

HADWIGER'S TRANSVERSAL THEOREM IN HIGHER DIMENSIONS

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1. INTRODUCTION

In 1957 H. Hadwiger proved the following theorem [5], which generalized previous results independently obtained by B. Grünbaum [3] and V. Klee [7]:

Hadwiger's Transversal Theorem. *Suppose $\mathcal{B} = \{B_1, \dots, B_n\}$ is a family of pairwise disjoint compact convex sets in the real affine plane \mathbf{A}^2 . Then the members of \mathcal{B} have a common transversal line if and only if there is a linear ordering of \mathcal{B} such that for every set of three indices i, j, k there is a directed line which meets B_i, B_j, B_k consistently with the ordering of \mathcal{B} .*

Since then, attempts to prove corresponding results about hyperplane transversals in higher dimensions have met with little success; the only result in the same spirit as Hadwiger's which we are aware of and which holds in arbitrary dimension is the following theorem of F. A. Valentine [8], whose "special position" hypothesis is rather restrictive (we state it in dimension 3, as does Valentine, who then remarks that a corresponding result holds in higher dimensions):

Valentine's Theorem. *Let \mathcal{F} be a family of compact convex sets in E_3 . Also suppose there exist three distinct planes P_i ($i = 1, 2, 3$) in E_3 containing a common line such that for each triple of members M_1, M_2, M_3 in \mathcal{F} each pair of the triple is strictly separated from the remaining member of the triple by a translate of either P_1 or P_2 or P_3 , and this correspondence is cyclic. If every four members of \mathcal{F} have a common intersecting plane, then all the members of \mathcal{F} have a common intersecting plane.*

The purpose of this paper is to extend Hadwiger's theorem to hyperplane transversals of compact convex sets in d dimensions, $d \geq 2$. The "ordering" of the sets is replaced by their "order type," and the condition that no two have

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a common point by the condition that the family be “separated.” (Our result is easily seen to include Valentine’s theorem as a special case.)

Theorem 1. *Suppose $\mathcal{B} = \{B_1, \dots, B_n\}$ is a separated family of compact convex sets in real affine d -space \mathbf{A}^d . Then the members of \mathcal{B} have a common transversal hyperplane if and only if there is a configuration of points $\mathcal{C} = \{x_1, \dots, x_n\}$ in \mathbf{A}^{d-1} such that for every set of $d + 1$ indices i_1, \dots, i_{d+1} there is an oriented hyperplane H which meets $B_{i_1}, \dots, B_{i_{d+1}}$ consistently with the order type of \mathcal{C} .*

A few definitions are in order. By a *separated* family of convex sets in \mathbf{A}^d we mean one for which no d members have a common transversal $d - 2$ -flat; e.g., separation in the plane just amounts to pairwise disjointness. (It is not hard to show that \mathcal{B} is a separated family in \mathbf{A}^d if and only if every set of k elements of \mathcal{B} can be separated from every set of $d - k$ other elements of \mathcal{B} by a hyperplane; however we will never use this fact.) By the *order type* of a numbered set of points $\mathcal{C} = \{x_1, \dots, x_n\}$ in \mathbb{R}^d we mean the family of orientations of its $d + 1$ -tuples, i.e., the family

$$\left(\operatorname{sgn} \det \begin{pmatrix} 1 & x_{i_0}^1 & \cdots & x_{i_0}^d \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{i_d}^1 & \cdots & x_{i_d}^d \end{pmatrix} \right)_{1 \leq i_0 < \cdots < i_d \leq n}$$

(see §1 of [4] for further details). Thus the condition of separation of \mathcal{B} assures that every set of representatives $\{b_{i_j} \in H \cap B_{i_j} \mid 1 \leq j \leq d + 1\}$ will have the same order type, and we can talk about the order type of $\{H \cap B_{i_1}, \dots, H \cap B_{i_{d+1}}\}$ itself.

In §2 we discuss the notion of a minimal Radon partition, which turns out to be crucial to the proof, and prove a key property which may be thought of as a “Radon exchange principle.” This is then used in §3 to prove Theorem 1. We conclude with a few remarks.

We express our appreciation to a referee for suggesting a shorter proof of Lemma 2 and for some salient comments about oriented matroids.

2. MINIMAL RADON PARTITIONS

In order to generalize Hadwiger’s theorem we make use of the notion of a minimal Radon partition, and we begin with a few elementary facts about these. For the purposes of this paper, a *minimal Radon partition* (MRP) in \mathbf{A}^d will mean a set of $d + 2$ points of \mathbf{A}^d in general position, together with a decomposition $S = T \cup V$ into two subsets with the property that

$$\operatorname{conv}(T) \cap \operatorname{conv}(U) \neq \emptyset.$$

Why “minimal”? Because it is easily seen that a set of $d + 1$ points in general position has no such decomposition (they form the vertices of a simplex, and any subset can be split from its complement by a suitable hyperplane); on the

other hand, Radon's theorem (see for example [2]) asserts that for $d + 2$ points such a decomposition always exists. Moreover, it is well known (and easy to see) that such a decomposition of S is necessarily unique, hence we may speak of *the* MRP associated to a set S of $d + 2$ points in general position. If we identify \mathbf{A}^d with \mathbb{R}^d , and write $S = \{x_1, \dots, x_{d+2}\}$, then the MRP $S = T \cup V$ is a decomposition of S into two sets, say $\{x_1, \dots, x_k\}$ and $\{x_{k+1}, \dots, x_{d+2}\}$, such that

$$\sum_{i=1}^k a_i x_i = \sum_{i=k+1}^{d+2} b_i x_i$$

for some choice of $a_i, b_i > 0$ with $\sum a_i = 1$ and $\sum b_i = 1$. In other words, an MRP arises from writing $\sum a_i x_i = 0$ with $\sum a_i = 0$, and letting T (resp. V) be the set of vertices having positive (resp. negative) coefficients.

Lemma 1. *If $S = T \cup V$ and $S' = T' \cup V'$ are each numbered configurations of points in general position in \mathbf{A}^d having the same order type, with T (resp. V) corresponding to T' (resp. V') in the numbering, then the first is a Radon partition if and only if the second is.*

Proof. The condition of $S = T \cup V$ not being a Radon partition entails the existence of a hyperplane spanned by d of the points of S which separates the remaining points of T and V . (One sees this by rotating any separating hyperplane about increasingly constrained $d - 2$ -flats until it contacts d of the points; the process is reversible.) Hence the condition of $S = T \cup V$ being a Radon partition is expressible purely in terms of the orientations of $d + 1$ -tuples of points of S , and this information is contained in the order type. \square

The following result can be viewed as an exchange principle for minimal Radon partitions. Surprisingly, it does not seem to appear in the literature surrounding Radon's theorem.

Lemma 2 (Radon exchange). *If $S = T \cup V$ is an MRP in \mathbf{A}^d and x is any point of \mathbf{A}^d in general position with respect to the points of S , then either*

- (1) $\exists t \in T$ such that $S \cup \{x\} \setminus \{t\} = (T \cup \{x\}) \setminus \{t\} \cup V$ is an MRP, or
- (2) $\exists v \in V$ such that $S \cup \{x\} \setminus \{v\} = (T \cup \{x\}) \cup (V \setminus \{v\})$ is an MRP.

(Note the asymmetry in T, V . This says that if we wish to add a new point to T , we can do so either by removing a point of T or, failing that, by removing a point of V .)

Proof. By hypothesis, there are constants $a_i, b_i > 0$ such that

$$(1) \quad \sum a_i t_i = \sum b_i v_i, \quad \sum a_i = \sum b_i.$$

Since x is affinely dependent on S , we may write

$$(2) \quad x + \sum c_i t_i = \sum d_i v_i, \quad 1 + \sum c_i = \sum d_i$$

for some c_i, d_i not necessarily positive. We wish to conclude from (1) and (2) that

$$x + \sum e_i t_i = \sum f_i v_i, \quad 1 + \sum e_i = \sum f_i,$$

with all $e_i, f_i \geq 0$ and some e_i or $f_i = 0$. But this follows immediately by subtracting k times (1) from (2), where k is the minimum of the ratios $c_i/a_i, d_i/b_i$. \square

(Note: It is not difficult to see that the point $t \in T$ or $v \in V$ which is replaced by x has the following geometric description. If C is the cell complex formed by all the hyperplanes joining sets of d points selected from S , then x lies in the interior of some cell Δ of C . If Δ is a bounded cell, precisely one vertex t of Δ will belong to T ; this is the point which x replaces. If Δ is an unbounded cell, then either x is adjacent to a (unique) vertex $t \in T$ of Δ , in which case this is again the point which x replaces, or else the antipodal cell $\bar{\Delta}$ of Δ will have precisely one vertex v belonging to V ; in the latter case it is v which x replaces.)

Lemma 3. *If K is a hyperplane in \mathbb{A}^d and $S = T \cup V$ is an MRP in K , then any hyperplane H which weakly separates T from V must be K itself.*

Proof. If the two closed halfspaces of \mathbb{A}^d determined by H contain T and V respectively, then the same is true of the two closed halfspaces of K determined by $H \cap K$ (if H and K are distinct hyperplanes); but this is impossible since $S = T \cup V$ is an MRP in K . \square

Lemma 4. *Given a set S of $d + 2$ points in general position in \mathbb{A}^d partitioned into three sets, $S = T' \cup V' \cup W$, suppose no hyperplane through the points of W separates T' from V' . Then $\exists T \supset T'$ and $V \supset V'$ with $T, V \subset S$ such that $S = T \cup V$ is an MRP.*

Proof. Let $A = \text{conv}(T' \cup W)$ and $B = \text{conv}(V' \cup W)$. Then $\text{relint}(A) \cap \text{relint}(B) \neq \emptyset$, since otherwise A and B would be weakly separated by a hyperplane which would necessarily pass through the points of W , contrary to hypothesis. This implies that

$$\sum a_i t'_i + \sum b_i w_i = \sum c_i v'_i + \sum d_i w_i, \quad \sum a_i + \sum b_i = \sum c_i + \sum d_i,$$

for some positive coefficients a_i, b_i, c_i, d_i , and—transposing the terms involving the w_i so that each w_i occurs only once—we get a relation of the desired form. (Notice that no w_i cancels out, since then we would have a Radon partition of fewer than $d + 2$ points, which is impossible.) \square

3. PROOF OF THEOREM 1

We begin as in [5]. For each $i = 1, \dots, n$, fix a point $c_i \in B_i$. For $t \in [0, 1]$, let $B_i(t)$ be the contraction of B_i by a factor of t about c_i ; i.e.,

$$B_i(t) = \{c_i + t(b - c_i) \mid b \in B_i\}.$$

If $t_1 < t_2$ and the hypothesis holds for $B_1(t_1), \dots, B_n(t_1)$, then clearly it holds for $B_1(t_2), \dots, B_n(t_2)$ as well. Hence, letting

$$\tau = \inf\{t \mid \text{the hypothesis holds for } B_1(t), \dots, B_n(t)\},$$

we see that the hypothesis holds for $t > \tau$ and fails for $t < \tau$.

In fact, it holds for $t = \tau$ as well, by the following argument. Fix i_1, \dots, i_{d+1} , let t_1, t_2, \dots be a decreasing sequence converging to τ , and—for each $i \in \{i_1, \dots, i_{d+1}\}$, $j \in \mathbf{Z}$ —let $y_i^j \in H_j \cap B_i(t_j)$ for a suitable oriented hyperplane H_j with positive unit normal vector v_i . Replacing t_1, t_2, \dots by a subsequence, if necessary, we may assume, by compactness, that $y_i^1, y_i^2, \dots \rightarrow y_i \in B_i$ for each $i \in \{i_1, \dots, i_{d+1}\}$, and $v_i \rightarrow v$. Hence—by continuity—the oriented hyperplane H with positive unit normal vector v passing through y_{i_1} , say, contains all of $y_{i_1}, \dots, y_{i_{d+1}}$, and their order type is weakly consistent with that of \mathcal{C} (i.e., consistent except for possible degeneracies). However, no degeneracies can occur by the assumption that the original family B_1, \dots, B_n is separated, and so the hypothesis holds for $t = \tau$.

If $\tau = 0$, the theorem is trivial since it is then immediate that all of the points c_1, \dots, c_n belong to a single hyperplane. Hence we may assume that the hypothesis is satisfied for the original convex sets B_i but not for any proper contractions $B_i(t)$, $t < 1$. It follows that for some set of $d + 1$ sets among B_1, \dots, B_n , say for B_1, \dots, B_{d+1} , there is an oriented hyperplane H which meets them consistently with the order type of \mathcal{C} , but not for any proper contraction $B_1(t), \dots, B_{d+1}(t)$, $t < 1$. We claim H meets all of B_1, \dots, B_n .

First notice that H must be a supporting hyperplane to each of B_1, \dots, B_{d+1} , i.e., that $H \cap B_i \neq \emptyset$, and B_i is contained in one of the closed halfspaces determined by H . This is seen as follows. Suppose H splits B_{d+1} . For each i , $1 \leq i \leq d$, we have both $B_i \cap H^+ \neq \emptyset$ and $B_i \cap H^- \neq \emptyset$, where, as usual, H^+ and H^- are the two closed halfspaces determined by H . Suppose first that no B_i lies entirely in H , so that (say) B_1, \dots, B_k meet $\text{int}(H^+)$ and B_{k+1}, \dots, B_d meet $\text{int}(H^-)$. Then letting b_i be an arbitrary point of $B_i \cap H$ for each i , we see that

$$\{b_1, \dots, b_d\} = \{b_1, \dots, b_k\} \cup \{b_{k+1}, \dots, b_d\}$$

cannot form a Radon partition in H , since (by the separation condition) these points are in general position in a $d - 1$ -plane, and there are only d of them. Thus some $d - 2$ -flat $F \subset H$ separates them, and we can rotate H around F slightly, to a new hyperplane H' , which now splits each of B_1, \dots, B_d and in addition continues to split B_{d+1} . (Figure 1 illustrates the case $d = 2$.) Hence we can contract B_1, \dots, B_d further about c_1, \dots, c_d , which is impossible.

Now suppose, more generally, that B_1, \dots, B_j are actually contained in H , and that B_{j+1}, \dots, B_k meet $\text{int}(H^+)$, while B_{k+1}, \dots, B_d meet $\text{int}(H^-)$. In this case, we let b_i be an arbitrary point of $B_i \cap H$ only for $i = j + 1, \dots, d$, and choose $F \subset H$ not only separating $\{b_{j+1}, \dots, b_k\}$ from $\{b_{k+1}, \dots, b_d\}$,

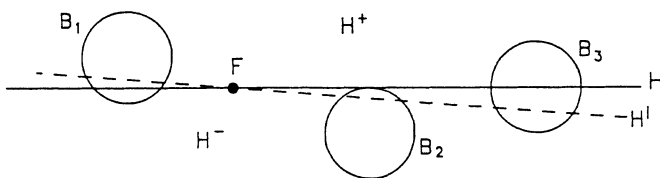


FIGURE 1

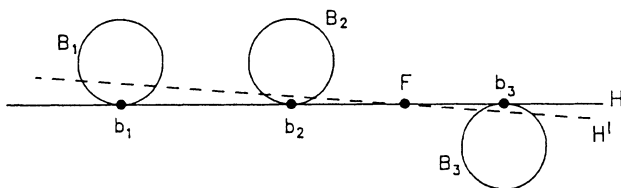


FIGURE 2

but also passing through the centers of contraction c_1, \dots, c_j (this can be done since $c_1, \dots, c_j, b_{j+1}, \dots, b_d$ are the vertices of a simplex in H). This guarantees that—after we rotate H about F as above—the new hyperplane will not only split B_{j+1}, \dots, B_{d+1} , but pass through c_1, \dots, c_j as well. Hence, as above, we can contract B_1, \dots, B_{d+1} further about c_1, \dots, c_{d+1} , which is impossible.

Since H supports B_1, \dots, B_{d+1} , we have $B_i \subset H^+$ or $B_i \subset H^-$ (or both, i.e. $B_i \subset H$) for each $i = 1, \dots, d+1$. Suppose first that none of B_1, \dots, B_{d+1} is contained in H , and that, without loss of generality, say $B_1, \dots, B_k \subset H^+$ and $B_{k+1}, \dots, B_{d+1} \subset H^-$, and—for each $i = 1, \dots, d+1$ —let $b_i \in B_i \cap H$. In this case, we claim that

$$\{b_1, \dots, b_{d+1}\} = \{b_1, \dots, b_k\} \cup \{b_{k+1}, \dots, b_{d+1}\}$$

is an MRP in H . Otherwise, as above, there would be a $d - 2$ -flat $F \subset H$ separating the first set from the second, and—by rotating H slightly about F in the proper direction—we could find a new hyperplane H' splitting each of B_1, \dots, B_{d+1} and still meeting them in the same order type, so that H' would still meet $B_1(t), \dots, B_{d+1}(t)$ correctly for some $t < 1$, which is impossible. (See Figure 2.)

It may happen, more generally, that not every one of B_1, \dots, B_{d+1} sticks out of H . Say $B_1, \dots, B_j \subset H$, $B_j, \dots, B_k \subset H^+$ (but not H^-), and $B_{k+1}, \dots, B_{d+1} \subset H^-$ (but not H^+). Let $b_i \in B_i \cap H$ for $i = j+1, \dots, d+1$. Then no $d - 2$ -flat F through c_1, \dots, c_j can separate $\{b_{j+1}, \dots, b_k\}$ from $\{b_{k+1}, \dots, b_{d+1}\}$, since otherwise we could again rotate H about F slightly to arrive at a new hyperplane H' splitting each of B_{j+1}, \dots, B_{d+1} and still meeting them correctly, and at the same time passing through c_1, \dots, c_j , so that H'

would still meet $B_1(t), \dots, B_{d+1}(t)$ correctly for some $t < 1$, which is again impossible. Hence, by Lemma 4, the points c_1, \dots, c_j may be partitioned into two sets, say $\{c_1, \dots, c_m\}$ and $\{c_{m+1}, \dots, c_j\}$, such that

$$\begin{aligned} & \{c_1, \dots, c_j, b_{j+1}, \dots, b_{d+1}\} \\ &= \{c_1, \dots, c_m, b_{j+1}, \dots, b_k\} \cup \{c_{m+1}, \dots, c_j, b_{k+1}, \dots, b_{d+1}\} \end{aligned}$$

is an MRP in H .

Now let p be any index $> d + 1$. We will show that H meets B_p . Suppose $B_p \subset H^+$. By Lemma 1, since the property of a union of two sets being a Radon partition is an invariant of the order type,

$$\{x_1, \dots, x_{d+1}\} = \{x_1, \dots, x_k\} \cup \{x_{k+1}, \dots, x_{d+1}\}$$

is a Radon partition. Let us apply Lemma 2 to this MRP and the point x_p . Then, for some $j = 1, \dots, d + 1$, either

$$\{x_1, \dots, \widehat{x}_j, \dots, x_{d+1}, x_p\} = \{x_1, \dots, \widehat{x}_j, \dots, x_k, x_p\} \cup \{x_{k+1}, \dots, x_{d+1}\}$$

or

$$\{x_1, \dots, \widehat{x}_j, \dots, x_{d+1}, x_p\} = \{x_1, \dots, x_k, x_p\} \cup \{x_{k+1}, \dots, \widehat{x}_j, \dots, x_{d+1}\}$$

is an MRP (depending on whether $j \leq k$ or $j \geq k + 1$). By hypothesis, there is a hyperplane K which meets $B_1, \dots, \widehat{B}_j, \dots, B_{d+1}, B_p$, and does so consistently with \mathcal{E} . If we let $b'_i \in K \cap B_i$ for each $i = 1, \dots, \widehat{j}, \dots, d + 1, p$, the points b'_i form an MRP in K , and moreover

$$b'_i \in H^+ \quad \text{for } i = 1, \dots, k \text{ and } i = p,$$

and

$$b'_i \in H^- \quad \text{for } i = k + 1, \dots, d + 1.$$

We can therefore invoke Lemma 3 and conclude that $H = K$, i.e., that H meets B_i as asserted. \square

4. REMARKS

1. Hadwiger's theorem holds not only for a finite family of compact convex sets in the plane, but for any countable family $\mathcal{B} = \{B_1, B_2, \dots\}$. The same simple compactness argument which Hadwiger uses in [5] to extend the result from finite to countable families works also in our case (we do not reproduce the details). The result is

Corollary. *If $\mathcal{B} = \{B_1, B_2, \dots\}$ is a countable separated family of compact convex sets in \mathbf{A}^{d-1} , then the members of \mathcal{B} have a common transversal hyperplane if and only if there is a configuration of points $\mathcal{E} = \{x_1, x_2, \dots\}$ in*

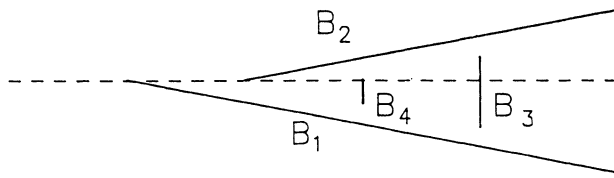


FIGURE 3

\mathbf{A}^{d-1} such that for any set of $d+1$ indices $i_1, \dots, i_{d+1} \in \mathbb{N}$ there is an oriented hyperplane which meets $B_{i_1}, \dots, B_{i_{d+1}}$ consistently with the order type of \mathcal{E} .

2. We have no examples indicating that the separation condition in Theorem 1 is necessary for the result to hold, even in dimension 2; it would therefore be interesting to know to what extent, if any, it can be weakened. [Added in proof: Rephael Wenger has recently shown that separation *can* be removed completely in dimension 2 (R. Wenger, *Hadwiger's theorem for intersecting convex sets*, *Discrete Comput. Geom.* (to appear)); the question remains open, however, in higher dimensions.] It is surprising to note, however, that the condition in Theorem 1 does not imply the existence of a common transversal meeting the members of \mathcal{B} in the *same* order type as \mathcal{E} , even though a casual reading of the statement would suggest that that is the case. Indeed, an example of G. Kertész [6] shows that even for $d = 2$ this conclusion would be false; see Figure 3, in which the sets B_1, B_2, B_3, B_4 have common transversals meeting every *three* in the corresponding order, but not all four.

3. Lemma 2 can also be proved for oriented matroids (see [1]), as Las Vergnas has pointed out. More generally, in fact, Theorem 1 still holds if we replace the configuration \mathcal{E} by a simple oriented matroid; the proof is virtually unchanged in that case.

4. In view of Theorem 1, a natural question to investigate would now be: Are there order and separation conditions for a family of compact convex sets in d -space to have a transversal k -flat?

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