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TWO ELEMENTARY PROOFS OF THE L^2 BOUNDEDNESS OF CAUCHY INTEGRALS ON LIPSCHITZ CURVES

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1. Introduction

Let $\Gamma = \{x + iA(x) \colon x \in \mathbf{R}\}$ be a Lipschitz curve in the complex plane $\mathbf{C}$ so that $A' \in L^\infty$. A well-known theorem (see [2]) asserts that the Cauchy integral operator is a bounded operator on $L^2(\Gamma)$. This result, first proved by Calderon [1] under the constraint $\|A'\|_{L^\infty} < \varepsilon_0$, now has myriad proofs. See, e.g., [4–6], as well as the books of Journé [9] and Murai [12]. In this note, we present two proofs which are considerably shorter and simpler than other previous proofs.

The first proof uses complex variables and is presented in this section and the next. The second approach gives not only estimates for the Cauchy integral, but also a simple proof of the $T(b)$ theorem [6, 11]. The formalism needed for this approach can be cast nicely in terms of martingales with respect to complex measures, or in terms of adapting the Haar functions on $\mathbf{R}$ to a complex measure. The two proofs run exactly in parallel. Lemma 1.1 corresponds to Lemma 3.1, and Lemma 1.2 corresponds to Lemma 3.2. The philosophy of our second proof, where the idea is to find a suitable frame for L^2, was heavily influenced by work of Tchamitchian and Meyer which they presented at the 1987 Conference on Fourier Analysis, El Escorial, Spain.

Define $\Omega_\pm = \{x + iA(x) \pm y \colon y > 0\}$ to be the two domains lying above or below Γ, and let

$$Cg(z) = \int_\Gamma \frac{g(\zeta)\, d\zeta}{z - \zeta}, \qquad z \in \Omega_+,$$

be the Cauchy integral of $g \in L^2(\Gamma)$. Then the theorem we shall prove is that the boundary values

$$Cg(z) = \lim_{y \to 0_+} Cg(z + iy), \qquad z \in \Gamma,$$

lie in L^2 and $\|Cg\|_{L^2(\Gamma)} \leq \text{Const}\, \|g\|_{L^2(\Gamma)}$. Our proof yields this inequality when both A' and g are C^∞ functions with compact support. Since all

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constants depend only on $\|A'\|_{L^\infty}$, the usual arguments (which we do not give) can be used to extend to the case of general A and g. The first proof uses only Hilbert space duality and two rather easy lemmata. Let $\mathscr{H}_+$ denote the Hilbert space of all complex valued, measurable functions on Ω_+ with norm

$$\|f\|_{\mathscr{H}_+} = \left(\iint_{\Omega_+} |f(z)|^2 d(z)\, dx\, dy \right)^{1/2},$$

where $d(z) = \text{distance}(z, \Gamma)$. Let $\langle\ ,\ \rangle_{\mathscr{H}_+}$ denote the inner product in $\mathscr{H}_+$. The Hilbert space $\mathscr{H}_-$ of functions on Ω_- is similarly defined. Our first lemma is a special case of a more general result due to Kenig [10]. A short proof is given in §2.

Lemma 1.1. *Suppose F is holomorphic in $\Omega_\pm$ and decays to zero at ∞. Then*

$$\|F\|_{L^2(\Gamma)} = \left(\int_\Gamma |F|^2\, ds \right)^{1/2} \le C(\|A'\|_{L^\infty}) \|F'(z)\|_{\mathscr{H}_+},$$

where ds denotes arclength.

The converse inequality in Lemma 1.1 is also true [10], and this follows from the argument given at the end of this section.

The idea of our proof is to use Lemma 1.1 to estimate $\|Cg\|_{L^2(\Gamma)}$.

To implement this philosophy we require some auxiliary functions whose $L^2(\Gamma)$ norms are rather simple to bound. The proof of our next lemma is also delayed until §2. It should be pointed out that this lemma is merely a disguised form of well-known results in the theory of Bergman spaces. See, e.g., [3] where calculations of this type are carried out in much greater generality.

Lemma 1.2. *Let $f \in \mathscr{H}_+$ and define*

$$Tf(\zeta) = \iint_{\Omega_+} \frac{f(z) d(z)}{(z-\zeta)^2}\, dx\, dy, \qquad \zeta \in \Gamma.$$

Then $\|Tf\|_{L^2(\Gamma)} \le C(\|A'\|_{L^\infty}) \|f\|_{\mathscr{H}_+}$.

The above two estimates can now be combined to yield a proof of the theorem. Let $B = \{f \in \mathscr{H}_+ : \|f\|_{\mathscr{H}_+} \le 1,\ f \text{ compactly supported in } \Omega_+\}$, so that for any $G \in \mathscr{H}_+$, $\|G\|_{\mathscr{H}_+} = \sup_{f\in B} |\langle G, f\rangle_{\mathscr{H}_+}|$. Fix $g \in L^2(\Gamma)$, define

$C'g(z) = dCg(z)/dz$, and invoke Lemma 1.1 plus Fubini to obtain

$$\begin{aligned}\left(\int_\Gamma |Cg|^2\,ds\right)^{1/2} &\le C\|C'g(z)\|_{\mathscr{H}_+} \\ &= C\sup_{f\in B}|\langle C'g, f\rangle_{\mathscr{H}_+}| \\ &= C\sup_{f\in B}\left|\iint_{\Omega_+}\left\{-\int_\Gamma \frac{g(\zeta)\,d\zeta}{(z-\zeta)^2}\right\}\overline{f(z)}d(z)\,dx\,dy\right| \\ &= C\sup_{f\in B}\left|\int_\Gamma g(\zeta)T(\bar f)(\zeta)\,d\zeta\right| \\ &\le C\sup_{f\in B}\|g\|_{L^2(\Gamma)}\|T(\bar f)\|_{L^2(\Gamma)} \\ &\le C'\|g\|_{L^2(\Gamma)},\end{aligned}$$

the final inequality following from Lemma 1.2.

2. Proving the lemmata

We first turn our attention to Lemma 1.1. Our proof is similar to Gamelin [7]. Let $\Phi\colon \mathbf{R}^2_+ \to \Omega_+$ denote a conformal mapping so that $\Phi(\mathbf{R}) = \Gamma$ and $\Phi(\infty) = \infty$. Then pulling back $\int_\Gamma |F|^2\,ds$ to $\mathbf{R}$ by Φ and invoking the Koebe $\frac{1}{4}$ theorem ($\Phi'(z)y \sim d(\Phi(z))$), we see that Lemma 1.1 is equivalent to the estimate

$$A \equiv \int_{\mathbf{R}} |G|^2|\Phi'|\,dx \le C\iint_{\mathbf{R}^2_+} |G'(z)|^2|\Phi'(z)|y\,dx\,dy \equiv CB, \tag{2.1}$$

for functions G dying at ∞. We first require some standard (and well-known) Littlewood-Paley estimates.

Lemma 2.1. *Let H and D be holomorphic on $\mathbf{R}^2_+$ and suppose $|D(z)| \le 1$, $z \in \mathbf{R}^2_+$. Then if H dies at ∞,*

$$\int_{\mathbf{R}} |H|^2\,dx = 4\iint_{\mathbf{R}^2_+} |H'(z)|^2 y\,dx\,dy \tag{2.2}$$

and

$$\iint_{\mathbf{R}^2_+} |H(z)D'(z)|^2 y\,dx\,dy \le \int_{\mathbf{R}} |H|^2\,dx. \tag{2.3}$$

Proof. The first equation is simply Green's theorem. To obtain (2.3) we use (2.2) to obtain

$$\int_{\mathbf{R}} |H|^2\,dx \ge \int_{\mathbf{R}} |HD|^2\,dx = 4\int_{\mathbf{R}^2_+} |H'D + HD'|^2 y\,dx\,dy.$$

The estimate now follows from the triangle inequality, (2.2), and the inequality $|H'D| \le |H'|$. □

Returning to the proof of (2.1), we use the standard trick that $A \leq CB$ is equivalent to the a priori estimate $A \leq C(B + A^{1/2}B^{1/2})$. We observe that since Γ is Lipschitz $\|\arg \Phi'\|_{L^\infty} < \pi/2 - \varepsilon$, where $\varepsilon = \operatorname{arccot}(\|A'\|_{L^\infty}) > 0$. Consequently,

$$\int_{\mathbf{R}} |G|^2 |\Phi'|\, dx \leq C \left| \int_{\mathbf{R}} |G|^2 \Phi'\, dx \right| ,$$

so by Green's theorem,

$$\begin{aligned} A = \int_{\mathbf{R}} |G|^2|\Phi'|\,dx &\leq C\left|\iint_{\mathbf{R}^2_+} \Delta(G\overline{G}\Phi')y\,dx\,dy\right| \\ &= 4C\left|\int_{\mathbf{R}^2_+} (|G'|^2\Phi' + G\overline{G}'\Phi'')y\,dx\,dy\right| \\ &\leq CB + C\iint_{\mathbf{R}^2_+} |G\overline{G}'\Phi''|y\,dx\,dy. \end{aligned}$$

Now let $\Phi' = e^V$ so that $\Phi'' = V'e^V = V'\Phi'$. Since $|\operatorname{Im} V(z)| < \pi/2$, $|\Phi''| \leq e^{\pi/2}|V'e^{iV}\Phi'| = e^{\pi/2}|D'\Phi'|$, where $D = e^{iV}$. Then by Cauchy-Schwarz

$$\begin{aligned} &\iint_{\mathbf{R}^2_+} |GG'\Phi''|y\,dx\,dy \\ &\leq \left(\iint_{\mathbf{R}^2_+} |G'|^2|\Phi'|y\,dx\,dy\right)^{1/2} \left(\iint_{\mathbf{R}^2_+} |G|^2|V'|^2|\Phi'|y\,dx\,dy\right)^{1/2} \\ &\leq B^{1/2}e^{\pi/2}\left(\iint_{\mathbf{R}^2_+} |G(\Phi')^{1/2}|^2|D'|^2y\,dx\,dy\right)^{1/2} \\ &\leq e^{\pi}B^{1/2}A^{1/2}, \end{aligned}$$

where the last inequality flows from (2.3) because $\|D\|_{H^\infty} \leq e^{\pi/2}$. □

The estimates needed to prove Lemma 1.2 are so crude that virtually any reasonable method will work. We give here a proof by Schur's lemma. By Lemma 1.1,

$$\|Tf\|_{L^2(\Gamma)} \leq C\|(Tf)'\|_{\mathscr{H}_-} \tag{2.4}$$

because Tf is holomorphic in Ω_-. Now

$$\begin{aligned} |Tf'(w)| &= \left|(-2)\iint_{\Omega_+} \frac{f(z)d(z)\,dx\,dy}{(z-w)^3}\right| \\ &\leq 2\iint_{\Omega_+} \frac{|f(z)|d(z)}{|z-w|^3}\,dx\,dy. \end{aligned}$$

Let $L^2_\pm$ denote the space of functions on $\Omega_\pm$ satisfying

$$\|F\|_{L^2_\pm} = \left(\iint_{\Omega_\pm} |F(z)|^2\,dx\,dy\right)^{1/2}.$$

Then by our last inequality, (2.4) follows from the boundedness (from L^2_+ to L^2_-) of the operator S defined by

$$SF(w) = d(w)^{1/2} \iint_{\Omega_+} \frac{F(z)d(z)^{1/2}}{|z-w|^3}\,dx\,dy = \iint_{\Omega_+} K(z,w)F(z)\,dx\,dy,$$

where $K(z,w) = d(z)^{1/2}d(w)^{1/2}|z-w|^{-3}$. Now for $w \in \Omega_-$ fixed, a simple calculus computation shows

$$\iint_{\Omega_+} K(z,w)\,dx\,dy \le 4\pi$$

because $d(z) \le |z-w|$. The same computation yields

$$\iint_{\Omega_-} K(z,w)\,dA(w) \le 4\pi.$$

By Schur's lemma (or interpolation between L^1 and L^∞), S is a bounded operator from L^2_+ to L^2_- and $\|S\|_{L^2_+,L^2_-} \le 4\pi$.

Our argument yields

$$\|Cg\|_{L^2(\Gamma)} \le C(1+\|A'\|_{L^\infty})^2\|g\|_{L^2(\Gamma)},$$

which is very close to the optimal estimate $C(1+\|A'\|_{L^\infty})^{3/2}$ due to Murai and David. (See, e.g., Murai's book [12].) This argument can be modified to yield David's theorem [4] that C is L^2 bounded if and only if Γ is an Ahlfors regular curve (see [8]).

3. The second approach

Let Γ be a rectifiable Jordan curve passing through ∞, and let $z(x)$ denote its arclength parameterization. We define the corresponding Cauchy integral operator $T = T_\Gamma$ by setting

$$Tf(x) = \lim_{\delta\to 0_+} \int_{-\infty}^{\infty} \frac{z'(y)}{z(y)-z(x)-i\delta z'(x)} f(y)\,dy.$$

We wish to show that if Γ is a chord-arc curve, T is a bounded operator on $L^2(\mathbf{R})$. (Recall that Γ is called a chord-arc curve if there is a constant k so that

$$|s-t| \le (1+k)|z(s)-z(t)|$$

for all s, $t \in \mathbf{R}$. Of course, any Lipschitz graph is a chord-arc curve.) Note that if f is a finite linear combination of characteristic functions, then Tf is defined almost everywhere. Furthermore, if such an f has $Tf \in L^1(\mathbf{R})$, Cauchy's theorem applied to one of the domains complementary to Γ yields

$$\int_{-\infty}^{\infty} Tf(x)z'(x)\,dx = 0.$$

To explain our approach we start with the case of the Hilbert transform, i.e., when $\Gamma = \mathbf{R}$, and use a variation of well-known methods for treating singular integrals via Littlewood-Paley theory. Let $\mathcal{F}$ denote the collection of all dyadic intervals in $\mathbf{R}$, and let h_I denote the Haar function corresponding to I. That is, $h_I(x) = |I|^{-1/2}$ on the left half of I and $h_I(x) = -|I|^{-1/2}$ on the right half of I. Then $\{h_I\}$ is an orthonormal basis for $L^2(\mathbf{R})$. We can then analyze the Cauchy integral operator T_0 by viewing it as a matrix operator relative to $\{h_I\}$, i.e.,

$$T_0 f = \sum_{I,J\in\mathcal{F}} \langle Th_I, h_J\rangle h_J \langle f, h_I\rangle .$$

Here $\langle\cdot,\cdot\rangle$ denotes the standard inner product on $L^2(\mathbf{R})$. Suppose we establish the inequality

$$\sup_I \left\{ \sum_{J\in\mathcal{F}} |\langle T_0 h_I, h_J\rangle| + \sum_{J\in\mathcal{F}} |\langle T_0 h_J, h_I\rangle| \right\} < \infty . \tag{3.1}$$

Then T_0 is a bounded operator because of Schur's lemma: a nonnegative matrix defines a bounded operator on l^2 if its row and column sums are uniformly bounded. The last inequality can be verified by an explicit computation.

For a general chord-arc curve Γ, we must modify the Haar system and the inner product $\langle\cdot,\cdot\rangle$. Fix a dyadic interval I and write $I = I_l \cup I_r$, where I_l and I_r are the left and right halves of I. Define

$$m(J) = \frac{1}{|J|}\int_J z'(x)\,dx$$

to be the mean value of z' over an interval J, so that $(1+k)^{-1} \le |m(J)| \le 1$, because of the chord-arc condition on Γ. Set

$$\beta_I(x) = |I|^{-1/2}\left(\frac{m(I_l)m(I_r)}{m(I)}\right)^{1/2}\left\{m(I_l)^{-1}\chi_{I_l}(x) - m(I_r)^{-1}\chi_{I_r}(x)\right\},$$

where the choice of the square root above is arbitrary. Also let $\langle\cdot,\cdot\rangle_\Gamma$ be the bilinear form defined by

$$\langle f, g\rangle_\Gamma = \int_{-\infty}^{\infty} f(x)g(x)z'(x)\,dx .$$

Then each β_I is supported on I and is constant on I_l and I_r. Furthermore,

$$\langle \beta_I, \beta_J\rangle_\Gamma = 0 \quad \text{if } I \ne J .$$

Our first lemma asserts that $\{\beta_I\}$ behaves like an orthonormal basis with respect to the weight $z'(x)\,dx$.

Lemma 3.1. *If $f \in L^2(\mathbf{R})$, then $f = \sum_I \langle f, \beta_I\rangle_\Gamma \beta_I$ and*

$$\frac{1}{C}\|f\|_2^2 \le \sum_I |\langle f, \beta_I\rangle_\Gamma|^2 \le C\|f\|_2^2 .$$

Our second lemma is the analogue of inequality (3.1).

Lemma 3.2.

$$\sup_I \sum_{J\in\mathscr{F}} \{|\langle T\beta_I, \beta_J\rangle_\Gamma| + |\langle T\beta_J, \beta_I\rangle_\Gamma|\} < \infty .$$

If we accept the above two lemmas, then by Schur's lemma T is bounded on $L^2(\mathbf{R})$.

Proof of Lemma 3.1. Let $\mathscr{F}_k$ denote the collection of dyadic intervals of length 2^{-k} and define the "expectation" operator E_k by

$$E_k f(x) = m(I)^{-1}|I|^{-1}\int_I f z'\,dt, \qquad x \in I \in \mathscr{F}_k .$$

Then standard reasoning shows that if $f \in L^2$, $E_k f \to f$ as $k \to +\infty$, and $E_k f \to 0$ as $k \to -\infty$. Setting $\Delta_k = E_{k+1} - E_k$ we obtain

$$f = \sum_{k=-\infty}^{\infty} \Delta_k f, \qquad f \in L^2 .$$

Since $\Delta_k f = \sum_{I\in\mathscr{F}_k} \langle f, \beta_I\rangle_\Gamma \beta_I$, the first conclusion of the lemma is verified.

Now consider the case where $\Gamma = \mathbf{R}$ and change notation so that E_k is replaced by P_k and Δ_k is replaced by Q_k. Then

$$Q_k f = \sum_{I\in\mathscr{F}_k} \langle f, h_I\rangle h_I$$

and since $\{h_I\}$ is an orthonormal basis for L^2,

$$\sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} |Q_k f|^2\,dx = \|f\|_2^2 . \tag{3.2}$$

Expanding out Δ_k, one sees that

$$\begin{aligned} |\Delta_k f| &= |E_{k+1} f - E_k f| \\ &= |P_{k+1}(z')^{-1} Q_k(z'f) - \frac{Q_k(z')}{P_k(z')P_{k+1}(z')} P_k(z'f)| \\ &\le C|Q_k(z'f)| + C|Q_k(z')||P_k(z'f)| , \end{aligned} \tag{3.3}$$

because by hypothesis $|P_k(z')| \ge c > 0$. To show that

$$\sum_I |\langle f, \beta_I\rangle_\Gamma|^2 \le C \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} |\Delta_k f|^2\,dx \le C\|f\|_2^2 ,$$

note that (3.3) plus an application of (3.2) to $z'f$ reduces us to verifying that

$$\sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} |Q_k(z')|^2 |P_k(z'f)|^2\,dx \le C\|f\|_2^2 . \tag{3.4}$$

Experts will note that this is simply Carleson's theorem on Carleson measures. We outline the well-known proof. The maximal function $F^*(x) = \sup_k |P_k F(x)|$ satisfies $\|F^*\|_{L^2} \le C\|F\|_{L^2}$. Then since $\|z'\|_{L^\infty} = 1$, (3.2) yields

$$\sum_{I \subseteq J} |\langle z', h_I \rangle|^2 \le |J|$$

for any interval $J \in \mathscr{F}$, and consequently

$$\sum_{I \subseteq \mathscr{O}} |\langle z', h_I \rangle|^2 \le |\mathscr{O}|$$

for any set $\mathscr{O} \subset \mathbf{R}$. Now let $\mathscr{O}_n = \{x : (z'f)^* > 2^n\}$ so that

$$\sum_{n=-\infty}^{\infty} 2^{2n} |\mathscr{O}_n| \sim \|z'f\|_2^2 = \|f\|_2^2 .$$

Then the left-hand side of (3.4) is bounded by

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} \sum_{I \in \mathscr{F}_k} 4 \sum_{n=-\infty}^{\infty} 2^{2n} \sum_{I \subseteq \mathscr{O}_n} |\langle z', h_I \rangle|^2
&= 4 \sum_{n=-\infty}^{\infty} 2^{2n} \sum_{\substack{I \in \mathscr{F} \\ I \subseteq \mathscr{O}_n}} |\langle z', h_I \rangle|^2 \\
&\le 4 \sum_{n=-\infty}^{\infty} 2^{2n} |\mathscr{O}_n| \le C\|f\|_2^2 .
\end{aligned}$$

Having shown that $\sum_I |\langle f, \beta_I \rangle|^2 \le C\|f\|_2^2$, the converse now follows from a standard polarization argument. Let $\|f\|_2 = 1$ and let $g = \bar{z}'\bar{f}$ so that $\|g\|_2 = 1$. Then

$$\begin{aligned}
1 = \|f\|_2^2 = \langle f, g \rangle_\Gamma &= \sum_I \langle f, \beta_I \rangle \langle g, \beta_I \rangle \\
&\le \left(\sum_I |\langle f, \beta_I \rangle_\Gamma|^2 \right)^{1/2} \left(\sum_I |\langle g, \beta_I \rangle_\Gamma|^2 \right)^{1/2} \\
&\le C \left(\sum_I |\langle f, \beta_I \rangle_\Gamma|^2 \right)^{1/2} . \quad \square
\end{aligned}$$

Notice that the operator E_k defined above is a conditional expectation in the sense of probability theory, but defined relative to the complex measure $z'(x)\,dx$. The point of the proof of Lemma 3.1 is that the expectation operator relative to a complex measure still has many of the same properties as in the case of a positive measure, although the proofs of some estimates (like the quadratic estimates in Lemma 3.1) are more involved.

Proof of Lemma 3.2. The computations needed for the proof are fairly standard and are similar to calculations in the proof of the $T(1)$ theorem of David and

Journé. Although the arguments we give will use the explicit form of the kernel of T, similar arguments can be given in much greater generality. The methods of this section can be used to give a new proof of the $T(b)$ theorem [6]. We shall discuss this more fully after proving Lemma 3.2.

Let us first collect some estimates for $T(\beta_I)$. We have

$$|T(\beta_I)(x)| \le C|x - x_I|^{-2}|I|^{3/2} \tag{3.5}$$

if $x \notin 2I$, and otherwise

$$|T(\beta_I)(x)| \le C|I|^{-1/2} \log \frac{10|I|}{\min\{|x - x_I|, |x - y_I|, |x - w_I|\}}, \tag{3.6}$$

where x_I, y_I, and w_I are the two endpoints and center of I.

To prove (3.5) we use $\int \beta_I(x) z'(x)\,dx = 0$ to obtain

$$|T(\beta_I)(x)| = \left| \int \left(\frac{1}{z(x) - z(y)} - \frac{1}{z(x_I) - z(y)} \right) z'(y)\beta_I(y)\,dy \right|.$$

An easy calculation using the definition of β_I and the chord-arc condition on Γ gives (3.5).

The proof of (3.6) can be obtained most directly by explicit computation, which we omit.

To prove Lemma 3.2, it suffices to show that

$$\sum_J |\langle T(\beta_{[0,1]}), \beta_J \rangle_\Gamma| \le C, \tag{3.7}$$

where C depends only on the chord-arc constant of Γ. That this is enough follows from symmetry and a rescaling argument.

When $|J| \ge \frac{1}{10}$, we get from (3.5) and (3.6) that

$$\begin{aligned} |\langle T(\beta_{[0,1]}), \beta_J \rangle_\Gamma| &\le C|J|^{-1/2} \quad \text{if } [0,1] \cap 2J \ne \varnothing \\ &\le C|J|^{3/2}|x_J|^{-2} \quad \text{if } [0,1] \cap 2J = \varnothing. \end{aligned}$$

From here it is easy to check that the $|J| \ge \frac{1}{10}$ piece of (3.7) is all right.

Assume now that $|J| \le \frac{1}{10}$ but $J \cap [-1, 2] = \varnothing$. We apply (3.5) with I replaced by J to get

$$|\langle T(\beta_{[0,1]}), \beta_J \rangle_\Gamma| = |\langle T(\beta_J), \beta_{[0,1]} \rangle_\Gamma| \le C|J|^{3/2}|x_J|^{-2}. \tag{3.8}$$

The corresponding piece of (3.7) is again easily checked to be bounded.

We are left with the case $|J| \le \frac{1}{10}$, $J \subseteq [-2, 3]$. We may as well assume that J is closer to 0 than to $\frac{1}{2}$ or 1, so that $J \subseteq [-2, \frac{3}{8}]$. (The other cases are similar.) By definition, $\beta_{[0,1]}$ is a linear combination of $\chi_{[0,1/2]}$ and $\chi_{[1/2,1]}$, and the contribution of the second can be controlled exactly as in (3.8). Thus, we may as well replace $\beta_{[0,1]}$ by $\chi_{[0,1/2]}$ for this estimate.

Suppose first that $J \subseteq [-2, 0]$. Using (3.5) with I replaced by J, we get

$$\begin{aligned} |\langle T(\chi_{[0,1/2]}), \beta_J \rangle_\Gamma| &= |\langle T(\beta_J), \chi_{[0,1/2]} \rangle_\Gamma| \\ &\le C|J|^{3/2}(|x_J| + |J|)^{-1}, \end{aligned} \tag{3.9}$$

at least if 0 is not an endpoint of J. Even if it is, (3.9) still holds. This can be seen by using (3.6) instead of (3.5) for the $\chi_{[0,|J|]}$ part of $\chi_{[0,1/2]}$.

Now suppose that $J \subseteq [0, \frac{3}{8}]$. Using Cauchy's theorem, it is not hard to see that $\langle T(1), \beta_J\rangle_\Gamma = \int_{-\infty}^{\infty} T(\beta_J)(x)z'(x)\,dx = 0$, so that

$$\langle T(\chi_{[0,1/2]}), \beta_J\rangle_\Gamma = -\langle T_\Gamma(\chi_{\mathbf{R}\setminus[0,1/2]}), \beta_J\rangle_\Gamma .$$

Using this, we can check that (3.9) holds also when $J \subseteq [0, \frac{3}{8}]$.

The $J \subseteq [-2, \frac{3}{8}]$ part of (3.7) can now be estimated using (3.9) and elementary computations. This completes the proof of Lemma 3.2.

Let us make some comments that relate the preceding proof of the L^2-boundedness of T to the $T(b)$ theorem.

With Δ_k as in the proof of Lemma 3.1, we have the identity

$$\sum \Delta_k^2 = I. \tag{3.10}$$

This follows from $\Delta_k^2 = \Delta_k$, which is easily seen from the interpretation of E_k as an expectation. Alternatively, it can be derived from the orthogonality properties of the β_I's.

The analysis of T could have been carried out in terms of the Δ_k's instead of the β_I's. Using the formula

$$T = \sum_{j,k} \Delta_k(\Delta_k T \Delta_j)\Delta_j ,$$

the L^2-boundedness of T is reduced to

$$\sup_j \sum_k \|\Delta_k T \Delta_j\| + \sup_k \sum_j \|\Delta_k T \Delta_j\| , \tag{3.11}$$

where the norm denotes the L^2 operator norm. The proof of this estimate is similar to the proof of Lemma 3.2.

It was not observed in [6] that one could find operators Δ_k as above that satisfy (3.10). Instead, different operators were built that had much the same qualitative properties as our Δ_k's, and the main difficulty was to find a substitute for (3.10). (Note, however, that the analogue of the Δ_k's in [6] had better smoothness properties on their kernels.)

Let us end with an indication of how the methods of this section can be used to give a new proof of the main theorem in [6]. For this we assume the reader is familiar with [6].

Let us say that $b \in L^\infty(\mathbf{R}^n)$ is dyadic pseudoaccretive if there is a $\delta > 0$ so that

$$\left| \frac{1}{|Q|} \int_Q b \right| \geq \delta$$

for all dyadic cubes Q. The operators E_k and Δ_k can be defined as before, and there are also versions of the Haar functions h_I and the β_I's. The $T(b)$ theorem can be proved for such a b in the same way as in [6], with the following minor technical changes. (Note that what we call Δ_k here was called E_k in [6].)

Let T be an operator associated to a standard kernel that satisfies the weak boundedness property and also $T(b) = 0$, $T^t(b) = 0$. We analyze T using the formula

$$T = \sum_{j,k} \Delta_{jk}(\Delta_k T M_b \Delta_j)\Delta_j M_b^{-1},$$

where $M_b f = bf$. A slightly more general version of the Schur criterion than the one we have used before tells us that T is bounded on L^2 if

$$\sup_j 2^{-\alpha j} \sum_k 2^{k\alpha} \|\Delta_k T M_b \Delta_j\| + \sup_k 2^{-\alpha k} \sum_j 2^{\alpha j} \|\Delta_k T M_b \Delta_j\| \tag{3.12}$$

is finite for some $\alpha \in \mathbf{R}$. In this case, the choice of α depends on the exponent of Hölder continuity of the second standard estimate on the kernel of T. (If that exponent is 1, as it is for the Cauchy integral, you can take $\alpha = 0$.)

The proof of (3.12) is like the proof of Lemma 3.12, but more technical. Similar estimates were obtained in [6], except that here we have discontinuities in the kernel of Δ_k. This difference is not serious.

In the case where $T(b) \neq 0$ or $T^t(b) \neq 0$, we reduce to the previous case by subtracting off paraproducts. These paraproducts are built out of the E_k 's and Δ_k 's, and hence they will not have standard kernels. However, their kernels will be close enough to standard so that the same proof techniques apply.

A more interesting issue arises when we try to deal with the case where b is merely para-accretive. This means that there are constants δ, $\varepsilon > 0$ so that for every dyadic cube Q there is a subcube Q_1 such that $|Q_1| \geq \delta |Q|$ and

$$\left| \frac{1}{|Q_1|} \int_{Q_1} b \right| \geq \varepsilon .$$

The following modification of our argument (observed by David) allows us to handle this case. The idea is to change the sequence of σ-algebras so that the same argument works.

Let $\mathscr{S}_k$ be the σ-algebra generated by the dyadic cubes of length 2^{-k}. Before we took $E_k(f)$ to be the conditional expectation of f relative to $\mathscr{S}_k$ and $b(x)\,dx$. This time we must be more careful in our choice of σ-algebras.

Choose $L \in \mathbf{Z}$ so that $2^{-L} \leq \delta$. Define σ-algebras $\mathscr{A}_j$ as follows. If Q is a dyadic cube of sidelength 2^{-Lj} such that

$$\left| \frac{1}{|Q_1|} \int_{Q} b \right| \geq \frac{1}{10} \delta^n \varepsilon ,$$

then we put Q in $\mathscr{A}_j$. Otherwise, we choose $Q_1 \subseteq Q$ as in the definition of para-accretivity, and we put Q_1 and $Q \backslash Q_1$ in $\mathscr{A}_j$. In this case, we have

$$\left| \frac{1}{|Q \backslash Q_1|} \int_{Q \backslash Q_1} b \right| \geq \frac{1}{2} \delta^n \varepsilon .$$

We define $\mathscr{A}_j$ to be the σ-algebra generated by these sets we have chosen.

The $\mathscr{A}_j$'s form an increasing sequence of σ-algebras in $\mathbf{R}^n$, and we can define expectation operators E_j relative to $\mathscr{A}_j$ and $b(x)\,dx$. Using these operators, the $T(b)$ theorem can be proved for this b as before.

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References

1. A. P. Calderón, *Cauchy integrals on Lipschitz curves and related operators*, Proc. Nat. Acad. Sci. U. S. A. **74** (1977), 1324–1327.
2. R. R. Coifman, A. McIntosh, and Y. Meyer, *L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes Lipschitziennes*, Ann. of Math. (2) **116** (1982), 361–388.
3. R. R. Coifman and R. Rochberg, *Representation theorems for holomorphic and harmonic functions in L^p*, Asterisque **77** (1980), 11–66.
4. G. David, *Opérateurs intégraux singuliers sur certaines courbes du plan complexe*, Ann. Sci. École Norm. Sup. (4) **17** (1984), 157–189.
5. G. David and J. L. Journé, *A boundedness criterion for generalized Calderón-Zygmund operators*, Ann. of Math. (2) **120** (1984), 371–397.
6. G. David, J. L. Journé, and S. Semmes, *Opérateurs de Calderón-Zygmund, fonctions para-aurétives et interpolation*, Rev. Mat. Iberoamericana **1** (1985), 1–56.
7. T. W. Gamelin, *Wolff's proof of the corona theorem*, Israel J. Math. **37** (1980), 113–119.
8. P. W. Jones, *Square functions, Cauchy integrals, analytic capacity, and harmonic measure*, Proc. Conf. on Fourier Analysis, El Escorial, 1987, Lecture Notes in Math., Springer, New York (to appear).
9. J. L. Journé, *Calderón-Zygmund operators, pseudo-differential operators, and the Cauchy integral of Calderón*, Lecture Notes in Math., no. 994, Springer-Verlag, Berlin and New York, 1983.
10. C. Kenig, *Weighted H^p spaces on Lipschitz domains*, Amer. J. Math. **102** (1980), 129–163.
11. A. McIntosh and Y. Meyer, *Algèbres d'opérateurs définis par des intégrales singulières*, C. R. Acad. Sci. Paris Sér. I Math. **301** (1985), 395–397.
12. T. Murai, *A real variable method for the Cauchy transform and applications to analytic capacity*, Lecture Notes in Math., no. 1307, Springer-Verlag, Berlin and New York, 1988.

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