THE INVARIANT TRACE FORMULA. II.
GLOBAL THEORY

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INTRODUCTION

The purpose of this article is to prove an explicit invariant trace formula. In the preceding paper \([1(j)]\), we studied two families of invariant distributions. Now we shall exhibit these distributions as terms on the two sides of the invariant trace formula. We refer the reader to the introduction of \([1(j)]\), which contains a general discussion of the problem. In this introduction, we shall describe the formula in more detail.

Let \( G \) be a connected reductive algebraic group over a number field \( F \), and let \( f \) be a function in the Hecke algebra on \( G(\mathbb{A}) \). We already have a "coarse" invariant trace formula

\[
\sum_{\sigma \in \mathcal{H}} I_{\sigma}(f) = \sum_{\chi \in \mathcal{X}} I_{\chi}(f),
\]

which was established in an earlier paper \([1(c)]\). This will be our starting point here. The terms on each side of (1) are invariant distributions, but as they stand, they are not explicit enough to be very useful. After recalling the formula (1) in \( \S 2 \), we shall study the two sides separately in \( \S \S 3 \) and 4. These two sections

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are the heart of the paper. Building on earlier investigations of non-invariant distributions [1(e), 1(g)], we shall establish finer expansions for each side of (1). The resulting identity

$$\sum_{M} |W_0^M| |W_0^G|^{-1} \sum_{\gamma \in (M(F))_{M,S}} a^M(S, \gamma) I_M(\gamma, f)$$

will be our explicit trace formula. The terms $I_M(\gamma, f)$ and $I_M(\pi, f)$ in (2) are essentially the invariant distributions studied in [1(j)]. The functions $a^M(S, \gamma)$ and $a^M(\pi)$ depend only on a Levi subgroup $M$, and are global in nature. They are strongly dependent on the discrete subgroup $M(F)$ of $M(A)$. We refer the reader to §§3 and 4 for more detailed description of these objects, as well as the sets $(M(F))_{M,S}$ and $\Pi(M, t)$.

The paper [1(c)] relied on certain hypotheses in local harmonic analysis. Some of these have since been resolved by the trace Paley-Wiener theorems. Others concern the density of characters in spaces of invariant distributions, and are not yet known in general. In fact, to even define the invariant distributions $I_M(\gamma, f)$ and $I_M(\pi, X, f)$, we had to introduce an induction hypothesis in [1(j)]. This hypothesis remained in force throughout [1(j)], and will be carried into this paper. We shall finally settle the matter in §5. We shall show that the invariant distributions in the trace formula are all supported on characters. Using [1(j), Theorem 6.1] we shall first establish in Lemma 5.2 that the distributions on the right-hand side of (2) have the required property. We shall then use formula (2) itself to deduce the same property of the distributions on the left (Theorem 5.1). This is a generalization of an argument introduced by Kazhdan in his Maryland lectures (see [8, 10]). Theorem 6.1 (in [1(j)]) and Theorem 5.1 (here) are actually simple versions of a technique that can be applied more generally. They provide a good introduction to the more complicated versions used for base change [2, §§II.10, II.17].

It is not known whether the right-hand side of (2) converges as a double integral over $t$ and $\pi$. It is a difficulty which originates with the Archimedean valuations of $F$. On the other hand, some result of this nature will definitely be required for many of the applications of the trace formula. In §6 we shall prove a weak estimate (Corollary 6.5) for the rate of convergence of the sum over $t$. It will be stated in terms of multipliers for the Archimedean part $\prod_{v \in S_\infty} G(F_v)$ of $G(A)$. One would then hope that by varying the multipliers, one could separate the terms according to their Archimedean infinitesimal character. For base change, this is in fact what happens. One can use the estimate to eliminate the problems caused by the Archimedean primes [2, §§II.15]. In general, Corollary 6.5 seems to be a natural device for isolating the contributions of a given infinitesimal character.
It is useful to have simple versions of the trace formula for functions

\[ f = \prod_v f_v \]

that are suitably restricted. Since the terms in (2) are all invariant distributions, we will be able to impose conditions on \( f \) strictly in terms of its orbital integrals. If at one place \( v \) the semisimple orbital integrals of \( f_v \) are supported on the elliptic set, then all the terms with \( M \neq G \) on the right-hand side of (2) vanish. If the same thing is true at two places, the terms with \( M \neq G \) on the left-hand side also vanish. These two assertions comprise Theorem 7.1. They are simple consequences of the descent and splitting formulas in [1(j), §§8-9]. We shall also see that in certain cases the remaining terms take a particularly simple form (Corollaries 7.3, 7.4, 7.5).

As with the preceding paper [1(j)], we shall conclude (§8) by discussing the example of \( GL(n) \). Groups related to \( GL(n) \) by inner twisting or cyclic base change are the simplest examples of general rank for which one can attempt a comparison of trace formulas. However, one must first establish some properties of the trace formula of \( GL(n) \) itself. By imposing less stringent conditions than those of §7, we shall establish more delicate vanishing properties. The resulting formula for \( GL(n) \) is then what should be compared with the twisted trace formula over a cyclic extension.

1. Assumptions on \( G \)

Let \( G \) be a connected component of a reductive algebraic group over a number field \( F \). We assume that \( G(F) \neq \emptyset \). As in previous papers, we shall write \( G^+ \) for the group generated by \( G \), and \( G^0 \) for the connected component of 1 in \( G^+ \). The component \( G/F \) will remain fixed throughout the paper except in §5.

We shall fix a minimal Levi subgroup \( M_0 \) of \( G^0 \) over \( F \). This was the point of view in the paper [1(g)], and we shall freely adopt the notation at the beginning of [1(g)]. In particular, we have the maximal \( F \)-split torus \( A_0 = A_{M_0} \) of \( G^0 \) and the real vector space \( a_0 = a_{M_0} \). On \( a_0 \), we fix a Euclidean norm which is invariant under the restricted Weyl group \( W_0 \) of \( G^0 \). We also have the finite collection \( \mathcal{L} = \mathcal{L}^G \) of (nonempty) Levi subsets \( M \subset G \) for which \( M^0 \) contains \( M_0 \), and the finite collection \( \mathcal{F} = \mathcal{F}^G \) of (nonempty) parabolic subsets \( P \subset G \) such that \( P^0 \) contains \( M_0 \). These collections can of course also be defined with \( G^0 \) in place of \( G \) in which case we shall write \( \mathcal{L}^0 = \mathcal{L}^{G^0} \) and \( \mathcal{F}^0 = \mathcal{F}^{G^0} \). Observe that \( M \rightarrow M^0 \) is a map from \( \mathcal{L} \) into \( \mathcal{L}^0 \) which is neither surjective nor injective. Finally, we have the maximal compact subgroup

\[ K = \prod_v K_v = \prod_v (K_v^+ \cap G^0(F_v)) \]
of $G^0(A)$. Set
\[ K_v^G = K_v^+ \cap G(F_v) \quad \text{and} \quad K^G = \prod_v K_v^G. \]

In [1(g)], we studied the geometric side of the (noninvariant) trace formula as a distribution on $C_c^\infty(G(A)^1)$. However, to deal with the other side of the trace formula, and to exploit the present knowledge of invariant harmonic analysis, we need to work with $K$-finite functions. This was the point of view of [1(i)] and [1(j)]. We shall also make use of the notation from §1 of these two papers, often without comment. In §11 of [1(i)] we defined the Hecke spaces $\mathcal{H}(G(F_S))$ and $\mathcal{H}_{ac}(G(F_S))$, where $S$ is any finite set of valuations of $F$ with the closure property. Recall that $\mathcal{H}_{ac}(G(F_S))$ consists of the Hecke functions $f$ on $G(F_S)$ of “almost compact” support, in the sense that for any $b \in C_c^\infty(a_{G,S})$, the function
\[ f^b(x) = f(x)b(H_G(x)), \quad x \in G(F_S), \]
belongs to $\mathcal{H}(G(F_S))$. Let $S_{\text{ram}}$ be the finite set of valuations of $F$ at which $G$ is ramified. (By agreement, $S_{\text{ram}}$ contains $S_{\infty}$, the set of Archimedean valuations of $F$.) Suppose that $S$ contains $S_{\text{ram}}$. We can multiply any function on $G(F_S)$ with the characteristic function of $\prod_{v \notin S} K_v^G$, thereby identifying it with a function on $G(A)$. This allows us to define the adelic Hecke spaces
\[ \mathcal{H}(G(A)) = \lim_S \mathcal{H}(G(F_S)) \]
and
\[ \mathcal{H}_{ac}(G(A)) = \lim_S \mathcal{H}_{ac}(G(F_S)). \]

Similarly, we can define the Hecke space
\[ \mathcal{H}(G(A)^1) = \lim_S \mathcal{H}(G(F_S)^1), \]
on $G(A)^1$. The terms in the trace formula are actually distributions on $\mathcal{H}(G(A)^1)$. However, the restriction map $f \to f^1$ sends $\mathcal{H}_{ac}(G(A))$ to $\mathcal{H}(G(A)^1)$, and we shall usually regard the terms as distributions on $\mathcal{H}(G(A))$ or $\mathcal{H}_{ac}(G(A))$ that factor through this map.

In §11 of [1(i)] we also defined function spaces $\mathcal{F}(G(F_S))$ and $\mathcal{F}_{ac}(G(F_S))$ on
\[ \Pi_{\text{temp}}(G(F_S)) \times a_{G,S}. \]

Let $\Pi(G(A))$ (respectively $\Pi_{\text{unit}}(G(A))$, $\Pi_{\text{temp}}(G(A))$) denote the set of equivalence classes of irreducible admissible (respectively unitary, tempered) representations of $G^+(A)$ whose restrictions to $G^0(A)$ remain irreducible. Observe that the disconnected group
\[ \Xi_A = \lim_S \Xi_S = \lim_S \text{Hom}(G^+(F_S)/G^0(F_S), \mathbb{C}^*), \]

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acts freely on each of these sets. We shall write \( \{\Pi(G(A))\} \), \( \{\Pi_{\text{unit}}(G(A))\} \), and \( \{\Pi_{\text{temp}}(G(A))\} \) for the sets of orbits. They correspond to the sets of representations of \( G^0(A) \) obtained by restriction. Suppose that \( S \) contains \( S_{\text{ram}} \). Then \( a_{G,S} = a_G \). We can identify any function \( \phi \) on \( \Pi_{\text{temp}}(G(F_S)) \times a_G \) with the function on \( \Pi_{\text{temp}}(G(A)) \times a_G \) whose value at
\[
(\pi, X), \quad \pi = \otimes_v \pi_v, \ X \in a_G,
\]
equals
\[
\phi \left( \bigotimes_v \pi_v, X \right) \prod_{v \not\in S} \text{tr} \left( \int_{K_v} \pi_v(k_v) dk_v \right).
\]
With this convention, we then define
\[
\mathcal{J}(G(A)) = \lim_S \mathcal{J}(G(F_S))
\]
and
\[
\mathcal{J}_{\text{ac}}(G(A)) = \lim_S \mathcal{J}_{\text{ac}}(G(F_S)).
\]
Keep in mind that any of our definitions can be transferred from \( G \) to a Levi component \( M \in \mathcal{L} \). In particular, we have spaces \( \mathcal{J}(M(A)) \) and \( \mathcal{J}_{\text{ac}}(M(A)) \). It is easy to see that the maps \( f \rightarrow f_M \) and \( f \rightarrow \phi_M(f) \), described in [1(i)] and [1(j)], extend to continuous maps from \( \mathcal{J}_{\text{ac}}(G(A)) \) to \( \mathcal{J}_{\text{ac}}(M(A)) \).

We are going to use the local theory of [1(j)] to study the trace formula. Because the Archimedean twisted trace Paley-Wiener theorem has not yet been established in general, the result of [1(j)] apply only if \( G \) equals \( G^0 \), or if \( G \) is an inner twist of a component
\[
G^* = \left( GL(n) \times \cdots \times GL(n) \right) \rtimes \theta^*.
\]
We shall therefore assume that \( G \) is of this form. However, we shall write the paper as if it applied to a general nonconnected group. With the exception of a Galois cohomology argument in the proof of Theorem 5.1, and a part of the appendix which relies on the Archimedean trace Paley-Wiener theorem, the arguments of this paper all apply in general.

Suppose that \( M \in \mathcal{L} \) and that \( S \) is a finite set of valuations of \( F \) with the closure property. In [1(j)] we defined invariant distributions
\[
I_M(\gamma, f) = J_M(\gamma, f) - \sum_{L \in \mathcal{L}_0(M)} \hat{I}_M^L(\gamma, \phi_L(f)), \quad \gamma \in M(F_S),
\]
and
\[
I_M(\pi, X, f) = J_M(\pi, X, f) - \sum_{L \in \mathcal{L}_0(M)} \hat{I}_M^L(\pi, X, \phi_L(f)),
\]
\( \pi \in \Pi(M(F_S)), \ X \in a_{M,S}, \) with \( f \in H_{\text{ac}}(G(F_S)) \). (Recall that \( \mathcal{L}_0(M) \) denotes the set of Levi subsets \( L \) of \( G \) with \( M \subset L \subseteq G \).) These definitions were contingent on an induction hypothesis which we must carry into this paper. We
assume that for any $S$, and for any elements $M \in \mathcal{L}$ and $L \in \mathcal{L}_0(M)$, the
distributions
\[ I_M^L(\gamma), \quad \gamma \in M(F_S), \]
on $\mathcal{H}(L(F_S))$ are all supported on characters. (A distribution attached to $G$ is
supported on characters, we recall, if it vanishes on every function $f$ such that $f_G = 0$.) Then the distributions $I_M(\gamma)$, and, thanks to Theorem 6.1 of [1(j)],
also the distributions $I_M(\pi, X)$, are well defined. In Corollary 5.3 we shall
complete the induction argument by showing that the condition holds when $L$
is replaced by $G$.

The distributions $I_M(\gamma)$ and $I_M(\pi, X)$ have many parallel properties. However, there is one essential difference between the two. If $\pi \in \Pi(M(A))$ and
$X \in a_M$, it is easy to see that $I_M(\pi, X)$ can be defined as a distribution on
$\mathcal{H}(G(A))$ or even $\mathcal{H}_{ac}(G(A))$. This is a consequence of the original definition
of $J_M(\pi, X)$ in terms of normalized intertwining operators, and in particular,
the property $(R_g)$ of [1(i), Theorem 2.1]. On the other hand, if $\gamma$ belongs to
$M(A)$, there seems to be no simple way to define $I_M(\gamma)$ as a distribution on
$\mathcal{H}(G(A))$. This circumstance is responsible for a certain lack of symmetry in
the trace formula. The terms on the geometric side depend on a suitably large
finite set $S$ of valuations, while the terms on the spectral side do not.

If $G(A)$ is replaced by $G(A)^1$, we can obviously define the sets $\Pi(G(A)^1)$,
$\Pi_{\text{unit}}(G(A)^1)$ and $\Pi_{\text{temp}}(G(A)^1)$ as above. The terms on the spectral side of
the trace formula will depend on elements $M \in \mathcal{L}$ and representations $\pi \in
\Pi_{\text{unit}}(M(A)^1)$. We shall generally identify a representation $\pi \in \Pi_{\text{unit}}(M(A)^1)$
with the corresponding orbit
\[ \{ \pi_\mu : \mu \in ia_M^* \} \]
of $ia_M^*$ in $\Pi_{\text{unit}}(M(A))$. With this convention, let us agree to write
\[ J_M(\pi, f) = J_M(\pi_\mu, 0, f) \]
and
\[ I_M(\pi, f) = I_M(\pi_\mu, 0, f), \quad f \in \mathcal{H}_{ac}(G(A)), \]
for the values of the distributions at $X = 0$. The two terms on the right are
independent of $\mu$, and are therefore well defined functions of $\pi$. They also
depend only on the restriction $f^1$ of $f$ to $G(A)^1$. This notation pertains
also to the map $f_G$. For if $\pi$ is an arbitrary representation in $\Pi(G(A))$ and
$X \in a_G$, we have
\[ f_G(\pi, X) = J_G(\pi, X, f) = I_G(\pi, X, f). \]
Therefore, if $\pi$ belongs to $\Pi_{\text{unit}}(G(A)^1)$, it makes sense to write
\[ f_G(\pi) = f_G(\pi_\mu, 0) = \text{tr} \pi(f^1), \quad \mu \in ia_G^*, f \in \mathcal{H}_{ac}(G(A)). \]
2. THE INVARIANT TRACE FORMULA: FIRST VERSION

The first version of the noninvariant trace formula is summarized in [1(b), §5] and [1(c), (2.5)]. (See also [7].) It is an identity

\begin{equation}
\sum_{\rho \in \mathcal{O}} J_{\rho}(f) = J(f) = \sum_{\chi \in \mathcal{H}} J_{\chi}(f), \quad f \in C_c^\infty(G(\mathbb{A}))^{1},
\end{equation}

in which a certain distribution \( J \) on \( C_c^\infty(G(\mathbb{A}))^{1} \) is expanded in two different ways. The sets \( \mathcal{O} = \mathcal{O}(G, F) \) and \( \mathcal{H} = \mathcal{H}(G, F) \) parametrize orbit theoretic and representation theoretic data respectively, but the corresponding terms are not given as explicitly as one would like.

Suppose that \( J_{\rho}(f) \) stands for one of the summands in (2.1). Then \( J_{\rho} \) is a distribution on \( C_c^\infty(G(\mathbb{A}))^{1} \) which behaves in a predictable way,

\[ J_{\rho}(f^y) = \sum_{Q \in \mathcal{F}} |W_0^Q||W_0^G|^{-1} J_{\rho}(f_{Q,y}), \quad y \in G^0(\mathbb{A}), \]

under conjugation [1(c), Theorem 3.2; 7]. Since we want to take \( f \) to be in \( \mathcal{H}^{ac}(G(\mathbb{A})) \), we cannot use this formula. However, as in the proof of Lemma 6.2 of [1(i)], we can easily transform it to an alternate formula

\begin{equation}
J_{\rho}(L_h f) = \sum_{Q \in \mathcal{F}} |W_0^M||W_0^G|^{-1} J_{\rho}(R_{Q,h} f),
\end{equation}

which makes sense for functions \( f \in \mathcal{H}^{ac}(G(\mathbb{A})) \) and \( h \in \mathcal{H}(G^0(\mathbb{A}))^{1} \). Let \( \mathcal{L}_0 \) denote the set of elements \( L \in \mathcal{L} \) with \( L \neq G \). We then define an invariant distribution

\[ I_{\rho}(f) = I_{\rho}(G)(f), \quad f \in \mathcal{H}^{ac}(G(\mathbb{A})), \]

inductively by setting

\begin{equation}
I_{\rho}(f) = J_{\rho}(f) - \sum_{M \in \mathcal{L}_0} |W_0^M||W_0^G|^{-1} I_{\rho}(M)(\phi_M(f)), \quad f \in \mathcal{H}^{ac}(G(\mathbb{A})).
\end{equation}

The invariance of \( I_{\rho} \) follows from (2.2) and the analogous formula [1(i), (12.2)] for \( \phi_M \) (see [1(c), Proposition 4.1]). Implicit in the definition is the induction assumption that for any \( L \in \mathcal{L}_0 \), the distribution \( I_L^{1} \) is defined and is supported on characters. This is what allows us to write \( I_{\rho}^{1} \). Observe that this induction hypothesis is our second of the paper. However, in §§3 and 4 we shall establish explicit formulas for \( I_{\rho} \) and \( I_{\chi} \) in terms of \( I_M(\gamma) \) and \( I_M(\pi) \) respectively. This will reduce the second induction hypothesis to the primary one adopted in §1.

It is a simple matter to substitute (2.3) for each of the terms in (2.1). The result is an identity

\begin{equation}
\sum_{\rho \in \mathcal{O}} I_{\rho}(f) = I(f) = \sum_{\chi \in \mathcal{H}} I_{\chi}(f), \quad f \in \mathcal{H}^{ac}(G(\mathbb{A})).
\end{equation}
in which the invariant distribution
\begin{equation}
I(f) = J(f) - \sum_{M \in \mathcal{L}_G} |W_0^M| |W_0^G|^{-1} \tilde{I}^M(\phi_M(f)), \quad f \in \mathcal{H}_c(G(\mathbb{A})),
\end{equation}
is expanded in two different ways (see [1(c), Proposition 4.2]). This is the first version of the invariant trace formula. It was established in [1(c)] modulo certain hypotheses in local harmonic analysis. In later papers [1(g)] and [1(e)], we found more explicit formulas for the terms \( J_{\sigma}(f) \) and \( J_{\chi}(f) \) in (2.1). The purpose of this paper is to convert these formulas into explicit expansions of each side of the invariant formula (2.4). In the process, we will establish the required properties of local harmonic analysis.

3. THE GEOMETRIC SIDE

We shall derive a finer expansion for the left-hand side of (2.4). The result will be a sum of terms, indexed by orbits in \( G(F) \), which separate naturally into local and global constituents. We shall first review the results of [1(g)], which provide a parallel expansion for the noninvariant distributions on the left-hand side of (2.1).

Recall that \( \mathcal{O} = \mathcal{O}(G, F) \) is the set of equivalence classes in \( G(F) \), in which two elements in \( G(F) \) are considered equivalent if their semisimple Jordan components belong to the same \( G^0(F) \)-orbit. The formulas in [1(g)] were stated in terms of another equivalence relation on \( G(F) \), which is intermediate between that of \( \mathcal{O} \) and \( G^0(F) \)-conjugacy. It depends on a finite set \( S \) of valuations of \( F \). The \((G, S)\)-equivalence classes are defined to be the sets \( G(F) \cap (\sigma U)^{G^0(F)} = \{ \delta^{-1} \sigma u \delta : \delta \in G^0(F), u \in U \cap G^0(F) \} \) in which \( \sigma \) is a semisimple element in \( G(F) \), and \( U \) is a unipotent conjugacy class in \( G_\sigma(F_S) \). Any class \( \sigma \) in \( \mathcal{O} \) breaks up into a finite set \((\sigma)_{G,S}\) of \((G, S)\)-equivalence classes. The first main result of [1(g)] is Theorem 8.1, an expansion
\begin{equation}
J_{\sigma}(f) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \sum_{\gamma \in (M(F) \cap \sigma)_{M,S}} a^M(S, \gamma) J_M(\gamma, f),
\end{equation}
for any \( \sigma \in \mathcal{O} \) and any \( f \in \mathcal{C}_c^\infty(G(F_S)^1) \). Here \( S \) is any finite set of valuations of \( F \) which contains a certain set \( S_\sigma \) determined by \( \sigma \). The distributions \( J_{\sigma}(\gamma, f) \) are purely local, in the sense that they depend only on \( \gamma \) as an element in \( M(F_S) \). The functions \( a^M(S, \gamma) \) are what carry the global information. These were defined by formula (8.1) of [1(g)] (and also Theorem 8.1 of [1(f)]), in the case that \( S \) contains \( S_\sigma \).

Suppose that \( M \in \mathcal{L} \). A semisimple element \( \sigma \in M(F) \) is said to be \( F \)-elliptic in \( M \) if the split component of the center of \( M_\sigma \) equals \( A_M \). Suppose that \( S \) is any finite set of valuations of \( F \) which contains \( S_\infty \). We shall write
\[ K_S^M = \prod_{\nu \in S} K_\nu^M = \prod_{\nu \in S} (K_\nu^+ \cap M(F_\nu)). \]
Suppose that $\gamma$ is an element in $M(F)$ with semisimple Jordan component $\sigma$. Set $i^M(S, \sigma)$ equal to 1 if $\sigma$ is $F$-elliptic in $M$, and if for every $v \notin S$, the set 
\[ \text{ad}(M^0(F_v))\sigma = \{m^{-1}\sigma m: m \in M^0(F_v)\} \]
intersects the compact set $K^M_v$. Otherwise set $i^M(S, \sigma)$ equal to 0. Then define
\[ (3.2) \quad a^M(S, \gamma) = i^M(S, \sigma)|i^M(\sigma)|^{-1} \sum_{\{u: \sigma u \sim \gamma\}} a^M(S, u), \]
in the notation of $[1(g), (8.1)]$. This definition matches the one in $[1(g)]$ in the special case that $S$ contains $S'$, where $\sigma$ is the class in $G$ which contains $\sigma$.

The second main result of $[1(g)]$ is Theorem 9.2, an expansion
\[ (3.3) \quad J(f) = \sum_{M \in \mathcal{M}} |W_0^M| |W_0^G|^{-1} \sum_{\gamma \in (M(\mathfrak{f}))_{M,S}} a^M(S, \gamma)J_M(\gamma, f), \]
for any $f \in C_1^\infty(G(F_S)^1)$. Here, $\Delta$ is a compact neighborhood in $G(\mathbb{A})^1$, and $S$ is any finite set of valuations of $\mathbb{F}$ which contains a certain set $S_\Delta$ determined by $\Delta$. This latter set is large enough so that $\Delta$ is the product of a compact neighborhood in $G(F_{S_\Delta})$ with the characteristic function of $\prod_{v \notin S_\Delta} K^G_v$, and by definition,
\[ C_1^\infty(G(F_S)^1) = C_1^\infty(G(\mathbb{A})^1) \cap C_1^\infty(G(F_S)^1). \]
In $[1(g)]$ we neglected to write down the general definition (3.2) for $a^M(S, \gamma)$. This is required for the expansion (3.3) to make sense.

**Proposition 3.1.** Suppose that $S$ is a finite set of valuations which contains $S_{\sigma}$, and that $f$ is a function in $\mathcal{H}_e(G(F_S))$. Then
\[ I_{\sigma}(f) = \sum_{M \in \mathcal{M}} |W_0^M| |W_0^G|^{-1} \sum_{\gamma \in (M(\mathfrak{f}))_{M,S}} a^M(S, \gamma)I_M(\gamma, f). \]

**Proof.** By definition, $I_{\sigma}(f)$ equals the difference between $J_{\sigma}(f)$ and
\[ \sum_{L \in \mathcal{L}_0} |W_0^L| |W_0^G|^{-1} \hat{I}_{\sigma}(\phi_L(f)). \]
We can assume inductively that if $L \in \mathcal{L}_0$, the proposition holds for $I^L_\sigma$. Since $\phi_L$ maps $\mathcal{H}_e(G(F_S))$ to $\mathcal{H}_e(L(F_S))$, we obtain
\[ \hat{I}^L_{\sigma}(\phi_L(f)) = \sum_{M \in \mathcal{M}^L} |W_0^M| |W_0^G|^{-1} \sum_{\gamma \in (M(\mathfrak{f}))_{M,S}} a^M(S, \gamma)I^L_M(\gamma, \phi_L(f)). \]
This is valid whenever $S$ contains the finite set $S_{\sigma}$ associated to $L$. A look at the conditions defining $S_{\sigma}$ on p. 203 of $[1(g)]$ reveals that $S_{\sigma}$ contains $S^L_{\sigma}$, so we can certainly take any $S \supset S_{\sigma}$. Combining this formula with (3.1), we write $I_{\sigma}(f)$ as
\[ \sum_{M \in \mathcal{M}} |W_0^M| |W_0^G|^{-1} \sum_{\gamma \in (M(\mathfrak{f}))_{M,S}} a^M(S, \gamma) \left( J_M(\gamma, f) - \sum_{L \in \mathcal{L}_0} \hat{I}^L_M(\gamma, \phi_L(f)) \right). \]
The expression in brackets on the right is just equal to \( I_M(\gamma, f) \), so we obtain the required formula for \( L_x(f) \). □

The original induction assumption of §1 implies that for any \( L \in \mathcal{L}_0 \), the distributions \( I_M^L(\gamma) \) are all supported on characters. The last proposition provides an expansion for \( I_x^L \) in terms of the distributions \( I_M^L(\gamma) \). Therefore, \( I_x^L \) is also supported on characters. Thus, half of the second induction hypothesis adopted in §2 is subsumed in the original assumption. In §4 we shall take care of the rest of the second induction hypothesis.

To be able to exploit the last proposition effectively, we shall establish an important support property of the distributions \( I_M(\gamma) \). Fix an element \( M \in \mathcal{L} \), a finite set \( S_1 \) of valuations containing \( S_{\text{ram}} \), and a compact neighborhood \( \Delta_1 \) in \( G(F_{S_1}) \). Let \( \mathcal{H}_{\Delta_1}(G(F_{S_1})) \) denote the set of functions in \( \mathcal{H}(G(F_{S_1})) \) which are supported on \( \Delta_1 \).

**Lemma 3.2.** There is a compact subset \( \Delta_1^M \) of \( M(F_{S_1}) \) such that for any finite set \( S \supset S_1 \), and any \( f \) in the image of \( \mathcal{H}_{\Delta_1}(G(F_{S_1})) \) in \( \mathcal{H}(G(F_S)) \), the function

\[
\gamma \mapsto I_M(\gamma, f), \quad \gamma \in M(F_S),
\]

is supported on the set

\[
\text{ad}(M_0(F_S))(\Delta_1^M K_S^M) = \{ m^{-1} cm : m \in M_0(F_S), c \in \Delta_1^M K_S^M \}.
\]

**Proof.** Suppose that

\[
\mathcal{M}_1 = \prod_{v \in S_1} M_v
\]

is a Levi subset of \( M \) defined over \( F_{S_1} \). Then for each \( v \in S_1 \), \( M_v \) is a Levi subset of \( M \) which is defined over \( F_v \). Let \( M_v(F_v) \) be the set of elements \( \gamma_v \in M_v(F_v) \) whose semisimple component \( \sigma_v \) satisfies the following two conditions.

(i) The connected centralizer \( M_{\sigma_v} \) of \( \sigma_v \) in \( M_0 \) is contained in \( M_0(F_v) \).

(ii) \( \sigma_v \) is an \( F_v \)-elliptic point in \( M_v \).

Set

\[
\mathcal{M}_1(F_{S_1})' = \prod_{v \in S_1} M_v(F_v)'.
\]

Consider the restriction of the map

\[
H_{\mathcal{M}_1} = \bigoplus_{v \in S_1} H_{M_v} : \mathcal{M}_1(F_{S_1}) - \rightarrow a_{\mathcal{M}_1} = \bigoplus_{v \in S_1} a_{M_v}
\]

to \( \mathcal{M}_1(F_{S_1})' \). The map is certainly constant on the orbit of

\[
\mathcal{M}_1^0(F_{S_1}) = \prod_{v \in S_1} M^0_v(F_v).
\]

The \( F_v \)-elliptic set in \( M_v(F_v) \) has a set of representatives which is compact modulo \( A_{M_v}(F_v) \). It follows easily that as a map on the space of...
\( \mathcal{M}_1^0(F_{S_1}) \)-orbits in \( \mathcal{M}_1(F_{S_1})' \), \( H_\mathfrak{f} \), is proper. To prove the lemma, we shall combine this fact with the descent and splitting properties of \( I_M(\gamma, f) \). The argument is quite similar to that of [1(c), Lemma 12.2].

We may assume that

\[
\Delta_1 = \prod_{v \in S_1} \Delta_v
\]

and

\[
f = \prod_{v \in S} f_v,
\]

so that \( f_v \) belongs to \( \mathcal{H}_\Delta^v(G(F_v)) \) if \( v \) belongs to \( S_1 \), and \( f_v \) equals the characteristic function of \( K_v^G \) if \( v \) belongs to the complement of \( S_1 \) in \( S \). Suppose that

\[
\gamma = \prod_{v \in S} \gamma_v
\]

is an element in \( M(F_S) \) such that \( I_M(\gamma, f) \neq 0 \). For each \( v \in S_1 \), let \( \sigma_v \) be the semisimple part of \( \gamma_v \), and let \( A_{\sigma_v} \) be the split component of the center of \( M_{\sigma_v} \). Set \( M_v \) equal to the centralizer of \( A_{\sigma_v} \) in \( M \). Then \( \gamma_v \) belongs to \( M_v(F_v)' \). In other words, if

\[
\mathcal{M}_1 = \prod_{v \in S_1} M_v,
\]

the element

\[
\gamma_1 = \prod_{v \in S_1} \gamma_v
\]

belongs to \( \mathcal{M}_1(F_{S_1})' \). If we were to replace \( \gamma \) by an \( M^0(F_{S_1}) \)-conjugate, \( \mathcal{M}_1 \) would be similarly conjugated, but \( I_M \) would remain nonzero. Now there are only finitely many \( M^0(F_{S_1}) \)-orbits of Levi subsets \( \mathcal{M}_1 \) over \( F_{S_1} \). It is therefore sufficient to fix \( \mathcal{M}_1 \), and to consider only those elements \( \gamma \) such that \( \gamma_1 \) belongs to \( \mathcal{M}_1(F_{S_1})' \).

For each valuation \( w \) in \( S - S_1 \), we set \( M_w = M \). We then define a Levi subset

\[
\mathcal{M} = \mathcal{M}_1 \times \left( \prod_{w \in S - S_1} M_w \right) = \prod_{v \in S} M_v
\]

of \( M \) over \( F_S \). Regarding \( \gamma \) as an element in \( \mathcal{M}(F_S) \), we can form the induced class

\[
\gamma^M = \prod_{v \in S} \gamma_v^M.
\]

But \( M_{v, \gamma_v} = M_{\gamma_v} \) for each \( v \), so \( \gamma^M \) is just the \( \mathcal{M}_1^0(F_S) \)-orbit of \( \gamma \). Applying Corollary 9.2 of [1(j)], we obtain

\[
I_\mathfrak{f}(\gamma, f) = \sum_{\mathcal{L} \in \mathcal{L}(\mathfrak{f})} d^G(\mathcal{L}, \mathcal{L}') \prod_{v \in S} i_{M_v}^L(\gamma_v, f_v, L_v) \neq 0.
\]
Recalling the definition of the constants $\sigma^G_M(M,\mathcal{L})$ in [1(j), §9], we find that we can choose
\[ \mathcal{L} = \prod_{v \in S} L_v, \quad L_v \in \mathcal{L}(M_v), \]
so that the natural map $\sigma^G_M(M,\mathcal{L}) \to \sigma^G_M \oplus \sigma^G_{\mathcal{L}}$ is an isomorphism, and so that
\[ (3.4) \quad \hat{I}^L_M(\gamma_v, f_v, L_v) \neq 0, \quad v \in S. \]

Suppose first that $w$ is a valuation in the complement of $S_1$ in $S$. Since $f_w$ is the characteristic function of $K_w^G$, Lemma 2.1 of [1(j)] tells us that
\[ \hat{I}^L_M(\gamma_w, f_w, L_w) = \hat{I}^L_M(\gamma_w, f_w, Q_w) = J^L_M(\gamma_w, f_w, Q_w) \]
for any $Q_w \in \mathcal{P}(L_w)$. The function on the right is a weighted orbital integral, and by Corollary 6.2 of [1(h)], it is the integral with respect to a measure on the induced class $\gamma_w^G$. Therefore, the class $\gamma_w^G$ must intersect $K_w^G$. Combining the definition of the induced class $\gamma_w^G$ with the standard properties of the special maximal compact group $K_w$, we find that the $M^0(F_w)$-orbit of $\gamma_w$ intersects $K_w^M$. Notice in particular that $H_M(\gamma_w) = 0$.

We turn, finally, to the valuations in $S_1$. It remains for us to show that the $M^0(F_{S_1})$-orbit of $\gamma_1$ intersects a compact subset $\Delta_1^M$ of $M(F_{S_1})$ which depends only on $\Delta_1$. For any $v \in S_1$, the distribution $\hat{I}^L_M(\gamma_v, f_v, L_v)$ depends only on the restriction of $f_v$ to the set $\{x_v \in G(F_v): H_{L_v}(x_v) = H_{L_v}(\gamma_v)\}$. It follows from (3.4) that $H_{L_v}(\gamma_v)$ belongs to $H_{L_v}(\Delta_v)$, the image of the support of $f_v$. In other words, $H_{\mathcal{L}_1}(\gamma_1)$ belongs to $\bigoplus_{v \in S_1} H_{L_v}(\Delta_v)$, a compact set which depends only on $\Delta_1$. This completes the proof of the lemma. \(\square\)
Suppose that \( f \) belongs to \( \mathcal{H}(G(\mathbb{A})) \). We shall write \( \text{supp}(f) \) for the support of \( f \). There exists a finite set \( S \) of valuations of \( F \), which contains \( S_{\text{ram}} \), such that \( f \) is the image of a function in \( \mathcal{H}(G(F_S)) \). We shall write \( V(f) \) for the minimal such set. If \( S \) is any such set and \( \gamma \) is a point in \( (M(F))_{M,S} \), we shall understand \( I_M(\gamma, f) \) to mean the value of the distribution \( I_M(\gamma) \) at \( f \), regarded as a function in \( \mathcal{H}(G(F_S)) \). Since we are thinking of \( I_M(\gamma) \) as a local object, this convention is quite reasonable. It simply means that when \( \gamma \in (M(F))_{M,S} \) parametrizes such a distribution, we should treat \( \gamma \) as a point in \( M(F_S) \) rather than \( M(F) \).

**Theorem 3.3.** Suppose that \( f \in \mathcal{H}(G(\mathbb{A})) \). Then
\[
I(f) = \sum_{M \in \mathcal{L}} |W_0^M| \sum_{\gamma \in (M(F))_{M,S}} a^M(S, \gamma) I_M(\gamma, f),
\]
where \( S \) is any finite set of valuations which is sufficiently large, in a sense that depends only on \( \text{supp}(f) \) and \( V(f) \). The inner series can be taken over a finite subset of \( (M(F))_{M,S} \) which also depends only on \( \text{supp}(f) \) and \( V(f) \).

**Proof.** By (2.4) and Proposition 3.1, we have
\[
I(f) = \sum_{\sigma \in \mathcal{O}} \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \sum_{\gamma \in (M(F) \cap \sigma)_{M,S}} a^M(S, \gamma) I_M(\gamma, f),
\]
where \( S \) is any finite set of valuations that contains \( S_{\sigma} \). We shall use Lemma 3.2 to show that the sum over \( \sigma \) is finite.

Choose any finite set \( S_1 \supset S_{\text{ram}} \), and a compact neighbourhood \( \Delta_1 \) in \( G(F_{S_1}) \), such that \( f \) belongs to \( \mathcal{H}(G(F_{S_1})) \). Assume that \( S \) contains \( S_1 \). Suppose that a class \( \sigma \) gives a nonzero contribution to the sum above. Then there is an \( M \in \mathcal{L} \), and an element \( \gamma \in (M(F) \cap \sigma)_{M,S} \) such that
\[
a^M(S, \gamma) I_M(\gamma, f) \neq 0.
\]
The nonvanishing of \( a^M(S, \gamma) \) implies that for each \( v \not\in S \), the image of \( \gamma \) in \( M(F_v) \) lies in
\[
\text{ad}(M^0(F_v))K_v^M.
\]
The image of \( \gamma \) in \( M(F_S) \) then lies in \( M(F_S)^1 \), and therefore belongs to a set
\[
\text{ad}(M^0(F_S))(\Delta_1^M K_S^M),
\]
by Lemma 3.2. It follows that the \( M^0(\mathbb{A}) \)-orbit of \( \gamma \) meets the compact set \( \Delta_1^M K^M \), and in particular that
\[
\text{ad}(G^0(\mathbb{A}))_\sigma \cap \Delta_1^M K^M \neq \emptyset.
\]
By Lemma 9.1 of [1(g)], \( \sigma \) must belong to a finite subset \( \mathcal{O}_1 \) of \( \mathcal{O} \). Since \( \Delta_1^M \) depends only on \( \Delta_1 \), \( \mathcal{O}_1 \) clearly depends only on \( \text{supp}(f) \) and \( V(f) \). The required expansion for \( I(f) \) then holds if \( S \) is any finite set which contains
the union of $S_1$ with the sets $S_\sigma$, as $\sigma$ ranges over $\mathcal{G}_1$. This establishes the first assertion of the theorem. The union over $\sigma \in \mathcal{G}_1$ of the sets

$$(M(F) \cap \sigma)_{M,S}$$

is certainly a finite subset of $(M(F))_{M,S}$, so the second assertion also follows. \(\square\)

4. THE SPECTRAL SIDE

We shall derive a finer expansion for the right-hand side of (2.4). The result will be a sum of terms, indexed by irreducible representations, which separate naturally into local and global constituents. Again, there is a parallel expansion for the noninvariant distributions on the right-hand side of (2.1). It is provided by the results of [1(e)] and [7]. However, these results are not immediately in the form we want, and it is necessary to review them in some detail.

The set $\mathcal{H} = \mathcal{H}(G,F)$ consists of cuspidal automorphic data [1(b), 7]. It is the set of orbits

$$\mathcal{H} = \{s_0(L_0, r_0) : s_0 \in W_0\} = \{s(L_0, r_0) : s \in W^G_0\},$$

where $L_0$ is a Levi subgroup in $L^0 = L^0(G)$, $r_0$ is an irreducible cuspidal automorphic representation of $L_0(\mathbb{A})^1$, and the pair $(L_0, r_0)$ is fixed by some element in the Weyl set $W^G_0$ of isomorphisms of $a_0$ induced from $G$. (We have indexed the Levi subgroup with the subscript 0 to emphasize that it need not be of the form $M_0$ for some $M \in \mathcal{L}$.) The set $\mathcal{H}$ has been used to describe the convergence of the spectral side, which is more delicate than that of the geometric side. However, for applications that involve a comparison of trace formulas, it is easier to handle the convergence by keeping track of Archimedean infinitesimal characters.

Set

$$F_\infty = F_{S_\infty} = \prod_{v \in S_\infty} F_v.$$

Regarding $G^0(F_\infty)$ as a real Lie group, we can define the Abelian Lie algebra

$$\mathfrak{h} = i\mathfrak{h}_K \oplus \mathfrak{h}_0$$

as in §3 of [1(d)]. Then $\mathfrak{h}_0$ is the Lie algebra of a fixed maximal real split torus in $M_0(F_\infty)$, and $\mathfrak{h}_K$ is a fixed Cartan subalgebra of the centralizer of $\mathfrak{h}_0$ in $K_\infty = \prod_{v \in S_\infty} K_v$.

The complexification $\mathfrak{h}_C$ is a Cartan subalgebra of the complex Lie algebra of $G^0(F_\infty)$, and the real form $\mathfrak{h}$ is invariant under the complex Weyl set $W^G$ of $G(F_\infty)$. (By definition, $W^G$ equals $\text{Ad}(\varepsilon)W$, where $\varepsilon$ is any element in $G(F_\infty)$ which normalizes $\mathfrak{h}_C$, and $W$ is the complex Weyl group of $G^0(F_\infty)$.)
with respect to $\mathfrak{h}$. It is convenient to fix a Euclidean norm $\| \cdot \|$ on $\mathfrak{h}$ which is invariant under $W^G$. We shall also write $\| \cdot \|$ for the dual Hermitian norm on $\mathfrak{h}^*_C$. To any representation $\pi \in \Pi(M(A))$ we can associate the induced representation $\pi^G$ of $G^+(A)$. Let $\nu_{\pi}$ denote the infinitesimal character of its Archimedean constituent; it is a $W$-orbit in $\mathfrak{h}^*_C$. We shall actually be more concerned with the case that $\pi$ is a representation in $\Pi(M(A)^1)$. Then $\nu_{\pi}$ is determined a priori only as an orbit of $a^*_{M,C}$ in $\mathfrak{h}^*_C$. However, this orbit has a unique point of smallest norm in $\mathfrak{h}^*_C$ (up to translation by $W$) and it is this point which we shall denote by $\nu_{\pi}$. If $t$ is a nonnegative real number, let $\Pi_{\text{unit}}(M(A)^1,t)$ denote the set of representations $\pi \in \Pi_{\text{unit}}(M(A)^1)$ such that
\[ \| \mathcal{F} m(\nu_{\pi}) \| = t, \]
where $\mathcal{F} m(\nu_{\pi})$ is the imaginary part of $\nu_{\pi}$ relative to the real form $\mathfrak{h}^*_C$. We adopt similar notation when $M$ is replaced by a group $L_0 \in \mathcal{L}^0$. In particular, if
\[ \chi = \{ s(L_0,r_0) : s \in W^G_0 \} \]
is any class in $\mathcal{K}$, we set $\nu_\chi = \nu_{r_0}$.

Suppose that $L_0$ is a Levi subgroup in $\mathcal{L}^0$. Set
\[ A_{L_0,\infty} = A_{L_0,0}(R)^0, \]
where $A_{L_0,0}$ is the split component of the center of the group obtained by restricting scalars from $F$ to $Q$. Let
\[ L^2_{\text{disc},t}(L_0(F)A_{L_0,\infty} \setminus L_0(A)) \]
be the subspace of $L^2(L_0(F)A_{L_0,\infty} \setminus L_0(A))$ which decomposes under $L_0(A)$ as a direct sum of representations in $\Pi_{\text{unit}}(L_0(A),t)$. For any group $Q_0$ in $\mathcal{G}^0(L_0)$ and a point $\Lambda \in a^*_{L_0,C}$, let
\[ \rho_{Q_0,t}(\Lambda) : x \rightarrow \rho_{Q_0,t}(\Lambda, x) \]
be the induced representation of $G^0(A)$ obtained from (4.1). If $Q_0'$ is another group in $\mathcal{G}^0(L_0)$, the theory of Eisenstein series provides an intertwining operator $M_{Q_0'Q_0}(\Lambda)$ from $\rho_{Q_0,t}(\Lambda)$ to $\rho_{Q_0',t}(\Lambda)$.

**Lemma 4.1.** The representation $\rho_{Q_0,t}(\Lambda)$ is admissible.

**Proof.** The assertion is that the restriction of $\rho_{Q_0,t}(\Lambda)$ to $K$ contains each irreducible representation with only finite multiplicity. Since admissibility is preserved under parabolic induction, it is enough to show that the representation of $L_0(A)$ on (4.1) is admissible. To this end, we may assume that
The assertion is then a consequence of Langlands’ theory of Eisenstein series [12, Chapter 7]. For one of the main results of [12] is a decomposition

$$L_{\text{disc},t}^2(G(F)A_{G,\infty} \setminus G(A)) = \bigoplus_{\chi} L_{\text{disc},\chi}^2(G(F)A_{G,\infty} \setminus G(A)),$$

where $\chi$ ranges over the data in $\mathcal{H}$ such that $\|\mathcal{F}m(\nu_{\chi})\|$ equals $t$, and each corresponding summand is an admissible $G(A)$-module. On the other hand, the set of all $\chi$ whose associated cuspidal representations contain the restrictions of a given $K$-type have discrete infinitesimal characters. That is, the associated points $\{\nu_{\chi}\}$ form a discrete subset of $B + ia_M^*$, with $B$ a compact ball about the origin in $a_M^*$. It follows that there are only finitely many modules $L_{\text{disc},\chi}^2$ in the direct sum above which contain a given $K$-type. The lemma follows.

The representation $\rho_{Q_0,t}(\Lambda)$ of $G^0(A)$ does not in general extend to the group generated by $G(A)$. However, suppose that $s$ is an element in $W_0^G$ with representative $w$ in $G(F)$. We can always translate functions on $G^0(A)$ on the right by elements in $G(A)$ if at the same time we translate on the left by $w^{-1}$. Therefore, if $y$ belongs to $G(A)$, we can define a linear map $\rho_{Q_0,t}(s, \Lambda, y)$ from the underlying Hilbert space of $\rho_{Q_0,t}(\Lambda)$ to that of $\rho_{sQ_0,t}(s\Lambda)$ such that

$$\rho_{Q_0,t}(s, \Lambda, y_1, y_2) = \rho_{sQ_0,t}(s\Lambda, y_1)\rho_{Q_0,t}(s, \Lambda, y)\rho_{Q_0,t}(\Lambda, y_1),$$

for any points $y_1$ and $y_2$ in $G^0(A)$. This map depends only on the image of $s$ in $W_0^G/W_0^{L_0}$. In particular, it is well defined for any element in $W_0^G(a_{L_0})$, the normalizer of $a_{L_0}$ in $W_0^G$. Suppose that $s$ is an element in $W_0^G(a_{L_0})$ which fixes $\Lambda$. If $f$ is a function in $\mathcal{H}(G(A))$, we write

$$\rho_{Q_0,t}(s, \Lambda, f^1) = \int_{G(A)} f(x)\rho_{Q_0,t}(s, \Lambda, x) \, dx.$$ 

Then

$$M_{Q_0|sQ_0}(\Lambda)\rho_{Q_0,t}(s, \Lambda, f^1)$$

is an operator of trace class on the underlying Hilbert space of $\rho_{Q_0,t}(\Lambda)$. According to (4.2), its trace is an invariant distribution, which by Lemma 4.1 can be written as a finite linear combination of irreducible characters

$$\text{tr } \pi(f^1) = f_G(\pi), \quad \pi \in \Pi_{\text{unit}}(G(A)^1_t).$$

Observe that each such irreducible character is determined in the expression only up to the orbit of $\pi$ under the group $\Xi_A$. As in §1, we write $\{\Pi_{\text{unit}}(G(A)^1_t)\}$ for the set of such orbits in $\Pi_{\text{unit}}(G(A)^1_t)$.

Consider the expression

$$\sum_{L_0 \in \mathcal{L}^0} |W_0^{L_0}| |W_0^G| \sum_s |\det(s - 1)_{a_{L_0}}|^{-1} \text{tr}(M_{Q_0|sQ_0}(0)\rho_{Q_0,t}(s, 0, f^1)),$$

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where $Q_0$ stands for any element in $\mathcal{R}^0(L_0)$ and $s$ is summed over the Weyl set

$$W^G(a_{L_0})_{\text{reg}} = \{ s \in W^G(a_{L_0}) : \det(s - 1)_{a_{L_0}}^* \neq 0 \}.$$  

This is just the "discrete part" of the formula for

$$\sum_{\{ \chi \in \mathcal{H} : \| \mathfrak{F} m(\nu_\chi) \| = t \}} J_\chi(f), \quad f \in \mathcal{H}(G(\mathbb{A})), $$

provided by Theorem 8.2 of [1(e)]. (For the case $G \neq G^0$, see the final lecture of [7].) According to the remarks above, we can rewrite (4.3) as

$$\sum_{\pi \in \{ \Pi_{\text{disc}}(G(\mathbb{A})^1, t) \}} a^G_{\text{disc}}(\pi)f_G(\pi),$$

a finite linear combination of characters. The complex valued function

$$a_{\text{disc}}(\pi) = a^G_{\text{disc}}(\pi), \quad \pi \in \Pi_{\text{unit}}(G(\mathbb{A})^1, t),$$

which is defined by the equality of (4.3) and (4.4), is the primary global datum for the spectral side.

It is convenient to work with a manageable subset of $\{ \Pi_{\text{unit}}(G(\mathbb{A})^1, t) \}$ which contains the support of $a^G_{\text{disc}}(\pi)$. Let $\Pi_{\text{disc}}(G, t)$ denote the subset of $\Xi_{\mathcal{A}}$-orbits in $\{ \Pi_{\text{unit}}(G(\mathbb{A})^1, t) \}$ which are represented by irreducible constituents of induced representations

$$\sigma^G_M, \quad M \in \mathcal{L}, \sigma \in \Pi_{\text{unit}}(M(\mathbb{A})^1, t), \lambda \in \mathfrak{i}a_M^*/\mathfrak{i}a_G^*,$$

where $\sigma_\lambda$ satisfies the following two conditions.

(i) $a^M_{\text{disc}}(\sigma) \neq 0$.

(ii) There is an element $s \in W^G(a_M)_{\text{reg}}$ such that $s \sigma_\lambda = \sigma_\lambda$.

Observe that the restriction to $G^0(\mathbb{A})$ of any representation in $\Pi_{\text{disc}}(G, t)$ is an irreducible constituent of an induced representation

$$\rho_{Q_0,t}(0), \quad Q_0 \in \mathcal{R}^0.$$  

From the last lemma we obtain

Lemma 4.2. Suppose that $\Gamma$ is a finite subset of $\Pi(K)$. Then there are only finitely many (orbits of) representations $\pi \in \Pi_{\text{disc}}(G, t)$ whose restrictions to $K$ contain an element in $\Gamma$. In particular, there are only finitely many orbits $\pi \in \{ \Pi(G(\mathbb{A})^1, t) \}$ which contain an element in $\Gamma$ and such that $a^G_{\text{disc}}(\pi) \neq 0$. □  

Before going on, we note the following lemma for future reference.
Lemma 4.3. Suppose that $\xi$ is a one dimensional character on $G^+(A)^1$ which is trivial on $G^0(F)$. Then
\[ a^G_{\text{disc}}(\xi \pi) = a^G_{\text{disc}}(\pi), \quad \pi \in \Pi_{\text{unif}}(G(A)^1, t), \]
where
\[ (\xi \pi)(x) = \xi(x)\pi(x), \quad x \in G^+(A). \]

Proof. If the character $\xi$ belongs to $\Xi$, the assertion of the lemma is of course part of the definition of $a^G_{\text{disc}}$. In general, observe that we can use $\xi$ to define a linear operator $\rho_{Q_0}(\xi)$ on the underlying Hilbert spaces of the representations $\rho_{Q_0, t}(0)$. It has the property that
\[ \rho_{Q_0}(\xi)^{-1} M_{Q_0} Q_0(0) \rho_{Q_0, t}(s, 0, f^1) \rho_{Q_0}(\xi) = M_{Q_0} Q_0(0) \rho_{Q_0, t}(s, 0, \xi f^1), \]
where
\[ (\xi f)(x) = \xi(x)f(x), \quad x \in G(A)^1. \]
Therefore, (4.3) remains unchanged if $f$ is replaced by $\xi f$. The lemma follows. \( \square \)

The remaining global ingredient is a function constructed from the global normalizing factors $[1(e), \S 6]$. We shall recall briefly how it is defined. Suppose that $M \in \mathcal{L}$ and that $\pi = \otimes_v \pi_v$ belongs to $\Pi_{\text{disc}}(M, t)$. The restriction of $\pi$ to $M^0(A)$ is an irreducible constituent of some representation $\rho_{Q_0, t}(0)$, where $L_0 \in \mathcal{L} M^0, R_0 \in \mathcal{R} M^0(L_0)$.

If $P \in \mathcal{P}(M)$, we can form the induced representation
\[ \mathcal{I}_P(\pi_\lambda), \quad \lambda \in a^*_M. \]
Its restriction to $G^0(A)$ is a subrepresentation of $\rho_{Q_0, t}(\lambda)$, where $Q_0$ is the group $P^0(R_0)$ in $\mathcal{R}^0(L_0)$ which is contained in $P^0$ and whose intersection with $M^0$ is $R_0$. If $P' \in \mathcal{P}(M)$ and $Q'_0 = (P')^0(R_0)$, the operator
\[ J_{P' \mid P}(\pi_\lambda) = \prod_v J_{P' \mid P}(\pi_{v, \lambda}), \]
defined as an infinite product of unnormalized intertwining operators, is therefore equivalent to the restriction of $M_{Q'_0}(\lambda)$ to an invariant subspace. The theory of Eisenstein series tells us that the infinite product converges for certain $\lambda$, and can be analytically continued to an operator valued function which is unitary when $\lambda \in ia^*_M$. But we also have the normalized intertwining operator
\[ R_{P' \mid P}(\pi_\lambda) = \prod_v R_{P' \mid P}(\pi_{v, \lambda}) = \prod_v (r_{P' \mid P}(\pi_{v, \lambda})^{-1} J_{P' \mid P}(\pi_{v, \lambda})), \]
described in $[1(i)]$. The infinite product reduces to a finite product at any smooth vector. It follows that the infinite product
\[ r_{P' \mid P}(\pi_\lambda) = \prod_v r_{P' \mid P}(\pi_{v, \lambda}). \]
of local normalizing factors converges for certain $\lambda$ and can be continued as a
meromorphic function which is analytic for $\lambda \in i\mathbb{R}$. Moreover,
\[ r_{P''|\mu}(\pi_{\lambda}) = r_{P''|\mu}(\pi_{\lambda})r_{P'|\pi}(\pi_{\lambda}), \]
if $P''$ is a third element in $\mathcal{P}(M)$.

For a fixed $P' \in \mathcal{P}(M)$, we define the $(G, M)$-family
\[ r_p(\nu, \pi_{\lambda}, P') = r_{P'|\pi}(\pi_{\lambda})^{-1} r_{P'|\pi}(\pi_{\lambda+\nu}), \quad P \in \mathcal{P}(M), \nu \in i\mathbb{R}. \]
Since
\[ r_{P'|\pi}(\pi_{\nu, \lambda}) = \prod_{\alpha \in \Sigma_{p} \cap \Sigma_{p'}} r_{\alpha}(\pi_{\nu}, \lambda(\nu)), \]
for each $\nu$ [1(i),§2], we have
\[ r_{P'|\pi}(\pi_{\lambda}) = \prod_{\alpha \in \Sigma_{p} \cap \Sigma_{p'}} r_{\alpha}(\pi, \lambda(\nu)), \]
where $r_{\alpha}(\pi, z)$ equals an infinite product
\[ \prod_{\nu} r_{\alpha}(\pi_{\nu}, z), \quad z \in \mathbb{C}, \]
which converges in some half-plane. Therefore, the $(G, M)$-family is of the
special sort considered in §7 of [1(e)]. In particular, if $L \in \mathcal{L}(M)$ and $Q \in
\mathcal{P}(L)$, the number
\[ r_{M}(\pi_{\lambda}) = \lim_{\nu \to 0} \sum_{\{P \in \mathcal{P}(M): P \subset Q\}} r_p(\nu, \pi_{\lambda}, P') \theta^Q_P(\nu)^{-1} \]
can be expressed in terms of logarithmic derivatives
\[ r_{\alpha}(\pi_{\lambda}, 0)^{-1} r_{\alpha}'(\pi_{\lambda}, 0), \quad \alpha \in \Sigma(L, A_M), \]
and is independent of $Q$ and $P'$ [1(e), Proposition 7.5]. As a function of $\lambda \in i\mathbb{R}$, it is a tempered distribution [1(e), Lemma 8.4].

For a given Levi subset $M \in \mathcal{L}$, let $\Pi(M, t)$ denote the disjoint union over
$M_1 \in \mathcal{L}$ of the sets
\[ \Pi_{M_1}(M, t) = \{ \pi = \pi_{1, \lambda}: \pi_1 \in \Pi_{\text{disc}}(M_1, t), \lambda \in i\mathbb{R}, i\mathbb{R} \}. \]
We define a measure $d\pi$ on $\Pi(M, t)$ by setting
\[ \int_{\Pi(M, t)} \phi(\pi) d\pi = \sum_{M_1 \in \mathcal{L}} |W_{1}| - |W_{0}|^{-1} \sum_{\pi_1 \in \Pi_{\text{disc}}(M_1, t)} \int_{i\mathbb{R}} \phi(\pi_{1, \lambda}) d\lambda, \]
for any suitable function $\phi$ on $\Pi(M, t)$. The global constituent of the spectral
side of the trace formula is the function
\[ a^M(\pi) = \sum_{M_1 \in \mathcal{L}} \pi_{1, \lambda} \pi_{M_1}(\pi_{1, \lambda}), \]
defined for any point
\[ \pi = \pi_{1, \lambda}, \quad \pi \in \Pi_{\text{disc}}(M_1, t), \lambda \in i\mathbb{R}, i\mathbb{R}. \]
in $\Pi_{M_1}(M,t)$. In our notation we should keep in mind that $\pi_1$ is a representation in $\Pi_{\text{unit}}(M_1(A)^1)$ (determined modulo $\Xi_A$), so that $\{\pi_{i,1}\}$ stands for the associated orbit of $i\alpha_{M_1}/\alpha_{M_1}$ in $\Pi_{\text{unit}}(M_1(A) \cap M(A)^1)$. In practice, however, we shall usually identify $\pi = \pi_{i,1}$ with the induced representation $\pi_{i,1}^M$ in $\{\Pi_{\text{unit}}(M(A)^1)\}$. In this sense, the invariant distribution

$$I_M(\pi, f) = I_M(\pi_{i,1}, 0, f), \quad \mu \in \alpha_{M_1}, f \in \mathcal{H}(G(A)),$$

studied in [1(j)] is defined. It will be the local constituent of the spectral side.

**Theorem 4.4.** Suppose that $f \in \mathcal{H}(G(A))$. Then

$$I(f) = \sum_{\mu \geq 0} \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \int_{\Pi(M,t)} a(M)(\pi, f) d\pi,$$

where the integral and outer sum each converge absolutely.

**Proof.** Set

$$J(f) = \sum_{\mu \geq 0} \sum_{M \in \mathcal{L}} J_M(f).$$

We shall apply the formula for $J(f)$ provided by Theorem 8.2 of [1(e)] (and the analogue in [7] for $G \neq G^0$). Then $J_M(f)$ equals the sum over $M_1 \in \mathcal{L}, L_0 \in \mathcal{L}^{M_0}$, and $s \in W^{M_1}(a_{L_0})_{\text{reg}}$, of the product of

$$|W_0^{L_0}| |W_0^G|^{-1} \det(s - 1)_{a_{L_0}}^{-1}$$

with

$$\int_{\alpha_{M_1}} \text{tr}(\mathcal{M}_{M_1}(\Lambda, Q_0) M_{Q_0} s Q_0(0) \rho_{Q_0,t}(s, \Lambda, f^1)) d\Lambda.$$

Here, $Q_0$ is an element in $\mathcal{P}^0(L_0)$, and the operator

$$\mathcal{M}_{M_1}(\Lambda, Q_0) = \lim_{\nu \to 0} \sum_{P \in \mathcal{P}(M_1)} \mathcal{M}_{P}(\nu, \Lambda, Q_0) \theta_{P}(\nu)^{-1}$$

is obtained from the $(G, M_1)$-family

$$\mathcal{M}_{P}(\nu, \Lambda, Q_0) = M_{P}^{0}(R_0) Q_0(0) \rho_{Q_0,(s, \Lambda)}(s, \Lambda, f^1),$$

for $P_1 \in \mathcal{P}(M_1)$ and $\nu \in \alpha_{M_1}^\ast$. As above, $R_0$ is a fixed parabolic subgroup of $M_1^0$ with Levi component $L_0$. We can assume that $Q_0 = P^0(R_0)$ for some fixed element $P$ in $\mathcal{P}(M_1)$.

The trace of the operator

$$\mathcal{M}_{M_1}(\Lambda, Q_0) M_{Q_0} s Q_0(0) \rho_{Q_0,t}(s, \Lambda, f)$$

vanishes except on an invariant subspace on which the representation $\rho_{Q_0,t}(\Lambda)$ reduces to a sum of induced representations

$$\mathcal{F}_{P}(\pi_{1,\Lambda}), \quad \pi_{1} \in \Pi_{\text{disc}}(M_1,t).$$
(Actually, \( \rho_{Q_0,I\Lambda}^A(\Lambda) \) is only a representation of \( G^0(A) \), so we really mean the restriction of \( \mathcal{S}_P(\pi_{1,\Lambda}) \) to this group.) With this interpretation, the intertwining operator \( M_{P_1|R_0|Q_0}^A(\Lambda) \) corresponds to a direct sum of operators
\[
J_{P_1|R_0|Q_0}(\pi_{1,\Lambda}) = r_{P_1|p}(\pi_{1,\Lambda})R_{P_1|P}(\pi_{1,\Lambda}), \quad \pi_1 \in \Pi_{\text{disc}}(M_1, t).
\]
Therefore, \( \mathcal{M}_{M_1}(\Lambda, Q_0) \) corresponds to a direct sum of operators
\[
\lim_{\nu \to 0} \sum_{P_1 \in \mathcal{P}(M_1)} r_{P_1}(\nu, \pi_{1,\Lambda}, P) \mathcal{R}_{P_1}(\nu, \pi_{1,\Lambda}, P) \theta_{P_1}(\nu)^{-1}.
\]
This last expression is obtained from a product of \((G, M)\)-families. By Corollary 6.5 of [1(c)] it equals
\[
\sum_{M \in \mathcal{M}(M_1)} r_{M_1}(\pi_{1,\Lambda}) \mathcal{R}_{M}(\pi_{1,\Lambda}, P).
\]

We now apply the definition of \( a^G_{\text{disc}} \). Given the observations above, we use the equality of (4.3) and (4.4) (with \( G \) replaced by \( M_1 \)) to rewrite \( J_\iota(f) \) as the sum over \( M_1 \in \mathcal{M} \) and \( M \in \mathcal{M}(M_1) \) of the product of \(|W_0^M||W_0^G|^{-1}\) with
\[
\sum_{\pi_1 \in \Pi_{\text{disc}}(M_1, t)} \int_{ia^*_{M_1}} a^M_{\text{disc}}(\pi_1) r_{M_1}(\pi_{1,\Lambda}) \mathcal{R}_{M}(\pi_{1,\Lambda}, P) \mathcal{J}_{P}(\pi_{1,\Lambda}, f) d\Lambda.
\]
Observe that \( r_{M_1}(\pi_{1,\Lambda}) \) depends only on the projection \( \lambda \) of \( \Lambda \) onto \( ia^*_{M_1}/ia^*_{M} \). Moreover, by the definition in [1(i),§7], we have
\[
\int_{ia^*_{M_1}} \mathcal{J}_{M}(\pi_{1,\Lambda+\mu}, f) d\mu = J_{M}(\pi_{1,\Lambda}, 0, f) = J_{M}(\pi_{1,\lambda}, f),
\]
if \( P \) is any element in \( \mathcal{P}(M) \). (Since \( \lambda \) stands for a coset of \( ia^*_{M} \) in \( ia^*_{M_1} \), it is understood that \( \pi_{1,\lambda}^M \) is a representation in \( \Pi_{\text{unit}}(M(A)) \). This justifies the notation of the last line.) Decomposing the original integral over \( \Lambda \) into a double integral of \((\lambda, \mu)\) in
\[
(ia^*_{M_1}/ia^*_{M}) \times (ia^*_{M}),
\]
we obtain
\[
J_{\iota}(f) = \sum_{M \in \mathcal{M}} \sum_{M_1 \in \mathcal{M}} |W_0^M||W_0^G|^{-1} \times \sum_{\pi_1 \in \Pi_{\text{disc}}(M_1, t)} \int_{ia^*_{M_1}} a^M_{\text{disc}}(\pi_1) r_{M_1}(\pi_{1,\lambda}) J_{M}(\pi_{1,\lambda}, f) d\lambda
= \sum_{M \in \mathcal{M}} |W_0^M||W_0^G|^{-1} \sum_{M_1 \in \mathcal{M}} \int_{\Pi_{\text{disc}}(M_1, t)} a^M(\pi) J_{M}(\pi, f) d\pi
= \sum_{M \in \mathcal{M}} |W_0^M||W_0^G|^{-1} \int_{\Pi(M, t)} a^M(\pi) J_{M}(\pi, f) d\pi.
\]
The convergence of the integral and the justification for our use of Fubini’s theorem follow from the fact that \( r^M_{M_1}(\pi_{1,\lambda}) \) is tempered.

Set
\[
I_t(f) = \sum_{\chi \in \mathcal{A}^* : \|\mathcal{F} m(\nu_{\chi})\| = t} I_{\chi}(f).
\]

Since the invariant \( \chi \) expansion converges absolutely to \( I(f) \), we have
\[
I(f) = \sum_{t \geq 0} I_t(f),
\]
the series converging absolutely. From the definition of \( I_{\chi}(f) \), we obtain
\[
I_t(f) = J_t(f) - \sum_{L \in \mathcal{L}_0} |W_0^L| |W_0^G|^{-1} \hat{I}_t^L(\phi_L(f)).
\]

Assume inductively that
\[
I_t^L(g) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \int_{\Pi(M,t)} a^M(\pi) I_M^L(\pi, g) \, d\pi
\]
for any \( L \in \mathcal{L}_0 \) and any \( g \in \mathcal{H}(L(A)) \). Combined with the formula above for \( J_t(f) \), this tells us that \( I_t(f) \) equals
\[
\sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \int_{\Pi(M,t)} a^M(\pi) \left( J_M(\pi, f) - \sum_{L \in \mathcal{L}_0(M)} \hat{I}_t^L(\phi_L(f)) \right) \, d\pi.
\]

It follows that
\[
I_t(f) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \int_{\Pi(M,t)} a^M(\pi) I_M(\pi, f) \, d\pi.
\]

The theorem follows immediately from (4.6) and (4.7).

The definitions in this paragraph have obvious analogues if the real number \( t \) is replaced by a fixed datum \( \chi \in \mathcal{A}^* \). In particular, if \( \|\mathcal{F} m(\nu_{\chi})\| = t \), we have a subrepresentation \( \rho_{Q_0,\chi}(\Lambda) \) of \( \rho_{Q_0,t}(\Lambda) \). As in earlier papers, we shall sometimes write \( \mathcal{A}^2_{Q_0,\chi} \) for the space of \( K \)-finite vectors in the underlying Hilbert space of \( \rho_{Q_0,\chi}(\Lambda) \). Then for any \( s \in W_0^G \) and \( f \in \mathcal{H}(G(A)) \), \( \rho_{Q_0,\chi}(s, \Lambda, f^1) \) is a map from \( \mathcal{A}^2_{Q_0,\chi} \) to \( \mathcal{A}^2_{Q_0,\chi} \). The definitions also provide functions \( a^M_{\text{disc},\chi} \) and \( a^M_\chi \) on respective subsets
\[
\Pi_{\text{disc}}(M_1, \chi) \subset \Pi_{\text{disc}}(M_1, t), \quad M_1 \in \mathcal{L},
\]
and
\[
\Pi(M, \chi) \subset \Pi(M, t), \quad M \in \mathcal{L}.
\]
The proof of Theorem 4.4 yields

**Corollary 4.5.** Suppose that \( f \in \mathcal{H}(G(\mathbb{A})) \) and \( \chi \in \mathcal{H} \). Then

\[
I_{\chi}(f) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \int_{\Pi(M, \chi)} a_M^G(\pi) I_M(\pi, f) d\pi. \quad \Box
\]

For any element \( L \in \mathcal{L}_0 \), the corollary provides an expansion for \( I_{\chi}^L \) in terms of the distributions

\[
I_{M}^L(\pi) = I_M^L(\pi_{\mu}, 0), \quad \mu \in \mathfrak{a}_M^*.
\]

But our original induction assumption of \( \S 1 \) implies that the distributions \( I_M^L(\pi_{\mu}, 0) \) are supported on characters. This is a consequence of Theorem 6.1 of [1(1)]. Therefore, the distributions \( I_{\chi}^L \) are also supported on characters. We have thus shown that the entire second induction assumption, adopted in \( \S 2 \), is subsumed in the original one.

5. COMPLETION OF THE INDUCTION ARGUMENT

We shall now show that all the distributions which occur in the invariant trace formula are supported on characters. These are local objects, so we shall not start off with the number field \( F \) that has been fixed up until now. Rather, we take a local field \( F_1 \) of characteristic 0, and a connected component \( G_1 \) of a reductive group over \( F_1 \), in which \( G_1(F_1) \neq \emptyset \). As usual, we shall assume either that \( G_1 = G_0 \), or that \( G_1 \) is an inner twist of a component

\[
G^* = (GL(n) \times \cdots \times GL(n)) \rtimes \theta^*.
\]

**Theorem 5.1.** For any \( G_1/F_1 \) as above, and any Levi subset \( M_1 \) of \( G_1 \) (with respect to \( F_1 \)), the distributions

\[
I_{M_1}(\gamma_1, f_1), \quad \gamma_1 \in M_1(F_1), f \in \mathcal{H}'(G_1(F_1)),
\]

are supported on characters.

**Proof.** Fix a positive integer \( N_1 \), and assume that the theorem is valid for any \( G_1/F_1 \) with \( \dim F_1(G_1) < N_1 \). Having made this induction assumption, we fix \( G_1 \) and \( F_1 \) such that \( \dim F_1(G_1) = N_1 \). If \( L_1 \in \mathcal{L}_0(M_1) \), the distributions \( I_{M_1}^L(\gamma_1) \) are by hypothesis supported on characters. This matches the induction assumption of \( \S 2 \) of [1(1)] that allowed us to define \( I_{M_1}(\gamma_1) \) in the first place.

Let \( f_1 \) be a fixed function in \( \mathcal{H}'(G_1(F_1)) \) such that

\[
f_1 \vert_{G_1} = 0.
\]

We must show that the distributions all vanish on \( f_1 \). It is convenient to fix \( M_1 \) and to make a second induction assumption that

\[
I_{L_1}(\delta_1, f_1) = 0, \quad \delta_1 \in L_1(F_1),
\]

for any \( L_1 \in \mathcal{L}(M_1) \) with \( L_1 \neq M_1 \). We must then show that \( I_{M_1}(\gamma_1, f_1) \) vanishes for each \( \gamma_1 \in M_1(F_1) \).
If \( \gamma_1 \) is an arbitrary point in \( M_1(F_1) \), we can write
\[
I_{M_1}(\gamma_1, f_1) = \lim_{a \to 1} \sum_{L_1 \in \mathcal{L}(M_1)} r^{L_1}_{M_1}(\gamma_1, a) I_{L_1}(a\gamma_1, f_1)
\]
\[
= \lim_{a \to 1} I_{M_1}(a\gamma_1, f_1),
\]
by (5.1) and [1(j), (2.2)]. Since \( a \) stands for a small regular point in \( A_M(F_1) \), we may assume without loss of generality that \( G_{1,\gamma_1} = M_{1,\gamma_1} \). But now we can apply [1(j), (2.3)]. This formula asserts that the function
\[
\gamma \mapsto I_{M_1}(\gamma, f_1)
\]
coincides with the orbital integral of a function on \( M_1(F_1) \), for all points \( \gamma \) whose semisimple part is close to that of \( \gamma_1 \). It is known that the orbital integral of a function on \( M_1(F_1) \) is completely determined by its values at regular semisimple points. For \( p \)-adic \( F_1 \), this is Theorem 10 of [9(c)]. If \( F_1 \) is Archimedean, the result is due also to Harish-Chandra. The proof, which was never actually published, uses the Archimedean analogues of the techniques of [9(c)]. In any case, it follows that if \( I_{M_1}(\gamma, f_1) \) vanishes whenever \( \gamma \) is \( G_1 \)-regular, it vanishes for all \( \gamma_1 \). We may therefore assume that \( \gamma_1 \) itself is \( G_1 \)-regular. We can also assume that \( \gamma_1 \) is an \( F_1 \)-elliptic point in \( M_1(F_1) \). For \( \gamma_1 \), would otherwise belong to a proper Levi subset \( M \) of \( M_1 \) defined over \( F_1 \), and we would be able to write
\[
I_{M_1}(\gamma_1, f_1) = \sum_{L \in \mathcal{L}(M)} d^G_M(M_1, L) \hat{I}^L_M(\gamma_1, f_1, L),
\]
by the descent property [1(j), Corollary 8.3]. Since \( d^G_M(M_1, L) = 0 \) unless \( L \) is properly contained in \( G \), the expression vanishes by our first induction assumption. Thus, it remains for us to show that \( I_{M_1}(\gamma_1, f_1) \) vanishes when \( \gamma_1 \) is a fixed point in \( M_1(F_1) \) which is \( G_1 \)-regular and \( F_1 \)-elliptic. For this basic case we shall use the global argument introduced by Kazhdan (see [8] and [10]).

Suppose that \( G \) is a component of a reductive group over some number field \( F \), with \( G(F) \neq \emptyset \), such that \( F_{v_1} \cong F_1 \) and \( G_{v_1} = G_1 \) for a valuation \( v_1 \) of \( F \). Then
\[
\dim_F(G) = \dim_{F_1}(G_1) = N_1.
\]
It follows from Corollary 9.3 of [1(j)] and our induction assumption on \( N_1 \) that for any \( S \), the distributions
\[
I^L_M(\gamma), \quad M \in \mathcal{L}, L \in \mathcal{L}_0(M), \gamma \in M(F_S),
\]
are all supported on characters. Therefore, \( G/F \) satisfies the conditions of §1, and we can apply the results of §§3 and 4.

**Lemma 5.2.** Suppose that
\[
f = \prod_v f_v, \quad f_v \in \mathcal{H}(G(F_v)),
\]
is a function in \( \mathcal{H}(G(\mathbb{A})) \) such that \( f_{v_1} = f_1 \). Then \( I(f) = 0 \).
Proof. Consider the spectral expansion

\[ I(f) = \sum_{t \geq 0} \sum_{M \in \mathcal{L}^2} |W_0^M|^{-1} \int_{\Pi(M, t)} a^M(\pi) I_M(\pi, f) d\pi \]

of Theorem 4.4. We shall show that the distributions

\[ I_M(\pi, f) = I_M(\pi, 0, f), \quad \mu \in i a_M^*, M \in \mathcal{L}, \pi \in \Pi(M, t), \]

which occur on the right, vanish. In doing this, we will make essential use of the fact that \( \pi \) is unitary.

It is clearly enough to establish the vanishing of the Fourier transform

\[ I_M(\pi, \mathcal{X}, f) = \int I_M(\pi_\Lambda, f) e^{-\Lambda(\mathcal{X})} d\Lambda, \]

where, for a large finite set \( S \) of valuations, \( \mathcal{X} \) belongs to the vector space of elements in \( \bigoplus_{v \in S} a_{M,v} \) whose components sum to 0. The integral is over the imaginary dual vector space. According to the splitting formula [1(j), Proposition 9.4], we can write \( I_M(\pi, \mathcal{X}, f) \) as a finite sum of products, over \( v \in S \), of distributions on the spaces \( \mathcal{H}(L(F_v)), L \in \mathcal{L}(M). \) But if \( L \in \mathcal{L}_0(M), \) our induction hypothesis, combined with Theorem 6.1 of [1(j)], tells us that the distributions

\[ I_M^L(\pi_1, X_1, f_{1,L}), \quad \pi_1 \in \Pi_{\text{unit}}(M(F_v)), X_1 \in a_{M,v_1}, \]

are well defined. They must then vanish, since \( f_{1,L} = 0. \) It is therefore enough to show that the distributions

\[ I_M(\pi_1, X_1, f_1), \quad \pi_1 \in \Pi_{\text{unit}}(M(F_v)), X_1 \in a_{M,v_1}, \]

vanish. (Recall that by an abuse of notation, we denoted these distributions by \( I_M(\pi_1, X_1, f_{1,G}) \) in the splitting formula.)

The formula [1(j), (3.2)] gives an expansion for \( I_M(\pi_1, X_1, f_1) \) in terms of the distributions associated to standard representations \( \rho \in \Sigma(M(F_v)). \) Only those \( \rho \) with \( \Delta(\rho, \pi_1) \neq 0 \) can occur in the expansion (see [1(i), §§5–6]). Since \( \pi_1 \) is unitary, this implies that \( \rho \) has a unitary central character. It is sufficient to establish that for any such \( \rho \) and any point \( \lambda \in a_M^* \) with a small real part, the distributions

\[ I_L(\rho_{\lambda}^L, h_L(X_1), f_1), \quad L \in \mathcal{L}(M), X_1 \in a_{M,v_1}, \tag{5.2} \]

all vanish. Since its central character is unitary, \( \rho \) must either be tempered or be induced from a proper parabolic subset of \( M. \) If \( \rho \) is tempered,

\[ I_L(\rho_{\lambda}^L, h_L(X_1), f_1) = \begin{cases} f_{1,G}(\rho_{\lambda}^G, h_G(X_1)), & \text{if } L = G, \\ 0, & \text{otherwise,} \end{cases} \]

by Lemma 3.1 of [1(j)]. But \( f_{1,G} = 0, \) so the distribution vanishes even if \( L = G. \) In the other case,

\[ \rho = \rho_{\lambda}^M, \quad M_1 \not\subset \mathcal{L}, \quad \rho_{\lambda} \in \Sigma(M_1(F_v)), \]

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and we can make use of the descent property \[1(j),\ Corollary 8.5\]. We obtain an expression for a Fourier transform of (5.2) in terms of the distributions

\[ \hat{f}_{M_2}^{M_1}(\rho_{\lambda}, Y_1, f_{1,M_2}), \quad M_2 \in \mathcal{L}_0(M_1), Y_1 \in \mathfrak{a}_{M_1,v_1}. \]

Since \( M_2 \neq G \), the distributions are well defined, and therefore vanish. Thus, the distribution (5.2) vanishes in all cases. In other words, the spectral expansion reduces to 0, and \( I(f) \) vanishes. \( \square \)

We must decide how to choose \( G, F \) and \( v_1 \) in order to prove the theorem. Our original element \( \gamma_1 \) in \( M_1(F_1) \) belongs to a unique "maximal torus"

\[ T_1 = T_{1,0} \gamma_1 \]

in \( M_1 \). By definition, \( T_{1,0} \) is the connected centralizer \( G_{1,\gamma_1} \) of \( \gamma_1 \) in \( G_1^0 \).

It is a torus in \( M_1^0 \) which is \( F_1 \)-anisotropic modulo \( A_{M_1} \). Let \( E_1 \supset F_1 \) be a finite Galois extension over which \( G_1 \) and \( T_1 \) split. Choose any number field \( E \), with a valuation \( w_1 \), such that \( E_{w_1} \cong E_1 \). The Galois group, \( \text{Gal}(E_1/F_1) \), can be identified with the decomposition group of \( E \) at \( w_1 \), and therefore acts on \( E \). Let \( F \) be the fixed field in \( E \) of this group, and let \( v_1 \) be a valuation of \( F \) which \( w_1 \) divides. Then \( F_1 \cong F_{v_1} \) and \( \text{Gal}(E_1/F_1) = \text{Gal}(E/F) \). We can therefore use \( G_1 \) to twist the appropriate Chevalley group and "maximal torus" over \( F \). We obtain a component \( G \) and "maximal torus" \( T \) defined over \( F \), with \( G(F) \) and \( T(F) \) not empty, such that \( G_1 = G_{v_1} \) and \( T_1 = T_{v_1} \).

Moreover, the construction is such that \( M_1 = M_{v_1} \) and \( a_{M_1} = a_M \), where \( M \) is a Levi subset of \( G \) which contains \( T \) and is defined over \( F \). It follows that

\[ I_{M_1}(\gamma_1, f_1) = I_M(\gamma_1, f_1). \]

But the set \( T(F) \) is dense in \( T(F_{v_1}) \). We can therefore approximate our \( G \)-regular point \( \gamma_1 \) by elements \( \gamma \in T(F) \). Since \( I_M(\gamma_1, f_1) \) is continuous in (regular) \( \gamma_1 \), we have only to show that \( I_M(\gamma, f_1) = 0 \) for any fixed \( G \)-regular element \( \gamma \) in \( T(F) \). We can use the trace formula to do this.

We shall choose a suitable function

\[ f = \prod_v f_v, \quad f_v \in \mathcal{H}(G(F_v)), \]

in \( \mathcal{H}(G(A)) \), and apply Lemma 5.2. Observe first that \( T \) is \( F_{v_1} \)-anisotropic modulo \( A_M \). This means that \( T \) is contained in no proper Levi subset of \( M \) (relative to \( F_{v_1} \)). We can always replace \( F \) by a finite extension in which \( v_1 \) splits completely. We may therefore assume that \( T \) is also \( F_{v_2} \)-anisotropic modulo \( A_M \), where \( v_2 \) is another valuation of \( F \). Let \( V = \{v_1, v_2, \ldots, v_k\} \) be a large finite set of valuations of \( F \) which contains \( v_1 \) and \( v_2 \), and outside of which \( G \) and \( T \) are unramified. At \( v = v_1 \), we have already been given our function \( f_{v_1} = f_1 \). If \( v \) is any of the other valuations in \( V \), let \( f_v \) be any function which is supported on a very small open neighborhood of \( \gamma \) in \( G(F_v) \), and such that

\[ \hat{I}_M^M(\gamma, f_v, M) = I_G(\gamma, f_v) = 1. \]
If \( v \) lies outside of \( V \), let \( f_v \) equal the characteristic function of \( K_v \). Then \( f = \prod_v f_v \) certainly belongs to \( \mathcal{A}(G(\mathbb{A})) \). It follows from Lemma 5.2 and Theorem 3.3 that

\[
(5.3) \quad \sum_{L \in \mathcal{L}} W_0^L \left| W_0^G \right|^{-1} \sum_{\delta \in (L(F))_{L,S}} a^L(S, \delta) I_L(\delta, f) = 0.
\]

Since \( V = V(f) \), the shrinking of the functions \( f_{v_1}, \ldots, f_{v_k} \) around \( \gamma \) does not increase \( V(f) \). Nor does it increase the support of \( f \). It follows that in (5.3), the set \( S \) may be chosen to be independent of \( f \), and the sums over \( \delta \) can be taken over finite sets which are also independent of \( f \).

Suppose that \( L \in \mathcal{L} \) and \( \delta \in (L(F))_{L,S} \). We apply the splitting formula [1(j), Corollary 9.2] to \( I_L(\delta, f) \). If \( L \subset L_1 \subsetneq G \), we have

\[
I^L_{L_1}(\delta, f_{v_1, L_1}) = 0,
\]

by assumption. It follows that

\[
I_L(\delta, f) = I_L(\delta, f_{v_1}) \cdot \prod_{v \neq v_1} I^L_{L_1}(\delta, f_{v, L}).
\]

Now the function \( f_{v_2} \) is supported on the \( F_{v_2} \)-anisotropic set in \( M(F_{v_2}) \). This means that \( f_{v_2, L} = 0 \) unless \( L \) contains a conjugate of \( M \). On the other hand, if \( L \) contains a conjugate

\[
wMw^{-1}, \quad w \in W_0,
\]

of \( M \), we can write

\[
I_L(\delta, f_{v_1}) = I_{w^{-1}Lw}(w^{-1} \delta w, f_{v_1}),
\]

by [1(j), (2.4*)]. If \( M \) is properly contained in \( w^{-1}Lw \), this vanishes by (5.1). Thus, the contribution of \( L \) to (5.3) vanishes unless \( L \) is conjugate to \( M \). Since the contributions from different conjugates of \( M \) are equal, we obtain

\[
(5.4) \quad \sum_{\delta \in (M(F))_{M,S}} a^M(S, \delta) \left( I_M(\delta, f_{v_1}) \prod_{v \neq v_1} I_G(\delta, f_v) \right) = 0.
\]

Once again, \( \delta \) can be summed over a finite set which is independent of how we shrink \( f \).

The orbital integrals

\[
I_G(\delta, f_v), \quad 2 \leq j \leq k,
\]

vanish unless \( \delta \) is close to the \( G^0(F_{v_j}) \)-orbit of \( \gamma \). In particular, the sum in (5.4) need only be taken over elements \( \delta \) which are regular semisimple. Consequently,

\[
a^M(S, \delta) = |M(\delta) \setminus M(F, \delta)|^{-1} \text{vol}(M(\delta) \setminus M(\mathbb{A})^1).
\]
by Theorem 8.2 of [1(g)]. Moreover, the \((M, S)\)-equivalence classes of regular semisimple elements in \(M(F)\) are just \(M^0(F)\)-orbits. It follows that

\[
\sum_{\delta} c(\delta) I_M(\delta, f_v) = 0,
\]

where \(\delta\) is summed over those \(M^0(F)\)-orbits in \(M(F)\) which are \(G^0(F_v)\)-conjugate to \(\gamma\) for \(2 \leq j \leq k\), and which meet \(K_v^G\) for \(v\) outside of \(V\), and where

\[
c(\delta) = |M_\delta(F) \setminus M(F, \delta)|^{-1} \text{vol}(M_\delta(F) \setminus M_\delta(\mathbb{A})^1) \cdot \prod_{v \in S-V} I_G(\delta, f_v).
\]

We must show that every such \(\delta\) is also \(G^0(F_v)\)-conjugate to \(\gamma\). As in [10, Appendix], we use an argument from Galois cohomology.

For the first time in this paper we shall explicitly invoke our limiting hypothesis on \(G\). If \(G\) is an inner twist of the component

\[G^* = (GL(n) \times \cdots \times GL(n)) \rtimes \theta^*,\]

then any two elements in \(G(F)\) which are in the same \(G^0\)-orbit are actually in the same \(G^0(F)\)-orbit. There is nothing further to prove in this case. We can assume therefore that \(G = G^0\). Then \(T\) is a maximal torus (in the usual sense) in \(G\). The set of \(G(F_v)\)-conjugacy classes in \(G(F_v)\) which are contained in the \(G\)-conjugacy class of \(\gamma\) is known to be in bijective correspondence with a subset of

\[
\cdot H^1(F_v, T) = H^1(\text{Gal}(\overline{F}_v/F_v), T(\overline{F}_v)).
\]

A similar assertion holds for \(G(F)\)-conjugacy classes. Let \(E/F\) be a finite Galois extension which is unramified outside \(V\), and over which \(T\) splits. Then \(H^1(F_v, T)\) equals \(H^1(\text{Gal}(E_w/F_v), T(E_w))\), and Tate-Nakayama theory provides an isomorphism between this group and

\[
\{\lambda^v \in X_*(T) : \text{Norm}_{E_w/F_v}(\lambda^v) = 0\}/\{\lambda^v - \sigma \lambda^v : \lambda^v \in X_*(T), \sigma \in \text{Gal}(E_w/F_v)\},
\]

and an isomorphism between

\[
H^1(\text{Gal}(E/F), T(\mathbb{A}_E)/T(E))
\]

and

\[
\{\lambda^v \in X_*(T) : \text{Norm}_{E/F}(\lambda^v) = 0\}/\{\lambda^v - \sigma \lambda^v : \lambda^v \in X_*(T), \sigma \in \text{Gal}(E/F)\}.
\]

Here \(w\) stands for a fixed valuation on \(E\) which lies above a given \(v\). Moreover, there is an exact sequence

\[
H^1(\text{Gal}(E/F), T(E)) \to \bigoplus_v H^1(\text{Gal}(E_w/F_v), T(E_w)) \to H^1(\text{Gal}(E/F), T(\mathbb{A}_E)/T(E)).
\]
The first map is compatible with the embedding of $G(F)$-conjugacy classes into $\prod_v G(F_v)$, and the second arrow is given by the natural map

$$\bigoplus_v \lambda_v \rightarrow \sum_v \lambda_v$$

from the direct sum of modules (5.6) into (5.7). Now, consider the conjugacy class of $\gamma$. Any $\delta$ which occurs in the sum (5.5) maps to an element $\bigoplus_v \lambda_v$ such that $\sum_v \lambda_v = 0$. If $v$ is one of the valuations $v_2, \ldots, v_k$, $\delta$ is $G(F_v)$-conjugate to $\gamma$, so that $\lambda_v = 0$. If $v$ lies outside $V$, $\delta$ is $M_0^0(F)$-conjugate to an element in $K_v^G$. Since $(G, T)$ is unramified at $v$, we again have $\lambda_v = 0$ [11(a), Proposition 7.1]. It follows that $\lambda_v = 0$. In other words, $\delta$ is $G(F_v)$-conjugate to $\gamma$, as we wanted to prove.

We are now done. For if $\delta$ is an element in $M(F)$ which is $G^0_0(F_v)$-conjugate to $\gamma$, we have $\delta = y^{-1} \gamma y$, for some element $y \in M_0^0(F_v)K_{v}$ which normalizes $M_0^0$. It follows from [1(4), (2.4*)] that

$$I_M(\delta, f_v) = I_M(\gamma, f_v).$$

But for any $\delta$ which occurs in the sum (5.5), the constant $c(\delta)$ is strictly positive. It follows from (5.5) that

$$I_M(\gamma, f_v) = 0.$$

As we noted earlier, this implies that

$$I_{M_1}(\gamma_1, f_1) = 0,$$

for our original point $\gamma_1 \in M_1(F)$ . Theorem 5.1 is proved. ☐

**Corollary 5.3.** Suppose that $G/F$ is as in §1. Then for any $S$ and any $M \in \mathcal{L}$, the distributions

$$I_M(\gamma), \quad \gamma \in M(F_S),$$

are supported on characters.

**Proof.** The corollary follows immediately from the theorem and Corollary 9.3 of [1(j)]. ☐

Corollary 5.3 justifies the primary induction assumption of §1. In particular, the distributions which occur in the invariant trace formula are all supported on characters. We have at last finished the extended induction argument, begun originally in [1(j)].

### 6. A Convergence Estimate

It is not known that the spectral expansion for $I(f)$ provided by Theorem 4.4 converges as a multiple integral over $t, M$ and $\pi$. The main obstruction
is the trace class problem. This is essentially the question of showing that the operators
\[ \bigoplus_{t \geq 0} \rho_{Q,t}(\Lambda, f), \quad Q \in \mathcal{F}^0, f \in \mathcal{H}(G(\mathbb{A})) , \]
are of trace class. We shall instead prove an estimate for the rate of convergence of the $\chi$-expansion. The estimate is an extension of some of the arguments used in the derivation of the trace formula. Although rather weak, it seems to be a natural tool for those applications which entail a comparison of trace formulas.

The estimate will be stated in terms of multipliers. Recall [1(d)] that multipliers are associated to elements in $\mathcal{E}(\mathfrak{h})^W$, the convolution algebra of compactly supported $W$-invariant distributions on $\mathfrak{h}$. For $\alpha \in \mathcal{E}(\mathfrak{h})^W$ and $f \in \mathcal{H}(G(\mathbb{A}))$, $f_\alpha$ is the new function in $\mathcal{H}(G(\mathbb{A}))$ such that
\[ \pi(f_\alpha) = \hat{\alpha}(\nu_\pi) \pi(f), \quad \pi \in \Pi(G(\mathbb{A})). \]
Similarly, for any function $\phi \in \mathcal{F}(G(\mathbb{A}))$, there is another function $\phi_\alpha \in \mathcal{F}(G(\mathbb{A}))$ such that
\[ \phi_\alpha(\pi) = \hat{\alpha}(\nu_\pi) \phi(\pi), \quad \pi \in \Pi_{\text{temp}}(G(\mathbb{A})). \]
(As in §11 of [1(i)], we shall sometimes regard $\phi$ as a function on $\Pi_{\text{temp}}(G(\mathbb{A}))$ instead of the product $\Pi_{\text{temp}}(G(\mathbb{A})) \times a_G$. Then two interpretations are of course related by the Fourier transform
\[ \phi(\pi, X) = \int_{ia_G^*} \phi(\pi, \lambda)e^{-\lambda(\pi - X)} d\lambda, \quad \lambda \in a_G^*. \]
on $ia_G^*$. Suppose that $\alpha$ belongs to the subalgebra $C_c^\infty(\mathfrak{h})^W$. Then we have
(6.1) \[ \phi_\alpha(\pi, X) = \int_{a_G} \phi(\pi, Z) \alpha_G(\pi, X - Z) dZ, \]
where
\[ \alpha_G(\pi, Z) = \int_{ia_G^*} \hat{\alpha}(\nu_\pi + \lambda)e^{-\lambda(\pi - Z)} d\lambda, \quad Z \in a_G. \]
Formula (6.1) is useful because it makes sense even if $\phi$ belongs to the larger space $\mathcal{F}_{\text{ac}}(G(\mathbb{A}))$. For if $X$ remains within a compact set, the function
\[ Z \mapsto \alpha_G(\pi, X - Z) \]
is supported on a fixed compact set. It follows that $\phi \to \phi_\alpha$ extends to an action of $C_c^\infty(\mathfrak{h})^W$ on $\mathcal{F}_{\text{ac}}(G(\mathbb{A}))$ such that (6.1) holds. Similarly, $f \to f_\alpha$ extends to an action of $C_c^\infty(\mathfrak{h})^W$ on $\mathcal{H}_{\text{ac}}(G(\mathbb{A}))$. Recall that if $f \in \mathcal{H}_{\text{ac}}(G(\mathbb{A}))$ and $X \in a_G$, $f^X$ is the restriction of $f$ to
\[ G(\mathbb{A})^X = \{ x \in G(\mathbb{A}): H_G(x) = X \}, \]
and
\[ \pi(f^X) = \int_{G(\mathbb{A})^X} f(x) \pi(x) dx, \quad \pi \in \Pi(G(\mathbb{A})). \]
Then we have

\[ \pi(f^X) = \pi((f^X)^X) = \int_{\mathfrak{g}} \pi(f^Z)\alpha_G(\pi, X - Z)\,dZ. \]

Setting \( X = 0 \), we obtain the formula

\[ \pi(f^\Pi) = \int_{\mathfrak{g}} \pi(f^Z)\alpha_G(\pi, -Z)\,dZ, \quad \pi \in \Pi(G(A)). \]

for the restriction \( f^\Pi \) of \( f^\alpha \) to \( G(A) \).

We do not want \( f \) to be an arbitrary function in \( \mathcal{H}_ac(G(A)) \). We must insist in some mild support and growth conditions on the functions \( f^Z \) as \( Z \) gets large. Fix a height function

\[ \|x\| = \prod_v \|x_v\|_v, \quad x \in G(A), \]

on \( G(A) \) as in §§2 and 3 of [1(d)]. We shall say that a function \( f \in \mathcal{H}_ac(G(A)) \) is moderate if there are positive constants \( c \) and \( d \) such that \( f \) is supported on

\[ \{x \in G(A) : \log \|x\| \leq c(\|H_G(x)\| + 1)\}, \]

and such that

\[ \sup_{x \in G(A)} (|\Delta f(x)| \exp\{-d\|H_G(x)\|\}) < \infty, \]

for any left invariant differential operator \( \Delta \) on \( G(F_{\infty}) \). In a similar fashion, one can define the notion of a moderate function in \( \mathcal{J}_ac(G(A)) \). (We shall recall the precise definition in the appendix.)

It is not hard to show that the map \( f \to f^G \) sends moderate functions in \( \mathcal{H}_ac(G(A)) \) to moderate functions in \( \mathcal{J}_ac(G(A)) \). Conversely, we have

**Lemma 6.1.** Suppose that \( \Gamma \) is a finite subset of \( \Pi(K) \) and that \( \phi \) is a moderate function in \( \mathcal{J}_ac(G(A))_\Gamma \). Then there is a moderate function \( f \in \mathcal{H}_ac(G(A))_\Gamma \) such that \( f^G = \phi \).

This lemma can be regarded as a variant of the trace Paley-Wiener theorem. We shall postpone its proof until the appendix.

We shall write \( C_\infty^N(\mathfrak{h})^W \), as usual, for the set of functions in \( C_\infty^\infty(\mathfrak{h})^W \) which are supported on the ball of radius \( N \).

**Lemma 6.2.** Suppose that \( f \) is a moderate function in \( \mathcal{H}_ac(G(A)) \). Then there is a constant \( c \) such that for any \( \alpha \in C_\infty^\infty(\mathfrak{h})^W \), with \( N > 0 \), the function \( f^\alpha \) is supported on

\[ \{x \in G(A) : \log \|x\| \leq c(\|H_G(x)\| + N + 1)\}. \]

**Proof.** We can use the direct product decomposition \( G(A) = G(A)^1 \times A_{G,\infty} \) to identify each of the restricted functions \( f^X, \ X \in \alpha_G \), with a function in \( \mathcal{H}(G(A)^1) \). The lemma then follows from Proposition 3.1 of [1(d)] and the appropriate variant of (6.2). \( \Box \)
We are now ready to state our convergence estimate. Fix a finite subset $\gamma$ of $\Pi(K)$. If $L_0 \in \mathcal{L}^0$ and $\chi \in \mathcal{H}(G,F)$, a variant of the definition of §4 provides a set $\Pi_{\text{disc}}(L_0,\chi)$ of irreducible representations of $L_0(A)$. Let $\Pi_{\text{disc}}(L_0,\chi)_{\Gamma}$ be the subset of representations in $\Pi_{\text{disc}}(L_0,\chi)$ which contain representations in the restriction of $\Gamma$ to $K \cap L_0(A)$.

**Lemma 6.3.** Suppose that $\phi$ is a moderate function in $\mathcal{F}_{\text{ac}}(G(A))_{\Gamma}$. Then there are constants $C$ and $k$ such that for any subset $\mathcal{I}_1$ of $\mathcal{H}(G,F)$ and any $\alpha \in C_N^{\infty}(\mathfrak{h})^W$, with $N > 0$, the expression

$$\sum_{\chi \in \mathcal{I}_1} |I_{\chi}(\phi_\alpha)|$$

is bounded by the supremum over $\chi \in \mathcal{I}_1$, $L_0 \in \mathcal{L}^0$, $\Lambda \in \text{ia}^*_L$, and $\sigma \in \Pi_{\text{disc}}(L_0,\chi)_{\Gamma}$ of

$$Ce^{kN}|\hat{\alpha}(\nu_\sigma + \Lambda)|.$$

**Proof.** By Lemma 6.1 there is a moderate function $f$ in $\mathcal{F}_{\text{ac}}(G(A))_{\Gamma}$ such that $f_G = \phi$. Then

$$\hat{I}_{\chi}(\phi_\alpha) = I_{\chi}(f_\alpha) = I_{\chi}(f_\alpha^1),$$

for $\chi \in \mathcal{H}$ and $\alpha \in C_N^{\infty}(\mathfrak{h})^W$. By Lemma 6.2, the function $f_\alpha^1$ is supported on a set

$$\{x \in G(A)^1 : \log \|x\| \leq c(1 + N)\},$$

where the constant $c$ depends only on $f$. We are first going to estimate the sum $\sum_{\chi \in \mathcal{I}_1} |J_{\chi}(f_\alpha)|$ of noninvariant distributions. We shall appeal to two results (Proposition 2.2 and Lemma A.1) of [1(d)] which apply to the case that $G = G^0$. The results for general $G$, which require slightly different notation, can be extracted from [7]. We shall simply quote them.

Fix a minimal parabolic subgroup $Q_0 \in \mathcal{Q}^0(M_0)$ for $G^0$. Proposition 2.2 of [1(d)] applies to the distribution $J_{\chi}^T(f_\alpha)$, where $T$ is a point in $\mathcal{a}_0$ such that the function

$$d_{Q_0}(T) = \min_{\alpha \in \Delta_{Q_0}} \{\alpha(T)\}$$

is suitably large. The assertion is that there is a constant $C_0$ such that if

(6.3) $$d_{Q_0}(T) > C_0 c(1 + N),$$

and if $f_\alpha$ is as above, then $J_{\chi}^T(f_\alpha)$ equals an expression

$$\sum_{\{Q \in \mathcal{Q}^0 : Q \supseteq Q_0\}} \int_{\text{ia}^*_Q/\text{ia}^*_G} \Psi_{Q,\chi}^T(\Lambda, f_\alpha) d\Lambda.$$

Here,

$$\Psi_{Q,\chi}^T(\Lambda, f_\alpha) = \left|\mathcal{D}(M_Q)\right|^{-1} \text{tr}(\Omega_{Q,s_Q,\chi}^T(s\Lambda) \rho_{Q,\chi}(s,\Lambda, f_\alpha^1)).$$
where \( s \) is any element in \( W_0^G \), \( \rho_{Q,x}(s, \Lambda, f^{1}_\alpha) \) is the linear map from \( \mathcal{A}_{Q,x} \) to \( \mathcal{A}_{Q,x} \) discussed in §4, and \( \Omega_{Q|s}^{T}(s, \Lambda) \) is the linear map from \( \mathcal{A}_{s}^{2} \) to \( \mathcal{A}_{s}^{2} \) such that for any pair of vectors \( \phi \in \mathcal{A}_{Q,x} \) and \( \phi_s \in \mathcal{A}_{s}^{2} \),

\[
(\Omega_{Q|s}^{T}(s, \Lambda) \phi_s, \phi)
\]
equals
\[
\int_{G(F)A_0 \backslash G_0(\mathbb{A})} \Lambda^{T} E_s Q(x, \phi_s, s \Lambda) \cdot \Lambda^{T} E_{Q}(x, \phi, \Lambda) \, dx.
\]

\( (E_{Q} \) stands for the Eisenstein series associated to \( Q \), and \( \Lambda^{T} \) is the truncation operator.) Therefore,

\[
\sum_{\chi \in \mathcal{A}_1} |J_{\chi}^{T}(f^{1}_\alpha)|
\]
is bounded by

\[
\sum_{\chi \in \mathcal{A}_1} \sum_{Q \supset Q_0} |\mathcal{P}(M_{Q})|^{-1} \int_{ia_0^G/ia_0^*} \|\Omega_{Q|s}^{T}(s, \Lambda, f^{1}_\alpha)\|_1 \, dA,
\]
where \( \| \cdot \|_1 \) denotes the trace class norm.

Suppose that \( f \) is bi-invariant under an open compact subgroup \( K_0 \) of \( G^0(\mathbb{A}_\text{fin}) \). According to Lemma A.1 of [1(d)], there are constants \( C_{K_0} \) and \( d_0 \) such that

\[
\sum_{\chi \in \mathcal{A}_1} \sum_{Q \supset Q_0} |\mathcal{P}(M_{Q})|^{-1} \int_{ia_0^G/ia_0^*} \|\rho_{Q,x}(\Lambda, \Delta^{m}_{\alpha})_{K_0}^{-1} \cdot \Omega_{Q|s}^{T}(s, \Lambda, \Delta^{m}_{\alpha} f^{1}_\alpha)\|_1 \, dA
\]
is bounded by

\[
C_{K_0}(1 + \|T\|^{d_0}),
\]
where \( \Delta^{m}_{\alpha} \) is a certain left invariant differential operator on \( G^0(F_{\infty})^{1} \) and \( (\cdot)_{K_0} \) denotes the restriction of a given operator to the space of \( K_0 \)-invariant vectors. In order to exploit this estimate, we note that

\[
\|\Omega_{Q|s}^{T}(s, \Lambda)\|_1 \rho_{Q,x}(s, \Lambda, f^{1}_\alpha)\|
\]
is no greater than

\[
\|\rho_{Q,x}(\Lambda, \Delta^{m}_{\alpha})_{K_0}^{-1} \cdot \Omega_{Q|s}^{T}(s, \Lambda)\|_1 \cdot \|\rho_{Q,x}(s, \Lambda, \Delta^{m}_{\alpha} f^{1}_\alpha)\|.
\]

It follows that (6.4) is bounded by the product of (6.5) with

\[
\sup_{\chi \in \mathcal{A}_1} \sup_{Q \supset Q_0} \sup_{\Lambda \in ia_0^G/ia_0^*} \|\rho_{Q,x}(s, \Lambda, \Delta^{m}_{\alpha} f^{1}_\alpha)\|.
\]

Now \( J_{\chi}^{T}(f^{1}_\alpha) \) is a polynomial in \( T \), and \( J_{\chi}(f^{1}_\alpha) \) is defined as its value at a fixed point \( T_0 \) [1(c), §2]. We can certainly interpolate \( J_{\chi}(f^{1}_\alpha) \) from the values of \( J_{\chi}^{T}(f^{1}_\alpha) \) in which \( T \) satisfies (6.3) [1(d), Lemma 5.2]. It follows that
there is a constant $C_{K_0}'$, depending only on $K_0$, such that the original sum $\sum_{\chi \in \mathcal{X}} |J_{\chi}(f_0)|$ is bounded by the product of (6.6) with $C_{K_0}'(1+N)^{d_0}$.

Consider the expression (6.6). For a given $Q$, write
\[ \rho_Q(x)(s, \Lambda, \Delta^m f_\alpha^1) = \bigoplus_{\sigma \in \Pi_{disc}(MQ, \chi)} \rho_{Q, \chi, \sigma}(s, \Lambda, \Delta^m f_\alpha^1), \]
where $\rho_{Q, \chi, \sigma}$ denotes the representation induced from the isotypical component of $\sigma$. Then
\[ \|\rho_{Q, \chi, \sigma}(s, \Lambda, \Delta^m f_\alpha^1)\| \leq \sup_{\{\sigma \in \Pi_{disc}(MQ, \chi)\}} \|\rho_{Q, \chi, \sigma}(s, \Lambda, \Delta^m f_\alpha^1)\|. \]

Since
\[ \Delta^m f_\alpha^1 = (\Delta^m f)^1, \]
the formula (6.2') leads to an inequality
\[ \|\rho_{Q, \chi, \sigma}(s, \Lambda, \Delta^m f_\alpha^1)\| \leq \int_{a_G} \|\rho_{Q, \chi, \sigma}(s, \Lambda, \Delta^m f^Z)\| \cdot |\alpha_G(\sigma^G_\Lambda, -Z)| dZ \]
\[ \leq \int_{a_G} \left( \int_{G(a)} |(\Delta^m f)(x)| \cdot \|\rho_{Q, \chi, \sigma}(s, \Lambda, x)\| dx \right) \cdot |\alpha_G(\sigma^G_\Lambda, -Z)| dZ. \]

The operator $\rho_{Q, \chi, \sigma}(s, \Lambda, x)$ is unitary, and has norm equal to 1. Observe also that the function $\alpha_G(\sigma^G_\Lambda, Z)$ vanishes unless $\|Z\| \leq N$. It follows that
\[ \|\rho_{Q, \chi, \sigma}(s, \Lambda, \Delta^m f_\alpha^1)\| \leq \left( \int_{G(a)_N} |\Delta^m f(x)| dx \right) \cdot \sup_{Z \in a_G} (|\alpha_G(\sigma^G_\Lambda, Z)|), \]
where
\[ G(a)_N = \{x \in G(a) : \|H_G(x)\| \leq N\}. \]
Since $f$ is moderate, the intersection of its support with $G(a)_N$ is contained in a set
\[ \{x \in G(a) : \log \|x\| \leq c(N + 1)\}, \]
whose volume depends exponentially on $N$. Moreover, the supremum of $|\Delta f(x)|$ on $G(a)_N$ is bounded by a function which also depends exponentially on $N$. It follows that
\[ \int_{G(a)_N} |\Delta^m f(x)| dx \leq C_0 e^{k_0 N}, \]
for constants $C_0$ and $k_0$ which are independent of $N$. On the other hand, we can write
\[ \sup_{Z \in a_G} |\alpha_G(\sigma^G_\Lambda, Z)| \leq \int_{ia_G} |\hat{\alpha}(\nu + \Lambda + \mu)| d\mu \]
\[ \leq C_G \sup_{\mu \in ia_G} \left( (1 + \|\Lambda + \mu\|^2)^{\dim a_v} |\hat{\alpha}(\nu + \Lambda + \mu)| \right). \]
where
\[
C_G = \int_{\mathcal{H}_G} (1 + \|\mu\|^2)^{-\dim a_G} \, d\mu.
\]
Combining these facts, we see that the expression (6.6) is bounded by the product of \( C_G C_0 e^{k_0 N} \) with the supremum over \( \chi \in \mathcal{H}_1, \, Q \supset Q_0, \, \Lambda \in \mathfrak{i}a_{Q} \) and \( \sigma \in \Pi_{\text{disc}}(M_Q, \chi)_\Gamma \) of
\[
(1 + \|\Lambda\|^2)^{\dim a_G} |\hat{\alpha}(\nu_\sigma + \Lambda)|.
\]

We can now state an estimate for (6.7)
\[
\sum_{\chi \in \mathcal{H}_1} |J_{\chi}(f_\alpha)|.
\]
In order to remove the dependence on \( Q_0 \), we shall replace the supremum over \( Q \) by one over \( L_0 \in \mathcal{L}_0 \). Choose positive constants \( C'_1 \) and \( k'_1 \) such that
\[
C'_0 (1 + N)^{d_0} C_G C_0 e^{k_0 N} \leq C'_1 e^{k'_1 N}.
\]
Then (6.7) is bounded by the supremum over \( \chi \in \mathcal{H}_1, \, L_0 \in \mathcal{L}_0, \, \Lambda \in \mathfrak{i}a_{L_0} \) and \( \sigma \in \Pi_{\text{disc}}(L_0, \chi)_\Gamma \) of
\[
C'_1 e^{k'_1 N} (1 + \|\Lambda\|^2)^{\dim a_G} |\hat{\alpha}(\nu_\sigma + \Lambda)|.
\]

To remove the factors \((1 + \|\Lambda\|^2)\) from the estimate, we require a simple lemma.

**Lemma 6.4.** For any integer \( m \geq 1 \) we can choose a bi-invariant differential operator \( z \) on \( G(F_\infty) \), and multipliers \( \alpha_1 \in C^m(\mathfrak{h})^W \) and \( \alpha_2 \in C^\infty(\mathfrak{h})^W \) such that \( f = (zf)_{\alpha_1} + f_{\alpha_2} \), for any function \( f \in \mathcal{B}_c(G(A)) \).

**Proof.** This follows from a standard argument, which was first applied to the trace formula by Duflo and Labesse (see for example [1(a), Lemma 4.1].) For any \( m \), one obtains a \( W \)-invariant differential operator \( \zeta \) with constant coefficients on \( \mathfrak{h} \), and functions \( \alpha_1 \in C^m(\mathfrak{h})^W \) and \( \alpha_2 \in C^\infty(\mathfrak{h})^W \), such that \( \zeta \alpha_1 + \alpha_2 \) is the Dirac measure at the origin in \( \mathfrak{h} \). Let \( z \) be the inverse image of \( \zeta \) under the Harish-Chandra map. Then
\[
f = f_{(\zeta \alpha_1 + \alpha_2)} = (zf)_{\alpha_1} + f_{\alpha_2},
\]
as required. \( \square \)

Returning to the proof of Lemma 6.3, we apply Lemma 6.4, with \( m \) large, to our moderate function \( f \). We see that (6.7) is bounded by
\[
\sum_{\chi \in \mathcal{H}_1} |J_{\chi}((zf)_{\alpha_1})| + \sum_{\chi \in \mathcal{H}_1} |J_{\chi}(f_{\alpha_2})|.
\]
Since the function $zf$ is also moderate, we can apply the estimate we have obtained to each of these sums. Notice that
\[
\sup_{\chi, L_0, \Lambda, \sigma} ((1 + \|\Lambda\|^2)^{\dim \mathcal{G}} |(\alpha_i * \alpha)^{\Lambda}(\nu_\sigma + \Lambda)|)
\]
\[\leq \sup((1 + \|\Lambda\|^2)^{\dim \mathcal{G}} |\hat{\alpha}_i(\nu_\sigma + \Lambda)| \cdot |\hat{\alpha}(\nu_\sigma + \Lambda)|)
\]
\[\leq \sup((1 + \|\nu_\sigma + \Lambda\|^2)^{\dim \mathcal{G}} |\hat{\alpha}_i(\nu_\sigma + \Lambda)|) \cdot \sup |\hat{\alpha}(\nu_\sigma + \Lambda)|.
\]
But the real parts of the points $\nu_\sigma$ lie in a fixed bounded set, and the functions $\hat{\alpha}_i$ decrease rapidly on cylinders (in a sense that depends on $m$). Therefore
\[
\sup((1 + \|\nu_\sigma + \Lambda\|^2)^{\dim \mathcal{G}} |\hat{\alpha}_i(\nu_\sigma + \Lambda)|) < \infty.
\]
It follows that there are positive constants $C_1$ and $k_1$ such that (6.7) is bounded by
\[
C_1e^{k_1N} \sup \sup_{\chi \in \mathcal{K}, L_0, \Lambda, \sigma} (|\hat{\alpha}(\nu_\sigma + \Lambda)|).
\]
We must convert this into an estimate for (6.8)
\[
\sum_{\chi \in \mathcal{K}} |\hat{I}_\chi(\phi_\alpha)|.
\]
Suppose that $M \in \mathcal{L}_0$. It follows from Corollary 12.3 of [1(i)] that the function $\phi_M(f)$ in $\mathcal{S}_c(M(A))$ is also moderate. Since
\[
\hat{I}_\chi^M(\phi_M(f)) = \hat{I}_\chi(\phi_M(f)),
\]
we can apply the lemma inductively to $\phi_M(f)$. We obtain constants $C_M$ and $k_M$, depending only on $f$, such that
\[
\sum_{\chi \in \mathcal{K}} |\hat{I}_\chi^M(\phi_M(f))|
\]
is bounded by
\[
C_Me^{k_MN} \sup \sup_{\chi \in \mathcal{K}, L_0, \Lambda, \sigma} (|\hat{\alpha}(\nu_\sigma + \Lambda)|).
\]
The required estimate for (6.8) then follows from the estimate for (6.7) and the formula
\[
\hat{I}_\chi(\phi_\alpha) = J_\chi(f_\alpha) - \sum_{M \in \mathcal{L}_0} |W_0^M||W_0^G|^{-1}\hat{I}_\chi^M(\phi_M(f_\alpha)).
\]
We shall restate the lemma in a simple form that is convenient for applications. Let $\mathfrak{h}_u^*$ denote the set of elements $\nu$ in $\mathfrak{h}_c^*/i\mathfrak{a}_G^*$ such that $\overline{\nu} = s\nu$ for some element $s \in W$ of order 2. Here $\overline{\nu}$ stands for the conjugation of $\mathfrak{h}_c^*$ relative to $\mathfrak{h}^*$. As is well known, the infinitesimal character $\nu_\pi$ of any unitary representation $\pi \in \Pi_{\text{unit}}(G^0(A)^1)$ belongs to $\mathfrak{h}_u^*$. Observe that if $r$ and $T$ are nonnegative real numbers, the set
\[
\mathfrak{h}_u^*(r, T) = \{\nu \in \mathfrak{h}_u^* : \|\Re(\nu)\| \leq r, \|\mathcal{S}(\nu)\| \geq T\}.
\]
is invariant under $W$. (An element $\nu \in h_u^*$ is only a coset of $ia_G^*$ in $h_c^*$, but $\|\nu\|$ is understood to be the minimum value of the norm on the coset.) Let $h^1$ be the orthogonal complement of $a_G$ in $h$. Then $h_u^*$ can be identified with a subset of the complex dual space of $h^1$.

**Corollary 6.5.** Choose any function $f \in \mathcal{F}_c(G(\mathbb{A}))$. Then there are positive constants $C, k$ and $r$ such that

$$
\sum_{\nu \in h_u^*} |I_t(f_\nu)| \leq Ce^{kN} \sup_{\nu \in h_u^* (r, T)} (|\hat{\alpha}(\nu)|),
$$

for any $T > 0$ and any $\alpha \in C^\infty_N(h^1)^W$, with $N > 0$.

**Proof.** Lemma 6.3 is stated for multipliers in $C^\infty_N(h)^W$, but it is equally valid if $\alpha$ belongs to $C^\infty_N(h^1)^W$. To see this, apply the lemma to the sequence

$$
\alpha_n(H + Z) = \alpha(H)\beta_n(Z), \quad H \in h^1, Z \in a_G,
$$

in $C^\infty_c(h)^W$, where $\beta_n \in C^\infty_c(a_G)$ approaches the Dirac measure at 1. The (upper) limits of each side of the resulting inequality give the analogous inequality for $\alpha$. Notice that $f^1_\alpha$ depends only on $f^1$, so that $f$ can indeed be an arbitrary function in $\mathcal{F}_c(G(\mathbb{A}))$.

We shall apply this version of the lemma to the given $\alpha$, with $\phi = f_G$, and with

$$
\mathcal{F}_1 = \{ \chi \in \mathcal{P}: \|\mathcal{F}m(\nu_\chi) > T\}.
$$

Then

$$
\sum_{\nu \in \mathcal{F}_1} |I_t(f_\nu)| = \sum_{\chi \in \mathcal{F}_1} |\hat{I}_\chi(\phi_\alpha)|.
$$

Choose a finite subset $\Gamma$ of $\Pi(K)$ such that $\phi$ belongs to $\mathcal{F}_c(G(\mathbb{A}))_{\Gamma}$. There is a positive number $r$ such that if $\pi$ is any representation in $\Pi_{\text{unit}}(G^0(\mathbb{A}))$ whose $K$-spectrum meets $\Gamma$, the point $\nu_\pi$ belongs to $\mathcal{F}_1 = \{ \nu \in h_u^*: \|\mathcal{F}e(\nu)\| \leq r\}$.

If $\chi$, $L_0$, $A$ and $\sigma$ are elements in $\mathcal{F}_1$, $\mathcal{L}^0$, $ia_{L_0}$ and $\Pi_{\text{disc}}(L_0, \chi)_\Gamma$, as in the inequality of the lemma, the point $\nu_\sigma + A$ then belongs to $h_u^*(r, T)$. The corollary follows. ◼  

**Remark.** Suppose that $h^2$ is any vector subspace of $h$ which contains $h^1$. Then there will be an obvious variant of Corollary 6.5 for multipliers $\alpha \in C^\infty_N(h^2)^W$. For this, $f$ must again be taken to be a moderate function in $\mathcal{F}_c(G(\mathbb{A}))$.

7. Simpler forms of the trace formula

The full trace formula is the identity

$$
\sum_{M \in \mathcal{J}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in (M(F))_{\mu,S}} a^M(S, \gamma)I_M(\gamma, f) = \sum_{M \in \mathcal{J}} \int_{\Pi(M, f)} a^M(\pi)I_M(\pi, f) d\pi, \quad f \in \mathcal{F}(G(\mathbb{A})),
$$

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given by the two expansions for $I(f)$ in Theorems 3.3 and 4.4. In this section we shall investigate how the formula simplifies if conditions are imposed on $f$. The conditions will be invariant, in the sense that they depend only on the image of $f$ in $\mathcal{F}(G(A))$. Equivalently, the conditions will depend only on the (invariant) orbital integrals of $f$.

We shall say that a function $f \in \mathcal{H}(G(A))$ is cuspidal at a valuation $v_1$ if $f$ is a finite sum of functions $\prod_v f_v$, $f_v \in \mathcal{H}(G(F_v))$, such that

$$f_{v_1,M} = 0, \quad M \in \mathcal{L}_0.$$ 

This is implied by the vanishing of the orbital integral $I_G(\gamma_1, f_{v_1})$, for any $G$-regular element $\gamma_1 \in G(F_{v_1})$ which is not $F_{v_1}$-elliptic.

**Theorem 7.1.** (a) If $f$ is cuspidal at one place $v_1$, we have

$$I(f) = \sum_{\ell \geq 0} \sum_{\pi \in \Pi_{\text{disc}}(G, t)} a^G_{\text{disc}}(\pi) I_G(\pi, f).$$

(b) If $f$ is cuspidal at two places $v_1$ and $v_2$, we have

$$I(f) = \sum_{\gamma \in (G(F))_{\gamma, S}} a^G(S, \gamma) I_G(\gamma, f).$$

**Proof.** We can assume that $f = \prod_v f_v$, with

$$f_{v_1,M} = 0, \quad M \in \mathcal{L}_0.$$ 

Part (a) will be a special case of the spectral expansion

$$I(f) = \sum_{\ell \geq 0} \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \int_{\Pi(M, t)} a^M(\pi) I_M(\pi, f) d\pi.$$ 

The main step is to show that if $M \in \mathcal{L}_0$, then

$$I_M(\pi, f) = 0, \quad \pi \in \Pi_{\text{unit}}(M(A)^1).$$

But this is very similar to the proof of Lemma 5.2. Using the splitting formula [1(j), Proposition 9.4], we reduce the problem to showing that

$$I_M(\pi_1, X_1, f_{v_1}) = 0, \quad \pi_1 \in \Pi_{\text{unit}}(M(F_{v_1})), \quad X_1 \in a_{M, v_1}, \quad M \in \mathcal{L}_0.$$ 

We then apply the expansion [1(j), (3.2)] into standard representations, and the descent formula [1(j), Corollary 8.5]. Since $\pi_1$ is unitary, the required vanishing formula follows as in Lemma 5.2. In particular, the terms with $M \neq G$ in the spectral expansion all vanish. Moreover,

$$\int_{\Pi(G, t)} a^G(\pi) I_G(\pi, f) d\pi = \sum_{M_t \in \mathcal{L}} |W_0^{M_t}| |W_0^G|^{-1} \sum_{\pi \in \Pi_{\text{disc}}(M_t, t)} \int_{\alpha_{M_t}/\alpha_G} a^G_{\text{disc}}(\pi) I^{G}_{M_t}(\pi_{1, \lambda}, f) d\lambda = \sum_{\pi \in \Pi_{\text{disc}}(G, t)} a^G_{\text{disc}}(\pi) I^{G}_{G}(\pi, f),$$

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since
\[ I_G(\pi^G_1, f) = I_{M_1}(\pi_1, f_{M_1}) = 0, \quad M_1 \neq G, \quad \pi_1 \in \Pi_{\text{unit}}(M_1(\mathbb{A}^1)). \]

Part (a) follows.

Suppose that \( f \) is also cuspidal at a second place \( v_2 \). Part (b) will be a special case of the geometric expansion
\[ I(f) = \sum_{M \in \mathcal{X}} |W_0^M| |W_0^G|^{-1} \sum_{\gamma \in \mathcal{Y}(M)} a^M(S, \gamma) I_M(\gamma, f). \]

The set \( S \) is large enough that it contains \( v_1 \) and \( v_2 \), and so that \( f \) belongs to \( \mathcal{H}(G(F_S)) \). Write
\[ f = f_1 f_2, \quad f_i \in \mathcal{H}(G(F_{S_i})), \]
where \( S_1 \) and \( S_2 \) are disjoint sets of valuations with the closure property, which contain \( v_1 \) and \( v_2 \) respectively, and whose union is \( S \). From the splitting formula [1(j), Proposition 9.1], we obtain
\[ I_M(\gamma, f) = \sum_{L_1, L_2 \in \mathcal{X}(M)} d^G_\mathcal{M}(L_1, L_2) I_{M_1}(\gamma, f_{L_1}) I_{M_2}(\gamma, f_{L_2}). \]

The distributions on the right vanish unless \( L_1 = L_2 = G \). Moreover, \( d^G_\mathcal{M}(G, G) = 0 \) unless \( M = G \). It follows that if \( M \neq G \), the distribution \( I_M(\gamma, f) \) equals \( 0 \), and the corresponding term in the geometric expansion vanishes. This gives (b). \( \square \)

**Corollary 7.2.** Suppose that \( f \) is cuspidal at two places. Then
\[ \sum_{\gamma \in \mathcal{Y}(G(F))} a^G(S, \gamma) I_G(\gamma, f) = \sum_{t \geq 0} \sum_{\pi \in \Pi_{\text{disc}}(G, t)} a^G_{\text{disc}}(\pi) I_G(\pi, f). \]

For simplicity, we shall assume that \( G = G^0 \) in the rest of \( \S 7 \). We shall also assume that \( f \in \mathcal{H}(G(\mathbb{A})) \) is such that
\[ f = \prod_v f_v, \quad f_v \in \mathcal{H}(G(\mathbb{F}_v)). \]

With additional invariant restrictions on \( f \) we shall be able to simplify the trace formula further.

**Corollary 7.3.** Suppose there is a place \( v_1 \) such that
\[ \text{tr}(\pi_1(f_{v_1})) = 0, \quad \pi_1 \in \Pi_{\text{unit}}(G(F_{v_1})), \]
whenever \( \pi_1 \) is a constituent of a (properly) induced representation
\[ \sigma_{v_1}^G, \quad \sigma_{v_1} \in \Pi_{\text{unit}}(M(F_{v_1})), \quad M \in \mathcal{X}_0. \]

Then
\[ I(f) = \sum_{t \geq 0} \text{tr}(R_{\text{disc}, t}(f)), \]
where \( R_{\text{disc}, t} \) denotes the representation of \( G(\mathbb{A}) \) on \( L_{\text{disc}, t}^2(G(F)A_{G, \infty} \backslash G(\mathbb{A})). \)
Proof. If $M$ belongs to $\mathcal{L}_0$, the condition implies that
\[ \text{tr}(\sigma_i^G(f_{\gamma})) = 0, \quad \sigma_i \in \Pi_{\text{temp}}(M(F_{\gamma})) , \]
so that $f_{\gamma,M} = 0$. Therefore $f$ is cuspidal at $\gamma$. Applying part (a) of the theorem, we obtain
\[ I(f) = \sum_{\gamma \in \mathcal{O}(\mathcal{R})} \sum_{\pi \in \Pi_{\text{disc}}(G, \pi)} a_{\text{disc}}^G(\pi) I_G(\pi, f) \]
\[ = \sum_{\gamma \in \mathcal{O}(\mathcal{R})} \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \]
\[ \times \sum_{s \in W(\mathcal{R})_{\text{reg}}} \det(s - 1)_{\alpha_M}^{-1} \text{tr}(M_{\alpha_Q}(0) \rho_{Q,s}(s, 0, f)) , \]
in the notation of §4. Here, $Q$ is any element in $\mathcal{R}(M)$. If $M \neq G$,
\[ \text{tr}(M_{\alpha_Q}(0) \rho_{Q,s}(s, 0, f)) \]
is a linear combination of characters of unitary induced representations. It vanishes by assumption. If $M = G$,
\[ \text{tr}(M_{\alpha_Q}(0) \rho_{Q,s}(s, 0, f)) = \text{tr}(\rho_{G,s}(0, f)) = \text{tr}(R_{\text{disc},s}(f)) , \]
by definition. The corollary follows. \[ \square \]

Corollary 7.4. Suppose there is a place $\gamma_1$ such that
\[ I_G(\gamma_1, f_{\gamma_1}) = 0 \]
for any element $\gamma_1 \in G(F_{\gamma_1})$ which is not semisimple and $F_{\gamma_1}$-elliptic. Suppose also that $f$ is cuspidal at another place $\gamma_2$. Then
\[ I(f) = \sum_{\gamma \in \mathcal{O}(G(F))} \text{vol}(G(F, \gamma) A_{G,\infty} \setminus G(A, \gamma)) \int_{G(A, \gamma) \setminus G(A)} f(x^{-1} \gamma x) \, dx , \]
where $\{G(F)_{\text{ell}}\}$ denotes the set of $G(F)$-conjugacy classes of $F$-elliptic elements in $G(F)$, and $G(F, \gamma)$ and $G(A, \gamma)$ denote the centralizers of $\gamma$ in $G(F)$ and $G(A)$.

Proof. The conditions imply that $f$ is cuspidal at $\gamma_1$ and $\gamma_2$. We can therefore apply the formula
\[ I(f) = \sum_{\gamma \in \mathcal{O}(G(F))} a^G(S, \gamma) I_G(\gamma, f) \]
of the theorem. If an element $\gamma \in G(F)$ is not $F$-elliptic, it is not $F_{\gamma_1}$-elliptic, and $I_G(\gamma, f) = 0$. The corollary then follows from Theorem 8.2 of [11(g)] and the definition of $I_G(\gamma, f)$. \[ \square \]

The conditions of Corollaries 7.3 and 7.4 sometimes arise naturally. For example, if $\gamma_1$ is discrete, Kottwitz [11(b)] has introduced a simple function
which satisfies the conditions of Corollary 7.4. Kottwitz also establishes a version of this corollary in [11(b)]. He imposes stronger conditions at \( v_2 \), but derives a formula without resorting to the invariant trace formula.

For another example, take \( G = GL(n) \). Suppose that \( f \) is cuspidal at \( v_1 \). Any element \( \gamma_1 \in G(F_{v_1}) \) which is not \( F_{v_1} \)-elliptic belongs to a \( G(F_{v_1}) \)-conjugacy class

\[
\delta_1^G, \quad \delta_1 \in M(F_{v_1}), \quad M \in \mathcal{L}_0.
\]

Consequently,

\[
I_G(\gamma_1, f_{v_1}) = i_M^M(\delta_1, f_{v_1}, M) = 0.
\]

Therefore, the first condition of Corollary 7.4 is satisfied. Moreover, it is known that any induced unitary representation

\[
\sigma_1^G, \quad \sigma_1 \in \Pi_{\text{unit}}(M(F_{v_1})), \quad M \in \mathcal{L},
\]

is irreducible ([3], [15]). Since

\[
\text{tr}(\sigma_1^G(f_{v_1})) = f_{v_1,M}(\sigma_1) = 0, \quad M \in \mathcal{L}_0,
\]

the condition of Corollary 7.3 also holds. Combining Corollaries 7.3 and 7.4, we obtain

**Corollary 7.5.** Assume that \( G = GL(n) \) and that \( f \) is cuspidal at two places \( v_1 \) and \( v_2 \). Then

\[
\sum_{\gamma \in \{G(F)_{\text{cl}}\}} \text{vol}(G(F, \gamma)A_{G, \infty} \setminus G(A, \gamma)) \int_{G(A, \gamma) \setminus G(A)} f(x^{-1} \gamma x) \, dx = \sum_{t \geq 0} \text{tr}(R_{\text{disc}, t}(f)). \quad \Box
\]

**8. The Example of GL(n). Global Vanishing Properties**

The simple versions of the trace formula were obtained by placing rather severe restrictions on \( f \). In many applications, one will need to prove that certain terms vanish for less severely restricted functions. We can illustrate this with the example of \( GL(n) \), begun in §10 of [1(j)].

Adopt the notation of [1(j), §10]. Then

\[
\eta: G \to G^* = \underbrace{(GL(n) \times \cdots \times GL(n))}_l \rtimes \theta^*
\]

is a given inner twist, and \( G' \) stands for the group \( GL(n) \), embedded diagonally in \((G^*)^0\). Let us write \( \mathcal{L}' \) for the set of Levi subgroups of \( G' \) which contain the group of diagonal matrices. For each \( L \in \mathcal{L}' \), we have the partition

\[
p(L) = (n_1, \ldots, n_r), \quad n_1 \geq n_2 \geq \cdots \geq n_r,
\]

of \( n \) such that

\[
L \cong GL(n_1) \times \cdots \times GL(n_r).
\]
Suppose that $p_1$ and $p_2$ are partitions of $n$. We shall write $p_1 \preceq p_2$, as in [1(c), §14], if there are groups $L_1 \subset L_2$ in $\mathcal{L}$ such that $p_1 = p(L_1)$ and $p_2 = p(L_2)$.

We shall assume that $\eta(M_0)$ is contained in a standard Levi subgroup of $(G^*)^0$, and that the restriction of $\eta$ to $A_{M_0}$ is defined over $F$. Then the map

$$M \to M' = \eta(M_0) \cap G',$$

is an injection of $\mathcal{L}$ into $\mathcal{L}'$. The image of this map is easy to describe. For as in [1(j), §10], we can assume that

$$G^0(E) = \prod_{l \mid \Gamma} GL_{\frac{n}{d}}(D \otimes E),$$

where $E/F$ is a cyclic extension of degree $l = l_1^{-1}$, $d$ is a divisor of $n$, and $D$ is a division algebra of degree $d^2$ over $F$. The minimal group $M'$ in the image corresponds to the partition $p(d) = (d, \ldots, d)$. The other groups in the image correspond to partitions $(n_1, \ldots, n_r)$ such that $d$ divides each $n_i$. For each valuation $v$, we shall write $d_v$ for the order of the invariant of the division algebra at $v$. Then $d$ is the least common multiple of the integers $d_v$.

In [1(j), §10], we described the norm mapping $\gamma \to \gamma'$ from (orbits in) $G(F)$ to (conjugacy classes in) $G'(F)$. It can be defined the same way for any element $M \in \mathcal{L}$. We also investigated certain functions on the local groups $G'(F_v)$. Let $f' = \prod_v f'_{v}$ be a fixed function in $\mathcal{H}(G'(\mathbb{A}))$ whose local constituents satisfy [1(j), (10.1)]. That is, the orbital integrals of $f'_{v}$ vanish at the $G'$-regular elements which are not local norms.

**Proposition 8.1.** Suppose that $L \in \mathcal{L}'$ and that $\delta \in L(F)$. Embed $\delta$ in $(L(F))_{L,S}$, where $S \supset S_{\text{ram}}$ is a large finite set of valuations. Then

$$I_L(\delta, f') = 0,$$

unless $L = M'$ and $\delta = \gamma'$, for elements $M \in \mathcal{L}$ and $\gamma \in M(F)$.

**Proof.** In the orbital integral, $\delta$ is to be considered as a point in $L(F_S)$. We must therefore regard $f' = \prod_{v \in S} f'_{v}$ as an element in $\mathcal{H}(G'(F_S))$. Assume that $I_L(\delta, f') \neq 0$. We must deduce that $L = M'$ and $\delta = \gamma'$.

The first part of the proof is taken from p. 73 of [1(c)]. Applying the splitting formula [1(j), Corollary 9.2], we obtain

$$(8.1) \quad I_L(\delta, f') = \sum_{\{L_v\}} d(\{L_v\}) \hat{I}_L(\delta, f'_{v,L_v}),$$

where the sum is taken over collections $\{L_v \in \mathcal{L}(L) : v \in S\}$, and $d(\{L_v\})$ is a constant which vanishes unless

$$(8.2) \quad a^G_L = \bigoplus_{v \in S} a^L_{v}. $$
By assumption, the left hand side of (8.1) is nonzero. Therefore, there is a collection \( \{ L_v \} \) for which (8.2) holds, and such that \( \hat{I}_L^G(\delta, f^I_{1,v,L_1}) \neq 0 \) for each \( v \in S \). This implies that

\[
p(\ell_v) \leq p(L_v), \quad v \in S.
\]

Our first task is to show that \( p(\ell) \leq p(L) \). Let \( p \) be any rational prime, and let \( p^k \) be the highest power of \( p \) which divides \( \ell \). Since \( \ell \) is the least common multiple of \( \{ \ell_v \} \), there is a valuation \( v \in S \) such that \( p^k \) divides \( \ell_v \). But the invariants of a central simple algebra sum to 0, so there must be a valuation \( w \in S \), distinct from \( v \), such that \( p^k \) also divides \( \ell_w \). It follows that \( p(p^k) \leq p(L_v) \) and \( p(p^k) \leq p(L_w) \). Since \( A_L \cap A_L^w = \{ 0 \} \), we can apply Lemma 14.1 of [1(c)]. The result is that \( p(p^k) \leq p(L) \). In other words, the integer \( p^k \) divides each of the numbers \( n_1, \ldots, n_r \) which make up the partition \( p(L) \). The same is therefore true of the integer \( \ell \), so that \( p(\ell) \leq p(L) \). In other words, \( L = M' \) for an element \( M \in \mathcal{L} \).

The next step is to show that \( \delta \) belongs to the set

\[
M'(F_v)^{M'} = \prod_{v \in S} M'(F_v)^{M'} = \prod_{v} \{ m_v \in M'(F_v) : \xi(m_v) \in N_{E_v/F_v}(E_v^*), \xi \in X(M')_F \}.
\]

Assume the contrary. Then there is a character \( \xi \in X(M')_F \) such that \( \xi(\delta) \) is not a local norm at some place. Consequently, \( \xi(\delta) \) is not a global norm. It follows from global class field theory that \( \xi(\delta) \) is not a local norm at two places \( v_1 \) and \( v_2 \). We can assume that \( v_1 \) and \( v_2 \) both belong to \( S \), and that the sets \( S_1 = S - \{ v_2 \} \) and \( S_2 = \{ v_2 \} \) both have the closure property. (In other words, if \( S \) contains an Archimedean valuation, so does \( S_1 \).) Define

\[
f'_i = \prod_{v \in S_i} f'_v, \quad i = 1, 2.
\]

Then by the splitting formula [1(j), Proposition 9.1], we have

\[
I_L(\delta, f^I) = \sum_{L_1, L_2 \in \mathcal{A}(L)} d^G_L(L_1, L_2) I^L_1(\delta, f^I_{1,1,L_1}) I^L_2(\delta, f^I_{2,L_2}).
\]

It follows that there is a pair \( L_1, L_2 \in \mathcal{A}(L) \) such that \( d^G_L(L_1, L_2) \neq 0 \), and

\[
I^L_i(\delta, f^I_{1,i,L_i}) \neq 0, \quad i = 1, 2.
\]

Now, by Lemma 10.1 of [1(j)], we can write

\[
\xi(\delta) = \xi_1(\delta) \xi_2(\delta), \quad \xi_1 \in X(L_1)_F, \ \xi_2 \in X(L_2)_F.
\]

Suppose that \( \xi_1(\delta) \) is a global norm. Then it is everywhere a local norm, so that \( \xi_2(\delta) \) is not a local norm at \( v_2 \). It follows without difficulty from the given property of \( f_{1,v_2} \) that \( I^L_2(\delta, f^I_{2,1,L_2}) \) vanishes. This is a contradiction. On the
other hand, if \( \xi_1(\delta) \) is not a global norm, it is not a local norm at two places in \( S \). At least one of these places must belong to \( S_1 \). It follows easily that \( \tilde{I}_L^I(\delta, f'_1, L) \) vanishes. This too is a contradiction. It follows that \( \delta \) belongs to the set \( M'(F_S)^M \).

The final step is to apply \([1(j), \text{Proposition 10.2}]\). This vanishing result was stated only for local fields, but by the splitting formula it extends immediately to \( G'(F_S) \). Since \( I_{M'}(\delta, f') \) does not vanish, and since \( \delta \) belongs to \( M'(F_S)^M \), the element \( \delta \) must belong to a smaller set

\[
M'(F_S)^M = \prod_{v \in S} M'(F_v)^M.
\]

(The set \( M'(F_v)^M \) was defined in the preamble to \([1(j), \text{Proposition 10.2}]\).)

Now, any element in \( M'(F_v)^M \) is the local norm of an element in \( M(F_v) \) \([1(j), \text{Lemma 10.4}]\). Since \( S \) is large, this implies that \( \delta \) is everywhere a local norm.

One can then show that \( \delta \) is the global norm of an element in \( M(F) \) (see \([2, \text{Lemma I.1.2}]\)). In other words, \( \delta = \gamma' \), for some element \( \gamma \in M(F) \). This completes the proof of the proposition. \( \square \)

**Proposition 8.2.** Suppose that \( L_1 \subset L \) are elements in \( \mathcal{L}' \) and that \( S \supset S_{\text{ram}} \) is a large finite set of valuations. Then

\[
I_L(\pi, Y, f') = 0,
\]

for any \( Y \in a_L \) and any induced representation

\[
\pi = \pi_1^L, \quad \pi_1 \in \Pi(L_1(F_S)),
\]

unless both \( L_1 \) and \( L \) are the images of elements in \( \mathcal{L} \).

**Proof.** Suppose that \( I_L(\pi, Y, f') \neq 0 \). Using the splitting formula \([1(j), \text{Proposition 9.4}]\), we first argue as at the beginning of the proof of the last proposition. This establishes that \( L = M' \) for some element \( M \in \mathcal{L} \). We then apply the local vanishing property \([1(j), \text{Proposition 10.3}]\). This proves that \( L_1 = M' \) for another \( M_1 \in \mathcal{L} \). \( \square \)

Propositions 8.1 and 8.2 are the first steps toward comparing the trace formulas of \( G \) and \( G' \). They assert that for functions \( f' \) on \( G'(A) \) as above, the distributions vanish at data which do not come from \( G \). The trace formula for \( G' \) becomes

\[
\sum_{M \in \mathcal{L}'} |W_0^M| |W_0^G|^{-1} \sum_{\gamma \in (M(F))_{M,S}} a^{M'}(S, \gamma') I_{M'}(\gamma', f')
\]

\[
= \sum_{t \geq 0} \sum_{M \in \mathcal{L}'} |W_0^M| |W_0^G|^{-1} \int_{\Pi(M', t)} a^{M'}(\pi) I_{M'}(\pi, f') \, d\pi.
\]

It is considerably harder to compare the terms which remain with the corresponding terms for \( G \). This problem will be one of the main topics of \([2]\).
Appendix. The trace Paley-Wiener theorems

We shall prove Lemma 6.1. The result can be extracted from the trace Paley-Wiener theorems [6(a), 6(b), 5 and 14] for real and \( p \)-adic groups. For implicit in these papers is the existence of a continuous section \( \phi \rightarrow f \) from \( \mathcal{F}(G(F_s))_\Gamma \) to \( \mathcal{H}(G(F_s))_\Gamma \), in which the growth and support properties of \( f \) can be estimated in terms of those of \( \phi \). I am indebted to J. Bernstein for explaining this to me in the \( p \)-adic case.

Suppose that \( S \) is any finite set of valuations of \( F \) with the closure property. The notion of a moderate function \( f \in \mathcal{H}_{ac}(G(F_s)) \) can be characterized in terms of the behavior of the functions

\[
 f^b(x) = f(x)b(H_G(x)), \quad b \in C_c^\infty(a_{G,S}).
\]

Indeed \( f \) will be moderate if and only if there are positive constants \( c \) and \( d \) such that for any \( N > 0 \), and any \( b \in C_c^\infty(a_{G,S}) \),

(i) \( f^b \in \mathcal{H}_{c(N+1)}(G(F_s)) \), the set of functions in \( \mathcal{H}(G(F_s)) \) supported on the ball of radius \( c(N + 1) \), and

(ii) \( \|f^b\| \leq \delta(b)d^N \).

Here,

\[
\|h\| = \sup_{x \in G(F_s)} |\Delta h(x)|, \quad h \in \mathcal{H}(G(F_s)),
\]

where \( \Delta \) is an arbitrary (but fixed) left invariant differential operator on \( G(F_{s,\infty} \cap S) \), while

\[
\delta(b) = \sum_{k=1}^r \sup_{x \in a_{G,S}} |D_k b(x)|, \quad b \in C_c^\infty(a_{G,S}),
\]

for invariant differential operators \( D_1, \ldots, D_r \) on \( a_{G,S} \) which depend only on \( \Delta \). (If \( S \) consists of one discrete valuation, we take \( \Delta \) and \( \{D_k\} \) to be constants.) The reader can check that this definition is equivalent to the one in §6. Similarly, the notion of a moderate function \( \phi \in \mathcal{F}_{ac}(G(F_s)) \) can be defined in terms of the behavior of the functions

\[
\phi^b(\pi, X) = \phi(\pi, X)b(X), \quad b \in C_c^\infty(a_{G,S}).
\]

More precisely, \( \phi \) is said to be moderate if there are positive constants \( c \) and \( d \) such that for any \( N > 0 \), and any \( b \in C_c^\infty(a_{G,S}) \),

(i) \( \phi^b \in \mathcal{F}_{c(N+1)}(G(F_s)) \), and

(ii) \( \|\phi^b\| \leq \delta(b)d^N \).

Recall [1(i)] that \( \mathcal{F}_{c(N+1)}(G(F_s)) \) is the set of \( \psi \in \mathcal{F}(G(F_s)) \) such that for every Levi subset \( \mathcal{M} = \prod_{v \in S} M_v \) of \( G \) over \( F_s \), and every representation

\[
\sigma = \bigotimes_v \sigma_v, \quad \sigma_v \in \Pi_{temp}(M_v(F_v)),
\]
the function
\[ \psi(\sigma, \mathcal{H}) = \int_{ia_{\mathcal{H}} G \setminus ia_{G} G} \psi(\sigma^G, h_G(\mathcal{H})) e^{-\Lambda(\mathcal{H})} d\Lambda, \quad \mathcal{H} \in a_{\mathcal{H}} G, \]
is supported on the ball of radius \( c(N + 1) \). In the second condition, it is understood that
\[ (A.3) \quad \|\psi\|^\prime = \sup_{\mathcal{H} \in a_{\mathcal{H}} G} |\Delta^\prime \psi(\sigma, \mathcal{H})|, \quad \psi \in \mathcal{I}(G(F_S)), \]
where \( \Delta^\prime \) is an arbitrary invariant differential operator on \( a_{\mathcal{H}} G \) for some fixed \( G \) and \( \sigma \), while \( \delta(b) \) is a seminorm on \( C^\infty_c(a_{G}, S) \) of the form (A.2) which depends only on \( \Delta^\prime \).

**Lemma A.1.** Suppose that \( \Gamma \) is a finite subset of \( \Pi(K) \). Then there is a continuous linear map
\[ h: \mathcal{I}(G(F_S))_\Gamma \to \mathcal{H}(G(F_S))_\Gamma \]
with the following four properties.

(a) \( h(\phi)_G = \phi, \quad \phi \in \mathcal{I}(G(F_S))_\Gamma \).

(b) \( h(\phi^{b}) = h(\phi)^b, \quad b \in C^\infty_c(a_{G,S}) \).

(c) There is a positive constant \( c \) such that for each \( N > 0 \), the image under \( h \) of \( \mathcal{I}_N(G(F_S))_\Gamma \) is contained in \( \mathcal{H}_{c(N+1)}(G(F_S))_\Gamma \).

(d) There is a positive constant \( d \) such that
\[ \|h(\phi)\| \leq \|\phi\|^\prime d^N, \quad \phi \in \mathcal{I}_N(G(F_S))_\Gamma, \quad N > 0, \]
where \( \| \cdot \| \) is an arbitrary seminorm of the form (A.1), while \( \| \cdot \|^\prime \) is a finite sum of seminorms (A.3) which depends only on \( \| \cdot \| \).

Lemma 6.1 follows easily from Lemma A.1. Take \( S \supset S_{\text{ram}} \) to be a large finite set of valuations of \( F \), and let \( \phi \) be a moderate function in \( \mathcal{I}_{ac}(G(F_S))_\Gamma \). Let \( \{ b_i \} \) be a smooth partition of unity for \( a_{G} \) and set
\[ f = \sum_i h(\phi^{b_i}). \]
Then \( f \) obviously belongs to \( \mathcal{H}_{ac}(G(F_S))_\Gamma \). We have
\[ f_G = \sum_i h(\phi^{b_i})_G = \sum_i \phi^{b_i} = \phi. \]

Suppose that \( N > 0 \) and that \( b \in C^\infty_c(a_{G}) \). Then
\[ f^b = \sum_i h(\phi^{b_i}) = h \left( \sum_i \phi^{b_i} \right) = h(\phi^b). \]
The required support and growth properties of \( f^b \) then follow from conditions (c) and (d) of Lemma A.1. □
The main point, then, is to establish Lemma A.1. It is evident that we can treat the valuations in $S$ separately. We shall therefore assume that $S$ consists of one valuation $v$. To simplify the notation, we shall also assume that $F$ itself is a local field (rather than a number field), so that $F = F_v = F_s$.

Suppose first that $F$ is non-Archimedean. In this case, the space $a_{G,v}$ is discrete, and the required condition (b) presents no problem. For if $h$ satisfies all the conditions but this one, and if

$$b_x(Z) = \begin{cases} 1, & Z = X, \\ 0, & Z \neq X, \end{cases}$$

for elements $X, Z \in a_{G,v}$, the map

$$\phi \mapsto \sum_{X \in a_{G,v}} h(\phi b_x)^{b_x}$$

will satisfy all the required conditions. It is therefore enough to construct a map $h$ for which the conditions (a), (c), and (d) hold.

The Bernstein center is a direct sum

$$\mathcal{Z}(G(F)) = \bigoplus_{x} \mathcal{Z}(G(F))_x$$

of components indexed by supercuspidal data $\chi$. Recall that a supercuspidal datum is a Weyl orbit

$$\{ s_0(L_0, r_0) : s_0 \in W_0 \} = \{ s(L_0, r_0) : s \in W_0^{G} \},$$

where $L_0$ is a Levi subgroup of $G^0$ and $r_0$ is an irreducible supercuspidal representation of $L_0(F)$ which is fixed by some element in $W_0^G$. The definition, in fact, is in precise analogy with that of a cuspidal automorphic datum, given in §4. We also recall that $\mathcal{Z}(G(F))_\chi = \mathcal{Z}(\overline{G^0}(F))_\chi$ is isomorphic to the algebra of finite Fourier series on the torus $\{ r_0, \Lambda : \Lambda \in \overline{a_{L_0}} \}$ which are invariant under the stabilizer of the torus in $W_0$. Let $\mathcal{Z}(F)_\Gamma$ denote the finite set of data $\chi$ such that $r_0$ contains a representation in the restriction of $\Gamma$ to $K \cap L_0(F)$. Then $\mathcal{Z}(F)_\Gamma$ is a finite set, and

$$\mathcal{Z}(G(F))_\Gamma = \bigoplus_{\chi \in \mathcal{Z}(F)_\Gamma} \mathcal{Z}(G(F))_\chi$$

is a finitely generated algebra over $C$. Let $z_1 = 1, z_2, \ldots, z_n$ be a fixed finite set of generators. There are actions $\phi \mapsto z\phi$ and $f \mapsto zf$ of $\mathcal{Z}(G(F))_\Gamma$ on $\mathcal{Z}(G(F))_\Gamma$ and $\mathcal{Z}(G(F))_\Gamma$, and the module $\mathcal{Z}(G(F))_\Gamma$ is finitely generated over $\mathcal{Z}(G(F))_\Gamma$. Let $\phi_1 = 1, \phi_2, \ldots, \phi_m$ be a generating set. Then any function $\phi \in \mathcal{Z}(G(F))_\Gamma$ can be written as a finite sum

$$\phi = \sum_{j=1}^{m} \sum_{\gamma} c_{\gamma}^j (z^\gamma \phi_j), \tag{A.4}$$
where \( \{c_j^\gamma\} \) are complex numbers, and where
\[ z^\gamma = z_1^{\gamma_1} \cdots z_n^{\gamma_n}, \]
for any \( n \)-tuple \( \gamma = (\gamma_1, \ldots, \gamma_n) \) of nonnegative integers. Assume that the functions
\[ X \to \phi_j(\pi, X), \quad \pi \in \Pi_{\text{temp}}(G(F)), \ X \in a_{\mathfrak{g}, v}, \]
are supported at \( X = 0 \). Then by the trace Paley-Wiener theorem, there are functions \( f_1, \ldots, f_m \) in \( \mathcal{H}(G(F)^1)_r \) such that \( (f_j)_G = \phi_j \). We are going to define
\[ (A.5) \quad h(\phi) = \sum_{j=1}^m \sum_{\gamma} c_j^\gamma (z^\gamma f_j). \]
However, the expansion \((A.4)\) for \( \phi \) is not unique. We must convince ourselves that it can be defined linearly in terms of \( \phi \) in a way which is sensitive to the growth and support properties of \( \phi \).

We can identify each \( \phi \in \mathcal{J}(G(F))_\Gamma \) with a collection of functions
\[ \phi_\sigma(\sigma_\Lambda) = \int_{a_{\mathfrak{g}, v}} \phi(\sigma_\Lambda^G, X) \, dX, \quad \Lambda \in ia_{\mathfrak{m}, \mathfrak{v}}^*, \]
in which
\[ \sigma = (M, \sigma), \quad M \in \mathcal{L}, \ \sigma \in \Pi_{\text{temp}}(M(F)^1), \]
ranges over a finite set of pairs which depends only on \( \Gamma \). Each \( \phi_\sigma \) is a finite Fourier series which is symmetric under the stabilizer \( W_\sigma \) of the orbit \( \{\sigma_\Lambda\} \) in \( W(a_M) \). The size of the support of \( \phi \) is determined by the largest degree of a nonvanishing Fourier coefficient. Let \( \|\phi\|' \) denote the largest absolute value of any of the Fourier coefficients. It is a continuous seminorm on \( \mathcal{J}(G(F))_\Gamma \) of the form \((A.3)\).

Let us embed \( \mathcal{J}(G(F))_\Gamma \) into the space \( \mathcal{J}(G(F))_\Gamma \) of collections
\[ \psi = \{\psi_\sigma(\sigma_\Lambda)\} \]
of finite Fourier series which have no symmetry condition. Then \( \mathcal{J}(G(F))_\Gamma \) is also a finite \( \mathcal{L}(G(F))_\Gamma \)-module. By averaging each function over \( W_\sigma \), we obtain a \( \mathcal{L}(G(F))_\Gamma \)-linear projection \( \psi \to \overline{\psi} \) from \( \mathcal{J}(G(F))_\Gamma \) onto \( \mathcal{J}(G(F))_\Gamma \). We can assume that our generating set for \( \mathcal{J}(G(F))_\Gamma \) is of the form
\[ (A.6) \quad \psi_\sigma = \sum_{\beta} b_{\beta, \sigma} y^\beta, \quad b_{\beta, \sigma} \in \mathbb{C}, \]
in which
\[ \beta = (\beta_1, \ldots, \beta_d), \quad d = d_\sigma = \dim a_M. \]
runs over $\mathbb{Z}^d$, and
\[ y^\beta = y_1^{\beta_1} \cdots y_d^{\beta_d} \]
denotes the function on $\{\sigma_n\}$ whose $\beta$th Fourier coefficient is 1 and whose other Fourier coefficients vanish. The functions $y_i$ and $y_i^{-1}$ of course belong to $\mathcal{S}(G(F))$, so we can define finite expansions
\[ y_i^{\pm 1} = \sum_j \sum_\gamma (\delta_{ij})^{\pm}(z^\gamma \psi_j), \quad (\delta_{ij})^{\pm} \in \mathbb{C}. \]
Substituting these expressions into the $\beta$th term of (A.6), and iterating $|\beta| = |\beta_1| + \cdots + |\beta_d|$ times, we obtain an expansion
\[ \psi_\sigma = \sum_j \sum_\gamma c_j^{\gamma}(z^\gamma \psi_j), \]
which is now well defined. If $\beta_{\max}$ and $\gamma_{\max}$ index the nonvanishing coefficients of greatest total degree in the expansions (A.6) and (A.7), one sees that
\[ |\gamma_{\max}| \leq c(|\beta_{\max}| + 1) \]
and
\[ \sup(|c_j^{\gamma}|) \leq \sup(|b_{\sigma, \gamma}|) \cdot d^{|\beta_{\max}|}, \]
for constants $c$ and $d$ which depend only on $\Gamma$. Finally, observe that if $\psi_\sigma$ equals an element $\phi_\sigma$ in $\mathcal{S}(G(F))$, we can project each side of (A.7) onto $\mathcal{S}(G(F))$. We obtain a canonical expansion
\[ \phi_\sigma = \sum_j \sum_\gamma c_j^{\gamma}(z^\gamma \phi_j). \]
We have shown how to define the expansion (A.4) in a way that depends linearly on $\phi$. Moreover, if $\phi$ belongs to $\mathcal{S}_N(G(F))$ and $\gamma_{\max}$ indexes the nonvanishing coefficient of highest degree in (A.4), we have
\[ |\gamma_{\max}| \leq c(N + 1) \]
and
\[ \sup(|c_j^{\gamma}|) \leq \|\phi\| \cdot d^N, \]
for fixed constants $c$ and $d$. We are thus free to define $h(\phi)$ by (A.5). It remains to check conditions (c) and (d) of Lemma A.1.

Let $K_0$ be an open compact subgroup of $G^0(F)$ which lies in the kernel of each of the representations in $\Gamma$. Set $g_0$ equal to the characteristic function of $K_0$ divided by the volume of $K_0$. Then $g_0$ acts by convolution on $\mathcal{S}(G(F))$ as the identity. The algebra $\mathcal{S}(G(F))$ acts on $\mathcal{S}(G^0(F))$, so we can set
\[ g_i = z_i g_0, \quad 1 \leq i \leq n. \]
These functions each belong to $\mathcal{S}(G^0(F))$, and they commute with each other under convolution. Consequently, for any $\gamma = (\gamma_1, \ldots, \gamma_n)$, the function
$g^\gamma = g^{\gamma_1} \cdots g^{\gamma_n}$ is well defined and belongs to $\mathcal{H}(G^0(F))$. Since $\mathcal{I}(G(F))_\Gamma$ acts as an algebra of multipliers on $\mathcal{H}(G(F))_\Gamma$, the function (A.5) can be written

$$h(\phi) = \sum_j \sum_\gamma c_\gamma^j (g^\gamma \ast f_j).$$

To estimate the support of $h(\phi)$, we use the inequalities

$$\text{supp}(g \ast h) \subset \text{supp}(g) \cdot \text{supp}(h), \quad g \in \mathcal{H}(G^0(F)), h \in \mathcal{H}(G(F)),$$

and

$$\|xy\| \leq \|x\| \cdot \|y\|,$$

both of which are easily established. It follows that $h(\phi)$ is supported on a set

$$\{x \in G(F) : \log \|x\| \leq c_1 (|\gamma_{\text{max}}| + 1)\},$$

where $c_1$ is a constant which is independent of $f$. The support condition (c) of the lemma then follows from (A.8). To establish the growth condition (d), we may assume that $\|\cdot\|$ is the supremum norm on $\mathcal{H}(G(F))$. Then

$$\|g \ast h\| \leq \|g\|_1 \|h\|, \quad g \in \mathcal{H}(G^0(F)), h \in \mathcal{H}(G(F)),$$

where $\|\cdot\|_1$ is the $L_1$-norm. Condition (d) then follows from (A.8) and (A.9). This completes the proof for non-Archimedean $F$.

Next, suppose that $F$ is Archimedean. If $G \neq G^0$, we must invoke our assumption that $G$ is an inner twist of $G^*= (GL(n) \times \cdots \times GL(n)) \rtimes \theta^*$, in order to have the trace Paley-Wiener theorem (see [2, Lemma I.7.1]). We shall say no more about this case. For one can obtain Lemma A.1 from the trace Paley-Wiener theorem by arguing as in the connected case below. We assume from now on that $G = G^0$. In this case the lemma is implicit in the work of Clozel-Delorme [6(a), 6(b)]. They construct a function $f = h(\phi)$ for every $\phi$, and they give an estimate for the support of $f$ which is stronger than our required condition (c). Our main tasks, then, are to convince ourselves that the map $\phi \to h(\phi)$ is well defined, and to check the growth conditions (d). We shall only sketch the argument.

The analogy between real and $p$-adic groups becomes clearer if we describe the steps of Clozel-Delorme in a slightly different order from that presented in [6(a)]. Let $\mathcal{D}_K(G(F)^1)_\Gamma$ be the space of distributions on $G(F)^1$ which are supported on $K$, and which transform under $K$ according to representations in $\Gamma$. For a typical example, take $\mu \in \Gamma$, and let $X$ be an element in $\mathcal{Z}(g(F)^1)^K$, the centralizer of $K$ in the universal enveloping algebra. Then the distribution

$$X_\mu : f \to \int_K (Xf)(k) \text{tr}(\mu(k)) \, dk, \quad f \in C_c^\infty(G(F)^1),$$

belongs to $\mathcal{D}_K(G(F)^1)_\Gamma$. Suppose that $D$ is any element in $\mathcal{D}_K(G(F)^1)_\Gamma$. Since it is a compactly supported distribution, it can be evaluated at a smooth
function from $G(F)$ to some vector space. In particular, one can evaluate $D$ on the function $\pi(x)$, for $\pi \in \Pi(G(F))$, to obtain an operator $\pi(D)$. Set

$$D_G(\pi) = \text{tr}(\pi(D)), \quad \pi \in \Pi_{\text{temp}}(G(F)).$$

Then $D_G$ is a scalar valued function on $\Pi_{\text{temp}}(G(F))$. Let us write $\Delta_G(G(F)^1)_\Gamma$ for the space of complex valued functions $\delta$ on $\Pi_{\text{temp}}(G(F))$ which satisfy the following two conditions.

(i) $\delta(\pi) = 0$, unless $\pi$ contains a representation in $\Gamma$.

(ii) For any Levi subgroup $M \in \mathcal{L}$, and any $\sigma \in \Pi_{\text{temp}}(M(F))$, the function

$$\Lambda \to \delta(\sigma^G_\Lambda), \quad \Lambda \in \mathfrak{a}^*_M, \mathbb{C},$$

is a polynomial which is invariant under $\mathfrak{a}^*_G, \mathbb{C}$.

It is easy to see that the map $D \to D_G$ sends $\mathcal{D}_K(G(F)^1)_\Gamma$ into $\Delta_K(G(F)^1)_\Gamma$. One of the main steps in the proof of Clozel-Delorme can be interpreted as an assertion that the map is surjective. In fact, any function $\delta \in \Delta_K(G(F)^1)_\Gamma$ is the image of a finite sum of distributions $X_\mu$. This is obtained by combining the characterization of the action of $\mathcal{U}(g(F)^1)_K$ on a minimal $K$-type ([6(a), Theorem 2] and [6(b), Theorem 2]) with the reduction argument based on Vogan’s theory of minimal $K$-types [6(a), p. 435].

Smooth multipliers on $G(F)$ map $\mathcal{D}_K(G(F)^1)_\Gamma$ to $\mathcal{H}(G(F))_\Gamma$. More precisely, if $D \in \mathcal{D}_K(G(F)^1)_\Gamma$ and $\alpha \in C_c^\infty(h)^W$, there is a unique function $D_\alpha$ in $\mathcal{H}(G(F))_\Gamma$ such that

$$\pi(D_\alpha) = \hat{\alpha}(\nu_\pi)\pi(D), \quad \pi \in \Pi(G(F))$$

(see [6(a), Lemma 6]). Observe also that if $\delta$ belongs to $\Delta_K(G(F)^1)_\Gamma$, the function

$$\delta_\alpha(\pi) = \delta(\pi)\hat{\alpha}(\nu_\pi), \quad \pi \in \Pi_{\text{temp}}(G(F)),$$

belongs to $\mathcal{F}(G(F))_\Gamma$. It is clear that $(D_\alpha)_G = D_{G,\alpha}$. The second main step of Clozel-Delorme can be interpreted as an assertion that over $C_c^\infty(h)^W$, the module $\mathcal{F}(G(F))_\Gamma$ has a finite set of generators in $\Delta_K(G(F)^1)_\Gamma$. In other words, there is a finite set $\delta_1, \ldots, \delta_m$ of elements in $\Delta_K(G(F)^1)_\Gamma$ with the property that any function $\phi \in \mathcal{F}(G(F))_\Gamma$ can be written

$$\phi = \delta_{1, \alpha_1} + \cdots + \delta_{m, \alpha_m},$$

for multipliers $\alpha_1, \ldots, \alpha_m$ in $C_c^\infty(h)^W$. Fix elements $D_1, \ldots, D_m$ in $\mathcal{D}_K(G(F)^1)_\Gamma$ such that

$$(D_j)_G = \delta_j, \quad 1 \leq j \leq m.$$

We are going to define

$$h(\phi) = D_{1, \alpha_1} + \cdots + D_{m, \alpha_m}.$$
However, we shall first indicate briefly how the expansion (A.11) can be defined in terms of $\phi$ so that it has the appropriate properties.

As in the $p$-adic case, we can identify each $\phi \in \mathcal{S}(G(F))_\Gamma$ with a collection of functions

$$\phi_\sigma(\Lambda) = \int_{a_G} \phi(\sigma^G, X) \, dX, \quad \Lambda \in ia_M^*,$$

in which

$$\sigma = (M, \sigma), \quad M \in \mathcal{S}, \quad \sigma \in \Pi_{\text{temp}}(M(F)^1),$$

ranges over a finite set of pairs. For each $\phi_\sigma$, one constructs a Paley-Wiener function $\Phi_\sigma$ on $\mathfrak{h}_C^*$ by following the procedure on p. 439 of [6(a)]. Clozel and Delorme then appeal to a result in [13], which asserts that

$$PW(\mathfrak{h}_C^*) = \hat{u}_1 PW(\mathfrak{h}_C^*)^W + \cdots + \hat{u}_d PW(\mathfrak{h}_C^*)^W,$$

for elements $u_1 = 1, u_2, \ldots, u_d$ in $S(\mathfrak{h})$, the symmetric algebra on $\mathfrak{h}^1$. Indeed, one need only take $\{u_i\}$ to be homogeneous elements which form a basis of the quotient field of $S(\mathfrak{h})$ over that of $S(\mathfrak{h})^W$. From the corollary of Lemma 11 of [9(a)], one can then construct continuous projections

$$PW(\mathfrak{h}_C^*) \to \hat{u}_i PW(\mathfrak{h}_C^*)^W, \quad 1 \leq i \leq d,$$

whose sum is the identity. Apply the decomposition to $\phi_\sigma$ and then restrict the functions obtained to the affine subspaces $\nu_\sigma + a_{M,C}^*$ of $\mathfrak{h}_C^*$. This provides a well-defined expansion (A.11) for $\phi_\sigma$. The expansion for $\phi$ is then the corresponding sum over $\sigma$. In particular, we take $\{\delta_1, \ldots, \delta_m\}$ to be the union over $\sigma$ of the sets of $d$ functions

$$\Lambda \to \hat{u}_i (\nu_\sigma + \Lambda), \quad \Lambda \in ia_M^*.$$

It follows that the expansion (A.11) is given by a well-defined linear map

(A.13) $\phi \to (\alpha_1, \ldots, \alpha_m), \quad \alpha_i \in C^\infty_c(\mathfrak{h})^W.$

The map $h$ is then determined by (A.12).

It is clear from the definitions that $h(\phi)_G$ equals $\phi$. The other conditions of the lemma come from properties of the map (A.13). For one can check that the map commutes with the natural action of $ia_G^*$ on $\mathcal{S}(G(F))_\Gamma$ and $C^\infty_c(\mathfrak{h})^W$. This gives the required condition (b). If $\phi \in \mathcal{S}_N(G(F))_\Gamma$, $N > 0$, it can be shown that each $\alpha_i$ belongs to $C^\infty_N(\mathfrak{h})^W$. Since the support of a function (or distribution) behaves well under the action of a multiplier, condition (c) follows. To prove (d), first note that a seminorm (A.1) is continuous on the Schwartz space of $G(F)$. It follows from the corollary of Theorem 13.1 of [9(b)] that the value of any such seminorm on $h(\phi)$ is bounded by a finite sum of continuous seminorms, evaluated at classical Schwartz functions

$$\lambda \to \mathcal{S}_p(\sigma_\lambda, h(\phi)), \quad \lambda \in ia_p^*.$$
Here $P \in \mathcal{F}(M_0)$, $\sigma \in \Pi_{\text{temp}}(M_p(F))$ and $\mathcal{F}_p(\sigma)$ is the induced representation of $G(F)$. We are assuming that $h(\phi)$ is given by (A.12), so that

$$\mathcal{F}_p(\sigma, h(\phi)) = \sum_{i=1}^{m} \hat{\alpha}_i(\nu_\sigma + \lambda) \mathcal{F}_p(\sigma, D_i).$$

But for any $k$ there is a seminorm $\| \cdot \|_k'$ on $\mathcal{F}(G(F))$ of the form (A.3) such that

$$\sup_i |\hat{\alpha}_i(\nu_\sigma + \lambda)| \leq \|\phi\|_k' e^{\|\nu_\sigma\| N (1 + \|\lambda\|)^{-k}},$$

for any $\lambda \in i\mathfrak{a}_p^*$ and any $\phi \in \mathcal{F}_N(G(F)), N > 0$. This is a consequence of the continuity properties of the map (A.13). The final condition (d) of the lemma follows. □

References

   (b) A trace formula for reductive groups. II: Applications of a truncation operator, Comp. Math. 40 (1980), 81–121.


(b) Harmonic analysis on real reductive groups. II. Wave packets in the Schwartz space, Invent. Math. 36 (1976), 1–55.

(c) Admissible invariant distributions on reductive \( p \)-adic groups, Queen’s Papers in Pure and Appl. Math. 48 (1978), 281–341.


(b) Tamagawa numbers, preprint.


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