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AN INJECTIVITY THEOREM WITH MULTIPLIER IDEAL SHEAVES OF SINGULAR METRICS WITH TRANSCENDENTAL SINGULARITIES

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Abstract

The purpose of this paper is to establish an injectivity theorem generalized to pseudo-effective line bundles with transcendental (non-algebraic) singular hermitian metrics and multiplier ideal sheaves. As an application, we obtain a Nadel type vanishing theorem. For the proof, we study the asymptotic behavior of the harmonic forms with respect to a family of regularized metrics, and give a method to obtain L^2-estimates of solutions of the $\overline{\partial}$-equation by using the de Rham-Weil isomorphism between the $\overline{\partial}$-cohomology and the Čech cohomology.

1. Introduction

The Kodaira vanishing theorem and its generalizations play an important role when we consider certain fundamental problems in higher dimensional algebraic geometry (in particular birational geometry). The following result proved by Kollár, the so-called injectivity theorem, is a celebrated generalization of the Kodaira vanishing theorem in algebraic geometry. In this paper, we study the injectivity theorem from the viewpoint of complex differential geometry and the theory of several complex variables. Our purpose is to establish an analytic version of the injectivity theorem formulated for pseudo-effective line bundles equipped with transcendental singular (hermitian) metrics by using multiplier ideal sheaves.

Theorem 1.1 ([19]; cf. [10])**.** *Let F be a semi-ample line bundle on a smooth projective variety X. Then, for a (non-zero) section s of a positive*

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multiple F^m *of the line bundle* F*, the multiplication map induced by the tensor product with* s*,*

$$\Phi_s\colon H^q(X, K_X \otimes F) \xrightarrow{\otimes s} H^q(X, K_X \otimes F^{m+1}),$$

is injective for any q*. Here* K_X *denotes the canonical bundle of* X*.*

In his paper [8], Enoki gave the following injectivity theorem. Kollár's proof is based on the Hodge theory. On the other hand, Enoki's proof is based on the theory of harmonic integrals, which enables us to approach the injectivity theorem from the viewpoint of complex differential geometry.

Theorem 1.2 ([8])**.** *Let* F *be a semi-positive line bundle on a compact Kähler manifold* X*. Then the same conclusion as in Theorem* 1.1 *holds.*

A semi-ample line bundle is always semi-positive (namely, it admits a "smooth" metric with semi-positive curvature), and thus Theorem 1.2 leads to Theorem 1.1. The above results can be regarded as a generalization of the Kodaira vanishing theorem to semi-ample (semi-positive) line bundles. On the other hand, the Kodaira vanishing theorem can be generalized to the Nadel vanishing theorem by using singular metrics with (strictly) positive curvature, which corresponds to the Kawamata-Viehweg vanishing theorem in algebraic geometry. Therefore, in the same direction, it is natural to generalize them to an injectivity theorem for singular metrics with semi-positive curvature.

The Kodaira vanishing { cpx. geometry: smooth positive metrics { alg. geometry: ample line bundles	semi-positivity $\longrightarrow$	The Kollár, Enoki injectivity { cpx. : smooth semi-positive metrics { alg. : semi-ample line bundles
$\downarrow$ singular metrics		$\downarrow$ singular metrics
The Nadel, Kawamata-Viehweg vanishing { cpx. : singular positive metrics { alg. : big line bundles	semi-positivity $\longrightarrow$	**Theorem 1.3** { cpx. : singular semi-positive metrics { alg. : pseudo-effective line bundles

The following theorem, which is the main result of this paper, is a common generalization of Kollár's (Enoki's) injectivity theorem and the Nadel (Kawamata-Viehweg) vanishing theorem. Moreover Theorem 1.3 is also a generalization of various results, for example, [8], [11], [19], [22], [26], [30], [31]. A (holomorphic) line bundle is said to be *pseudo-effective* if it admits a "singular" metric with semi-positive curvature, and thus Theorem 1.3 can be seen as an injectivity theorem for pseudo-effective line bundles.

Theorem 1.3 (The main result)**.** *Let* F *be a* (*holomorphic*) *line bundle on a compact Kähler manifold* X *and* h *be a singular* (*hermitian*) *metric with semi-positive curvature on* F*. Then, for a* (*non-zero*) *section* s *of a positive multiple* F^m *satisfying* $\sup_X |s|_{h^m} < \infty$*, the multiplication map*

$$\Phi_s\colon H^q(X, K_X \otimes F \otimes \mathcal{I}(h)) \xrightarrow{\otimes s} H^q(X, K_X \otimes F^{m+1} \otimes \mathcal{I}(h^{m+1}))$$

is (well-defined and) injective for any q. Here $\mathcal{I}(h)$ denotes the multiplier ideal sheaf associated to the singular metric h.

Remark 1.4.

(1) We can show that the multiplication map from $H^q(X, K_X \otimes F^\ell \otimes \mathcal{I}(h^\ell))$ to $H^q(X, K_X \otimes F^{\ell+m} \otimes \mathcal{I}(h^{\ell+m}))$ is also injective for $\ell > 0$ by applying Theorem 1.3 to F^ℓ and $s^\ell \in H^0(X, F^{\ell m})$.

(2) The multiplication map is well-defined thanks to the assumption that $\sup_X |s|_{h^m} < \infty$. We can always apply this theorem to a pseudo-effective line bundle F since a metric $h_{\min}$ with minimal singularities on F satisfies $\sup_X |s|_{h^m_{\min}} < \infty$ for any section s of F^m (see [3] for the definition of metrics with minimal singularities).

It is important to emphasize that a singular metric h in our formulation may have transcendental (non-algebraic) singularities. To handle singular metrics with transcendental singularities, we have to take a more analytic approach to the cohomology groups with coefficients in $K_X \otimes F \otimes \mathcal{I}(h)$, which gives a generalization of techniques of [8], [11], [22], [24], [30].

Our formulation is motivated by the problem of extending sections from subvarieties to the ambient space. When we attempt to extend sections by the vanishing (injectivity) theorem or the Ohsawa-Takegoshi extension theorem, we need a suitable singular metric h. In many cases, the metric h is constructed by taking the limit of suitable metrics h_m. For example, this strategy plays a crucial role in the proof of the invariance of pluri-genera or the extension theorem for pluri-canonical sections (see [6], [27], [29]). However it seems to be quite hard to investigate the regularity (smoothness) of the limit h, even if h_m has algebraic singularities. Therefore for important applications it is worth formulating Theorem 1.3 for arbitrary singular metrics.

Thanks to this advantage, as applications of Theorem 1.3, we can obtain an injectivity theorem for nef and abundant line bundles (Corollary 4.1) and Nadel type vanishing theorems (Theorem 4.5, Corollary 4.2), by considering metrics with minimal singularities (which do not always have algebraic singularities). Theorem 4.5 is a generalization of [22], [24]. Moreover, we prove some extension theorems for pluri-canonical sections of log pairs motivated by the abundance conjecture in [13].

At the end of this section, we briefly explain the proof of Theorem 1.3. First we recall Enoki's proof of Theorem 1.2 (the special case where h is smooth). In this case, the cohomology group $H^q(X, K_X \otimes F)$ can be represented by the space of harmonic forms with respect to h

$$\mathcal{H}^{n,q}_h(F) := \{u \,|\, u \text{ is an } F\text{-valued } (n,q)\text{-form on } X \text{ with } \overline{\partial} u = 0 \text{ and } \overline{\partial}^*_h u = 0\},$$

where $\overline{\partial}^*_h$ is the adjoint operator of the $\overline{\partial}$-operator. For an arbitrary harmonic form $u \in \mathcal{H}^{n,q}_h(F)$, we can show that su is also harmonic with respect to h^{m+1} from semi-positivity of the curvature of h. It implies that the multiplication map Φ_s induces the map from $\mathcal{H}^{n,q}_h(F)$ to $\mathcal{H}^{n,q}_{h^{m+1}}(F^{m+1})$, and then the injectivity is obvious. This method heavily depends on semi-positivity of the curvature.

In our situation, we cannot directly use the theory of harmonic integrals since a given singular metric h may have transcendental singularities. For this reason, in Step 1, we first approximate the metric h by singular metrics $\{h_\varepsilon\}_{\varepsilon>0}$ that are smooth on a Zariski open set Y. Then a given cohomology class can be represented by the associated harmonic form u_ε with respect to h_ε on Y. We want to show that su_ε is harmonic by the same argument as in Enoki's proof. However, the same argument fails since the curvature of h_ε is no longer semi-positive. Therefore, we investigate the asymptotic behavior of the harmonic form u_ε. This asymptotic analysis contains a new ingredient. In Step 2, by generalizing Enoki's method, we prove that the L^2-norm $\|\overline{\partial}^*_{h^{m+1}_\varepsilon} su_\varepsilon\|$ converges to zero as ε tends to zero. In Step 3, we construct a solution v_ε of the $\overline{\partial}$-equation $\overline{\partial} v_\varepsilon = su_\varepsilon$ such that the L^2-norm $\|v_\varepsilon\|$ is uniformly bounded, by using the Čech complex. The above arguments yield

$$\|su_\varepsilon\|^2 = \langle\!\langle su_\varepsilon, \overline{\partial} v_\varepsilon \rangle\!\rangle \leq \|\overline{\partial}^*_{h^{m+1}_\varepsilon} su_\varepsilon\| \|v_\varepsilon\| \to 0 \quad \text{as } \varepsilon \to 0.$$

In Step 4, from these observations, we prove that u_ε weakly converges to zero, and this completes the proof.

This paper is organized as follows: In Section 2, we summarize the fundamental facts used in this paper. We prove the main result in Section 3 and give several applications of the main result in Section 4. Compared to [24], the crucial technique established in this paper is to construct a solution v_ε of the $\overline{\partial}$-equation $\overline{\partial} v_\varepsilon = su_\varepsilon$ with uniformly bounded L^2-norm, by applying the de Rham-Weil isomorphism between the $\overline{\partial}$-cohomology and the Čech cohomology. In Section 5, we explain this construction after we study the topology of the Čech complex induced by the local L^2-norms of singular metrics. This technique is rather complicated, but it seems to have more applications.

2. Preliminaries

In this section, we summarize the fundamental results for the proof of the main result. Unless otherwise mentioned, X denotes a compact Kähler manifold of dimension n and F denotes a (holomorphic) line bundle on X.

2.1. Singular metrics and multiplier ideal sheaves. We first recall the definition of singular metrics and curvatures. Fix a smooth (hermitian) metric g on F.

Definition 2.1 (Singular metrics and curvatures)**.**

(1) For an L^1-function φ on X, the metric h defined by $h := ge^{-2\varphi}$ is called a *singular hermitian metric* on F. Further φ is called the *weight* of h with respect to the fixed smooth metric g.

(2) The *curvature* $\sqrt{-1}\Theta_h(F)$ of h is defined by

$$\sqrt{-1}\Theta_h(F) = \sqrt{-1}\Theta_g(F) + 2\sqrt{-1}\partial\overline{\partial}\varphi,$$

where $\sqrt{-1}\Theta_g(F)$ is the Chern curvature of g.

In this paper, the singular hermitian metric is often written simply as the singular metric. The Levi form $\sqrt{-1}\partial\overline{\partial}\varphi$ is taken in the sense of distributions, and thus the curvature is a $(1,1)$-current but not always a smooth $(1,1)$-form. The curvature $\sqrt{-1}\Theta_h(F)$ of h is said to be *semi-positive* if $\sqrt{-1}\Theta_h(F) \geq 0$ in the sense of currents. When a singular metric h satisfies $\sqrt{-1}\Theta_h(F) \geq \gamma$ for some smooth $(1,1)$-form γ, the weight φ of h becomes a quasi-plurisubharmonic (quasi-psh for short) function. In particular φ is upper semi-continuous, and thus it is bounded above.

Definition 2.2 (Multiplier ideal sheaves)**.** Let h be a singular metric on F such that $\sqrt{-1}\Theta_h(F) \geq \gamma$ for some smooth $(1,1)$-form γ on X. Then the ideal sheaf $\mathcal{I}(h)$ defined to be

$$\mathcal{I}(h)(U) := \mathcal{I}(\varphi)(U) := \{f \in \mathcal{O}_X(U) \mid |f|e^{-\varphi} \in L^2_{\mathrm{loc}}(U)\}$$

for every open set $U \subset X$ is called the *multiplier ideal sheaf* associated to h.

2.2. Equisingular approximations. In the proof, we apply the equisingular approximation to a given singular metric. In this subsection, we reformulate [7, Theorem 2.3] with our notation and give an additional property.

Theorem 2.3 ([7, Theorem 2.3])**.** *Let ω be a positive $(1,1)$-form on X and h be a singular metric on F with semi-positive curvature. Then there exist singular metrics $\{h_\varepsilon\}_{1\gg\varepsilon>0}$ on F with the following properties:*

(a) *h_ε is smooth on $X \setminus Z_\varepsilon$, where Z_ε is a subvariety on X.*
(b) *$h_{\varepsilon_2} \leq h_{\varepsilon_1} \leq h$ holds for any $0 < \varepsilon_1 < \varepsilon_2$.*
(c) *$\mathcal{I}(h) = \mathcal{I}(h_\varepsilon)$.*
(d) *$\sqrt{-1}\Theta_{h_\varepsilon}(F) \geq -\varepsilon\omega$.*

Moreover, if the set $\{x \in X \mid \nu(h,x) > 0\}$ is contained in a subvariety Z, then we can add the property that Z_ε is contained in Z for every $\varepsilon > 0$. Here $\nu(h,x)$ is the Lelong number at x of the weight of h.

Proof. By applying [7, Theorem 2.3] to the weight function φ of h, we obtain quasi-psh functions φ_ν with equisingularities. If we take a sufficiently large $\nu = \nu(\varepsilon)$ for a given $\varepsilon > 0$, the metric h_ε defined by $h_\varepsilon := ge^{-2\varphi_{\nu(\varepsilon)}}$ satisfies properties (a), (b), (c), (d).

The latter conclusion follows from the proof in [7]. We shortly see this fact by using the notation in [7]. In their proof, they locally approximate φ by $\varphi_{\varepsilon,\nu,j}$ with a logarithmic pole. From inequality (2.5) in [7], the Lelong number of $\varphi_{\varepsilon,\nu,j}$ is less than or equal to that of φ. It follows that $\varphi_{\varepsilon,\nu,j}$ is smooth on $X \setminus Z$ since $\varphi_{\varepsilon,\nu,j}$ has a logarithmic pole. Since φ_ν is obtained from Richberg's regularization of the supremum of these functions (see around (2.10)), we obtain the latter conclusion. □

2.3. The theory of harmonic integrals. In this subsection, we recall the L^2-space of differential forms and the theory of harmonic integrals. Throughout this subsection, let Y be a (not necessarily compact) complex manifold with a positive $(1,1)$-form $\widetilde{\omega}$ and E be a (holomorphic) line bundle on Y with a smooth metric h. In the proof of Theorem 1.3, the manifold Y is a Zariski open set of X and E is the restriction of F to Y.

For E-valued (p,q)-forms u and v, the point-wise inner product $\langle u, v\rangle_{h,\widetilde{\omega}}$ can be defined, and the (global) inner product $\langle\!\langle u, v\rangle\!\rangle_{h,\widetilde{\omega}}$ can also be defined by

$$\langle\!\langle u, v\rangle\!\rangle_{h,\widetilde{\omega}} := \int_Y \langle u, v\rangle_{h,\widetilde{\omega}}\, dV_{\widetilde{\omega}},$$

where $dV_{\widetilde{\omega}} := \widetilde{\omega}^n/n!$ and n is the dimension of Y. The Chern connection D_h on E determined by the holomorphic structure and the hermitian metric h can be written as $D_h = D'_h + \overline{\partial}$ with the $(1,0)$-connection D'_h and the $(0,1)$-connection $\overline{\partial}$ (the $\overline{\partial}$-operator). The connections D'_h and $\overline{\partial}$ are regarded as a densely defined closed operator on the L^2-space $L^{p,q}_{(2)}(Y,E)_{h,\widetilde{\omega}}$ defined by

$$L^{p,q}_{(2)}(Y,E)_{h,\widetilde{\omega}} := \{u \mid u \text{ is an } E\text{-valued } (p,q)\text{-form with } \|u\|_{h,\widetilde{\omega}} < \infty\}.$$

The formal adjoints D'^*_h and $\overline{\partial}^*_h$ agree with the Hilbert space adjoints in the sense of von Neumann if $\widetilde{\omega}$ is a *complete* form on Y (see [4, (3.2) Theorem in Chapter VIII]). The following proposition can be obtained from the Bochner-Kodaira-Nakano identity and the density lemma.

Proposition 2.4. *Let $\widetilde{\omega}$ be a complete Kähler form on Y and h be a smooth metric on E such that $\sqrt{-1}\Theta_h(E) \geq -C\widetilde{\omega}$ for some constant $C > 0$. Then, for every $u \in \operatorname{Dom}\overline{\partial}^*_h \cap \operatorname{Dom}\overline{\partial} \subset L^{n,q}_{(2)}(Y,E)_{h,\widetilde{\omega}}$, the following equality holds:*

$$\|\overline{\partial}^*_h u\|^2_{h,\widetilde{\omega}} + \|\overline{\partial} u\|^2_{h,\widetilde{\omega}} = \|D'^*_h u\|^2_{h,\widetilde{\omega}} + \langle\!\langle \sqrt{-1}\Theta_h(E)\Lambda_{\widetilde{\omega}} u, u\rangle\!\rangle_{h,\widetilde{\omega}}.$$

Here $\Lambda_{\widetilde{\omega}}$ denotes the adjoint of the wedge product $\widetilde{\omega} \wedge \bullet$.

The following lemmas are proved by straightforward computations. For the reader's convenience, we give a proof of Lemma 2.6.

Lemma 2.5. *Let ω and $\widetilde{\omega}$ be positive $(1,1)$-forms with $\widetilde{\omega} \geq \omega$. If u is an (n,q)-form, then $|u|^2_{\widetilde{\omega}}\, dV_{\widetilde{\omega}} \leq |u|^2_{\omega}\, dV_{\omega}$. Further, if u is an $(n,0)$-form, then $|u|^2_{\widetilde{\omega}}\, dV_{\widetilde{\omega}} = |u|^2_{\omega}\, dV_{\omega}$.*

Lemma 2.6. *Let ω be a positive $(1,1)$-form and u, v be differential forms.*

(1) *There exists a positive constant C (depending only on the degree of u, v) such that $|u \wedge v|_{\omega} \leq C|u|_{\omega}|v|_{\omega}$.*

(2) *If $\widetilde{\omega}$ is a positive $(1,1)$-form with $\widetilde{\omega} \geq \omega$, then we have $|u|^2_{\widetilde{\omega}} \leq |u|^2_{\omega}$. In particular, we have $|u \wedge v|_{\widetilde{\omega}} \leq C|u|_{\widetilde{\omega}}|v|_{\omega}$.*

Proof of Lemma 2.6. For a given point x, we choose a local coordinate $(z_1, z_2, \ldots, z_n)$ such that

$$\omega = \frac{\sqrt{-1}}{2}\sum_{j=1}^{n} dz_j \wedge d\overline{z_j} \quad \text{and} \quad \widetilde{\omega} = \frac{\sqrt{-1}}{2}\sum_{j=1}^{n} \lambda_j dz_j \wedge d\overline{z_j} \quad \text{at } x.$$

When the differential forms u and v are written as $u = \sum_{I,J} u_{I,J} dz_I \wedge d\overline{z}_J$ and $v = \sum_{K,L} v_{K,L} dz_K \wedge d\overline{z}_L$ in terms of this coordinate, it is easy to see that

$$|u|^2_{\omega} = \sum_{I,J} |u_{I,J}|^2 \quad \text{and} \quad |u|^2_{\widetilde{\omega}} = \sum_{I,J} |u_{I,J}|^2 \frac{1}{\prod_{(i,j)\in(I,J)} \lambda_i \lambda_j} \quad \text{at } x.$$

Here I, J, K, L are ordered multi-indices and $dz_I := dz_{i_1} \wedge dz_{i_2} \wedge \cdots \wedge dz_{i_p}$ for $I = \{i_1 < i_2 < \cdots < i_p\}$. The second claim follows from $\lambda_i \geq 1$. The inequalities $|u_{I,J}| \leq |u|_{\omega}$ and $|v_{K,L}| \leq |v|_{\omega}$ yield

$$\begin{aligned} |u \wedge v|_{\omega} &= \Big| \sum_{I,J,K,L} u_{I,J}\, v_{K,L}\, dz_I \wedge d\overline{z}_J \wedge dz_K \wedge d\overline{z}_L \Big|_{\omega} \\ &\leq \sum_{I,J,K,L} |u_{I,J}||v_{K,L}| \leq \sum_{I,J,K,L} |u|_{\omega}|v|_{\omega} = C|u|_{\omega}|v|_{\omega}. \end{aligned}$$

Here C is a constant depending only on the degree of u, v. □

2.4. Fréchet spaces. In this subsection, for the reader's convenience, we see that the following theorem leads to Proposition 2.8.

Theorem 2.7 (The open mapping theorem)**.** *Let $\pi : D \to E$ be a linear map between Fréchet spaces D and E. If π is continuous and surjective, then π is an open map.*

Proposition 2.8. *Let $\pi : D \to E$ be a continuous linear map between Fréchet spaces D and E. If the cokernel of π is finite dimensional, then the image $\operatorname{Im}\pi$ of π is closed in E.*

Proof. We first consider the case where $\pi : D \to E$ is injective. We take a finite dimensional subspace E_1 of E such that the quotient map $p : E_1 \to E/\operatorname{Im}\pi$ is isomorphic, and consider a continuous map $\pi_1 : D \oplus E_1 \to E$ defined to be $\pi_1(d,e) := \pi(d) + e$ for every $(d,e) \in D \oplus E_1$. Since π_1 is surjective (and injective) and continuous, the inverse map $\pi_1^{-1} : E \to D \oplus E_1$ is also continuous by the open mapping theorem. By composing π_1^{-1} with the second projection $D \oplus E_1 \to E_1$, we obtain the continuous map $\pi_2 : E \to E_1$. It is easy to see that the kernel of π_2 agrees with the image of π, which implies that the image of π is closed. When $\pi : D \to E$ is not injective, by considering the linear map $\overline{\pi} : D/\operatorname{Ker}\pi \to E$, we can obtain the conclusion. □

3. Proof of the main result

In this section, we give a proof of the main result. The proof is based on a technical combination of the theory of harmonic integrals and the L^2-method for the $\overline{\partial}$-equation.

Theorem 3.1 (=Theorem 1.3). *Let F be a line bundle on a compact Kähler manifold X and h be a singular metric with semi-positive curvature on F. Then, for a* (*non-zero*) *section s of a positive multiple F^m satisfying* $\sup_X |s|_{h^m} < \infty$*, the multiplication map*

$$\Phi_s \colon H^q(X, K_X \otimes F \otimes \mathcal{I}(h)) \xrightarrow{\otimes s} H^q(X, K_X \otimes F^{m+1} \otimes \mathcal{I}(h^{m+1}))$$

is (*well-defined and*) *injective for any q.*

Proof of Theorem 3.1. The case $q = 0$ is obvious, and thus we assume $q > 0$. The proof can be divided into four steps. In Step 1, we approximate a given singular metric h by singular metrics $\{h_\varepsilon\}_{\varepsilon>0}$ that are smooth on a Zariski open set. In this step, we fix the notation to apply the theory of harmonic integrals and explain the sketch of the proof. For a given cohomology class in $H^q(X, K_X \otimes F \otimes \mathcal{I}(h))$ that goes to zero in $H^q(X, K_X \otimes F^{m+1} \otimes \mathcal{I}(h^{m+1}))$ by Φ_s, we take the associated harmonic form u_ε with respect to h_ε. In Step 2, we study the asymptotic behavior of u_ε and su_ε as ε tends to zero. In Step 3, we construct a suitable solution v_ε of the $\overline{\partial}$-equation $\overline{\partial} v_\varepsilon = su_\varepsilon$. In Step 4, we show that u_ε converges to zero in a suitable sense.

Step 1 (Equisingular approximation of h)**.** Throughout the proof, let ω be a Kähler form on X. For the proof, we want to apply the theory of harmonic integrals, but a given singular metric h may not be smooth. For this reason, we approximate h by singular metrics $\{h_\varepsilon\}_{\varepsilon>0}$ that are smooth on a Zariski open set. By Theorem 2.3, we obtain singular metrics $\{h_\varepsilon\}_{\varepsilon>0}$ on F with the following properties:

(a) h_ε is smooth on $X \setminus Z_\varepsilon$, where Z_ε is a subvariety on X.
(b) $h_{\varepsilon_2} \le h_{\varepsilon_1} \le h$ holds for any $0 < \varepsilon_1 \le \varepsilon_2$.

(c) $\mathcal{I}(h) = \mathcal{I}(h_\varepsilon)$.
(d) $\sqrt{-1}\Theta_{h_\varepsilon}(F) \geq -\varepsilon\omega$.

For the weight function φ (resp. φ_ε) of the singular metric h (resp. h_ε) with respect to a smooth metric g, we may assume $\varphi_\varepsilon \leq 0$ by adding a constant, since φ_ε is bounded above on X. Hence we have

$$g \leq h_\varepsilon = ge^{-2\varphi_\varepsilon} \leq h = ge^{-2\varphi}.$$

Since the point-wise norm $|s|_{h^m}$ is bounded on X, there exists a constant C such that $\log|s| \leq m\varphi + C$, where s is locally regarded as a holomorphic function under a local trivialization of F. It implies that the Lelong number of $m\varphi$ is less than or equal to that of $\log|s|$. In particular, the set $\{x \in X \mid \nu(h,x) > 0\}$ is contained in the subvariety Z defined by $Z := \{x \in X \mid s(x) = 0\}$, and thus we may assume a stronger property than property (a), namely

(e) h_ε is smooth on $Y := X \setminus Z$, where $Z := \{x \in X \mid s(x) = 0\}$.

Now we construct a "complete" Kähler form on Y with suitable potential function. Take a quasi-psh function ψ on X such that ψ has a logarithmic pole along Z and ψ is smooth on Y. Since the function ψ is bounded above on X, we may assume $\psi \leq -e$ by adding a constant. We define the $(1,1)$-form $\widetilde{\omega}$ on Y by

$$\widetilde{\omega} := \ell\omega + \sqrt{-1}\partial\overline{\partial}\Psi,$$

where ℓ is a positive number and $\Psi := 1/\log(-\psi)$. Then we can show that the $(1,1)$-form $\widetilde{\omega}$ satisfies the following properties for a sufficiently large $\ell > 0$:

(A) $\widetilde{\omega}$ is a complete Kähler form on Y.
(B) Ψ is bounded on X.
(C) $\widetilde{\omega} \geq \omega$.

Indeed, properties (B), (C) follow from the definition of Ψ, $\widetilde{\omega}$, and property (A) follows from straightforward computations. See [11, Lemma 3.1] for the precise proof of property (A). In the proof, we mainly consider F-valued differential forms on Y (not X) and the L^2-norm with respect to h_ε and $\widetilde{\omega}$ (not h and ω).

Let $L^{n,q}_{(2)}(Y,F)_{h_\varepsilon,\widetilde{\omega}}$ be the L^2-space of F-valued (n,q)-forms u on Y with respect to the inner product $\|\bullet\|_{h_\varepsilon,\widetilde{\omega}}$ defined by

$$\|u\|^2_{h_\varepsilon,\widetilde{\omega}} := \int_Y |u|^2_{h_\varepsilon,\widetilde{\omega}}\, dV_{\widetilde{\omega}}.$$

Then, by Proposition 5.8, we obtain the following orthogonal decomposition:

$$L^{n,q}_{(2)}(Y,F)_{h_\varepsilon,\widetilde{\omega}} = \operatorname{Im}\overline{\partial} \oplus \mathcal{H}^{n,q}_{h_\varepsilon,\widetilde{\omega}}(F) \oplus \operatorname{Im}\overline{\partial}^*_{h_\varepsilon}.$$

As explained in subsection 2.3, the operator $\overline{\partial}^*_{h_\varepsilon}$ (resp. $D'^*_{h_\varepsilon}$) denotes the formal adjoint of the densely defined closed operator $\overline{\partial}$ (resp. D'_{h_ε}), and they agree with the Hilbert space adjoints since $\widetilde{\omega}$ is complete. (See [4, §3, Chapter VIII] for a comparison of the formal adjoint and the Hilbert space adjoint.) Strictly speaking, the $\overline{\partial}$-operator also depends on h_ε, $\widetilde{\omega}$ since the domain and range of $\overline{\partial}$ depend on them. We will write it simply as $\overline{\partial}$ when no confusion can arise. Here $\mathcal{H}^{n,q}_{h_\varepsilon,\widetilde{\omega}}(F)$ is the space of harmonic forms with respect to h_ε and $\widetilde{\omega}$, namely

$$\mathcal{H}^{n,q}_{h_\varepsilon,\widetilde{\omega}}(F) = \{u \mid u \text{ is an } F\text{-valued } (n,q)\text{-form with } \overline{\partial}u = 0 \text{ and } \overline{\partial}^*_{h_\varepsilon}u = 0\}.$$

A harmonic form in $\mathcal{H}^{n,q}_{h_\varepsilon,\widetilde{\omega}}(F)$ is smooth by elliptic regularity (for example see [4, (3.2), Theorem, Chapter VIII]). These results seem to be known to specialists. The precise proof for them can be found in [4], [11, Claim 1], and Section 5.

It follows that $|u|^2_{h_\varepsilon,\widetilde{\omega}}\, dV_{\widetilde{\omega}} \leq |u|^2_{h_\varepsilon,\omega}\, dV_\omega$ for an F-valued (n,q)-form u from Lemma 2.5 and property (C), which leads to the inequality $\|u\|_{h_\varepsilon,\widetilde{\omega}} \leq \|u\|_{h_\varepsilon,\omega}$. From this inequality and property (b) of h_ε, we obtain

$$\|u\|_{h_\varepsilon,\widetilde{\omega}} \leq \|u\|_{h_\varepsilon,\omega} \leq \|u\|_{h,\omega} \tag{3.1}$$

for an F-valued (n,q)-form u. These inequalities play a crucial role in the proof. In this paper $\| \bullet \|_{\widetilde{\omega}}$ denotes the L^2-norm on Y (not X) and $\| \bullet \|_\omega$ denotes the L^2-norm on X (not Y) if otherwise mentioned. Strictly speaking $\|u\|_{h_\varepsilon,\widetilde{\omega}}$ is the norm of the restriction of u to Y, but we will omit the notation of the restriction.

For the L^2-space $L^{n,q}_{(2)}(X,F)_{h,\omega}$ of F-valued (n,q)-forms on X with respect to the inner product $\|\bullet\|_{h,\omega}$, we have the standard de Rham-Weil isomorphism

$$H^q(X, K_X \otimes F \otimes \mathcal{I}(h)) \cong \frac{\operatorname{Ker}\overline{\partial}}{\operatorname{Im}\overline{\partial}} \text{ of } L^{n,q}_{(2)}(X,F)_{h,\omega},$$

where the right hand side is the $\overline{\partial}$-cohomology group defined by the closed operator $\overline{\partial}$ between L^2-spaces $L^{n,\bullet}_{(2)}(X,F)_{h,\omega}$. By this isomorphism, we can represent a given cohomology class by an F-valued (n,q)-form u with $\|u\|_{h,\omega} < \infty$. In order to prove that the multiplication map Φ_s is injective, we assume that the cohomology class of su is zero in $H^q(X, K_X \otimes F^{m+1} \otimes \mathcal{I}(h^{m+1}))$. Our final goal is to show that the cohomology class of u is actually zero, that is, $u \in \operatorname{Im}\overline{\partial} \subset L^{n,q}_{(2)}(X,F)_{h,\omega}$.

It follows that $u \in L^{n,q}_{(2)}(Y,F)_{h_\varepsilon,\widetilde{\omega}}$ for every $\varepsilon > 0$ from inequality (3.1). By the above orthogonal decomposition, there exist $u_\varepsilon \in \mathcal{H}^{n,q}_{h_\varepsilon,\widetilde{\omega}}(F)$ and $w_\varepsilon \in \operatorname{Dom}\overline{\partial} \subset L^{n,q-1}_{(2)}(Y,F)_{h_\varepsilon,\widetilde{\omega}}$ such that

$$u = \overline{\partial}w_\varepsilon + u_\varepsilon.$$

Note that the component of $\operatorname{Im}\overline{\partial}^*_{h_\varepsilon}$ is zero since u is $\overline{\partial}$-closed.

At the end of this step, we explain the strategy of the proof. In Step 2, we show that $\|\overline{\partial}^*_{h_\varepsilon^{m+1}} su_\varepsilon\|_{h_\varepsilon^{m+1},\widetilde{\omega}}$ converges to zero as ε tends to zero. We have already known that there is a solution v_ε of the $\overline{\partial}$-equation $\overline{\partial} v_\varepsilon = su_\varepsilon$ since the cohomology class of su is assumed to be zero. However, for our goal, we need L^2-estimates for v_ε. In Step 3, we construct a solution v_ε of the $\overline{\partial}$-equation $\overline{\partial} v_\varepsilon = su_\varepsilon$ such that the norm $\|v_\varepsilon\|_{h_\varepsilon^{m+1},\widetilde{\omega}}$ is uniformly bounded. By Step 2 and Step 3, we can obtain that

$$\|su_\varepsilon\|^2_{h_\varepsilon^{m+1},\widetilde{\omega}} \leq \|\overline{\partial}^*_{h_\varepsilon^{m+1}} su_\varepsilon\|_{h_\varepsilon^{m+1},\widetilde{\omega}} \|v_\varepsilon\|_{h_\varepsilon^{m+1},\widetilde{\omega}} \to 0 \quad \text{as } \varepsilon \to 0.$$

In Step 4, from this convergence, we prove that u_ε converges to zero in a suitable sense, and this completes the proof.

Step 2 (A generalization of Enoki's argument for the injectivity theorem)**.** The aim of this step is to prove the following proposition, which can be seen as a generalization of Enoki's proof of Theorem 1.2.

Proposition 3.2. *As ε tends to zero, the norm $\|\overline{\partial}^*_{h_\varepsilon^{m+1}} su_\varepsilon\|_{h_\varepsilon^{m+1},\widetilde{\omega}}$ converges to zero.*

Proof of Proposition 3.2. The key to prove the proposition is the following inequalities:

$$\|u_\varepsilon\|_{h_\varepsilon,\widetilde{\omega}} \leq \|u\|_{h_\varepsilon,\widetilde{\omega}} \leq \|u\|_{h,\omega} < \infty. \tag{3.2}$$

The first inequality follows from the definition of u_ε and the second inequality follows from inequality (3.1). The important point here is that the right hand side is independent of ε. By applying Proposition 2.4 to u_ε, we obtain

$$0 = \langle\!\langle \sqrt{-1}\Theta_{h_\varepsilon}(F)\Lambda_{\widetilde{\omega}} u_\varepsilon, u_\varepsilon \rangle\!\rangle_{h_\varepsilon,\widetilde{\omega}} + \|D'^*_{h_\varepsilon} u_\varepsilon\|^2_{h_\varepsilon,\widetilde{\omega}} \tag{3.3}$$

since u_ε is harmonic with respect to h_ε and $\widetilde{\omega}$. Let A_ε be the first term and B_ε be the second term of the right hand side of equality (3.3). We first show that the first term A_ε and the second term B_ε converge to zero. Let g_ε be the integrand of A_ε, that is,

$$g_\varepsilon := \langle \sqrt{-1}\Theta_{h_\varepsilon}(F)\Lambda_{\widetilde{\omega}} u_\varepsilon, u_\varepsilon \rangle_{h_\varepsilon,\widetilde{\omega}}.$$

Then there exists a constant $C > 0$ (independent of ε) such that

$$g_\varepsilon \geq -\varepsilon C |u_\varepsilon|^2_{h_\varepsilon,\widetilde{\omega}}. \tag{3.4}$$

This inequality follows from simple computations. Indeed, let $\lambda_1^\varepsilon \leq \lambda_2^\varepsilon \leq \cdots \leq \lambda_n^\varepsilon$ be the eigenvalues of $\sqrt{-1}\Theta_{h_\varepsilon}(F)$ with respect to $\widetilde{\omega}$. For every point $y \in Y$, there exists a local coordinate $(z_1, z_2, \ldots, z_n)$ centered at y such that

$$\sqrt{-1}\Theta_{h_\varepsilon}(F) = \frac{\sqrt{-1}}{2}\sum_{j=1}^n \lambda_j^\varepsilon dz_j \wedge d\overline{z_j} \quad \text{and} \quad \widetilde{\omega} = \frac{\sqrt{-1}}{2}\sum_{j=1}^n dz_j \wedge d\overline{z_j} \quad \text{at } y.$$

When we locally write u_ε as $u_\varepsilon = \sum_{|K|=q} u^\varepsilon_K \, dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_K$, we can easily see that

$$g_\varepsilon = \sum_{|K|=q} \Big(\sum_{j \in K} \lambda^\varepsilon_j \Big) |u^\varepsilon_K|^2_{h_\varepsilon}$$

by straightforward computations. On the other hand, from property (C) of $\widetilde{\omega}$ and property (d) of h_ε, we have $\sqrt{-1}\Theta_{h_\varepsilon}(F) \geq -\varepsilon\omega \geq -\varepsilon\widetilde{\omega}$. It leads to $\lambda^\varepsilon_j \geq -\varepsilon$, and thus we obtain inequality (3.4). From inequality (3.4) and equality (3.3), we have

$$0 \geq A_\varepsilon = \int_Y g_\varepsilon \, dV_{\widetilde{\omega}} \geq -\varepsilon C \int_Y |u_\varepsilon|^2_{h_\varepsilon, \widetilde{\omega}} \, dV_{\widetilde{\omega}} \geq -\varepsilon C \|u\|^2_{h,\omega}.$$

The last inequality follows from inequality (3.2). Therefore A_ε converges to zero. Further it follows that B_ε also converges to zero from equality (3.3).

To apply Proposition 2.4 to su_ε, we need to prove that $su_\varepsilon \in L^{n,q}_{(2)}(Y, F^{m+1})_{h^{m+1}_\varepsilon, \widetilde{\omega}}$ and $su_\varepsilon \in \operatorname{Dom} \overline{\partial}^*_{h^{m+1}_\varepsilon}$. It can be proven that $su_\varepsilon \in \operatorname{Dom} \overline{\partial}^*_{h^{m+1}_\varepsilon}$ from [23, Proposition 2.2]. By the assumption, the point-wise norm $|s|_{h^m}$ with respect to h^m is bounded. Further we have $|s|_{h^m_\varepsilon} \leq |s|_{h^m}$ from property (b) of h_ε. They imply

$$\|su_\varepsilon\|_{h^{m+1}_\varepsilon, \widetilde{\omega}} \leq \sup_X |s|_{h^m_\varepsilon} \|u_\varepsilon\|_{h_\varepsilon, \widetilde{\omega}} \leq \sup_X |s|_{h^m} \|u\|_{h,\omega} < \infty.$$

Hence we know $su_\varepsilon \in L^{n,q}_{(2)}(Y, F^{m+1})_{h^{m+1}_\varepsilon, \widetilde{\omega}}$. Note that the right hand side is independent of ε. By applying Proposition 2.4 to su_ε, we obtain

$$\begin{aligned} &\|\overline{\partial}^*_{h^{m+1}_\varepsilon} su_\varepsilon\|^2_{h^{m+1}_\varepsilon, \widetilde{\omega}} \\ =&\langle\!\langle \sqrt{-1}\Theta_{h^{m+1}_\varepsilon}(F^{m+1})\Lambda_{\widetilde{\omega}} su_\varepsilon, su_\varepsilon \rangle\!\rangle_{h^{m+1}_\varepsilon, \widetilde{\omega}} + \|D'^*_{h^{m+1}_\varepsilon} su_\varepsilon\|^2_{h^{m+1}_\varepsilon, \widetilde{\omega}}. \end{aligned} \tag{3.5}$$

Here we used $\overline{\partial} su_\varepsilon = s\overline{\partial} u_\varepsilon = 0$. From now on, we prove that the second term of the right hand side converges to zero. It is easy to see that $D'^*_{h^{m+1}_\varepsilon} su_\varepsilon = sD'^*_{h_\varepsilon} u_\varepsilon$ holds since s is a holomorphic section and $D'^* = - * \overline{\partial} *$, where $*$ is the Hodge star operator with respect to $\widetilde{\omega}$. Therefore we have

$$\|D'^*_{h^{m+1}_\varepsilon} su_\varepsilon\|^2_{h^{m+1}_\varepsilon, \widetilde{\omega}} \leq \sup_X |s|^2_{h^m_\varepsilon} \int_Y |D'^*_{h_\varepsilon} u_\varepsilon|^2_{h_\varepsilon, \widetilde{\omega}} \, dV_{\widetilde{\omega}} \leq \sup_X |s|^2_{h^m} B_\varepsilon.$$

Since $|s|^2_{h^m}$ is bounded and B_ε converges to zero, the second term $\|D'^*_{h^{m+1}_\varepsilon} su_\varepsilon\|^2_{h^{m+1}_\varepsilon, \widetilde{\omega}}$ converges to zero.

For the proof of the proposition, it remains to show that the first term of the right hand side of equality (3.5) converges to zero. Now we investigate A_ε in detail. By the definition of A_ε, we have

$$A_\varepsilon = \int_{\{g_\varepsilon \geq 0\}} g_\varepsilon \, dV_{\widetilde{\omega}} + \int_{\{g_\varepsilon \leq 0\}} g_\varepsilon \, dV_{\widetilde{\omega}}.$$

Let A^{+}_{ε} be the first term and A^{-}_{ε} be the second term of the right hand side. Then inequalities (3.2) and (3.4) lead to

$$0 \geq A^{-}_{\varepsilon} \geq -\varepsilon C \int_{\{g_\varepsilon \leq 0\}} |u_\varepsilon|^2_{h_\varepsilon,\widetilde{\omega}}\, dV_{\widetilde{\omega}} \geq -\varepsilon C \int_{Y} |u_\varepsilon|^2_{h_\varepsilon,\widetilde{\omega}}\, dV_{\widetilde{\omega}} \geq -\varepsilon C \|u\|^2_{h,\omega}.$$

Hence A^{+}_{ε} and A^{-}_{ε} converge to zero since $A_\varepsilon = A^{+}_{\varepsilon} + A^{-}_{\varepsilon}$ converges to zero. On the other hand, we have

$$\begin{aligned}
&\langle\!\langle \sqrt{-1}\Theta_{h_\varepsilon^{m+1}}(F^{m+1})\Lambda_{\widetilde{\omega}} s u_\varepsilon, s u_\varepsilon \rangle\!\rangle_{h_\varepsilon^{m+1},\widetilde{\omega}} \\
=&(m+1)\int_Y |s|^2_{h_\varepsilon^m} g_\varepsilon\, dV_{\widetilde{\omega}} \\
=&(m+1)\Big\{ \int_{\{g_\varepsilon \geq 0\}} |s|^2_{h_\varepsilon^m} g_\varepsilon\, dV_{\widetilde{\omega}} + \int_{\{g_\varepsilon \leq 0\}} |s|^2_{h_\varepsilon^m} g_\varepsilon\, dV_{\widetilde{\omega}} \Big\}.
\end{aligned}$$

Then it is easy to see the following inequalities:

- $$0 \leq \int_{\{g_\varepsilon \geq 0\}} |s|^2_{h_\varepsilon^m} g_\varepsilon\, dV_{\widetilde{\omega}} \leq \sup_X |s|^2_{h_\varepsilon^m} \int_{\{g_\varepsilon \geq 0\}} g_\varepsilon\, dV_{\widetilde{\omega}} \leq \sup_X |s|^2_{h^m}\ A^{+}_{\varepsilon},$$
- $$0 \geq \int_{\{g_\varepsilon \leq 0\}} |s|^2_{h_\varepsilon^m} g_\varepsilon\, dV_{\widetilde{\omega}} \geq \sup_X |s|^2_{h_\varepsilon^m} \int_{\{g_\varepsilon \leq 0\}} g_\varepsilon\, dV_{\widetilde{\omega}} \geq \sup_X |s|^2_{h^m} A^{-}_{\varepsilon}.$$

Therefore the right hand side of equality (3.5) converges to zero. We obtain the conclusion of Proposition 3.2. □

Step 3 (A construction of solutions of the $\overline{\partial}$-equation)**.** In this step, we prove Proposition 3.4 by using Theorem 5.9. The proof of Theorem 5.9 is given in Section 5.

Proposition 3.3. *There exists an F-valued $(n, q-1)$-form w_ε on Y with the following properties:*

(1) $\overline{\partial} w_\varepsilon = u - u_\varepsilon$. (2) *The norm $\|w_\varepsilon\|_{h_\varepsilon,\widetilde{\omega}}$ is uniformly bounded.*

Proof. It is easy to see that $U_\varepsilon := u - u_\varepsilon$ satisfies the assumptions of Theorem 5.9. Indeed, it follows that

$$\|U_\varepsilon\|_{h_\varepsilon,\widetilde{\omega}} \leq \|u\|_{h_\varepsilon,\widetilde{\omega}} + \|u_\varepsilon\|_{h_\varepsilon,\widetilde{\omega}} \leq 2\|u\|_{h,\omega} < \infty$$

from inequality (3.2) and that $U_\varepsilon = u - u_\varepsilon \in \operatorname{Im} \overline{\partial} \subset L^{n,q}_{(2)}(Y, F)_{h_\varepsilon,\widetilde{\omega}}$ from the definition of u_ε. □

Proposition 3.4. *There exists an F^{m+1}-valued $(n, q-1)$-form v_ε on Y with the following properties:*

(1) $\overline{\partial} v_\varepsilon = s u_\varepsilon$. (2) *The norm $\|v_\varepsilon\|_{h_\varepsilon^{m+1},\widetilde{\omega}}$ is uniformly bounded.*

Proof. Since the cohomology class of su is assumed to be zero in $H^q(X, K_X \otimes F^{m+1} \otimes \mathcal{I}(h^{m+1}))$, there exists an F^{m+1}-valued $(n, q-1)$-form v such that $\overline{\partial} v = su$ and $\|v\|_{h^{m+1},\omega} < \infty$. If we take w_ε satisfying the properties in Proposition 3.3 and put $v_\varepsilon := -sw_\varepsilon + v$, then we have $\overline{\partial} v_\varepsilon = su_\varepsilon$. Further an easy computation yields

$$\begin{aligned}\|v_\varepsilon\|_{h_\varepsilon^{m+1},\widetilde{\omega}} &\leq \|sw_\varepsilon\|_{h_\varepsilon^{m+1},\widetilde{\omega}} + \|v\|_{h_\varepsilon^{m+1},\widetilde{\omega}} \\ &\leq \sup_X |s|_{h^m} \|w_\varepsilon\|_{h_\varepsilon,\widetilde{\omega}} + \|v\|_{h^{m+1},\widetilde{\omega}}.\end{aligned}$$

By Lemma 2.5 and property (B), we have $\|v\|_{h^{m+1},\widetilde{\omega}} \leq \|v\|_{h^{m+1},\omega} < \infty$. Since the norm $\|w_\varepsilon\|_{h_\varepsilon,\widetilde{\omega}}$ is uniformly bounded, the right hand side can be estimated by a constant independent of ε. □

Step 4 (Limit of the harmonic forms)**.** In this step, we show that u_ε converges to zero in a suitable sense and this completes the proof. We first consider the following proposition obtained by Step 2 and Step 3.

Proposition 3.5. *As ε tends to zero, the norm $\|su_\varepsilon\|_{h_\varepsilon^{m+1},\widetilde{\omega}}$ converges to zero.*

Proof. For $v_\varepsilon \in L^{n,q-1}_{(2)}(Y, F^{m+1})_{h_\varepsilon^{m+1},\widetilde{\omega}}$ satisfying the properties in Proposition 3.4, it is easy to see that

$$\begin{aligned}\|su_\varepsilon\|^2_{h_\varepsilon^{m+1},\widetilde{\omega}} &= \langle\!\langle su_\varepsilon, \overline{\partial} v_\varepsilon \rangle\!\rangle_{h_\varepsilon^{m+1},\widetilde{\omega}} \\ &= \langle\!\langle \overline{\partial}^*_{h_\varepsilon^{m+1}} su_\varepsilon, v_\varepsilon \rangle\!\rangle_{h_\varepsilon^{m+1},\widetilde{\omega}} \\ &\leq \|\overline{\partial}^*_{h_\varepsilon^{m+1}} su_\varepsilon\|_{h_\varepsilon^{m+1},\widetilde{\omega}} \|v_\varepsilon\|_{h_\varepsilon^{m+1},\widetilde{\omega}}.\end{aligned}$$

The norm $\|v_\varepsilon\|_{h_\varepsilon^{m+1},\widetilde{\omega}}$ is uniformly bounded by Proposition 3.4. On the other hand, the norm $\|\overline{\partial}^*_{h_\varepsilon^{m+1}} su_\varepsilon\|_{h_\varepsilon^{m+1},\widetilde{\omega}}$ converges to zero by Proposition 3.2. Hence the norm $\|su_\varepsilon\|_{h_\varepsilon^{m+1},\widetilde{\omega}}$ converges to zero. □

We want to take the limit of $u_\varepsilon \in L^{n,q}_{(2)}(Y, F)_{h_\varepsilon,\widetilde{\omega}}$, but the L^2-space $L^{n,q}_{(2)}(Y, F)_{h_\varepsilon,\widetilde{\omega}}$ depends on ε. For this reason we fix a small number $\varepsilon_0 > 0$ and consider the fixed L^2-space $L^{n,q}_{(2)}(Y, F)_{h_{\varepsilon_0},\widetilde{\omega}}$. For every number ε with $0 < \varepsilon < \varepsilon_0$, we obtain

$$\|u_\varepsilon\|_{h_{\varepsilon_0},\widetilde{\omega}} \leq \|u_\varepsilon\|_{h_\varepsilon,\widetilde{\omega}} \leq \|u\|_{h,\omega}$$

by property (b) of h_ε and inequality (3.2), which says that the norm of u_ε with respect to h_{ε_0} is uniformly bounded. In particular, there exists a subsequence of $\{u_\varepsilon\}_{\varepsilon>0}$ that converges to some $u_0 \in L^{n,q}_{(2)}(Y, F)_{h_{\varepsilon_0},\widetilde{\omega}}$ with respect to the weak L^2-topology. For simplicity we continue to use the same notation $\{u_\varepsilon\}_{\varepsilon>0}$ for this subsequence. The following proposition is proved by Proposition 3.5.

Proposition 3.6. *The weak limit u_0 of $\{u_\varepsilon\}_{\varepsilon>0}$ in $L^{n,q}_{(2)}(Y, F)_{h_{\varepsilon_0},\widetilde{\omega}}$ is zero.*

Proof. For every positive number $\delta > 0$, we define the subset Y_δ by $Y_\delta := \{x \in Y \mid |s|^2_{h^m_{\varepsilon_0}} > \delta \text{ at } x.\}$. Since the weight φ_{ε_0} of h_{ε_0} is upper semi-continuous, the norm $|s|^2_{h^m_{\varepsilon_0}}$ is lower semi-continuous. In particular, the subset Y_δ is an open set of Y. A simple computation yields

$$\|su_\varepsilon\|^2_{h^{m+1}_\varepsilon,\widetilde{\omega}} \geq \|su_\varepsilon\|^2_{h^{m+1}_{\varepsilon_0},\widetilde{\omega}} \geq \int_{Y_\delta} |s|^2_{h^m_{\varepsilon_0}} |u_\varepsilon|^2_{h_{\varepsilon_0},\widetilde{\omega}}\, dV_{\widetilde{\omega}} \geq \delta \int_{Y_\delta} |u_\varepsilon|^2_{h_{\varepsilon_0},\widetilde{\omega}}\, dV_{\widetilde{\omega}} \geq 0$$

for every $\delta > 0$. Since the left hand side converges to zero by Proposition 3.5, the norm $\|u_\varepsilon\|_{Y_\delta, h_{\varepsilon_0},\widetilde{\omega}}$ on Y_δ also converges to zero as ε tends to zero. We can easily see that $u_\varepsilon|_{Y_\delta}$ converges to $u_0|_{Y_\delta}$ with respect to the weak L^2-topology in $L^{n,q}_{(2)}(Y_\delta, F)_{h_{\varepsilon_0},\widetilde{\omega}}$. Here $u_\varepsilon|_{Y_\delta}$ (resp. $u_0|_{Y_\delta}$) denotes the restriction of u_ε (resp. u_0) to Y_δ. Indeed, for an arbitrary $w \in L^{n,q}_{(2)}(Y_\delta, F)_{h_{\varepsilon_0},\widetilde{\omega}}$, the inner product $\langle\!\langle u_\varepsilon|_{Y_\delta}, w\rangle\!\rangle_{Y_\delta} = \langle\!\langle u_\varepsilon, \widetilde{w}\rangle\!\rangle_Y$ converges to $\langle\!\langle u_0, \widetilde{w}\rangle\!\rangle_Y = \langle\!\langle u_0|_{Y_\delta}, w\rangle\!\rangle_{Y_\delta}$, where $\widetilde{w}$ denotes the zero extension of w to Y. Since $u_\varepsilon|_{Y_\delta}$ weakly converges to $u_0|_{Y_\delta}$ and the norm is lower semi-continuous with respect to the weak L^2-topology, we obtain

$$\|u_0|_{Y_\delta}\|_{Y_\delta, h_{\varepsilon_0},\widetilde{\omega}} \leq \liminf_{\varepsilon \to 0} \|u_\varepsilon|_{Y_\delta}\|_{Y_\delta, h_{\varepsilon_0},\widetilde{\omega}} = 0.$$

Hence $u_0|_{Y_\delta} = 0$ for every $\delta > 0$. Since the union of $\{Y_\delta\}_{\delta > 0}$ is equal to $Y = X \setminus Z$ by the definition of Y_δ, the weak limit u_0 is zero on Y. □

By using Proposition 3.6, we complete the proof of Theorem 3.1. By the definition of u_ε, we have

$$u = u_\varepsilon + \overline{\partial} w_\varepsilon.$$

Proposition 3.6 asserts that $\overline{\partial} w_\varepsilon$ converges to u with respect to the weak L^2-topology in the fixed L^2-space. By the orthogonal decomposition, it is easy to show that u is a $\overline{\partial}$-exact form in the fixed L^2-space (that is, $u \in \operatorname{Im} \overline{\partial} \subset L^{n,q}_{(2)}(Y, F)_{h_{\varepsilon_0},\widetilde{\omega}}$). Indeed, for every $w = w_1 + \overline{\partial}^*_{h_{\varepsilon_0}} w_2 \in \mathcal{H}^{n,q}_{h_{\varepsilon_0},\widetilde{\omega}}(F) \oplus \operatorname{Im} \overline{\partial}^*_{h_{\varepsilon_0}}$, we have $\langle\!\langle u, w\rangle\!\rangle = \lim_{\varepsilon \to 0} \langle\!\langle \overline{\partial} w_\varepsilon, w_1 + \overline{\partial}^*_{h_{\varepsilon_0}} w_2\rangle\!\rangle = 0$.

From $u \in \operatorname{Im} \overline{\partial} \subset L^{n,q}_{(2)}(Y, F)_{h_{\varepsilon_0},\widetilde{\omega}}$ and property (c), we can show that $u \in \operatorname{Im} \overline{\partial} \subset L^{n,q}_{(2)}(Y, F)_{h,\omega}$, which says that the cohomology class $\{u\}$ is zero. To clarify our argument, let $\overline{\partial}_{h,\omega}$ (resp. $\overline{\partial}_{h_{\varepsilon_0},\widetilde{\omega}}$) be the closed operator $\overline{\partial}$ between L^2-spaces $L^{n,\bullet}_{(2)}(X, F)_{h,\omega}$ (resp. $L^{n,\bullet}_{(2)}(Y, F)_{h_{\varepsilon_0},\widetilde{\omega}}$). We consider the Dolbeault cohomology group and the de Rham-Weil isomorphism. Then we have the following commutative diagram:

$$\begin{array}{ccc}
\dfrac{\operatorname{Ker} \overline{\partial}_{h,\omega}}{\operatorname{Im} \overline{\partial}_{h,\omega}} & \xrightarrow{\quad j \quad} & \dfrac{\operatorname{Ker} \overline{\partial}_{h_{\varepsilon_0},\widetilde{\omega}}}{\operatorname{Im} \overline{\partial}_{h_{\varepsilon_0},\widetilde{\omega}}} \\
\overline{f}_1 \Big\downarrow \cong & & \overline{f}_2 \Big\downarrow \cong \\
\check{H}^q(X, K_X \otimes F \otimes \mathcal{I}(h)) & = & \check{H}^q(X, K_X \otimes F \otimes \mathcal{I}(h_{\varepsilon_0})).
\end{array}$$

Here j is the map induced by the natural map from $L^{n,\bullet}_{(2)}(X,F)_{h,\omega}$ to $L^{n,\bullet}_{(2)}(Y,F)_{h_{\varepsilon_0},\widetilde{\omega}}$, and $\overline{f}_i$ is the de Rham-Weil isomorphism to the Čech cohomology group. (See Section 5 or [11, Claim 1] for the construction of $\overline{f}_2$.) The below equality is obtained from $\mathcal{I}(h_\varepsilon) = \mathcal{I}(h)$. Here we essentially used property (c). It follows that the cohomology class $\{u\}$ represented by $u \in L^{n,q}_{(2)}(X,F)_{h,\omega}$ goes to zero by j from $u \in \operatorname{Im}\overline{\partial} \subset L^{n,q}_{(2)}(Y,F)_{h_{\varepsilon_0},\widetilde{\omega}}$. Therefore we can obtain that $u \in \operatorname{Im}\overline{\partial} \subset L^{n,q}_{(2)}(Y,F)_{h,\omega}$ by chasing the above diagram. □

4. Applications

In this section, we give two corollaries of Theorem 1.3 and their proof. One is an injectivity theorem for nef and abundant line bundles, and the other is a Nadel type vanishing theorem.

It is reasonable to expect the same conclusion as in Theorem 1.1 to hold for nef line bundles, but there exist counterexamples to the injectivity theorem for nef line bundles. However, it follows from [18, Proposition 2.1] (cf. [25], [28, Corollary 1]) that a metric $h_{\min}$ with minimal singularities on F satisfies $\mathcal{I}(h_{\min}^m) = \mathcal{O}_X$ for any $m > 0$ if F is nef and abundant (that is, the numerical dimension agrees with the Kodaira dimension). Therefore Theorem 1.3 leads to the following corollary. (On projective varieties, a similar conclusion was proved in [9] and [10] by different methods.) It is worth pointing out that Theorem 1.2 is not sufficient to obtain Corollary 4.1. This is because the above metric $h_{\min}$ is not smooth and does not always have algebraic singularities even if F is nef and abundant (for example, see [12, Example 5.2]).

Corollary 4.1. *Let F be a nef and abundant line bundle on a compact Kähler manifold X. Then the same conclusion as in Theorem* 1.1 *holds. That is, for a* (*non-zero*) *section s of a positive multiple F^m of the line bundle F, the multiplication map induced by the tensor product with s,*

$$\Phi_s : H^q(X, K_X \otimes F) \xrightarrow{\otimes s} H^q(X, K_X \otimes F^{m+1}),$$

is injective for any q.

As another application, we can obtain a Nadel type vanishing (Theorem 4.5), which leads to the following corollary.

Corollary 4.2 (cf. [1], [24])**.** *Let F be a line bundle on a smooth projective variety X and $h_{\min}$ be a metric with minimal singularities on F. Then*

$$H^q(X, K_X \otimes F \otimes \mathcal{I}(h_{\min})) = 0 \quad \textit{for any } q > n - \kappa(F).$$

Here $\kappa(F)$ denotes the Kodaira dimension of F.

This result is non-trivial even when the line bundle F is big (that is, $\kappa(F) = n$). In his paper [1], Cao proved the celebrated vanishing theorem for cohomology groups with coefficients in $K_X \otimes F \otimes \mathcal{I}_+(h)$. It is relatively easier to handle $\mathcal{I}_+(h)$ than $\mathcal{I}(h)$ (see [2] for the precise definition). If $h_{\min}$ has algebraic singularities, we can easily see that $\mathcal{I}_+(h_{\min})$ agrees with $\mathcal{I}(h_{\min})$, but unfortunately $h_{\min}$ does not always have algebraic singularities. Thanks to Theorem 1.3, we can obtain Corollary 4.2 without the assumption of algebraic singularities.

Remark 4.3. Three months after we finished writing our preprint, Guan and Zhou proved the strong openness conjecture in [15]. Other proofs were given by Hiệp in [16] and by Lempert in [20]. Although their celebrated result and Cao's theorem lead to Corollary 4.2, we believe that it is worth displaying our techniques. This is because our techniques are quite different from theirs and give a new viewpoint to prove the vanishing theorem via the asymptotic vanishing theorem.

At the end of this section, we prove Theorem 4.5 by using Theorem 1.3. First we give the following definition.

Definition 4.4. Let F be a line bundle on a compact complex manifold X and h be a singular metric on F.

(1) We define $H^0_{\mathrm{bdd},h}(X, F)$ by the space of sections of F with bounded norm with respect to h. That is,

$$H^0_{\mathrm{bdd},h}(X, F) := \{s \in H(X, F) \mid \sup_X |s|_h < \infty\}.$$

(2) The *generalized Kodaira dimension* $\kappa_{\mathrm{bdd}}(F, h)$ of (F, h) is defined to be $-\infty$ if $H^0_{\mathrm{bdd},h^m}(X, F^m) = 0$ for any $m > 0$. Otherwise, $\kappa_{\mathrm{bdd}}(F, h)$ is defined by

$$\kappa_{\mathrm{bdd}}(F, h) := \sup\{k \in \mathbb{Z} \mid \limsup_{m\to\infty} \dim H^0_{\mathrm{bdd},h^m}(X, F^m)/m^k > 0\}.$$

If $h_{\min}$ is a metric with minimal singularities on F, the norm $|s|_{h^m_{\min}}$ is bounded on X for any section $s \in H^0(X, F^m)$. (For example see [3] or [24].) It implies that $H^0_{\mathrm{bdd},h^m_{\min}}(X, F^m)$ is isomorphic to $H^0(X, F^m)$ for every $m \geq 0$. In particular, $\kappa_{\mathrm{bdd}}(F, h_{\min})$ agrees with the usual Kodaira dimension $\kappa(F)$. Therefore the following theorem leads to Corollary 4.2.

Theorem 4.5. *Let F be a line bundle on a smooth projective variety X and h be a singular metric on F with semi-positive curvature. Then*

$$H^q(X, K_X \otimes F \otimes \mathcal{I}(h)) = 0 \quad \textit{for any } q > n - \kappa_{\mathrm{bdd}}(F, h).$$

Proof. For a contradiction, we assume that there exists a non-zero cohomology class $\alpha \in H^q(X, K_X \otimes F \otimes \mathcal{I}(h))$. If sections $\{s_i\}_{i=1}^N$ in $H^0_{\mathrm{bdd},h^m}(X, F^m)$ are linearly independent, then $\{s_i\alpha\}_{i=1}^N$ are also linearly independent in

$H^q(X, K_X \otimes F^{m+1} \otimes \mathcal{I}(h^{m+1}))$. Indeed, if $\sum_{i=1}^N c_i s_i \alpha = 0$ for some $c_i \in \mathbb{C}$, then we obtain $\sum_{i=1}^N c_i s_i = 0$ by Theorem 1.3. Since $\{s_i\}_{i=1}^N$ are linearly independent, we have $c_i = 0$ for every $i = 1, 2, \dots, N$. Therefore we obtain

$$\dim H^0_{\mathrm{bdd}, h^m}(X, F^m) \leq \dim H^q(X, K_X \otimes F^{m+1} \otimes \mathcal{I}(h^{m+1})).$$

On the other hand, by [22, Theorem 4.1], we have the asymptotic vanishing theorem

$$\dim H^q(X, K_X \otimes F^m \otimes \mathcal{I}(h^m)) = O(m^{n-q}) \quad \text{as } m \to \infty$$

for any $q \geq 0$ (cf. [3, (6.18), Lemma]). If $q > n - \kappa_{\mathrm{bdd}}(F, h)$, it is a contradiction. □

5. Čech complex and de Rham-Weil isomorphism

The aim of this section is to prove Theorem 5.9, which gives solutions of the $\overline{\partial}$-equation with suitable L^2-estimates in Step 3 of Section 3. The Čech complex and the de Rham-Weil isomorphism play a crucial role in the proof of Theorem 5.9.

5.1. On the space of cochains. In this subsection, for the proof of Theorem 5.3, we study the space of cochains with the topology induced by the local L^2-norms with respect to singular metrics, which is used when we prove Theorem 5.9. We first recall the following result on holomorphic functions, which can be proved by the division theorem. See [14, Section D, Chapter II] for the proof.

Theorem 5.1 ([14, Theorem 2, Section D, Chapter II]). *Let $G_1, G_2, \dots, G_N$ be holomorphic functions on an open set B in $\mathbb{C}^n$. If holomorphic functions $\{G_i\}_{i=1}^N$ generate the stalk $\mathcal{I}_p$ at $p \in B$ of an ideal sheaf $\mathcal{I} \subset \mathcal{O}_B$, then there exist a neighborhood $L_p \Subset B$ of p and a constant $C_p > 0$ with the following property:*

For every holomorphic function F on L_p whose germ at p belongs to $\mathcal{I}_p$, there exist holomorphic functions $\{h_j\}_{j=1}^N$ on L_p such that

$$F = \sum_{j=1}^N h_j G_j \quad \textit{and} \quad \sup_{L_p} |h_j| \leq C_p \sup_{L_p} |F|.$$

This theorem leads to the following lemma. In his paper [1], Cao proved the former conclusion of the lemma when a quasi-psh function φ has analytic singularities. For our purpose, we need a generalization of his result and the stronger conclusion (the latter conclusion of the lemma).

Lemma 5.2. *Let φ be a quasi-psh function on an open set B in $\mathbb{C}^n$ and $G_1, G_2, \ldots, G_N$ be holomorphic functions on B that generate the stalk of the multiplier ideal sheaf $\mathcal{I}(\varphi)$ at every point in B. Consider a sequence of holomorphic functions $\{f_k\}_{k=1}^{\infty}$ satisfying the following properties:*

(1) *f_k belongs to $H^0(B, \mathcal{I}(\varphi))$ (that is, $|f_k|e^{-\varphi}$ is locally L^2-integrable on B).*
(2) *$\{f_k\}_{k=1}^{\infty}$ uniformly converges to f on every relatively compact set in B.*

Then the limit f belongs to $H^0(B, \mathcal{I}(\varphi))$. Moreover, for every relatively compact set $K \Subset B$, the (local) L^2-norm

$$\int_K |f_k - f|^2 e^{-2\varphi}$$

converges to zero as k tends to infinity.

Proof. For an arbitrary point $p \in B$, there exist a neighborhood $L_p \Subset B$ of p and a positive constant C_p with the property in Theorem 5.1. Since the germ of f_k belongs to the stalk $\mathcal{I}(\varphi)_p$, there exist holomorphic functions $\{h_{k,j}\}_{j=1}^{N}$ on L_p such that

$$f_k = \sum_{j=1}^{N} h_{k,j} G_j \quad \text{and} \quad \sup_{L_p} |h_{k,j}| \leq C_p \sup_{L_p} |f_k|.$$

The sup-norm $\sup_{L_p} |f_k|$ on L_p is uniformly bounded by property (2). The above inequality implies that the sup-norm $\sup_{L_p} |h_{k,j}|$ is also uniformly bounded, and thus by Montel's theorem there exists a subsequence $\{h_{k_\ell,j}\}_{\ell=1}^{\infty}$ that uniformly converges to a holomorphic function h_j on every relatively compact set in L_p. For every point x in L_p we have

$$f(x) = \lim_{\ell \to \infty} f_{k_\ell}(x) = \lim_{\ell \to \infty} \sum_{j=1}^{N} h_{k_\ell,j}(x) G_j(x) = \sum_{j=1}^{N} h_j(x) G_j(x).$$

Therefore the germ of f belongs to $\mathcal{I}(\varphi)_p$ since the germ of G_j belongs to $\mathcal{I}(\varphi)_p$.

Finally, we prove the latter conclusion. We have already known that the germ of $f_k - f$ belongs to $\mathcal{I}(\varphi)_p$. By Theorem 5.1, there exist a relatively compact set $L_p \Subset B$, a positive constant C_p, and holomorphic functions $\{g_{k,j}\}_{j=1}^{N}$ on L_p such that

$$f_k - f = \sum_{j=1}^{N} g_{k,j} G_j \quad \text{and} \quad \sup_{L_p} |g_{k,j}| \leq C_p \sup_{L_p} |f_k - f| \to 0.$$

On the other hand, an easy computation yields

$$\begin{aligned}\int_{L_p} |f_k - f|^2 e^{-2\varphi} &\leq \int_{L_p} \Big(\sum_{j=1}^{N} |g_{k,j}|^2\Big)\Big(\sum_{j=1}^{N} |G_j|^2\Big) e^{-2\varphi} \\ &\leq \Big(\sum_{j=1}^{N} \sup_{L_p} |g_{k,j}|^2\Big) \int_{L_p} \sum_{j=1}^{N} |G_j|^2 e^{-2\varphi}.\end{aligned}$$

The right hand side converges to zero since the integral of $|G_j|^2 e^{-2\varphi}$ is finite and $g_{k,j}$ uniformly converges to zero on L_p. For a given relatively compact set $K \Subset B$, by taking a finite cover $\{L_{p_\nu}\}_{\nu=1}^m$ of K, we can see that

$$\int_K |f_k - f|^2 e^{-2\varphi} \leq \sum_{\nu=1}^{m} \int_{L_{p_\nu}} |f_k - f|^2 e^{-2\varphi} \to 0.$$

This completes the proof. $\square$

To prove Theorem 5.3, we recall the notation on the space of cochains. Let G be a line bundle on a complex manifold X and h be a singular metric on G satisfying $\sqrt{-1}\Theta_h(G) \geq \gamma$ for some smooth $(1,1)$-form γ. We take a Stein cover $\mathcal{U} := \{B_i\}_{i \in I}$ of X with the following properties:

- G admits a local trivialization on B_i.
- There are holomorphic functions on B_i that generate the stalk of the multiplier ideal sheaf $\mathcal{I}(h)$ at every point in B_i.

Note that we can take such an open cover since the multiplier ideal sheaf $\mathcal{I}(h)$ is coherent by a theorem of Nadel. Let $C^q(\mathcal{U}, G \otimes \mathcal{I}(h))$ be the space of q-cochains with coefficients in $G \otimes \mathcal{I}(h)$. For a q-cochain $\alpha = \{\alpha_{i_0 \ldots i_q}\}_{i_0 \ldots i_q} \in C^q(\mathcal{U}, G \otimes \mathcal{I}(h))$, we often omit the notation of the subscript, such as "$i_0 \ldots i_q$", and regard $\alpha_{i_0 \ldots i_q}$ as a holomorphic function under the trivialization of G on B_i. The semi-norm $p_{K_{i_0 \ldots i_q}}(\bullet)$ is defined by

$$p_{K_{i_0 \ldots i_q}}(\alpha)^2 := \int_{K_{i_0 \ldots i_q}} |\alpha_{i_0 \ldots i_q}|_h^2$$

for a relatively compact set $K_{i_0 \ldots i_q} \Subset B_{i_0 \ldots i_q} := B_{i_0} \cap \cdots \cap B_{i_q}$. At the end of this section, we show that $C^q(\mathcal{U}, G \otimes \mathcal{I}(h))$ is a Fréchet space with respect to these semi-norms.

Theorem 5.3. *In the above situation, the space of q-cochains $C^q(\mathcal{U}, G \otimes \mathcal{I}(h))$ is a Fréchet space.*

Proof. For a given Cauchy sequence $\big\{\{\alpha_{k, i_0 \ldots i_q}\}\big\}_{k=1}^{\infty}$ in $C^q(\mathcal{U}, G \otimes \mathcal{I}(h))$, we put $\alpha_k := \alpha_{k, i_0 \ldots i_q}$ and $B := B_{i_0 \ldots i_q}$. Further we regard α_k as a holomorphic function on B. For the proof, it is sufficient to show that there exists a holomorphic function α on B such that

$$\int_K |\alpha_k - \alpha|_h^2 \to 0$$

for every relatively compact set $K \Subset B$.

Since $\{\alpha_k\}_{k=1}^{\infty}$ is a Cauchy sequence with respect to the semi-norms, the L^2-norm $\int_K |\alpha_k|_h^2$ of α_k on K is uniformly bounded. Since the local weight φ of h is quasi-psh, φ is upper semi-continuous. In particular φ is bounded above, and thus the L^2-norm $\int_K |\alpha|^2\, dV_\omega$ is also uniformly bounded. By Montel's theorem, there exists a subsequence $\{\alpha_{k_\ell}\}_{\ell=1}^{\infty}$ of $\{\alpha_k\}_{k=1}^{\infty}$ that uniformly converges to a holomorphic function α on every relatively compact set in B. Since this subsequence $\{\alpha_{k_\ell}\}_{\ell=1}^{\infty}$ satisfies the assumptions of Lemma 5.2, it can be shown that the limit α also belongs to $\mathcal{I}(h)$. Moreover, we have

$$p_K(\alpha_{k_\ell} - \alpha) = \int_K |\alpha_{k_\ell} - \alpha|_h^2 \to 0$$

for every relatively compact set $K \Subset B$. Since $\{\alpha_k\}_{k=1}^{\infty}$ is a Cauchy sequence, the semi-norm $p_K(\alpha_k - \alpha)$ also converges to zero. □

5.2. De Rham-Weil isomorphisms. In this subsection, we observe the construction of the de Rham-Weil isomorphism between the $\overline{\partial}$-cohomology and the Čech cohomology in detail. The content of this subsection is essentially contained in [11].

Let ω be a Kähler form on a compact Kähler manifold X and h be a singular metric on F satisfying $\sqrt{-1}\Theta_h(F) \geq -a\omega$ for some constant $a > 0$. Further let Z be a subvariety on X and let $\widetilde{\omega}$ be a Kähler form on the Zariski open set $Y := X \setminus Z$ with the following properties:

(B) For every point p in X, there exist an open neighborhood B of p and a bounded function Φ on B such that $\widetilde{\omega} = \sqrt{-1}\partial\overline{\partial}\Phi$ on $B \setminus Z$.

(C) $\widetilde{\omega} \geq \omega$.

The important point is that $\widetilde{\omega}$ locally admits a "bounded" potential on a neighborhood of every point p in X (not Y). Note that the Kähler form $\widetilde{\omega}$ constructed in Step 1 satisfies these properties. When we construct the de Rham-Weil isomorphism, we locally solve the $\overline{\partial}$-equation with L^2-estimate by using the following lemma.

Lemma 5.4 (cf. [5, 4.1 Théorème]). *Under the same situation as above, we assume that B is a Stein open set in X with property (B). Then, for an arbitrary $\alpha \in L^{n,q}_{(2)}(B \setminus Z, F)_{h,\widetilde{\omega}}$ with $\overline{\partial}\alpha = 0$, there exist*

$$\beta \in L^{n,q-1}_{(2)}(B \setminus Z, F)_{h,\widetilde{\omega}}$$

and a positive constant C (*depending only on* a, Φ, q) *such that*

$$\overline{\partial}\beta = \alpha,$$
$$\int_{B\setminus Z} |\beta|^2_{h,\widetilde{\omega}}\, dV_{\widetilde{\omega}} \leq C \int_{B\setminus Z} |\alpha|^2_{h,\widetilde{\omega}}\, dV_{\widetilde{\omega}}.$$

Proof. For a bounded function Φ on B with $\widetilde{\omega} = \sqrt{-1}\partial\overline{\partial}\Phi$, we define the metric H on F by $H := he^{-(1+a)\Phi}$. Then it follows that the curvature of H satisfies

$$\sqrt{-1}\Theta_H(F) = \sqrt{-1}\Theta_h(F) + (1+a)\sqrt{-1}\partial\overline{\partial}\Phi \geq -a\widetilde{\omega} + (1+a)\widetilde{\omega} \geq \widetilde{\omega}$$

from $\widetilde{\omega} \geq \omega$ and $\sqrt{-1}\Theta_h(F) \geq -a\omega$. The L^2-norm $\|\alpha\|_{H,\widetilde{\omega}}$ with respect to H is finite since the function Φ is bounded and $\|\alpha\|_{h,\widetilde{\omega}}$ is finite. We remark that $\widetilde{\omega}$ is not a complete form on $B \setminus Z$, but $B \setminus Z$ admits a complete Kähler form. Therefore, from the standard L^2-method for the $\overline{\partial}$-equation (for example see [5, 4.1 Théorème]), we obtain a solution β of the $\overline{\partial}$-equation $\overline{\partial}\beta = \alpha$ with

$$\|\beta\|^2_{H,\widetilde{\omega}} \leq \frac{1}{q}\|\alpha\|^2_{H,\widetilde{\omega}}.$$

By putting $C_1 := \inf_B e^{-(a+1)\Phi}$ and $C_2 := \sup_B e^{-(a+1)\Phi}$, we have

$$C_1\|\beta\|^2_{h,\widetilde{\omega}} \leq \|\beta\|^2_{H,\widetilde{\omega}} \quad \text{and} \quad \|\alpha\|^2_{H,\widetilde{\omega}} \leq C_2\|\alpha\|^2_{h,\widetilde{\omega}}.$$

These inequalities lead to the L^2-estimate in the lemma. □

From now on, we fix a Stein finite cover $\mathcal{U} := \{B_i\}_{i\in I}$ of X such that $\widetilde{\omega}$ admits a bounded potential function on B_i, and we consider the space of q-cochains $C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$ with coefficients in $K_X \otimes F \otimes \mathcal{I}(h)$ with the topology induced by the semi-norms $p_{K_{i_0\dots i_q}}(\bullet)$ defined as follows: For every $\alpha = \{\alpha_{i_0\dots i_q}\} \in C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$ and a relatively compact set $K_{i_0\dots i_q} \Subset B_{i_0\dots i_q}$, the semi-norm $p_{K_{i_0\dots i_q}}(\alpha)$ of α is defined by

$$p_{K_{i_0\dots i_q}}(\alpha)^2 := \int_{K_{i_0\dots i_q}} |\alpha_{i_0\dots i_q}|^2_{h,\omega}\, dV_\omega.$$

This semi-norm is independent of the choice of ω by Lemma 2.5. We remark that it is a Fréchet space (that is, it is complete with respect to these semi-norms) by Theorem 5.3. In this subsection, we observe the following de Rham-Weil isomorphism :

$$\frac{\operatorname{Ker}\overline{\partial}}{\operatorname{Im}\overline{\partial}} \text{ of } L^{n,q}_{(2)}(Y,F)_{h,\widetilde{\omega}} \xrightarrow{\cong} \check{H}^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$$
$$:= \frac{\operatorname{Ker}\delta}{\operatorname{Im}\delta} \text{ of } C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h)).$$

Here δ is the coboundary operator defined as follows: For every q-cochain $\{\alpha_{i_0\ldots i_q}\}_{i_0\ldots i_q}$,

$$\delta(\{\alpha_{i_0\ldots i_q}\}_{i_0\ldots i_q}) := \{\sum_{\ell=0}^{q+1}(-1)^\ell \alpha_{i_0\ldots \hat{i_\ell}\ldots i_{q+1}} |_{B_{i_0\ldots i_{q+1}}}\}_{i_0\ldots i_{q+1}},$$

where $B_{i_0\ldots i_{q+1}} := B_{i_0}\cap\cdots\cap B_{i_{q+1}}$. We will omit the notation of the restriction in the right hand side.

Proposition 5.5. *Under the same situation as above, there exist continuous maps*

$$f : \operatorname{Ker}\overline{\partial} \textit{ in } L^{n,q}_{(2)}(Y,F)_{h,\widetilde{\omega}} \to \operatorname{Ker}\delta \textit{ in } C^q(\mathcal{U}, K_X\otimes F\otimes \mathcal{I}(h)),$$
$$g : \operatorname{Ker}\delta \textit{ in } C^q(\mathcal{U}, K_X\otimes F\otimes \mathcal{I}(h)) \to \operatorname{Ker}\overline{\partial} \textit{ in } L^{n,q}_{(2)}(Y,F)_{h,\widetilde{\omega}},$$

satisfying the following properties:

- f *induces the isomorphism*

$$\overline{f}\colon \frac{\operatorname{Ker}\overline{\partial}}{\operatorname{Im}\overline{\partial}} \textit{ of } L^{n,q}_{(2)}(Y,F)_{h,\widetilde{\omega}} \xrightarrow{\ \cong\ } \frac{\operatorname{Ker}\delta}{\operatorname{Im}\delta} \textit{ of } C^q(\mathcal{U}, K_X\otimes F\otimes \mathcal{I}(h)).$$

- g *induces the isomorphism*

$$\overline{g}\colon \frac{\operatorname{Ker}\delta}{\operatorname{Im}\delta} \textit{ of } C^q(\mathcal{U}, K_X\otimes F\otimes \mathcal{I}(h)) \xrightarrow{\ \cong\ } \frac{\operatorname{Ker}\overline{\partial}}{\operatorname{Im}\overline{\partial}} \textit{ of } L^{n,q}_{(2)}(Y,F)_{h,\widetilde{\omega}}.$$

- $\overline{f}$ *is the inverse map of* $\overline{g}$.

Proof. We first define $f(U) \in \operatorname{Ker}\delta \subset C^q(\mathcal{U}, K_X\otimes F\otimes \mathcal{I}(h))$ for a given $U \in \operatorname{Ker}\overline{\partial} \subset L^{n,q}_{(2)}(Y,F)_{h,\widetilde{\omega}}$.

For the 0-cochain $\alpha^0 := \{\alpha_{i_0}\}$ defined by $\alpha_{i_0} := U|_{B_{i_0}\setminus Z}$, by applying Lemma 5.4 to α_{i_0}, it is shown that the $\overline{\partial}$-equation $\overline{\partial}\beta_{i_0} = \alpha_{i_0}$ on $B_{i_0}\setminus Z$ has a solution. The solution β_{i_0} whose L^2-norm $\|\beta_{i_0}\|_{h,\widetilde{\omega}}$ is minimum among all solutions satisfies the L^2-estimate $\|\beta_{i_0}\|^2_{h,\widetilde{\omega}} \leq C\|\alpha_{i_0}\|^2_{h,\widetilde{\omega}} \leq C\|U\|^2_{h,\widetilde{\omega}}$. Here the constant C does not depend on h and U. In the proof, C denotes (possibly) different positive constants independent of h and U.

For the 1-cochain α^1 defined by $\alpha^1 := \{\alpha_{i_0i_1}\} := \delta\{\beta_{i_0}\}$, we have

$$\begin{aligned}\|\alpha^1\|^2_{h,\widetilde{\omega}} &:= \sum_{i_0,i_1\in I}\int_{B_{i_0i_1}\setminus Z} |\alpha_{i_0i_1}|^2_{h,\widetilde{\omega}}\, dV_{\widetilde{\omega}} \\ &\leq \sum_{i_0,i_1\in I}(\|\beta_{i_0}\|_{h,\widetilde{\omega}} + \|\beta_{i_1}\|_{h,\widetilde{\omega}})^2 \leq C\|U\|^2_{h,\widetilde{\omega}}\end{aligned}$$

for some constant C. Further we have

$$\overline{\partial}\alpha^1 = \overline{\partial}\delta\{\beta_{i_0}\} = \delta\overline{\partial}\{\beta_{i_0}\} = \delta\{\alpha_{i_0}\} = 0.$$

Therefore, by applying Lemma 5.4 to α^1 again, we can take the solution of the $\overline{\partial}$-equation $\overline{\partial}\beta_{i_0i_1} = \alpha_{i_0i_1}$ on $B_{i_0i_1}\setminus Z$ whose L^2-norm $\|\beta_{i_0i_1}\|_{h,\widetilde{\omega}}$ is minimum among all solutions. Note that we have $\|\beta_{i_0 i_1}\|^2_{h,\widetilde{\omega}} \leq C\|\alpha_{i_0 i_1}\|^2_{h,\widetilde{\omega}}$. Similarly, by putting $\alpha^2 := \{\alpha_{i_0 i_1 i_2}\} := \delta\{\beta_{i_0 i_1}\}$, we can check that

$$\|\alpha^2\|^2_{h,\widetilde{\omega}} := \sum_{i_0, i_1 i_2 \in I} \int_{B_{i_0 i_1 i_2} \setminus Z} |\alpha_{i_0 i_1 i_2}|^2_{h,\widetilde{\omega}} \, dV_{\widetilde{\omega}} \leq C\|U\|^2_{h,\widetilde{\omega}}.$$

By repeating this process, we can obtain the k-cochain $\{\beta_{i_0 \dots i_k}\}$ with coefficients in the F-valued $(n, q-k-1)$-forms with the following equalities:

$$(*) \begin{cases} \overline{\partial}\beta_{i_0} & = U|_{B_{i_0} \setminus Z}, \\ \overline{\partial}\{\beta_{i_0 i_1}\} & = \delta\{\beta_{i_0}\}, \\ \overline{\partial}\{\beta_{i_0 i_1 i_2}\} & = \delta\{\beta_{i_0 i_1}\}, \\ & \vdots \\ \overline{\partial}\{\beta_{i_0 \dots i_{q-1}}\} & = \delta\{\beta_{i_0 \dots i_{q-2}}\}. \end{cases}$$

It follows that

$$\|\beta_{i_0 \dots i_k}\|^2_{h,\widetilde{\omega}} \leq C\|U\|^2_{h,\widetilde{\omega}}$$

from the construction. Now $\alpha^q := \{\alpha_{i_0 \dots i_q}\} := \delta\{\beta_{i_0 \dots i_{q-1}}\}$ is a q-cocycle with coefficients in the F-valued $(n, 0)$-forms and satisfies

$$\overline{\partial}\alpha^q = \overline{\partial}\delta\{\beta_{i_0 \dots i_{q-1}}\} = \delta\overline{\partial}\{\beta_{i_0 \dots i_{q-1}}\} = \delta\delta\{\beta_{i_0 \dots i_{q-2}}\} = 0.$$

In particular, $\alpha_{i_0 \dots i_q}$ can be regarded as a holomorphic function with bounded L^2-norm since it is a $\overline{\partial}$-closed F-valued $(n, 0)$-form and it satisfies $\|\alpha_{i_0 \dots i_q}\|_{h,\omega} = \|\alpha_{i_0 \dots i_q}\|_{h,\widetilde{\omega}} < \infty$ by Lemma 2.5. Then $\alpha_{i_0 \dots i_q}$ can be extended from $B_{i_0 \dots i_q} \setminus Z$ to the $\overline{\partial}$-closed F-valued $(n, 0)$-form on $B_{i_0 \dots i_q}$ by the Riemann extension theorem. Therefore it determines $\alpha^q \in \operatorname{Ker}\delta \subset C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$. We define $f(U)$ by $f(U) := \alpha^q = \delta\{\beta_{i_0 \dots i_{q-1}}\}$. It follows that f is continuous from the construction of f and the L^2-estimate

$$\|f(U)\|^2_{h,\widetilde{\omega}} = \sum_{i_0 \dots i_q \in I} \int_{B_{i_0 \dots i_q} \setminus Z} |\alpha_{i_0 \dots i_q}|^2_{h,\widetilde{\omega}} \, dV_{\widetilde{\omega}} \leq C\|U\|^2_{h,\widetilde{\omega}}.$$

Next we define $g(\alpha^q) \in \operatorname{Ker}\overline{\partial} \subset L^{n,q}_{(2)}(Y, F)_{h,\widetilde{\omega}}$ for a given $\alpha^q = \{\alpha_{i_0 \dots i_q}\} \in \operatorname{Ker}\delta \subset C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$. For the $(q-1)$-cochain $\alpha^{q-1} := \{\alpha_{i_0 \dots i_{q-1}}\}$ defined by

$$\alpha_{i_0 \dots i_{q-1}} := \sum_{k \in I} \rho_k \alpha_{k i_0 \dots i_{q-1}} ,$$

we can easily check $\delta\alpha^{q-1} = \alpha^q$ from $\delta\alpha^q = 0$, and thus we have $\delta\overline{\partial}\alpha^{q-1} = \overline{\partial}\delta\alpha^{q-1} = \overline{\partial}\alpha^q = 0$. When we define $\alpha^{q-2} := \{\alpha_{i_0 \dots i_{q-2}}\}$ by

$$\alpha_{i_0 \dots i_{q-2}} := \sum_{k \in I} \rho_k \overline{\partial}\alpha_{k i_0 \dots i_{q-2}} ,$$

we can easily check $\delta\alpha^{q-2} = \overline{\partial}\alpha^{q-1}$ from $\delta\overline{\partial}\alpha^{q-1} = 0$ again. By repeating this process, we obtain the k-cochain α^k with coefficients in the F-valued $(n, q-k-1)$-forms. Then $\overline{\partial}\alpha^0$ determines the $\overline{\partial}$-closed F-valued form globally defined on X by $\delta\overline{\partial}\alpha^0 = \overline{\partial}\delta\alpha^0 = \overline{\partial}\overline{\partial}\alpha^1 = 0$. We define $g(\alpha^q)$ by $g(\alpha^q) := \overline{\partial}\alpha^0$. The properties in Proposition 5.5 can be proved by the standard argument, and thus we omit it. □

Remark 5.6.

(1) The map g is linear, but f is not linear. This is because the norm of $\beta_1 + \beta_2$ is not necessarily minimum even if β_i is the solution of $\beta_i = \overline{\partial}\alpha_i$ whose L^2-norm is minimum. The induced maps $\overline{f}$ and $\overline{g}$ are linear.

(2) For the proof of Theorem 5.9, we remark that

$$\begin{aligned} &g(\alpha^q) \\ =&\overline{\partial}\Bigg(\sum_{k_q\in I}\rho_{k_q}\overline{\partial}\Bigg(\sum_{k_{q-1}\in I}\rho_{k_{q-1}}\cdots\overline{\partial}\Big(\sum_{k_3\in I}\rho_{k_3}\overline{\partial}\big(\sum_{k_2\in I}\rho_{k_2}\overline{\partial}\big(\sum_{k_1\in I}\rho_{k_1}\alpha_{k_1\ldots k_q i_0}\big)\big)\Big)\Bigg)\Bigg) \\ =&\sum_{k_q\in I}\overline{\partial}\rho_{k_q}\wedge\sum_{k_{q-1}\in I}\overline{\partial}\rho_{k_{q-1}}\wedge\cdots\sum_{k_3\in I}\overline{\partial}\rho_{k_3}\wedge\sum_{k_2\in I}\overline{\partial}\rho_{k_2}\wedge\overline{\partial}\big(\sum_{k_1\in I}\rho_{k_1}\alpha_{k_1\ldots k_q i_0}\big) \end{aligned}$$

holds on B_{i_0} by the construction and the Leibnitz rule.

Proposition 5.5 leads to the following lemma and proposition.

Lemma 5.7. *Under the same situation as above, the space of q-cocycles $Z^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h)) := \operatorname{Ker}\delta$ and the space of q-coboundaries*

$$B^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h)) := \operatorname{Im}\delta$$

are closed subspaces in $C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$ (in particular Fréchet spaces).

Proof. We can easily check that the coboundary operator δ from $C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$ to $C^{q+1}(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$ is continuous. This implies that $\operatorname{Ker}\delta$ is a closed subspace. Now we consider the following coboundary operator:

$$\delta : C^{q-1}(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h)) \to Z^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h)).$$

The cokernel of δ is isomorphic to $H^p(X, K_X \otimes F \otimes \mathcal{I}(h))$, whose dimension is finite. The open mapping theorem implies that $\operatorname{Im}\delta$ is a closed subspace (see Proposition 2.8). □

Proposition 5.8. *Under the same situation as above, the ranges $\operatorname{Im}\overline{\partial} \subset L^{n,q}_{(2)}(Y, F)_{h,\widetilde{\omega}}$ and $\operatorname{Im}\overline{\partial}^* \subset L^{n,q}_{(2)}(Y, F)_{h,\widetilde{\omega}}$ are closed subspaces in $L^{n,q}_{(2)}(Y, F)_{h,\widetilde{\omega}}$.*

Proof. For a given sequence $\{U_k\}_{k=1}^{\infty}$ in $\operatorname{Im}\overline{\partial}$ that converges to some U, it is shown that $\overline{f}([U_k])$ converges to $\overline{f}([U])$ from Proposition 5.5. Here $[\bullet]$ denotes the $\overline{\partial}$-cohomology class. We have $\overline{f}([U]) = 0$ from $\overline{f}([U_k]) = 0$ since the Čech cohomology is a separated topological space by Lemma 5.7. It follows that

$U \in \operatorname{Im} \overline{\partial}$ since $\overline{f}$ is an isomorphism. It is shown that $\operatorname{Im} \overline{\partial}^*$ is also closed from this fact (see [17, Theorem 1.1]). □

5.3. Proof of the key theorem. In this subsection, we prove the following theorem:

Theorem 5.9. *Under the same situation as in subsection* 5.2*, we consider a family of singular metrics* $\{h_\varepsilon\}_{1\gg\varepsilon>0}$ *on* F *with* $h_\varepsilon \leq h$, $\mathcal{I}(h_\varepsilon) = \mathcal{I}(h)$, *and* $\sqrt{-1}\Theta_{h_\varepsilon}(F) \geq -\omega$. *Then, for* $U_\varepsilon \in \operatorname{Im} \overline{\partial} \subset L^{n,q}_{(2)}(Y,F)_{h_\varepsilon,\widetilde{\omega}}$ *such that the* L^2*-norm* $\|U_\varepsilon\|_{h_\varepsilon,\widetilde{\omega}}$ *is uniformly bounded, there exists an* F*-valued* $(n, q-1)$*-form* V_ε *with the following properties:*

(1) $\overline{\partial} V_\varepsilon = U_\varepsilon$. (2) *The norm* $\|V_\varepsilon\|_{h_\varepsilon,\widetilde{\omega}}$ *is uniformly bounded.*

Proof. The strategy of the proof is as follows: The main idea of the proof is to convert the $\overline{\partial}$-equation $\overline{\partial} V_\varepsilon = U_\varepsilon$ to the equation $\delta\gamma_\varepsilon = f_\varepsilon(U_\varepsilon)$ of the coboundary operator δ in the space of cochains $C^\bullet(K_X \otimes F \otimes \mathcal{I}(h_\varepsilon))$, by using the Čech complex and pursuing the de Rham-Weil isomorphism. Here f_ε is the continuous map constructed for h_ε in Proposition 5.5. The important point is that $C^\bullet(K_X \otimes F \otimes \mathcal{I}(h_\varepsilon))$ is independent of ε thanks to the property of h_ε although the L^2-space $L^{n,\bullet}_{(2)}(Y,F)_{h_\varepsilon,\widetilde{\omega}}$ depends on ε. Since $\|U_\varepsilon\|_{h_\varepsilon,\widetilde{\omega}}$ is uniformly bounded, it is proven that $f_\varepsilon(U_\varepsilon)$ converges to some q-coboundary in $C^q(K_X \otimes F \otimes \mathcal{I}(h))$ with the topology induced by the local L^2-norms with respect to h. Further it is shown that the coboundary operator δ is an open map. Then, by these observations, we can construct a solution γ_ε of the equation $\delta\gamma_\varepsilon = f_\varepsilon(U_\varepsilon)$ with uniformly bounded norm. Finally, by using a partition of unity (the map g_ε constructed in Proposition 5.5), we conversely construct $V_\varepsilon \in L^{n,q}_{(2)}(Y,F)_{h_\varepsilon,\widetilde{\omega}}$ with the properties in Theorem 5.9.

Fix a Stein finite cover $\mathcal{U} := \{B_i\}_{i\in I}$ of X such that $\widetilde{\omega}$ admits a bounded potential function on B_i. By applying the argument in Proposition 5.5 to U_ε, we can obtain $\{\beta_{\varepsilon,i_0\ldots i_k}\}$ satisfying equality $(*)$. By the assumption of h_ε, we have

$$\begin{aligned}\alpha^q_\varepsilon := f_\varepsilon(U_\varepsilon) := \delta\{\beta_{\varepsilon,i_0\ldots i_{q-1}}\} \in &C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h_\varepsilon)) \\ = &C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h)).\end{aligned}$$

Then we prove the following claim.

Claim 5.10. In the above situation, we have the following:

- $f_\varepsilon(U_\varepsilon) \in \operatorname{Im}\delta \subset C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$ for every $\varepsilon > 0$.
- $\{f_\varepsilon(U_\varepsilon)\}_{\varepsilon>0}$ has a subsequence that converges to a q-cochain α^q.
- The limit $\alpha^q \in \operatorname{Im}\delta \subset C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$.

Proof of Claim 5.10. It follows that

$$f_\varepsilon(U_\varepsilon) \in \operatorname{Im}\delta \subset C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h_\varepsilon))$$

from the assumption $U_\varepsilon \in \operatorname{Im}\overline{\partial} \subset L^{n,q}_{(2)}(Y,F)_{h_\varepsilon,\widetilde{\omega}}$ and Proposition 5.5. By the assumption of h_ε, we obtain the first conclusion.

Now we prove that each component $\alpha_{\varepsilon,i_0\dots i_q}$ of $f_\varepsilon(U_\varepsilon)$ has a subsequence that converges to some F-valued $(n,0)$-form. By the construction of $\alpha^q_\varepsilon = f_\varepsilon(U_\varepsilon) = \delta(\{\beta_{\varepsilon,i_0\dots i_{q-1}}\})$, we have

$$\|\alpha_{\varepsilon,i_0\dots i_q}\|^2_{h_\varepsilon,\widetilde{\omega}} \leq \|f_\varepsilon(U_\varepsilon)\|^2_{h_\varepsilon,\widetilde{\omega}} \leq C\|U_\varepsilon\|^2_{h_\varepsilon,\widetilde{\omega}}.$$

Here the above constant C does not depend on $U_\varepsilon, h_\varepsilon$, and thus the right hand side can be estimated by a constant independent of ε. In particular, $\alpha_{\varepsilon,i_0\dots i_q}$ can be regarded as a holomorphic function with uniformly bounded L^2-norm. (Note that $\alpha_{\varepsilon,i_0\dots i_q}$ is a $\overline{\partial}$-closed F-valued $(n,0)$-form.) Then the sup-norm $\sup_{K_{i_0\dots i_q}} |\alpha_{\varepsilon,i_0\dots i_q}|$ is also uniformly bounded for every relatively compact set $K_{i_0\dots i_q} \Subset B_{i_0\dots i_q}$. Therefore, by Montel's theorem, we obtain a subsequence of $\{\alpha_{\varepsilon,i_0\dots i_q}\}_{\varepsilon>0}$ that uniformly converges to some F-valued $(n,0)$-form $\alpha_{i_0\dots i_q}$ on every relatively compact set in $B_{i_0\dots i_q}$. Lemma 5.2 asserts that this subsequence converges to $\alpha_{i_0\dots i_q}$ with respect to the semi-norms $\{p_{K_{i_0\dots i_q}}(\bullet)\}_{K_{i_0\dots i_q}\Subset B_{i_0\dots i_q}}$. Hence we can find a subsequence of $f_\varepsilon(U_\varepsilon)$ that converges to some q-cochain α^q in $C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$. The latter conclusion follows from Lemma 5.7. □

We will construct a solution γ_ε of the equation $\delta\gamma_\varepsilon = f_\varepsilon(U_\varepsilon)$ with uniformly bounded norm. For simplicity we use the same notation $\{f_\varepsilon(U_\varepsilon)\}_{\varepsilon>0}$ for the subsequence obtained in Claim 5.10. Note that the space of q-coboundaries $B^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h)) := \operatorname{Im}\delta$ is also a Fréchet space by Lemma 5.7. The coboundary operator

$$\delta : C^{q-1}(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h)) \to B^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$$

is a continuous and surjective linear map between Fréchet spaces, and thus this coboundary operator is an open map by the open mapping theorem.

By Claim 5.10, there exists $\gamma \in C^{q-1}(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$ such that $\delta\gamma = \alpha^q$. For a given family $K := \{K_{i_0\dots i_{q-1}}\}$ of relatively compact sets $K_{i_0\dots i_{q-1}} \Subset B_{i_0\dots i_{q-1}}$, we define the open bounded neighborhood Δ_K of γ in

$$C^{q-1}(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$$

by

$$\Delta_K := \{\beta \in C^{q-1}(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h)) \mid\ p_{K_{i_0\dots i_{q-1}}}(\beta - \gamma) < 1\}.$$

Then $\delta(\Delta_K)$ is an open neighborhood of the limit α^q in $B^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$ by the above observation. Therefore $f_\varepsilon(U_\varepsilon)$ belongs to $\delta(\Delta_K)$ for a sufficiently small $\varepsilon > 0$ since $f_\varepsilon(U_\varepsilon)$ converges to α^q. By the definition of Δ_K, we can obtain $\gamma_\varepsilon =: \{\gamma_{\varepsilon, i_0 \dots i_{q-1}}\} \in C^{q-1}(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$ such that

$$\delta\gamma_\varepsilon = f_\varepsilon(U_\varepsilon), \tag{5.1}$$

$$p_{K_{i_0 \dots i_{q-1}}}(\gamma_\varepsilon)^2 = \int_{K_{i_0 \dots i_{q-1}}} |\gamma_{\varepsilon, i_0 \dots i_{q-1}}|^2_{h,\omega}\, \omega^n \leq C_K \tag{5.2}$$

for some positive constant C_K. The above constant C_K depends on the choice of K, γ, but does not depend on ε.

We will construct an F-valued (n,q)-form V_ε from γ_ε and $f_\varepsilon(U_\varepsilon)$ by using g_ε. The strategy is as follows: It follows that $g_\varepsilon(\delta\gamma_\varepsilon) = \overline{\partial} v_\varepsilon$ and $g_\varepsilon(f_\varepsilon(U_\varepsilon)) = U_\varepsilon + \overline{\partial}\widetilde{v}_\varepsilon$ for some v_ε and $\widetilde{v}_\varepsilon$ since $\overline{g}_\varepsilon$ gives the isomorphism in Proposition 5.5. On the other hand, we have $U_\varepsilon = \overline{\partial}(v_\varepsilon - \widetilde{v}_\varepsilon)$ by equality (5.1). Then we can concretely compute v_ε and $\widetilde{v}_\varepsilon$ by using a partition of unity $\{\rho_i\}_{i\in I}$, and thus we obtain the L^2-estimate for them.

Claim 5.11. There exists an F-valued $(n, q-1)$-form v_ε on X satisfying the following properties:

(1) $\overline{\partial} v_\varepsilon = g_\varepsilon(\delta\gamma_\varepsilon)$. (2) The norm $\|v_\varepsilon\|_{h,\widetilde{\omega}}$ is uniformly bounded.

Proof. We observe that

$$\gamma_{\varepsilon, k_2 \dots k_q i_0} + \sum_{\ell=2}^{q} (-1)^{\ell-1} \gamma_{\varepsilon, k_1 \dots \hat{k_\ell} \dots k_q i_0} + (-1)^q \gamma_{\varepsilon, k_1 \dots k_q}$$

and the construction of g_ε (see Remark 5.6).

Argument 1.

Firstly we consider the first term $\gamma_{\varepsilon, k_2 \dots k_q i_0}$. It is easy to see that

$$\overline{\partial} \sum_{k_1 \in I} \rho_{k_1} \gamma_{\varepsilon, k_2 \dots k_q i_0} = \overline{\partial} \gamma_{\varepsilon, k_2 \dots k_q i_0} = 0$$

since $\gamma_{\varepsilon, k_2 \dots k_q i_0}$ does not depend on k_1. Here we used $\sum_{k_1 \in I} \rho_{k_1} = 1$. We can conclude that this term does not affect $g_\varepsilon(\delta\gamma_\varepsilon)$ from Remark 5.6.

Argument 2.

Secondly we consider the second term $\gamma_{\varepsilon, k_1 \dots \hat{k_\ell} \dots k_q i_0}$. For an integer ℓ with $2 \leq \ell \leq q$, by the Leibniz rule, we can show that

$$\begin{aligned}
&\sum_{k_q \in I} \overline{\partial}\rho_{k_q} \wedge \cdots \wedge \sum_{k_\ell \in I} \overline{\partial}\rho_{k_\ell} \wedge \sum_{k_{\ell-1} \in I} \overline{\partial}\rho_{k_{\ell-1}} \wedge \cdots \wedge \sum_{k_1 \in I} \overline{\partial}(\rho_{k_1} \gamma_{\varepsilon, k_1 .. \hat{k_\ell} .. k_{q-1} i_0}) \\
&= \sum_{k_q \in I} \overline{\partial}\rho_{k_q} \wedge \cdots \wedge \overline{\partial} \sum_{k_{\ell-1} \in I} \overline{\partial}\rho_{k_{\ell-1}} \wedge \cdots \wedge \sum_{k_1 \in I} \overline{\partial}(\rho_{k_1} \gamma_{\varepsilon, k_1 .. \hat{k_\ell} .. k_{q-1} i_0}) = 0.
\end{aligned}$$

Here we used $\overline{\partial}\,\overline{\partial} = 0$ and $\sum_{k_\ell \in I} \rho_{k_\ell} = 1$. Therefore the second term does not affect $g_\varepsilon(\delta\gamma_\varepsilon)$.

Argument 3.

Finally we consider the third term $(-1)^q\gamma_{\varepsilon,k_1\dots k_q}$. If v_ε is defined by

$$\begin{aligned}v_\varepsilon :=&(-1)^q \sum_{k_1,\dots,k_q\in I} \rho_{k_q}\overline{\partial}\rho_{k_{q-1}}\wedge\overline{\partial}\rho_{k_{q-1}}\wedge\cdots\wedge\overline{\partial}\rho_{k_2}\wedge\overline{\partial}(\rho_{k_1}\wedge\gamma_{\varepsilon,k_1\dots k_q})\\ =&(-1)^q \sum_{k_1,\dots,k_q\in I} \rho_{k_q}\overline{\partial}\rho_{k_{q-1}}\wedge\overline{\partial}\rho_{k_{q-1}}\wedge\cdots\wedge\overline{\partial}\rho_{k_2}\wedge\overline{\partial}\rho_{k_1}\wedge\gamma_{\varepsilon,k_1\dots k_q}.\end{aligned}$$

Then v_ε determines the F-valued $(n,q-1)$-form on X since $\gamma_{\varepsilon,k_1\dots k_q}$ is independent of i_0. The second equality follows from the Leibnitz rule and $\overline{\partial}\gamma_{\varepsilon,k_1\dots k_q}=0$. We have $g_\varepsilon(\delta\gamma_\varepsilon)=\overline{\partial}v_\varepsilon$ by the definition of v_ε and Arguments 1, 2. For the proof, it is sufficient to show that the norm $\|v_\varepsilon\|_{h,\widetilde{\omega}}$ is uniformly bounded. When we define the $(0,q-1)$-form $\eta_{k_1\dots k_q}$ on X by

$$\eta_{k_1\dots k_q} := \rho_{k_q}\overline{\partial}\rho_{k_{q-1}}\wedge\overline{\partial}\rho_{k_{q-2}}\wedge\cdots\wedge\overline{\partial}\rho_{k_1},$$

we have

$$v_\varepsilon = \sum_{k_1,\dots,k_q\in I}\eta_{k_1\dots k_q}\wedge\gamma_{\varepsilon,k_1\dots k_q}.$$

Since the support of $\eta_{k_1\dots k_q}$ is relatively compact in $B_{k_1\dots k_q}$, there exists $K:=\{K_{k_1\dots k_q}\}$ such that Supp $\eta_{k_1\dots k_q}\Subset K_{k_1\dots k_q}\Subset B_{k_1\dots k_q}$. For the family $K=\{K_{k_1\dots k_q}\}$, we may assume that the q-cochain γ_ε satisfies inequality (5.2). By Lemma 2.6, there exists a positive constant $C>0$ such that

$$|v_\varepsilon|_{h,\widetilde{\omega}} \le \sum_{k_1,\dots,k_q\in I}|\eta_{k_1\dots k_q}\wedge\gamma_{\varepsilon,k_1\dots k_q}|_{h,\widetilde{\omega}} \le C\sum_{k_1,\dots,k_q\in I}\chi_{K_{k_1\dots k_q}}|\gamma_{\varepsilon,k_1\dots k_q}|_{h,\widetilde{\omega}},$$

where $\chi_{K_{k_1\dots k_q}}$ is the characteristic function of $K_{k_1\dots k_q}$. Note that C depends on the choice of $\{\rho_i\}_{i\in I}$, but does not depend on ε. Therefore we have

$$\|v_\varepsilon\|_{h,\widetilde{\omega}} \le C\sum_{k_1,\dots,k_q\in I} p_{K_{k_1\dots k_q}}(\gamma_\varepsilon)$$

from the fundamental inequality $(\sum_{i=1}^N|a_i|)^2\le 2^{N-1}\sum_{i=1}^N|a_i|^2$. The right hand side can be estimated by a constant independent of ε by inequality (5.2). This completes the proof. □

The proof of the following claim is based on an argument similar to that of Claim 5.11. To avoid confusion, we use the following notation in the proof.

Definition 5.12. Let a_ε and b_ε be F-valued (n,k)-forms on Y. We write $a_\varepsilon\equiv b_\varepsilon$, if there exists an F-valued $(n,k-1)$-form c_ε on Y such that $\overline{\partial}c_\varepsilon = a_\varepsilon - b_\varepsilon$ and the norm $\|c_\varepsilon\|_{h_\varepsilon,\widetilde{\omega}}$ is uniformly bounded.

Claim 5.13. There exists an F-valued $(n,q-1)$-form $\widetilde{v}_\varepsilon$ on Y satisfying the following properties:

(1) $\overline{\partial}\widetilde{v}_\varepsilon + U_\varepsilon = g_\varepsilon(f_\varepsilon(U_\varepsilon))$. (2) The norm $\|\widetilde{v}_\varepsilon\|_{h_\varepsilon,\widetilde{\omega}}$ is uniformly bounded.

Proof. For $\beta_{\varepsilon,i_0\dots i_{q-1}}$ with equality $(*)$, we have $f_\varepsilon(U_\varepsilon)=\delta\{\beta_{\varepsilon,i_0\dots i_{q-1}}\}$. We observe

$$\beta_{\varepsilon,k_2\dots k_q i_0}+\sum_{\ell=2}^{q}(-1)^{\ell-1}\beta_{\varepsilon,k_1\dots\hat{k_\ell}\dots k_q i_0}+(-1)^q\beta_{\varepsilon,k_1\dots k_q}.$$

Argument 4.

Firstly we consider the second term. For an integer ℓ with $2\le\ell\le q$, the second term $\beta_{\varepsilon,k_1\dots\hat{k_\ell}\dots k_q i_0}$ is independent of k_ℓ. By the same reason as Argument 2 in Claim 5.11, we can conclude that this term does not affect $g_\varepsilon(f_\varepsilon(U_\varepsilon))$.

Argument 5.

Secondly we consider the third term. Our aim in Argument 5 is to show that

$$\sum_{k_q\in I}\overline{\partial}\rho_{k_q}\wedge\sum_{k_{q-1}\in I}\overline{\partial}\rho_{k_{q-1}}\wedge\cdots\wedge\sum_{k_2\in I}\overline{\partial}\rho_{k_2}\wedge\sum_{k_1\in I}\overline{\partial}(\rho_{k_1}\beta_{\varepsilon,k_1\dots k_q})\equiv 0.$$

When we define $\eta_{k_2\dots k_{k_q}}$ by

$$\eta_{k_2\dots k_{k_q}}:=\rho_{k_q}\overline{\partial}\rho_{k_{q-1}}\wedge\overline{\partial}\rho_{k_{q-2}}\cdots\wedge\overline{\partial}\rho_{k_2},$$

the left hand side agrees with

$$\overline{\partial}\Big(\sum_{k_1,\dots,k_q\in I}\eta_{k_2\dots k_{k_q}}\wedge\big(\overline{\partial}\rho_{k_1}\wedge\beta_{\varepsilon,k_1\dots k_q}+\rho_{k_1}\wedge\overline{\partial}\beta_{\varepsilon,k_1\dots k_q}\big)\Big)$$

by the Leibniz rule. By Lemma 2.6 and the fundamental inequality $|a+b|^2\le 2(|a|^2+|b|^2)$, we obtain

$$\begin{aligned}&\Big|\sum_{k_1,\dots,k_q\in I}\eta_{k_2\dots k_{k_q}}\wedge\big(\overline{\partial}\rho_{k_1}\wedge\beta_{\varepsilon,k_1\dots k_q}+\rho_{k_1}\wedge\overline{\partial}\beta_{\varepsilon,k_1\dots k_q}\big)\Big|^2_{h_\varepsilon,\widetilde{\omega}}\\ \le& C\big(\big|\beta_{\varepsilon,k_1\dots k_q}\big|^2_{h_\varepsilon,\widetilde{\omega}}+\big|\overline{\partial}\beta_{\varepsilon,k_1\dots k_q}\big|^2_{h_\varepsilon,\widetilde{\omega}}\big)\end{aligned}$$

for some positive constant $C>0$. The norms $\|\beta_{\varepsilon,k_1\dots k_q}\|_{h_\varepsilon,\widetilde{\omega}}$ and $\|\overline{\partial}\beta_{\varepsilon,k_1\dots k_q}\|_{h_\varepsilon,\widetilde{\omega}}$ can be estimated by a constant independent of ε by the construction (see the proof of Proposition 5.5). Therefore we have

$$\sum_{k_q\in I}\overline{\partial}\rho_{k_q}\wedge\sum_{k_{q-1}\in I}\overline{\partial}\rho_{k_{q-1}}\wedge\cdots\wedge\sum_{k_2\in I}\overline{\partial}\rho_{k_2}\wedge\sum_{k_1\in I}\overline{\partial}(\rho_{k_1}\beta_{\varepsilon,k_1\dots k_q})\equiv 0.$$

Argument 6.

Finally we consider the first term. Our aim is to show that

$$\sum_{k_q\in I}\overline{\partial}\rho_{k_q}\wedge\sum_{k_{q-1}\in I}\overline{\partial}\rho_{k_{q-1}}\wedge\cdots\wedge\sum_{k_2\in I}\overline{\partial}\rho_{k_2}\wedge\sum_{k_1\in I}\overline{\partial}(\rho_{k_1}\beta_{\varepsilon,k_2\dots k_q i_0})\equiv U_\varepsilon.$$

Since $\beta_{\varepsilon,k_2\dots k_q i_0}$ does not depend on k_1, we have

(5.3)

$$\sum_{k_1\in I}\overline{\partial}(\rho_{k_1}\beta_{\varepsilon,k_2\dots k_q i_0})=\overline{\partial}\beta_{\varepsilon,k_2\dots k_q i_0}$$
$$=\beta_{\varepsilon,k_3\dots k_q i_0}+\sum_{\ell=3}^{q}(-1)^{\ell}\beta_{\varepsilon,k_2\dots\hat{k_\ell}\dots k_q i_0}+(-1)^{q+1}\beta_{\varepsilon,k_2\dots k_q}.$$

Note that the second equality follows from equality $(*)$. The second term of the right hand side of (5.3) does not affect, by the same reason as Argument 2 (Argument 4). Moreover, for the third term of the right hand side, we can show that

$$\sum_{k_q\in I}\overline{\partial}\rho_{k_q}\wedge\sum_{k_{q-1}\in I}\overline{\partial}\rho_{k_{q-1}}\wedge\dots\wedge\sum_{k_2\in I}\overline{\partial}(\rho_{k_2}\beta_{\varepsilon,k_2\dots k_q})\equiv 0$$

by the same method as Argument 5. In summary, we have proved that

$$\sum_{k_q\in I}\overline{\partial}\rho_{k_q}\wedge\sum_{k_{q-1}\in I}\overline{\partial}\rho_{k_{q-1}}\wedge\dots\wedge\sum_{k_2\in I}\overline{\partial}\rho_{k_2}\wedge\sum_{k_1\in I}\overline{\partial}(\rho_{k_1}\beta_{\varepsilon,k_2\dots k_q i_0})$$
$$\equiv\sum_{k_q\in I}\overline{\partial}\rho_{k_q}\wedge\sum_{k_{q-1}\in I}\overline{\partial}\rho_{k_{q-1}}\wedge\dots\wedge\sum_{k_2\in I}\overline{\partial}\rho_{k_3}\wedge\sum_{k_1\in I}\overline{\partial}(\rho_{k_2}\beta_{\varepsilon,k_3\dots k_q i_0}).$$

By repeating this procedure, we obtain

$$\sum_{k_q\in I}\overline{\partial}\rho_{k_q}\wedge\sum_{k_{q-1}\in I}\overline{\partial}\rho_{k_{q-1}}\wedge\dots\wedge\sum_{k_2\in I}\overline{\partial}\rho_{k_2}\wedge\sum_{k_1\in I}\overline{\partial}(\rho_{k_1}\beta_{\varepsilon,k_2\dots k_q i_0})$$
$$\equiv\sum_{k_q\in I}\overline{\partial}\rho_{k_q}\wedge\sum_{k_{q-1}\in I}\overline{\partial}(\rho_{k_{q-1}}\beta_{\varepsilon,k_q i_0})$$
$$=\sum_{k_q\in I}\overline{\partial}\rho_{k_q}\wedge\overline{\partial}\beta_{\varepsilon,k_q i_0}$$
$$=\sum_{k_q\in I}\overline{\partial}\big(\rho_{k_q}(\beta_{\varepsilon,i_0}-\beta_{\varepsilon,k_q})\big).$$

The last equality follows from equality $(*)$. The norm of β_{ε,k_q} can be estimated by a constant independent of ε, and $\overline{\partial}\beta_{\varepsilon,i_0}=U_\varepsilon$ holds on $B_{i_0}\setminus Z$ by the construction. Hence we have

$$\overline{\partial}\sum_{k_q\in I}\big(\rho_{k_q}\beta_{\varepsilon,k_q}\big)\equiv 0\quad\text{and}\quad\overline{\partial}\Big(\sum_{k_q\in I}\rho_{k_q}\beta_{\varepsilon,i_0}\Big)=\overline{\partial}\beta_{\varepsilon,i_0}=U_\varepsilon.$$

This completes the proof. □

From Claims 5.11 and 5.13 we can obtain the conclusion. Indeed, if we put $V_\varepsilon:=v_\varepsilon-\widetilde{v}_\varepsilon$, then V_ε satisfies the properties in Theorem 5.9. □

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References

[1] Junyan Cao, *Numerical dimension and a Kawamata-Viehweg-Nadel-type vanishing theorem on compact Kähler manifolds*, Compos. Math. **150** (2014), no. 11, 1869–1902, DOI 10.1112/S0010437X14007398. MR3279260

[2] Jean-Pierre Demailly, Lawrence Ein, and Robert Lazarsfeld, *A subadditivity property of multiplier ideals*, Michigan Math. J. **48** (2000), 137–156, DOI 10.1307/mmj/1030132712. Dedicated to William Fulton on the occasion of his 60th birthday. MR1786484

[3] Jean-Pierre Demailly, *Analytic methods in algebraic geometry*, Surveys of Modern Mathematics, vol. 1, International Press, Somerville, MA; Higher Education Press, Beijing, 2012. MR2978333

[4] J.-P. Demailly, *Complex analytic and differential geometry*, lecture notes on the webpage of the author.

[5] Jean-Pierre Demailly, *Estimations L^2 pour l'opérateur $\bar{\partial}$ d'un fibré vectoriel holomorphe semi-positif au-dessus d'une variété kählérienne complète* (French), Ann. Sci. École Norm. Sup. (4) **15** (1982), no. 3, 457–511. MR690650

[6] Jean-Pierre Demailly, Christopher D. Hacon, and Mihai Păun, *Extension theorems, non-vanishing and the existence of good minimal models*, Acta Math. **210** (2013), no. 2, 203–259, DOI 10.1007/s11511-013-0094-x. MR3070567

[7] Jean-Pierre Demailly, Thomas Peternell, and Michael Schneider, *Pseudo-effective line bundles on compact Kähler manifolds*, Internat. J. Math. **12** (2001), no. 6, 689–741, DOI 10.1142/S0129167X01000861. MR1875649

[8] Ichiro Enoki, *Kawamata-Viehweg vanishing theorem for compact Kähler manifolds*, Einstein metrics and Yang-Mills connections (Sanda, 1990), Lecture Notes in Pure and Appl. Math., vol. 145, Dekker, New York, 1993, pp. 59–68. MR1215279

[9] Lawrence Ein and Mihnea Popa, *Global division of cohomology classes via injectivity*, Michigan Math. J. **57** (2008), 249–259, DOI 10.1307/mmj/1220879407. Special volume in honor of Melvin Hochster. MR2492451

[10] Hélène Esnault and Eckart Viehweg, *Lectures on vanishing theorems*, DMV Seminar, vol. 20, Birkhäuser Verlag, Basel, 1992. MR1193913

[11] Osamu Fujino, *A transcendental approach to Kollár's injectivity theorem*, Osaka J. Math. **49** (2012), no. 3, 833–852. MR2993068

[12] Osamu Fujino, *A transcendental approach to Kollár's injectivity theorem II*, J. Reine Angew. Math. **681** (2013), 149–174. MR3181493

[13] Yoshinori Gongyo and Shin-ichi Matsumura, *Versions of injectivity and extension theorems* (English, with English and French summaries), Ann. Sci. Éc. Norm. Supér. (4) **50** (2017), no. 2, 479–502. MR3621435

[14] Robert C. Gunning and Hugo Rossi, *Analytic functions of several complex variables*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1965. MR0180696

[15] Qi'an Guan and Xiangyu Zhou, *Effectiveness of Demailly's strong openness conjecture and related problems*, Invent. Math. **202** (2015), no. 2, 635–676. MR3418242

[16] Pham Hoang Hiep, *The weighted log canonical threshold* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **352** (2014), no. 4, 283–288, DOI 10.1016/j.crma.2014.02.010. MR3186914

[17] Lars Hörmander, *L^2 estimates and existence theorems for the $\bar{\partial}$ operator*, Acta Math. **113** (1965), 89–152. MR0179443

[18] Y. Kawamata, *Pluricanonical systems on minimal algebraic varieties*, Invent. Math. **79** (1985), no. 3, 567–588, DOI 10.1007/BF01388524. MR782236

[19] János Kollár, *Higher direct images of dualizing sheaves. I*, Ann. of Math. (2) **123** (1986), no. 1, 11–42, DOI 10.2307/1971351. MR825838

[20] L. Lempert, *Modules of square integrable holomorphic germs*, preprint, arXiv:1404.0407v2.

[21] Robert Lazarsfeld, *Positivity in algebraic geometry. I, Classical setting: line bundles and linear series*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48, Springer-Verlag, Berlin, 2004. MR2095471

[22] Shin-ichi Matsumura, *A Nadel vanishing theorem via injectivity theorems*, Math. Ann. **359** (2014), no. 3-4, 785–802, DOI 10.1007/s00208-014-1018-6. MR3231016

[23] Shin-ichi Matsumura, *Some applications of the theory of harmonic integrals*, Complex Manifolds **2** (2015), 16–25, DOI 10.1515/coma-2015-0003. MR3370350

[24] Shin-ichi Matsumura, *A Nadel vanishing theorem for metrics with minimal singularities on big line bundles*, Adv. Math. **280** (2015), 188–207, DOI 10.1016/j.aim.2015.03.019. MR3350216

[25] Noboru Nakayama, *The lower semicontinuity of the plurigenera of complex varieties*, Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., vol. 10, North-Holland, Amsterdam, 1987, pp. 551–590. MR946250

[26] Takeo Ohsawa, *On a curvature condition that implies a cohomology injectivity theorem of Kollár-Skoda type*, Publ. Res. Inst. Math. Sci. **41** (2005), no. 3, 565–577. MR2153535

[27] Mihai Păun, *Siu's invariance of plurigenera: a one-tower proof*, J. Differential Geom. **76** (2007), no. 3, 485–493. MR2331528

[28] Francesco Russo, *A characterization of nef and good divisors by asymptotic multiplier ideals*, Bull. Belg. Math. Soc. Simon Stevin **16** (2009), no. 5, Linear systems and subschemes, 943–951. MR2574371

[29] Yum-Tong Siu, *Invariance of plurigenera*, Invent. Math. **134** (1998), no. 3, 661–673, DOI 10.1007/s002220050276. MR1660941

[30] Kensho Takegoshi, *On cohomology groups of nef line bundles tensorized with multiplier ideal sheaves on compact Kähler manifolds*, Osaka J. Math. **34** (1997), no. 4, 783–802. MR1618661

[31] S. G. Tankeev, *n-dimensional canonically polarized varieties, and varieties of basic type* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 31–44. MR0277528

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