JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 2, Number 1, January 1989

A CRITERION FOR THE ABSOLUTE CONTINUITY OF THE HARMONIC MEASURE ASSOCIATED WITH AN ELLIPTIC OPERATOR

R. FEFFERMAN

INTRODUCTION

In this note, we consider elliptic operators L of the form

$$L = \sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j) \tag{1}$$

where the coefficients a_{ij} are defined in some bounded open set $\Omega \subseteq \mathbb{R}^n$, are measurable, and satisfy the ellipticity and boundedness condition

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \lambda^{-1}|\xi|^2 \tag{2}$$

for some $\lambda > 0$ and for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$. We also assume $a_{ij} = a_{ji}$.

For such operators, the Dirichlet problem is solvable in Ω if and only if it is solvable for the Laplace operator, according to a theorem of Littman, Stampacchia and Weinberger [1]. This means that if $\Omega \subseteq \mathbb{R}^n$ is a sufficiently nice bounded region (the unit ball, B, is an example) and f is a given continuous function on the boundary of Ω, then there exists a unique function u, continuous on $\overline{\Omega}$, so that $L(u) = 0$ in Ω and $u = f$ on $\partial\Omega$. Let us assume, for convenience, that the origin belongs to Ω. Then the mapping $f \in \mathscr{C}(\partial\Omega) \to u(0)$ is a positive linear functional so there exists a unique nonnegative measure ω on $\partial\Omega$ such that for every $f \in \mathscr{C}(\partial\Omega)$,

$$\int_{\partial\Omega} f\, d\omega = u(0).$$

This measure ω is called the harmonic measure associated to L. It is often important for applications to know whether or not ω is absolutely continuous with respect to the surface measure $d\sigma$ on the boundary of Ω. If this is the case, it is also of interest to know how nice the Radon-Nikodym derivative $d\omega/d\sigma$ (the Poisson kernel) is.

In recent years, several results have been found to answer these questions. First, according to a result of Caffarelli, Fabes, and Kenig [2] there exist elliptic

Received by the editors May 22, 1988.
1980 *Mathematics Subject Classification*. Primary 35J25.

operators L of the form (1) such that the measure ω associated to L is not absolutely continuous with respect to σ. Later, Fabes, Jerison, and Kenig [3] proved the following theorem:

Suppose $L = \sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j)$, *where the* a_{ij} *satisfy* (2). *Suppose further that the coefficient matrices* $A(x) = (a_{ij}(x))$ *are continuous on* $\overline{B}$. *Let the modulus of continuity* η *be defined for* $0 < t < 1$ *by*

$$\eta(t) = \sup_{\substack{x \in \partial B \\ 0 < s \leq t}} \|A(x - sx) - A(x)\|.$$

Assume that $\int_0^1 \eta^2(t)dt/t < \infty$. *Then* ω *is absolutely continuous with respect to* σ. *In addition, if* $d\omega = k\,d\sigma$ *then* k *satisfies the reverse Hölder inequality*

$$\left(\frac{1}{\sigma(\Delta)} \int_\Delta k^2\,d\sigma\right)^{1/2} \leq C \frac{1}{\sigma(\Delta)} \int_\Delta k\,d\sigma \tag{3}$$

for all surface balls $\Delta \subseteq \partial B$.

Then in [4], Dahlberg extended this result to the case where the coefficients of L are discontinuous, and to a more general setting. To describe his result, let us recall several definitions. If $\Delta \subseteq \partial B$ is a surface ball centered at $x \in \partial B$, of radius r, then we set $S(\Delta) = B(x; r) \cap B$. A nonnegative measure μ in B is called a Carleson measure if and only if $\mu(S(\Delta)) \leq C\sigma(\Delta)$ for all surface balls $\Delta \subseteq \partial B$. We say that μ is a Carleson measure with vanishing trace provided there exists a function $h(r)$ for $0 < r < 1$, such that $\lim_{r \to 0} h(r) = 0$ and $\mu(S(\Delta)) \leq h(r)\sigma(\Delta)$ for all surface balls Δ of radius r, and all $0 < r < 1$. If $f(x)$ is a nonnegative function on ∂B, then we say that $f \in B^p$ for some $1 < p < \infty$ if and only if for all surface balls Δ,

$$\left(\frac{1}{\sigma(\Delta)} \int_\Delta f^p\,d\sigma\right)^{1/p} \leq C \left(\frac{1}{\sigma(\Delta)} \int_\Delta f\,d\sigma\right). \tag{4}$$

The smallest C for which (4) is valid is called the "B^p norm" of f.

In [4], Dahlberg proves the following theorem:

Suppose L_0 *and* L_1 *are two operators of the form* (1), *with coefficient matrices* $A_0(x)$ *and* $A_1(x)$ *respectively. For* $z \in B$ *set*

$$a(z) = \sup_{x \in B(z; \delta(z)/2)} \|A_0(x) - A_1(x)\|,$$

where $\delta(z)$ *denotes the distance from* z *to* ∂B. *Assume that* $a^2 dz/\delta$ *is a Carleson measure with vanishing trace. Assume that the harmonic measure* ω_0 *associated with* L_0 *is absolutely continuous with respect to* σ *and that* $d\omega_0 = k_0 d\sigma$ *where* $k_0 \in B^p$, $1 < p < \infty$. *Then the same must be true of the harmonic measure* ω_1 *associated to* L_1, *i.e.,* $\omega_1 = k_1 d\sigma$ *for some* $k_1 \in B^p$.

To state our result, let us recall that a function $f(x) \geq 0$ on ∂B belongs to A^∞ if and only if for $d\nu = f\,d\sigma$, and any subset $E \subseteq \Delta$, Δ a surface ball, we have

$$\frac{\nu(E)}{\nu(\Delta)} \leq C\left[\frac{\sigma(E)}{\sigma(\Delta)}\right]^{\theta}, \qquad \text{for some } \theta > 0.$$

Then it is well known, [5], that $f \in A^{\infty}$ if and only if $f \in B^{p}$ for some $p > 1$.

The result of this article will give a criterion which, on the formal level, looks very much like the Dini condition of Fabes, Jerison, and Kenig which guarantees that if the difference of the coefficient matrices A_0 and A_1 of two elliptic operators L_0 and L_1 meets the criterion, then $\omega_0 = k_0 d\sigma$, $k_0 \in A^{\infty}$ implies that $\omega_1 = k_1 d\sigma$, with $k_1 \in A^{\infty}$. We are not able to prove that the B^p condition is preserved for a given p. Thus the conclusion of the theorem of Dahlberg is stronger than ours here. The significance of our result comes from the fact that, unlike the results in [3] and [4] our hypothesis does *not* require that the coefficients of L_0 be uniformly close to those of L_1 as we approach the boundary of B, in order to guarantee that the good properties of the harmonic measure associated with L_0 are inherited by that of L_1.

Statement of Theorem

Suppose that L_0 and L_1 are elliptic operators in divergence form (1) with bounded measureable coefficients defined in the unit ball, B. Suppose, as above, $A_0(x)$ and $A_1(x)$ denote their coefficient matrices, and ω_0 and ω_1 the associated harmonic measures.

As above, for $z \in B$ set

$$a(z) = \sup_{x \in B\left(z;\delta(z)/2\right)} \|A_0(x) - A_1(x)\|$$

where $\delta(z)$ is the distance of z to the boundary of B. Then we have the following:

Theorem. *Suppose, for each* $x \in \partial B$,

$$\int_0^1 a^2((1-t)x)\frac{dt}{t} \leq C,$$

for some constant C *independent of* x. *Assume that* $\omega_0 = k_0 d\sigma$ *where* $k_0 \in A^{\infty}$. *Then* ω_1 *is absolutely continuous with respect to* σ *and* $\omega_1 = k_1 d\sigma$ *where* $k_1 \in A^{\infty}$.

The proof of our theorem follows the method of Dahlberg in [4]. There, Dahlberg considers the family of operators L_t, $0 \leq t \leq 1$, given by $L_t = (1-t)L_0 + tL_1$. Let ω_t denote the harmonic measure associated to L_t and $Q(t)$ the B^p norm of ω_t. He proves the differential inequality $|\dot{Q}(t)| \leq CQ(t)^N$ where $\dot{Q}(t)$ is the t derivative of $Q(t)$, C depends on the Carleson measure constant of $a^2 dz/\delta$, and λ, the ellipticity constant. N is some large positive integer. This differential inequality shows that if $a^2 dz/\delta$ is a Carleson measure of vanishing trace (so that essentially C can be taken as small as desired) and

$Q(0) < \infty$, then $Q(1) < \infty$. The smallness of C is obviously crucial, since the solution of the equation $|\dot{Q}(t)| = CQ(t)^N$ has a singularity when $N > 1$. Our aim here is to replace the differential inequality above with $|\dot{Q}(t)| \leq CQ(t)$, i.e., $N = 1$. C can then be as large as we like.

As one application of our theorem, we introduce the following notion, which we call "region of arbitrary perturbation." Let $\Omega \subseteq \mathbb{R}^n$ be a bounded region in which the classical Dirichlet problem is solvable, and let $\Omega_0 \subseteq \Omega$ be a subregion. We call Ω_0 a region of arbitrary perturbation provided whenever L_0 and L_1 are two elliptic operators of the form (1) with coefficients $A_0(x)$ and $A_1(x)$, then if $A_0(x) - A_1(x)$ is supported in Ω_0, and if the harmonic measure associated with L_0 is A^∞ on $\partial\Omega$, then this implies that the same is true for the harmonic measure associated with L_1. Now set $B_+ \subseteq \mathbb{R}^{n+1}$ to be

$$B_+ = \{(x, y) \mid x \in \mathbb{R}^n, y > 0, \text{ and } |x|^2 + y^2 < 1\}$$

and

$$B_+(\tfrac{1}{2}) = \{(x, y) \mid x \in \mathbb{R}^n, y > 0 \text{ and } |x|^2 + y^2 < \tfrac{1}{2}\}.$$

Suppose $\varphi(x)$ is a function defined on the ball centered at 0 of radius $\frac{1}{4}$ in $\mathbb{R}^n$ whose graph is contained in $B_+(\frac{1}{2})$ and which is slowly oscillating in the sense that there exists a constant C so that for each $x_0 \in B_{1/4}(0)$, we have $\frac{1}{2}\varphi(x_0) \leq \varphi(x) \leq 2\varphi(x_0)$ whenever $x \in B_{|x_0|/C}(x_0) \cap B_{1/4}(0)$. Let

$$\Omega_0 = \{(x, y) \mid |x|^2 + y^2 < \tfrac{1}{4} \text{ and } \varphi(x) < y < 2\varphi(x)\}.$$

Then Ω_0 is a region of arbitrary perturbation, as can easily be seen from our theorem.

The notion of regions of arbitrary perturbation can be used to yield information on the harmonic measure associated with some basic examples of elliptic operators. A discussion of these will appear elsewhere.

Proof of the Theorem. Consider L_0 and L_1 as in the statement of our theorem, and let $L_t = (1-t)L_0 + tL_1$ with associated harmonic measure ω_t. In [4] it is shown that we may assume that the coefficients of the L_t are C^∞ in B. Let $\omega_t = k_t d\sigma$, and Δ be a surface ball on ∂B, Δ centered at x_0 of radius r. Let $A = B(x_0; 2r) \cap B$. We are trying to show that if $k_0 \in A^\infty$ then the same is true of k_1. To this end, let $\tilde{\sigma}$ be the normalized surface measure on Δ, i.e., $\tilde{\sigma} = \sigma/\sigma(\Delta)$. We shall show that

$$(5) \qquad \|k_1\|_{L(\log L)(d\tilde{\sigma})} \leq C\|k_1\|_{L^1(d\tilde{\sigma})}$$

and this shows that $k_1 \in A^\infty$. To do this, we require a trivial lemma, which the reader will notice is essentially just exploiting the duality of H^1 with BMO.

Lemma. *Fix $t_0 \in [0, 1]$, and $\Delta \subseteq \partial B$ a surface ball centered at x_0 of radius r. There exists a function $f(x)$ defined on the boundary of B, which is continuous,* *nonnegative, supported in the surface ball centered at* x_0 *of radius* $\frac{3}{2}r$, $\widetilde{\Delta}$, *such that*

$$\|f\|_{\mathrm{BMO}(d\sigma)} + \|f\|_{L^1\left(d\sigma/\sigma(\Delta)\right)} \leq C_n$$

and

$$\int_{\widetilde{\Delta}} f k_{t_0}\, d\tilde{\sigma} \geq c_n \|k_{t_0}\|_{L(\log L)(d\tilde{\sigma})}.$$

(*Here* C_n, c_n *depend only on* n.)

Proof of Lemma. We clearly may assume, by homogeneity, that $\|k_{t_0}\|_{L^1(d\tilde{\sigma})} = 1$. By a well-known theorem of E. M. Stein [6],

$$\|k_{t_0}^*\|_{L^1(d\tilde{\sigma})} \geq A_n \|k_{t_0}\|_{L(\log L)(d\tilde{\sigma})}$$

where $k_{t_0}^*$ denotes the Hardy-Littlewood (dyadic) maximal function of k_{t_0} taken over cubes contained in Δ. Perform the Calderòn-Zygmund decomposition of the k_{t_0} at heights B^j, $j = 1, 2, \ldots$, where B is as follows:

Let the Calderòn-Zygmund cubes at height B^j be called Q_ℓ^j. Then choose B so large that

$$\sigma\left(Q_\ell^j \cap \left[\bigcup_m Q_m^{j+1}\right]\right) < \frac{1}{2}\sigma(Q_\ell^j).$$

Then let φ_ℓ^j be a nonnegative smooth bump function which is 1 on Q_ℓ^j and 0 off of the dilate of Q_ℓ^j by $3/2$. It is trivial to show that the function $f = \sum_{j,\ell} \varphi_\ell^j + \varphi$ belongs to $\mathrm{BMO}(d\sigma)$ ($\varphi \in C^\infty(\partial B)$), $\varphi \geq 0$, $\varphi \equiv 1$ on Δ, and is supported in the concentric dilate of Δ by $3/2$) and

$$\|f\|_{\mathrm{BMO}(d\sigma)} + \|f\|_{L^1(d\tilde{\sigma})} \leq C_n,$$

while

$$\begin{aligned}
\int f k_{t_0} d\sigma &\geq \int_\Delta k_{t_0} d\tilde{\sigma} + \sum_{j,\ell} \int_{Q_\ell^j} k_{t_0} d\tilde{\sigma} \\
&\geq \frac{1}{\sigma(\Delta)}\left[\sigma(\Delta)\left(\frac{1}{\sigma(\Delta)}\int_\Delta k_{t_0} d\sigma\right) + \sum_{j,\ell} \sigma(Q_\ell^j)\left(\frac{1}{\sigma(Q_\ell^j)}\int_{Q_\ell^j} k_{t_0} d\sigma\right)\right] \\
&\geq \frac{1}{\sigma(\Delta)} c_n' \left[\sigma(\Delta)\left(\frac{1}{\sigma(\Delta)}\int_\Delta k_{t_0} d\sigma\right) + \sum_{j,\ell} B^j \sigma(Q_\ell^j)\right] \\
&\geq c_n'' \int_\Delta k_{t_0}^* d\tilde{\sigma} \geq c_n'' A_n \|k_{t_0}\|_{L \log L(d\tilde{\sigma})}.
\end{aligned}$$

This proves the lemma.

Now, to show (5), we fix $t_0 \in [0, 1]$ and Δ, and we select f as in the lemma, and estimate

$$\int_{\widetilde{\Delta}} f k_{t_0}\, d\sigma \Big/ \int_\Delta k_{t_0}\, d\sigma, \tag{6}$$

and this ratio is shown in Dahlberg [4] to be equivalent to a quantity V as follows: Let u_t be the solution of $L_t u_t = 0$ in A and $u_t = f$ on ∂A. Let $\psi(z) \in C^\infty(\mathbb{R}^n)$, ψ supported inside the ball $B(0; \frac{1}{2}r)$, $\psi \equiv 2$ inside $B(0; \frac{1}{4}r)$. Let $\bar{z} = (1 - \frac{3}{2}r)x_0$ and $\psi(z) = \psi((s - \bar{z})/r)r^{-n}$. Put $V = \int_A u_{t_0} \psi\, dz$. We have $c_1 V \leq (6) \leq c_2 V$, so we estimate (6) by estimating V. Following [4] we control $\int_A u_t \psi\, dz$, via a differential inequality. We consider the modified Green's function h_t defined by $L_t h_t = \psi$ in A and $h_t \equiv 0$ on ∂A. Then as in [4],

$$\left| \int_A u_t \psi\, dz \right| \leq C \int_A \varepsilon |\nabla u_t| |\nabla h_t|\, dz$$

where $\varepsilon(z) = \|A_0(z) - A_1(z)\|$.

Now, our assumption on a implies that $\|\varepsilon\|_{L^\infty} \leq C$. Let $A_1 = B(x_0; \frac{3}{2}r) \cap B$ and $A_2 = A - A_1$. Then

$$\int_{A_2} \varepsilon |\nabla u_t| |\nabla h_t|\, dz \leq C \left(\int_{A_2} |\nabla u_t|^2 dz \right)^{1/2} \left(\int_{A_2} |\nabla h_t|^2 dz \right)^{1/2}$$

and this is easily seen to be

$$\leq C \int_{\widetilde{\Delta}} f k_t\, d\sigma ,$$

where in this case and from here on ω_t will stand for the harmonic measure associated with L_t *in* A taken at the point $\bar{z}$, and $\omega_t = k_t d\sigma$. Then

$$\begin{aligned} \int_{\widetilde{\Delta}} f k_t d\sigma &\leq C\sigma(\Delta) \int_{\widetilde{\Delta}} f k_t \frac{d\sigma}{\sigma(\widetilde{\Delta})} \\ &\leq C\sigma(\Delta) \int_{\widetilde{\Delta}} |f - f_{\widetilde{\Delta}}| k_t \frac{d\sigma}{\sigma(\widetilde{\Delta})} + C\sigma(\Delta) f_{\widetilde{\Delta}} \int_{\widetilde{\Delta}} k_t \frac{d\sigma}{\sigma(\widetilde{\Delta})} \end{aligned}$$

(where $f_{\widetilde{\Delta}} = \int_{\widetilde{\Delta}} f\, d\sigma / \sigma(\widetilde{\Delta})$)

$$\begin{aligned} &\leq C\sigma(\Delta) \|f - f_{\widetilde{\Delta}}\|_{\exp(d\tilde{\sigma})} \|k_t\|_{L \log L(d\tilde{\sigma})} + C \\ &\leq C \|f\|_{\mathrm{BMO}(d\sigma)} Q(t) \|k_t\|_{L^1(d\sigma/\sigma(\Delta))} \sigma(\Delta) + C \leq C' Q(t) \end{aligned}$$

where

$$Q(t) = \sup_{\Delta \subseteq \partial B} \frac{\|k_t\|_{L \log L \left(d\sigma/\sigma(\Delta) ; \Delta \right)}}{\|k_t\|_{L^1 (d\sigma/\sigma(\Delta) ; \Delta)}}$$

the sup being taken over all surface balls $\Delta \subseteq \partial B$.

Now we must estimate $\int_{A_1} \varepsilon |\nabla u_t| |\nabla h_t|\, dz$. For each dyadic surface cube $Q \subseteq \widetilde{\Delta}$, we let

$$\widehat{Q} = \{ z \in B \mid z/\|z\| \in Q ,\ c_n \ell \leq \delta(z) < 2c_n \ell ,\ \ell = \text{ side length of } Q \} , \quad c_n = 10\sqrt{n} .$$

Then if $a_Q = \sup\{\varepsilon(z) | z \in \widehat{Q}\}$,

$$\begin{aligned}(7)\qquad \int_{A_1} \varepsilon|\nabla u_t||\nabla h_t|\,dz &\le \sum_{\substack{Q\subseteq\widetilde{\Delta}\\ Q \text{ dyadic}}} \int_{\widehat{Q}} \varepsilon|\nabla u_t||\nabla h_t|\,dz \\ &\le \sum_{\substack{Q\subseteq\widetilde{\Delta}\\ Q \text{ dyadic}}} a_Q \left(\int_{\widehat{Q}} |\nabla u_t|^2 dz\right)^{1/2} \left(\int_{\widehat{Q}} |\nabla h_t|^2 dz\right)^{1/2}.\end{aligned}$$

By Caccioppoli's inequality,

$$\left(\int_{\widehat{Q}} |\nabla h_t|^2 dz\right)^{1/2} \le C \left(\int_{\widehat{Q}^{\sim}} h_t^2 dz\right)^{1/2} \delta(Q)^{-1}$$

where $\widehat{Q}^{\sim}$ denotes the concentric dilate of $\widehat{Q}$ by $(1+\frac{1}{100})$, and $\delta(Q)$ denotes the side length of Q.

By standard estimates on Green's function [7], if $z \in \widehat{Q}^{\sim}$, $h_t(z) \le C\delta(Q)^{2-n}\omega_t(Q)$, so that

$$\left(\int_{\widehat{Q}^{\sim}} h_t^2 dz\right)^{1/2} \le C\delta(Q)^{2-n/2}\omega_t(Q)$$

so that

$$\begin{aligned}(8)\qquad \int_{A_1} \varepsilon|\nabla u_t||\nabla h_t|\,dz &\le \sum_{\substack{Q\subseteq\widetilde{\Delta}\\ Q \text{ dyadic}}} a_Q \left(\int_{\widehat{Q}} |\nabla u_t|^2 dz\right)^{1/2} \delta^{1-n/2}(Q)\omega_t(Q) \\ &\le \sum_{\substack{Q\subseteq\widetilde{\Delta}\\ Q \text{ dyadic}}} a_Q \left(\int_{\widehat{Q}} |\nabla u_t|^2 \delta^{2-n}(z)\,dz\right)^{1/2} \omega_t(Q).\end{aligned}$$

Define $F(x) \in \ell^2$ for $x \in \partial B$, by letting $F(x) = \{a_Q\}_{\substack{x\in Q\\ Q \text{ dyadic}}}$. Define $G(x) \in \ell^2$ for $x \in \partial B$ by

$$G(x) = \left\{\left(\int_{\widehat{Q}} |\nabla u_t|^2 \delta^{2-n}(z)\,dz\right)^{1/2}\right\}_{\substack{x\in Q\\ Q \text{ dyadic}}}.$$

Then $(8) \le \int_{\partial B} F\cdot G(x)\,d\omega_t(x)$. Also, our assumption on a implies that $F \in L^\infty(\ell^2)$ and $|G(x)|_{\ell^2} = S(f)(x)$.

We therefore see that

$$(8) \le C\int_{\partial B} S(f)(x)\,d\omega_t(x) \le C\left[\int_{\partial B} S^2(f)(x)\,d\omega_t(x)\right]^{1/2}.$$

By a result of Dahlberg, Jerison, and Kenig [8], S is bounded on $L^2(d\omega_t)$ with a bound depending only on n and the ellipticity constant of L_t. Thus this last quantity is bounded by

(9)

$$C'\left(\int_{\partial B} f^2 d\omega_t\right)^{1/2} = C'\left(\int_{\widetilde{\Delta}} f^2 d\omega_t\right)^{1/2} \le C'\left(\int_{\widetilde{\Delta}} |f - \overline{f}_{\widetilde{\Delta}}|^2 \, d\omega_t\right)^{1/2} + C'|\overline{f}_{\widetilde{\Delta}}|$$

where

$$\overline{f}_{\widetilde{\Delta}} = \frac{1}{\omega_t(\widetilde{\Delta})} \int_{\widetilde{\Delta}} f \, d\omega_t .$$

At this point, we observe that ω_t satisfies a doubling condition on surface balls [7], with doubling constants depending only on the ellipticity constants of L_t, and so

$$\left(\frac{1}{\omega_t(\widetilde{\Delta})} \int_{\widetilde{\Delta}} |f - \overline{f}_{\widetilde{\Delta}}|^2 \, d\omega_t\right)^{1/2} \le C'' \sup_{\Delta \subseteq \partial B} \left(\frac{1}{\omega_t(\Delta)} \int_{\Delta} |f - \overline{f}_{\Delta}| \, d\omega_t\right)$$
$$\equiv C'' \|f\|_{\mathrm{BMO}(d\omega_t)}$$

where C'' depends only on n, C, and the ellipticity constant of L_0 and L_1, and where $\overline{f}_{\Delta} = (1/\omega_t(\Delta)) \int_{\Delta} f \, d\omega_t$.

Next, we claim that $\|f\|_{\mathrm{BMO}(d\omega_t)} \le C'''Q(t)$. In fact, if Q is a surface cube on ∂B, and if $f_Q = (1/\sigma(Q)) \int_Q f \, d\sigma$, then

$$\begin{aligned}
\frac{1}{\omega_t(Q)} \int_Q |f - f_Q| \, d\omega_t &= \frac{\sigma(Q)}{\omega_t(Q)} \int_Q |f - f_Q| k_t \frac{d\sigma}{\sigma(Q)} \\
&\le \frac{\sigma(Q)}{\omega_t(Q)} \|f - f_Q\|_{\exp(d\sigma/\sigma(Q);Q)} \|k_t\|_{L \log L(d\sigma/\sigma(Q);Q)} \\
&\le C \frac{\sigma(Q)}{\omega_t(Q)} \|f\|_{\mathrm{BMO}(d\sigma)} Q(t) \|k_t\|_{L^1(d\sigma/\sigma(Q);Q)} \\
&\le C'''Q(t),
\end{aligned}$$

proving the claim.

Finally

$$(10) \qquad (9) \le \overline{C}Q(t) + C' \frac{1}{\omega_t(\widetilde{\Delta})} \int_{\widetilde{\Delta}} f \, k_t \, d\sigma \le \overline{C}Q(t) + C' \frac{\sigma(\widetilde{\Delta})}{\omega_t(\widetilde{\Delta})} \int_{\widetilde{\Delta}} f \, k_t \frac{d\sigma}{\sigma(\widetilde{\Delta})} .$$

But

$$\int_{\widetilde{\Delta}} f \, k_t \frac{d\sigma}{\sigma(\widetilde{\Delta})} \le \int_{\widetilde{\Delta}} |f - f_{\widetilde{\Delta}}| k_t \frac{d\sigma}{\sigma(\widetilde{\Delta})} + f_{\widetilde{\Delta}} \int_{\widetilde{\Delta}} k_t d\sigma$$

(where $f_{\widetilde{\Delta}} = (1/\sigma(\widetilde{\Delta})) \int_{\widetilde{\Delta}} f \, d\sigma \le C$)

$$\begin{aligned}
&\le \|f\|_{\mathrm{BMO}(d\sigma)} \|k_t\|_{L \log L(d\sigma/\sigma(\widetilde{\Delta});\widetilde{\Delta})} + C \\
&\le CQ(t) \|k_t\|_{L^1(d\sigma/\sigma(\widetilde{\Delta});\widetilde{\Delta})} + C .
\end{aligned}$$

From this it follows that $(10) \le CQ(t)$. This proves that $|\int_A u_t \psi \, dz| \le CQ(t)$.

Now we have shown that

$$\int_A u_{t_0}\psi\,dz \le \int_A u_0\psi\,dz + \int_0^{t_0}\left(\int_A u_t\psi\,dz\right)dt \le C\left[Q(0) + \int_0^{t_0} Q(t)\,dt\right],$$

while $Q(t_0) \le C\int_A u_{t_0}\psi\,dz$, if we choose Δ correctly, so that

$$Q(t_0) \le C\left[Q(0) + \int_0^{t_0} Q(t)\,dt\right]$$

and this implies the bound on $Q(1)$ we require to finish the proof of the theorem.

References

1. W. Littman, G. Stampacchia, and H. Weinberger, *Regular points for elliptic equations with discontinuous coefficients*, Ann. Scuola Norm. Sup. Pisa **17** (1963), 45–79.
2. L. Caffarelli, E. Fabes, and C. Kenig, *Completely singular elliptic harmonic measures*, Indiana Univ. Math. J. **30** (1981), 917–924.
3. E. Fabes, D. Jerison, and C. Kenig, *Necessary and sufficient conditions for absolute continuity of elliptic-harmonic measure*, Ann. of Math. **119** (1984), 121–141.
4. B. Dahlberg, *On the absolute continuity of elliptic measures*, Amer. J. Math. **108** (5) (1986), 1119–1138.
5. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. **51** (1974), 241–250.
6. E. M. Stein, *Note on the class* $L\log L$, Studia Math. **31** (1969), 305–310.
7. L. Caffarelli, E. Fabes, S. Mortella, and S. Salsa, *Boundary behavior of non-negative solutions of elliptic operators in divergence form*, Indiana Univ. Math. J. **30** (1981), 621–640.
8. B. Dahlberg, D. Jerison, and C. Kenig, *Area integral estimates for elliptic differential operators with non-smooth coefficients*, Arkiv fur Matematik **22** (1984), 97–108.
9. F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426.

Department of Mathematics, University of Chicago, 5734 S. University Avenue, Chicago, Illinois 60637