JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 4, Number 3, July 1991

PERVERSE SHEAVES AND $\mathbb{C}^*$-ACTIONS

VICTOR GINSBURG

1. Main theorem

Let X be a smooth complex projective variety. Assume we are given an algebraic $\mathbb{C}^*$-action on X with isolated fixed points. Let W denote the fixed point set. For each $w \in W$ one defines, following [BB], its attracting set:

$$X_w = \left\{ x \in X \mid \lim_{t \to 0} t \cdot x = w \right\}, \qquad t \in \mathbb{C}^*,$$

where $t \cdot x$ denotes the action of t on x. The set X_w is known to be a locally-closed $\mathbb{C}^*$-stable algebraic subvariety of X isomorphic to an affine space. The pieces X_w form a cell decomposition $X = \bigcup_{w \in W} X_w$, $w \in W$, the so-called Bialynicki-Birula decomposition. We assume this decomposition to be an algebraic stratification of X (the closure of a cell may not be a union of cells, in general).

Let $\mathfrak{L}_w$ be the intersection cohomology complex on the closure of a cell X_w extended by 0 to the whole of X. The hypercohomology $H^{\cdot}\mathfrak{L}_w$ has a natural structure of a graded $H^{\cdot}(X, \mathbb{C})$-module. Further, for any $x, y \in W$, there is a natural morphism of graded spaces:

$$\operatorname{Ext}^{\cdot}(\mathfrak{L}_x, \mathfrak{L}_y) \to \operatorname{Hom}^{\cdot}_{H^{\cdot}(X, \mathbb{C})}(H^{\cdot}\mathfrak{L}_x, H^{\cdot}\mathfrak{L}_y) \tag{1.1}$$

(the Ext-groups on the left are taken in $D^b(X)$, the bounded derived category of constructible complexes on X; the map (1.1) says simply that taking hypercohomology is a functor on $D^b(X)$).

The main result of this paper is the following

Theorem. *Assume that the $\mathbb{C}^*$-action on X satisfies condition* (1.2) *below. Then, for any $x, y \in W$, the morphism* (1.1) *is an isomorphism.*

The significance of the theorem is in giving a recipe for computing the Ext-groups on the left of (1.1), provided one knows the hypercohomology on the right of (1.1). The special case of the theorem, where $X = G/B$ is the flag manifold, is especially important because of its connection with the proof of the Koszul duality conjecture (see [BG] and also [BGS]). We will briefly explain that connection in the next section.

Received by the editors June 25, 1990.
1991 *Mathematics Subject Classification.* Primary 14F32, 22E47.

We now turn to the extra-condition (1.2) mentioned in the theorem. The condition may be thought of as providing a certain democracy among the fixed points. Specifically, assume that the group $\mathbb{C}^*$ is embedded into a complex torus T, and that the $\mathbb{C}^*$-action is extended to an algebraic T-action on X with the same fixed point set W. Assume further that for each $w \in W$ the following holds:

(1.2) *There exists a one-parameter subgroup $T_w \subset T$ which contracts a small neighborhood of w to w as the parameter of the subgroup tends to zero.*

Condition (1.2) is equivalent to the requirement that all the weights of the T-action on the tangent space at w belong to an open half-space of the real vector space spanned by the weight lattice.

Among the interesting examples of varieties X satisfying (1.2) are generalized flag manifolds G/P, smooth toric varieties with simplicial nondegenerate fans, and complete symmetric varieties [DP].

2. Soergel's theorem

Let G be a connected complex semisimple Lie group, B a Borel subgroup in G, T a maximal torus in B, and $X = G/B$ the flag manifold. Then the T-fixed points in X are in 1-1 correspondence with elements of W, the Weyl group of the pair (G, T). Each B-orbit in X contains exactly one T-fixed point. Hence, B-orbits form a decomposition of X indexed by W, the Bruhat decomposition. This decomposition coincides with the Bialynicki-Birula decomposition $X = \bigcup_w X_w$, relative to an appropriately chosen subgroup $\mathbb{C}^* \subset T$. Thus, we are in the setup of the theorem.

Further, let $g \supset b \supset t$ be the Lie algebras of the groups $G \supset B \supset T$ and $U(g) \supset U(b)$ be the enveloping algebras. Let $\mathscr{O}$ be the Bernstein-Gel′fand-Gel′fand category of finitely-generated $U(g)$-modules which are:

(i) $U(b)$-locally-finite;
(ii) t-semisimple.

Let $\rho \in t^*$ be the half-sum of positive roots (i.e., of the weights of the adjoint t-action on b) and let $\lambda \in t^*$ be a dominant integral weight. The group W acts on t^* and we let P_w denote the indecomposable projective object in $\mathscr{O}$ generated by a weight vector of weight $w(\lambda + \rho) - \rho$.

The Koszul duality conjecture of [BG], in down-to-earth terms, amounts to isomorphisms (for each pair $y, w \in W$)

$$\operatorname{Ext}^{\bullet}(\mathfrak{L}_y, \mathfrak{L}_w) = \operatorname{Hom}_{U(g)}(P_y, P_w) \tag{2.1}$$

which should be compatible with composition (that is, the isomorphisms (2.1), for all pairs (y, w) combined together, should lead to an *algebra isomorphism* $\operatorname{Ext}(\mathfrak{L}, \mathfrak{L}) = \operatorname{Hom}_{U(g)}(P, P)$, where we used the notation $\mathfrak{L} = \bigoplus_{w \in W} \mathfrak{L}_w$ and $P = \bigoplus_{w \in W} P_w$).

The isomorphisms (2.1) were constructed by Soergel [S]. We shall now outline his argument.

The first step, which has been done already [BG], is to prove that

(2.2) *both sides in* (2.1) *have the same dimension.*

Next, let $\mathbb{C}[t]$ denote the algebra of polynomial functions on the Lie algebra t, and I the ideal in $\mathbb{C}[t]$ generated by all W-invariant polynomials without constant term. Put $C = \mathbb{C}[t]/I$. It is well known that $H^*(X)$, the cohomology of the flag manifold, is isomorphic to C. For any $w \in W$, Soergel has produced a Kazhdan-Lusztig type algorithm for computing the hypercohomology $H^{\cdot}\mathfrak{L}_w$, viewed as a C-module.

Further, let w_0 be the longest element in W and $P_0 := P_{w_0(\lambda+\rho)-\rho}$ be the "antidominant projective." One can construct a natural algebra homomorphism: $C \to \mathrm{Hom}_{U(g)}(P_0, P_0)$ (which is actually an isomorphism, but we do not use that). Now, on the category $\mathscr{O}$ Soergel defines the functor

$$V(\cdot) = \mathrm{Hom}_{U(g)}(P_0, \cdot).$$

For any $M \in \mathscr{O}$, the space $V(M)$ has a C-module structure arising from the natural $\mathrm{Hom}_{U(g)}(P_0, P_0)$-action on $\mathrm{Hom}_{U(g)}(P_0, M)$ via composition. Again, for each $w \in W$, Soergel gives an algorithm for computing $V(P_w)$ as a C-module. Moreover, it turned out that the algorithm is identical to that for computing $H^{\cdot}\mathfrak{L}_W$. Hence, one obtains a C-module isomorphism

$$H^{\cdot}\mathfrak{L}_w \cong V(P_w). \tag{2.3}$$

Next, by functoriality, we have a natural morphism

$$\mathrm{Hom}_{U(g)}(P_w, P_y) \to \mathrm{Hom}_C(V(P_w), V(P_y)). \tag{2.4}$$

It is relatively easy to prove (see [S]) that

(2.5) *the morphisms* (1.1) *and* (2.4) *are injective.*

The final and the most difficult part of Soergel's argument is to show that

(2.6) *the morphism* (2.4) *is surjective.*

Now, it is clear that isomorphism (2.1) follows from statements (2.2), (2.3), (2.5), and (2.6). It is equally clear that our theorem can replace statement (2.6) (and part of (2.5)).

3. Proof of the theorem

Let v be the vector field on X generating the S^1-action on X arising from the $\mathbb{C}^*$-action, by restriction to the unit circle. Let ω be an S^1-equivariant Kähler 2-form on X (it exists since X is projective), and let $i_v\omega$ be the 1-form obtained by contraction. This 1-form is exact since $H^1(X) = 0$. Hence, there is a smooth function f on X such that $i_v\omega = df$. The function f is known to be a Morse function whose critical points are exactly the fixed points of the $\mathbb{C}^*$-action. Moreover, the cell decomposition of X associated to f via the Morse theory is the Bialynicki-Birula decomposition $X = \bigcup X_w$.

Let $c_0 < c_1 < \cdots$ be the critical values of f and set $X_n = f^1((-\infty, c_n])$. The sets X_n, $n = 0, 1, 2, \ldots$, form an increasing filtration of X by closed subvarieties. Put $U_n := X_n \backslash X_{n-1}$. We have natural inclusions

$$X_n \overset{i_n}{\hookrightarrow} X, \qquad X_{n-1} \overset{v}{\hookrightarrow} X_n \overset{u}{\hookleftarrow} U_n.$$

Clearly, the X_n's and U_n's are unions of the strata X_w, $w \in W$, and U_n is a Zariski-open part of X_n. We shall assume, for simplicity, that U_n consists of a single stratum. Otherwise, make a refined filtration of X by adjoining to X_{n-1} the components of U_n one by one.

Now, fix some $x \in W$ and set $L_n := i_n^* \mathfrak{L}_x$, $n = 0, 1, 2, \ldots$. We have the following distinguished triangles:

$$(3.1) \qquad u_! \cdot u^! L_n \to L_n \to v_* \cdot v^* L_n, \qquad v_! \cdot v^! L_n \to L_n \to u_* \cdot u^* L_n.$$

We begin the proof of the theorem with the following technical result.

Proposition 3.2. *All connecting homomorphisms in the long exact sequences of cohomology associated to the triangles* (3.1) *vanish.*

We record two immediate corollaries of the proposition which will be used later.

Corollary 3.3. *There is a natural short exact sequence*

$$0 \to H^{\cdot}(u_! \cdot u^! L_n) \to H^{\cdot} L_n \to H^{\cdot} L_{n-1} \to 0.$$

Corollary 3.4. *The natural restriction morphism* $H^{\cdot} L_n \to H^{\cdot}(u^* L_n)$ *is surjective.*

In the course of the proof of Proposition 3.2 we shall use the language of "weights." To have a right to do so, we shall view the complex $\mathfrak{L}_x$ as a pure Hodge module (say, of weight 0) in the sense of Saito [Sa].

Let $j_w : \{w\} \hookrightarrow X$ denote the inclusion of a fixed point.

Lemma 3.5. *The cohomology* $H^{\cdot}(j_w^* \mathfrak{L}_x)$ *is pure.*

Proof of the lemma. Let T_w be the one-parameter subgroup of T which contracts a neighbourhood of w to w (see (1.2)). Let V be the cell containing w in the Bialynicki-Birula cell decomposition with respect to the group T_w. Clearly, V is a Zariski-open part of X isomorphic to a vector space, and $\mathfrak{L}_{x|V}$ is a T_w-equivariant complex on V. For such a complex one has an isomorphism $H^{\cdot}(V, \mathfrak{L}_x) = H^{\cdot}(j_w^* \mathfrak{L}_x)$ (see, e.g., [Sp]). But the functor $H^{\cdot}(V, \cdot)$ increases the weights while the functor $H^{\cdot} j_w^*$ decreases the weights. Hence, $H^{\cdot} j_w^* \mathfrak{L}_x$ is pure. □

Proof of Proposition 3.2. The part of the proposition concerning the first triangle in (3.1) follows from Lemma 3.5 and [S, Lemma 19]. To prove the second part, recall the set V introduced in the proof of Lemma 3.5 and let $\varepsilon\colon X\backslash V \hookrightarrow X$ denote the embedding. Write the long exact sequence of cohomology:

$$\cdots \to H^i(X, \mathfrak{L}_\chi) \to H^i(V, \mathfrak{L}_x) \xrightarrow{[1]} H^{i+1}(\varepsilon^! \mathfrak{L}_x)) \to \cdots .$$

The connecting homomorphism in this sequence is trivial since $H^i(V, \mathfrak{L}_x)$ is pure of weight i (Lemma 3.5) and the weights of $H^{i+1}(\varepsilon^! \mathfrak{L}_x)$ are $\geq i+1$. Hence, the map $H^\cdot(X, \mathfrak{L}_x) \to H^\cdot(V, \mathfrak{L}_x)$ is surjective.

Now, decompose the restriction from X to $w = w_n$ in two different ways as follows:

$$\begin{tikzcd} & H^\cdot(V, \mathfrak{L}_x) \arrow[dr, "\beta"] & \\ H^\cdot(X, \mathfrak{L}_x) \arrow[ur, "\alpha"] \arrow[dr] & & H^\cdot(j_w^* \mathfrak{L}_x) \\ & H^\cdot L_n \xrightarrow{\gamma} H^\cdot(u^* L_n) \arrow[ur, "\delta"] & \end{tikzcd}$$

The composition of the maps α and β is surjective by the preceding paragraph. The map δ is an isomorphism. This yields surjectivity of the map γ, and the proposition follows. □

We now fix the second complex $\mathfrak{L}_y$, $y \in W$, and set $M_n := i_n^! \mathfrak{L}_y$, $n = 0, 1, 2, \ldots$. There are analogues of results 3.2–3.5 for M_n instead of L_n.

Proposition 3.6. *For any* n:
(i) $\operatorname{Ext}^\cdot(L_n, M_n)$ is pure.
(ii) There is a natural short exact sequence

$$0 \to \operatorname{Ext}^\cdot(L_{n-1}, M_{n-1}) \to \operatorname{Ext}^\cdot(L_n, M_n) \to \operatorname{Ext}^\cdot(u^* L_n, u^* M_n) \to 0.$$

Proof. For $n = 1$ there is nothing to prove. Assume $n > 1$ and use induction in n. Write the distinguished triangle $v_! \cdot v^! M_n \to M_n \to u_* \cdot u^* M_n$. It yields a long exact sequence:

$$\cdots \to \operatorname{Ext}(L_n, v_! M_{n-1}) \to \operatorname{Ext}(L_n, M_n) \to \operatorname{Ext}(L_n, u_* \cdot u^* M_n) \to \cdots . \tag{3.7}$$

The third term can be rewritten, by adjunction, as $\operatorname{Ext}^\cdot(u^* L_n, u^* M_n)$; this space is pure by Lemma 3.5. Further, by adjunction, $\operatorname{Ext}^\cdot(u_! \cdot u^! L_n, v_! M_{n-1}) = \operatorname{Ext}^\cdot(u^! L_n, u^! \cdot v_! M_{n-1})$, and the latter space is zero because of the identity $u^! \cdot v_!(\cdot) = 0$. Hence, the exact triangle $u_! \cdot u^! L_n \to L_n \to v_* L_{n-1}$ yields $\operatorname{Ext}(L_n, v_! M_{n-1}) = \operatorname{Ext}^\cdot(L_{n-1}, M_{n-1})$, and the latter Ext-group is pure by the induction hypothesis. Now, the purity of the second term follows from (3.7). This proves part (i); part (ii) follows from (i) and (3.7). □

We have completed preparatory stages of the proof of the theorem. Now we arrive at its crucial point.

In addition to the Bialynicki-Birula decomposition $X = \bigcup X_w$, which is often called the *plus*-decomposition, we introduce the *minus*-decomposition

$X = \bigcup X_w^-$, where X_w^- stands for the expanding set of w, that is, $X_w^- = \{x \in X \mid \lim_{t\to\infty} t\cdot x = w\}$. The minus-decomposition coincides with the cell decomposition associated with the Morse function $(-f)$ via the Morse theory.

Recall our filtration $X_0 \subset X_1 \subset \cdots$. Let w be the fixed point in $X_n \backslash X_{n-1}$, so that $X_n \backslash X_{n-1} = X_w$. Then $\overline{X_w^-}$, the closure of X_w^-, has empty intersection with X_{n-1} (because, for any $x \in X_{n-1}$ and $x^- \in \overline{X_w^-}$ $(x^- \neq w)$, one has strict inequalities $f(x) < f(w) < f(x^-)$). Furthermore, $\overline{X_w^-}$ meets X_n transversally in the single point w.

Let $c_n \in H^\cdot(X, \mathbb{C})$ be the cohomology class dual to the fundamental cycle $[\overline{X_w^-}]$ (w as above) via the Poincaré duality. We have $\langle c_n, [X_n]\rangle = 1$. Furthermore, multiplication by c_n annihilates hypercohomology of any complex supported on X_{n-1}. Hence, the action of c_n on $H^\cdot L_n$ is completely described by the following commutative diagram:

(3.8a)
$$\begin{array}{ccccccc} & & & & H^\cdot L_n & \overset{3.4}{\twoheadrightarrow} & H^\cdot(u^* L_n) \\ & & & & \downarrow c_n & & \downarrow \wr c_n \\ 0 & \leftarrow & H^\cdot L_{n-1} & \twoheadleftarrow & H^\cdot L_n & \overset{3.3}{\hookleftarrow} & H^\cdot_c(u^* L_n) \end{array}$$

(The right vertical arrow here is an isomorphism which is essentially the familiar isomorphism $H^0(\mathbb{R}^d) \xrightarrow{\sim} H_c^d(\mathbb{R}^d)$.)

One also has the dual diagram:

(3.8b)
$$\begin{array}{ccccccc} 0 & \rightarrow & H^\cdot M_{n-1} & \rightarrow & H^\cdot M_n & \twoheadrightarrow & H^\cdot(u^* M_n) \\ & & & & \downarrow c_n & & \downarrow \wr c_n \\ & & & & H^\cdot M_n & \hookleftarrow & H^\cdot_c(u^* M_n) \end{array}$$

We record the following isomorphisms which are immediate from diagrams (3.8a,b) and which will be used later:

(3.9.1) $$\operatorname{coker}(c_n, H^\cdot L_n) \cong H^\cdot L_{n-1},$$

(3.9.2) $$H^\cdot L_n / \ker(c_n, H^\cdot L_n) \cong H^\cdot(u^* L_n),$$

(3.9.3) $$H^\cdot M_n / \ker(c_n, H^\cdot M_n) \cong H^\cdot(u^* M_n),$$

(3.9.4) $$\ker(c_n, H^\cdot M_n) \cong H^\cdot M_{n-1},$$

where $\ker(c_n, H^\cdot L_n)$ denotes the kernel of the operator "*multiplication by* c_n": $H^\cdot L_n \to H^\cdot L_n$, etc.

Now, we complete the proof of the theorem by proving the following

Proposition 3.10. *For any* n, *the natural morphism*

$$\operatorname{Ext}^\cdot(L_n, M_n) \to \operatorname{Hom}_{H^\cdot(X)}(H^\cdot L_n, H^\cdot M_n)$$

is an isomorphism.

Proof. For $n = 1$ the statement is trivial. Assume that $n > 1$ and use induction on n. Let $h \in \operatorname{Hom}_{H^\cdot(X)}(H^\cdot L_n, H^\cdot M_n)$. Clearly, h maps $\ker(c_n, H^\cdot L_n)$ into

$\ker(c_n, H^{\cdot} M_n)$, hence, induces a morphism

$$H^{\cdot} L_n / \ker(c_n, H^{\cdot} L_n) \to H^{\cdot} M_n / \ker(c_n, H^{\cdot} M_n).$$

This morphism can be viewed, by (3.9.2)–(3.9.3), as a morphism $\bar{h}: H^{\cdot}(u^* L_n) \to H^{\cdot}(u^* M_n)$. The assignment $h \to \bar{h}$ defines an arrow $\mathrm{Hom}_{H^{\cdot}(X)}(H^{\cdot} L_n, H^{\cdot} M_n) \to \mathrm{Hom}_{H^{\cdot}(X)}(H^{\cdot}(u^* L_n), H^{\cdot}(u^* M_n))$.

Further, the projection $H^{\cdot} L_n \to H^{\cdot} L_{n-1}$ (cf. (3.3)) and the dual injection $H^{\cdot} M_{n-1} \to H^{\cdot} M_n$ give rise to an injective arrow $\mathrm{Hom}_{H^{\cdot}(X)}(H^{\cdot} L_{n-1}, H^{\cdot} M_{n-1}) \to \mathrm{Hom}_{H^{\cdot}(X)}(H^{\cdot} L_n, H^{\cdot} M_n)$. This arrow, combined with that of the preceding paragraph and with Proposition 3.6, yields the following natural commutative diagram:

(3.11)

$$\begin{array}{ccccccc}
0 \to & \mathrm{Ext}(L_{n-1}, M_{n-1}) & \to & \mathrm{Ext}(L_n, M_n) & \to & \mathrm{Ext}(u^* L_n, u^* M_n) & \to 0 \\
& \downarrow ① & & \downarrow ② & & \downarrow ③ & \\
0 \to & \mathrm{Hom}(H^{\cdot} L_{n-1}, H^{\cdot} M_{n-1}) & \to & \mathrm{Hom}(H^{\cdot} L_n, H^{\cdot} M_n) & \xrightarrow{\pi} & \mathrm{Hom}(H^{\cdot} u^* L_n, H^{\cdot} u^* M_n) &
\end{array}$$

(in the bottom row of the diagram "Hom" always stands for graded Hom over $H^{\cdot}(X)$).

We will now show that the bottom row of the diagram is exact at the middle term. Let $h \in \mathrm{Hom}(H^{\cdot} L_n, H^{\cdot} M_n)$, and suppose that $\pi(h) = 0$. Then the exact sequence in the top row of diagram (3.8b) implies that $\mathrm{im}\, h \subset H^{\cdot} M_{n-1}$. Hence, $\mathrm{im}\, h \subset \ker(c_n H^{\cdot} M_n)$ by (3.9.4). Hence, h vanishes on $\mathrm{im}(c_n, H^{\cdot} L_n)$. Thus, h comes from a morphism $\mathrm{coker}(c_n, H^{\cdot} L_n) \to H^{\cdot} M_{n-1}$ and the claim follows from (3.9.1).

Now, the vertical arrow ③ in (3.11) is obviously an isomorphism. The arrow ① is an isomorphism, by the induction hypothesis. Hence, the arrow ② is an isomorphism also. This completes the proof of the theorem. □

Remark. Our proof works in some cases which are not formally covered by the theorem. It can be applied, for instance, to flag manifolds and Grassmannians associated with a Kac-Moody algebra. In particular, this provides a proof of [Gi, Theorem 10.3]. Our theorem might also be useful in proving an analogue of the Koszul duality conjecture [BG] for affine Lie algebras.

References

[BG] A. Beilinson and V. Ginsburg, *Mixed categories, Ext-duality and representations*, preprint, Moscow, 1986.

[BGS] A. Beilinson, V. Ginsburg, and W. Soergel, *Parabolic Koszul duality*, in preparation.

[BB] A. Bialynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. of Math. (2) **98** (1973), 480–497.

[DP] C. De Concini and C. Procesi, *Complete symmetric varieties*, Lecture Notes in Math, vol. 996, Springer, 1983, pp. 1–30.

[Gi] V. Ginsburg, *Perverse sheaves on a loop group and Langlands' duality*, preprint, Moscow, 1989.

[Sa] M. Saito, *Modules de Hodge polarisable*, preprint, RIMS, 1986.

[S] W. Soergel, *Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe*, J. Amer. Math. Soc. **3** (1990), 421–445.

[SP] T. A. Springer, *A purity result for fixed point variety in flag manifolds*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **31** (1984), 271–282.

INSTITUTE DES HAUTES ÉTUDES SCIENTIFIQUES, 91440-BURES SUR YVETTE, FRANCE

Current address: Vinokurova 5/6-2-5, 117449 Moscow, USSR