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INVERSE SPECTRAL RESULTS ON TWO-DIMENSIONAL TORI

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1. Introduction

Let Γ be a two-dimensional lattice, let M be the two-torus, $\mathbf{R}^2/\Gamma$, and let V be a C^∞ function on M. Eskin, Ralston, and Trubowitz have shown (in [3] and [4]) that the inverse spectral problem for the Schroedinger operator

$$-\Delta_M + V \tag{1.1}$$

is much more complicated than the corresponding problem on surfaces of genus ≥ 2. On one hand, for certain classes of potentials, one can reconstruct V from the spectrum of (1.1); on the other hand, there are potentials that have (lots of) isospectral deformations. In their analysis of (1.1), an important element is the decomposition of the Fourier series of V into "primitive summands," and we begin here by saying a few words about this decomposition since it will also be important in the results to be described. An element of a lattice is said to be *primitive* if it is not a positive integer multiple of another element of the lattice. Now let

$$\sum a_\omega e^{2\pi i\langle\omega, x\rangle}, \qquad \omega \in \Gamma^*,$$

be the Fourier series expansion of V, and for each primitive element δ of Γ^*, let

$$\mathscr{Q}_\delta(V) = \sum a_{n\delta} e^{2\pi i n\langle\delta, x\rangle}. \tag{1.2}$$

Then assuming that the zeroth Fourier coefficient of V is zero, one can write

$$2V = \sum \mathscr{Q}_\delta(V) \tag{1.3}$$

summed over the set of primitive elements of Γ^* (with $\mathscr{Q}_\delta = \mathscr{Q}_{-\delta}$). This we will call the *decomposition of V into primitive summands.* The main observation of the Eskin-Ralston-Trubowitz paper is that the heat trace of (1.1) has encoded into it lots of information about the individual terms in this sum (in fact, enough information to enable one to decide in certain cases that V is spectrally rigid).

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Eskin, Ralston, and Trubowitz also consider in their paper the "Bloch spectra" of the operator (1.1). A rather complicated (but appealing) way of describing Bloch spectra is as follows. Let $L \to M$ be a line bundle and ∇ a connection on L. Then one can define a Laplace operator (usually denoted $\nabla \cdot \nabla$) by means of the connection, obtaining an operator

$$-\nabla \cdot \nabla + V \tag{1.4}$$

similar to (1.1) but now operating on sections of L. Let $\mathscr{B}$ be the set of all (L, ∇) pairs for which $\operatorname{curv}(\nabla) = 0$. It can be shown that

$$\mathscr{B} \cong \mathbf{R}^2/\Gamma^*.$$

The Bloch eigenvalues of (1.1) are defined to be the eigenvalues of (1.4) associated with points (L, ∇) on this dual torus. One of the main theorems Eskin, Ralston, and Trubowitz assert is that the spectrum of (1.1) determines *all* the Bloch spectrum (modulo some assumptions about V and Γ; see [3, §6]). In other words, if $\operatorname{curv}(\nabla) = 0$, the spectrum of (1.4) does not contain any information about V that is not already encoded in the spectrum of (1.1).

In this article we describe what happens when this assumption is dropped. It might appear that by dropping this assumption one makes the inverse spectral problem for (1.4) a lot harder; however, in some sense the opposite is true: The more "twisted" the topology of L the easier is the reconstruction problem. In this article we assume that the topology of L is as twisted as possible in the sense that $c_1(L)[M]$ is equal to ± 1. We prove that this hypothesis, together with some parity assumptions on ∇ and V and curvature bounds on ∇, enables one to reconstruct *both* ∇ and V from spectral data associated with the wave trace of (1.4). The data involved in this reconstruction are called "band data" and have been used previously in connection with inverse spectral problems on the two-sphere (e.g. in [6, 7, 13]). We suspect that "band spectra techniques" (see §2 for details) will eventually turn out to have lots of other inverse spectral applications besides those to be described.

Before turning to these applications, we need to make a few preliminary remarks about lattices, tori, and line bundles over tori.

1. The lattice Γ is said to have *simple length spectrum* if, for every pair of elements v and w either $|v| \neq |w|$ or $v = \pm w$. It is clear that if Γ has this property, it cannot be a rectangular lattice or regular hexagonal lattice; so, in particular, its symmetry group has to be $\mathbf{Z}_2$ and is generated by rotation through 180°, i.e., by the involution

$$\sigma : \mathbf{R}^2 \to \mathbf{R}^2, \qquad (x, y) \to (-x, -y). \tag{1.5}$$

From now on we assume that Γ is such a lattice.

The action of $\mathbf{Z}_2$ on Γ induces an action of $\mathbf{Z}_2$ on the torus $M = \mathbf{R}^2/\Gamma$, and the fixed point set of this action is the image in $\mathbf{R}^2/\Gamma$ of $\Gamma/2$. (In particular, $\mathbf{Z}_2$ has exactly *four* fixed points on M. The same is also true of the induced action of $\mathbf{Z}_2$ on $\mathbf{R}^2/\Gamma^*$.)

2. Given a line bundle $L \to M$ and a connection ∇ on L, we would like to know if the action of $\mathbf{Z}_2$ on M can be lifted to an action of $\mathbf{Z}_2$ on L that preserves ∇. A necessary condition for this to be the case is that $\operatorname{curv}(\nabla)$ be $\mathbf{Z}_2$ invariant. Conversely, if Ω is an integral two-form on M that is $\mathbf{Z}_2$ invariant, it is easy to see [11, p. 133] that there exists a line bundle-connection pair (L, ∇) with $\operatorname{curv}(\nabla) = \Omega$ and a $\mathbf{Z}_2$ action on L that preserves ∇. Unfortunately, the pair (L, ∇) is not unique, as one can see by considering the extreme case $\operatorname{curv}(\nabla) = 0$. As was pointed out above, the set $\mathscr{B}$ of all line bundle-connection pairs (L, ∇) with $\operatorname{curv}(\nabla) = 0$ can be canonically identified with the dual torus $\mathbf{R}^2/\Gamma^*$, and, hence (up to equivalence) there are exactly *four* line bundle-connection pairs that are $\mathbf{Z}_2$ invariant (corresponding to the four fixed points of the $\mathbf{Z}_2$ action on $\mathscr{B}$). More generally, given any $\mathbf{Z}_2$ invariant line bundle-connection pair (L, ∇) with $\operatorname{curv}(\nabla) = \Omega$, one gets another such pair by tensoring (L, ∇) by a flat $\mathbf{Z}_2$-invariant line bundle-connection pair. Hence, to each Ω correspond exactly four $\mathbf{Z}_2$-invariant line bundle-connection pairs (L, ∇) with $\operatorname{curv}(\nabla) = \Omega$.

3. Let ω be a translation invariant two-form on M (which we will normalize by requiring that its integral over M be one). For every integer N there exist, by the remarks above, four distinct $\mathbf{Z}_2$-invariant line bundle-connection pairs (L, ∇_0) with $\operatorname{curv}(\nabla_0) = N\omega$. Let α be a $\mathbf{Z}_2$-invariant one-form on M. Then the operator

$$\nabla: C^\infty(L) \to C^\infty(L \otimes T^*M)$$

defined by $\nabla s = \nabla_0 s + s \otimes \alpha$ is a $\mathbf{Z}_2$-invariant connection on L with curvature form $\Omega = N\omega + d\alpha$.

4. Since we are assuming that $c_1(L)[M]$ is ± 1, the "N" in this equality is ± 1, and hence,

$$\operatorname{curv}(\nabla) = \pm\omega + d\alpha. \tag{1.6}$$

Furthermore, by Hodge theory

$$\alpha = \mu + *df + dg,$$

μ being a translation invariant one-form and f and g C^∞ functions. Since the decomposition above is $\mathbf{Z}_2$ equivariant, both f and g are $\mathbf{Z}_2$-invariant and $\mu = 0$. By "change of gauge" we can also make $g = 0$, and hence the identity above reduces to

$$\alpha = *df. \tag{1.7}$$

Thus, to specify the data (L, ∇) amounts to specifying the function f in (1.7).

5. The inverse problem that we will be concerned with in this paper is the following: Given $\mathbf{Z}_2$-invariant functions f and V, to what extent does the spectrum of the Schroedinger operator (1.4) determine these functions? The mains results that we will prove are the the following: Let

$$\sum a_\omega e^{2\pi i\langle\omega, x\rangle}, \qquad \omega \in \Gamma^*,$$

be the Fourier series of f, and, for each primitive element δ of Γ^*, let

$$\mathscr{Q}_\delta(f) = \sum a_{N\delta} e^{2\pi i N\langle\delta, x\rangle}. \tag{1.8}$$

Theorem I. *Suppose that*

$$\sum N^2 |a_{N\delta}| < (|\delta|^2 \operatorname{Area} M)^{-1}. \tag{1.9}$$

Then the spectrum of (1.4) *determines* Q_δ.

Corollary. *If the* H_3 *norm of* f *is small enough, the spectrum of* (1.4) *determines* f.

Remark. In the proof of Theorem I, we will make use of data that can be read off from the *leading* symbol of the wave trace. Since this data does not depend on V, the same is true of the results just stated.

Turning to the reconstruction problem for the potential V, suppose that f is now fixed and satisfies (1.9) for all primitive elements δ of Γ^*. These hypotheses will enable us to prove

Thereom II. *From the spectrum of* (1.4), *one can reconstruct* V.

2. Band invariants

Let M be a compact Riemannian manifold, $L \to M$ a Hermitian line bundle, ∇ a connection on L, and $V: M \to \mathbf{R}$ a C^∞ function. Without loss of generality, one can assume that the operator

$$-\nabla \cdot \nabla + V \tag{2.1}$$

is positive and, hence, that its square root

$$A = (-\nabla \cdot \nabla + V)^{1/2} \tag{2.2}$$

is well defined. Let

$$\exp \sqrt{-1} tA, \qquad -\infty < t < \infty, \tag{2.3}$$

be the one-parameter group of unitary transformations generated by A. We list below a few properties of (2.3). These properties are all well known. (More detailed discussions can be found in the third and fourth volumes of Hörmander [9], in the two foundational papers on Fourier integral operators [10, 2], in Treves' book [12], and in the author's paper with Duistermaat [1].)

1. The symbol of A is just the function

$$\sigma(A)(x, \xi) = |\xi|.$$

Let Ξ be the Hamiltonian vector field defined by this function and

$$\exp t\Xi, \qquad -\infty < t < \infty,$$

the geodesic flow that it generates on $T^*M - O$. Then according to [10], *the operator* (2.3) *is a zeroth order Fourier integral operator and its underlying canonical transformation is* $\exp t\Xi$.

2. Let $E(x, y, t)$ be the Schwartz kernel of (2.3). Viewed as a generalized function on $M \times M \times \mathbf{R}$, $E(x, y, t)$ is a Fourier integral distribution and its wave front set is the set of all triples $(x, \xi, y, \eta, t, \tau)$ satisfying

$$(y, -\eta) = (\exp t\Xi)(x, \xi)$$

and

$$\tau = \sigma(A)(x, \xi).$$

3. When we ignore Maslov factors, the symbol of $E(x, y, t)$ at the point $(x, \xi, y, \eta, t, \tau)$ is the holonomy of the connection ∇ along the curve

$$\gamma(s) = \pi((\exp s\Xi)(x, \xi)), \qquad 0 \le s \le t,$$

joining x to y. Here $\pi\colon T^*M \to M$ is the standard cotangent fibration. We will denote this holonomy by the symbol

$$e^{i\kappa(x, \xi, t)}. \tag{2.4}$$

By definition, it is a map of L_x onto L_y (or in the special case where $(x, \xi) = (y, \eta)$, of L_x onto itself, i.e., in this case, we can view (2.4) as a complex number of modulus one).

The Maslov adjustment to (2.4) requires that (2.4) be multiplied by a factor $e^{i\sigma_\gamma}$, which can also be interpreted as a holonomy mapping; namely, the Maslov bundle is equipped with a canonical flat connection and the Maslov factor above is just the holonomy of this connection along the curve

$$(\exp s\Xi)(x, \xi), \qquad 0 \le s \le t.$$

To avoid cluttering up the formulas below with Maslov factors, we will henceforth let $\kappa(x, \xi, t) = \kappa(x, \xi, t) + \sigma_\gamma$.

4. The trace of $\exp\sqrt{-1}tA$ is well defined as a distribution on the real line, and its singular support is contained in the *period spectrum* of M: if T is in the singular support of $\operatorname{trace}(\exp\sqrt{-1}tA)$, there exists a periodic geodesic on M of period T.

5. Let S^*M be the unit cosphere bundle of M (which we will think of as the subset of T^*M defined by $\sigma(A) = 1$). To say that T is in the period spectrum of M is the same as saying that the map

$$\exp T\Xi\colon S^*M \to S^*M$$

has a nonempty fixed point set. Let us denote this fixed point set by W_T. One says that W_T is *clean* if it is a submanifold, and if, in addition, for each $p \in W_T$, the tangent space to W_T at p is the fixed point set of the linear mapping $d(\exp T\Xi)_p$ on the tangent space to S^*M at p. Assume now that W_T is clean. Then the *trace formula* [1, §4] says that for t close to T

$$\operatorname{trace}(\exp\sqrt{-1}tA) = \left(\int e^{i\kappa(x, \xi, T)}\, d\mu_T\right) e_m(t - T) + \cdots, \tag{2.5}$$

where $2m-1=\dim W_T$, $e_m(t)=(t+i0+)^{-m}$, and the remainder term indicated by dots is a distribution that has a singularity at $t=T$ that is milder than $e_m(t-T)$. The integral in parentheses is over W_T, and μ_T is a smooth measure on W_T intrinsically associated with the mapping $\exp T\Xi$, i.e., not depending on ∇ and V. (2.5) says that this integral is a *spectral invariant of the operator* (2.1).

6. Some properties of this invariant will be needed below. The mapping $\exp t\Xi$ maps W_T into itself and as a function of t is periodic of period T. Suppose that it is *simply* periodic of period T; i.e., suppose that for all points (x, ξ) on W_T

$$(\exp t\Xi)(x, \xi) \neq (x, \xi)$$

except when t is an integer multiple of T. Then

$$(\exp t\Xi)\colon W_T \to W_T, \qquad t \in \mathbf{R}(\operatorname{mod} T)$$

is a free action of the circle group $\mathbf{R} \operatorname{mod} T$ on W_T. Since W_T is compact, this implies that there exists a smooth, compact, $2(m-1)$-dimensional manifold B_T and a fibration

$$\pi_T\colon W_T \to B_T \tag{2.6}$$

whose fibers are the trajectories of W_T. We will denote by ν_T the push-forward of μ_T by the mapping (2.6). It is clear that the function (2.4) is constant along the fibers of (2.6); so it defines a function on B_T that we will denote by $e^{i\kappa_T}$. With this notation the coefficient of $e_m(t-T)$ in formula (2.5) can be rewritten as an integral over B_T:

$$\int e^{i\kappa_T}\, d\nu_T. \tag{2.7}$$

7. Denote by $\sigma(M)$ the period spectrum of M. Notice that if T is in $\sigma(M)$, so are all its integer multiples. T is said to be *primitive* if it is not itself the integer multiple of another element of $\sigma(M)$ of absolute value less than $|T|$. In the discussion below, let T be a fixed primitive element in the period spectrum. Suppose that for every integer multiple $T' = nT$ of T the following are true.

(2.8)
(a) The fixed point set $W_{T'}$ of $\exp T'\Xi$ is clean and is equal to W_T.
(b) The intrinsic measure $\mu_{T'}$ on this fixed point set is a constant times μ_T.

Then the spectral invariant (2.7) associated with T' is the integral over B_T, $\int e^{in\kappa_T}\, d\nu_T$, times the same constant. Hence, for every trigonometric polynomial

$$p(e^{i\theta}) = \sum a_k e^{ik\theta}, \qquad -N \le k \le N,$$

the integral

$$\int p(e^{i\kappa_T})\, d\nu_T \tag{2.9}$$

is a spectral invariant. Regarding $e^{i\kappa_T}$ as a map from B_T into S^1, (2.9) is just the integral over B_T of the pull-back of $p(e^{i\theta})$ with respect to this map. Since the set of $p(e^{i\theta})$'s is dense in the space of continuous functions on S^1, this implies that the push-forward of the measure ν_T with respect to this map, i.e., the measure

$$(e^{i\kappa_T})_* \nu_T \tag{2.10}$$

is a spectral invariant of the operator (2.1). To summarize, we have proved

Theorem. *Let T be a primitive element in the period spectrum of M. Suppose that for all iterates $T' = nT$ of T the hypotheses* (2.8) *hold. Then the measure on the unit circle defined by* (2.10) *is a spectral invariant of* (2.1).

We will refer to these measures collectively as the *band* invariants of ∇.

Apropos of the hypotheses (2.8), Uribe and the author have shown recently that these hypotheses hold if the dynamical system

$$\exp t\Xi, \qquad -\infty < t < \infty,$$

is *completely integrable.* (See [8]; in particular, apropos of (2.8b) see Proposition 4.6 in [8].)

8. So far, the potential term V in (2.1) has not played any role in this discussion. Let us, however, now fix the connection part of (2.1) and treat the eigenvalues of the operator $\nabla \cdot \nabla$ as part of the given data of our problem. Let

$$A_0 = (-\nabla \cdot \nabla)^{1/2}.$$

Since

$$A = A_0 + \tfrac{1}{2} A_0^{-1} V + \cdots ,$$

one gets by variation of constants

$$\exp \sqrt{-1} tA = (\exp \sqrt{-1} tA_0) + (\exp \sqrt{-1} tA_0)\mathscr{Q}, \tag{2.11}$$

$\mathscr{Q}$ being a pseudodifferential operator of order -1 having the same leading symbol as

$$A_0^{-1} \int_0^t \exp(-\sqrt{-1} sA_0) V \exp(\sqrt{-1} sA_0)\, ds. \tag{2.12}$$

Therefore, by Egorov's theorem the leading symbol of $\mathscr{Q}$ is

$$\sigma(\mathscr{Q}) = \sigma(A)^{-1} \int_0^t (\exp s\Xi)^* \pi^* V\, ds, \tag{2.13}$$

π being, as above, the cotangent fibration.

Let us now compare the traces of the two sides of (2.11); i.e., consider

$$\operatorname{trace}(\exp \sqrt{-1} tA - \exp \sqrt{-1} tA_0). \tag{2.14}$$

As was pointed out in §4, (2.14) is well defined as a distributional function of t, and its singular support is contained in the period spectrum of M. Moreover, if T satisfies the hypotheses of §5, then by (2.11), the expression (2.14) is equal in a small neighborhood of T to

$$\left(\int \sigma(\mathscr{Q})(x, \xi) e^{i\kappa(x, \xi, T)}\, d\mu_T\right) e_{m-1}(t-T) + \cdots . \tag{2.15}$$

Suppose, in addition, that the hypotheses of §6 are satisfied. Let $V_T \nu_T$ be the pushforward of the measure $V\mu_T$ with respect to the fibration (2.6); i.e., let V_T be the fiber integral of V over the fibers of (2.6). Then by (2.13) the coefficient in front of $e_{m-1}(t-T)$ in (2.15) can be rewritten as the integral over B_T:

$$\int e^{i\kappa_T} V_T\, d\nu_T. \tag{2.16}$$

In particular, this expression is a spectral invariant of the wave trace.

9. Finally, suppose that the hypotheses of §7 are satisfied. Then for every iterate $T' = nT$ of T, the integral over B_T, $\int e^{in\kappa_T} V_T\, d\nu_T$, is a spectral invariant of the wave trace, and hence, the measure on S^1 defined by

$$(e^{i\kappa_T})_* (V_T \nu_T) \tag{2.17}$$

is a spectral invariant of the wave trace.

We will call these measures the *band invariants of* V *associated with the primitive periods* T in $\sigma(M)$.

3. Band invariants on the two-torus

We will now compute these invariants for the two-torus $M = \mathbf{R}^2/\Gamma$. To begin with, the period spectrum of M is just the set

$$\sigma(M) = \{\pm|v|,\ v \in \Gamma\}. \tag{3.1}$$

The assumption that Γ has simple period spectrum amounts to saying that for $T \in \sigma(M)$ there exist just two vectors, v and $-v$, in Γ with $T = \pm|v|$. The subset W_T of S^*M corresponding to $T \in \sigma(M)$ has two components, namely,

$$W_T^+ = \left\{(x, \xi),\ x \in M,\ \xi = \frac{v}{|v|}\right\} \tag{3.2_+}$$

and

$$W_T^- = \left\{(x, \xi),\ x \in M,\ \xi = -\frac{v}{|v|}\right\}, \tag{3.3_-}$$

and these two components are interchanged by the symmetry T^*M induced by (1.5). Moreover, the cotangent fibration $\pi\colon T^*M \to M$ maps W_T^+ and W_T^- diffeomorphically onto M; so there are canonical identifications

$$W_T^+ \cong M \quad \text{and} \quad W_T^- \cong M.$$

Modulo these identifications, geodesic flow on $W_T^\pm$ is the linear flow

$$x \to x \pm tv, \qquad -\infty < t < \infty. \tag{3.4}$$

Since the data above are invariant under translations, the canonical measures $\mu_T^{\pm}$ on $W_T^{\pm}$ are up to constant multiple just Lebesgue measure. Moreover, since these data are invariant under the involution σ (rotation through $180°$), σ also interchanges μ_T^+ and μ_T^-.

Suppose now that T is a primitive element of the period spectrum (or, equivalently, that v is a primitive element of the lattice Γ). In the dual lattice, consider the set

$$\{\omega \in \Gamma^*, \langle v, \omega\rangle = 0\}. \tag{3.5}$$

This set is a one-dimensional sublattice of Γ, and hence, up to sign, there is a unique primitive element δ of Γ^* that generates (3.4).

(3.1) **Lemma.** *There exist vectors $w \in \Gamma$ and $\nu \in \Gamma^*$ with the property that (v, w) is a basis of Γ and (ν, δ) the corresponding dual basis of Γ^*.*

Proof. Let v_1 and v_2 be a pair of generators of Γ. Since v is primitive, $v = kv_1 + lv_2$, the integers k and l being mutually prime, i.e,. $(k, l) = 1$. Hence there exist integers m and n with $mk + nl = 1$. Now set $w = -nv_1 + mv_2$. Since the determinant relating v_1 and v_2 to v and w is 1, the vectors v and w are also a basis of Γ. Let v^* and w^* be the dual basis. Then $\langle w^*, v\rangle = 0$, so w^* is an element of the set (3.4) and hence a generator of this set. Replacing w by $-w$ if necessary, we can assume that $w^* = \delta$. Q.E.D.

Without loss of generality, we can assume that the pair (v, w) had the same orientation as the standard basis of $\mathbf{R}^2$. Then the lemma above implies

Corollary. *Let ω be the standard volume form on $\mathbf{R}^2$ normalized so that the integral of ω over M is one. Then $\omega(v, w) = 1$.*

We will identify the quotient space $\mathbf{R}^2/\{cv, c \in \mathbf{R}\}$ with $\mathbf{R}$ via the mapping

$$t \to \text{projection of } tw \text{ onto } \mathbf{R}^2/\{cv,\ c \in \mathbf{R}\}. \tag{3.6}$$

Consider now the fibration (2.6). This is just the fibration

$$\mathbf{R}^2/\Gamma \to (\mathbf{R}^2/\{cv + nw,\ c \in \mathbf{R} \text{ and } n \in \mathbf{Z}\}), \tag{3.7}$$

so the two components $B_T^{\pm}$ of B_T can be identified with $\mathbf{R}/\mathbf{Z}$ via (3.6) and (3.7). Moreover, since $\sigma(w) = -w$, the diagram

$$\begin{array}{ccc} B_T^+ & \xrightarrow{\sigma} & B_T^- \\ \downarrow & & \downarrow \\ \mathbf{R}/\mathbf{Z} & \longrightarrow & \mathbf{R}/\mathbf{Z} \end{array} \tag{3.8}$$

commutes (the bottom arrow being the map $t \to -t$) and

$$\sigma^* \nu_T^- = \nu_T^+, \tag{3.9}$$

both these measures being identified via the vertical arrows in (3.8) with standard Lebesgue measure dt on $\mathbf{R}/\mathbf{Z}$.

Now let $L \to M$ be a line bundle on M, and let ∇ be a connection on L whose curvature is given by the formulas (1.6) and (1.7), i.e.,

$$\text{curv}(\nabla) = \omega + d\alpha, \qquad \alpha = *df, \tag{3.10}$$

f being invariant under σ. Let

$$\sum a_\omega e^{2\pi i \langle \omega, x\rangle}, \qquad \omega \in \Gamma^*, \tag{3.11}$$

be the Fourier series of f. Since f is real and $\sigma^* f = f$, one has

$$a_\omega = \bar{a}_\omega = a_{-\omega} = \bar{a}_{-\omega}. \tag{3.12}$$

Let $Q_\delta = Q_\delta(f)$ be, as in §1, the sum $\sum a_{N\delta} e^{2\pi i N\langle\delta, x\rangle}$. Under the mapping (3.6), Q_δ pulls back to the function

$$Q_\delta(tw) = \sum a_{N\delta} e^{2\pi i N t} \stackrel{\text{def}}{=} q_\delta(t). \tag{3.13}$$

Let $e^{i\kappa_T}$ be the holonomy mapping in formula (2.7). We will prove

(3.2) **Proposition.** *Modulo the identifications in* (3.8), $e^{i\kappa_T}$ *is the same on both components of* B_T *and is equal to*

$$\exp 2\pi i(t + \langle *\delta, v\rangle \dot{q}_\delta(t)). \tag{3.14}$$

Proof. The fact that, modulo the identification in (3.8), $e^{i\kappa_T}$ is the same on both components of B_T follows easily from the fact that $\sigma^*\nabla = \nabla$. Therefore, it suffices to compute $e^{i\kappa_T}$ on B_T^+. Identifying B_T^+ with $\mathbf{R}/\mathbf{Z}$ and W_T^+ with $\mathbf{R}^2/\Gamma$, the fibration (2.6) is just the linear mapping

$$\mathbf{R}^2/\Gamma \to \mathbf{R}/\mathbf{Z}$$

described in (3.6), the fiber over $a \in \mathbf{R}/\mathbf{Z}$ being the geodesic

$$aw + sv, \qquad 0 \le s \le 1. \tag{3.15_a}$$

By definition, $e^{i\kappa_T(a)}$ is the "shift in phase" caused by transporting the fiber of L parallel to itself along the curve (3.15_a). It can be computed by determining separately the contributions of each of the two terms in the sum (3.10) and then taking their product. We begin by computing the contribution of the first term. It is easy to see that for $a = 0$ this contribution is the trivial holonomy 1 since σ maps the curve (3.15_o) into the same curve traced in the opposite direction. Then for $a = t$, the contribution is $e^{2\pi i A}$, where A is the integral of ω over the region bounded by the curves (3.15_o) and (3.15_a). This is clearly equal to $t\omega(v, w)$, and, hence, by the corollary to Lemma 3.1 is just t. Thus, the contribution of the first term in (3.10) to $e^{i\kappa_T}$ is

$$e^{2\pi i t}. \tag{3.16}$$

Next we compute the contribution to $e^{i\kappa_T}$ of the second term in (3.10). This is $e^{2\pi i h(t)}$, where $h(t)$ is the integral of $*df$ over the curve (3.15_t). By (3.11),

$$*df = 2\pi i \sum (a_\omega e^{2\pi i\langle\omega, x\rangle}) * \omega,$$

so the contribution of each term in this sum to the integral is

$$2\pi i a_\omega e^{2\pi i\langle \omega , w\rangle t}\left(\int_0^1 e^{2\pi i\langle \omega , v\rangle s}\,ds\right)\langle *\omega , v\rangle.$$

However, the term in parentheses is zero unless $\langle \omega , v\rangle = 0$, i.e., unless ω belongs to the sublattice (3.4), or in other words, unless $\omega = N\delta$. Thus, the integral of $*df$ over (3.15_t) is given by the sum $2\pi i\langle *\delta , v\rangle \sum k a_{k\delta} e^{2\pi i k t}$, which is just $\langle *\delta , v\rangle \dot{q}_\delta(t)$, and the contribution to $e^{i\kappa_T}$ is

$$\exp 2\pi i\langle *\delta , v\rangle \dot{q}_\delta(t). \tag{3.17}$$

Taking the product of (3.16) and (3.17), we get for $e^{i\kappa_T}$ the expression (3.14). Q.E.D.

A simple computation, which we omit, shows that

$$\langle *\delta , v\rangle = (\operatorname{Area} M)|\delta|^2. \tag{3.18}$$

Therefore, one gets as a corollary of this proposition

Corollary. *If* $Q_\delta(f)$ *satisfies* (1.9), $e^{i\kappa_T}$ *is a diffeomorphism of* $\mathbf{R}/\mathbf{Z}$ *onto* S^1.

We now compute the band invariant associated with T. This is also the sum of two pieces, a "plus" piece coming from B_T^+ and a "minus" piece coming from B_T^-; however, by Proposition (3.2), these pieces are equal and their sum is

$$2(e^{i\kappa_T})_* \,dt, \tag{3.19}$$

where $e^{i\kappa_T}$ is the mapping (3.14). The main question of interest for us is: "To what extent does (3.19) determine κ_T (and hence $\mathscr{Q}_\delta$)?" The only diffeomorphisms of $\mathbf{R}/\mathbf{Z}$ that preserve dt are the "translation plus shifts"

$$t \to \pm t + a. \tag{3.20}$$

Hence, if $\mathscr{Q}_\delta$ satisfies (1.9), $e^{i\kappa_T}$ is determined by the measure (3.19) up to such a mapping.

In other words, the band invariant (3.9) does not tell us (or, at least for the moment, does not appear to tell us) what the function $t + \dot{q}_\delta(t)$ is, but tells us simply that this function belongs to a collection of functions

$$g^\pm_{a,n} = g(\pm t + a) + 2\pi n, \qquad n \in \mathbf{Z},\ a \in [0, 1), \tag{3.21}$$

g being a function of the form "t plus an odd real-valued periodic function." On the other hand, however, it is easy to see that if g is of this form, it is the *only* function of this form in the collection (3.21) since the zero Fourier coefficient of an odd periodic function is zero. Thus, the band invariant does, in fact, determine $t+\dot{q}_\delta(t)$ (and hence Q_δ) unambiguously. This proves Theorem I.

Turning to the proof of Theorem II, the band invariant for V associated with the primitive element T in $\sigma(M)$ is the sum of two terms, a "plus term" coming from B_T^+ and a "minus term" coming from B_T^-. However, in view of Proposition (3.2) and the fact that $\sigma^* V = V$, these terms are both the same and are equal to $(e^{i\kappa_T})_* V_T\, dt$. If (1.9) holds, $e^{i\kappa_T}$ is a diffeomorphism of $\mathbf{R}/\mathbf{Z}$ onto S^1 (corollary to Proposition (3.2)) so this band invariant determines V_T. Therefore, to prove Theorem II, we only have to show that the V_T's determine V. Let $V = \sum b_\omega e^{2\pi i\langle\omega, x\rangle}$. Then by (3.15_t),

$$V_T(t) = \sum b_\omega e^{2\pi i\langle\omega, tw\rangle}\left(\int_0^1 e^{2\pi i\langle\omega, sv\rangle}\, ds\right).$$

As above, most of the integrals in parentheses vanish, and we are left with

$$V_T(t) = \sum b_{N\delta} e^{2\pi i N t}.$$

This shows that the Tth band invariant of V is essentially just $\mathscr{Q}_\delta(V)$, and hence, these band invariants clearly determine V.

Acknowledgments

The genesis of this article is some conversations the author had with Alex Uribe and Steve Zelditch at Oberwolfach two years ago in which they pointed out that if one thinks of ∇ and V as the data for an "electromagnetic field" on M, the set of all (∇, V)'s has an intrinsic symplectic structure. This suggested that the heat invariants of (1.2) might give rise to isospectral flows on this space. This paper is the result of a long and painful process of realizing that this is *not* the case (in the course of which comments by Uribe and Zelditch were extremely helpful).

The author also owes a debt of gratitude to the referee for his careful refereeing. (He not only caught a number of embarrassing gaffes but also suggested some substantial improvements in the formulation of Theorem I.)

The referee also called the author's attention to Eskin's paper [5]. This paper covers somewhat the same ground as is covered here. It also deals with the inverse spectral problem for periodic "vector potentials." However, in place of the assumption $c_1(L)[M] = \pm 1$, Eskin makes the assumption $c_1(L)[M] = 0$, which has the effect of making the vector potential problem much closer in spirit to the *scalar* potential problem than to the problem above.

Bibliography

1. J. J. Duistermaat and V. W. Guillemin, *The spectrum of positive elliptic operators and periodic bicharacteristics*, Invent. Math. **29** (1975), 39–79.
2. J. J. Duistermaat and L. Hörmander, *Fourier integral operators*. II, Acta Math. **128** (1972), 183–269.
3. G. Eskin, J. Ralston, and E. Trubowitz, *On isospectral periodic potentials in* $\mathbf{R}^n$, Comm. Pure Appl. Math. **37** (1984), 647–676.
4. ——, *On isospectral periodic potentials in* $\mathbf{R}^n$. II, Comm. Pure Appl. Math. **37** (1984), 715–753.
5. G. Eskin, *Inverse spectral problems for the Schroedinger equation with a periodic vector potential*, Comm. Math. Phys. **125** (1989), 263–300.

6. V. Guillemin, *Band asymptotics in two dimensions*, Adv. in Math. **42** (1981), 248–282.
7. V. Guillemin and A. Uribe, *Clustering theorems with twisted spectra*, Math. Ann. **273** (1986), 479–507.
8. ——, *Monodromy in the quantum spherical pendulum*, Comm. Math. Phys. **122** (1989), 563–574.
9. L. Hörmander, *The analysis of linear partial differential operators.* III and IV, Springer-Verlag, Berlin and New York, 1985.
10. ——, *Fourier integral operators.* I, Acta Math. **127** (1971), 79–183.
11. B. Kostant, *Quantization and unitary representations*, Lecture Notes in Math., vol. 170, Springer-Verlag, Berlin and New York, 1970, pp. 87–208.
12. F. Treves, *Introduction to pseudodifferential and Fourier integral operators*, vols. 1 and 2, Plenum Press, New York, 1980.
13. A. Weinstein, *Asymptotics of eigenvalue clusters for the Laplacian plus a potential*, Duke Math. J. **44** (1977), 883–892.

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