.

Corollary 4. Keep the same notation and assume that \( \pi \mid K \) has finite multiplicities, then the mapping \( \Theta \) defined by equation (7) is surjective.

Corollaries 2 and 4 allow to complete the proof of Conjecture 1.1. We have the following:

Theorem 4. Let \( G = \exp \mathfrak{g} \) be a connected and simply connected nilpotent Lie group. Then Conjecture 1.1 holds. That is, when \( \pi \mid K \) has finite multiplicities, the mapping \( \Theta \) gives by passing to the quotient an isomorphism of algebras from \( D_\pi(G)^K \) to the algebra \( \mathbb{C}[\Omega(\pi)]^K \) of the \( K \)-invariant polynomial functions on the orbit \( \Omega(\pi) \).

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References


A PROOF OF THE POLYNOMIAL CONJECTURE FOR RESTRICTIONS 633


