REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 26, Pages 1063–1079 (October 7, 2022)
https://doi.org/10.1090/ert/626

RELATIONS BETWEEN CUSP FORMS SHARING HECKE EIGENVALUES

DIPENDRA PRASAD AND RAVI RAGHUNATHAN

ABSTRACT. In this paper we consider the question of when the set of Hecke eigenvalues of a cusp form on $\mathrm{GL}_n(\mathbb{A}_F)$ is contained in the set of Hecke eigenvalues of a cusp form on $\mathrm{GL}_m(\mathbb{A}_F)$ for $n \leq m$. This question is closely related to a question about finite dimensional representations of an abstract group, which also we consider in this work.

CONTENTS

1. Introduction 1063
2. Preliminaries 1066
3. The results 1069
4. Isobaric types of $\mathrm{Sym}^6(\pi)$ 1072
5. Group theoretic analogues 1074
6. Virtually non-abelian groups 1076
Acknowledgments 1078
References 1078

1. INTRODUCTION

Let F be a number field. Each automorphic representation π of $\mathrm{GL}_d(\mathbb{A}_F)$ gives rise to Hecke eigenvalues (also called the Satake parameter), a d-tuple of (unordered) nonzero complex numbers $H(\pi_v) = (a_{1v}, \cdots, a_{dv})$, at each place v of F where π is unramified, and thus at almost all finite places of F.

Let π_1 and π_2 be two irreducible automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ which are written as isobaric sums:

$$\begin{aligned}\pi_1 &= \pi_{11} \boxplus \pi_{12} \boxplus \cdots \boxplus \pi_{1\ell},\\ \pi_2 &= \pi_{21} \boxplus \pi_{22} \boxplus \cdots \boxplus \pi_{2\ell'},\end{aligned}$$

where π_{1j} and π_{2k} are irreducible cuspidal automorphic representations of $\mathrm{GL}_{d_j}(\mathbb{A}_F)$ and $\mathrm{GL}_{d_k}(\mathbb{A}_F)$ respectively. Then by the strong multiplicity one theorem due to Jacquet and Shalika cf. [JS1], [JS2], if π_1 and π_2 have the same Hecke eigenvalues

Received by the editors December 7, 2021, and, in revised form, July 16, 2022.
2020 *Mathematics Subject Classification.* Primary 11F70; Secondary 22E55.
The first author was supported by the Science and Engineering Research Board of the Department of Science and Technology, India through the JC Bose National Fellowship of the Govt. of India, project number JBR/2020/000006. His work was also supported by a grant of the Government of the Russian Federation for the state support of scientific research carried out under the agreement 14.W03.31.0030 dated 15.02.2018.

$H(\pi_{1,v}) = H(\pi_{2,v})$ at almost all finite places v of F where π_1, π_2 are unramified, then $\ell = \ell'$, and up to a permutation of indices, $\pi_{1j} = \pi_{2j}$.

In this paper, we will consider a variant of the strong multiplicity one theorem, identified in the following definition.

Definition. Given automorphic representations π_1 on $\mathrm{GL}(n, \mathbb{A}_F)$ and π_2 on $\mathrm{GL}(m, \mathbb{A}_F)$, we say that π_1 is **immersed** in π_2, written $\pi_1 \preceq \pi_2$, if the Hecke eigenvalues of π_1 (counted with multiplicity) are contained in the Hecke eigenvalues of π_2 (counted with multiplicity) for almost all primes of the number field F. On the other hand, we say that π_1 is **embedded** in π_2, written as $\pi_1 \subset \pi_2$ if there is an automorphic representation π_3 such that,

$$\pi_2 = \pi_1 \boxplus \pi_3.$$

The following is the motivating question for this paper.

Question.

(1) Can it happen for distinct cuspidal representations that $\pi_1 \preceq \pi_2$?
(2) If yes, can we classify all such pairs of cuspidal representations $\pi_1 \preceq \pi_2$?

One would have liked to assert that for cuspidal representations $\pi_1 \preceq \pi_2$ never happens if $n < m$, but that is not true. For example, let π be a cuspidal non-CM automorphic representation of $\mathrm{PGL}_2(\mathbb{A}_F)$. At any unramified place v of F, if (a_v, a_v^{-1}) are the Hecke eigenvalues of π_v, then for the automorphic representation $\mathrm{Sym}^2(\pi)$ of $\mathrm{PGL}_3(\mathbb{A}_F)$, the Hecke eigenvalues at the place v of F, are $(a_v^2, 1, a_v^{-2})$. Thus the Hecke eigenvalues of the trivial representation of $\mathrm{GL}_1(\mathbb{A}_F)$ are contained in the set of Hecke eigenvalues of the cuspidal automorphic representation $\mathrm{Sym}^2(\pi)$ of $\mathrm{PGL}_3(\mathbb{A}_F)$ at each unramified place of π.

This paper is written in the hope that although an automorphic representation π_2 may be immersed in a cuspidal automorphic representation π_2, without π_1 being the same as π_2, this happens rarely, and only for pairs of representations (π_1, π_2) which are related in some well-defined way.

We begin by proving Proposition 1.

Proposition 1. *Let π_1 (resp. π_2) be an irreducible cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ (resp. $\mathrm{GL}_{n+1}(\mathbb{A}_F)$). Then π_1 cannot be immersed in π_2.*

Proof. The proof is a simple consequence of the strong multiplicity one theorem of Jacquet-Shalika recalled at the beginning of this paper. Let ω_1 (resp. ω_2) be the central character of π_1 (resp. π_2); these are Grössencharacters of $\mathrm{GL}_1(\mathbb{A}_F)$. It is easy to see that, if $H(\pi_{1,v})$ is contained in $H(\pi_{2,v})$ at almost all places v of F, then,

$$\pi_1 \boxplus (\omega_2/\omega_1) = \pi_2,$$

which is not allowed by the strong multiplicity one theorem since π_2 is cuspidal. $\square$

Here is another similarly 'negative' result, this time proved with considerably more effort.

Proposition 2. *Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_4(\mathbb{A}_F)$. Then $H(\pi_v)$ cannot contain 1 at almost all finite places v of F where π is unramified.*

Proof. We will prove the proposition by contradiction, so assume that $H(\pi_v)$ contains 1 at almost all places of F where π is unramified. Observe that to say that

$H(\pi_v)$ contains 1 is equivalent to saying that $\det(1 - H(\pi_v)) = 0$, which translates into the following identity (assuming that $H(\pi_v)$ operates on a 4-dimensional vector space V):

$$1 - V + \Lambda^2(V) - \Lambda^3(V) + \Lambda^4(V) = 0.$$

(One way to think of this identity is in the Grothendieck group of representations of an abstract group G which comes equipped with a 4-dimensional representation V of G such that the action of any $g \in G$ on V has a nonzero fixed vector.)

Thus we get the identity:

$$1 + \Lambda^4(V) + \Lambda^2(V) = V + \Lambda^3(V).$$

Let the central character of π be $\omega : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$. Since we know by the work of Kim, cf. [Kim], that $\Lambda^2(\pi)$ is automorphic, by the strong multiplicity one theorem, we get an identity of the isobaric sum of automorphic representations:

$$1 \boxplus \omega \boxplus \Lambda^2(\pi) = \pi \boxplus \omega \cdot \pi^\vee.$$

Observe that the right hand side of this equality is a sum of two cuspidal representations on $\mathrm{GL}_4(\mathbb{A}_F)$, whereas there are two one dimensional characters of $\mathbb{A}_F^\times/F^\times$ on the left hand side. This is not allowed by the strong multiplicity one theorem, completing the proof of the proposition. □

Question 1 lies at the basis of this work.

Question 1. Let π_1 (resp. π_2) be an irreducible cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ (resp. $\mathrm{GL}_{n+2}(\mathbb{A}_F)$). Suppose that π_1 is immersed in π_2. Then, is there an automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ with central character $\omega : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$, and a character $\chi : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$, such that,

$$\pi_1 = \chi \cdot \omega \otimes \mathrm{Sym}^{n-1}(\pi),$$
$$\pi_2 = \chi \otimes \mathrm{Sym}^{n+1}(\pi)?$$

We will provide an affirmative answer to this question for $n = 1, 2, 3$ in this paper. On the other hand, in section 5, we will provide counter-examples to this question using the strong Artin conjecture for all pairs of integers $(q-1, q+1)$ where $q \geq 5$ is a prime power. The work [Ca] proves strong Artin conjecture for certain cases for $q = 5$, which allows us to construct an unconditional counter-example for the pair $(\mathrm{GL}_4, \mathrm{GL}_6)$ over $\mathbb{Q}$ in section 5.

The question studied in this paper can be studied from a purely group theoretic point of view, and is discussed in section 5 from this perspective. We are unaware of this group theoretic point of view to have been put to use earlier; it seems of interest for connected reductive groups too. In section 6, we prove that the group theoretic question has an affirmative answer for groups which are not virtually abelian, i.e., do not contain a subgroup of finite index which is abelian. Thus in the automorphic context, when one of the representations π_1, or π_2 is Steinberg at a finite place, or has regular infinitesimal character at one of the archimedean places of F, our question should have an affirmative answer, but for the moment, we do not know how to deal with it.

Remark 1. Here is a geometric analogue of the questions being discussed in this paper.[1] Let A and B be abelian varieties over a number field F with A simple. For

[1]This question has now been settled in a recent work of Khare and Larsen, cf. [KL].

v any finite place of F where both A and B have good reduction, let A_v, B_v denote their reductions mod v (thus A_v, B_v are abelian varieties over finite fields). Assume that there are isogenies from A_v to B_v (not surjective as we are not assuming $\dim(A) = \dim(B)$) for almost all places v of F where A and B have good reduction. Then the question is if there is an isogeny from A to B? If $\dim(A) = \dim(B)$, this is a consequence of the famous theorem of Faltings.

Remark 2. The paper was inspired by the notion of *relevance* introduced in [GGP], and to understand whether two global A-parameters which are locally relevant at all places must be globally relevant. This is not the case, and exactly for the reason discussed in this paper: that the Hecke eigenvalues of the cuspidal representation π_1 may be contained in the set of Hecke eigenvalues of the cuspidal representation π_2 at almost all places of the number field without π_1 being the same as π_2.

Remark 3. Most of the paper deals with cusp forms π_1 on $\mathrm{GL}_n(\mathbb{A}_F)$, and π_2 on $\mathrm{GL}_m(\mathbb{A}_F)$ for the restricted pairs (n, m) with $m = n + 2$, as the first non-obvious case beyond $m = n$ and $m = n + 1$, π_1 is immersed in π_2. However, one might begin at the other extreme $(n, m) = (1, m)$, and try to classify cusp forms π_2 on $\mathrm{GL}_m(\mathbb{A}_F)$ such that the Hecke eigenvalues of π_2 at almost all places of F contain the eigenvalue 1. By Proposition 3, there is a nice answer for $(n, m) = (1, 3)$, whereas by Proposition 2, there are none in the case $(n, m) = (1, 4)$. It is easy to see that the cuspidal representations π_2 of $\mathrm{GL}_6(\mathbb{A}_F)$ which arise as the basechange of a cuspidal selfdual representation of $\mathrm{PGL}_3(\mathbb{A}_E)$ for E/F quadratic, have Hecke eigenvalue 1 at almost all places of F. Using Theorem 2 of the paper [Yam] and using a similar identity as in the proof of Proposition 2 which this time would be:

$$1 + \Lambda^2 + \Lambda^4(V) + \Lambda^6(V) = V + \Lambda^3(V) + \Lambda^5(V),$$

a cuspidal representation π_2 of $\mathrm{GL}_6(\mathbb{A}_F)$ having Hecke eigenvalue 1 at almost all places of F arises as the automorphic induction of a cuspidal representation of $\mathrm{GL}_3(\mathbb{A}_E)$ for E/F quadratic (which is most likely selfdual, and on $\mathrm{PGL}_3(\mathbb{A}_F)$, but this we have not proved). We have not investigated the situation for general pairs $(1, m)$.

We end the introduction by remarking that the last two sections of the paper, sections 5 and 6, are written for finite groups and Lie groups respectively, and are quite independent of the earlier sections. Section 5 eventually has implications for automorphic representations through the known cases of (strong) Artin's conjecture. Since automorphic representations on $\mathrm{GL}_n(\mathbb{A}_F)$, F a function field, are characterised by their Galois representations by the work of L. Lafforgue, these sections also construct both counter-examples to Question 1, and to assert that the only counter-examples come from potentially abelian automorphic representations.

2. Preliminaries

For automorphic representations π_1, π_2 on $\mathrm{GL}_a(\mathbb{A}_F), \mathrm{GL}_b(\mathbb{A}_F)$, we denote by $\pi_1 \boxplus \pi_2$ the isobaric sum of π_1, π_2, which is an automorphic form on $\mathrm{GL}_{a+b}(\mathbb{A}_F)$. If $H(\pi_{1,v})$ and $H(\pi_{2,v})$ are the Hecke eigenvalues of π_1 and π_2, then the Hecke eigenvalues of $\pi_1 \boxplus \pi_2$ are the union (with multiplicities) of $H(\pi_{1,v}), H(\pi_{2,v})$.

We will also use the notation $A \boxplus B$ where A (resp. B) is any collection of a (resp. b) nonzero complex numbers defined for almost all finite places v of F. In this generality, we will use partial L function $L(s, A)$, where the Euler product is taken outside a finite set S of places, S containing the places at infinity.

In the same spirit, for A, B as in the last para, we will define $A \boxtimes B$ to be a collection of $a \cdot b$ nonzero complex numbers for almost all finite places v of F, and the associated partial Rankin product L function $L(s, A \boxtimes B)$, again where the Euler product is taken outside a finite set S of places, S containing the places at infinity.

Lemma 1. *Suppose C, D are automorphic representations on $\mathrm{GL}_c(\mathbb{A}_F), \mathrm{GL}_d(\mathbb{A}_F)$, and χ a Grössencharacter on F. Suppose A is any collection of $c + d - 1$ nonzero complex numbers defined for almost all finite places v of F such that*

$$A \boxplus \chi = C \boxplus D.$$

Suppose that $L(\chi^{-1}A, s)$ is known to have meromorphic continuation to the entire complex plane with no zero at $s = 1$. Then there is an automorphic representation π_3 of $\mathrm{GL}_{c+d-1}(\mathbb{A}_F)$ whose Hecke eigenvalues are equal to A at almost all finite places v of F.

Proof. Expand $\chi^{-1}(C \boxplus D)$ as an isobaric sum of cusp forms, and note that for any cusp form π on $\mathrm{GL}_m(\mathbb{A}_F)$, $L(\pi, s)$, the partial L-function without regard to omitted set of places has a pole at $s = 1$ if and only if $m = 1$, and $\pi = 1$. Therefore by what is given for $L(\chi^{-1}A, s)$, the isobaric sum decomposition of $\chi^{-1}(C \boxplus D)$ in terms of cusp forms must contain the trivial representation of $\mathrm{GL}_1(\mathbb{A}_F)$, omitting which from $\chi^{-1}(C \boxplus D)$ defines $\chi^{-1}A$ as an automorphic representation. $\square$

Lemma 2. *Suppose π_1 is a cuspidal automorphic representation on $\mathrm{GL}_n(\mathbb{A}_F)$ and π_2 is a cuspidal automorphic representation on $\mathrm{GL}_{n+2}(\mathbb{A}_F)$ such that at almost all unramified places of π_1 and π_2, $H(\pi_{1,v}) \subseteq H(\pi_{2,v})$. Let ω_1 (resp. ω_2) be the central character of π_1 (resp. π_2) which is a Grössencharacter on $\mathrm{GL}_1(\mathbb{A}_F)$. Suppose that $\Lambda^2(\pi_2), \mathrm{Sym}^2(\pi_1)$ are known to be automorphic. Then,*

1. *The Rankin product $\pi_1 \boxtimes \pi_2$ is automorphic.*
2. *We have the isobaric decomposition of automorphic representations:*

$$\pi_2 \boxtimes \pi_1 \boxplus \omega_2/\omega_1 = \Lambda^2(\pi_2) \boxplus \mathrm{Sym}^2(\pi_1).$$

Proof. We first prove the identity expressed in (2), i.e., that the two sides have the same Hecke eigenvalues at almost all the primes of F. This task is made more transparent by looking at vector spaces V, W, A with $V = W + A$ with A two dimensional, and noting the identity:

$$\begin{aligned} V \otimes W + \Lambda^2(A) &= W \otimes W + A \otimes W + \Lambda^2(A) \\ &= \Lambda^2(W) + A \otimes W + \Lambda^2(A) + \mathrm{Sym}^2(W) \\ &= \Lambda^2(V) + \mathrm{Sym}^2(W). \end{aligned}$$

Now, Lemma 1 proves the automorphy of $\pi_2 \boxtimes \pi_1$ since its L-function is known to be entire and non-vanishing on the line $\mathrm{Re}(s) = 1$ by the Rankin-Selberg theory, see Theorem 5.2 of Shahidi [Sha]. $\square$

Lemma 3. *Suppose π_1, π_2, π_3 are cuspidal automorphic representations on $\mathrm{GL}_{n_i}(\mathbb{A}_F)$ (for $i = 1, 2, 3$). Suppose that the Rankin products $\pi_1 \boxtimes \pi_2$ and $\pi_2 \boxtimes \pi_3$ are known to be automorphic. Then in the isobaric sum decomposition of $\pi_1 \boxtimes \pi_2$, $\pi_3^\vee \subset \pi_1 \boxtimes \pi_2$ if and only if in the isobaric sum decomposition of $\pi_2 \boxtimes \pi_3$, $\pi_1^\vee \subset \pi_2 \boxtimes \pi_3$.*

Proof. Since the Rankin product $\pi_1 \boxtimes \pi_2$ is given to be automorphic, by Jacquet-Shalika, the L-function,

$$L(s, \pi_1 \boxtimes \pi_2 \boxtimes \pi_3),$$

has a pole at $s = 1$ if and only if $\pi_3^\vee \subset \pi_1 \boxtimes \pi_2$. Same triple-product L-function dictates $\pi_1^\vee \subset \pi_2 \boxtimes \pi_3$. □

Besides the strong multiplicity one theorem of Jacquet-Shalika, we will use the symmetric square lift of Gelbart-Jacquet cf. [GJ] which we state in the form we will use. The work of Gelbart-Jacquet was to establish the symmetric square lift from GL_2 to GL_3; for characterising the image of the symmetric square lift, we refer to [Ra1].

Theorem 1. *Let π_2 be a cuspidal automorphic representation of $\mathrm{PGL}_3(\mathbb{A}_F)$ with $\pi_2 \cong \pi_2^\vee$. Then π_2 arises as the adjoint lift of an automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$, i.e.,*

$$\pi_2 \cong \mathrm{Ad}(\pi) = \omega^{-1} \otimes \mathrm{Sym}^2(\pi),$$

where ω is the central character of π, a Grössencharacter on $\mathrm{GL}_1(\mathbb{A}_F)$.

Corollary 1. *Let π_2 be a cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A}_F)$ with $\pi_2 \cong \chi \otimes \pi_2^\vee$ for χ a Grössencharacter on $\mathrm{GL}_1(\mathbb{A}_F)$. Then π_2 can be written as*

$$\pi_2 = \lambda \otimes \mathrm{Sym}^2(\pi),$$

where π is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$, and λ a Grössencharacter on $\mathrm{GL}_1(\mathbb{A}_F)$.

Proof. Let ω_2 be the central character of π_2. Comparing the central characters for the given isomorphism:

$$\pi_2 \cong \chi \otimes \pi_2^\vee, \tag{1}$$

it follows that,

$$\omega_2^2 = \chi^3.$$

Therefore for $\mu = \chi/\omega_2$, the isomorphism in (1) can be rewritten as:

$$\pi_2 \cong \mu^{-2} \otimes \pi_2^\vee, \tag{2}$$

or,

$$(\mu \otimes \pi_2) \cong (\mu \otimes \pi_2)^\vee. \tag{3}$$

Therefore, the representation $\mu \otimes \pi_2$ of $\mathrm{GL}_3(\mathbb{A}_F)$ is selfdual. Comparing the central characters on the two sides of equation (2), we find that the central character ω of the representation $\mu \otimes \pi_2$ of $\mathrm{GL}_3(\mathbb{A}_F)$ is quadratic. Twisting the representation $\mu \otimes \pi_2$ of $\mathrm{GL}_3(\mathbb{A}_F)$ by ω, we find that the representation $(\omega\mu) \otimes \pi_2$ of $\mathrm{GL}_3(\mathbb{A}_F)$ is both selfdual and of trivial central character, so Theorem 1 applies, proving that π_2 is a symmetric square up to a twist. □

3. The results

We introduce the following notation keeping Question 1 in mind. Suppose π_1 is a cuspidal automorphic representation on $\mathrm{GL}_n(\mathbb{A}_F)$ and π_2 is a cuspidal automorphic representation on $\mathrm{GL}_m(\mathbb{A}_F)$ such that at almost all unramified places of π_1 and π_2, $H(\pi_{1,v}) \subseteq H(\pi_{2,v})$, we write $\pi_1 \preceq \pi_2$.

We observe that we may twist the pair (π_1, π_2) appearing in Question 1 by a Grössencharacter. Accordingly, in Proposition 3 that provides an affirmative answer to Question 1 for $n = 1$ we may assume that $\pi_1 = 1$.

Proposition 3. *Let* 1 *denote the trivial representation of* $\mathrm{GL}_1(\mathbb{A}_F)$ *and suppose that* π_2 *is a cuspidal automorphic representation on* $\mathrm{GL}_3(\mathbb{A}_F)$ *such that* $1 \preceq \pi_2$. *Then* π_2 *is a self-dual representation of* $\mathrm{PGL}_3(\mathbb{A}_F)$, *and arises as* $\omega^{-1}\cdot\mathrm{Sym}^2(\pi)$ *(the adjoint lift) of a cuspidal automorphic form* π *on* $\mathrm{GL}_2(\mathbb{A}_F)$ *with central character* $\omega : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$.

Proof. The proof will be a simple consequence of the strong multiplicity one theorem of Jacquet-Shalika recalled in the beginning of this paper and Theorem 1 due to Gelbart-Jacquet. Let ω_2 be the central character of π_2 which is a Grössencharacter on $\mathrm{GL}_1(\mathbb{A}_F)$. By Lemma 2 in this case for $(\pi_1, \pi_2) = (1, \pi_2)$, it follows that

$$\Lambda^2(\pi_2) \boxplus 1 = \pi_2 \boxplus \omega_2.$$

Therefore, by the strong multiplicity one theorem, we deduce that:

(1) $\omega_2 = 1$,
(2) $\Lambda^2(\pi_2) = \pi_2$.

By (1) and (2), we find that

$$\pi_2 \cong \pi_2^\vee.$$

Therefore, by Theorem 1 due to Gelbart-Jacquet, π_2 arises as the adjoint lift from a cuspidal automorphic form π on $\mathrm{GL}_2(\mathbb{A}_F)$, i.e.,

$$\pi_2 = \omega^{-1} \cdot \mathrm{Sym}^2(\pi),$$

proving the proposition. ∎

Proposition 4 provides an affirmative answer to Question 1 for $n = 2$.

Proposition 4. *Suppose that* π_1 *is a cuspidal automorphic representation of* $\mathrm{GL}_2(\mathbb{A}_F)$ *with central character* $\omega_1 : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$, *and that* π_2 *is a cuspidal automorphic representation of* $\mathrm{GL}_4(\mathbb{A}_F)$, *and that* $\pi_1 \preceq \pi_2$. *Then,*

(1) π_1 *cannot be CM (a CM representation is one defined using a Grössencharacter of a quadratic extension* E *of* F*).*
(2) $\pi_2 = \omega_1^{-1} \otimes \mathrm{Sym}^3(\pi_1)$.

Proof. Let ω_2 be the central character of π_2 which is a Grössencharacter of $\mathrm{GL}_1(\mathbb{A}_F)$. By Lemma 2,

$$\pi_2 \boxtimes \pi_1 \boxplus \omega_2/\omega_1 = \Lambda^2(\pi_2) \boxplus \mathrm{Sym}^2(\pi_1), \tag{1}$$

where all the terms appearing above are automorphic: $\Lambda^2(\pi_2)$ by Kim [Kim], $\mathrm{Sym}^2(\pi_1)$ by Gelbart-Jacquet [GJ], and $\pi_2 \boxtimes \pi_1$ by Lemma 2.

We first assume that π_1 is CM. Observe that since π_1 is a cusp form on $\mathrm{GL}_2(\mathbb{A}_F)$ and π_2 a cusp form on $\mathrm{GL}_4(\mathbb{A}_F)$, $\pi_2 \boxtimes \pi_1$ cannot contain any Grössencharacter. Therefore, by the isobaric decomposition (1), exactly one of the two terms $\Lambda^2(\pi_2)$ or $\mathrm{Sym}^2(\pi_1)$ may contain a Grössencharacter. Since we have assumed that π_1 is

CM, $\mathrm{Sym}^2(\pi_1)$ contains a Grössencharacter, and therefore, $\Lambda^2(\pi_2)$ cannot contain a Grössencharacter if π_1 is CM.

Since π_1 is CM, we can write,

$$\mathrm{Sym}^2(\pi_1) = \pi_3 \boxplus \chi_3,$$

where χ_3 is a Grössencharacter, and π_3 must be cuspidal (because the left hand side of (1) has only one Grössencharacter in its isobaric decomposition). By (1), $\chi_3 = \omega_2/\omega_1$, and we can simplify (1) to

$$\pi_2 \boxtimes \pi_1 = \Lambda^2(\pi_2) \boxplus \pi_3.$$

From Lemma 3, it follows that,

$$\pi_2 = \pi_1 \boxtimes \pi_3^\vee.$$

Therefore,

$$\Lambda^2(\pi_2) = [\mathrm{Sym}^2(\pi_1) \boxtimes \Lambda^2(\pi_3^\vee)] \boxplus [\Lambda^2(\pi_1) \boxtimes \mathrm{Sym}^2(\pi_3^\vee)].$$

Since $\mathrm{Sym}^2(\pi_1) = \pi_3 \boxplus \chi_3$, therefore as $\Lambda^2(\pi_3^\vee)$ is a Grössencharacter, we find that $\Lambda^2(\pi_2)$ contains a Grössencharacter which is not allowed, proving that π_1 cannot be a CM form.

Now we turn to the case when π_1 is not CM in which case it is known by Gelbart-Jacquet that $\mathrm{Sym}^2(\pi_1)$ is a cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A}_F)$. Therefore by the strong multiplicity one theorem applied to (1), we make the following conclusions:

(1) the character $\omega_2/\omega_1 : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ must belong to the isobaric sum decomposition of $\Lambda^2(\pi_2)$, in particular, $\pi_2 \cong \pi_2^\vee \otimes (\omega_2/\omega_1)$, i.e., π_2 has parameter in the symplectic similitude group, and considering the similitude factor, we find:

$$(\omega_2/\omega_1)^2 = \omega_2, \quad \text{i.e., } \omega_2 = \omega_1^2. \tag{2}$$

(2) $\mathrm{Sym}^2(\pi_1)$ must be contained in the isobaric sum decomposition of $\pi_2 \boxtimes \pi_1$.

Since by [KS1], [KS2], $\pi_2 \boxtimes \pi_1$ and $\mathrm{Sym}^2(\pi_1) \boxtimes \pi_1$ are known to be automorphic, we can apply Lemma 3, and conclude that:

$$\pi_2^\vee = \pi_2 \otimes (\omega_1/\omega_2) \subset \pi_1 \boxtimes \mathrm{Sym}^2(\pi_1^\vee) = \pi_1 \boxtimes \omega_1^{-2} \boxtimes \mathrm{Sym}^2(\pi_1). \tag{3}$$

It is easy to see that,

$$\pi_1 \boxtimes \mathrm{Sym}^2(\pi_1) = (\omega_1 \otimes \pi_1) \boxplus \mathrm{Sym}^3(\pi_1),$$

therefore we can write (3) as:

$$\pi_2 \otimes (\omega_1/\omega_2) \subset (\omega_1^{-1} \otimes \pi_1) \boxplus \omega_1^{-2} \boxtimes \mathrm{Sym}^3(\pi_1). \tag{4}$$

Since π_2 is a cuspidal automorphic representation of $\mathrm{GL}_4(\mathbb{A}_F)$, applying the strong multiplicity one theorem to (4), the only option we have (after using (2)) is that:

$$\pi_2 = \omega_1^{-1} \otimes \mathrm{Sym}^3(\pi_1),$$

proving the proposition. $\square$

Proposition 5 provides an affirmative answer to Question 1 for $n = 3$. The proof of this proposition will use the unproved cases of functorialty for $\Lambda^2(\pi_2)$ where π_2 is a cusp form on $\mathrm{GL}_5(\mathbb{A}_F)$, as well as $\mathrm{Sym}^6(\pi)$ for π a cusp form on $\mathrm{GL}_2(\mathbb{A}_F)$. It may be mentioned that although automorphy of $\mathrm{Sym}^2(\pi_1)$ for π_1 a cusp form on $\mathrm{GL}_3(\mathbb{A}_F)$ is not known, in our context below, it will be applied to π_1 which is selfdual up to a twist, and hence is a symmetric square of a cusp form on $\mathrm{GL}_2(\mathbb{A}_F)$ up to a twist by Corollary 1 which allows one to conclude automorphy of $\mathrm{Sym}^2(\pi_1)$ using known cases of functoriality for $\mathrm{Sym}^4(\pi)$.

Proposition 5. *Suppose that π_1 is a cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A}_F)$ and that π_2 is a cuspidal automorphic representation of $\mathrm{GL}_5(\mathbb{A}_F)$ such that $\pi_1 \preceq \pi_2$. Then, there exists a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ of central character ω such that up to simultaneous twisting of the pair (π_1, π_2) by a Grössencharacter, we have:*

$$\pi_1 = \mathrm{Sym}^2(\pi),$$
$$\pi_2 = \omega^{-1} \otimes \mathrm{Sym}^4(\pi).$$

Proof. Let ω_1 (resp. ω_2) be the central character of π_1 (resp. π_2) which is a Grössencharacter on $\mathrm{GL}_1(\mathbb{A}_F)$. By Lemma 2, for any Grössencharacter χ on F:

$$L(s, \pi_2 \times \pi_1 \times \chi)L(s, \omega_2/\omega_1 \times \chi) = L(s, [\Lambda^2(\pi_2) \oplus \mathrm{Sym}^2(\pi_1)] \times \chi), \tag{1}$$

Since π_2 is a cuspidal automorphic representation of $\mathrm{GL}_5(\mathbb{A}_F)$, it is known by Jacquet-Shalika, cf. [JS], that $L(s, \Lambda^2(\pi_2))$ cannot have a pole at $s = 1$. Therefore for $\chi = \omega_1/\omega_2$, since the left hand side of the product of L-functions in (1) has a simple pole at $s = 1$, right hand side of (1) too must have a simple pole, contributed therefore by $L(s, \mathrm{Sym}^2(\pi_1) \otimes \omega_1/\omega_2)$. In particular, $\pi_1^\vee \cong \pi_1 \boxtimes \omega_1/\omega_2$. By Corollary 1, such representations of $\mathrm{GL}_3(\mathbb{A}_F)$ arise as a twist of a symmetric square:

$$\pi_1 \cong \lambda \otimes \mathrm{Sym}^2(\pi),$$

for a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ of central character ω, and λ a Grössencharacter on $\mathbb{A}_F^\times$. Twisting the pair (π_1, π_2) by λ^{-1}, we assume that $\pi_1 \cong \mathrm{Sym}^2(\pi)$.

Since $\pi_1 = \mathrm{Sym}^2(\pi)$, and it is easy to see that,

$$\mathrm{Sym}^2(\pi_1) = \mathrm{Sym}^2(\mathrm{Sym}^2(\pi)) = \omega^2 + \mathrm{Sym}^4(\pi),$$

and therefore by Kim, cf. [Kim], since $\mathrm{Sym}^4(\pi)$ is known to be automorphic, so is $\mathrm{Sym}^2(\pi_1)$. Since we are assuming that $\Lambda^2(\pi_2)$ is known to be automorphic, Lemma 2 applies, allowing us to conclude that $\pi_2 \boxtimes \pi_1$ is automorphic and we have the isobaric decomposition:

$$\pi_2 \boxtimes \pi_1 \boxplus \omega_2/\omega_1 = \Lambda^2(\pi_2) \boxplus \mathrm{Sym}^2(\pi_1). \tag{2}$$

Therefore, by the strong multiplicity one theorem applied to (2), we conclude $\mathrm{Sym}^2(\pi_1)$ must be contained in the isobaric sum decomposition of $\pi_2 \boxtimes \pi_1$ as a direct summand.

Applying Lemma 3 to (2), we conclude that:

$$\pi_2 \subset \pi_1^\vee \boxtimes \mathrm{Sym}^2(\pi_1), \tag{3}$$

again as a direct summand, since as we will see now, $\pi_1^\vee \boxtimes \mathrm{Sym}^2(\pi_1)$ is automorphic by our assumption that $\mathrm{Sym}^6(\pi)$ is automorphic.

Since $\pi_1 = \mathrm{Sym}^2(\pi)$, and $\mathrm{Sym}^2(\mathrm{Sym}^2(\pi)) = \omega^2 + \mathrm{Sym}^4(\pi)$, we find that:

$$\begin{aligned}\pi_1 \boxtimes \mathrm{Sym}^2(\pi_1) &= \mathrm{Sym}^2(\pi) \boxtimes (\omega^2 \boxplus \mathrm{Sym}^4(\pi)),\\ &= \omega^2\,\mathrm{Sym}^2(\pi) \boxplus \mathrm{Sym}^2(\pi) \boxtimes \mathrm{Sym}^4(\pi),\\ &= \omega^2\,\mathrm{Sym}^2(\pi) \boxplus \omega^2\,\mathrm{Sym}^2(\pi) \boxplus \omega\,\mathrm{Sym}^4(\pi) \boxplus \mathrm{Sym}^6(\pi),\end{aligned}$$

if particular, if $\mathrm{Sym}^6(\pi)$ is automorphic, so is $\pi_1^\vee \boxtimes \mathrm{Sym}^2(\pi_1)$.

Since $\pi_2 \subset \pi_1^\vee \boxtimes \mathrm{Sym}^2(\pi_1) = \omega^{-2}\pi_1 \boxtimes \mathrm{Sym}^2(\pi_1)$, we find that:

$$\pi_2 \subset \mathrm{Sym}^2(\pi) \boxplus \mathrm{Sym}^2(\pi) \boxplus \omega^{-1}\,\mathrm{Sym}^4(\pi) \boxplus \omega^{-2}\,\mathrm{Sym}^6(\pi). \tag{4}$$

Now π_2 is a cuspidal representation on $\mathrm{GL}_5(\mathbb{A}_F)$, and by Proposition 6 proved in the next section, isobaric decomposition of $\mathrm{Sym}^6(\pi)$ cannot have a cuspidal representation of $\mathrm{GL}_5(\mathbb{A}_F)$. Therefore applying the strong multiplicity one theorem to (4), we find that the only option we have is that $\mathrm{Sym}^4(\pi)$ is cuspidal, and

$$\pi_2 = \omega^{-1}\,\mathrm{Sym}^4(\pi),$$

proving the proposition. □

Remark 4. The identity proved in Lemma 2:

$$\pi_2 \otimes \pi_1 + \omega_2/\omega_1 = \Lambda^2(\pi_2) + \mathrm{Sym}^2(\pi_1), \tag{1}$$

holds, as the proof shows, among any two representations (π_1, V_1) and (π_2, V_2) of an abstract group G when $\dim(\pi_2) - \dim(\pi_1) = 2$ and when for any $g \in G$, the set of eigenvalues of $\pi_1(g)$ acting on V_1, counted with multiplicity, is contained in the set of eigenvalues of $\pi_2(g)$ acting on V_2, counted with multiplicity. All the proofs in this section of Propositions 3, 4, 5 giving an affirmative answer to Question 1 for the pair $(\mathrm{GL}_n, \mathrm{GL}_{n+2})$ for $n = 1, 2, 3$ use this identity (1) crucially, answering Question 1 for any group G, and then we had to carefully transport that proof (for arbitrary group G) to the world of automorphic forms using the strong multiplicity one theorem about isobaric decomposition of automorphic forms, and instances of functoriality. However, when in section 6, we answer Question 1 in the affirmative for general groups which are not virtually abelian, we do not rely on the identity (1).

4. Isobaric types of $\mathrm{Sym}^6(\pi)$

For an automorphic representation π_1 of $\mathrm{GL}_n(\mathbb{A}_F)$ with isobaric decomposition

$$\pi_1 = \pi_{11} \boxplus \pi_{12} \boxplus \cdots \boxplus \pi_{1\ell},$$

where π_{1j} are irreducible cuspidal automorphic representations of $\mathrm{GL}_{d_j}(\mathbb{A}_F)$, we call the set of un-ordered integers $\{d_j\}$ which forms a partition of n to be the isobaric type of π_1.

Proposition 6. *Let π be a cuspidal non-CM automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ with central character $\omega : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$. Assume that $\mathrm{Sym}^i(\pi)$ are automorphic for $i \leq 6$, and that $\mathrm{Sym}^6(\pi)$ is not cuspidal. Then the isobaric type of $\mathrm{Sym}^6(\pi)$ is $(3,3,1)$ if π is tetrahedral, $(4,2,1)$ if π is octahedral, and of type $(4,3)$ otherwise.*

Proof. We will split the proof according to the different cases for π.

Tetrahedral case: In this case, one knows that $\mathrm{Sym}^3(\pi)$ is reducible, and by Theorem 2.2.2 of Kim-Shahidi [KS3],

$$\mathrm{Sym}^3(\pi) = \chi_1\pi \boxplus \chi_2\pi, \tag{1}$$

for certain Grössencharacters χ_1, χ_2 of F. Since,

$$\mathrm{Sym}^2 \mathrm{Sym}^3(\pi) = \mathrm{Sym}^6(\pi) \boxplus \omega^2 \mathrm{Sym}^2(\pi), \tag{2}$$

equation (1) gives:

$$\mathrm{Sym}^6(\pi) \boxplus \omega^2 \mathrm{Sym}^2(\pi) = \chi_1^2 \mathrm{Sym}^2(\pi) \boxplus \chi_2^2 \mathrm{Sym}^2(\pi) \boxplus \chi_1\chi_2\pi \boxtimes \pi. \tag{3}$$

Since we are assuming that π is non-CM, $\mathrm{Sym}^2(\pi)$ is a cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A}_F)$, and thus the only option for the isobaric type of $\mathrm{Sym}^6(\pi)$ is (3,3,1).

Octahedral case: In this case, one knows that $\mathrm{Sym}^2(\pi), \mathrm{Sym}^3(\pi)$ are irreducible, but $\mathrm{Sym}^4(\pi)$ is reducible, and by Theorem 3.3.7(3) of Kim-Shahidi [KS3],

$$\mathrm{Sym}^4(\pi) = \chi_1\pi \boxplus \chi_2 \mathrm{Sym}^2(\pi). \tag{4}$$

Since,

$$\Lambda^2 \mathrm{Sym}^4(\pi) = \omega \mathrm{Sym}^6(\pi) \boxplus \omega^2 \mathrm{Sym}^2(\pi), \tag{5}$$

using (4) and (5) we have,

$$\omega \mathrm{Sym}^6(\pi) \boxplus \omega^2 \mathrm{Sym}^2(\pi) = \chi_1^2\Lambda^2(\pi) \boxplus \chi_2^2\Lambda^2 \mathrm{Sym}^2(\pi) \boxplus \chi_1\chi_2\pi \boxtimes \mathrm{Sym}^2(\pi). \tag{6}$$

On the other hand,

$$\pi \boxtimes \mathrm{Sym}^2(\pi) = \mathrm{Sym}^3(\pi) \boxplus \omega\pi. \tag{7}$$

Using (6) and (7) we find that the isobaric type of $\mathrm{Sym}^6(\pi)$ is (4,2,1).

Rest of the cases when $\mathrm{Sym}^6(\pi)$ is not cuspidal: (Although one expects this case to consist exactly of icosahedral representations, this seems not known. Such automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$ are called *quasi-icosahedral* in [Ra]. For our proof below, this lack of knowledge does not matter.)

By Theorem A′ of Ramakrishnan [Ra], if we are not in the above two cases, and $\mathrm{Sym}^6(\pi)$ is not cuspidal, then $\mathrm{Sym}^i(\pi)$ are cuspidal for all $i \leq 5$, and

$$\mathrm{Sym}^5(\pi) = \chi\pi \boxtimes \mathrm{Sym}^2(\pi'), \tag{8}$$

where χ is a Grössencharacter on F, and π' is a cuspidal automorphic representation on $\mathrm{GL}_2(\mathbb{A}_F)$ such that $\mathrm{Sym}^2(\pi)$ and $\mathrm{Sym}^2(\pi')$ are not twist equivalent.

We use the identity:

$$\pi \boxtimes \mathrm{Sym}^5(\pi) = \mathrm{Sym}^6(\pi) \boxplus \omega \mathrm{Sym}^4(\pi). \tag{9}$$

Therefore using (8) we have,

$$\chi\pi \boxtimes \pi \boxtimes \mathrm{Sym}^2(\pi') = \mathrm{Sym}^6(\pi) \boxplus \omega \mathrm{Sym}^4(\pi), \tag{10}$$

or,

$$\chi[\omega \boxplus \mathrm{Sym}^2(\pi)] \boxtimes \mathrm{Sym}^2(\pi') = \mathrm{Sym}^6(\pi) \boxplus \omega \mathrm{Sym}^4(\pi), \tag{11}$$

which under the assumption that $\mathrm{Sym}^6(\pi)$ is automorphic, is an identity of isobaric automorphic representations.

Since $\mathrm{Sym}^2(\pi')$ is a cuspidal representation on $\mathrm{GL}_3(\mathbb{A}_F)$, there is a GL_3-cuspform in the isobaric decomposition on the left hand side of the identity (11). Further, we are forced to have a GL_5-cuspform on the left hand side of the identity (11) to account for $\mathrm{Sym}^4(\pi)$ on the right (which is given to be cuspidal), so the possible isobaric types on the left hand side of the identity (11) are (3,5) + a partition of 4. Thus we deduce that the isobaric type for $\mathrm{Sym}^6(\pi)$, an automorphic representation on $\mathrm{GL}_7(\mathbb{A}_F)$, is 3 + a partition of 4. However, $\mathrm{Sym}^6(\pi)$ cannot have, in its isobaric decomposition, a representation of GL_1 or a representation of GL_2 as follows from the isobaric decomposition (9) above. For example, if $\mathrm{Sym}^6(\pi)$ had the shape $\pi_2 + \pi_4$ with π_2 an automorphic representation of GL_2 and π_4 on GL_4, then clearly the Rankin product of the right hand side of (9) with $\pi_2^\vee$ will have a pole (at $s = 1$) whereas the left hand side of the identity (11) which will be $\pi \boxtimes \mathrm{Sym}^5(\pi) \boxtimes \pi_2^\vee$ does not have a pole since in our case, $\mathrm{Sym}^5(\pi)$ is a cuspidal representation. Thus the only option for the isobaric decomposition of $\mathrm{Sym}^6(\pi)$ is $(4, 3)$ □

5. Group theoretic analogues

In this paper we have answered the Question 1 for $(\mathrm{GL}_n, \mathrm{GL}_{n+2})$ for $n = 1, 2, 3$ in the positive. Thus the first case left unsettled is $(\mathrm{GL}_4, \mathrm{GL}_6)$.

In this section, in Example 1 we construct instances where our Question 1 has a positive answer (assuming strong Artin conjecture) for $(\mathrm{GL}_4, \mathrm{GL}_6)$, a case not treated by our work so far in this paper, and then in the final remark of the section, we construct instances where our Question 1 has a negative answer, using Calegari's work in [Ca], for $(\mathrm{GL}_4, \mathrm{GL}_6)$. We begin with some generalities on finite groups, focusing eventually on $\mathrm{GL}_2(\mathbb{F}_q)$.

For representations V_1 and V_2 of a group G, define a relationship $V_1 \preceq V_2$ (to be read as V_1 **immersed** in V_2) if for each element $g \in G$, the set of eigenvalues of the action of g on V_1 (counted with multiplicities) is contained in the set of eigenvalues of g acting on V_2 (counted with multiplicities). Thus if $V_1 \subset V_2$ as representations of G, then $V_1 \preceq V_2$. If $V_1 \preceq V_2$ and $\dim(V_1) = \dim(V_2)$, then of course, $V_1 \cong V_2$ as G-modules, whereas just as in Proposition 1, if $\dim(V_1) + 1 = \dim(V_2)$, then also $V_1 \preceq V_2$ implies $V_1 \subset V_2$ as G-modules. However, it is not true in general that if $V_1 \preceq V_2$, then $V_1 \subset V_2$ as G-modules as we will now see.

Proposition 7. *Let $G = \mathrm{GL}_2(\mathbb{F}_q)$. Let C be an irreducible cuspidal representation of $\mathrm{GL}_2(\mathbb{F}_q)$ of dimension $(q-1)$, and P an irreducible principal series representation of $\mathrm{GL}_2(\mathbb{F}_q)$ of dimension $(q+1)$. Assume that the central character of C and P are the same, which is $\omega : Z = \mathbb{F}_q^\times \to \mathbb{C}^\times$. Then,*

$$C \preceq P.$$

Proof. One knows that:

(1) The restriction of C to the diagonal torus $T = \mathbb{F}_q^\times \times \mathbb{F}_q^\times$ in $\mathrm{GL}_2(\mathbb{F}_q)$ is the set of all characters with multiplicity 1 of T whose restriction to the center Z is ω. These characters of T are also contained in the restriction of P to T (there are two characters $\{\chi_1 \times \chi_2, \chi_2 \times \chi_1\}$ of T appearing in the restriction of P to T with multiplicity 2 which are the characters $\{\chi_1 \times \chi_2, \chi_2 \times \chi_1\}$ of T used to define the principal series P).

(2) The restriction of C to the anisotropic torus $S = \mathbb{F}_{q^2}^\times$ in $\mathrm{GL}_2(\mathbb{F}_q)$ is the set of all characters with multiplicity 1 of S whose restriction to the center Z is ω except that the two characters $\{\chi, \bar{\chi}\}$ of S used to define the cuspidal representation C do not appear. On the other hand, the restriction of P to

S is the set of all characters of S with multiplicity 1 whose restriction to the center Z is ω.

(3) The restriction of C to the upper triangular unipotent group $U = \mathbb{F}_q$ in $\mathrm{GL}_2(\mathbb{F}_q)$ is the regular representation of U except that the trivial representation of U does not appear in C. On the other hand, the restriction of P to U is the regular representation of U, except that the trivial representation appears twice.

It follows that $C \preceq P$, with $\dim P - \dim C = 2$. □

Now we note Proposition 8 whose obvious proof will be omitted. Using this proposition, and the example of $C \preceq P$, we get counter-examples to Question 1 at the beginning of the paper. (The group $\mathrm{GL}_2(\mathbb{F}_q)$, $q \neq 2, 3, 4$ has no two dimensional irreducible (projective) representation, thus we cannot realize C and P as symmetric powers of a two dimensional representation of a central cover of $\mathrm{GL}_2(\mathbb{F}_q)$.)

Proposition 8. *If G is a finite group realized as a Galois group of number fields $G = \mathrm{Gal}(E/F)$, thus any two representations of G, V_1 of dimension n and V_2 of dimension m gives rise to Artin L-functions, which assuming the strong Artin conjecture give rise to cuspidal automorphic representations π_1 of $\mathrm{GL}_n(\mathbb{A}_F)$ and π_2 of $\mathrm{GL}_m(\mathbb{A}_F)$. If $V_1 \preceq V_2$, then the Hecke eigenvalues of the automorphic representation π_1 are contained in the Hecke eigenvalues of the automorphic representation π_2.*

Example 1. Here is a nice example to illustrate the use of finite groups for Question 1. The details of the example are taken from Lemma 5.1 and 5.3 of Kim [Kim2] whose notation we will follow. The group $\mathrm{SL}_2(\mathbb{F}_5)$ has two 2-dimensional irreducible representations σ, σ^τ (favorite of representation theorists, the odd Weil representation!). These have character values in $\mathbb{Q}(\sqrt{5})$, and are Galois conjugate. We have $\mathrm{Sym}^3(\sigma) \cong \mathrm{Sym}^3(\sigma^\tau)$, an irreducible 4 dimensional representation of $\mathrm{SL}_2(\mathbb{F}_5)$ extending to a cuspidal representation C of $\mathrm{GL}_2(\mathbb{F}_5)$ of non-trivial central character. Further, we have,

$$\mathrm{Sym}^5(\sigma) \cong \mathrm{Sym}^5(\sigma^\tau) \cong \mathrm{Sym}^2(\sigma) \otimes \sigma^\tau \cong \mathrm{Sym}^2(\sigma^\tau) \otimes \sigma,$$

giving the unique irreducible representation of $\mathrm{SL}_2(\mathbb{F}_5)$ of dimension 6 extending to a principal series representation P of $\mathrm{GL}_2(\mathbb{F}_5)$ of non-trivial central character, same as that of C. In particular, for the automorphic representations π_1 of $\mathrm{GL}_4(\mathbb{A}_F)$ associated to $\mathrm{Sym}^3(\sigma)$ and for the automorphic representations π_2 of $\mathrm{GL}_6(\mathbb{A}_F)$ associated to $\mathrm{Sym}^5(\sigma)$, for which since $\mathrm{Sym}^3(\sigma) \preceq \mathrm{Sym}^5(\sigma)$, Proposition 8 applies, constructing an instance where Question 1 has an affirmative answer (using the group $G = \mathrm{SL}_2(\mathbb{F}_5)$.)

Remark 5. By the work of Calegari, cf. [Ca], irreducible representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ of dimension 4 and 6 factoring through a Galois extension E of $\mathbb{Q}$ with $\mathrm{Gal}(E/\mathbb{Q}) = S_5 \cong \mathrm{PGL}_2(\mathbb{F}_5)$ which arise by taking a cuspidal representation of $\mathrm{PGL}_2(\mathbb{F}_5)$ of dimensions $4 = q-1$, and a principal series of dimension $6 = q+1$, give rise to cuspidal automorphic representations Π and Π' of $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$ and $\mathrm{GL}_6(\mathbb{A}_\mathbb{Q})$ respectively.

Considerations of this section will then give a counter-example to the Question 1 for the pair of cuspidal automorphic representations $\pi_1 = \Pi$ and $\pi_2 = \Pi'$ of $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$ and $\mathrm{GL}_6(\mathbb{A}_\mathbb{Q})$, since by Lemma 4, the automorphic representation Π of $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$ does not arise as $\mathrm{Sym}^3(\pi)$ of an automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$.

We leave checking that results of this section are in conformity with the earlier results in the paper for $q = 2, 3, 4$ to the reader as a curious exercise!

The proof of Lemma 4 is due to F. Calegari.

Lemma 4. *Let* Π *be an automorphic representation of* $\mathrm{GL}_4(\mathbb{A}_F)$ *which is an Artin representation coming from the standard 4-dimensional irreducible representation of* S_5 *(realized as the Galois group of an extension* E/F*). Then* Π *cannot be written as* $\mathrm{Sym}^3(\pi)$ *of an automorphic representation* π *of* $\mathrm{GL}_2(\mathbb{A}_F)$*.*

Proof. Choose a place v of F such that the Frobenius conjugacy class for the extension E/F in S_5 is the conjugacy class of a transposition in S_5, and therefore, the Hecke eigenvalues of Π at the place v is:

$$(1, 1, 1, -1).$$

Suppose $\Pi_v = \mathrm{Sym}^3(\pi_v)$, with Hecke eigenvalues of π_v being (α_v, β_v). Therefore we have the equality of un-ordered quadruples $(1, 1, 1, -1)$ and $(\alpha_v^3, \alpha_v^2\beta_v, \alpha_v\beta_v^2, \beta_v^3)$. Assume without loss of generality that $\alpha_v^3 = 1$, and one of $\alpha_v^2\beta_v, \alpha_v\beta_v^2$ is also 1. Thus either $\alpha_v = \beta_v$ or $\alpha_v = -\beta_v$. Neither is an option if the un-ordered quadruples $(1, 1, 1, -1)$ and $(\alpha_v^3, \alpha_v^2\beta_v, \alpha_v\beta_v^2, \beta_v^3)$ are the same, and $\alpha_v^3 = 1$. □

Question 2 was posed by F. Calegari.

Question 2 (F. Calegari)**.** If the Hecke eigenvalues of an automorphic representation Π of $\mathrm{GL}_{n+1}(\mathbb{A}_F)$ are of the form $(\alpha_v^n, \alpha_v^{n-1}\beta_v, \cdots, \alpha_v\beta_v^{n-1}, \beta_v^n)$ at almost all places v of F, then is $\Pi = \mathrm{Sym}^n(\pi)$ for an automorphic form π of $\mathrm{GL}_2(\mathbb{A}_F)$?

6. Virtually non-abelian groups

In this section we prove that the group theoretic analogue of the Question 1 has an affirmative answer as long as the group is not 'virtually abelian', i.e., does not contain a finite index subgroup which is abelian.

In Proposition 9, we call a connected reductive group Q of type A_1 if its derived subgroup is $\mathrm{PGL}_2(\mathbb{C})$ or $\mathrm{SL}_2(\mathbb{C})$.

Proposition 9. *Let* G *be a connected reductive algebraic group over* $\mathbb{C}$*. Let* π_1 *and* π_2 *be two finite dimensional representations of* G *with* $\pi_1 \preceq \pi_2$ *such that*

$$\dim(\pi_2) - \dim(\pi_1) = 2.$$

Then either

$$\pi_2 = \pi_1 + \lambda + \mu,$$

where λ, μ *are one dimensional representations of* G*, or* G *has a reductive quotient* Q *of type* A_1 *and* π_1', π_2' *irreducible representations of* Q *of dimensions* d *and* $d+2$ *respectively* ($d = 0$ *allowed*) *such that,*

$$\pi_1 = \pi + \pi_1',$$
$$\pi_2 = \pi + \pi_2',$$

for a finite dimensional representation π *of* G*.*

Proof. Let T be a maximal torus in G, and W its Weyl group. Clearly, $\pi_1 \preceq \pi_2$ if and only if the weights of π_1 for the torus T are contained in the weights of π_2 (with multiplicity). Since weights are W-invariant, if $\pi_1 \preceq \pi_2$ with $\dim(\pi_2) - \dim(\pi_1) = 2$, we see that there is a set of two (not necessarily distinct) weights of T (that of

$\pi_2 - \pi_1$) which is W-invariant. By Lemma 5, this means that either these are weights of T invariant under W, hence arise from characters $\lambda, \mu : G \to \mathbb{C}^\times$, or the group G has a quotient Q (obtained by dividing G by all normal simple groups except one which is $\mathrm{PGL}_2(\mathbb{C})$ or $\mathrm{SL}_2(\mathbb{C})$), with a quotient S of T as a maximal torus in Q. Further, by the same Lemma 5, these two characters of T are pulled back of characters of S via the map $T \to S$. In the first case, i.e., when these are two characters $\lambda, \mu : G \to \mathbb{C}^\times$, the two representations of G, π_2 and $\pi_1 + \lambda + \mu$, have the same characters on T, therefore must be isomorphic.

In the second case, we appeal to the elementary fact that for a reductive group Q of type A_1, with maximal torus S, any set of two distinct characters of S of the form $\{\chi, \chi^w\}$ where w is the unique non-trivial element of the Weyl group of Q, $\chi + \chi^w$ is the difference of two irreducible representations π_1', π_2' of Q of dimensions d and $d+2$ respectively ($d = 0$ allowed), therefore, we have

$$\pi_2 - \pi_1 = \pi_2' - \pi_1',$$

as T-modules, and therefore

$$\pi_2 + \pi_1' = \pi_2' + \pi_1,$$

as G-modules, and the conclusion of the proposition follows. □

Lemma 5. *Let G be a connected reductive group with T a maximal torus and W its Weyl group. Then if χ is a character of T whose W-orbit has ≤ 2 elements, then either χ is the restriction of a one dimensional representation of G to T, or G has a quotient Q of type A_1 with S a maximal torus of Q which is a quotient of T, and χ is a character of T factoring through S.*

Proof. It suffices to prove the lemma for semisimple groups where it easily reduces to a simple group. The lemma for a simple group reduces to the assertion that if G is a simple group which is not $\mathrm{PGL}_2(\mathbb{C})$ or $\mathrm{SL}_2(\mathbb{C})$, and if χ is a character of T whose W-orbit has ≤ 2 elements, then character must be trivial. For this, observe the well-known fact, cf. [Hum], Lemma B of section 10.3, that the stabilizer of χ (in W), an element in the character group $X^\star(T)$, which we assume without loss of generality to belong to fundamental Weyl chamber, is generated by the simple reflections fixing χ, hence in particular, it is the Weyl group W_X of the associated Levi subgroup. Now $|W/W_X| \geq 3$ can be easily proved for groups G of rank ≥ 2, by an easy reduction to rank 2 where it is clear. □

Corollary 2 follows by an application of Clifford theory (applied to the normal subgroup G_0 of G) combined with Proposition 9 applied to G_0, we omit its proof.

Corollary 2. *Let G be an algebraic group over $\mathbb{C}$, with G_0, the connected component of identity, a non-abelian reductive group. Assume G has irreducible finite dimensional representations π_1 and π_2, with $\pi_1 \preceq \pi_2$ (when restricted to G_0) such that*

$$\dim(\pi_2) - \dim(\pi_1) = 2,$$

and the action of G on $\pi_1 + \pi_2$ is faithful. Then, both π_1, π_2 remain irreducible when restricted to G_0, and their restriction to G_0 arises from a quotient Q (not necessarily semi-simple) of G of type A_1.

Remark 6. By proposition 9, there are no relations $\pi_1 \preceq \pi_2$ among irreducible representations of a connected simple algebraic group with $\dim(\pi_2) - \dim(\pi_1) = 2$, other than the obvious ones for $G = \mathrm{SL}_2(\mathbb{C})$, and $G = \mathrm{PGL}_2(\mathbb{C})$. Without the constraint on $\dim(\pi_2) - \dim(\pi_1) = 2$, there are naturally many more representations, such as the pair of representations $\Lambda^k(\mathbb{C}^n), \mathrm{Sym}^k(\mathbb{C}^n)$ of $\mathrm{GL}_n(\mathbb{C})$. It seems interesting to classify all possible pairs of irreducible representations (π_1, π_2) with $\pi_1 \preceq \pi_2$ for connected simple algebraic groups with $\dim \pi_2 - \dim \pi_1$, a fixed integer.

Acknowledgments

The authors thank F. Calegari and D. Ramakrishnan for helpful correspondences, and the referee for careful reading of our paper and several useful comments.

References

[Ca] Frank Calegari, *The Artin conjecture for some S_5-extensions*, Math. Ann. **356** (2013), no. 1, 191–207, DOI 10.1007/s00208-012-0839-4. MR3038126

[GGP] Wee Teck Gan, Benedict H. Gross, and Dipendra Prasad, *Branching laws for classical groups: the non-tempered case*, Compos. Math. **156** (2020), no. 11, 2298–2367, DOI 10.1112/S0010437X20007496. MR4190046

[GJ] Stephen Gelbart and Hervé Jacquet, *A relation between automorphic representations of* GL(2) *and* GL(3), Ann. Sci. École Norm. Sup. (4) **11** (1978), no. 4, 471–542. MR533066

[Hum] James E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, Vol. 9, Springer-Verlag, New York-Berlin, 1972. MR0323842

[JS] Hervé Jacquet and Joseph Shalika, *Exterior square L-functions*, Automorphic forms, Shimura varieties, and L-functions, Vol. II (Ann Arbor, MI, 1988), Perspect. Math., vol. 11, Academic Press, Boston, MA, 1990, pp. 143–226. MR1044830

[JS1] H. Jacquet and J. A. Shalika, *On Euler products and the classification of automorphic representations. I*, Amer. J. Math. **103** (1981), no. 3, 499–558, DOI 10.2307/2374103. MR618323

[JS2] H. Jacquet and J. A. Shalika, *On Euler products and the classification of automorphic forms. II*, Amer. J. Math. **103** (1981), no. 4, 777–815, DOI 10.2307/2374050. MR623137

[KL] Chandrashekhar B. Khare and Michael Larsen, *Abelian varieties with isogenous reductions* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **358** (2020), no. 9-10, 1085–1089, DOI 10.5802/crmath.129. MR4196779

[Kim] Henry H. Kim, *Functoriality for the exterior square of* GL_4 *and the symmetric fourth of* GL_2, J. Amer. Math. Soc. **16** (2003), no. 1, 139–183, DOI 10.1090/S0894-0347-02-00410-1. With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak. MR1937203

[Kim2] Henry H. Kim, *An example of non-normal quintic automorphic induction and modularity of symmetric powers of cusp forms of icosahedral type*, Invent. Math. **156** (2004), no. 3, 495–502, DOI 10.1007/s00222-003-0340-5. MR2061327

[KS1] Henry H. Kim and Freydoon Shahidi, *Symmetric cube L-functions for* GL_2 *are entire*, Ann. of Math. (2) **150** (1999), no. 2, 645–662, DOI 10.2307/121091. MR1726704

[KS2] Henry H. Kim and Freydoon Shahidi, *Functorial products for* $\mathrm{GL}_2 \times \mathrm{GL}_3$ *and the symmetric cube for* GL_2, Ann. of Math. (2) **155** (2002), no. 3, 837–893, DOI 10.2307/3062134. With an appendix by Colin J. Bushnell and Guy Henniart. MR1923967

[KS3] Henry H. Kim and Freydoon Shahidi, *Cuspidality of symmetric powers with applications*, Duke Math. J. **112** (2002), no. 1, 177–197, DOI 10.1215/S0012-9074-02-11215-0. MR1890650

[Ra] Dinakar Ramakrishnan, *Remarks on the symmetric powers of cusp forms on* GL(2), Automorphic forms and L-functions I. Global aspects, Contemp. Math., vol. 488, Amer. Math. Soc., Providence, RI, 2009, pp. 237–256, DOI 10.1090/conm/488/09570. MR2522032

[Ra1] Dinakar Ramakrishnan, *An exercise concerning the selfdual cusp forms on* GL(3), Indian J. Pure Appl. Math. **45** (2014), no. 5, 777–785, DOI 10.1007/s13226-014-0088-1. MR3286086

[Sha] Freydoon Shahidi, *On certain L-functions*, Amer. J. Math. **103** (1981), no. 2, 297–355, DOI 10.2307/2374219. MR610479

[Yam] Shunsuke Yamana, *On poles of the exterior cube L-functions for* GL_6, Math. Z. **279** (2015), no. 1-2, 267–270, DOI 10.1007/s00209-014-1366-7. MR3299852

Indian Institute of Technology Bombay, Powai, Mumbai-400076, India; and St Petersburg State University, St Petersburg, Russia

Email address: prasad.dipendra@gmail.com

Indian Institute of Technology Bombay, Powai, Mumbai-400076, India

Email address: ravir@math.iitb.ac.in