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A PROOF OF THE POLYNOMIAL CONJECTURE FOR RESTRICTIONS OF NILPOTENT LIE GROUPS REPRESENTATIONS

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This work is dedicated to the memory of Takaaki Nomura

Abstract. Let G be a connected and simply connected nilpotent Lie group, K an analytic subgroup of G and π an irreducible unitary representation of G whose coadjoint orbit of G is denoted by $\Omega(\pi)$. Let $\mathscr{U}(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{g}$ designating the Lie algebra of G. We consider the algebra $D_\pi(G)^K \simeq (\mathscr{U}(\mathfrak{g})/\ker(\pi))^K$ of the K-invariant elements of $\mathscr{U}(\mathfrak{g})/\ker(\pi)$. It turns out that this algebra is commutative if and only if the restriction $\pi|_K$ of π to K has finite multiplicities (cf. Baklouti and Fujiwara [J. Math. Pures Appl. (9) 83 (2004), pp. 137-161]). In this article we suppose this eventuality and we provide a proof of the polynomial conjecture asserting that $D_\pi(G)^K$ is isomorphic to the algebra $\mathbb{C}[\Omega(\pi)]^K$ of K-invariant polynomial functions on $\Omega(\pi)$. The conjecture was partially solved in our previous works (Baklouti, Fujiwara, and Ludwig [Bull. Sci. Math. 129 (2005), pp. 187-209]; J. Lie Theory 29 (2019), pp. 311-341).

1. Introduction

Let $G = \exp \mathfrak{g}$ be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and $K = \exp \mathfrak{k}$ an analytic subgroup of G. We denote by $\mathfrak{g}^*$ (resp. $\mathfrak{k}^*$) the dual vector space of $\mathfrak{g}$ (resp. $\mathfrak{k}$). Then, G (resp. K) acts on $\mathfrak{g}^*$ (resp. $\mathfrak{k}^*$) by the coadjoint action whose orbit space realizes by the orbit method [8], [12], [21] the unitary dual $\hat{G}$ (resp. $\hat{K}$) of G (resp. K). We denote by $\theta_G : \mathfrak{g}^* \to \hat{G}$ the Kirillov map and by $\Omega(\pi) = \Omega_G(\pi) = \theta_G^{-1}(\pi)$ the coadjoint orbit of G associated to $\pi \in \hat{G}$. Although we use the notation $\simeq$ for the unitary equivalence, we often identify an irreducible unitary representation with its equivalence class.

We know in the nilpotent case the branching laws of induced and restricted representations ([15], [16]). Let $p : \mathfrak{g}^* \to \mathfrak{k}^*$ be the restriction mapping. For $\pi \in \hat{G}$, we consider a finite measure μ_π on $\mathfrak{g}^*$ equivalent to the canonical measure on the orbit $\Omega_G(\pi)$ which is regarded as a measure on $\mathfrak{g}^*$. Put $\nu_\pi = (\theta_K \circ p)_*(\mu_\pi)$. The restriction $\pi|_K$ of π to K is disintegrated as:

$$\pi|_K \simeq \int_{\hat{K}}^{\oplus} m_\sigma^\pi \sigma d\nu_\pi(\sigma),$$

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where the multiplicities m_σ^π are obtained as the number of the K-orbits contained in $\Omega_G(\pi) \cap p^{-1}(\Omega_K(\sigma))$ (cf. [11] and [17]).

In other respects, it is well known ([2], [10], [11]) that in these situations the multiplicities are either uniformly bounded almost everywhere or equal to the infinity almost everywhere. According to these two eventualities, we say that the representation $\pi|_K$ has either finite or infinite multiplicities.

We denote by $\mathscr{U}(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$ and let $\ker(\pi)$ be the primitive ideal of $\mathscr{U}(\mathfrak{g})$ associated to π. We introduce the algebra

$$\mathscr{U}_\pi(\mathfrak{g})^{\mathfrak{k}} = \{A \in \mathscr{U}(\mathfrak{g}); [A, \mathfrak{k}] \subset \ker(\pi)\}$$

and its image

$$D_\pi(G)^K \cong \mathscr{U}_\pi(\mathfrak{g})^{\mathfrak{k}} / \ker(\pi) \cong \left(\mathscr{U}(\mathfrak{g}) / \ker(\pi)\right)^K,$$

where the last member designates the quotient algebra of K-invariant elements. The algebra $D_\pi(G)^K$ was the object of our three previous works [4], [5] and [6]. In particular, we proved [5] that our algebra $D_\pi(G)^K$ is commutative if and only if the restricted representation $\pi|_K$ has finite multiplicities (cf. [19]). We then substantiated in [6] Conjecture 1.1 (cf. [17]):

Conjecture 1.1 (cf. [17]). *Let G be a connected and simply connected nilpotent Lie group, K an analytic subgroup of G. Let $\pi \in \hat{G}$ be a unitary and irreducible representation of G such that $\pi|_K$ is of finite multiplicities. Then the algebra $D_\pi(G)^K$ is isomorphic to the algebra $\mathbb{C}[\Omega(\pi)]^K$ of the K-invariant polynomial functions on $\Omega(\pi)$.*

We positively proved Conjecture 1.1 in many settings, especially when K is a normal subgroup of G or where the orbit $\Omega(\pi)$ is flat in [6] and further, the case where K is abelian or where $\Omega(\pi)$ admits a normal polarizing subgroup [7]. The aim of the present paper is to provide a proof of Conjecure 1.1.

The outline of the paper is as follows: We introduce in the next section some backgrounds about the algebra $D_\pi(G)^K$ and some algebraic tools to describe its generators in term of the enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. This makes use of Pedersen's construction of the kernel $\ker(\pi)$, π being the Kirillov's model associated to $\Omega(\pi)$ (cf. [21]). Section 3 is devoted to prepare the ingredients to prove the main result, mainly an algorithm which allows to define a rational function P_W on $\Omega(\pi)$, for a given $W \in \mathscr{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$. Sections 4 and 5 are devoted to prove Conjecture 1.1.

2. Backgrounds

2.1. Let G be a connected and simply connected nilpotent Lie group. We consider a unipotent representation of G on a real vector space V of finite dimension. Let $v \in V$ be an invariant vector by the action of G, i.e. $g \cdot v = v$ for all $g \in G$. Put for $x \in V$ arbitrarily fixed, $L_x = \{x + tv;\ t \in \mathbb{R}\}$, the straight line passing through x and having the direction of v. Then, there are two possibilities: either $L_x \cap G{\cdot}x = L_x$ or $L_x \cap G{\cdot}x = \{x\}$. According to these two possibilities, we shall say that the orbit $G{\cdot}x$ is either saturated or non-saturated in the direction $\mathbb{R}v$. We shall utilize in what follows this fact applied to the coadjoint representation of G (or a subgroup K of G), where the invariant vector v will be a linear form which vanishes on an ideal $\mathfrak{g}'$ of codimension 1 of $\mathfrak{g}$. In this situation, we shall say that the orbit in question is either saturated or non-saturated with respect to $\mathfrak{g}'$.

2.2. Let

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n = \mathfrak{g} \tag{1}$$

be a Jordan-Hölder sequence of $\mathfrak{g}$, i.e. an increasing sequence of ideals of $\mathfrak{g}$ such that $\dim(\mathfrak{g}_j) = j,\ j = 0, \dots, n$. Let $\{Y_1, \dots, Y_n\}$ be a Jordan-Hölder basis of $\mathfrak{g}$, associated to this Jordan-Hölder sequence, and $\{Y_1^*, \dots, Y_n^*\}$ the basis of $\mathfrak{g}^*$ such that $Y_i^*(Y_j) = \delta_{i,j}, 1 \leq i, j \leq n$. Let $p_i : \mathfrak{g}^* \to \mathfrak{g}_i^*$ be the canonical projection which intertwines the actions of G on $\mathfrak{g}^*$ and $\mathfrak{g}_i^*$. For $\ell \in \mathfrak{g}^*$, we put $e_i(\ell) = \dim G{\cdot}p_i(\ell)$, $e(\ell) = (e_1(\ell), \dots, e_n(\ell))$ and $\mathscr{E} = \{e(\ell), \ell \in \mathfrak{g}^*\}$. For $e \in \mathscr{E}$, we define the G-invariant layer $U_e = \{\ell \in \mathfrak{g}^* :\ e(\ell) = e\}$. Putting $e_0 = 0$, we define also

$$S(e) = \{i :\ e_i = 1 + e_{i-1}\},\ \mathfrak{g}_S^* = \mathbb{R}\!-\operatorname{vect}\{Y_i^* :\ i \in S(e)\}$$

$$T(e) = \{i :\ e_i = e_{i-1}\},\ \mathfrak{g}_T^* = \mathbb{R}\!-\operatorname{vect}\{Y_i^* :\ i \in T(e)\}.$$

Then we have $\mathfrak{g}^* = \mathfrak{g}_S^* \oplus \mathfrak{g}_T^*$. There exists an order among the elements of $\mathscr{E} = \{e^{(1)} > \cdots > e^{(k)}\}$ in such a manner that $U_{e^{(1)}}$ and $\cup_{j \leq i} U_{e^{(j)}}$ are Zariski open sets of $\mathfrak{g}^*$ for every i. In this way all the layers U_e are semi-algebraic set, i.e. difference of two Zariski open sets of $\mathfrak{g}^*$. Let U_e be an arbitrary layer, we write $S(e) = \{j_1 < \cdots < j_r\}$ where r designates the dimension of the G-orbits in U_e. Then there exist some functions $R_j^e : U_e \times \mathbb{R}^r \to \mathbb{R},\ j = 1, \dots, n$ such that:

(a) For $f \in U_e$ fixed, $x = (x_1, \dots, x_r) \mapsto R_j^e(f, x) : \mathbb{R}^r \to \mathbb{R}$ is a polynomial function in x and the coefficients are G-invariant functions on U_e;

(b) $R_j^e(f, x) = x_k$ for $j = j_k \in S(e),\ f \in U_e$;

(c) If $j_k \leq j < j_{k+1}$, then $R_j^e(f, x)$ depends only on $x_1, \dots, x_k$;

(d) For any $f \in U_e$, the coadjoint orbit $G{\cdot}f$ is given by:

$$G{\cdot}f = \{\sum_{j=1}^{n} R_j^e(f, x) Y_j^*; x \in \mathbb{R}^r\},$$

(see [22]).

Let $r_j^e(f)$ be the image in $\mathscr{U}(\mathfrak{g})$ by the symmetrization of the element

$$R_j^e(f, -iY_{j_1}, \dots, -iY_{j_r})$$

in the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}_{\mathbb{C}}$, namely, we replace the variable x_k in $R_j^e(f, x)$ by $-iX_{j_k}$. Notice in particular that $r_{j_k}^e(f) = -iY_{j_k}$. Let V_e be the subspace of $S(\mathfrak{g})$ spanned by the elements of the form $Y_{j_1}^{\alpha_1} \cdots Y_{j_r}^{\alpha_r}$, $\alpha_1, \dots, \alpha_r \in \mathbb{N}$, and let F_e be the image in $\mathscr{U}(\mathfrak{g})$ of V_e by the symmetrization. On the other hand, let E_e be the subspace of $\mathscr{U}(\mathfrak{g})$ spanned by the elements of the form $Y_{j_1}^{\alpha_1} \cdots Y_{j_r}^{\alpha_r}$, $\alpha_1, \dots, \alpha_r \in \mathbb{N}$. If $S(e) = \emptyset$, we put $V_e = F_e = E_e = \mathbb{C}{\cdot}1$. Pedersen proved that the primitive ideal $\ker(\pi)$, where $\pi \in \hat{G}$ such that $f \in \Omega(\pi)$ is generated by the elements

$$u_j^e(f) = Y_j - ir_j^e(f),\ j \in T(e)$$

and that

$$\mathscr{U}(\mathfrak{g}) = \ker(\pi) \oplus E_e = \ker(\pi) \oplus F_e$$

(see Theorem 2.1.1 and Theorem 2.2.1 in [22]). In the same way, the actions of π on E_e and F_e are faithful (see Lemma 2.2.12 and Lemma 2.2.13 in [22]). In this way, identifying E_e and F_e à $\mathscr{U}(\mathfrak{g})/\ker(\pi)$ and abusing notations, we have

$$D_\pi(G)^K \simeq E_e^K \simeq F_e^K \simeq \mathbb{C}[Y_{j_1}, \dots, Y_{j_r}]^K.$$

These isomorphisms are simply isomorphisms of vector spaces.

2.3. In [13], Corwin and Greenleaf showed that Pedersen's construction of the kernel $\ker(\pi_\ell)$, where π_ℓ designates the Kirillov's model [21] which represents the class $\theta_G(\ell)$, for $\ell \in U_e$ leads to construct e-central elements (cf. Theorem 3.1 in [13]). These are elements A of the enveloping algebra $\mathscr{U}(\mathfrak{g})$ such that the operators $\pi_\ell(A)$ are scalars for $\ell \in U_e$. Then $\pi_{\ell'}(A) = \pi_\ell(A)$ for all $\ell' \in G \cdot \ell$. More precisely, let $U_e \subset \mathfrak{g}^*$ be one of the layers constructed above. Then there exists a Zariski open set $Z \subset \mathfrak{g}^*$ such that $Z \cap U_e$ is non-empty G-invariant and for all $j \in T(e)$ there exists an e-central element $A_j \in \mathscr{U}(\mathfrak{g}_j)$ on $Z \cap U_e$, i.e. the operators $\pi_\ell(A_j)$ are scalars for all $\ell \in Z \cap U_e$ with the following properties:

(1) $A_j = P_jY_j + Q_j$, where P_j, Q_j are in $\mathscr{U}(\mathfrak{g}_{j-1})$.

(2) P_j is e-central on $Z \cap U_e$ and does not belong to $\ker(\pi_\ell)$.

(3) $\pi_\ell(A_j) = \phi_j(\ell) Id$ for $\ell \in Z \cap U_e$, where $\phi_j(\ell) = \tilde{p}_j(\tilde{\ell})\ell(Y_j) + \tilde{q}_j(\tilde{\ell})$, $\tilde{p}_j$ and $\tilde{q}_j$ being non-singular rational functions on $Z \cap U_e$ depending only on $(\ell(Y_1), \dots, \ell(Y_{j-1}))$. While the rational function $\tilde{p}_j(\tilde{\ell})$ is G-invariant and never vanishes on $Z \cap U_e$. Moreover, we easily see that the system $\{A_j;\ j \in T(e)\}$ of these e-central elements separates the orbits in $Z \cap U_e$.

Having given the construction of A_j, Corwin-Greenleaf [13] remarked the following: Dropping out the Zariski open set $Z \cap U_e$ from U_e, we notice that, $U_e \backslash Z$ being G-invariant and semi-algebraic, the parametrization of the orbits in U_e is carried out and retains all its properties on this sub-layer in U_e. We are able to repeat the whole process starting from $U_e \backslash Z$. Since U_e is semi-algebraic, the ascendent chain condition for the ideals in $\mathbb{C}[\mathfrak{g}^*]$ assures that the process terminates after a finite number of steps. So, patching the pieces together, we may suppose that $Z \cap U_e = U_e$.

Let ρ be a unitary representation of G. We denote by $\mathscr{H}_\rho$, $\mathscr{H}_\rho^\infty$ and $\mathscr{H}_\rho^{-\infty}$ respectively the space of ρ, that of its differentiable vectors and the anti-dual of $\mathscr{H}_\rho^\infty$ (cf. [9] and [23]). For $a \in \mathscr{H}_\rho^{\pm\infty}$ and $b \in \mathscr{H}_\rho^{\mp\infty}$, we denote by $\langle a, b\rangle$ the image of b by a, so that $\langle a, b\rangle = \overline{\langle b, a\rangle}$. Being given a subgroup H of G and its unitary character χ, put

$$\left(\mathscr{H}_\rho^{-\infty}\right)^{H,\chi} = \left\{a \in \mathscr{H}_\rho^{-\infty}; \rho(h)a = \chi(h)a,\ \forall h \in H\right\}.$$

3. First preparations to the proof of Conjecture 1.1

3.1. Recall once again our situation. Let $G = \exp \mathfrak{g}$ be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, $K = \exp \mathfrak{k}$ an analytic subgroup of G and π an irreducible unitary representation of G whose coadjoint orbit is denoted by $\Omega(\pi)$. For $\ell \in \Omega(\pi)$, we designate by $\mathfrak{b}[\ell|_{\mathfrak{k}}]$ a polarization of $\mathfrak{k}$ at $\ell|_{\mathfrak{k}} \in \mathfrak{k}^*$. We know [5] that $\pi|_K$ has finite multiplicities if and only if $\mathfrak{b}[\ell|_{\mathfrak{k}}] + \mathfrak{g}(\ell)$ is a Lagrangian subspace for the bilinear form $B_\ell : (X, Y) \mapsto \ell([X, Y])$, at μ_π-almost all ℓ in $\Omega(\pi)$.

At the flag of ideals (1) of $\mathfrak{g}$, let $\mathscr{I} = \{i_1 < \cdots < i_d\}$ where $d = \dim \mathfrak{k}$ be the set of indices $1 \le i \le n$ such that $\mathfrak{k} \cap \mathfrak{g}_i \neq \mathfrak{k} \cap \mathfrak{g}_{i-1}$ and put

$$\mathscr{J} = \{j_1 < \cdots < j_q\} = \{1, 2, \dots, n\} \setminus \mathscr{I}$$

with $q = \dim(\mathfrak{g}/\mathfrak{k})$. Putting $\mathfrak{k}_d = \mathfrak{k}$ and $\mathfrak{k}_{d+r} = \mathfrak{k} + \mathfrak{g}_{j_r}$ for $1 \le r \le q$, we obtain a sequence of subalgebras of $\mathfrak{g}$:

$$\mathfrak{k} = \mathfrak{k}_d \subset \mathfrak{k}_{d+1} \subset \cdots \subset \mathfrak{k}_{n-1} \subset \mathfrak{k}_n = \mathfrak{g},\ \dim(\mathfrak{k}_r/\mathfrak{k}_{r-1}) = 1. \quad (2)$$

Furthermore, considering $\mathfrak{k}_s = \mathfrak{k} \cap \mathfrak{g}_{i_s}$ $(1 \le s \le d)$, we get a flag of ideals of $\mathfrak{k}$:

$$\{0\} = \mathfrak{k}_0 \subset \mathfrak{k}_1 \subset \cdots \subset \mathfrak{k}_{d-1} \subset \mathfrak{k}_d = \mathfrak{k},\ \dim \mathfrak{k}_s = s. \tag{3}$$

3.2. Let $\ell \in \Omega(\pi)$. Taking there a real polarization $\mathfrak{b}[\ell]$ of $\mathfrak{g}$, we realize π as $\pi = \mathrm{ind}_{B[\ell]}^G \chi_\ell$ with $B[\ell] = \exp(\mathfrak{b}[\ell])$ and χ_ℓ is the unitary character of $B[\ell]$ whose differential is $i\ell|_{\mathfrak{b}[\ell]}$. On the other hand, by means of the flag (3), we construct [8] the Vergne polarization $\mathfrak{b}[\ell|_\mathfrak{k}]$ of $\mathfrak{k}$ at $\ell|_\mathfrak{k} \in \mathfrak{k}^*$. Put $B[\ell|_\mathfrak{k}] = \exp(\mathfrak{b}[\ell|_\mathfrak{k}])$. It is easy to verify [6] that the formula

$$\langle a_\ell^K, \varphi\rangle = \langle a_\ell, \varphi\rangle = \int_{B[\ell|_\mathfrak{k}]/(B[\ell|_\mathfrak{k}]\cap B[\ell])} \overline{\varphi(b)\chi_\ell(b)}\,d\dot{b} \quad (\forall \varphi \in \mathscr{H}_\pi^\infty)\,, \tag{4}$$

$d\dot{b}$ designating an invariant measure on the homogeneous space $B[\ell|_\mathfrak{k}]/(B[\ell|_\mathfrak{k}]\cap B[\ell])$, gives us a semi-invariant generalized vector a_ℓ in $(\mathscr{H}_\pi^{-\infty})^{B[\ell|_\mathfrak{k}],\chi_\ell}$.

Suppose that $\pi|_K$ has finite multiplicities. This would say as in the case of the monomial representations, that $\mathfrak{b}[\ell|_\mathfrak{k}] + \mathfrak{g}(\ell)$ is a Lagrangian subspace of $\mathfrak{g}$ for B_ℓ at almost all $\ell \in \Omega(\pi)$ with respect to the invariant measure. Then, it results μ_π-almost everywhere in $\Omega(\pi)$ that a_ℓ is an eigen vector for all the elements of $D_\pi(G)^K$ acting on $\mathscr{H}_\pi^{-\infty}$ by continuity. This also means that for every $W \in \mathscr{U}_\pi(\mathfrak{g})^\mathfrak{k}$ we have

$$W \cdot a_\ell := \pi(W)a_\ell = \lambda_\ell(W)a_\ell$$

with a certain scalar $\lambda_\ell(W)$ (cf. [6]). Remark that this scalar $\lambda_\ell(W)$ does not depend on the choice of the polarization $\mathfrak{b}[\ell]$ and of the flag (3) (cf. [15], Proposition 3).

Further, we also have the

Theorem 1 ([6], Theorem 3.4)**.** *Suppose that $\pi|_K$ has finite multiplicities. The homomorphism $\mathscr{U}_\pi(\mathfrak{g})^\mathfrak{k} \ni W \mapsto P_W : \ell \mapsto \lambda_\ell(W)$ defines an imbedding of $D_\pi(G)^K$ into the field $\mathbb{C}(\Omega(\pi))^K$ of rational K-invariant functions on $\Omega(\pi)$.*

We can say even more. Aligning the two sequences (2) and (3), we have a sequence of subalgebras of $\mathfrak{g}$:

$$\{0\} = \mathfrak{k}_0 \subset \mathfrak{k}_1 \subset \cdots \subset \mathfrak{k}_d = \mathfrak{k} \subset \cdots \subset \mathfrak{k}_{n-1} \subset \mathfrak{k}_n = \mathfrak{g}. \tag{5}$$

Relatively to this sequence, let us extract again a vector $X_k \in \mathfrak{k}_k \setminus \mathfrak{k}_{k-1}$ and put $\ell_k = \ell(X_k)$ for $1 \le k \le n$. Consider the action of K on the sequence (5) and define two sets $S_K,\ T_K$ of jump and non-jump indices. Namely, we denote by $e_j^K(\ell)$ the dimension of the K-orbit of $\ell|_{\mathfrak{k}_j} \in \mathfrak{k}_j^*$ for every $1 \le j \le n$. Then we agree $e_0^K(\ell) = 0$. For each index j, the same possibility of the alternative $e_j^K(\ell) = e_{j-1}^K(\ell) + 1$ or $e_j^K(\ell) = e_{j-1}^K(\ell)$ happens μ_π-almost everywhere on $\Omega(\pi)$. We denote by S_K the set of the indices $1 \le j \le n$ which verify the first eventuality and by T_K that of indices of the second eventuality. Put $\mathscr{U}_\pi(\mathfrak{k}_j)^\mathfrak{k} = \mathscr{U}_\pi(\mathfrak{g})^\mathfrak{k} \cap \mathscr{U}(\mathfrak{k}_j)$. Theorem 2 is proved in [5].

Theorem 2. *We keep the same notations and hypotheses. Then:*

(1) *If $j \in S_K$, then $\mathscr{U}_\pi(\mathfrak{k}_j)^\mathfrak{k} = \mathscr{U}_\pi(\mathfrak{k}_{j-1})^\mathfrak{k} + \mathscr{U}(\mathfrak{k}_j)\,(\mathscr{U}(\mathfrak{k}_{j-1}) \cap \ker(\pi))$.*

(2) *If $j \in T_K$, then $\mathscr{U}_\pi(\mathfrak{k}_j)^\mathfrak{k} \ne \mathscr{U}_\pi(\mathfrak{k}_{j-1})^\mathfrak{k} + \mathscr{U}(\mathfrak{k}_j)\,(\mathscr{U}(\mathfrak{k}_{j-1}) \cap \ker(\pi))$ and there exists $W_j \in \mathscr{U}_\pi(\mathfrak{k}_j)^\mathfrak{k}$ having the form $W_j = aX_j + b$ $(a, b \in \mathscr{U}(\mathfrak{k}_{j-1}))$, $a \in \mathscr{U}_\pi(\mathfrak{k}_{j-1})^\mathfrak{k}$ with $\pi(a) \ne 0$.*

(3) *For $j \in T_K$ and $\ell \in \Omega(\pi)$, $P_{W_j}(\ell) = \varphi_j(\ell)\ell_j + \psi_j(\ell)$, where $\varphi_j(\ell),\ \psi_j(\ell)$ are two rational functions of $\ell_1, \ldots, \ell_{j-1}$.*

As a direct consequence of this result, we obtain as in [6]:

Proposition 1.

(1) *Let A be an element of $\mathscr{U}_\pi(\mathfrak{k}_m)^{\mathfrak{k}}$ for $1 \le m \le n$ satisfying $\pi(A) \neq 0$. Then there exists two non-zero polynomials β_A and γ_A of the elements $\{W_j; j \in T_K,\ j \le m\}$ such that $\beta_A A \equiv \gamma_A$ modulo* $\ker(\pi)$.

(2) *The functions $\left\{P_{W_j}(\ell); j \in T_K\right\}$ rationally generate the field $\mathbb{C}\left(\Omega(\pi)\right)^K$.*

3.3. **On the coordinates of the coadjoint orbit.** As in Section 2, we start from the flag of ideals (1) of $\mathfrak{g}$ to parameterize the orbit $\Omega = \Omega(\pi)$ and denote there by S_Ω and T_Ω respectively the sets of jump and non-jump indices. Let $\{Y_1, \dots, Y_n\}$ be a Malcev basis adapted to the flag (1), $\ell_j = \ell(Y_j)$ $(1 \le j \le n)$ for $\ell \in \Omega$, $S_\Omega = \{s_1 < \cdots < s_r\}$, $r = \dim \Omega$ and $x_k = \ell_{s_k}$ for $1 \le k \le r$. Describe as in Section 2 the orbit Ω by the polynomial relations

$$\ell_j = F_j(x_1, \dots, x_k),\ s_k < j < s_{k+1}, \tag{6}$$

where $x = (x_1, \dots, x_r)$ runs through $\mathbb{R}^r$. In these circumstances the rational functions on Ω are nothing but the rational functions of the variables $(x_1, \dots, x_r)$.

For $1 \le k \le r$, let $I^{(k)}$ be the set of the K-invariant polynomial functions on Ω, which depend only on the variables $\{x_i; i \le k\}$. The arguments developed in the pages 60–61 of [24] make us see that every R in $\mathbb{C}(\Omega)^K$ verifying

$$\frac{\partial R}{\partial x_k} \neq 0 \text{ and } \frac{\partial R}{\partial x_i} = 0\ (i > k)$$

is written in the form P/Q, where P and Q belong to $I^{(k)}$. Therefore, the existence of such an element R means that $I^{(k-1)}$ is strictly contained in $I^{(k)}$. Next, let $Q = \sum_{i=0}^m Q_i x_k^i$ $(m > 0)$ be an element of $I^{(k)} \backslash I^{(k-1)}$, where Q_i $(0 \le i \le m)$ designate polynomials of $(x_1,\ \dots, x_{k-1})$ verifying $Q_m \neq 0$. We then confirm that Q_m and $mQ_m x_k + Q_{m-1}$ are K-invariant polynomials.

4. Proof of Conjecture 1.1: First part

We keep all our notations. We first define the following:

Definition 1.

(1) We say that $W \in \mathscr{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ is K-diagonal, if

$$\pi(W) a_\ell \quad = \quad P_W(\ell) a_\ell$$

for a certain scalar $P_W(\ell) \in \mathbb{C}$ independent of the polarizations chosen to describe the distribution a_ℓ and $\ell \mapsto P_W(\ell)$ extends to a rational function on Ω.

(2) Let $\mathscr{U}$ be the set of K-diagonal elements of $\mathscr{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$. Let

$$\Theta : \mathscr{U} \ni W \mapsto P_W \tag{7}$$

Remark 1.

(1) From ([1], Theorem 4.1), any K-diagonal element of $\mathscr{U}(\mathfrak{g})$ belongs to $\mathscr{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$.

(2) Definition 1 is posed independently from the fact that $\pi|_K$ has finite multiplicities or not. In the case of finiteness, any element of $\mathscr{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ is K-diagonal (cf. Theorem 1).

Next, we can easily adapt the arguments of ([6], Lemma 3.2) to prove the following:

Lemma 1. *Let $W \in \mathscr{U}(\mathfrak{g})$ be K-diagonal. Then P_W is identically zero if and only if $W \in \ker(\pi)$.*

Proof. If $W \in \ker(\pi)$, $P_W(\ell) \equiv 0$ because $a_\ell \in \mathscr{H}_\pi^{-\infty}$. Suppose that $P_W(\ell) = 0$ almost everywhere on Ω and let us prove that $W \in \ker(\pi)$ by induction on $\dim G$.

Let $p : \mathfrak{g}^* \to (\mathfrak{k}_{n-1})^*$ be the restriction mapping and $K_{n-1} = \exp(\mathfrak{k}_{n-1})$. If Ω is non-saturated with respect to $\mathfrak{k}_{n-1}$, there exists in $\ker(\pi)$ an element A having the form $A = X_n + V$ with a certain $V \in \mathscr{U}(\mathfrak{k}_{n-1})$. Making use of A to kill from W the part which is found outside of $\mathscr{U}(\mathfrak{k}_{n-1})$, we can suppose that $W \in \mathscr{U}(\mathfrak{k}_{n-1})$. Since $p(\Omega)$ is a K_{n-1}-orbit, the induction hypothesis gives us immediately the desired result.

Suppose now that Ω is saturated with respect to $\mathfrak{k}_{n-1}$. This implies that W belongs to $\mathscr{U}(\mathfrak{k}_{n-1})$. The restriction $\pi|_{K_{n-1}}$ is disintegrated as $\pi|_{K_{n-1}} \simeq \int_{\mathbb{R}}^{\oplus} \pi_t dt$ into a one parameter family $\{\pi_t\}_{t\in\mathbb{R}}$ of irreducible unitary representations of K_{n-1} and accordingly the restriction $p(\Omega) = \Omega|_{\mathfrak{k}_{n-1}}$ is decomposed as $p(\Omega) = \sqcup_{t\in\mathbb{R}}\omega_t$, where ω_t is the coadjoint orbit of K_{n-1} associated to π_t. Then, the induction hypothesis says that W belongs to $\ker(\pi_t)$ for almost all $t \in \mathbb{R}$ and hence $W \in \ker(\pi)$. ☐

The first step to prove Conjecture 1.1 consists in proving Theorem 3:

Theorem 3. *Let $\pi \in \hat{G}$ and let $W \in \mathscr{U}(\mathfrak{g})$ be K-diagonal. The function P_W extends to a K-invariant polynomial function on Ω.*

The proof of Theorem 3 will be achieved through different steps. Let us start with the following:

4.1. **A preliminary inductive proof.** In order to prove Theorem 3, we proceed by induction on $\delta(G,K) = \dim G + \dim G/K$. For small $\delta(G,K)$, G turns out to be abelian and the answer is immediate. Consider the flag of algebras (5) of $\mathfrak{g}$ and for the sake of simplicity of notation, denote $\mathfrak{g}' = \mathfrak{k}_{n-1}$ which contains $\mathfrak{k}$. Put $G' = \exp\mathfrak{g}'$ and suppose that Theorem 3 holds for G'.

4.1.1. *Case where the ideal $\mathfrak{g}'$ is of non-saturation.* Suppose that the orbit Ω is non-saturated with respect to $\mathfrak{g}'$, namely that $n \in T_\Omega$. Then the projection $pr : \mathfrak{g}^* \to \mathfrak{g}'^*$ turns out to be a K-equivariant homeomorphism between Ω and $\omega = pr(\Omega)$ which is a G'-orbit. Hence, $\mathbb{C}[\Omega]^K \cong \mathbb{C}[\omega]^K$. On the other hand, $\pi' = \pi|_{G'}$ is irreducible and there exists in $\ker(\pi)$ an element W' having the form $W' = X_n + A$ with $A \in \mathscr{U}(\mathfrak{g}')$ which allows us to identify $D_\pi(G)^K$ with $D_{\pi'}(G')^K$. Since ω is the coadjoint orbit of G' associated to π' and since $a_\ell = a_{\ell|_{\mathfrak{g}'}}$, the induction hypothesis proves Theorem 3 in this case.

4.1.2. *Case where the ideal $\mathfrak{g}'$ is of saturation.* Suppose now that Ω is saturated with respect to $\mathfrak{g}'$, namely that $n \in S_\Omega$. We have Lemma 2:

Lemma 2 ([2], [6, Lemma 4.1])**.** *There exists one and only one index $2 \le j \le n-1$ belonging to S_Ω and $b \in \mathscr{U}(\mathfrak{g}_{j-1})$ such that $Y_j + b \in \mathscr{U}_\pi(\mathfrak{g}_j)^{\mathfrak{g}'}$.*

Likewise, if $j = s_i$ $(1 \le i \le r-1)$, there exists a G'-invariant polynomial function

$$\alpha = x_i + \varphi(x_1, \dots, x_{i-1}) \tag{8}$$

on Ω, which separates the G'-orbits $w_\alpha = \{\ell \in \Omega : \alpha(\ell) = \alpha\}$ contained in $pr(\Omega)$. This means that $pr(\Omega) = \coprod_{\alpha\in\mathbb{R}} \omega_\alpha$, the disjoint union of G'-orbits ω_α. Accordingly,

$$\pi|_{G'} \simeq \int_{\mathbb{R}}^{\oplus} \pi_\alpha d\alpha \tag{9}$$

with $\pi_\alpha = \theta_{G'}(\omega_\alpha)$ for all $\alpha \in \mathbb{R}$.

Since the orbit Ω is saturated with respect to $\mathfrak{g}'$, for any $\ell \in \Omega$ there exists then a polarization $\mathfrak{b}[\ell]$ at ℓ contained in $\mathfrak{g}'$, which is also a polarization at $\ell|_{\mathfrak{g}'}$. Furthermore we can suppose that $W \in \mathscr{U}(\mathfrak{g}')$, since $\pi(W)a_\ell = P_W(\ell)a_\ell, \ell \in \Omega$, and $\pi_\ell = \mathrm{ind}_{G'}^G \pi_{\ell|_{\mathfrak{g}'}}$. It follows then from the definition of $a_\ell, \ell \in \Omega$, that

$$P_W^G(\ell) \quad = \quad P_W^{G'}(\ell_{|\mathfrak{g}'}), \ell \in \Omega,$$

where the index ^{G} (resp. $^{G'}$) indicates the action of W on a_ℓ (resp. on $a_{\ell|_{\mathfrak{g}'}}$). We apply the induction hypothesis to W and G'. Then it follows that the function P_W, which is rational on Ω restricts to the G'-orbits $\omega_\alpha, \alpha \in \mathbb{R}$, as a polynomial function. Let ℓ be a point of Ω, for each real number t, let α be such that $Ad^*(\exp tX_n)\ell \in \omega_\alpha$, then:

$$P_W(Ad^*(\exp(tX_n)g')\ell) = P_W(\alpha, g') \quad = \quad \frac{A(\alpha, g')}{B(\alpha, g')}, g' \in G',$$

for two polynomial functions A, B. Since $P_{W|\omega_\alpha}$ is polynomial, we have that B is independent of the variable g' and so P_W is given by a polynomial function A devided by a polynomial function in α.

The following consequence is then immediate.

Corollary 1. *Suppose that $\mathfrak{k}$ is contained in an ideal of codimension 2. Then for every K-diagonal $W \in \mathscr{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$, the function $\ell \mapsto P_W(\ell)$ is polynomial.*

Proof. Let $\mathfrak{h}_j, j = 1, 2$ be two distinct ideals of codimension 1 containing $\mathfrak{k}$. Accordingly to Subsection 4.1.1, we can assume that the orbit Ω is satured with respect to $\mathfrak{h}_j, j = 1, 2$. We fix the flag (1) such that $\mathfrak{g}_{n-2} = \mathfrak{h}_1 \cap \mathfrak{h}_2$, and $\mathfrak{g}_{n-1} = \mathfrak{h}_1$, thus, if w is the G_{n-2}-orbit of $l|\mathfrak{g}_{n-2}$, the set of jump indices are $S_\Omega = S_w \cup \{n-1, n\}$, or $S_\Omega = S_w \cup \{i, k, n-1, n\}$. In the first case, we have $n - 1 = s_i$, $n = s_{i+1}$, by (8), there is a H_1-invariant polynomial function $\alpha_1 = x_i + \varphi_1(x_1, ...x_{i-1})$ separating the H_1-orbits, and replacing $\mathfrak{h}_1$ by $\mathfrak{h}_2$ in the flag (1), there is a H_2-invariant polynomial function $\alpha_2 = x_{i+1} + \varphi_2(x_1, ...x_{i-1})$ separating the H_2-orbits. Moreover, for any complex numbers c_j, there is no common divisor for $\alpha_1 + c_1$ and $\alpha_2 + c_2$. In the second case, suppose the jump indices for the H_1-orbit w_1 of $l|\mathfrak{h}_1$ are $S_w \cup \{i, n-1\}$, with $i = s_{i_1}$, and by (8), there is a G_{n-2}-invariant polynomial function $\beta_1 = x_{i_1} + \varphi_1(x_1, ...x_{i_1-1})$ separating the G_{n-2}-orbits in the H_1-orbit w_1. Suppose X_n be in $\mathfrak{h}_2 \setminus \mathfrak{h}_1$, and $k = s_{i_2}$, by (8) there is $\alpha_2 = x_{i_2} + \varphi_2(x_1, ...x_{i_2-1})$ separating the H_2-orbits in Ω. Fix X_n such that $\alpha_2(\exp(tX_n)l) = t$ for each l in Ω such that $\alpha_2(l) = 0$. Finally put: $\alpha_1(\exp(tX_n)l) = \beta_1(l|\mathfrak{h}_1)$ or

$$\alpha_1(x_i) = (e^{-\alpha_2(x_i)ad^*(X_n)}\beta_1)(x_i) = \sum_m \frac{(-\alpha_2(x_i))^m}{m!}\beta_1((ad^*(X_n))^m(x_i)).$$

The function α_2, polynomial on Ω is H_1-invariant and separates the H_1 orbits in Ω. Moreover, since for any complex numbers c_1 and c_2,

$$\alpha_1 + c_1 = e^{c_2 ad^*(X_n)}\beta_1 + c_1 + \sum_{m>0} \frac{(-\alpha_2 - c_2)^m}{m!} e^{c_2 ad^*(X_n)}\beta_1((ad^*(X_n))^m \cdot),$$

and $e^{c_2 ad^*(X_n)}\beta_1 = x_{i_1} + \psi(x_1, \ldots, x_{i_1-1})$, there is no common divisor for $\alpha_1 + c_1$ and $\alpha_2 + c_2$. In both cases, applying the induction hypothesis to $W$and H_j , we can write P_W as a quotient of a polynomial function A_j by a function $B_j(\alpha_j)$, polynomial in α_j . Thus: $P_W = \frac{A_1}{B_1(\alpha_1)} = \frac{A_2}{B_2(\alpha_2)}$, and

$$B_2(\alpha_2)A_1 = B_1(\alpha_1)A_2.$$

Since $\alpha_1 + c_1$ and $\alpha_2 + c_2$ have no common divisor, P_W itself is a polynomial function. □

On the other hand, let $W \in \mathscr{U}_\pi(\mathfrak{k}_v)^{\mathfrak{k}}$. If $v \leq d$, W belongs to $\mathscr{U}_\pi(\mathfrak{k})^{\mathfrak{k}}$ and the operator $\sigma(W)$ is a scalar for almost all $\sigma \in \hat{K}$ with respect to the measure ν_π used in the irreducible decomposition of $\pi|_K$. Then, we can apply Theorem 2.1.1 in [19] to get:

Proposition 2. *For any $W \in \mathscr{U}_\pi(\mathfrak{k})^{\mathfrak{k}}$, the function $\ell \mapsto P_W(\ell)$ is polynomial on Ω.*

4.2. **Proof of Theorem 3.** As usual, we can assume that the center $\mathfrak{z}$ of $\mathfrak{g}$ has dimension 1, that $\pi(= \pi_\ell)$ is not 0 on $\mathfrak{z}$ and that $\mathfrak{z} \subset \mathfrak{k}$. Also according to 4.1.1, we can assume that for every subalgebra $\mathfrak{g}'$ of codimension one containing $\mathfrak{k}$, that Ω is saturated with respect to $\mathfrak{g}'$. In particular a polarization $\mathfrak{b}[\ell]$ with $B[\ell] = \exp(\mathfrak{b}[\ell])$ of ℓ can always be found in $\mathfrak{g}'$ and $W \in \mathscr{U}(\mathfrak{g}')$.

We make now a further induction on j_0, the smallest index $j \in \{1, \ldots, n\}$, such that $W \in \mathscr{U}(\mathfrak{k}_{j_0})$. If $j_0 \leq d$, then W is an e-central element of Corwin-Greenleaf for the projection of Ω on $\mathfrak{k}^*$ and hence the function $P_W(\ell)$ is polynomial as in Proposition 2. We can therefore assume that $j_0 \geq d + 1$.

Let now $\mathfrak{l} = \mathfrak{k}_{d+1}$ and $L = \exp \mathfrak{l}$. If the generic L-orbits in $\Omega|_{\mathfrak{l}}$ are non-saturated with respect to $\mathfrak{k}$, there exists a $\nu = aX_{d+1} + b,\ a, b \in \mathscr{U}(\mathfrak{k})$ which is e-central for $\Omega|_{\mathfrak{l}}$. Applying W, ν to the Penney distribution $a_\ell(\ell \in \Omega)$, we see that they commute modulo $\ker(\pi)$ and so W is also L-invariant. If we use the Penney distributions a_ℓ^L (as in formula (4)) and if $\mathfrak{b}[\ell|_{\mathfrak{l}}] \cap \mathfrak{b}[\ell] = \mathfrak{b}[\ell|_{\mathfrak{k}}] \cap \mathfrak{b}[\ell]$, we see that for some $S \in (\mathfrak{l}(\ell|_{\mathfrak{l}}) \cap \ker(\ell)) \setminus \mathfrak{k}$, we have for any $\varphi \in \mathscr{H}_\pi^\infty$:

$$\begin{aligned}
\langle W \cdot a_\ell^L, \varphi \rangle &= \int_{B[\ell|_{\mathfrak{l}}]/(B[\ell|_{\mathfrak{l}}] \cap B[\ell])} \overline{\pi(W^*)\varphi(b)\chi_\ell(b)} d\dot{b} \\
&= \int_{\mathbb{R}} \int_{B[\ell|_{\mathfrak{k}}]/(B[\ell|_{\mathfrak{k}}] \cap B[\ell])} \overline{\pi(W^*)\varphi(\exp(sS)b)\chi_\ell(\exp(sS)b)} d\dot{b} ds \\
&= \int_{\mathbb{R}} \int_{B[\ell|_{\mathfrak{k}}]/(B[\ell|_{\mathfrak{k}}] \cap B[\ell])} \overline{\pi(W^*)(\pi(\exp(-sS))\varphi)(b)\chi_\ell(\exp b)} d\dot{b} ds \\
&= \int_{\mathbb{R}} P_W^K(\ell_{|\mathfrak{k}}) \int_{B[\ell|_{\mathfrak{l}}]/(B[\ell|_{\mathfrak{l}}] \cap B[\ell])} \overline{\varphi(\exp(sS)b)\chi_\ell(\exp b)} d\dot{b} ds \\
&= P_W^K(\ell) \langle a_\ell^L, \varphi \rangle.
\end{aligned}$$

Therefore W is L-diagonal. Since $\delta(G,L) < \delta(G,K)$, the induction hypothesis implies that $P_W^K = P_W^L$ is polynomial.

Recall now that we are in the situation where the orbit Ω is saturated with respect to $\mathfrak{k}_{n-1}$. There exists then by Lemma 2, a unique index $2 \leq r_0 \leq n-1$ belonging to S_Ω and $b \in \mathscr{U}(\mathfrak{g}_{r_0-1})$ such that

$$\kappa = Y_{r_0} + b \in \mathscr{U}_\pi(\mathfrak{g}_{r_0})^{\mathfrak{g}'} \tag{10}$$

and $[X_n, \kappa] \neq 0 \bmod \ker(\pi)$. The polynomial function P_κ on $\Omega_{|\mathfrak{k}_{n-1}}$ then separates the K_{n-1}-orbits $\omega_y, y \in \mathbb{R}$ and, as we have seen in 4.1.2, W belongs to $\mathscr{U}(\mathfrak{k}_{n-1})$ and P_W can be written as $\frac{A}{B}$ for a polynomial function A on Ω divided by a polynomial B in the variable P_κ.

Let now $\widetilde{\mathfrak{g}}$ be another ideal of $\mathfrak{g}$ of codimension 1. If $\mathfrak{k} \subset \widetilde{\mathfrak{g}}$, then Theorem 3 holds by Corollary 1. Hence we assume that $\mathfrak{k} \not\subset \widetilde{\mathfrak{g}}$. Let us treat first the case where Ω is not saturated with respect to $\widetilde{\mathfrak{g}}$. Write $\mathfrak{g} = \mathbb{R}\widetilde{X} + \widetilde{\mathfrak{g}}$ and $\widetilde{G} = \exp \widetilde{\mathfrak{g}}$. We can again assume as in 4.1.1 that $W \in \mathscr{U}(\widetilde{\mathfrak{g}})$. Let $\widetilde{\mathfrak{k}} = \mathfrak{k} \cap \widetilde{\mathfrak{g}}$ and $\widetilde{K} = \exp \widetilde{\mathfrak{k}}$. If $\mathfrak{b}[\ell|_\mathfrak{k}] \subset \widetilde{\mathfrak{g}}$ almost everywhere on Ω, then $a_\ell = a_{\ell_{|\widetilde{\mathfrak{g}}}}$ and the induction hypothesis tells us that $P_W(\ell)$ is a polynomial function on the $\widetilde{G}$-orbit $\widetilde{\Omega} = \widetilde{p}(\Omega)$, where $\widetilde{p} : \mathfrak{g}^* \to (\widetilde{\mathfrak{g}})^*$ is the restriction map. Hence P_W is also a polynomial function on Ω.

If $\mathfrak{b}[\ell|_\mathfrak{k}] \not\subset \widetilde{\mathfrak{g}}$ for almost all $\ell \in \Omega$, let us write $\mathfrak{b}[\ell|_\mathfrak{k}] = \mathbb{R}\widetilde{X(\ell)} + \mathfrak{b}[\widetilde{\ell}|_{\widetilde{\mathfrak{k}}}]$, where $\widetilde{\ell} = \widetilde{p}(\ell)$. We remark that we can take $\mathfrak{b}[\widetilde{\ell}|_{\widetilde{\mathfrak{k}}}]$ to be the Vergne polarisation at $\widetilde{\ell}|_{\widetilde{\mathfrak{k}}} \in (\widetilde{\mathfrak{k}})^*$ built from a Jordan-Hölder sequence $\mathscr{S} \cap \widetilde{\mathfrak{g}}$ of $\widetilde{\mathfrak{g}}$, $\mathscr{S}$ denoting the flag (1) of $\mathfrak{g}$. As W is K-invariant, we see that

$$\langle W \cdot a_\ell, \varphi \rangle = \int_\mathbb{R} \langle W \cdot a_{\widetilde{\ell}}, \varphi(\exp(t\widetilde{X(\ell)})\cdot)\rangle dt \ (\ell \in \Omega)$$

for $\varphi \in \mathscr{H}_\pi^\infty$. We identify $\mathscr{H}_\pi^{-\infty}$ with $\mathscr{H}_{\widetilde{\pi}}^{-\infty}$. Fixing a generic $\ell \in \Omega$ and taking a Malcev basis in $\mathfrak{g}$ relative to $\mathfrak{b}[\ell]$, which contains a Malcev basis in $\mathfrak{b}[\ell|_\mathfrak{k}]$ relative to $\mathfrak{b}[\ell|_\mathfrak{k}] \cap \mathfrak{b}[\ell]$, we identify the space $\mathscr{H}_\pi$ of π with $\mathbb{R}^m, m = \dim(\mathfrak{g}/\mathfrak{b}[\ell])$. Since

$$B[\ell|_\mathfrak{k}]/(B[\ell|_\mathfrak{k}] \cap B[\ell]) \simeq B[\ell|_\mathfrak{k}]B[\ell]/B[\ell] = \exp(\mathbb{R}\widetilde{X(\ell)})B[\widetilde{\ell}|_{\widetilde{\mathfrak{k}}}]B[\ell]/B[\ell],$$

we finally get the following two eventualities: either

$$B[\ell|_\mathfrak{k}]/(B[\ell|_\mathfrak{k}] \cap B[\ell]) \simeq B[\widetilde{\ell}|_{\widetilde{\mathfrak{k}}}]/(B[\widetilde{\ell}|_{\widetilde{\mathfrak{k}}}] \cap B[\widetilde{\ell}])$$

or

$$\begin{aligned} B[\ell|_\mathfrak{k}]/(B[\ell|_\mathfrak{k}] \cap B[\ell]) &\simeq \exp(\mathbb{R}\widetilde{X(\ell)})B[\widetilde{\ell}|_{\widetilde{\mathfrak{k}}}]B[\ell]/B[\ell] \\ &\simeq \exp(\mathbb{R}\widetilde{X(\ell)}) \times B[\widetilde{\ell}|_{\widetilde{\mathfrak{k}}}]/(B[\widetilde{\ell}|_{\widetilde{\mathfrak{k}}}] \cap B[\widetilde{\ell}]). \end{aligned}$$

In the first case, the distribution a_ℓ associated to π can be identified with the generalized vector $a_{\widetilde{\ell}}$ of $\widetilde{\pi} = \pi|_{\widetilde{G}}$. In the second case, for $\varphi \in \mathscr{H}_\pi^\infty$ satisfying

$$\varphi(\exp(t\widetilde{X(\ell)})\widetilde{g}) = \phi(t)\psi(\widetilde{g}),\ t \in \mathbb{R}, \widetilde{g} \in \widetilde{G},$$

with $\phi \in C_c(\mathbb{R}), \psi \in \mathscr{H}_{\widetilde{\pi}}^\infty$, the $\mathfrak{k}$-invariance of W implies that

$$\begin{aligned} \langle W \cdot a_\ell, \varphi \rangle &= (\int_\mathbb{R} \overline{\phi(t)e^{it\ell(\widetilde{X(\ell)})}} dt)\langle W \cdot a_{\widetilde{\ell}}, \psi \rangle \\ &= P_W(\ell)(\int_\mathbb{R} \overline{\phi(t)e^{it\ell(\widetilde{X(\ell)})}} dt)\langle a_{\widetilde{\ell}}, \psi \rangle. \end{aligned}$$

In both cases we see that $W\cdot a_{\widetilde{\ell}}=P_W(\ell)a_{\widetilde{\ell}}$. According to the induction hypothesis $P_W(\ell)=P_W(\widetilde{\ell})$ is a polynomial function on $\widetilde{\Omega}$ and hence also on Ω.

We can now assume, as we have seen before, that Ω is saturated with respect to $\widetilde{\mathfrak{g}}$, that for generic $\ell\in\Omega$, the L-orbits of $\ell|_{\mathfrak{l}}$ are saturated with respect to $\mathfrak{k}$ and that $\mathfrak{k}\not\subset\widetilde{\mathfrak{g}}$.

Recall again $\widetilde{\mathfrak{k}}:=\mathfrak{k}\cap\widetilde{\mathfrak{g}}$. If $W\in\mathscr{U}(\widetilde{\mathfrak{g}})$, then the last computation tells us that

$$W\cdot a^{\widetilde{K}}_{\ell_{|\widetilde{\mathfrak{g}}}}=P_W(\ell_{|\widetilde{\mathfrak{g}}})a^{\widetilde{K}}_{\ell_{|\widetilde{\mathfrak{g}}}}.$$

Since $\delta(\tilde{G},\tilde{K})<\delta(G,K)$, by the induction hypothesis, $P_W(\ell_{|\widetilde{\mathfrak{g}}})$ is a polynomial on the $\tilde{G}$-orbit of $\tilde{\ell}$.

Suppose that $P_\kappa(\ell)\neq 0$ and $ad^*(X_n)P_\kappa=1$. Let $\widetilde{\kappa}_1=Y_{\widetilde{r_0}}+\widetilde{U},\widetilde{U}\in\mathscr{U}(\mathfrak{g}_{\widetilde{r_0}-1})$ be the e-central element of Corwin-Greenleaf in $\mathscr{U}(\widetilde{\mathfrak{g}})$ associated to $\mathfrak{k}_{n-1}\cap\widetilde{\mathfrak{g}}$ and $\tilde{G}$-orbit of $\tilde{\ell}$ as in (10). Then as in the proof of Corollary 1, we conclude that the denominator of the rational function P_W is a polynomial in $P_{\widetilde{\kappa}_1}(Ad^*(\exp(-P_\kappa(\ell)X_n))\ell)$. Since the denominator is also a polynomial in $P_\kappa(\ell)$, it follows that P_W is in fact a polynomial function.

Therefore we can finally assume that W is not contained in $\mathscr{U}(\widetilde{\mathfrak{g}})$. This means that $\mathfrak{b}[\ell|_{\mathfrak{k}}]\not\subset\widetilde{\mathfrak{k}}$ for generic $\ell\in\Omega$. This being assumed, we suppose that the denominator of the rational function $P_W(\ell)$ is not trivial. We are brought to the case where this denominator is equal to $P_{\kappa-c}(\ell)$ for some $c\in\mathbb{C}$. Take $\widetilde{X}$ in $\mathfrak{k}$. In these circumstances, there exists in $\mathscr{U}(\mathfrak{k})$ an element

$$\sigma=\bar{a}\widetilde{X}+\bar{b},\ \bar{a},\bar{b}\in\mathscr{U}(\widetilde{\mathfrak{k}}) \tag{11}$$

which is e-central for $\Omega|_{\mathfrak{k}}$. If W is of degree m relatively to $\widetilde{X}$ with the dominant term $w_m\widetilde{X}^m, w_m\in\mathscr{U}(\widetilde{\mathfrak{g}})$, we saw in Subsection 3.3 that w_m and $\bar{a}$ are $\mathfrak{k}$-invariant. Then, applying $\bar{a}$ and w_m to $a_\ell(\ell\in\Omega)$, we see that they commute each other modulo $\ker(\pi)$. Thus,

$$W_1=\bar{a}^mW-w_m\sigma^m \tag{12}$$

is of degree inferior to m relatively to $\widetilde{X}$. Repeating this process, we build an element $\widetilde{W}\in\mathscr{U}(\widetilde{\mathfrak{g}})$ such that $P_{\widetilde{W}}(\ell)$ is a polynomial function on Ω. This means that α is a factor of $\bar{a}$.

Recall that j_0 is the smallest index such that $W\in\mathscr{U}(\mathfrak{k}_{j_0})$ modulo $\ker(\pi)$. We now prove the following:

Lemma 3. *There exists a K-diagonal element*

$$\nu=\beta X_{j_0}+\gamma,\ \beta,\gamma\in\mathscr{U}(\mathfrak{k}_{j_0-1}),$$

in $\mathscr{U}_\pi(\mathfrak{k}_{j_0})^{\mathfrak{k}}$ such that $P_\nu(\ell)$ extends to a polynomial function on Ω and such that β is not divisible by α modulo $\ker(\pi)$.

Proof. We proceed by induction on $\dim\mathfrak{k}$. Let first $\dim\mathfrak{k}=1$, namely $\mathfrak{k}$ is abelian. At each point $\ell\in\Omega$, the Penney's distribution a_ℓ is nothing but the Dirac measure at the unit element of G. Put $\mathfrak{b}=\cap_{\ell\in\Omega}\mathfrak{b}[\ell]$, which is an ideal of $\mathfrak{g}$. Then, the existence of W allows us to take X_{j_0} in $\mathfrak{b}$. This being done, $\nu=X_{j_0}$ suits us. Suppose now that $\dim\mathfrak{k}>1$. Let us repeat the above construction of the element $\widetilde{W}\in\mathscr{U}_\pi(\widetilde{\mathfrak{g}})^{\mathfrak{k}}$ such that $P_{\widetilde{W}}(\ell)=\widetilde{P}_{\widetilde{W}}(\widetilde{\ell})$ extends to a polynomial function on Ω. Here, $\widetilde{\ell}=\ell|_{\widetilde{\mathfrak{g}}}$ and $\widetilde{P}$ designates the object obtained from the pair $(a_{\widetilde{\ell}},\widetilde{\mathfrak{k}})$.

In the first step of construction, if $w_m \notin \mathscr{U}(\widetilde{\mathfrak{k}}_{j_0-1})$, then we put $W' = w_m$ which belongs to $\mathscr{U}_\pi(\widetilde{\mathfrak{g}})^{\mathfrak{k}}$ but not in $\mathscr{U}(\widetilde{\mathfrak{k}}_{j_0-1})$, where $\widetilde{\mathfrak{k}}_{j_0-1} = \mathfrak{k}_{j_0-1} \cap \widetilde{\mathfrak{g}}$. Otherwise, the element W_1 defined in equation (12) does not belong to $\mathscr{U}(\mathfrak{k}_{j_0-1})$, and we replace W by W_1, and continue the construction of $\widetilde{W}$. At the end of this process, we get an element W' in $\mathscr{U}_\pi(\widetilde{\mathfrak{g}})^{\mathfrak{k}}$ but not in $\mathscr{U}(\widetilde{\mathfrak{k}}_{j_0-1})$.

Hence, by the induction hypothesis, there exists a $\widetilde{K}$-diagonal element

$$\widetilde{\nu} = \widetilde{a}X_{j_0} + \widetilde{b},\ \widetilde{a}, \widetilde{b} \in \mathscr{U}(\widetilde{\mathfrak{k}}_{j_0-1}),$$

in $\mathscr{U}_\pi(\widetilde{\mathfrak{k}}_{j_0})^{\widetilde{\mathfrak{k}}}$ such that $\widetilde{P}_{\widetilde{\nu}}(\widetilde{\ell})$ extends to a polynomial function on Ω and that $\widetilde{a}$ is not divisible by α modulo $\ker(\pi)$. Since σ is e-central for $\Omega|_{\mathfrak{k}}$, it gives us the polynomial function $P_\sigma(\ell)$ when it is applied to Penney's distributions for $\widetilde{\mathfrak{k}}$. It follows that $[\sigma, \widetilde{\nu}] \in \ker(\pi)$. Thus, $\widetilde{\nu}$ turns out to be $\mathfrak{k}$-invariant and $P_{\widetilde{\nu}}(\ell) = \widetilde{P}_{\widetilde{\nu}}(\widetilde{\ell})$. □

We continue the proof of Theorem 3. Let us write

$$W = \sum_{j=0}^{r} w_j X_{j_0}^j,\ w_j \in \mathscr{U}(\mathfrak{k}_{j_0-1})(0 \le j \le r). \tag{13}$$

We go now to engage a double induction on the index $j_0 > d$ and on the degree r of X_{j_0} in the expression of W. As w_r is $\mathfrak{k}$-invariant, it follows from the induction hypothesis that $w_r{\cdot}a_\ell = P_{w_r}(\ell)a_\ell$ for $\ell \in \Omega$ with a function $P_{w_r}(\ell)$ which extends into a polynomial function on Ω. Next, in the expression (13), let us suppose our assertion established for the elements whose degree relative to X_{j_0} is inferior or equal to $r-1$. We see that

$$\widetilde{W} = \beta^r W - w_r \nu^r$$

is of degree inferior to r relative to X_{j_0} and hence $P_{\widetilde{W}}(\ell)$ is a polynomial function on Ω. One deduces from this that $P_W(\ell)$ is polynomial because β is not divisible by α.

□

Corollary 2. *Suppose that $\pi|_K$ has finite multiplicities. Then the rational function $\ell \mapsto P_W(\ell) = \Theta(W)(\ell)$ extends to a polynomial function on Ω, where Θ is defined as in equation* (7) .

5. Proof of Conjecture 1.1: Second part

Recall first the flag of subalgebras (2), where $\mathfrak{k} = \mathfrak{k}_d$, $j_0 \ge d+1$ the smallest index such that $W \in \mathscr{U}(\mathfrak{k}_{j_0})$ and α as given in equation (8). Let us first prove the following result, which could be regarded as a substitute to Lemma 3. Repeating this process, we get the element $\widetilde{\nu}$ in Lemma 3.

Proposition 3. *Let $m \le d$ such that the generic K_m-orbits in $\Omega|_{\mathfrak{k}_m}$ are saturated with respect to $\mathfrak{k}_{m-1}$. Write $\mathfrak{k}_m = \mathbb{R}X_m + \mathfrak{k}_{m-1}$ for some $X_m \in \mathfrak{k}_m \setminus \mathfrak{k}_{m-1}$ and let*

$$\tau_m = a'_m X_{k_m} + b'_m, a'_m, b'_m \in \mathscr{U}(\mathfrak{k}_{k_m-1})$$

be an e-central element for $\Omega|_{\mathfrak{k}_{m-1}}$ which is not e-central for $\Omega|_{\mathfrak{k}_m}$ with the index k_m as small as possible. Then:

(1) *τ_m and $[X_m, \tau_m]$ can be choosen in a way that they are not divisible by α modulo* $\ker(\pi)$.

(2) *Suppose that* $\mathfrak{h}'_m = \mathfrak{k}_m + \mathfrak{g}_{j_0-1}$ *is strictly included in* $\mathfrak{h}_m = \mathfrak{k}_m + \mathfrak{g}_{j_0}$ *and there exists* $W_m \in \mathscr{U}_\pi(\mathfrak{h}_m)^{\mathfrak{k}_m}$ *such that* $W_m \notin \mathscr{U}_\pi(\mathfrak{h}'_m)^{\mathfrak{k}_m}$*, which gives us a rational function on* Ω *when it is applied to Penney's distributions for* $\mathfrak{k}_m$*, then there exists an element*

$$\nu_m = a_m X_{j_0} + b_m,\ a_m, b_m \in \mathscr{U}(\mathfrak{h}'_m),$$

where $\mathfrak{g}_{j_0} = \mathbb{R}X_{j_0} + \mathfrak{g}_{j_0-1}$*, which is* $\mathfrak{k}_m$*-invariant and gives us a polynomial function on* Ω *when it is applied to Penney's distributions for* $\mathfrak{k}_m$ *and such that* a_m *is not divisible by* α *modulo* $\ker(\pi)$.

Proof. Let us proceed by induction on $\dim \mathfrak{k}$. The claim is trivial when $\dim \mathfrak{k} \leq 3$. We prove both the assertions at the same time in case of saturation. Let $4 \leq m \leq d$ and suppose that the generic orbits by $K_m = \exp(\mathfrak{k}_m)$ in $\Omega|_{\mathfrak{k}_m}$ are saturated with respect to $\mathfrak{k}_{m-1}$. Let

$$\tau_m = a'_m X_{k_m} + b'_m, a'_m, b'_m \in \mathscr{U}(\mathfrak{k}_{k_m-1})$$

be a e-central element for $\Omega|_{\mathfrak{k}_{m-1}}$ which is not e-central for $\Omega|_{\mathfrak{k}_m}$ and which is not divisible by α. Choose the index k_m as small as possible.

Replacing $\mathfrak{k}$ by $\mathfrak{k}_m$, Lemma 3 gives us the element $\tau_m = a'_m X_{k_m} + b'_m$, with $a'_m, b'_m \in \mathscr{U}(\mathfrak{k}_{k_m-1})$ and a'_m is not divisible by α modulo $\ker(\pi)$. Now $[X_m, \tau_m]$ is by construction in $\mathscr{U}(\mathfrak{k}_{k_m-1})$, thus it is not divisible by α modulo $\ker(\pi)$.

This being done, suppose that there exists $W_m \in \mathscr{U}_\pi(\mathfrak{h}_m)^{\mathfrak{k}_m} \backslash \mathscr{U}_\pi(\mathfrak{h}'_m)$, where $\mathfrak{h}_m = \mathfrak{k}_m + \mathfrak{g}_{j_0}$, which gives us a rational function on Ω when it is applied to Penney's distributions for $\mathfrak{k}_m$ and let us build the element ν_m with the properties cited in the proposition.

By the saturation argument, we see that $W_m \in \mathscr{U}(\mathfrak{h}_{m-1})$ and that the Penney's distributions for $\mathfrak{k}_m$ are the same as those for $\mathfrak{k}_{m-1}$. Therefore, by the induction hypothesis, there exists a K_{m-1}-diagonal element

$$\nu_{m-1} = a_{m-1} X_{j_0} + b_{m-1},\ a_{m-1}, b_{m-1} \in \mathscr{U}(\mathfrak{h}'_{m-1})$$

in $\mathscr{U}(\mathfrak{h}_{m-1})$ which gives us a polynomial function on Ω when it is applied to Penney's distributions for $\mathfrak{k}_{m-1}$ and such that a_{m-1} is not divisible by α. If ν_{m-1} is $\mathfrak{k}_m$-invariant, it is qualified as our desired ν_m. Suppose that ν_{m-1} is not $\mathfrak{k}_m$-invariant and retake the construction of our ν introduced in [7]. For a sufficiently large integer $v \in \mathbb{N}$, we consider

$$\psi = \nu_{m-1} + F(\tau_m),$$

where $F(t)$ is a polynomial in one variable t of degree $2v$. For $k \in \mathbb{N}$, put

$$\psi_0 = \psi,\ \psi_k = (\mathrm{ad}(X_m))^k(\psi).$$

Remark that $[X_m, [X_m, \tau_m]] \in \ker(\pi)$. Therefore, if v is sufficiently large, then

$$\psi_{2v} \notin \ker(\pi),\ \psi_{2v+1} \in \ker(\pi).$$

We now build an element of $\mathscr{U}(\mathfrak{h}_{m-1})^{\mathfrak{k}_m}$ by the formula

$$\begin{aligned}\nu_m =& (\psi_0\psi_{2v} + \psi_{2v}\psi_0) - (\psi_1\psi_{2v-1} + \psi_{2v-1}\psi_1) + \cdots \\ &+ (-1)^{v-2}(\psi_{v-2}\psi_{v+2} + \psi_{v+2}\psi_{v-2}) \\ &+ (-1)^{v-1}(\psi_{v-1}\psi_{v+1} + \psi_{v+1}\psi_{v-1}) + (-1)^v \psi_v^2.\end{aligned}$$

Remark once again the fact that v is sufficiently large. This assures that ν_m is of degree 1 with respect to X_{j_0}. Moreover, $[X_m, \nu_{m-1}]$ applied to a_ℓ gives us a polynomial function on Ω. Indeed, we see by definition that

$$P_{[X_m,\nu_{m-1}]}(\ell) = \frac{d}{dt} P_{\nu_{m-1}}(\exp(tX_m)\cdot \ell)\big|_{t=0},\ \ell \in \Omega.$$

It follows that $\nu_m \cdot a_\ell = P_{\nu_m}(\ell) a_\ell$ for generic $\ell \in \Omega$ with a polynomial function $P_{\nu_m}(\ell)$ on Ω.

Finally, since, for any k, $(\mathrm{ad}(X_n))^k F(\tau_m)$ belongs to $\mathscr{U}(\mathfrak{h}'_m)$, we can choose the polynomial F such that $\nu_m = a_m X_{j_0} + b_m,\ a_m, b_m \in \mathscr{U}(\mathfrak{h}'_m)$ and a_m not divisible by α modulo $\ker(\pi)$. Indeed, let

$$F(t) = \lambda_0 + \lambda_1 t + \cdots + \lambda_{2v-1} t^{2v-1} + \lambda_{2v} t^{2v},\ \lambda_j \in \mathbb{C}\ (0 \le j \le 2v).$$

Suppose that $(\mathrm{ad}(X_m))^k(a_{m-1})$ $(0 \le k \le n_0)$ are not divisible by α modulo $\ker(\pi)$, but that $(\mathrm{ad}(X_m))^{n_0+1}(a_{m-1})$ and hence all the elements $(\mathrm{ad}(X_m))^k(a_{m-1}),\ k \ge n_0+1$ are divisible by α modulo $\ker(\pi)$.

Considering the λ_j as variables, and supposing that for any choice of these variables, the coefficient a_m of X_{j_0} in ν_m is divisible by α modulo $\ker(\pi)$, thus for any j, the coefficient of $\lambda_j X_{j_0}$ of ν_m is divisible by α modulo $\ker(\pi)$. Remark now that the terms $\lambda_{2v-n_0} X_{j_0}$ in ν_m appear only in the sum:

$$\sum_{k \ge n_0} (-1)^k (\psi_k \psi_{2v-k} + \psi_{2v-k}\psi_k) \equiv 2 \sum_{k \ge n_0} (-1)^k \psi_{2v-k}\psi_k \pmod{\ker(\pi)}$$

and they are modulo $\ker(\pi)$:

$$\Big(\sum_{k \ge n_0} c_k (\mathrm{ad}X_m)^{2v-k}({\tau_m}^{2v-n_0})(\mathrm{ad}X_m)^k a_{m-1}\Big)\lambda_{2v-n_0} X_{j_0},$$

where c_k is a numerical constant. Each term in this sum is divisible by α except the first one, by definition of n_0. This proves that there is a polynomial F such that the conditions of the proposition hold for ν_m.

Now, suppose that the generic K_m-orbits in $\Omega|_{\mathfrak{k}_m}$ are non-saturated with respect to $\mathfrak{k}_{m-1}$. Then, there exists an element

$$\sigma_m = c_m X_m + d_m,\ c_m, d_m \in \mathscr{U}(\mathfrak{k}_{m-1})$$

which is e-central for $\Omega|_{\mathfrak{k}_m}$. If

$$W_m = v_r X_m^r + v_{r-1} X_m^{r-1} + \cdots + v_1 X_m + v_0,\ v_j \in \mathscr{U}(\mathfrak{h}_{m-1})(0 \le j \le r)$$

with $v_r \notin \ker(\pi)(r>0)$, $W'_m = c_m^r W_m - \sigma_m^r v_r$ is $\mathfrak{k}_m$-invariant and of degree smaller or equal to $r-1$ relative to X_m because v_r, c_m are also $\mathfrak{k}_m$-invariant and commute each other modulo $\ker(\pi)$. Repeating these manipulations if necessary, we arrive to a $\mathfrak{k}_m$-invariant element $W_{m-1} \in \mathscr{U}(\mathfrak{h}_{m-1})$ which gives us a rational function on Ω when it is applied to Penney's distributions. From the induction hypothesis there exists a $\mathfrak{k}_{m-1}$-invariant element ν_{m-1} which satisfies the required conditions as above. Applying ν_{m-1}, σ_m to Penney's distributions for $\mathfrak{k}_{m-1}$, we confirm that they commute each other modulo $\ker(\pi)$. In this way, ν_{m-1} turns out to be $\mathfrak{k}_m$-invariant and is qualified as our desired ν_m. □

Corollary 3. *Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{k}$ and $\mathfrak{h}'$ an ideal of codimension 1 in $\mathfrak{h}$ such that the generic orbits by $H = \exp\mathfrak{h}$ in $\Omega|_{\mathfrak{h}}$ are saturated with respect to $\mathfrak{h}'$. Let*

$$\tau = a' X_{k'} + b',\ a', b' \in \mathscr{U}(\mathfrak{k}_{k'-1}),\ a' \notin \ker(\pi),$$

be a e-central element for $\Omega|_{\mathfrak{h}'}$*, which is not e-central for* $\Omega|_{\mathfrak{h}}$ *for which* k' *is minimal. Then* τ *and* $[X,\tau]$ *can be chosen in a way that they are not divisible by* α*, where* $\mathfrak{h} = \mathbb{R}X + \mathfrak{h}'$*.*

We now look at the surjectivity of the homomorphism Θ defined by equation (7). We first record the following, which will be of use later

Proposition 4 ([7, Proposition 4.4])**.** *Keep the same notations and hypotheses and let us denote by* y' *the variable corresponding to the polynomial function defined as in equation (8). Then for every polynomial* $\zeta(x) \in \mathbb{C}[\Omega]^K$*, there exists a polynomial* $s(y')$ *of* y' *such that the product* $s(y')\zeta(x)$ *is in the image of* Θ*.*

Let $\mathscr{V}$ be the set of K-diagonal elements $W \in \mathscr{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ such that $W\cdot a_\ell = P_W(\ell)a_\ell$ with a function $P_W(\ell)$ which extends to a polynomial function on Ω. We consider the image M of the mapping

$$\Theta_{\mathscr{V}} : \mathscr{V} \ni W \mapsto P_W \in \mathbb{C}[\Omega]^K.$$

We now prove the following:

Proposition 5. *Let* $q(\ell) \in \mathbb{C}[\Omega]^K$*. If there exists* $0 \neq u(\ell) \in M$ *such that the product* $u(\ell)q(\ell)$ *belongs to* M*, then the function* $q(\ell)$ *itself belongs to* M*.*

Proof. We proceed by induction on $\dim G + \dim(G/K)$. Let $u(\ell) = P_{W_1}(\ell)$ and $u(\ell)q(\ell) = P_{W_2}(\ell)$ with $W_1, W_2 \in \mathscr{V}$. Examine first the case where $\mathfrak{k} = \{0\}$. Put $\mathfrak{b} = \cap_{\ell\in\Omega}\mathfrak{b}[\ell]$, which is an ideal of $\mathfrak{g}$. It is seen that $\mathscr{U}(\mathfrak{b})$ is identified modulo $\ker(\pi)$ to the symmetric algebra $S(\mathfrak{b})$ of $\mathfrak{b}$ because $[\mathfrak{b},\mathfrak{b}] \subset \ker(\pi)$. Then, W_1, W_2 belong to $\mathscr{U}(\mathfrak{b}) \simeq S(\mathfrak{b})$ and W_2 is divisible by W_1, namely that there exists $W \in S(\mathfrak{b}) \simeq \mathscr{U}(\mathfrak{b})$ such that $W_2 = W_1W$. It is clear that $W \in \mathscr{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$ and $P_W(\ell) = q(\ell)$. In sum, $q(\ell) \in M$.

Suppose that $\dim\mathfrak{k} \geq 1$. Keep the notations introduced before. When Ω is non-saturated with respect to $\mathfrak{k}_{n-1}$, W_1, W_2 are taken in $\mathscr{U}_\pi(\mathfrak{k}_{n-1})^{\mathfrak{k}}$ and the result derives immediately from the induction hypothesis.

Suppose that Ω is saturated with respect to $\mathfrak{k}_{n-1}$. It follows that $W_1, W_2 \in \mathscr{U}(\mathfrak{k}_{n-1})$ and that $q(\ell)$ depends only on $\ell' = \ell|_{\mathfrak{k}_{n-1}}$. For almost all $t \in \mathbb{R}$, there exists by the induction hypothesis an element $W_t \in \mathscr{U}_{\pi_t}(\mathfrak{k}_{n-1})^{\mathfrak{k}}$ verifying $P_{W_t}(\ell') = q(\ell')$ for almost all $\ell' \in \omega_t$. Here, W_t depends rationally on $t \in \mathbb{R}$. By Proposition 4, there exists a polynomial $s(y')$ of $y' = P_\kappa(\ell)$ such that $s(y')q(\ell) \in M$.

Now take an ideal $\widetilde{\mathfrak{g}} \neq \mathfrak{k}_{n-1}$ of codimension 1 in $\mathfrak{g}$. Suppose first that Ω is non-saturated with respect to $\widetilde{\mathfrak{g}}$. Then W_1, W_2 are in $\mathscr{U}(\widetilde{\mathfrak{g}})$ modulo $\ker(\pi)$. If $\mathfrak{k} \subset \widetilde{\mathfrak{g}}$, the induction hypothesis provides us the result. If $\mathfrak{k} \not\subset \widetilde{\mathfrak{g}}$, put $\widetilde{\mathfrak{k}} = \mathfrak{k} \cap \widetilde{\mathfrak{g}}$ and $\widetilde{K} = \exp\widetilde{\mathfrak{k}}$. The induction hypothesis assures that there exists a $\widetilde{K}$-diagonal $\widetilde{W} \in \mathscr{U}_\pi(\widetilde{\mathfrak{g}})^{\widetilde{\mathfrak{k}}}$ so that we have $\widetilde{P}_{\widetilde{W}}(\ell) = q(\ell)$. Since $q(\ell)$ is $\mathfrak{k}$-invariant, $\widetilde{W}$ turns out to be $\mathfrak{k}$-invariant and hence $P_{\widetilde{W}}(\ell) = q(\ell)$. In this way, $q(\ell) \in M$.

Recall now our previous notations: $\mathfrak{g}' = \mathfrak{k}_{n-1}$, κ its corresponding e-central element and y' as in equation (8). Suppose that Ω is saturated with respect to $\widetilde{\mathfrak{g}}$. If $\mathfrak{k} \subset \widetilde{\mathfrak{g}}$, W_1, W_2 belong to $\mathscr{U}(\widetilde{\mathfrak{g}})$. As above, there exists a polynomial $\widetilde{s}(\widetilde{y})$ of $\widetilde{y} = P_{\widetilde{\kappa}}(\ell)$ such that $\widetilde{s}(\widetilde{y})q(\ell) \in M$. Let $s(y')q(\ell) = P_{W'}(\ell)$ and $\widetilde{s}(\widetilde{y})q(\ell) = P_{\widetilde{W}}(\ell)$ for some $W', \widetilde{W} \in \mathscr{V}$. Then, $\widetilde{s}(\widetilde{\kappa})W' \equiv s(\kappa)\widetilde{W}$ modulo $\ker(\pi)$. Therefore, W' must be divisible modulo $\ker(\pi)$ by $s(\kappa)$ and $W' \equiv s(\kappa)W$ modulo $\ker(\pi)$ with a certain K-diagonal $W \in \mathscr{U}_\pi(\mathfrak{g})^{\mathfrak{k}}$. Thus, $q(\ell) = P_W(\ell)$.

Finally, suppose that $\mathfrak{k}$ is not found in $\widetilde{\mathfrak{g}}$. We shall argue similarly as in the proof of Theorem 3. If $\mathfrak{b}[\ell|_{\mathfrak{k}}] \subset \widetilde{\mathfrak{k}}$ almost everywhere on Ω, W_1, W_2 belong to $\mathscr{U}(\widetilde{\mathfrak{g}})$ and hence $q(\ell)$ depends only on $\ell|_{\widetilde{\mathfrak{g}}}$. From the induction hypothesis applied to $\widetilde{\mathfrak{k}}$, there exists a $\widetilde{K}$-diagonal $\widetilde{W} \in \mathscr{U}_\pi(\widetilde{\mathfrak{g}})^{\widetilde{\mathfrak{k}}}$ such that $q(\ell) = \widetilde{P}_{\widetilde{W}}(\ell)$. Since $q(\ell)$ is $\mathfrak{k}$-invariant, $\widetilde{W}$ is $\mathfrak{k}$-invariant too and $\widetilde{P}_{\widetilde{W}}(\ell) = P_{\widetilde{W}}(\ell)$. Therefore, $q(\ell) \in M$.

We place in the last possibility where $\mathfrak{b}[\ell|_{\mathfrak{k}}] \not\subset \widetilde{\mathfrak{k}}$ almost everywhere on Ω. It is sufficient for us to treat the case where $s(y') = \alpha$ which is a polynomial in y' of degree 1.

Let j_0 be the smallest index such that $q(\ell)$ belongs to the symmetric algebra $S(\mathfrak{k}_{j_0}) = \mathbb{C}[\mathfrak{k}_{j_0}^*]$ of $\mathfrak{k}_{j_0}$ with respect to the sequence (5) of subalgebras. Aligning back to Subsection 3.3, let $\{Y_k\}_{k=1}^n$ be a Jordan-Hölder basis of $\mathfrak{g}$ adapted to the flag (1) and let

$$S = \{s_1 < \cdots < s_r\}$$

be the set of jump indices for Ω with respect to the flag (1) which appear in $\mathfrak{k}_{j_0}$. Set $x_i = \ell(Y_{s_i})$ for $1 \leq i \leq r$, where $Y_{s_r} = X_{j_0}$ changing the ordering. So, $q(\ell)$ depends on $\{x_1, \ldots, x_r\}$. Write

$$q(\ell) = \sum_{j=0}^{v} q_j(\ell) x_r^j,$$

where $q_j(\ell)(0 \leq j \leq v)$ are polynomial functions of $x_1, \ldots, x_{r-1}$.

Everything as in the proof of Lemma 3, we now prove by induction on the dimension of $\mathfrak{k}$ that there exists in M an element

$$\nu(\ell) = \beta(\ell)x_r + \gamma(\ell),$$

where $\beta(\ell), \gamma(\ell)$ are polynomials of $\{x_1, \ldots, x_{r-1}\}$ and where $\beta(\ell) \in M$ is not divisible by α. Indeed, assume first that $j_0 > d$. Making use of the e-central element σ for $\Omega|_{\mathfrak{k}}$ as in equation (11), one finds in $S((\mathfrak{k}_{j_0} \cap \tilde{\mathfrak{g}})) \cap \mathbb{C}[\Omega]^{\tilde{K}}$, an element $\tilde{q}(\ell)$ outside $S(\mathfrak{k}_{j_0 - 1})$ such that $\alpha\tilde{q}(\ell) \in \tilde{M}$, the corresponding set for $\tilde{\mathfrak{k}}$. By the induction hypothesis, there exists in $\tilde{M}$ an element

$$\tilde{\nu}(\ell) = \tilde{\beta}(\ell)x_r + \tilde{\gamma}(\ell),$$

where $\tilde{\beta}(\ell), \tilde{\gamma}(\ell)$ are polynomials of $\{x_1, \ldots, x_{r-1}\}$ and where $\tilde{\beta}(\ell) \in \tilde{M}$ is not divisible by α. Now, using the element σ as in (11), $\tilde{\nu}(\ell)$ turns out to be K-invariant and hence belongs to M as is to be shown.

When $j_0 \leq d$, we first prove Lemma 4.

Lemma 4. *We regard the symmetric algebra $S(\mathfrak{k})$ of $\mathfrak{k}$ as the algebra of polynomial functions on $\Omega|_{\mathfrak{k}}$ through the evaluation $\Omega|_{\mathfrak{k}} \ni \ell \mapsto \sqrt{-1}\ell(X)$ for $X \in \mathfrak{k}$. Let $\zeta : S(\mathfrak{k}) \to \mathscr{U}(\mathfrak{k})$ be the symmetrization map. Then, $\zeta(q)$ is K-diagonal and*

$$\zeta(q) \cdot a_\ell = q(\ell) a_\ell, \ \ell \in \Omega|_{\mathfrak{k}}.$$

Proof. We proceed by induction on $\dim \mathfrak{k}$. When $\dim \mathfrak{k} = 1$, the claim is trivial. Let $\mathfrak{z}(\mathfrak{k})$ be the center of $\mathfrak{k}$. If $\dim \mathfrak{z}(\mathfrak{k}) = 1$, $\mathfrak{z}(\mathfrak{k})$ is nothing but the center $\mathfrak{z}$ of $\mathfrak{g}$. As $\pi|_{\mathfrak{z}} \neq 0$, $q \in S(\mathfrak{k}')$ where $\mathfrak{k}'$ denotes the centralizer of $\mathfrak{k}_2$ in $\mathfrak{k}$, where $\mathfrak{k}_2$ is as in the flag (3). Since $\mathfrak{b}[\ell|_{\mathfrak{k}}] \subset \mathfrak{k}'$ for $\ell \in \Omega$, we can apply the induction hypothesis to $\mathfrak{k}'$. Suppose $\dim \mathfrak{z}(\mathfrak{k}) \geq 2$. For $\ell \in \Omega|_{\mathfrak{k}}$, we put $\mathfrak{a} = \mathfrak{z}(\mathfrak{k}) \cap \ker(\ell)$, $\bar{\mathfrak{k}} = \mathfrak{k}/\mathfrak{a}$ and $\bar{\ell} \in (\bar{\mathfrak{k}})^*$ such that $\bar{\ell} \circ p = \ell$ with the canonical projection $p : \mathfrak{k} \to \bar{\mathfrak{k}}$. Let $a_{\bar{\ell}}$ be the Penney distribution of $\bar{\mathfrak{k}}$ at $\bar{\ell}$. Then, we have $\bar{\zeta}(\bar{q}) \cdot a_{\bar{\ell}} = \bar{q}(\bar{\ell}) a_{\bar{\ell}}$ from the induction hypothesis applied to $\bar{\mathfrak{k}}$. Here, $\bar{\zeta} : S(\bar{\mathfrak{k}}) \to \mathscr{U}(\bar{\mathfrak{k}})$ denotes the symmetrization map and $\bar{q} \in S(\bar{\mathfrak{k}})$ is such that $\bar{q} \circ p = q$. Thus, we get the claim. □

Now if $j_0 > d$, we use assertion 2 of Proposition 3 to argue similarly as in the previous case.

We now utilize a new induction on the degree v of q relatively to x_r. If so,

$$\beta(\ell)^v q(\ell) - q_v(\ell)\nu(\ell)^v$$

is of degree smaller than v relatively to x_r and hence belongs to M. Thus, $\beta(\ell)^v q(\ell) \in M$. Let

$$\beta(\ell)^v = P_{W_3}(\ell),\ \beta(\ell)^v q(\ell) = P_{W_4}(\ell)$$

with $W_3, W_4 \in \mathscr{V}$. Then,

$$\alpha\beta(\ell)^v q(\ell) = P_{W'}(\ell)P_{W_3}(\ell) = P_{\kappa'}(\ell)P_{W_4}(\ell),$$

where κ' is the polynomial in κ of degree 1 such that $P_{\kappa'}(\ell) = \alpha$. In other words,

$$W'W_3 \equiv \kappa' W_4$$

modulo $\ker(\pi)$. Because W_3 is not divisible by κ', W' must be divisible by κ'. Consequently, $q(\ell)$ belongs to M. □

Remark 2. It is worthnoting here that by a result of M. Duflo (cf. [14]), for any $\sigma \in \hat{K}$ and any $q \in S(\mathfrak{k})^K$, $\sigma(\zeta(q)) = q(\ell)id$ for any ℓ in the orbit associated to σ in $\mathfrak{k}^*$. It remains unclear to us whether this results provides directly a proof of Lemma 4.

Corollary 4. *Keep the same notation and assume that $\pi|_K$ has finite multiplicities, then the mapping Θ defined by equation* (7) *is surjective.*

Corollaries 2 and 4 allow to complete the proof of Conjecture 1.1. We have the following:

Theorem 4. *Let $G = \exp\mathfrak{g}$ be a connected and simply connected nilpotent Lie group. Then Conjecture* 1.1 *holds. That is, when $\pi|_K$ has finite multiplicities, the mapping Θ gives by passing to the quotient an isomorphism of algebras from $D_\pi(G)^K$ to the algebra $\mathbb{C}[\Omega(\pi)]^K$ of the K-invariant polynomial functions on the orbit $\Omega(\pi)$.*

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