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AN ESTIMATE FOR CHARACTER SUMS

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In this note, we give estimates for a class of character sums that occur as eigenvalues of adjacency matrices of certain graphs constructed by F. R. K. Chung. Her situation is as follows. We are given a finite field F, an integer $n \geq 1$, an extension field E of F of degree n, and an element x in E that generates E over F, i.e., an element x such that E is $F(x)$.

Theorem 1. *Let χ be any nontrivial complex-valued multiplicative character of $E^{\times}$ (extended by zero to all of E), and x in E any element that generates E over F. Then*

$$\left\| \sum_{t \in F} \chi(t-x) \right\| \leq (n-1)\sqrt{\#(F)}.$$

It turns out to be easier to consider the following more general situation. F is a finite field, $n \geq 1$ is an integer, and B is a finite etale F-algebra of dimension n over F (i.e., over a finite extension K of F, there exists an isomorphism of K-algebras $B \otimes_F K \simeq K \times K \times \cdots \times K$). We assume given an element x in B that is regular in the sense that its characteristic polynomial $\det_F(T - x \mid B)$ in the regular representation of B on itself has n distinct eigenvalues. (In terms of the above isomorphism $B \otimes_F K \simeq K \times K \times \cdots \times K$, x is regular if and only if $x \otimes 1 \simeq (x_1, \ldots, x_n)$ with all distinct components x_i. Or equivalently, x is regular if and only if B is equal to the F-subalgebra $F[x]$ generated by x. In the special case when B is a field F, the element x is regular if and only if $F(x) = E$.)

Theorem 2. *Let χ be any nontrivial complex-valued multiplicative character of $B^{\times}$ (extended by zero to all of B), and x in B any regular element. Then*

$$\left\| \sum_{t \in F} \chi(t-x) \right\| \leq (n-1)\sqrt{\#(F)}.$$

Proof. The basic idea is that the theorem is an immediate consequence of Weil's estimates for one-variable character sums in the case when the F-algebra B is completely split, and that one can reduce to this case by thinking geometrically about suitable Lang torsors.

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We begin by explaining how to view the problem geometrically. Given any finite-dimensional commutative F-algebra A, we denote by $\mathbb{A}$ the smooth affine scheme over F given by "A as algebraic group over F"; concretely, for any F-algebra R, the group $\mathbb{A}(R)$ of R-valued points of $\mathbb{A}$ is $A \otimes_F R$. We denote by $\mathbb{A}^\times$ the open subscheme of $\mathbb{A}$ given by "$A^\times$ as algebraic group over F"; concretely, for any F-algebra R, the group $\mathbb{A}^\times(R)$ of R-valued points of $\mathbb{A}$ is $(A \otimes_F R)^\times$. These concepts will be applied to the cases $A = B$ and $A = F$. It will be important in what follows to think of $\mathbb{A}^\times$ as a smooth commutative group scheme over F, but to think of $\mathbb{A}$ only as an ambient scheme (not as a group scheme) containing $\mathbb{A}^\times$ as an open subscheme.

Because $\mathbb{B}^\times$ is a smooth, geometrically connected commutative group scheme over the finite field F, the Lang isogeny $1 - \mathrm{Frob}_F : \mathbb{B}^\times \to \mathbb{B}^\times$ makes $\mathbb{B}^\times$ into a $B^\times$-torsor over itself, the "Lang torsor" $\mathbb{L}$. Let us now fix a prime number $l \neq \mathrm{char}(F)$, an algebraic closure $\overline{Q}_l$ of Q_l, and an isomorphism of fields $C \simeq \overline{Q}_l$. This isomorphism allows us to view χ as a $\overline{Q}_l$-valued character of $B^\times$, by which it makes sense to push out the Lang torsor $\mathbb{L}$ to obtain a lisse rank one $\overline{Q}_l$-sheaf $\mathbb{L}_\chi$ on $\mathbb{B}^\times$ which is pure of weight zero. If we denote by $j : \mathbb{B}^\times \to \mathbb{B}$ the inclusion, we may form the extension by zero $j_! \mathbb{L}_\chi$ on $\mathbb{B}$. Now consider the morphism of F-schemes of $f : \mathbb{F} \to \mathbb{B}$ defined by $f(t) := t - x$, and the pullback sheaf $\mathscr{F} := f^*(j_! \mathbb{L}_\chi)$ on $\mathbb{F}$. The sheaf $\mathscr{F}$ is lisse of rank one and pure of weight zero on the open set $f^{-1}(\mathbb{B}^\times)$, and zero outside. The sheaf $\mathscr{F}$ is everywhere tamely ramified, simply because on $f^{-1}(\mathbb{B}^\times)$ it is lisse of order dividing that of χ, hence of order prime to the characteristic of F.

In terms of this data, the character sum in question is given by

$$\sum_{t \in F} \chi(t - x) = \sum_{t \in f^{-1}(\mathbb{B}^\times)(F)} \mathrm{Trace}(\mathrm{Frob}_{t,F} \mid \mathscr{F}),$$

and by the Lefschetz Trace Formula this last sum is equal to

$$\sum_i (-1)^i \mathrm{Trace}(\mathrm{Frob}_F \mid H^i_{\mathrm{comp}}(f^{-1}(\mathbb{B}^\times) \otimes_F \overline{F}, \mathscr{F})).$$

By Weil (but expressed in the language of Deligne's paper [De]) we know that the above cohomology groups H^i_{comp} are mixed of weight $\leq i$. For dimension reasons, H^i_{comp} vanishes for $i > 2$, and H^0_{comp} vanishes because $\mathscr{F}$ is lisse on the incomplete curve $f^{-1}(\mathbb{B}^\times) \otimes_F \overline{F}$. It thus remains only to establish the following two facts:

(a) $H^2_{\mathrm{comp}}(f^{-1}(\mathbb{B}^\times) \otimes_F \overline{F}, \mathscr{F}) = 0$,

(b) $\dim H^1_{\mathrm{comp}}(f^{-1}(\mathbb{B}^\times) \otimes_F \overline{F}, \mathscr{F}) = n - 1$.

Both of these facts are geometric, i.e., they concern the situation over the algebraic closure of F, and hence it suffices to verify them universally in the case when the F-algebra B is completely split. (The key point here is that our hypothesis that χ is nontrivial is stable under finite extension of scalars.

Indeed, after extension of scalars from F to any finite extension field K, the pullback to $(\mathbb{B}^\times)\otimes_F K$ of $\mathbb{L}_\chi$ is $\mathbb{L}_{\tilde{\chi}}$, where $\tilde{\chi}$ is the character of $(B\otimes_F K)^\times$ obtained from χ by composition with the norm homomorphism $\mathrm{Norm}_{K/F}$ from $(B\otimes_F K)^\times$ to $B^\times$. Because this norm map is surjective, the character $\tilde{\chi}$ is nontrivial provided that χ is nontrivial.)

Suppose now that B is simply the n-fold self product of F with itself. Then a nontrivial character χ of $B^\times$ is simply an n-tuple $(\chi_1, \dots, \chi_n)$ of characters of $F^\times$, not all of which are trivial, the regular element x is just an n-tuple $(x_1, \dots, x_n)$ with all distinct components x_i, the open set $f^{-1}(\mathbb{B}^\times)$ is just the complement $\mathbb{F} - \{x_1, \dots, x_n\}$ of the n distinct points x_i in $\mathbb{F}$, the sheaf $\mathscr{F}$ is just the tensor product of the sheaves $[t\mapsto t-x_i]^*\mathbb{L}_{\chi_i} \mid \mathbb{F} - \{x_1, \dots, x_n\}$, and the sum in question is

$$\sum_{t\in F-\{x_1,\dots,x_n\}} \chi_1(t-x_1)\chi_2(t-x_2)\cdots\chi_n(t-x_n).$$

By assumption, at least one of the χ_i is nontrivial. For such an index i, the sheaf $[t\mapsto t-x_i]^*\mathbb{L}_{\chi_i}$ is tamely but nontrivially ramified at x_i, while all the other factors $[t\mapsto t-x_j]^*\mathbb{L}_{\chi_j}$ with $j\neq i$ are lisse at x_i (by the hypothesis that all the x_j are distinct). Therefore, the sheaf $\mathscr{F}$ is nontrivially ramified at the point x_i. Because $\mathscr{F}$ is lisse of rank one on $\mathbb{F} - \{x_1, \dots, x_n\}$, its coinvariants under the inertia group I_{x_i} must vanish, and a fortiori its covariants under the entire π_1^{geom} of $\mathbb{F} - \{x_1, \dots, x_n\}$ must also vanish, i.e., its H^2_{comp} vanishes. Once we have the vanishing of all the H^i_{comp} save for $i=1$, the asserted dimension formula $\dim H^1_{\mathrm{comp}} = n-1$ is then equivalent to the Euler characteristic formula

$$\sum_i (-1)^i \dim H^i_{\mathrm{comp}}((\mathbb{F} - \{x_1, \dots, x_n\})\otimes_F \overline{F}, \mathscr{F}) = 1-n,$$

which holds because $\mathscr{F}$ is lisse of rank one and everywhere tame on the open curve $(\mathbb{F} - \{x_1, \dots, x_n\})\otimes_F \overline{F}$, whose Euler characteristic is $1-n$. Q.E.D.

Remarks and Questions. (1) If we drop the hypothesis that the element x be regular, then Theorem 2 remains valid for characters χ of $B^\times$ whose restriction to $F^\times$ is nontrivial. The proof proceeds along the same lines as above, reducing to the completely split case in which χ is simply an n-tuple $(\chi_1, \dots, \chi_\eta)$ of characters of $F^\times$, with the property that their product $\prod_i \chi_i$ is nontrivial on $F^\times$. Now one gets the vanishing of H^2_{comp} by observing that the sheaf $\mathscr{F}$ is nontrivially ramified at ∞ (as an I_∞-representation, $\mathscr{F}$ is isomorphic to $\mathbb{L}_{\prod_i \chi_i}$), and the constant " $n-1$ " actually improves to "(the number of distinct x_i)-1." Indeed, in the case of the choice $x := 0$, the character sum in question is exactly $\sum_{t\in F^\times} \chi(t)$. (Alternately, one could apply Theorem 2 directly to the (automatically finite etale) subalgebra $B_0 := F[x]$ of B generated by x over F, to the regular element x of B_0, and to the nontrivial (because nontrivial on $F^\times$) character $\chi \mid (B_0)^\times$.)

(2) What happens if we also drop the hypothesis that B be etale? Suppose that we are given an arbitrary n-dimensional commutative F-algebra A, a multiplicative character χ of $A^\times$ (extended by zero to all of A) whose restriction to $F^\times$ is nontrivial, and an element x in A. It seems plausible that the estimate

$$\left\|\sum_{t\in F}\chi(t-x)\right\| \leq (n-1)\sqrt{\#(F)}$$

should still hold. For example, in the case when A is the algebra of dual numbers $F[x]/(x^2)$, the character sums in question are none other than the usual Gauss sums attached to the field F.

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