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EXTREMALS FOR THE SOBOLEV INEQUALITY ON THE HEISENBERG GROUP AND THE CR YAMABE PROBLEM

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1. INTRODUCTION

A *CR structure* on a real manifold M is a distinguished complex subbundle $\mathscr{H}$ of the complex tangent bundle $\mathbb{C}TM$, satisfying $\mathscr{H} \cap \overline{\mathscr{H}} = 0$ and $[\mathscr{H}, \mathscr{H}] \subset \mathscr{H}$. For example, the complex structure of $\mathbb{C}^{n+1}$ induces a natural biholomorphically invariant CR structure on any real hypersurface: $\mathscr{H}$ is the space of vectors in the span of $\partial/\partial z^1, \ldots, \partial/\partial z^{n+1}$ which are tangent to the hypersurface. An abstract CR manifold M is said to be of *hypersurface type* if $\dim_{\mathbb{R}} M = 2n+1$ and $\dim_{\mathbb{C}} \mathscr{H} = n$; all our CR manifolds will be of this type.

If M is oriented, then there is a globally defined real one-form θ that annihilates $\mathscr{H}$ and $\overline{\mathscr{H}}$. The *Levi form*, given by $L_\theta(V, \overline{W}) = -2id\theta(V \wedge \overline{W})$, is a hermitian form on $\mathscr{H}$. We will assume that the CR structure is *strictly pseudoconvex*: for some choice of θ, the Levi form L_θ is positive definite on $\mathscr{H}$. In this case θ defines a contact structure on M and we call θ a contact form associated with the CR structure.

The Levi form plays a role similar to that of the metric in Riemannian geometry. However, the CR structure only determines the Levi form up to a conformal multiple; this multiple is fixed by the choice of a contact form. A CR structure with a given choice of contact form is called a *pseudohermitian structure.* Thus there is an analogy between pseudohermitian and CR manifolds on the one hand and Riemannian and conformal manifolds on the other. In particular, Webster [W1, W2] and Tanaka [T] have defined a pseudohermitian scalar curvature associated to L_θ. The CR Yamabe problem is: *given a compact, strictly pseudoconvex CR manifold, find a choice of contact form for which the pseudohermitian scalar curvature is constant.*

Suppose M is a strictly pseudoconvex CR manifold of dimension $2n+1$. Solutions to the CR Yamabe problem are precisely the critical points of the *CR*

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Yamabe functional:

$$\mathscr{Y}(\theta) = \frac{\int_M R\theta \wedge d\theta^n}{\left(\int_M \theta \wedge d\theta^n\right)^{2/p}},$$

in which $p = 2 + 2/n$, θ is any contact form associated to the CR structure of M, and R is its pseudohermitian scalar curvature. We set

$$\lambda(M) = \inf_\theta \mathscr{Y}(\theta).$$

The main result of [JL2] is that the CR Yamabe problem has a solution on a compact strictly pseudoconvex CR manifold M provided that $\lambda(M) < \lambda(S^{2n+1})$, where S^{2n+1} is the sphere in $\mathbb{C}^{n+1}$. In order to show that most CR manifolds M satisfy $\lambda(M) < \lambda(S^{2n+1})$, it is crucial to identify the extremal contact forms for the Yamabe functional on the sphere.

The restriction to TS^{2n+1} of $\hat{\theta} = \frac{i}{2}(\overline{\partial} - \partial)|z|^2$ is a contact form for S^{2n+1}. The form $\hat{\theta}$ and its images under CR automorphisms of the sphere (which are all induced by biholomorphisms of the unit ball in $\mathbb{C}^{n+1}$) have constant pseudohermitian scalar curvature $\hat{R} = n(n+1)$. In [JL2] we conjectured that these are the *only* solutions to the Yamabe problem on the sphere. The purpose of this paper is to confirm this conjecture. Our main result is

Theorem A. *If θ is a contact form associated with the standard CR structure on the sphere which has constant pseudohermitian scalar curvature, then θ is obtained from a constant multiple of the standard form $\hat{\theta}$ by a CR automorphism of the sphere.*

In [JL2] we proved the weaker result that this conclusion holds provided θ is invariant under some conjugate of $U(n)$ in the CR automorphism group of the sphere. That proof used a variant of the same methods we will employ in this paper.

We showed in [JL2] that the minimum $\lambda(S^{2n+1})$ of the Yamabe functional is actually achieved by a smooth contact form (of constant scalar curvature). Thus

Corollary B. $\lambda(S^{2n+1}) = \mathscr{Y}(\hat{\theta}) = 2\pi n(n+1)$; *this minimum value is achieved only by constant multiples of $\hat{\theta}$ and its images under CR automorphisms.*

There is an equivalent formulation of this result on the Heisenberg group $\mathbf{H}^n$. We consider $\mathbf{H}^n$ as the set $\mathbb{C}^n \times \mathbb{R}$ with coordinates (z, t) and group law

$$(z,t)(\zeta,\tau) = \left(z + \zeta, t + \tau + 2\mathrm{Im}\sum_{\alpha=1}^{n} z^\alpha \overline{\zeta}^\alpha\right) \quad \text{for } (z,t), (\zeta,\tau) \in \mathbb{C}^n \times \mathbb{R}.$$

The CR structure of $\mathbf{H}^n$ is given by the bundle $\mathscr{H}$ spanned by the left-invariant vector fields $Z_\alpha = \partial/\partial z^\alpha + i\overline{z}^\alpha \partial/\partial t$, $\alpha = 1, \dots, n$. The standard (left-invariant) contact form on $\mathbf{H}^n$ is $\Theta = dt + \sum(iz^\alpha d\overline{z}^\alpha - i\overline{z}^\alpha dz^\alpha)$. It follows from the work of Folland and Stein [FS] that there exists a positive constant

C such that the following Sobolev-type inequality holds for all functions u for which both sides of the inequality are finite:

$$\left(\int_{\mathbb{H}^n} |u|^p \Theta \wedge d\Theta^n\right)^{2/p} \leq C \int_{\mathbb{H}^n} \sum_{\alpha=1}^{n} (|Z_\alpha u|^2 + |\overline{Z}_\alpha u|^2) \Theta \wedge d\Theta^n, \tag{1.1}$$

with $p = 2 + 2/n$. (Note that $\Theta \wedge d\Theta^n = 4^n n!\, dt \wedge dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n$ is bi-invariant under group multiplication.) As shown in [JL2], if $u > 0$ the ratio of the two sides of (1.1) is exactly $(2C/p)\mathscr{Y}(u^{2/n}\Theta)$.

The Cayley transform (see [JL2]) gives a CR equivalence between $\mathbb{H}^n$ and S^{2n+1} minus a point, under which $\hat{\theta}$ corresponds to $2|w + i|^{-2}\theta$, with $w = t + i|z|^2$. Corollary B implies that $\mathscr{Y}(v^{2/n}\hat{\theta}) \geq 2\pi n(n+1)$ for any smooth positive function v on the sphere. By continuity, this can be extended to any $v \geq 0$ for which the numerator and denominator of $\mathscr{Y}$ are finite. Applying the Cayley transform, $\mathscr{Y}(u^{2/n}\Theta) \geq 2\pi n(n+1)$ for all $u \geq 0$ on $\mathbb{H}^n$ for which both sides of (1.1) are finite. By taking the absolute value of u, one sees easily that (1.1) holds with the same constant for complex-valued u. By the regularity theory of [JL2], the extremals in (1.1) must be constant multiples of images under the Cayley transform of extremals on the sphere. Thus

Corollary C. *The best constant in the Sobolev inequality* (1.1) *is* $C = 1/2\pi n^2$. *Equality is attained only by the functions*

$$u(z,t) = K|w + z \cdot \mu + \lambda|^{-n}, \tag{1.2}$$

with $K, \lambda \in \mathbb{C}$, $\operatorname{Im}\lambda > |\mu|^2/4$, *and* $\mu \in \mathbb{C}^n$. (*These are obtained from the function* $K|w+i|^{-n}$ *by left translations and dilations* $(z,t) \mapsto (\delta z, \delta^2 t)$ *on the Heisenberg group.*)

The proof of Theorem A is based on the same idea as Obata's proof [O] of the analogous result in Riemannian geometry: the only Riemannian metrics on the sphere that are conformal to the standard one and have constant scalar curvature are obtained from the standard metric by a conformal diffeomorphism of the sphere.

The idea of Obata's proof can be sketched as follows (see also [LP]). Suppose g is any constant scalar curvature metric on S^N conformal to the standard metric $\overline{g}$. Writing $\overline{g} = \varphi^{-2} g$ and using the fact that $\overline{g}$ is Einstein allows one to express the traceless Ricci tensor B of g in terms of the covariant Hessian of φ. Then the contracted Bianchi identity and the fact that g has constant scalar curvature imply that

$$\operatorname{div}(B^{jk}\varphi_j \partial_k) = \varphi |B|^2. \tag{1.3}$$

Integrating over the sphere shows that $B = 0$, so g is Einstein. But it is easy to describe all Einstein metrics conformal to the standard metric on the sphere.

Let us emphasize the key feature of this argument: a function φ which *a priori* satisfies a single nonlinear equation (expressing the fact that g has constant scalar curvature) is shown to satisfy a *system* of equations $B = 0$.

The naive generalization of (1.3) to CR manifolds does not work because the pseudohermitian Bianchi identities involve extra torsion terms. On the Heisenberg group this reflects the nontrivial commutation relations. One might hope, therefore, for any identity like (1.3) where the right-hand side involves both the traceless Ricci B and the torsion A. Unfortunately there is no such identity. One is thus forced to look for a formula involving not only $|B|^2$ and $|A|^2$, but also higher order terms such as $|\operatorname{div} A|^2$. Once these higher order terms are introduced, the number of possible identities becomes enormous, and a systematic search is required.

Our approach was to write the most general such formula with undetermined coefficients, in which the tensors on the right-hand side are formed from combinations of B, A, and $\operatorname{div} A$. Equating like terms leads to a system of linear equations for the coefficients. One seeks a solution for which the right-hand side is positive. Integrating over the sphere will then prove that the right-hand side vanishes identically. This approach led us to a 25×25 variable-coefficient system which we solved using the computer algebra program MACSYMA. Surprisingly, we then found a three-dimensional family of solutions with positive right-hand side.

In §3, we verify the simplest of these identities (3.1) and prove Theorem A. In §4, we state without proof three formulas on the Heisenberg group which are linearly independent when $n > 1$, and explain how they provide an independent proof of Theorem A purely in terms of the Heisenberg group. We have chosen to omit the proofs in §4 because the Bianchi identities make the proof of (3.1) much shorter.

The most important application of this result is, of course, to the solution of the CR Yamabe problem on an arbitrary compact strictly pseudoconvex CR manifold M. In a separate paper [JL3], we will show that the CR Yamabe problem has a solution provided M has dimension greater than 3 and is not locally CR equivalent to the sphere. The proof makes use of the extremal functions from Corollary C as test functions in order to show that $\lambda(M) < \lambda(S^{2n+1})$. The remaining cases will require the construction of a global test function, as they do in the Riemannian case. In those cases as well, we expect that knowledge of the extremals for the sphere will be of crucial importance.

An interesting (but vaguely defined) problem raised by this work is to find an "explanation" for the existence of divergence formulas such as (1.2) and (3.1). Is there a theoretical framework that would predict the existence and the structure of such formulas, so that they could be discovered more systematically?

2. Pseudohermitian invariants

We begin with a brief review of the formalism of pseudohermitian geometry. For more details, see [JL2] and [L2].

Given a contact form θ on a $2n+1$-dimensional CR manifold M, let T be the characteristic vector field defined by $\theta(T) = 1$, $T \lrcorner\, d\theta = 0$. An *admissible coframe* is a set of $(1,0)$-forms $\theta^1, \dots, \theta^n$ whose restrictions to $\mathscr{H}$ form a

basis for $\mathscr{H}^*$, and such that $\theta^\alpha(T) = 0$. We can write

$$d\theta = ih_{\alpha\bar\beta}\theta^\alpha \wedge \theta^{\bar\beta}$$

for a matrix of smooth functions $h_{\alpha\bar\beta}$. We will observe the summation convention, and use the matrix $h_{\alpha\bar\beta}$ and its inverse $h^{\alpha\bar\beta}$ as usual to raise and lower indices. Let $R_{\alpha\bar\beta\rho\bar\gamma}$ be the Webster curvature tensor and $A_{\alpha\beta}$ the Webster torsion tensor determined by the coframe $\{\theta^\alpha\}$ (see [W2]). The pseudohermitian connection induces covariant differentiation of functions and tensors, which we will indicate with indices preceded by a comma, as in $Q_{\alpha\bar\beta,\rho\bar\gamma}$. (For derivatives of a scalar function we will sometimes omit the comma.) A zero index denotes covariant differentiation by the characteristic vector field T. The invariant $A_{\beta\alpha,}{}^{\alpha}$ (the divergence of the torsion tensor) will play an essential role in the identities that follow.

The *pseudohermitian Ricci tensor* is the contraction $R_{\rho\bar\sigma} = R_\alpha{}^\alpha{}_{\rho\bar\sigma}$ of the pseudohermitian curvature tensor, and the *pseudohermitian scalar curvature* is $R = R_\alpha{}^\alpha$. It is convenient also to introduce the traceless Ricci tensor $B_{\alpha\bar\beta} = R_{\alpha\bar\beta} - \frac{1}{n}Rh_{\alpha\bar\beta}$. We say θ is *pseudo-Einstein* if $B_{\alpha\bar\beta} = 0$.

The second and third covariant derivatives of a scalar function u satisfy the following commutation relations:

$$\begin{gathered} u_{\alpha\beta} - u_{\beta\alpha} = 0, \qquad u_{\alpha\bar\beta} - u_{\bar\beta\alpha} = ih_{\alpha\bar\beta}u_0, \qquad u_{0\alpha} - u_{\alpha 0} = A_{\alpha\beta}u^\beta, \\ u_{\alpha\beta\bar\gamma} - u_{\alpha\bar\gamma\beta} = ih_{\beta\bar\gamma}u_{\alpha 0} + R_\alpha{}^\rho{}_{\beta\bar\gamma}u_\rho. \end{gathered} \tag{2.1}$$

The curvature and torsion satisfy the *Bianchi identities*:

$$R_{,\alpha} - R_{\alpha\bar\beta,}{}^{\bar\beta} = (n-1)iA_{\alpha\beta,}{}^{\beta} \tag{2.2}$$

$$R_{,0} = A_{\alpha\beta,}{}^{\beta\alpha} + A_{\bar\alpha\bar\beta,}{}^{\bar\beta\alpha}. \tag{2.3}$$

If θ is a given contact form on M, suppose f is a smooth function and consider $\tilde\theta = e^{2f}\theta$. In [L2] it is shown that the basic pseudohermitian invariants transform as follows:

$$\tilde A_{\alpha\beta} = A_{\alpha\beta} + 2if_{\alpha\beta} - 4if_\alpha f_\beta, \tag{2.4}$$

$$\tilde R_{\alpha\bar\beta} = R_{\alpha\bar\beta} - (n+2)(f_{\alpha\bar\beta} + f_{\bar\beta\alpha}) - (f_\gamma{}^\gamma + f_{\bar\gamma}{}^{\bar\gamma} + 4(n+1)f_\gamma f^\gamma)h_{\alpha\bar\beta}, \tag{2.5}$$

$$\tilde R = e^{-2f}(R - 2(n+1)(f_\gamma{}^\gamma + f_{\bar\gamma}{}^{\bar\gamma}) - 4n(n+1)f_\gamma f^\gamma), \tag{2.6}$$

when $\tilde R_{\alpha\bar\beta}$ and $\tilde A_{\alpha\beta}$ are evaluated with respect to the admissible coframe $\{\tilde\theta^\alpha = \theta^\alpha + 2if^\alpha\theta\}$. The calculations below will be somewhat simpler if we write $\tilde\theta = \varphi^{-1}\theta$ and substitute $e^{2f} = \varphi^{-1}$ in the above formulas:

$$\tilde A_{\alpha\beta} = A_{\alpha\beta} - i\varphi^{-1}\varphi_{\alpha\beta}, \tag{2.7}$$

$$\begin{aligned}\tilde{R}_{\alpha\bar{\beta}} = R_{\alpha\bar{\beta}} &+ \frac{n+2}{2}(\varphi^{-1}(\varphi_{\alpha\bar{\beta}} + \varphi_{\bar{\beta}\alpha}) - 2\varphi^{-2}\varphi_\alpha\varphi_{\bar{\beta}}) \\ &+ \frac{1}{2}(\varphi^{-1}(\varphi_\gamma{}^\gamma + \varphi_{\bar{\gamma}}{}^{\bar{\gamma}}) - 2(n+2)\varphi^{-2}\varphi_\gamma\varphi^\gamma)h_{\alpha\bar{\beta}},\end{aligned} \tag{2.8}$$

$$\tilde{R} = \varphi R + (n+1)(\varphi_\beta{}^\beta + \varphi_{\bar{\beta}}{}^{\bar{\beta}}) - (n+1)(n+2)\varphi^{-1}\varphi_\beta\varphi^\beta. \tag{2.9}$$

These also imply

$$\tilde{B}_{\alpha\bar{\beta}} = B_{\alpha\bar{\beta}} + (n+2)(\varphi^{-1}\varphi_{\alpha\bar{\beta}} - \varphi^{-2}\varphi_\alpha\varphi_{\bar{\beta}}) - \frac{n+2}{n}(\varphi^{-1}\varphi_\gamma{}^\gamma - \varphi^{-2}\varphi_\gamma\varphi^\gamma)h_{\alpha\bar{\beta}}. \tag{2.10}$$

3. Proof of the main theorem

The following proposition is the key to the proof of Theorem A. For simplicity of notation, we will often write the norm induced by the Levi form as $|\ |^2$, even when we denote a tensor by writing its components with respect to a local coframe. For example, we write $|A_{\alpha\beta}|^2$ for the scalar invariant whose local expression is $A_{\alpha\beta}A^{\alpha\beta}$.

Proposition 3.1. *Let S^{2n+1} denote the sphere with its standard CR structure. Suppose θ is an associated contact form on S^{2n+1} which has constant pseudohermitian scalar curvature $R \equiv n(n+1)$. Write the standard contact form on the sphere as $\hat{\theta} = \varphi^{-1}\theta$, where φ is a smooth positive function, and define the tensors*

$$D_{\alpha\beta} = -iA_{\alpha\beta}, \qquad D_{\alpha\beta\bar{\gamma}} = \varphi^{-1}\varphi_{\bar{\gamma}}D_{\alpha\beta}, \qquad D_\alpha = D_{\alpha\beta}{}^\beta,$$
$$E_{\alpha\bar{\beta}} = -\frac{1}{n+2}B_{\alpha\bar{\beta}}, \qquad E_{\alpha\bar{\beta}\gamma} = \varphi^{-1}\varphi_\gamma E_{\alpha\bar{\beta}}, \qquad E_\alpha = E_{\alpha\bar{\beta}}{}^{\bar{\beta}},$$
$$U_\alpha = -\frac{2}{n+2}iA_{\alpha\beta,}{}^\beta,$$

where all pseudohermitian invariants and covariant derivatives are understood to be with respect to θ. Let $g = 1 + \varphi + \varphi^{-1}\varphi_\beta\varphi^\beta + i\varphi_0$. Then we have the following identity on S^{2n+1}:

(3.1)
$$\begin{aligned}&\mathrm{Re}(gD_\alpha + \bar{g}E_\alpha - 3i\varphi_0 U_\alpha)^{,\alpha} \\ &\quad = (1+\varphi)(|D_{\alpha\beta}|^2 + |E_{\alpha\bar{\beta}}|^2) \\ &\qquad + \varphi(|U_\alpha - D_\alpha + E_\alpha|^2 + |U_\alpha - D_\alpha|^2 + |U_\alpha + E_\alpha|^2 + |D_{\alpha\beta\bar{\gamma}} + E_{\alpha\bar{\gamma}\beta}|^2).\end{aligned}$$

Proof. Since $\hat{A}_{\alpha\beta}$ and $\hat{B}_{\alpha\bar{\beta}}$ vanish, transformation laws (2.7) and (2.10) yield expressions for $A_{\alpha\beta}$ and $B_{\alpha\bar{\beta}}$ in terms of φ:

$$A_{\alpha\beta} = i\varphi^{-1}\varphi_{\alpha\beta}, \tag{3.2}$$

$$(3.3)\quad B_{\alpha\bar\beta} = -(n+2)(\varphi^{-1}\varphi_{\alpha\bar\beta} - \varphi^{-2}\varphi_\alpha\varphi_{\bar\beta}) + \frac{n+2}{n}(\varphi^{-1}\varphi_\gamma{}^\gamma - \varphi^{-2}\varphi_\gamma\varphi^\gamma)h_{\alpha\bar\beta}.$$

It follows that

$$(3.4)\qquad A_{\alpha\beta,}{}^\beta = i\varphi^{-1}\varphi_{\alpha\beta}{}^\beta - i\varphi^{-2}\varphi_{\alpha\beta}\varphi^\beta.$$

From (2.9), since $R = \hat{R} = n(n+1)$,

$$(3.5)\qquad \varphi_\beta{}^\beta = \frac{n}{2} - \frac{n}{2}\varphi + \frac{n+2}{2}\varphi^{-1}\varphi_\beta\varphi^\beta + \frac{n}{2}i\varphi_0$$

and so, using the commutation relations (2.1),

$$\begin{aligned}\varphi_{\alpha\beta}{}^\beta &= \varphi_\beta{}^\beta{}_\alpha + i\varphi_{\alpha 0} + R_{\alpha\bar\beta}\varphi^{\bar\beta}\\ &= -\frac{n}{2}\varphi_\alpha - \frac{n+2}{2}\varphi^{-2}\varphi_\alpha\varphi_\beta\varphi^\beta + \frac{n+2}{2}\varphi^{-1}\varphi_{\beta\alpha}\varphi^\beta + \frac{n+2}{2}\varphi^{-1}\varphi^{\bar\beta}\varphi_{\bar\beta\alpha}\\ &\quad + \frac{n+2}{2}i\varphi_{0\alpha} - iA_{\alpha\beta}\varphi^\beta + R_{\alpha\bar\beta}\varphi^{\bar\beta}.\end{aligned}$$

Writing $R_{\alpha\bar\beta} = B_{\alpha\bar\beta} + \frac{1}{n}Rh_{\alpha\bar\beta}$, using (3.3) and (3.5), and simplifying yields

$$R_{\alpha\bar\beta}\varphi^{\bar\beta} = \frac{n+2}{2}(-2\varphi^{-1}\varphi_{\alpha\bar\beta}\varphi^{\bar\beta} + 3\varphi^{-2}\varphi_\alpha\varphi_\beta\varphi^\beta + \varphi^{-1}\varphi_\alpha + i\varphi^{-1}\varphi_0\varphi_\alpha) + \frac{n}{2}\varphi_\alpha.$$

Substituting these formulas into (3.4), we obtain

(3.6)
$$A_{\alpha\beta,}{}^\beta = \frac{n+2}{2}i(i\varphi^{-1}\varphi_{0\alpha} + \varphi^{-2}\varphi_{\alpha\beta}\varphi^\beta - \varphi^{-2}\varphi_{\alpha\bar\beta}\varphi^{\bar\beta} + 2\varphi^{-3}\varphi_\alpha\varphi_\beta\varphi^\beta + \varphi^{-2}\varphi_\alpha).$$

From (3.2), (3.3), and (3.6) we see that

$$\begin{aligned}D_{\alpha\beta} &= \varphi^{-1}\varphi_{\alpha\beta},\\ E_{\alpha\bar\beta} &= (\varphi^{-1}\varphi_{\alpha\bar\beta} - \varphi^{-2}\varphi_\alpha\varphi_{\bar\beta}) - \tfrac{1}{2}(\varphi^{-1} - 1 + \varphi^{-2}\varphi_\gamma\varphi^\gamma + i\varphi^{-1}\varphi_0)h_{\alpha\bar\beta},\\ U_\alpha &= i\varphi^{-1}\varphi_{0\alpha} + \varphi^{-2}\varphi_{\alpha\beta}\varphi^\beta - \varphi^{-2}\varphi_{\alpha\bar\beta}\varphi^{\bar\beta} + 2\varphi^{-3}\varphi_\alpha\varphi_\beta\varphi^\beta + \varphi^{-2}\varphi_\alpha.\end{aligned}$$

Now from the Bianchi identities (2.2) and (2.3) and the fact that R is constant it follows that

$$D_{\alpha\beta,}{}^\alpha = \frac{n+2}{2}U_\beta, \qquad E_{\alpha\bar\beta,}{}^\alpha = \frac{1-n}{2}U_{\bar\beta}, \qquad \operatorname{Re} iU_{\alpha,}{}^\alpha = 0.$$

Thus when we expand the left-hand side of (3.1) we get

$$\begin{aligned}
\mathrm{LHS} &= \mathrm{Re}(\varphi^\alpha - \varphi^{-2}\varphi_\beta\varphi^\beta\varphi^\alpha + \varphi^{-1}\varphi_\beta{}^\alpha\varphi^\beta + \varphi^{-1}\varphi_\beta\varphi^{\beta\alpha} + i\varphi_0{}^\alpha)D_\alpha \\
&\quad + \mathrm{Re}(-g\varphi^{-2}\varphi^\alpha\varphi^\beta D_{\alpha\beta}) + \mathrm{Re}\, g\varphi^{-1}\varphi^{\beta\alpha}D_{\alpha\beta} + \mathrm{Re}\left(\frac{n+2}{2}g\varphi^{-1}\varphi^\beta U_\beta\right) \\
&\quad + \mathrm{Re}(\varphi^\alpha - \varphi^{-2}\varphi_\beta\varphi^\beta\varphi^\alpha + \varphi^{-1}\varphi_\beta{}^\alpha\varphi^\beta + \varphi^{-1}\varphi_\beta\varphi^{\beta\alpha} - i\varphi_0{}^\alpha)E_\alpha \\
&\quad + \mathrm{Re}\,\overline{g}(-\varphi^{-2}\varphi^\alpha\varphi^{\overline{\beta}} + \varphi^{-1}\varphi^{\overline{\beta}\alpha})E_{\alpha\overline{\beta}} \\
&\quad + \mathrm{Re}\left(\frac{1-n}{2}\overline{g}\varphi^{-1}\varphi^{\overline{\beta}}U_{\overline{\beta}}\right) - \mathrm{Re}\, 3i\varphi_0{}^\alpha U_\alpha \\
&= \mathrm{Re}(-2\varphi^{-2}\varphi_\beta\varphi^\beta\varphi^\alpha + \varphi^{-1}\varphi^{\alpha\overline{\beta}}\varphi_{\overline{\beta}} + \varphi^{-1}\varphi^{\beta\alpha}\varphi_\beta + i\varphi_0{}^\alpha - \varphi^{-1}\varphi^\alpha)D_\alpha \\
&\quad + \mathrm{Re}(\varphi^\alpha - \varphi^{-2}\varphi_\beta\varphi^\beta\varphi^\alpha + \varphi^{-1}\varphi^{\alpha\overline{\beta}}\varphi_{\overline{\beta}} + i\varphi^{-1}\varphi_0\varphi^\alpha \\
&\qquad\qquad + \varphi^{-1}\varphi_\beta\varphi^{\beta\alpha} - i\varphi_0{}^\alpha)E_\alpha \\
&\quad + 3\mathrm{Re}\left(-i\varphi_0{}^\alpha + \tfrac{1}{2}\varphi^{-1}\varphi^\alpha + \tfrac{1}{2}\varphi^\alpha + \tfrac{1}{2}\varphi^{-2}\varphi_\gamma\varphi^\gamma\varphi^\alpha + \tfrac{i}{2}\varphi^{-1}\varphi_0\varphi^\alpha\right)U_\alpha \\
&\quad + (1 + \varphi + \varphi^{-1}\varphi_\gamma\varphi^\gamma)(D^{\alpha\beta}D_{\alpha\beta} + E^{\alpha\overline{\beta}}E_{\alpha\overline{\beta}}) \\
&= \mathrm{Re}\,\varphi(2D^\alpha - U^\alpha)D_\alpha + \mathrm{Re}\,\varphi(2E^\alpha + U^\alpha)E_\alpha + 3\mathrm{Re}\,\varphi(U^\alpha - D^\alpha + E^\alpha)U_\alpha \\
&\quad + (1 + \varphi + \varphi^{-1}\varphi_\gamma\varphi^\gamma)(D^{\alpha\beta}D_{\alpha\beta} + E^{\alpha\overline{\beta}}E_{\alpha\overline{\beta}}).
\end{aligned}$$

One can check easily that this is equal to the expansion of the right-hand side. □

Proof of Theorem A. If θ is the given contact form, write $\hat{\theta} = \varphi^{-1}\theta$ as above. Multiplying θ by a constant, we may assume θ has scalar curvature $n(n+1)$, so the previous proposition applies. Integrating (3.1) over the sphere and applying the divergence theorem (see [L2]), we find that $D_{\alpha\beta} = 0$ and $E_{\alpha\overline{\beta}} = 0$, and hence θ is pseudo-Einstein and torsion-free.

The next step is to show that φ can be written $\varphi = |v|^2$ for some function v which is *CR holomorphic* on S^{2n+1}. This is equivalent to showing that $\log\varphi$ is the real part of a holomorphic function.

From the formula for $E_{\alpha\overline{\beta}}$ in terms of φ, it follows that

$$\begin{aligned}
0 &= (\varphi^{-1}\varphi_{\alpha\overline{\beta}} - \varphi^{-2}\varphi_\alpha\varphi_{\overline{\beta}}) - \tfrac{1}{n}(\varphi^{-1}\varphi_\gamma{}^\gamma - \varphi^{-2}\varphi_\gamma\varphi^\gamma)h_{\alpha\overline{\beta}} \\
&= (\log\varphi)_{\alpha\overline{\beta}} - \tfrac{1}{n}(\log\varphi)_\gamma{}^\gamma h_{\alpha\overline{\beta}}.
\end{aligned}$$

It is shown in [B] (see also [L2]) that if $n > 1$ this is a necessary and sufficient condition for $\log\varphi$ to be the real part of a CR holomorphic function (since S^{2n+1} is simply connected). In case $n = 1$, however, we have to use the full strength of (3.1): observe that since D_α and E_α vanish, (3.1) implies that $U_\alpha = 0$ also.

From [BF] and [L2], when $n = 1$ the condition for a real function u to be the real part of a CR holomorphic function is $P_\alpha u = 0$, where

$$P_\alpha u = u_{\overline{\beta}}{}^{\overline{\beta}}{}_\alpha + iA_{\alpha\beta}u^\beta .$$

Since $A_{\alpha\beta} = \varphi_{\alpha\beta} = 0$, we have

$$P_\alpha(\log\varphi) = (\log\varphi)_{\overline{\beta}}{}^{\overline{\beta}}{}_\alpha = \varphi^{-1}\varphi_{\overline{\beta}}{}^{\overline{\beta}}{}_\alpha - 2\varphi^{-2}\varphi_{\overline{\beta}}{}^{\overline{\beta}}\varphi_\alpha + 2\varphi^{-3}\varphi_{\overline{\beta}}\varphi^{\overline{\beta}}\varphi_\alpha .$$

The commutation relations (2.1) show that

$$\varphi_{\overline{\beta}}{}^{\overline{\beta}}{}_\alpha = \varphi_{\beta\alpha}{}^\beta - 2i\varphi_{0\alpha} - R_{\alpha\overline{\beta}}\varphi^{\overline{\beta}} = -2i\varphi_{0\alpha} - 2\varphi_\alpha .$$

Moreover, using the fact that $\varphi_{\alpha\overline{\beta}} = \varphi_\gamma{}^\gamma h_{\alpha\overline{\beta}}$ when $n = 1$, $U_\alpha = 0$ implies

$$-2i\varphi^{-1}\varphi_{0\alpha} = -2\varphi^{-2}\varphi_\beta{}^\beta\varphi_\alpha + 4\varphi^{-3}\varphi_\alpha\varphi_\beta\varphi^\beta + 2\varphi^{-2}\varphi_\alpha ,$$

and so

$$P_\alpha(\log\varphi) = -2\varphi^{-2}(\varphi_\beta{}^\beta + \varphi_{\overline{\beta}}{}^{\overline{\beta}})\varphi_\alpha - 2\varphi^{-1}\varphi_\alpha + 6\varphi^{-3}\varphi_\alpha\varphi_\beta\varphi^\beta + 2\varphi^{-2}\varphi_\alpha .$$

Substituting (3.5) then shows that $P_\alpha(\log\varphi) = 0$, so again we conclude that $\log\varphi$ is the real part of a holomorphic function.

Therefore in any dimension there exists a nonvanishing CR-holomorphic function v on S^{2n+1} such that $\varphi = |v|^2$.

Next, we transfer our attention to the Heisenberg group. With respect to the contact form Θ on $\mathbb{H}^n$, the characteristic vector field is $T = \partial/\partial t$, and we will always use the standard holomorphic frame $\{Z_\alpha\}$ and dual admissible coframe $\{dz^\alpha\}$. Covariant derivatives on $\mathbb{H}^n$ are given by $u_{j_1\cdots j_r} = Z_{j_r}\cdots Z_{j_1}u$, for $j_i = \alpha, \overline{\alpha}$, or 0, with the conventions that $Z_0 = T = \partial/\partial t$ and $Z_{\overline{\alpha}} = \overline{Z_\alpha}$.

We will identify $\mathbb{H}^n$ with S^{2n+1} minus a point by means of the Cayley transform (see [JL2]), and thus consider θ and $\hat{\theta}$ as one-forms on $\mathbb{H}^n$. It is easy to check that $\hat{\theta} = 2|w+i|^{-2}\Theta$, where $w = t + i|z|^2$, and therefore $\theta = 2|v|^2|w+i|^{-2}\Theta$. A straightforward calculation shows that the images of $\hat{\theta}$ under CR automorphisms of the sphere correspond to the one-forms $u^{2/n}\Theta$ on the Heisenberg group, where u is of the form (1.3) in the introduction. We will complete the proof by showing that θ is also of this form.

With v as above, set $g = (w+i)/v$ and $\psi = \frac{1}{2}|g|^2$ on $\mathbb{H}^n$. The preceding argument shows that g is holomorphic and $\theta = \psi^{-1}\Theta$. Since θ and Θ are torsion-free, the transformation law (2.7) for torsion implies that $\psi_{\alpha\beta} = 0$, and hence $g_{\alpha\beta} = 0$.

Using the fact that $g_{\overline{\alpha}} = 0$, we have

$$g_{0\overline{\alpha}} = g_{\overline{\alpha}0} = 0;$$

$$g_{0\alpha} = -\tfrac{i}{n}(g_\beta{}^\beta{}_\alpha - g_{\overline{\beta}}{}^{\overline{\beta}}{}_\alpha) = -\tfrac{i}{n}(g_{\beta\alpha}{}^\beta - ig_{\alpha 0}) = -\tfrac{1}{n}g_{0\alpha} .$$

Thus g_0 is constant. Now consider the CR-holomorphic function $k = g - g_0 w$. Observe that $\partial k/\partial t = k_0 = 0$ since $\partial w/\partial t = 1$, and so k is a holomorphic function of $\{z^\alpha\}$. This implies

$$\frac{\partial^2 k}{\partial z^\alpha \partial z^\beta} = k_{\alpha\beta} = g_{\alpha\beta} - g_0 w_{\alpha\beta} = 0,$$

and therefore k is a linear polynomial in z. Thus we can write

$$g = g_0(w + z \cdot \overline{\mu} + \lambda)$$

for some $\mu \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$. Since g vanishes nowhere, we must have $\operatorname{Im}\lambda > |\mu|^2/4$, and so $\theta = 2|g|^{-2}\Theta$ is indeed of the required form. This completes the proof. □

4. Divergence formulas on the Heisenberg group

Let Θ denote the standard contact form on $\mathbb{H}^n$, as in the introduction. With respect to the standard holomorphic frame $\{Z_\alpha\}$ and dual admissible coframe $\{dz^\alpha\}$, we have $h_{\alpha\overline{\beta}} = 2\delta_{\alpha\overline{\beta}}$, and as in §3 the covariant derivative $f_{j_1\cdots j_r}$ of a function f is just the ordinary derivative $Z_{j_r}\cdots Z_{j_1} f$.

Proposition 4.1. *Suppose f is a smooth real solution on $\mathbb{H}^n$ to the equation*

$$\operatorname{Re} f_{\alpha\overline{\alpha}} + n f_\alpha f_{\overline{\alpha}} + n e^{2f} = 0. \tag{4.1}$$

Define the tensors

$$\begin{aligned} D_{\alpha\beta} &= f_{\alpha\beta} - 2 f_\alpha f_\beta, & D_\alpha &= D_{\alpha\beta} f_{\overline{\beta}}, \\ E_{\alpha\overline{\beta}} &= f_{\alpha\overline{\beta}} - \tfrac{1}{n} f_{\gamma\overline{\gamma}} \delta_{\alpha\overline{\beta}}, & E_\alpha &= E_{\alpha\overline{\beta}} f_\beta, \\ G_\alpha &= i f_{0\alpha} - i f_0 f_\alpha + e^{2f} f_\alpha + f_\beta f_{\overline{\beta}} f_\alpha. \end{aligned}$$

Then f satisfies the identities

(4.2)

$$\begin{aligned} \operatorname{Re} Z_{\overline{\alpha}}(e^{2(n-1)f}(&f_\rho f_{\overline{\rho}} f_{\alpha\beta} f_{\overline{\beta}} + f_\rho f_{\overline{\rho}} f_{\alpha\overline{\beta}} f_\beta - f_\rho f_{\overline{\rho}} f_\beta f_{\overline{\beta}} f_\alpha + 2 i f_0 f_\rho f_{\overline{\rho}} f_\alpha \\ &+ e^{2f} f_{\alpha\beta} f_{\overline{\beta}} + e^{2f} f_{\alpha\overline{\beta}} f_\beta - 2 e^{2f} i f_0 f_\alpha + e^{4f} f_\alpha - 2 i f_0 f_{\alpha\beta} f_{\overline{\beta}} \\ &+ 2 i f_0 f_{\alpha\overline{\beta}} f_\beta + 3 f_0 f_{0\alpha} - f_0^2 f_\alpha)) \\ = e^{2nf}&(|D_{\alpha\beta}|^2 + |E_{\alpha\overline{\beta}}|^2) \\ &+ e^{2(n-1)f}(|G_\alpha|^2 + |G_\alpha + D_\alpha|^2 + |G_\alpha - E_\alpha|^2 + |D_{\alpha\beta} f_{\overline{\gamma}} + E_{\alpha\overline{\gamma}} f_\beta|^2). \end{aligned}$$

(4.3)

$$\begin{aligned}
\operatorname{Re} Z_{\bar\alpha}(e^{2(n-1)f}&((n+2)f_\rho f_{\bar\rho} f_{\alpha\bar\beta} f_\beta + (1-n) f_\rho f_{\bar\rho} f_\beta f_{\bar\beta} f_\alpha \\
&- (n^2-4n+2) i f_0 f_\rho f_{\bar\rho} f_\alpha + (2-n) e^{2f} f_\rho f_{\bar\rho} f_\alpha + n e^{2f} f_{\alpha\beta} f_{\bar\beta} \\
&+ e^{2f} f_{\alpha\bar\beta} f_\beta + (n^2-n-2) e^{2f} i f_0 f_\alpha + e^{4f} f_\alpha - 2ni f_0 f_{\alpha\beta} f_{\bar\beta} \\
&+ 2i f_0 f_{\alpha\bar\beta} f_\beta + (n+2) f_0 f_{0\alpha} - (n^2+n-1) f_0^2 f_\alpha \\
&\qquad + n f_{\beta\rho} f_{\bar\beta} f_{\bar\rho} f_\alpha - n f_{\beta\bar\rho} f_{\bar\beta} f_\rho f_\alpha)) \\
&= e^{2(n-1)f}((n+2)|E_{\alpha\bar\beta} f_\gamma|^2 + |E_\alpha|^2 + (n-2)|D_\alpha|^2 \\
&\quad + (n+1)|G_\alpha + D_\alpha|^2 + |G_\alpha - D_\alpha - E_\alpha|^2) \\
&\quad + e^{2nf}(|E_{\alpha\bar\beta}|^2 + n(|D_{\alpha\beta}|^2).
\end{aligned}$$

(4.4)

$$\begin{aligned}
\operatorname{Re} Z_{\bar\alpha}(e^{2(n-1)f}&(f_\rho f_{\bar\rho} f_{\alpha\beta} f_{\bar\beta} - 2 f_\rho f_{\bar\rho} f_{\alpha\bar\beta} f_\beta - 4 f_\rho f_{\bar\rho} f_\beta f_{\bar\beta} f_\alpha + (6n+2) i f_0 f_\rho f_{\bar\rho} f_\alpha \\
&- 6e^{2f} f_\rho f_{\bar\rho} f_\alpha + e^{2f} f_{\alpha\beta} f_{\bar\beta} - 2e^{2f} f_{\alpha\bar\beta} f_\beta + 4e^{2f} i f_0 f_\alpha - 2e^{4f} f_\alpha \\
&\quad + (1-3n) i f_0 f_{\alpha\beta} f_{\bar\beta} + (3n+2) i f_0 f_{\alpha\bar\beta} f_\beta + 3n f_0 f_{0\alpha} + 2 f_0^2 f_\alpha)) \\
&= e^{2(n-1)f}(|D_{\alpha\beta} f_{\bar\gamma}|^2 - 2|E_{\alpha\bar\beta} f_\gamma|^2 + 3n|G_\alpha|^2 - 2|E_\alpha|^2 + |D_\alpha|^2 \\
&\quad - (3n+2)\operatorname{Re} E_\alpha G_{\bar\alpha} + (3n-1)\operatorname{Re} D_\alpha G_{\bar\alpha} - \operatorname{Re} D_\alpha E_{\bar\alpha}) \\
&\quad + e^{2nf}(|D_{\alpha\beta}|^2 - 2|E_{\alpha\bar\beta}|^2).
\end{aligned}$$

These formulas can be checked by a very long, routine calculation. Note that formula (4.2) is similar to (3.1).

Corollary 4.2. *Let $\theta = u^{2/n}\Theta$ be a smooth contact form on $\mathbf{H}^n$ which has constant scalar curvature, and suppose $u \in L^p(\mathbf{H}^n)$, $p = 2 + 2/n$. Then u is of the form* (1.2) *given in the introduction.*

Proof. Multiplying θ by a constant, we may assume the scalar curvature of θ is $R = 2n(n+1)$. If we define f by $e^{2f} = u^{2/n}$, then the transformation law (2.6) for scalar curvature and the fact that Θ has vanishing scalar curvature imply that f satisfies (4.1). Therefore by Proposition 4.1 f also satisfies, say, (4.2). Before integrating by parts, we must check that f and its derivatives have sufficient decay at infinity.

By considering the CR inversion $(z,t) \mapsto (z/w, -t/|w|^2)$, one can show exactly as in the proof of Theorem 7.8 of [JL2] that the function

$$\tilde u(z,t) = |w|^{-n} u(z/w, -t/|w|^2)$$

extends smoothly across the origin. This implies that

$$|e^{2f}| \le C|w|^{-2}, \qquad |D^k T^l f| \le C_{k,l}|w|^{-l-k/2},$$

where D^k is a monomial in Z_α and $Z_{\bar\alpha}$ of degree k.

Multiplying (4.2) by a compactly supported cutoff function, integrating over $\mathbb{H}^n$, and letting the cutoff approach 1, it is easy to check that the conditions above insure that the boundary terms vanish. Thus the right-hand side of (4.2) vanishes identically.

If $n > 1$, $E_{\alpha\bar{\beta}} = 0$ implies that f is the real part of a CR holomorphic function on $\mathbb{H}^n$. If $n = 1$, on the other hand, we have

$$f_{\bar{\beta}\beta\alpha} = -n(f_{\bar{\beta}}f_\beta + e^{2f} + if_0)_\alpha = -n(G_\alpha + D_\alpha + E_\alpha) = 0.$$

Thus, as in §3, we conclude again that f is the real part of a holomorphic function. The rest of the argument proceeds exactly as in §3. □

This corollary immediately yields an alternative proof of Theorem A, since if the contact form $\theta = u^{2/n}\Theta$ on $\mathbb{H}^n$ is the pullback of a smooth contact form on the sphere, then u automatically satisfies the hypothesis of the corollary.

When $n = 1$, the three divergence formulas (4.2)–(4.4) are identical. For $n > 1$, however, it is remarkable that the formula (4.2) used in the proof above is not unique. Both (4.2) and (4.3) have sums of squares on the right-hand sides. Thus any convex combination of these two formulas would work equally well. Moreover, although the right-hand side of (4.4) is not positive, any sufficiently small multiple of it can be added to the other two formulas to produce yet another useful identity. As we mentioned in the introduction, the reason for the existence of these formulas is a mystery.

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