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LATTICES OF MINIMAL COVOLUME IN SL_2: A NONARCHIMEDEAN ANALOGUE OF SIEGEL'S THEOREM $\mu \geq \pi/21$

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A classical result of Siegel [Si] asserts that if $G = SL_2(\mathbf{R})$ and μ is its Haar measure (appropriately normalized) then $\min_\Gamma \mu(\Gamma\backslash G) = \pi/21$, where Γ runs over all the discrete subgroups of G. The minimum is obtained for the triangle group $(2, 3, 7)$ which is a cocompact lattice in G (see also [Gr]). For $G = SL_2(\mathbf{C})$, Meyerhoff [Me] has shown recently that among the nonuniform lattices in G (that is, discrete subgroups of finite covolume but not cocompact) the minimum volume is obtained with $\Gamma = SL_2(\mathscr{O}_3)$ where $\mathscr{O}_3$ is the ring of integers of $\mathbf{Q}(\sqrt{-3})$, and it is approximately 0.0863. (But, there are cocompact lattices with smaller covolumes and the minimum is not known, see [EGM, Th]).

In this paper, we address the analogue question for locally compact nonarchimedean fields. Our main result is: Let F be a locally compact field of characteristic $p > 0$ with residue field of order $q = p^\alpha$. (Hence F is isomorphic to $\mathbf{F}_q((1/t))$, the field of Laurent formal power series in $1/t$).

Theorem 1. *Let $G = SL_2(F)$ and μ be its Haar measure normalized so that $\mu(K) = 1$ for all maximal compact subgroups of G. Then*

$$\min\{\mu(\Gamma\backslash G) | \Gamma \text{ a lattice in} G\} = (q-1)^{-2}(q+1)^{-1}.$$

This minimum is obtained for $\Gamma_0 = SL_2(\mathbf{F}_q[t])$, the characteristic p modular group (as it is called by Weil [We]).

Theorem 1 is somewhat surprising in that the minimum is obtained for a nonuniform lattice. It should be interesting to understand what the typical situation is: the Siegel Theorem or our Theorem 1. Namely, for a semisimple Lie group over a local field, is a lattice of minimal covolume a cocompact or nonuniform lattice?

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For cocompact lattices we have the stronger lower bounds:

Theorem 2.

$$\min\{\mu(\Gamma\backslash G) | \Gamma \textit{ a cocompact lattice of } G\} = \begin{cases} \frac{1}{24} & \textit{if } q = 5, \\ \frac{1}{48} & \textit{if } q = 7, \\ \frac{1}{120} & \textit{if } q = 11, 19, 29, \\ \frac{1}{q+1} & \textit{if } q = 2^\alpha, \\ \frac{1}{2(q+1)} & \textit{otherwise.} \end{cases}$$

Recall that for a local nonarchimedean field F of characteristic 0, every lattice is cocompact (cf. [S2]). For these we have:

Theorem 3. *Let p be a prime, $\mathbf{Q}_p$ the field of p-adic numbers, $G = SL_2(\mathbf{Q}_p)$, and μ normalized to give maximal compact subgroups measure 1. Then*

$$\min\{\mu(\Gamma\backslash G) | \Gamma \textit{ a lattice in } G\} = \begin{cases} \frac{1}{12} & \textit{for } p = 2, \\ \frac{1}{24} & \textit{for } p = 3, 5, \\ \frac{1}{48} & \textit{for } p = 7, \\ \frac{1}{120} & \textit{for } p = 11, 19, 29, \\ \frac{1}{2(p+1)} & \textit{for all other } p. \end{cases}$$

For all cases of Theorems 2 and 3 we explicitly construct lattices with minimal covolume in §3. Section 2 is devoted to the proof of Theorem 1, and we end in §4 with some open problems.

1. Preliminaries and notation

In this section, we review some basic properties of $G = SL_2(F)$ and the tree associated with it. We also set some notation that will be used freely in the next sections.

Let F be a locally compact nonarchimedean field, with valuation ring D and a finite residue field $\mathbf{F}_q$. Hence if $\mathrm{char}(F) = 0$ then F is a finite extension of Q_p where $q = p^\alpha$, and if $\mathrm{char}(F) = p > 0$ then F is (isomorphic to) $\mathbf{F}_q((1/t))$, the field of Laurent formal power series over $\mathbf{F}_q$. Let π be a uniformizer of F (so we take $\pi = 1/t$ if $\mathrm{char}(F) > 0$). Set $G = SL_2(F)$, $K_1 = SL_2(D)$, and $K_2 = \left\{\left(\begin{smallmatrix} a & \pi b \\ \pi^{-1} c & d \end{smallmatrix}\right) | \left(\begin{smallmatrix} a & d \\ c & d \end{smallmatrix}\right) \in K_1\right\}$. Both K_1 and K_2 are maximal open compact subgroups of G and every compact subgroup of G is conjugate to a subgroup of either K_1 or K_2. Let $I = K_1 \cap K_2$ and be of index $q+1$ in both. Any Haar measure of G, therefore, gives the same volume to K_1 and K_2. We will normalize the Haar measure to be 1 on K_1 and K_2.

Let T be the Bruhat–Tits tree associated with G (cf. [S2, Chapter II]). The vertices $V(T)$ of T are the left cosets $G/K_1 \cup G/K_2$ and the edges are G/I where gI connects gK_1 to gK_2. We denote $v_1 = 1K_1$ and $v_2 = 1K_2$, so $d(v_1, v_2) = 1$ where d is the distance function on T. The cosets of K_1 (resp. K_2) will be called the *blue* (resp. *red*) vertices. A more concrete realization of T is given in [S2, Chapter II] via D-lattices in F^2. It is proved there that T is a $(q+1)$-regular tree on which G acts through its action on left cosets. There are two orbits of vertices: the blue ones and the red ones, while the action on the edges is transitive.

A *ray* in T (with origin in a vertex x) is an infinite path in T (beginning in x) and without backtracking. A *line* is an infinite path—infinite to both directions and without backtracking. Two rays are said to be equivalent if their intersetion is infinite. An equivalent class of rays is called an *end* of T. The set of ends ∂T is called the boundary of T. From the description of T in [S2] one can easily see that ∂T is identified with the projective line $\mathbf{P}^1(F)$. Given $x \in V(T)$ and $\varepsilon \in \partial T$, there is a unique ray *from* x *toward* ε, that is, a unique ray beginning in x and in the equivalent class of ε.

Now fix an end ε on the boundary. For two vertices $x, y \in V(T)$ let r_x and r_y be rays extending from them toward ε. Take $v \in r_x \cap r_y$. We say that x and y are in the same horosphere (with respect to ε) if $d(x, v) = d(y, v)$ (this is independent of the choice of v). We denote by $S(\varepsilon, x)$ the horosphere of x around ε. All vertices in $S(\varepsilon, x)$ have the same color as x.

We say that the horosphere $S(\varepsilon, x)$ is closure to ε than $S(\varepsilon, y)$ if $d(y, v) \geq d(x, v)$ for $v \in r_x \cap r_y$. Now choose some horosphere to be the zero horosphere denoted $S^0(\varepsilon)$. Let $x \in S^0(\varepsilon)$, $y \in V(T)$, and $v \in r_x \cap r_y$. Then the horosphere $S(\varepsilon, y)$ will be labeled $S^n(\varepsilon)$ if $d(x, v) = d(y, v) + n$. It is easy to see that the labeling is well defined and unique up to translation. The horosphere $S^n(\varepsilon)$ is closure to ε than $S^m(\varepsilon)$ iff $n \geq m$. Also, for every $n \in \mathbf{Z}$ there is a map $\varphi = \varphi_n \colon S^n(\varepsilon) \to S^{n+1}(\varepsilon)$ such that for $x \in S^n(\varepsilon)$, $\varphi(x)$ is the unique element of $S^{n+1}(\varepsilon)$ which is adjacent to x. The map φ can be regarded as a map from $V(T)$ to $V(T)$ and, therefore, we can also consider its powers. If $x, y \in S^n(\varepsilon)$ then for sufficiently large k, $\varphi^k(x) = \varphi^k(y)$. In particular, φ is not one to one. In fact, for every $x \in V(T)$, $\varphi^{-1}(x)$ is a set of cardinality q. The horoball $B(\varepsilon, x)$ is the union of all the horospheres that are closer to ε than $S(\varepsilon, x)$, that is, $B(\varepsilon, x) = \bigcup_{k\geq 0} S(\varepsilon, \varphi^k(x))$.

The group G acts on ∂T and after the identification with $\mathbf{P}^1(F)$ the action is by Möbius transformations, that is, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(\varepsilon) = (a\varepsilon + b)/(c\varepsilon + d)$. When G acts on a set X, $x \in X$, and H is a subgroup of G, we denote $H_x = \mathrm{Stab}_H(x) = \{h \in H | hx = x\}$. So when $\varepsilon \in \partial T$, G_ε is a Borel subgroup, for example, $\varepsilon = \infty$ and $G_\infty = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\}$. The map φ commutes with the action of G_ε. Also, G_ε acts on the horospheres. We denote

$$\begin{aligned} A_\varepsilon &= \{g \in G_\varepsilon | g(S^n(\varepsilon)) = S^n(\varepsilon) \text{ for every } n \in \mathbf{Z}\} \\ &= \{g \in G_\varepsilon | g(S^n(\varepsilon)) = S^n(\varepsilon) \text{ for some } n \in \mathbf{Z}\}. \end{aligned}$$

A_ε contains the unipotent radical U_ε of G_ε. Also, $G_\varepsilon = T_\varepsilon U_\varepsilon$ when T_ε is a torus, for example, when $\varepsilon = \infty$, $T_\varepsilon = \left\{\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) | a \in F^*\right\}$, $U_\varepsilon = \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) | b \in F\}$, and $A_\varepsilon = T_\varepsilon^1 U_\varepsilon$ when $T_\varepsilon^1 = \{\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) | a \in D^*\}$.

T_ε^1 is a compact group. The torsion subgroup of T_ε^1 is of order $q-1$. The group U_ε acts simply transitive on $\partial T - \{\varepsilon\}$, so every $u \in U_\varepsilon$ has a *unique* fixed point (ε) on the boundary. Moreover, $\partial T - \{\varepsilon\} = \varprojlim_{\varphi} S^n(\varepsilon)$ and hence U_ε acts transitively on every horosphere $S^n(\varepsilon)$.

If $\mathrm{char}(F) = 0$ then U_ε is a torsionfree group, but if $\mathrm{char}(F) = p$, then every element of U_ε is of order p. Moreover, every element of G of order p (or p-power) is in U_ε for some ε.

2. Proof of Theorem 1

The proof of the theorem will be split into a sequence of propositions and lemmas, some of them of independent interest. We keep the notation of §1, and assume here that $\mathrm{char}(F) = p > 0$.

If $x \in V(T)$ is a blue (resp. red) vertex then G_x is conjugate to K_1 (resp. K_2). Also $K_1 \simeq K_2 \simeq SL_2(D)$ so for every $x \in V(T)$ there is an epimorphism $G_x \to SL_2(q)$ whose kernel $N(x)$ is well defined. $N(x)$ is a pro-p group.

Definition 2.1. A nontrivial unipotent element $u \in G_x$ is called a *dominant unipotent at* x if it is in $N(x)$.

Lemma 2.2. *A unipotent element $u \in G_x$ is dominant if and only if $u(y) = y$ for every $y \in V(T)$ satisfying $d(y, x) = 1$.*

Proof. The set of $q + 1$ vertices adjacent to x can be identified with $\mathbf{P}^1(q)$, the projective line over $\mathbf{F}_q$. The action of G_x on it factors through the action of $SL_2(q)$ on $\mathbf{P}^1(q)$. $PSL_2(q)$ acts faithfully, so the set of elements of G_x acting trivially on the neighbors of x is a group N^1 containing $N(x)$ as a subgroup of index 2 if $(2, q) = 1$ and is equal to $N(x)$ if $q = 2^\alpha$. These observations suffice to prove the lemma.

Proposition 2.3. *Let H be a finite subgroup of G. Then*

(i) *$H \subseteq G_x$ for some (not necessarily unique) $x \in V(T)$.*

(ii) *The order of H is $p^\beta m$ when $(m, p) = 1$ and $m | q^2 - 1$.*

(iii) *If $|H| = p^\beta m$ and $p^\beta > p^\alpha = q$ then H contains a dominant unipotent with respect to x. In particular, if $|H| > q(q^2 - 1)$ then H contains a dominant unipotent.*

(iv) *If $H \subseteq G_x$ and contains a dominant unipotent, then there exists a unique end $\varepsilon \in \mathbf{P}^1(F)$ that is fixed by H, that is, $H \subseteq G_\varepsilon$.*

Proof. (i) is clear. (ii) and (iii) follow from the fact that $N(x)$ is a pro-p group and $|SL_2(q)| = q(q^2 - 1)$. To prove (iv): $H \cap N(x)$ is a nontrivial finite p-group so all its elements are unipotent. Each $u \in H \cap N(x)$ fixes some unique end, but as $Z(H \cap N(x)) \neq \{1\}$ (since every finite p-group has a nontrivial center) all have the same fixed end ε. Now, H normalizes $H \cap N(x)$ and hence fixes ε as well.

Remark. Proposition 2.3(iv) is not valid if H contains a unipotent but not a dominant unipotent. This can be seen by looking at $H = SL_2(q) \subseteq SL_2(D) = K_1 = G_{v_1}$. H has many unipotent elements with different fixed ends as well as some elements which fix no end.

Lemma 2.4. *Let $x \in V(T)$ and $u \in G_x$ a unipotent element fixing the end ε. Then*:

(i) *For every $y \in B(\varepsilon, x)$, $u(y) = y$.*
(ii) *If u is dominant w.r.t. x then it is dominant w.r.t. every $y \in B(\varepsilon, x)$.*

Proof. (i) Let $S(\varepsilon, x)$ be the horosphere of x. Every $y \in S(\varepsilon, x)$ can be written as $y = vx$ for some $v \in U_\varepsilon$. As U_ε is abelian, we have $u(y) = u(vx) = v(ux) = vx = y$. Now if $z \in B(\varepsilon, x)$ then $z = \varphi^k(y)$ for some $k \in \mathbf{N}$ and $y \in S(\varepsilon, x)$. Hence $u(z) = u(\varphi^k(y)) = \varphi^k(u(y)) = \varphi^k(y) = z$. (ii) If u is a dominant unipotent and $S(\varepsilon, x) = S^l(\varepsilon)$ for some l, then from Lemma 2.2 and Lemma 2.4(i) it follows that u fixes every vertex of $S^{l-1}(\varepsilon)$. Hence u is dominant for every $y \in S^k(\varepsilon)$ for every $k \geq l$.

Proposition 2.5. *Let Γ be a discrete subgroup of G and $u \in \Gamma_x$ a dominant unipotent w.r.t. x fixing an end ε. Then for $y, z \in B(\varepsilon, x)$, if $\gamma \in \Gamma$ satisfies $\gamma(y) = z$ then $\gamma \in \Gamma_\varepsilon$.*

Proof. Look at $d(\gamma^{-1}u\gamma y, y) = d(u\gamma y, \gamma y) = d(uz, z) = 0$, whence $\gamma^{-1}u\gamma$ is in Γ_y. Γ_y is a finite group (since $\Gamma_y = \Gamma \cap G_y$, Γ is discrete and G_y is compact) and contains the dominant unipotent u. u is a dominant unipotent at y as well by 2.4(ii). By 2.3(iv), $\gamma^{-1}u\gamma$ also fixes ε which forces $\gamma \in \Gamma_\varepsilon$.

Before continuing, we will make some careful remarks about calculating $\mu(\Gamma\backslash G)$ for Γ discrete.

In general, if K is an open compact subgroup of G, and S is a system of double coset representatives for $\Gamma\backslash G/K$, and for each $s \in S$ let $g(s)$ be the order of the finite group $\Gamma_s = \Gamma \cap sKs^{-1}$, then $\mu(\Gamma\backslash G) = \mu(K)\sum_{s\in S} 1/g(s)$ (cf. [S2, II 1.5, p. 84]). Now, in our case there are two natural possibilities for K, either K_1 or K_2. The double cosets $\Gamma\backslash G/K_1$ (resp. $\Gamma\backslash G/K_2$) are exactly the blue (resp. red) orbits of Γ on T. So let $Y = \Gamma\backslash T$ and E be a fundamental domain for Γ, that is, E is a subtree of T whose vertices are mapped bijectively to those of Y. Denote by $V_b(E)$ (resp. $V_r(E)$), the blue (resp. red) vertices of E. Then

$$\mu(\Gamma\backslash G) = \sum_{x\in V_b(E)} \frac{1}{|\Gamma_x|} = \sum_{x\in V_r(E)} \frac{1}{|\Gamma_x|}. \tag{2.6}$$

Example 2.7. Let $\overline{\Gamma} = SL_2(\mathbf{F}_q[t]) \subseteq G = SL_2(\mathbf{F}_q((1/t)))$. Then $\overline{\Gamma}$ is a nonuniform lattice in G with a fundamental domain, a ray $x_0, x_1, x_2, \dots$, where $\{x_{2n} | n \in \mathbf{N}\}$ are blue and $\{x_{2n+1} | n \in \mathbf{N}\}$ are red. Also, $|\overline{\Gamma}_{x_0}| = q(q^2 - 1)$ while $|\overline{\Gamma}_{x_n}| = (q-1)q^{n+1}$ for $n \geq 1$. Hence $\mu(\overline{\Gamma}\backslash G) = 1/((q-1)(q^2-1))$ (see [S2, II.1.6, Exercise 6, p. 89] but notice the difference between our normalization and the normalization there).

Most of the rest of this section is devoted to proving that for any other lattice of G, $\mu(\Gamma\backslash G) \geq \mu(\overline{\Gamma}\backslash G)$. Cocompact lattices can easily be eliminated using the following easy lemma (in §3 we will give much better results on cocompact lattices).

Lemma 2.8. *Let Γ be a cocompact lattice in G. Then $\mu(\Gamma\backslash G) \geq 1/(q^2-1) \geq \mu(\overline{\Gamma}\backslash G)$.*

Proof. The fundamental domain for Γ contains at least one blue vertex x (and one red vertex). Γ_x does not contain elements of order p since cocompact lattices do not contain unipotent elements (cf. [GGPS, p. 10]) hence $(|\Gamma_x|, p) = 1$, and by Lemma 2.3(ii) it is of order at most q^2-1. Hence $\mu(\Gamma\backslash G) \geq 1/(q^2-1)$.

From now on we can assume that Γ is a nonuniform lattice, that is, of finite covolume but not cocompact. Therefore if E is a fundamental domain for Γ on T, then E is infinite but

$$\sum_{x\in E} \frac{1}{|\Gamma_x|} = 2\mu(\Gamma\backslash G) < \infty.$$

Definition 2.9. An end $\varepsilon \in \partial T$ is called a *cusp* of Γ if for some $x \in V(T)$, Γ_x contains a dominant unipotent fixing ε.

Since Γ is nonuniform, there is $x \in V(T)$ with $|\Gamma_x| > q(q^2-1)$. From Lemma 2.3(iv) we deduce that Γ has a cusp, call it ε.

Lemma 2.10. *Assume $x \in V(T)$ and Γ is a nonuniform lattice in G containing a dominant unipotent w.r.t. x fixing an end ε (so ε is a cusp for Γ). Then*

(i) $|\Gamma_\varepsilon\backslash S(\varepsilon, x)| < \infty$.

(ii) *For some* $k \in \mathbf{N}$, $|\Gamma_\varepsilon\backslash S(\varepsilon, \varphi^k(x))| = 1$.

Proof. (i) First we note that from Proposition 2.5 it follows that $\Gamma_\varepsilon\backslash S(\varepsilon, x)$ is embedded in $Y = \Gamma\backslash X$. We will prove that if $z \in S(\varepsilon, x)$ then Γ_z is bounded. So, if there are infinitely many Γ_ε orbits in $S(\varepsilon, x)$ we get in Y infinitely many vertices with bounded stabilizers. This would contradict the finiteness of the volume of Y. To prove that Γ_z is bounded: First recall from Lemma 2.4(ii) and Proposition 2.3(iv) that $\Gamma_z \subseteq G_\varepsilon$. Secondly, Γ_z is a finite subgroup of $G_\varepsilon = T_\varepsilon U_\varepsilon$. The only torsion in T_ε is of order $q-1$ so $[\Gamma_z : \Gamma_z \cap U_\varepsilon] \leq q-1$. Moreover, there exists $v \in U_\varepsilon$ such that $z = vx$ and hence $\Gamma_z = \{\gamma \in \Gamma | \gamma v x = vx\} = \{\gamma \in \Gamma | v^{-1}\gamma v x = x\}$. Hence, $v\Gamma_z v^{-1} \in G_x$; but because $\Gamma_z \cap U_\varepsilon$ commutes with v, $\Gamma_z \cap U_\varepsilon \subseteq \Gamma_x$. This proves that $|\Gamma_z| \leq (q-1)|\Gamma_x|$ and (i) is proven.

To prove (ii): Let $y_1, \dots, y_r$ be representatives for $\Gamma_\varepsilon\backslash S(\varepsilon, x)$. Then for some $k \in \mathbf{N}$, $\varphi^k(y_1) = \varphi^k(y_2) = \cdots = \varphi^k(y_r)$. Since φ commutes with Γ_ε, it follows that $|\Gamma_\varepsilon\backslash S(\varepsilon, \varphi^k(x))| = 1$.

Now let ε be a cusp of Γ and $x \in V(T)$ such that Γ_x contains a dominant unipotent fixing ε. Assume $x \in S^n(\varepsilon)$. We claim that for some m, for no $y \in S^{n-m}(\varepsilon)$, Γ_y contains a dominant unipotent fixing ε. Otherwise, if y contains such an element we can assume $\varphi^m(y) = x$. Hence $\Gamma_y \subseteq \Gamma_x$. From Proposition 2.5 it follows that the ray from y toward ε injects into $Y = \Gamma\backslash T$. This implies $m/|\Gamma_x| < \mu(\Gamma\backslash G)$. Hence m must be bounded.

Now let

$$l = \min\{k \in \mathbf{Z} | \exists y \in S^k(\varepsilon) \textit{ such that } \Gamma_y \textit{ contains a dominant unipotent fixing } \varepsilon\}.$$

The ball $B^l(\varepsilon) = \bigcup_{k\geq l} S^k(\varepsilon)$ will be called the *ε-tritory* w.r.t. Γ. By Proposition 2.5, $\Gamma_\varepsilon\backslash B^l(\varepsilon)$ injects into $Y = \Gamma\backslash T$. For future use, we remark here that for two different cusps of Γ the corresponding tritories are disjoint.

We are now going to analyze the contribution of $\Gamma_\varepsilon\backslash B^l(\varepsilon)$ to the volume of Y.

Lemma 2.11. *If $x \in S^l(\varepsilon)$, then $|\Gamma_x| \leq (q-1)q^2$.*

Proof. First note that $\Gamma_x \subseteq G_\varepsilon$, hence it fixes $\varphi(x)$. Let $J = \varphi^{-2}(x)$, then $|J| = q^2$ and the abelian p-sylow subgroup P of Γ_x acts on J. If $|P| > q^2$ then some nontrivial element $u \in P$ fixes some $y_0 \in J$ and by 2.4(i) it fixes every $y \in J$. Thus u is a dominant unipotent in Γ_z for every $z \in \varphi^{-1}(x) = \varphi(J)$. This contradicts the minimality of l. Now Γ_x/P is a finite subgroup of the torus $T \simeq F^*$, whence its order is at most $q-1$. This prove the lemma.

Now fix $x \in S^l(\varepsilon)$ and let $r = \{x = x_1, x_2, x_3, \ldots\}$ be the ray from x toward ε. By Proposition 2.5, r injects into Y. We have just proved that $\Gamma_{x_1} \leq (q-1)q^2$. We will use it to continue by induction to prove that $\Gamma_{x_n} \leq (q-1)q^{n+1}$. Indeed, assume this is true for $n-1$. Now, $\Gamma_{x_{n-1}} \subseteq \Gamma_{x_n}$ (since $\Gamma_{x_{n-1}} \subseteq \Gamma_\varepsilon$) and the index is at most q (since Γ_{x_n} fixes x_{n+1} and acts on the set of q elements $\varphi^{-1}(x_n)$). Hence $\Gamma_{x_n} \leq (q-1)q^{n+1}$.

The ray r injects into Y, as previously mentioned. Assume that x_1 is a blue (resp. red) vertex and compute the volume of Y (or equivalently of a fundamental domain containing this ray) along the blue (resp. red) vertices. This volume is at least $\sum_{n \text{ odd}} 1/((q-1)(q^{n+1})) = 1/((q-1)(q^2-1))$. Thus $\mu(\Gamma|G) \geq 1/((q-1)(q^2-1)) = \mu(\overline{\Gamma}|G)$ and the proof of Theorem 1 is complete.

Some additional corollaries can be deduced from the proof.

Corollary 2.12. *If Γ is a nonuniform lattice in G with c equivalent classes of cusps then*

$$\mu(\Gamma\backslash G) \geq \frac{c}{2}\frac{1}{(q-1)^2 q}.$$

Proof. We remarked in the proof that the tritories of the cusps are disjoint. We have to remark, however, that *not* every (equivalent class of a) cusp contributes the amount of $1/((q^2-1)(q-1))$: some of them are "blue cusps" and some are "red cusps," that is, for some cusps ε, $S^l(\varepsilon)$ (where l is like in the proof) is a blue horosphere and for others it might be red. We can instead sum over all vertices blue and red: so, $2\mu(\Gamma\backslash G) \geq c\,.\ \sum_{n=1}^{\infty} 1/((q-1)q^{n+1})$. This finishes the proof of the corollary.

The next corollary is a special case of a result that is true for every nonuniform lattice in any rank one semisimple group over F (see [R., Lu]). We have already proved almost all that is necessary for our G; we introduce this corollary from which we will deduce another result on covolumes of lattices.

Corollary 2.13. *Let Γ be a nonuniform lattice in G. Then $Y=\Gamma\backslash T$ is a graph built up from a finite set of vertices plus finitely many infinite rays.*

Proof. We show in the proof (Lemma 2.10(ii)), that for every cusp ε the image of some horoball around ε in Y is just a ray. From the previous corollary, it follows that there are only finitely many equivalent classes of cusps. It remains only to show that if E is a fundamental domain for Γ, then any ray in E is going toward a cusp of Γ. Let $r=\{x_1, x_2, \dots\}$ be such a ray in E going toward ε in E. Since $\sum_{i=1}^{\infty} 1/|\Gamma_{x_i}| < \infty$, it follows that, for almost all i, $|\Gamma_{x_i}| > (q+1)!$ So by slightly changing the ray, we can assume this for every i. This implies: (i) Γ_{x_i} contains a dominant unipotent and so Γ_{x_i} fixes an end ε_i; (ii) Γ_{x_i} and $\Gamma_{x_{i+1}}$ have a nontrivial intersection whence $\varepsilon_i=\varepsilon_{i+1}$ and by induction they all fix the same end ε'.

We claim that $\varepsilon=\varepsilon'$. If not, the ray r' from x_1 to ε' satisfies $r\cap r' = \{x_1, \dots, x_N\}$. Hence for $i\geq N$, as Γ_{x_i} fixes ε', Γ_{x_i} is a subgroup of Γ_{x_N}. This is impossible since the orders of Γ_{x_i} are unbounded. This proves that the only way to go to infinity in E is toward cusps of Γ and the corollary is proved.

Corollary 2.14. *For every lattice Γ in G, $\mu(\Gamma\backslash G)$ is a rational number.*

Proof. This is clear for cocompact lattices. For nonuniform lattices we claim that the contribution of a ray going toward a cusp must be a sum of geometric progression. Indeed, if $x_1, x_2, \dots$ is such a ray then for sufficiently large i, $\Gamma_{x_i}\subseteq\Gamma_{x_{i+1}}$ and the only neighbors of x_i in Y are x_{i-1} and x_{i+1}. By [S2, I.5.3] we must have: $[\Gamma_{x_i}:\Gamma_{x_{i-1}}]+1=q+1$. So $[\Gamma_{x_i}:\Gamma_{x_{i-1}}]=q$. Hence the contribution of every cusp is rational and the rest follows from (2.13).

It might be interesting to say more on the set of numbers which appears as covolumes of lattices in G. In the case of $G=SL_2(\mathbf{C})$ a very interesting situation occurs (see [Th, Chapter 6]). On the other hand, H. Bass showed that for $G=\mathrm{Aut}(X_k)$, X_k the k-regular tree ($k\geq 3$), every positive real number is the covolume of some lattice in G.

3. COCOMPACT LATTICES

Again let F be a local nonarchimedean field of characteristic $l \geq 0$, with a residue field of order $q = p^\alpha$ (where $p = l$ if $l > 0$). We begin with a Lemma that will enable us later to construct cocompact lattices in $G = SL_2(F)$. We continue the previous notation. In particular, T is the tree associated with G.

Lemma 3.1. *Let x_1 and x_2 be two adjacent vertices of T and A_i $(i = 1, 2)$ a finite subgroup of G_{x_i} such that*

(a) *A_i acts transitively on the set of $q+1$ neighbors of x_i,*
(b) $\mathrm{Stab}_{A_i}(x_{3-i}) = A_1 \cap A_2$.

Then $\Gamma = \langle A_1, A_2 \rangle$ the group generated by A_1 and A_2 is a cocompact lattice in G with a fundamental domain on T consisting of two vertices x_1 and x_2. Moreover, Γ is isomorphic to the free product with amalgamation $\Gamma \simeq A_1 *_{A_1 \cap A_2} A_2$ and $\Gamma_{x_i} = A_i$.

Proof. Let Δ be the abstract free product with amalgamation $\Delta = A_1 *_{A_1 \cap A_2} A_2$ and φ the obvious homomorphism from Δ to Γ. Let $a_1, \dots, a_r$ (resp. $b_1, \dots, b_s$) be representative of the nontrivial cosets of $A_1 \cap A_2$ in A_1 (resp. A_2). In fact, $r = s = q$ but we do not need it now. A word $w \in \Delta$ has a normal form $w = a_{i_1} b_{j_1} \cdots a_{i_t} b_{j_t} c$ where $c \in A_1 \cap A_2$ (and with the possibility that $a_{i_1} = 1$ or $b_{i_t} = 1$).

Induction on t shows that if e is in the edge between x_1 and x_2 then $d(we, e) \geq t$ (when distance between edges is defined in the obvious way). In particular $w(e) \neq e$ unless $w \in A_1 \cap A_2$. This proves that Γ is isomorphic to Δ and that $\Gamma_{x_i} = A_i$. Hence Γ is also discrete. Moreover, from general nonsense, it follows that the set $E = \{e' | e' \text{ an edge of } T \text{ and } d(e', e) \leq d(\gamma e', e),\ \forall s \in \Gamma\}$ forms the set of edges of a connected subtree which contains a representative from every orbit of edges of T. But E contains e and none of the edges adjacent to e (by condition (a)). Hence $E = \{e\}$ and so Γ acts transitively on the edges of T. This finishes the proof of the lemma.

Definition 3.2. Let $P = \{q | q \text{ is a prime power in } N\}$. Define the function $\lambda: P \to \mathbf{Q}$ by

(i) $\lambda(q) = 1/(2(q+1))$ if $(q, 2) = 1$ and $q \neq 5, 7, 11, 19, 29$,
(ii) $\lambda(q) = 1/(q+1)$ if $q = 2^\alpha$,
(iii) $\lambda(q) = 1/120$ for $q = 11, 19, 29$,
(iv) $\lambda(q) = 1/48$ for $q = 7$,
(v) $\lambda(q) = 1/24$ for $q = 5$.

Theorem 3.3. *Let F be a local nonarchimedean field with a residue field of order $q = p^\alpha$. Then,*

(i) *$G = SL_2(F)$ has a cocompact lattice of covolume $\lambda(q)$.*
(ii) *If $\mathrm{char}(F) = p = l > 0$ then every cocompact lattice in G has covolume greater-equal $\lambda(q)$.*

For the proof of the theorem, we need the following result (see [La, Theorem 2.3, p. 185], [F2, Theorem 1, p. 201] and [S4, p. 281]).

Proposition 3.4. *Let F be a field of characteristic l and S a finite subgroup of $SL_2(F)$ of order prime to l. Let H be the image of S in $PSL_2(F)$. Then*

(a) *One of the following cases occurs:*

  (i) *H is cyclic and S is contained in a Cartan subgroup;*
  (ii) *H is dihedral and S is contained in the normalizer of a Cartan subgroup but not in the Cartan subgroup itself; or*
  (iii) *H is isomorphic to A_4, S_4, or A_5.*

(b)

  (i) *If $\mathrm{char}(F) > 2$ then $PSL_2(F)$ contains a subgroup isomorphic to A_4.*
  (ii) *If $F = \mathbf{F}_q$ and $\mathrm{char}(F) \neq 2, 3$ then $PSL_2(F)$ contains a subgroup isomorphic to S_4 if and only if $q \equiv \pm 1 \pmod 8$.*
  (iii) *If $F = \mathbf{F}_q$ and $\mathrm{char}(F) \neq 2, 3, 5$ then $PSL_2(F)$ contains a subgroup isomorphic to A_5 if and only if $q \equiv \pm 1 \pmod 5$.*

The reader is referred to the above references for the definition of split and nonsplit Cartan subgroups.

Lemma 3.5. *For every $q \in \mathbf{P}$, $SL_2(q)$ has a subgroup M_q of order $\lambda(q)^{-1}$ that acts transitively on the projective line $\mathbf{P}^1(q)$.*

Proof. If $q \neq 2^\alpha$, 5, 7, 11, 19, 29 take M_q to be the normalizer of a nonsplit Cartan subgroup. This is a group of order $2(q+1)$ whose intersection with "the" Borel subgroup B_q of $SL_2(q)$ is $\pm I$. Since B_q is the stabilizer of $\infty \in \mathbf{P}^1(q)$, this shows that the action of M_q is transitive. Similarly, for $q = 2^\alpha$ we take as M_q a nonsplit Cartan subgroup. It is of order $q+1$ and has trivial intersection with B_q, and hence acts transitively on $\mathbf{P}^1(q)$.

Now if $q = 11, 19, 29$, then $PSL_2(q)$ contains A_5 by (3.4). Take M_q to be the preimage of A_5 in $SL_2(q)$. So $|M_q| = 120$ and $|B_q| = (q-1)q$. Hence $(|M_q|, |B_q|) = 10, 6, 4$ if $q = 11, 19, 29$ respectively. So $|M_q \cap B_q| \leq 10, 6, 4$ respectively. But this bound is exactly equal to $120/(q+1)$ in all these three cases which proves that the action of A_5 on $\mathbf{P}^1(q)$ is transitive in these cases.

Similarly, for $q = 7$, take M_q to a preimage of S_4 (which exists by (3.4)), so $|M_q| = 48$. As $(48, (q-1)q) = 6 = 48/(q+1)$, M_q acts transitively on $\mathbf{P}^1(q)$. Finally, for $q = 5$, let M_q be a preimage of A_4. So $|M_q| = 24$, $(24, (q-1)q) = 4 = 24/(q+1)$ and the lemma is proven.

Incidentally, we note that in all cases, we took M_q to contain $\pm I = \mathrm{Ker}(SL_2(q) \to PSL_2(q))$. Of course if $q = 2^\alpha$ then $-I = I$.

Proof of Theorem 3.3. Let $K_1 = G_{v_1}$ and $K_2 = G_{v_2}$ as in §1. Then $K_1 \simeq K_2 \simeq SL_2(D)$ and the projection $\varphi\colon K_i \to SL_2(q)$ has a kernel that is a pro-p

group. If M is a subgroup of $SL_2(q)$ of order prime to p, then by the Schur–Zassenhaus Theorem [Su, p. 235] applied to the pro-finite group $\varphi^{-1}(M)$, M can be lifted to K_i. (If $\mathrm{char}(F) > 0$, $SL_2(q)$ is anyway a subgroup of K_i). In particular, take the M_q presented in (3.5). Because $(\lambda(q)^{-1}, q) = 1$, M_q can be considered as a subgroup of K_1. Moreover, it has an intersection with K_2 which is of order equal to $M_q \cap B_q$. This is because the set of neighbors of x_1 can be identified with $\mathbf{P}^1(q)$ and so the stabilizer of x_2 in M_q is like the stabilizer of a point of $\mathbf{P}^1(q)$. A case by case checking shows that a group M_q' can be chosen in K_2 of order $\lambda(q)^{-1}$, acting transitively on the neighbors of x_2 and such that $M_q' \cap K_1 = M_q \cap K_2 = M_q' \cap M_q$. (Only the last property needs some verification which is left to the reader.)

We can now take $\Gamma = \langle M_q, M_q' \rangle$. Lemma 3.1 shows that Γ is a cocompact lattice of covolume $|M_q|^{-1} = \lambda(q)$ and part (i) of the theorem is proven.

To prove part (ii) we recall again that a cocompact lattice does not contain unipotent elements ([GGPS, p. 10]) and that an element of order p in G is unipotent. Hence if E is a fundamental domain of a cocompact lattice Γ and x is a blue vertex in E then $\mathrm{Vol}(\Gamma\backslash G) \geq 1/\Gamma_x$ and Γ_x is a subgroup of $K_1 \simeq SL_2(D)$ of order prime to p, whence mapped injectively into $SL_2(q)$. By (3.4a) the maximal subgroups of $SL_2(q)$ of order prime to p are of orders $2(q+1)$ (normalizers of nonsplit Cartan subgroups), $2(q-1)$ (normalizers of split Cartan subgroups), $120, 48$ *and* 24 (pre-images of A_5, S_4, and A_4 respectively). This is for odd p, for $p = 2$ the possibilities are $q+1$ or $q-1$. Hence in all cases if $q \geq 59$, $|\Gamma_x| \leq \lambda(q)^{-1}$ and we are done.

If $30 < q < 59$, then a lattice Γ can have a covolume $< \lambda(q)$ only if a fundamental domain E for Γ has only one blue vertex and one red vertex (that is, Γ is transitive on edges) and $|\Gamma_x| = 120$ for every $x \in E$. But in such a case, if $E = \{x_1, x_2\}$ is a fundamental domain and e the edge between x_1 and x_2 then $[\Gamma_{x_i} : \Gamma_e]$ should be equal to $q+1$ for $i = 1, 2$ (cf. [S2, I.5.3]). This is impossible since Γ_e contains $\pm I$ and A_5 does not have a subgroup of index $q+1$ for $30 < q < 59$.

Hence we can assume $q < 29$. If q is divisible by 2 or 3, Γ_x cannot be of the type described in (3.4a(iii)), and for $q = 11, 19, 29$ we might have $|\Gamma_x| = 120$ and indeed we constructed a lattice of covolume $1/120$. This leaves us to consider $q = 5, 7, 13, 17, 23, 25$. If $q = 25$ then A_5 is impossible as Γ_x and so $|\Gamma_x| \leq 2(q+1)$ and again we are done. For $q = 23$, $PSL_2(q)$ does not contain A_5 by (3.4b(iii)) and so again $|\Gamma_x| \leq 2(q+1)$. Similarly for $q = 13$, $PSL_2(q)$ contains neither A_5 nor S_4 and hence $|\Gamma_x| \leq 2(q+1)$. For $q = 17$ an additional argument is needed: Indeed $PSL_2(q)$ contains S_4 but for having a lattice of covolume less than $1/(2(q+1))$ we need that such Γ would act transitively on edges and S_4 should have a subgroup of index 18 which is impossible.

If $q = 7$, $PSL_2(q)$ contains S_4 and indeed we have a lattice of covolume

$1/48$. Since any other subgroup of order prime to 7 is of order less than 48, this is the minimal possible covolume. Similarly, for $q = 5$ and A_4 and now the theorem is completely proven.

We turn our attention now to the situation when $\mathrm{char}(F) = 0$. Here the list of finite subgroups of $SL_2(F)$ is also very limited. Still, for general local field F, $SL_2(F)$ might contain a large finite cyclic p-group of size larger than $2(q+1)$ (see [F1, 3.5]). We do not know if typically $1/(2(q+1))$ is the minimal covolume since these p-groups (which are *not* unipotent) can in principle appear in cocompact lattices. It is not even clear to us that the minimal covolume is obtained with lattices whose fundamental domain has only two vertices. (Note that this was the situation in all cases when $\mathrm{char}(F) > 0$, but we deduce it only using case by case analysis. It will be interesting to understand it from a more conceptual point of view. The theory developed in [Ba, BK] might be useful for that purpose.)

Anyway for the most interesting fields of characteristic 0, that is $F = \mathbf{Q}_p$ the p-adic numbers, we can give a definite result:

Theorem 3.6. *Let $F = \mathbf{Q}_p$ the field of p-adic numbers, and $G = SL_2(F)$. Then*

$$\min\{\mu(\Gamma\backslash G)|\Gamma \text{ a lattice in } G\} = \begin{cases} \lambda(p) & \text{if } p \neq 2, 3, \\ \frac{1}{24} & \text{if } p = 3, \\ \frac{1}{12} & \text{if } p = 2. \end{cases}$$

Proof. Every lattice in G is cocompact by a result of Tamagawa (cf. [S2, p. 84]).

If $p \neq 2, 3$ we claim that $SL_2(\mathbf{Q}_p)$ does not contain elements of order p. Indeed, if α is such an element then its eigenvalues β and β^{-1} satisfy $\beta^p = 1$. But as α is a 2×2 matrix, the irreducible polynomial of β over $\mathbf{Q}_p$ is of degree ≤ 2. Just like over the rationals, the Eisenstein criterion implies that this is impossible for $p \neq 2, 3$.

Thus, every finite subgroup H of $SL_2(\mathbf{Q}_p)$ is of order prime to p and it is also isomorphic to a subgroup of $SL_2(p)$. So all the discussion in the proof of Theorem 3.3(ii) is valid also in this case and so volume $(\Gamma\backslash G) \geq \lambda(p)$. By 3.3(i), lattices with covolume $\lambda(p)$ do exist and the case $p \neq 2, 3$ is done.

The case $p = 3$: We claim that a finite subgroup S of $SL_2(\mathbf{Q}_p)$ is of order at most 24. Indeed such a subgroup can be assumend (by (2.3)(i)) to be in $SL_2(\mathbf{Z}_3)$. Let $N = \mathrm{Ker}(SL_2(\mathbf{Z}_3) \to SL_2(\mathbf{F}_3))$. Then N is a pro-3 group. If A is a matrix in N different from the identity and satisfying $A^3 = 1$, then its eigenvalues are $\{\lambda, \lambda^{-1}\}$ the set of two nontrivial cubic roots of the identity. Hence $\mathrm{tr}(A) = -1$ and $\det(A) = 1$. Thus $A^2 + A + 1 = 0$. On the other hand $A \equiv 1 \pmod 3$ and so

$$9|(A-1)^2 = A^2 - 2A + 1 = A^2 - 2A + 1 - (A^2 + A + 1) = -3A.$$

Hence $3|A$ which is a contradiction. This proves that N is torsionfree and so S must inject into $SL_2(\mathbb{F}_3)$. The order of the later is 24 whence $|S| \leq 24$.

Now we claim that $SL_2(\mathbf{Q}_3)$ does have a subgroup of order 24. (Note that $SL_2(\mathbf{Q})$ does not have such a subgroup!) Here is a way to see it:

Let H be the Hamiltonian quaternion algebra. Then H splits over all odd primes and hence $H(\mathbf{Q}_p) \simeq M_2(\mathbf{Q}_p)$. The isomorphism takes the group $H'(\mathbf{Q}_p)$ of elements of norm one in $H(\mathbf{Q}_p)$ to $SL_2(\mathbf{Q}_p)$. Now, $H'(\mathbf{Q})$ has a subgroup of order 24, the group $\{\pm 1, \pm i, \pm j, \pm k\}\cdot\langle\frac{1}{2}(1+i+j+k)\rangle$ and hence $SL_2(\mathbf{Q}_p)$ does as well. (A more conceptual way to prove such a result is to use the theory of Schur indices and to apply it to the representations of $SL_2(\mathbf{F}_3)$ over $\mathbf{Q}_3$—see [F3] or [S3].)

The above discussion shows that the map $SL_2(\mathbf{Z}_3) \to SL_2(\mathbf{F}_3)$ splits and so $G_{x_1} = SL_2(\mathbf{Z}_3)$ has a subgroup of order 24 which acts transitively on the neighbors of x_1 and we can use, as before, Lemma 3.1 to construct a lattice of volume $1/24$ in $SL_2(\mathbf{Q}_3)$.

The case $p = 2$: Denote $N = \operatorname{Ker}(SL_2(\mathbf{Z}_2) \to SL_2(\mathbf{F}_2))$. This time N has torsion $\{\pm I\}$. We claim that this is the only torsion. Indeed if $A \in N$ is an element of order two then $A^2 = 1$ and so the eigenvalues are $\{\pm 1\}$. Since $\det(A) = 1$, A must be either I or $-I$. Moreover, if $A \in N$ is of order 4, and $\det A = 1$ we know that the eigenvalues of A must be $\{\pm i\}$ where $i = \sqrt{-1}$ and so $\operatorname{tr} A = 0$. Hence A satisfies the characteristic polynomial $A^2 + 1 = 0$. Since $A \equiv I \pmod 2$, we have $4|(A-1)^2 = A^2 - 2A + 1 = -2A$; hence $2|A$, a contradiction. The group N is a pro-2 group, so every finite subgroup of it is a 2-group. The above argument shows that a finite subgroup of N is of exponent 2 and hence it is a subgroup of $\{\pm I\}$. This together with the fact that $|PSL_2(\mathbf{F}_2)| = 6$, shows that a finite subgroup of $SL_2(\mathbf{Z}_2)$ is of order at most 12.

There is in $SL_2(\mathbf{Z}_2)$ a subgroup of order 12. (Again, this is not the case for $SL_2(\mathbf{Z})$ or $SL_2(\mathbf{Q})$!) A quick way to see it: $\mathbf{Z}_2$ contains a cubic root of unity w (by Hensel lemma). The group generated by $-I$, $\left(\begin{smallmatrix} w & 0 \\ 0 & w^2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ is of order 12. This group is mapped onto $SL_2(\mathbb{F}_2)$ and again one can use (3.1) to build a lattice of covolume $1/12$, which is the minimum possible.

Remark. For every odd p, we have an arithmetic lattice in $SL_2(\mathbf{Q}_p)$ constructed using the Hamiltonian quaternion algebra (see [GVP, §IX.1]). This lattice has covolume $1/24$ so for $p = 3, 5$ this is of minimal possible covolume.

We end with a corollary: Hurwitz Theorem asserts that if S_g is a Riemann surface of genus g then $|\operatorname{Aut}(S)| \le 84(g-1)$. This is equivalent to: Let Γ be a torsionfree cocompact subgroup of $PSL_2(\mathbf{R})$ and assume Γ is isomorphic to $\pi_1(S_g)$ (note that every such Γ satisfies this assumption for some $g \ge 2$) then $[N_G(\Gamma)\colon \Gamma] \le 84(g-1)$. Hurwitz theorem can be deduced from Siegel's theorem (cf. [Gr]), and similarly we have:

Theorem 3.7. *Let F be a local nonarchimedean field with residue field of order q and $M_F = \min\{\operatorname{Vol}(\Gamma\backslash SL_2(F)) | \Gamma$ a cocompact lattice in $SL_2(F)\}$.*

Let Γ be a torsion free cocompact lattice in $SL_2(F)$. Then:
(i) *Γ is a free group on say g generators; and*
(ii) $[N_G(\Gamma):\Gamma] \leq (g-1)/(M_F(q-1))$.

Note that we have computed M_F for all fields of positive characteristic and for $F = \mathbf{Q}_p$. For $q \geq 60$ and odd we had $M_F = 1/(2(q+1))$ and hence $[N_G(\Gamma):\Gamma] \leq 2(q+1)(g-1)/(q-1)$.

Proof. (i) is due to Ihara (cf. [S2, p. 82]). To prove (ii) note that $\Gamma\backslash T$ is a regular $(q+1)$-graph so if it has n vertices it has $\frac{1}{2}(q+1)n$ edges. The fundamental group of $\Gamma\backslash T$ is free on $g = \frac{1}{2}(q+1)n - (n-1) = \frac{1}{2}(q-1)n + 1$ generators. Hence, $n = 2(g-1)/(q-1)$. Half of them are blue and half are red, so $\mathrm{Vol}(\Gamma\backslash G) = (g-1)/(q-1)$. Thus $M_F \leq \mathrm{Vol}(N_G(\Gamma)\backslash G) = \mathrm{Vol}(\Gamma\backslash G)/[N_G(\Gamma):\Gamma] = (g-1)(q-1)\cdot[N_G(\Gamma):\Gamma]^{-1}$ and so $[N_G(\Gamma):\Gamma] \leq (g-1)/((q-1)M_F)$.

4. Questions

The most interesting question we leave open is the one mentioned in the introduction; namely, given a semisimple Lie group G, let Γ be a lattice of minimal covolume in G. For which groups G, is Γ cocompact and for which groups is it a nonuniform lattice? We do not know the answer for any other groups but those mentioned in this paper. Even the case $G = SO(n, 1)$ for $n \geq 4$ is open and seems to suggest an interesting geometric question. On the other hand for higher rank groups, in light of Margulis's arithmeticity theorem and the recent formula of Prasad [Pr], our problem gives an interesting number theoretic problem.

Note that our Theorems 1 and 2 show that for $G = SL_2(\mathbf{F}_2((1/t))$ the minimal covolume of lattices is $\frac{1}{3}$ and this minimum is obtained by a cocompact lattice as well as by a nonuniform lattice. Are there other Lie groups with this property?

A second problem left open is computing the minimal covolume of lattices in $SL_2(F)$ for fields F that are finite extensions of Q_p.

The uniqueness of the lattices of minimal covolume was not studied. It is also of interest, in comparison to the beautiful result for $SL_2(\mathbf{C})$ (see [Th]) to understand the subset of $\mathbf{R}$ of those numbers that are covolumes of lattices in $SL_2(\mathbf{F}_q((1/t))$.

More results and questions on covolumes of lattices can be found in [Bo, BP, Me, EGM, MR] and references therein.

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