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ON THE ENVELOPE OF HOLOMORPHY OF A 2-SPHERE IN $\mathbb{C}^2$

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1. Introduction

We let Γ be a smooth 2-sphere imbedded in $\mathbb{C}^2$, or more generally, in a 2-dimensional Stein manifold. We consider here the problem of finding a 3-dimensional ball B such that $\partial B = \Gamma$ and B is foliated by complex disks. Our motivation for this is to give a description of the envelope of holomorphy and polynomial hull of a 2-sphere in $\mathbb{C}^2$. Gromov [G] and Eliashberg [E1] have shown how this problem is related to problems of symplectic geometry on almost complex manifolds. Recently, Eliashberg [E2] has observed how this may be applied to obtain topological information on 2-dimensional Stein manifolds.

The tangent plane to Γ at a point is either totally real (which occurs when the tangent plane is in general position) or a complex tangency (when the tangent space is a complex subspace). Near a generic complex tangency, Γ may be written (after a holomorphic change of coordinates) as

$$\Gamma = \{w = z\overline{z} + \lambda \Re e z^2 + \sigma_3(z, \overline{z})\}$$

where $\lambda \geq 0$, and σ_3 vanishes to third or higher order. The complex tangency is *elliptic* if $0 \leq \lambda < 1$ and *hyperbolic* if $1 < \lambda$; the *parabolic* case $\lambda = 1$ does not occur for a generic 2-manifold. If Γ is oriented, then we will call a complex tangency p *positive* or *negative* according to whether the orientation of $T_p\Gamma$ agrees or disagrees with the canonical orientation induced from $\mathbb{C}^2$. We may define the *positive/negative index* to be

$$i^{\pm} = e^{\pm} - h^{\pm}$$

where $e^{\pm}$ is the number of positive/negative elliptic points, and $h^{\pm}$ is the number of positive/negative hyperbolic points.

It was shown by Bishop [Bi1] that for a generic 2-manifold Γ imbedded in $\mathbb{C}^2$, $i^+ + i^- = \chi(\Gamma)$ holds, where χ is the Euler characteristic. Formulas for i^+ and i^- in the more general case where Γ is imbedded in a complex manifold X are given in [La, EH]. Thus, if Γ is a generic 2-sphere imbedded

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in $\mathbb{C}^2$, then Γ must have at least two elliptic points; and Γ has a locally generated envelope given by a one-parameter family of complex disks at each elliptic point. We note that there are surfaces of higher genus in $\mathbb{C}^2$ which are holomorphically convex [GW]. However, by a theorem of A. Browder [Br], no compact, orientable 2-manifold in $\mathbb{C}^2$ is polynomially convex.

Examples show (cf. [BG]) that the envelope of a general 2-sphere can be rather complicated, so we consider imbeddings that are restricted by convexity conditions.

Theorem 1. *Let X be a 2-dimensional Stein manifold, and let $\Omega \subset\subset X$ be a strongly pseudoconvex domain with smooth boundary, and let Γ be a 2-sphere smoothly imbedded in $\partial\Omega$. Then there is a small, C^2 perturbation Γ' of Γ with the property that Γ' is swept out by the boundaries of complex disks. Further, there is a smooth 3-ball B' that is the disjoint union of complex disks, such that $\partial B' = \Gamma'$, and $\overline{B}'$ is the envelope of holomorphy of Γ'.*

In our first version of this paper, we obtained Theorem 3 below as the main result. The present formulation of Theorem 1 was suggested to us by Y. Eliashberg, who outlined to us the main steps in making the passage from the case where Γ is a graph, as in Theorem 3, to the more general geometric situation in Theorems 1 and 2. We wish to thank Eliashberg for his contributions to this paper.

The next result involves hyperbolic points that are "good" (see the definition just before Lemma 3.2).

Theorem 2. *Let $\Omega \subset \mathbb{C}^2$ be a bounded, strictly pseudoconvex domain with smooth boundary, and let $\Gamma \subset \partial\Omega$ be a 2-sphere for which all complex tangencies are either elliptic points or good hyperbolic points. Then there is a 3-dimensional topological ball B that is the disjoint union of complex disks such that $\partial B = \Gamma$. Further, $\overline{B}$ is the envelope of holomorphy of Γ.*

In order to deal with 2-spheres for which the complex tangencies can be arbitrary, we will consider 2-spheres, which are given as graphs in the following way. We use the variables (z, w) on $\mathbb{C}^2$, with $z = x + iy$ and $w = u + iv$. Let $D \subset \mathbb{C} \times \mathbb{R}$ be a contractible, smoothly bounded open set such that $D \times i\mathbb{R}$ is strongly pseudoconvex. We will consider 2-spheres that have the form

$$\Gamma(\varphi) = \{v = \varphi(z, u) : (z, u) \in \overline{D}\}$$

for some function Φ defined on $\overline{D}$ that coincides with φ on ∂D. We will refer to $\hat{\Gamma}(\Phi) \cap D \times i\mathbb{R}$ as the *interior* of $\hat{\Gamma}(\Phi)$. Our result for graphs is as follows:

Theorem 3. *Given a function $\varphi \in C^2(\partial D)$, there exists a unique function $\Phi \in \mathrm{Lip}(\overline{D})$ with $\Phi|_{\partial D} = \varphi$ and such that the graph $\hat{\Gamma}(\Phi)$ has the following*

properties:

(i) *The interior of* $\hat{\Gamma}(\Phi)$ *has a foliation* $\mathscr{M}$ *whose leaves are complex manifolds. If* $g : \Delta \to \mathbb{C}^2$ *is a holomorphic mapping with* $g(\Delta) \subset \operatorname{int} \hat{\Gamma}(\Phi)$, *then* $g(\Delta)$ *is contained in a leaf of* $\mathscr{M}$. *If* p *is a totally real point of* $\Gamma(\varphi)$, *then there is a neighborhood* U *of* p *such that if* M *is a leaf of* $\mathscr{M} \cap U$ *then* $\overline{M}$ *is smooth at* $\overline{M} \cap \Gamma(\varphi) \cap U$; *and the sets* $\overline{M} \cap \Gamma(\varphi) \cap U$ *are smooth curves that foliate* $\Gamma(\varphi) \cap U$.

(ii) $\hat{\Gamma}(\Phi)$ *is the envelope of holomorphy of* $\Gamma(\varphi)$.

(iii) $\hat{\Gamma}(\Phi)$ *is the polynomial hull of* $\Gamma(\varphi)$.

(iv) *If* $f : \Delta \to \mathbb{C}^2$ *is a bounded holomorphic mapping with* $f(\zeta) \in \Gamma(\varphi)$ *for a.e.* $\zeta \in \partial\Delta$, *then* $f(\Delta) \subset \hat{\Gamma}(\Phi)$.

These theorems also have an interpretation as giving the solution to a complex Plateau problem when the surface is smooth: since the hypersurface $\hat{\Gamma}(\Phi)$ is Levi flat, the mean curvature in the holomorphic tangential direction vanishes (see [DG]). The equations describing this condition, however, are degenerate, and we are not able to use them in our analysis.

Our approach to this problem is to give a global construction of analytic disks. The simplest case of these theorems is where Γ has no hyperbolic points and thus only two elliptic points. The result is obtained by showing that there is a one-parameter family of analytic disks with boundaries that fill out Γ by starting at one elliptic point and sweeping smoothly to the other. This was done in [BG] in the case where $\Gamma = \Gamma(\varphi)$ is a graph; later Gromov [G] obtained this result in the more general context of almost complex manifolds tamed by symplectic structures. The regularity of $\hat{\Gamma}(\Phi)$ at the elliptic points themselves is rather subtle and was shown in [KW, MW, Mo].

If Γ contains hyperbolic points, this approach becomes more complicated since a bifurcation will occur at a hyperbolic point: a 1-parameter family of disks may "split" or two families may "join." The construction of disks in this case is essentially global since Γ is locally polynomially convex at a hyperbolic point (see [Fr1, Fr2, S]). Theorem 3 was proved in [Be] in the special case where φ is close in $C^2(\partial D)$ to the boundary values of a smooth solution $\Phi_0 \in C^2(\overline{D})$.

In the disk method of finding hulls and envelopes, it is essential to consider the boundary regularity of a complex disk with its boundary in Γ. If we write Γ locally as $\{w = \psi(z, \overline{z})\}$, then a complex disk is given as $\{w = f(z) : z \in \omega\}$, where f is holomorphic on $\omega \in \mathbb{C}$, continuous on $\omega \cup \gamma$ for some $\gamma \subset \partial\omega$ and $f(z) = \psi(z, \overline{z})$ for $z \in \gamma$. Here γ is a curve in the boundary of $\partial\omega$ corresponding to the portion of the boundary of the complex disk lying in Γ. Thus we may view this as a free boundary value problem for an analytic function in a region $\omega \subset \mathbb{C}$. Regularity for this problem was established by H. Lewy [L] in the case where $\partial\psi/\partial\overline{z} \neq 0$, which corresponds to the case of Γ totally real. In order for us to deal with hyperbolic points, we give a treatment of the case $\psi = z\overline{z} + \lambda \Re e z^2$, $1 < \lambda$, with $0 \in \gamma$. In §3 we obtain regularity for a disk passing through a hyperbolic point which is sufficient to enable us to handle the bifurcation of disks.

2. Preliminaries

Our proof of the theorem will involve several known results, which we gather in this section. Let us denote by Γ^* the set of totally real points of Γ. By a *complex disk* we will mean a holomorphic mapping $f : \Delta \to \mathbb{C}^2$ from the unit disk Δ that extends continuously to the closure $\overline{\Delta}$. Sometimes when we speak of a complex disk we will refer instead to the image $f(\Delta)$. A *hyperbolic disk* is a complex disk with a finite set of points $\{\zeta_1, \dots, \zeta_k\} \subset \partial\Delta$ such that $f(\partial\Delta - \{\zeta_1, \dots, \zeta_k\}) \subset \Gamma^*$, and $f(\zeta_j)$ is a hyperbolic point of Γ for $1 \le j \le k$.

The 2-sphere Γ disconnects $\partial\Omega$ into two pieces $(\partial\Omega)_1$ and $(\partial\Omega)_2$. Making small C^2 perturbations of $\partial\Omega$, we may obtain strongly pseudoconvex domains Ω_1 and Ω_2 so that $\Omega_j \supset (\partial\Omega)_j$ for $j = 1, 2$ and $\Gamma \subset \partial\Omega_j$ for $j = 1, 2$, but the outward normal to $\partial\Omega$ at $p \in \Gamma$ lies strictly inside the convex hull of the outward normals to $\partial\Omega_j$ at p. Thus $\Omega_1 \cap \Omega_2$ is pseudoconvex, and $\Gamma \in \partial(\Omega_1 \cap \Omega_2)$. The sets Ω_1 and Ω_2 serve as barriers for the envelope of holomorphy of Γ.

Locally, they are like the barriers used in [BG], and so we obtain

Lipschitz estimate. *For any point $p \in \Gamma^*$ there is a local coordinate system (ξ, η) on Γ^* in a neighborhood U of p such that $p = (0, 0)$ and with the property: if σ is a complex disk with boundary in Γ^* and if $\sigma \subset \Omega_1 \cap \Omega_2$, then every portion of $\partial\sigma \cap U$ is uniformly Lipschitz as a graph over the ξ-axis.*

(See [BG, §6] for details.) As a consequence of the Lipschitz estimate, we obtain the following result on the convergence of boundaries of disks.

Lemma 2.1. *Let $\sigma(t)$, $0 < t < \infty$, denote a one-parameter family of complex disks with boundaries in Γ. If $U \subset \Gamma^*$, and if $\bigcup_{0<t<\infty} U \cap \partial\sigma(t)$ is relatively compact in Γ^*, then the curves $U \cap \partial\sigma(t)$ converge locally uniformly on compact subsets of U as $t \to \infty$ to a Lipschitz curve $\gamma(\infty)$.*

The η-axis for the Lipschitz estimate above is chosen so that it is tangent at $p = (0, 0)$ to $T_p\Gamma \cap H_p(\partial\Omega)$, where $H_p(\partial\Omega)$ denotes the holomorphic tangent space to $\partial\Omega$. This was called the *characteristic direction* in [E1].

The characteristic direction is not defined at a complex tangency. Since Ω is strongly pseudoconvex, however, the characteristic directions at a complex tangency are generated by a vector field of the form

$$V = \Im m e^{ia}(z + \lambda\overline{z})\partial_z + O(|z|^2).$$

The geometric form of the Lipschitz estimate is thus: *the tangent to the curve $\gamma = \partial\sigma$ must make a uniformly positive angle $\kappa > 0$ with V.* At a hyperbolic point, this means that the tangent to γ must lie inside a field of cones such as is illustrated in Figure 3 of [Be]. From this angle condition on a curve γ, we obtain

Lemma 2.2. *The length of γ is locally finite.*

And for a family $\{\gamma_t\}$ of curves whose tangents lie in the cone field we have

Lemma 2.3. *Let γ_t be a one-parameter family of disjoint, simple curves that converge to the curve γ. Then locally we have* $\sup_t$ *Length* $(\gamma_t) < \infty$.

We omit the proofs since these Lemmas will not be used below.

Regularity of complex disks. The condition that Γ is totally real imposes strong regularity properties on a complex disk with boundary in Γ. A good version of this result is given by Chirka [C].

Regularity Theorem. *Let $f : \Delta \to \mathbb{C}^2$ be a complex disk such that $f \in C(\overline{\Delta})$, $f(e^{i\theta_0})$ is a totally real point of Γ, and $f(e^{i\theta}) \in \Gamma$ for θ near θ_0. If Γ is smooth of class $C^{m,\alpha}$ for some $0 < \alpha < 1$, then f is also of class $C^{m,\alpha}$ near $e^{i\theta_0}$.*

Uniqueness of complex disks. We note that any real analytic curve $\gamma \subset \Gamma$ may be complexified locally to give a piece of a complex disk containing γ in its boundary. This shows that a uniqueness theorem can hold only under special circumstances. We may write Γ locally as a graph over $\mathbb{C} \times \mathbb{R}$, and we may obtain uniqueness in terms of the projection $\tilde{\pi} : \mathbb{C}^2 \to \mathbb{C} \times \mathbb{R}$, defined by $\tilde{\pi}(z, w) = (z, u)$. The following result is a restatement of Theorem 7.4 of [BG].

Uniqueness Theorem. *Let $R_j = f_j(\Delta)$, $\partial R_j \subset \Gamma^*$, $j = 1, 2$ be smooth, complex disks in $D \times i\mathbb{R}$. Suppose also that $p \in \partial R_1 \cap \partial R_2$ and that there is a neighborhood U of p such that $U \cap \partial R_1$ lies to one side of $U \cap \partial R_2$. If $\tilde{\pi} R_1$ lies to one side of $\tilde{\pi} R_2$, then $R_1 = R_2$.*

Klingenberg [K] has obtained a uniqueness theorem for hyperbolic disks that extend analytically through the hyperbolic point and that intersect Γ nicely. But hyperbolic disks are not in general smooth at the hyperbolic point.

A hyperbolic point will have disks approaching from opposite sides, and a disk approaching it from one side is quite unrelated to the disk approaching it from the other side. In this sense, there can be no uniqueness. But in Lemma 3.5 we obtain a uniqueness result analogous to the Uniqueness Theorem above. That is, there cannot be another hyperbolic disk just "above" or "below" the given one. And this is the uniqueness result needed at hyperbolic points.

Disks as local graphs. Let us consider a one-parameter family of complex disks $\sigma(t)$, $0 < t < \infty$, with boundaries in Γ, such that $\partial\sigma(t)$ fills out an open subset of Γ in a monotone fashion, i.e., the boundaries $\partial\sigma(t)$ are simple closed curves, and $\partial\sigma(t') \cap \partial\sigma(t'') = \varnothing$ for $t' \neq t''$. Suppose that on compact subsets of Ω, $\{\sigma(t)\}$ converges in the Hausdorff distance as $t \to \infty$ to a complex disk $\sigma(\infty)$ with $\partial\sigma(\infty) \subset \Gamma$.

Let us suppose that the origin $(0, 0)$ is a limit point of the set $\{\partial\sigma(t) : t < \infty\}$, and let us assume that $(0, 0) \in \Gamma^*$. It follows from Lemma 2.1, then, that the curves $\partial\sigma(t)$ converge uniformly in a neighborhood of $(0, 0)$ to a Lipschitz

curve $\gamma(\infty)$. We may choose coordinates $z = x + iy$ and $w = u + iv$ near $(0, 0)$ so that the v-axis is the characteristic direction, and the x-axis is the outward normal to Ω at $(0, 0)$. We may assume, too, that $\Omega \subset \{x < 0\}$ and $\Gamma \cap \{x > -\varepsilon\}$ is a small neighborhood of the origin in Γ. For $\varepsilon > 0$ sufficiently small, the Lipschitz estimate will hold for the curve $\partial\sigma(t) \cap \{x > -\varepsilon\}$ in the (v, y)-coordinates.

Let π denote the projection $\pi(z, w) = z$. We may choose a small ball B centered at $(0, 0)$ such that the above holds on the open set $\Gamma \cap B$ and such that $\Gamma \cap B$ is a graph over $\pi(\Gamma \cap B)$.

Lemma 2.4. *Let $\{\sigma(t)\}$, $\sigma(\infty)$, and B be as above. Then for t sufficiently large, there are a domain $\omega_t \subset \mathbb{C}$ and a function $\psi_t \in C(\overline{\omega_t})$ such that $\sigma(t) \cap B = \{(z, \psi_t(z)) : z \in \omega_t\}$. Further, $\partial\sigma(t) \cap B$ converges to $\partial\sigma(\infty) \cap B$ as $t \to \infty$.*

Proof. Let $\omega_t = \pi(\sigma(t) \cap \{\Re e\, z > -\varepsilon\})$. By the Lipschitz estimate, the portion $\partial\sigma(t) \cap \{\Re e\, z > -\varepsilon\}$ projects to a simple curve in $\mathbb{C}$, i.e. without self-intersections. Consider now the whole curve $\pi\partial\sigma(t)$ in $\mathbb{C}$. It consists of two pieces, one of which is $\pi\partial\sigma(t) \cap \{\Re e\, z > -\varepsilon\}$. By the argument principle, then, $\pi\partial\sigma(t)$ has winding number $+1$ about any point of ω_t. Thus the projection $\pi : \sigma(t) \cap \{\Re e\, z > -\varepsilon\} \to \omega_t$ is one-to-one. The existence of the function ψ_t follows from this.

Remark. It follows that for any compact subset $K \subset \omega_\infty \cup \pi(\partial\sigma(\infty) \cap B)$ the functions ψ_t converge uniformly to ψ_∞ on $\omega_t \cap K$. For this it suffices to assume that $\psi_\infty = 0$ and show that ψ_t converges to 0 as $t \to \infty$. This follows by applying standard estimates on harmonic measure to $\log|\psi_t|$.

Lemma 2.5. *If B is a small ball and $\sigma(t)$ is a complex disk with boundary in Γ, then $B \cap \sigma(t)$ is a union of complex disks.*

Proof. Since B is a ball, it is polynomially convex. Thus there is a plurisubharmonic function r on Ω such that $B \cap \Omega = \{r < 0\}$. Let $f : \Delta \to \Omega$ denote a holomorphic mapping such that $f(\Delta) = \sigma(t)$. Then $r \circ f$ is subharmonic on the disk, so the set $\Delta_0 := \{\zeta \in \Delta : r \circ f(\zeta) < 0\}$ is simply connected by the maximum principle. Uniformizing the components of Δ_0, we may represent the components of $B \cap \sigma(t)$ as complex disks.

Hyperbolic chains. We consider the model hyperbolic point $\{w = z\overline{z} + \lambda \Re e z^2\}$, $\lambda > 1$. We let $\omega^\pm \subset \mathbb{C}$ denote the connected components of $\{z\overline{z} + \lambda \Re e z^2 < 0\}$, where the $\pm$ is chosen so that ω^+ intersects the positive y-axis. We want to consider hyperbolic disks whose behavior is close to that of ω^+ and that are in some sense the best possible hyperbolic disks. We say that a hyperbolic disk $f : \Delta \to \mathbb{C}^2$ has *good approach* at the hyperbolic point (0,0) if there is a neighborhood U of (0,0) in $\mathbb{C}^2$ such that

(i) There is a domain $\omega \subset \mathbb{C}$ that is asymptotic at 0 to one of the domains ω^+ or ω^-.

(ii) There is an analytic function $g(z)$ on ω such that $U \cap f(\Delta) = \{w = g(z) : z \in \omega\}$.

(iii) $g \in C^2(\overline{\omega})$, and $g(0) = g'(0) = g''(0) = 0$.

We define a *hyperbolic chain* $\mathscr{C} = \bigcup_{j=1}^{k} R_j$ as a finite set of analytic disks $\{R_j = f_j(\Delta)\}$, $1 \le j \le k$, with the properties:

(i) $\overline{\mathscr{C}}$ is connected.

(ii) Each disk R_j has good asymptotic approach to all of the hyperbolic points in ∂R_j.

(iii) If R_i and R_j both approach H, then they do it through opposite approach regions.

(This definition is slightly stronger than the definition of *regular chain* in [Be].)

We will say that the chain $\mathscr{C}$ is *saturated* if for every hyperbolic point $H \in \overline{\mathscr{C}}$, both approach regions of H are used by disks of $\mathscr{C}$. Let us observe that the following results are easy topological consequences of the definition of chain.

Lemma 2.6. *Let $\mathscr{C}$ be a hyperbolic chain consisting of k disks. If $\overline{\mathscr{C}}$ is not simply connected, then $\overline{\mathscr{C}}$ contains at least k hyperbolic points.*

Lemma 2.7. *Let $\mathscr{C}$ be a chain consisting of k disks. If $\mathscr{C}$ is not saturated, then $\overline{\mathscr{C}}$ contains at least k hyperbolic points.*

Flattening. We have already noted that if two hyperbolic disks approach a hyperbolic point H through opposite approach regions, they do not necessarily "fit together" to provide analytic continuations of each other. A related concept is the flattening of Γ at a hyperbolic point. We say that Γ can be *flattened* at H if there is a holomorphic function $h(z, w)$ in a neighborhood of H such that $\Gamma \subset \{\Im m h = 0\}$ holds in a neighborhood of H. The observation that Γ cannot in general be flattened at a hyperbolic point was made in [Be, Appendix; MW].

Because of this situation, we will define the more useful concept of almost flattening. First let us write a hyperbolic chain as $\mathscr{C} = \{R_1, \dots, R_k\}$. We will use the notation $\{H_1, \dots, H_t\}$ to denote the set of hyperbolic points contained in $\overline{\mathscr{C}}$, and we let $\mathscr{C}^\varepsilon$ denote an ε-neighborhood of $\mathscr{C}$. Under a sufficiently small C^2 perturbation, each H_j will be moved slightly to a nearby hyperbolic point H_j'. We will say that the chain $\mathscr{C}$ can be *almost flattened* if for every $\varepsilon > 0$ there exists a closed, complex submanifold $\mathscr{R}$ of $\mathscr{C}^\varepsilon$ and a perturbation Γ' of Γ such that

(i) Γ' is arbitrarily close to Γ in the C^2 norm. (Thus we may assume that the new hyperbolic points H_j' lie arbitrarily close to the old ones H_j.)

(ii) There exists a chain $\mathscr{C}'$ that is contained in $\mathscr{R}$ such that $\partial\mathscr{C}' \subset \Gamma'$ and $\{H_1', \dots, H_t'\} \subset \partial\mathscr{C}'$.

(iii) Γ' coincides with Γ outside an ε-neighborhood of $\mathscr{C}'$.

(iv) For some $\delta > 0$, there is a holomorphic imbedding $H = (h_1, h_2) : \mathscr{R} \times \{|w| < \delta\} \to \mathbb{C}^2$ such that $H(\mathscr{R} \times \{0\}) \supset \mathscr{C}'$, and $H(\mathscr{R} \times (-\delta, \delta))$ contains a neighborhood of $\mathscr{C}'$ in Γ'.

(v) For $-\delta < t < \delta$, the set $R(t) = \{\zeta \in \mathscr{R} : H(\zeta, t) \notin \Gamma'\}$ has only one component that is not relatively compact, and the relatively compact components are disks.

We note that the effect of (v) is to fill a neighborhood of $\partial\mathscr{C}'$ in Γ' with the boundaries of complex disks; the disks themselves are given by $H(\cdot, t) : \omega \to \mathbb{C}^2$, where ω is a relatively compact component of $R(t)$.

The following result is a combination of Lemmas 7.6 and 8.1 of [Be].

Flattening Theorem. *Let $\mathscr{C}$ be a saturated hyperbolic chain such that $\overline{\mathscr{C}}$ is simply connected. Then $\mathscr{C}$ may be almost flattened.*

3. Regularity of hyperbolic disks

Here we study the behavior of a hyperbolic disk σ at a hyperbolic point H that lies in $\partial\sigma$. We may assume that $H = (0, 0)$ is the origin and $T_H\Gamma = \{w = 0\}$. In fact we will consider the special case where Γ is exactly quadratic in a neighborhood of (0,0), i.e., $\Gamma \cap U$ is given by

$$w = \psi(z) := z\overline{z} + \tfrac{\lambda}{2}(z^2 + \overline{z}^2). \tag{1}$$

We show (Lemma 3.2) that for special values of $\lambda > 1$ the complex disk σ extends to a holomorphic variety in a neighborhood of (0,0). As a consequence, we see (Lemma 3.4) that σ has good approach at H. At the end of this section, we show (Lemma 3.5) that σ has a uniqueness property.

Consider the proper holomorphic map $p : \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$p(z, w) = (z, zw + \tfrac{\lambda}{2}(z^2 + w^2)).$$

It is clear that p is generically 2 to 1 and defines a two-fold branched covering of $\mathbb{C}^2$. The preimage of Γ under p consists of two totally real linear subspaces of $\mathbb{C}^2$:

$$L_1 = \{w = \overline{z}\}, \qquad L_2 = \{w = -\overline{z} - \tfrac{2}{\lambda}z\}.$$

We define $R_1 : \mathbb{C}^2 \to \mathbb{C}^2$ to be the antiholomorphic reflection about L_1. To compute it explicitly, we note that

$$L_1 = \{\Im m(z + w) = 0, \Im m(iz - iw) = 0\}.$$

In the holomorphic coordinates, $W_1 = z + w$, $W_2 = iz - iw$, R_1 is given by $R_1(W_1, W_2) = (\overline{W}_1, \overline{W}_2)$. In the original coordinates, this map may be written in matrix form as

$$R_1 = \rho \circ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where ρ denotes complex conjugation. A similar analysis for the antiholo-

morphic reflection about L_2 gives

$$R_2 = \rho \circ \begin{bmatrix} -2/\lambda & -1 \\ (2/\lambda)^2 - 1 & 2/\lambda \end{bmatrix}.$$

Since R_1 and R_2 commute with ρ, we will disregard ρ and view R_1, R_2 as elements of $GL(2, \mathbb{R})$.

Lemma 3.1. *There is a dense subset of values λ in $(1, \infty)$ for which the group generated by R_1, R_2 is isomorphic to the dihedral group D_{2n} for some n, and in particular these reflections generate a finite group.*

Proof. We first make a change of coordinates that leaves R_1 fixed and makes R_2 orthogonal. Any matrix $T = \left[\begin{smallmatrix} t_1 & t_2 \\ t_2 & t_1 \end{smallmatrix}\right]$ commutes with R_1. Setting $a = 2/\lambda$, we calculate

$$\begin{aligned} \tilde{R}_2 &= T^{-1} R_2 T \\ &= \frac{1}{t_1^2 - t_2^2} \begin{bmatrix} -at_1^2 - a^2 t_1 t_2 - at_2^2 & -t_1^2 - 2at_1 t_2 + (1 - a^2 t_2)^2 \\ (a^2 - 1)t_1^2 + 2t_1 t_2 a + t_2^2 & at_1^2 + a^2 t_1 t_2 + at_2^2 \end{bmatrix}. \end{aligned}$$

Since the trace of $\tilde{R}_2$ is zero, it follows that $\tilde{R}_2$ is orthogonal if it is symmetric. The condition for $\tilde{R}_2$ to be symmetric is

$$at_1^2 + 4t_1 + a = 0. \tag{2}$$

This equation has real solutions since $0 < a < 2$.

The axis of reflection of $\tilde{R}_2$ is given by the solution of the equation

$$\tilde{R}_2 \begin{pmatrix} \nu \\ 1 \end{pmatrix} = \begin{pmatrix} \nu \\ 1 \end{pmatrix}. \tag{3}$$

Using our expression for $\tilde{R}_2$, we solve for ν in the first component of (3) to obtain

$$\nu = \frac{-t_1^2 - 2at_1 + 1 - a^2}{(1 + a)t_1^2 + a^2 t_1 + a - 1}. \tag{4}$$

The final point of our calculations is to remark that ν is a nonconstant algebraic function of the variable $\lambda = 2/a$.

The group generated by R_1 and $\tilde{R}_1$ is generated by orthogonal reflections about the vectors $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \nu \\ 1 \end{smallmatrix}\right)$. If the angle between these vectors is a rational multiple of π, say $p\pi/q$ with p and q relatively prime, then R_1 and $\tilde{R}_2$ generate the dihedral group D_{2q}. Since this must occur for a dense set of $1 < \lambda < \infty$, the lemma is proved.

A hyperbolic point H in Γ will be called *good* if (1) holds near H and if the invariant λ satisfies the conclusion of Lemma 3.1.

Lemma 3.2. *Let σ be a hyperbolic disk, and let $H = (0, 0) \in \partial\sigma$ be a good hyperbolic point of Γ. Let η denote a component of σ in a neighborhood of H so that η has the form $\{w = g(z) : z \in \omega\}$ with $g \in \mathscr{O}(\omega) \cap C(\overline{\omega})$. Then it follows that η is contained in a germ of a complex variety passing through the origin, and for some $m \in \mathbb{N}$, g is given on ω by a Puiseux expansion of the form*

$$g(z) = \sum a_k z^{k/m},$$

where the sum is taken over $k \geq 2m$.

Proof. We write $\Delta^+ = \{\zeta \in \mathbb{C} : |\zeta| < 1, \Im m\zeta > 0\}$. By hypothesis, there is a holomorphic map $f : \Delta^+ \to \mathbb{C}^2$ such that $f \in C(\overline{\Delta^+})$, $f(\Delta^+) = \eta$, and $f(0) = H$. Further, we may assume that $f(x) \in \overline{\eta} \cap \Gamma$ for $-1 \leq x \leq 1$.

Let U be a neighborhood of H such that η is a subvariety of $U - \Gamma$. It follows that $p^{-1}(\eta)$ is a complex subvariety of $p^{-1}(U - \Gamma)$. Since H is a good hyperbolic point, the group G generated by the reflections R_1 and R_2 about L_1 and L_2 is finite. Let us set

$$\begin{aligned}\tilde{U} &= \bigcap_{R \in G} Rp^{-1}(U - \Gamma) \\ &= \bigcap_{R \in G} R((p^{-1}U) - L_1 \cup L_2).\end{aligned}$$

Thus $V_0 = \tilde{U} \cap [\bigcup_{R \in G} Rp^{-1}\eta]$ is an analytic subvariety of $\tilde{U}$.

For $R, R' \in G$, $R(L_1) \cap R'(L_2) = \{(0, 0)\}$. Thus it follows from the reflection principle for varieties that V_0 extends as a subvariety through $\tilde{U} \cap (R(L_j) - \{(0, 0)\})$. If $\overline{V}_0$ denotes the closure of V_0 in $\tilde{U}$, then $\overline{V}_0 - \{(0, 0)\}$ is a subvariety of $\tilde{U} - \{(0, 0)\}$. Thus, $\overline{V}_0$ is a subvariety of $\tilde{U}$. Since p is a proper mapping, it follows that $p(\overline{V}_0)$ defines a germ of a variety at the origin.

Now $\{g(z) = w\}$ lies in a complex variety passing through the origin, and so it follows that $g(z)$ has a Puiseux expansion at the origin. Since $g(z) = \psi(z)$ holds for $z \in \partial\omega$, it follows that all of the terms must be $O(|z|^2)$.

Lemma 3.3. *In the notation of Lemma 3.2, the Puiseux expansion has the form*

$$g(z) = \sum_{k > 2m} g_k z^{k/m}. \tag{5}$$

Proof. Let us continue with the notation of the proof of Lemma 3.2. We will show that the coefficient g_{2m} in the Puiseux expansion must vanish. Let us suppose to the contrary that it does not vanish and write

$$g(z) = \alpha z^2 + O(|z|^{2+1/m}). \tag{6}$$

For $\varepsilon > 0$ small, $\partial\omega \cap \{|z| < \varepsilon\}$ consists of two parts

$$\gamma_\pm = f(x \lessgtr 0) \cap \{|z| < \varepsilon\}.$$

Since $\gamma_- \cup \gamma_+ \subset \{\Im m g = 0\}$, each $\gamma_\pm$ is a 1-dimensional real analytic set. By (6) γ_+ and γ_- are each tangential to one of the lines in $\{\Im m \alpha z^2 = 0\}$ at $z = 0$. Thus, $\partial\omega$ has an asymptotic opening of angle κ at $z = 0$, where κ is a multiple of $\pi/2$. We consider the cases separately.

Case (i). $\kappa > \pi$. We recall that the disk $f(\Delta)$ lies inside the open set $\tilde{D} = D \times \mathbb{R}$, and a complex affine coordinate change was introduced so that $H = (0, 0)$ and the new z-axis is the complex tangent space to Γ at H. There is a pluri-subharmonic function r such that $\tilde{D} = \{r < 0\}$, and in the new coordinates near H, we have

$$r(z, w) = -Re w + z\bar{z} + \gamma(z^2 + \bar{z}^2) + O(|z|^3 + |zw| + |w|^2).$$

It follows that $r(z, g(z))$ is a negative subharmonic function on ω which vanishes at 0. If the opening κ is at least π, then it follows from the Hopf Lemma that $\frac{\partial}{\partial z} r(z, g(z)) \neq 0$ at $z = 0$, which is a contradiction.

Case (ii). $\kappa = \pi/2$. Since $g(z)$ is real on $\gamma_\pm$, it follows that $g(z) = \pm|\alpha z^2| + O[|z|^{2+1/m}]$ holds for $z \in \partial\omega$, and since γ_- is tangential to γ_+ at $z = 0$; g must have opposite signs on γ_+ and γ_-. If at the origin γ_- and γ_+ are tangential to lines with angles μ_+ and $\mu_- = \mu_+ + \pi/2$ respectively, then this means that in the notation of (1) we must have $\psi[e^{i\mu_+}] = -\psi[e^{i[\mu_+ + \pi/2]}]$. However, this means that we have a solution to

$$1 + \lambda\cos(2\mu_+) = -1 - \lambda\cos(2(\mu_+ + \pi/2)),$$

which is impossible.

Case (iii). $\kappa = 0$. We note that there must be a point $z \in \omega$ where $\Im m g(z) \neq 0$. Since $\Im m g = \Im m(\alpha z^2 + O(|z|^{2+1/m}))$, it follows that for every $\varepsilon > 0$, there is a sector of angular opening $\pi/2 - \varepsilon$ inside $\{\Im m g \neq 0\}$. If follows that γ_1 and γ_2 must be separated by an angle of at least $\pi/2$.

Since none of these cases can occur, we conclude that $\alpha = 0$, which completes the proof.

We will write

$$\psi(z) = r^2\chi(\theta) = r^2(1 + \lambda\cos(2\theta)) \tag{7}$$

where $r = |z|$. Thus there is an angle $0 < \mu < \pi/4$ such that

$$\{\chi = 0\} = \{\theta = \tfrac{\pi}{2} \pm \mu\}.$$

Let us define

$$\nu = -\chi'(\tfrac{\pi}{2} - \mu) = \chi'(\tfrac{\pi}{2} + \mu) > 0, \tag{8}$$

and let us write

$$g(z) = \alpha z^{k/m} + \beta z^{k'/m} + o(|z|^{k'/m}).$$

We consider a point $z = re^{i\theta(r)} \in \partial\omega$. Since $g(z) = o(r^{k/m})$ and

$$g(z) = \psi(z) = r^2\chi(\theta(r)), \tag{9}$$

we see that

$$\chi(\theta(r)) = O[r^{k/m-2}].$$

From this we conclude that

$$\lim_{\substack{z\to 0\\ z\in\gamma_\pm}} \theta(r) = \tfrac{\pi}{2} \mp \mu,$$

and in fact

$$\theta(r) = \tfrac{\pi}{2} \mp \mu + o(r^{k/m-2}). \tag{10}$$

In particular we have shown the following, since it follows that $\partial\omega$ is tangential to the lines $\{\theta = \pi/2 \pm \mu\}$ at $z = 0$.

Lemma 3.4. *A hyperbolic disk at a good hyperbolic point has good approach.*

Now to continue with our discussion, we remark that since $\{\theta = \pi/2 \pm \mu\} = \{\Im m\alpha z^{k/m} = 0\}$, it follows that α is real and $2\mu = \frac{m}{k}\pi$. It also follows that for $z \in \gamma_\pm$,

$$g(z) = \pm\alpha|z|^{k/m} + O(|z|^{k'/m}). \tag{11}$$

Appealing again to (9) and (11), we deduce that for $z \in \gamma_\pm$

$$\theta(r) = \tfrac{\pi}{2} \mp \mu - \tfrac{\alpha}{\nu} r^{k/m-2} + O[r^{(k+1)/m-2}]. \tag{12}$$

We note that the contants μ, ν, and k/m are determined by λ. Inspection of the influence of $\Im m\beta z^{k'/m}$ on the curve $\gamma_\pm$ reveals that $k' = 2k(k-m)$, although we will not need to use this fact.

Important for us, however, will be the following uniqueness result.

Lemma 3.5. *Let σ_1, σ_2 be hyperbolic disks at a good hyperbolic point H. If, for some $\varepsilon > 0$, $\omega_1 \cap \{|z| < \varepsilon\} \subset \omega_2 \cap \{|z| < \varepsilon\}$, and if $\Re e\, g_1(z) \le \Re e\, g_2(z)$ for $z \in \omega_1 \cap \{|z| < \varepsilon\}$, then $\sigma_1 = \sigma_2$.*

Proof. We first consider the Puiseux expansion $g_j(z) = \alpha_j z^{k/m} + \cdots$ for $j = 1, 2$. Since the boundary of ω_j is given by (12), and since near the origin ω_1 is contained in ω_2, it follows that $\alpha_1 = \alpha_2$.

Next we consider the difference

$$g_2(z) - g_1(z) = \sum_{s>k} c_s z^{s/m}.$$

If g_1 is not identically equal to g_2, then there will be a first nonvanishing

coefficient c_q. Now we have

$$0 < \Re e(g_2(z) - g_1(z)) = \Re e c_q z^{q/m} + o(|z|^{(q+1)/m})$$

on $\omega_1 \cap \{|z| < \varepsilon\}$. But this is not possible, since $\Re e c_q z^{q/m}$ can be positive only on a sector of angle $\pi m/q$ whereas $\partial\omega_1$ forms a strictly larger angle at $z = 0$.

4. Hyperbolic chains

In this section we discuss the procedure of sliding complex disks and the global behavior ("joining"/"splitting") of chains. In the following arguments leading to Lemma 4.5, we will assume that all complex tangencies are either elliptic or good hyperbolic points. If $\sigma \subset X$ is a complex disk with boundary in Γ^*, then there is an integer index $\operatorname{ind}(\Gamma, \sigma)$ of the pair Γ and σ. (See [F1] for this, and also [Be, BG], where the quantity $\operatorname{ind}(\Gamma, \sigma) + 1$ is called a "winding number".) If $\operatorname{ind}(\Gamma, \sigma) = 0$, then there is a one-parameter family of disks $\sigma(t)$, $-\varepsilon < t < \varepsilon$, depending smoothly on t, with boundary in Γ and with $\sigma = \sigma(0)$. The sets $\overline{\sigma(t)}$ are disjoint, and the curves $\partial\sigma(t)$ sweep out an open neighborhood of $\partial\sigma$ in Γ.

Now we want to review the procedure of sliding a one-parameter family of complex disks with boundaries in Γ^*. This was carried out for the case where Γ is a graph without complex tangencies in §8 of [BG]. The case where a graph Γ is allowed to have hyperbolic points does not add any essential difficulties and was considered in [Be]. Here we show what modifications need to be made if Γ is not a graph. (See [G, E1] for a discussion of how to pass to the more general case of almost complex manifolds.)

We consider a family of complex disks $\sigma(t)$ for $0 \le t < \infty$. One possibility is that the closure of $\bigcup_{0<t<\infty} \partial\sigma(t)$ contains an elliptic point E. By Bishop [Bi1] there is a one-parameter family of disks $\sigma'(s)$ growing out of E. Applying the Uniqueness Theorem to the families $\sigma(t)$ and $\sigma'(s)$, we see that they must coincide (after a change of parameter) and so $\lim_{t\to\infty} \sigma(t) = E$.

Let us write $\Sigma = \bigcup_{0\le t<\infty} \sigma(t)$ and $\sigma(\infty) = (\overline{\Sigma} - \Sigma) \cap \Omega$.

Lemma 4.1. *The limit set $\sigma(\infty)$ is a union of complex disks in Ω.*

Proof. The first step of the proof is to show that $\sigma(\infty)$ is a complex subvariety of Ω. To this end we will show that the area of $\sigma(t)$ remains bounded for $t < \infty$. Thus by the theorem of Bishop [Bi2] $\sigma(\infty)$ is a variety. If we set $\eta = \frac{i}{2}(z_1 d\overline{z}_1 + z_2 d\overline{z}_2)$, then the Kähler form for the standard Euclidean metric on $\mathbb{C}^2$ is $d\eta$, which is exact. By the Wirtinger Inequality, we have Area $(\sigma(t)) = \int_{\sigma(t)} d\eta$. Now let $\Gamma(t) \subset \Gamma$ be a region such that $\partial\Gamma(t) = \partial\sigma(t) \cup \gamma_0$, where γ_0 is a curve that is independent of t. Thus

$$\text{Area } (\sigma(t)) = \int_{\partial\sigma(t)} \eta - \int_{\gamma_0} \eta = \int_{\Gamma(t)} d\eta - \int_{\gamma_0} \eta.$$

Since $\int_{\Gamma(t)} d\eta$ is bounded above by Area (Γ), we see that Area $(\sigma(t))$ is bounded above.

Next we show that σ is nonsingular. If $z_0 \in \sigma(\infty)$ is a singular point, we may take a ball B of radius δ and center z_0, and we may assume that $\gamma = \overline{\sigma} \cap \partial B$ is a smooth curve. If z_0 is a singular point, then γ is knotted in the 3-sphere ∂B (cf. [Mi]). If t_0 is sufficiently large, then $\gamma(t_0) = \sigma(t_0) \cap \partial B$ will be a small deformation of γ inside ∂B. It will suffice to show that there is a isotopy deforming $\gamma(t_0)$ to a point inside ∂B. From this it will follow that $\gamma(t_0)$, and thus γ, is not knotted.

We may replace B by a small perturbation so that the intersection of ∂B with the family $\sigma(t)$ is generic. That is, for all but a finite number of values of t, $\gamma(t)$ is a smooth curve. And for the exceptional values of t, $\gamma(t)$ has only one singular point, which can have only one of two possible forms:

(i) a component of $\gamma(t)$ is a point; or

(ii) $\gamma(t)$ contains a "hyperbolic" singular point p, where two disks of $\sigma(t) \cap B$ "join," i.e., there are exactly two complex disks $\sigma'(t)$, $\sigma''(t)$ that are the connected components of $\sigma(t) \cap B$ containing p in their boundaries.

For $t > 0$ sufficiently small, $\gamma(t) = \varnothing$. Let us consider first a special case. Suppose that the only singular values are $0 < t_2', t_2'' < t_1 < t_0$. Suppose that t_1 corresponds to case (ii), and t_2', t_2'' correspond to (i). Thus $\sigma(t_1) \cap B$ splits into two complex disks $\sigma'(t_1)$ and $\sigma''(t_1)$ with boundaries in ∂B, and $p = \overline{\sigma'(t_1)} \cap \overline{\sigma''(t_1)}$. We may extend $\sigma'(t_1)$ to a one-parameter family of complex disks $\{\sigma'(t) : t_2' < t < t_1\}$.

The family of curves $\gamma'(t) := \overline{\sigma'(t)} \cap \partial B$, $t_2' < t < t_1$ gives an isotopy inside ∂B deforming $\gamma'(t_1)$ to a point. We may replace the curves $\gamma'(t)$ by an isotopy $\tau(s)$, $0 < s < 1$, such that $p \in \tau(s)$ for all s, and $\tau(s)$ deforms $\gamma'(t_1)$ to the point p. It is now evident that for $\varepsilon > 0$ small, there is an isotopy $\{\tilde{\gamma}(s) : 0 < s < 1\} \subset \partial B$ deforming $\gamma(t_1 + \varepsilon)$ to $\overline{\sigma''(t_1 - \varepsilon)} \cap \partial B$.

To complete the second step, we proceed by induction on the number of singular points. If there are no singular points of type (ii), then the family $\gamma(t)$ gives an isotopy deforming $\gamma(t_0)$ to a point. Otherwise, we let $0 < t_1 < t_0$ be the smallest value of t for which $\gamma(t)$ has a singular point of type (ii). Now there are exactly two components of $\gamma(t_1) \cap B$, which we denote by $\sigma'(t_1)$ and $\sigma''(t_1)$, which contain p in their closure; and we may construct smooth one-parameter families $\sigma'(t)$ for $t_2' < t < t_1$ and $\sigma''(t)$ for $t_2'' < t < t_1$. The only singularities that can occur in these families are of type (i), so these families vanish to a point. Thus, as in the previous paragraph, we may create an isotopy deforming $\sigma'(t_1) \cup \sigma''(t_1)$ to a point. The remainder has fewer singular points, and the proof of the second step is complete.

For the third step of the proof we need to show that the components of σ are disks. Since $\sigma(t)$ is a disk for all $t < \infty$, we conclude by Lemma 2.5, that each component of σ will be equivalent to a disk, possibly with a finite number of punctures. But there is a psh function ρ such that $\Omega = \{\rho < 0\}$.

If a component σ' of σ were equivalent to a punctured disk, however, $\rho|_{\sigma'}$ would extend to be subharmonic and zero on the puncture, which would violate the maximum principle. Thus there can be no punctures.

By the Regularity Theorem, $\sigma(\infty)$ is smooth at all points of $\Gamma^* \cap \overline{\sigma}(\infty)$, and by Lemma 3.4, $\sigma(\infty)$ has good approach at the complex tangencies (which are assumed to be good hyperbolic points). Thus the components of $\sigma(\infty)$ are complex disks, which completes the proof.

Lemma 4.2. *If $\partial\sigma(\infty)$ contains a hyperbolic point, then $\sigma(\infty)$ is a hyperbolic chain.*

Proof. By Lemma 4.1, $\sigma(\infty)$ is a union of complex disks, and it is clear that $\overline{\sigma(\infty)}$ is connected. Given a hyperbolic point H, we may assume that $H = (0, 0)$, and Γ is given in a neighborhood U of H as a graph $\{w = \psi(z, \overline{z})\}$. We may assume that for each t, $\sigma(t) \cap U$ is given as a graph $\{w = f(z, t) : z \in \omega(t)\}$, where $\omega(t)$ is a smoothly bounded Jordan domain in a neighborhood of $0 \in \mathbb{C}$. Since the curves $\{\partial\sigma(t)\}$ are disjoint, $\{\omega(t)\}$ is a monotone family of domains converging to $\omega(\infty)$, and $\partial\omega(\infty)$ is smooth away from the origin. Thus each component of $\omega(\infty)$ is a Jordan domain, and by Lemma 3.3, $f(z, \infty)$ is of class $C^2(\overline{\omega(\infty)})$ and $f(0, \infty) = f'(0, \infty) = f''(0, \infty) = 0$. By Lemma 3.4, $\sigma(\infty)$ has good approach to $(0, 0)$, and by Lemma 3.5, each approach region can be used only once. Thus $\sigma(\infty)$ is a hyperbolic chain, which completes the proof.

Let $\{\sigma(t) : 0 < t < T_0\}$ be a one-parameter family of complex disks with boundaries in Γ^*. If $\lim_{t \to T_0} \sigma(t)$ is again a complex disk with boundary in Γ^*, then the family may be extended to a one-parameter family for $0 < t < T_0 + \varepsilon$. We will say that the family $\{\sigma(t) : t < T_0\}$ is *maximal* if it cannot be extended further. It is natural to write a maximal family as $\{\sigma(t) : t < \infty\}$.

In this terminology, Lemma 4.2 says that *a maximal family converges either to an elliptic point or to a hyperbolic chain.* We note that a family $\{\sigma_t : t < T\}$ assigns an orientation to Γ in the following manner. We let dt denote a 1-form on Γ which corresponds to the direction of increasing t, and we let $d\theta$ be induced by the natural orientation of $\partial\sigma(t)$. We say that the family $\{\sigma(t) : t < T\}$ is *positive* if the orientation assigned by $dt \wedge d\theta$ agrees with the orientation already on Γ.

Let us consider further the situation in Lemma 4.1 and how $\sigma(t)$ may approach σ near a hyperbolic point H as t approaches infinity. By Lemma 2.4, we may represent the disk $\sigma(t)$ locally as a graph near H. That is, we may choose coordinates (z, ω) near H so that $H = (0, 0)$, and there is a neighborhood U of $(0, 0)$ such that $\sigma(t) \cap U = \{\omega = g(z, t) : z \in \omega(t)\}$ where $\omega(t) \subset \mathbb{C}$ is open, and $g(z, t)$ is analytic in z.

Since the boundaries $\partial\sigma(t)$ are disjoint, we may choose U small so that the domains $\omega(t)$ are either monotone increasing or decreasing with t. We say that the family of disks $\sigma(t)$ *approaches σ from inside at H* if the family $\omega(t)$ is monotone increasing in t. Otherwise $\sigma(t)$ *approaches σ from outside*

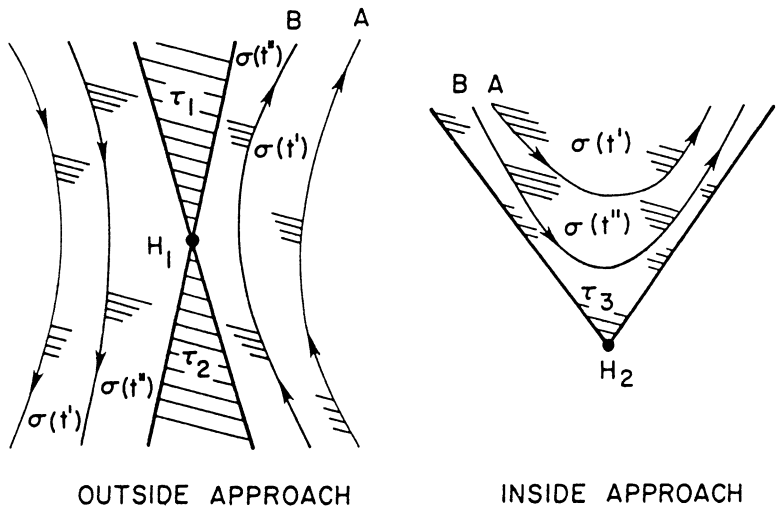

FIGURE 1

at H. A basic observation is

Lemma 4.3. *If* $\{\sigma(t) : t < T\}$ *is a positive family that approaches a positive hyperbolic point* H, *then it must approach from the inside. If a positive family approaches* H *from the outside, then* H *is a negative point.*

Proof. This is evident from checking the orientation at a totally real boundary point of a hyperbolic disk.

If we write the limit chain $\sigma(\infty)$ as $\sigma = \{\tau_1, \dots, \tau_p\}$ where each τ_i is a hyperbolic disk, then there are two cases. Either σ contains only one disk τ_1 with $H \in \partial\tau_1$ or it contains two disks τ_1, τ_2 with $H \in \partial\tau_1 \cap \partial\tau_2$. By Lemma 3.4, τ_j has good approach at H for $j = 1, 2$. Thus in the case of inside approach, the family $\omega(t)$ behaves near H as in Figure 1. Now we show that the case of outside approach also corresponds to Figure 1.

Lemma 4.4. *If a family* $\{\sigma(t)\}$ *of complex disks approaches a good hyperbolic point* H *from the outside, then the limit chain* $\sigma(\infty)$ *fills both approach regions at* H.

Proof. As was noted above, the disks $\sigma(t)$ converge near H to either a single hyperbolic disk τ_1 or $\tau_1 \cup \tau_2$. The domains $\omega(t)$ are decreasing, and the two cases above correspond to (i) $\omega_1 = \operatorname{int}(\bigcap \omega(t))$ and (ii) $\omega_1 \cup \omega_2 = \operatorname{int}(\bigcap \omega(t))$. It will suffice to show that (i) cannot occur.

Let us assume, to the contrary, that we are in case (i). We may assume that $\Gamma = \{w = z\bar{z} + \lambda \Im m z^2\}$ near $H = (0, 0)$ and that ω_1 is asymptotic to $\{z\bar{z} + \lambda \Im m z^2 < 0,\ \Re e z \geq 0\}$ at $z = 0$. The curves $\partial\omega(t)$ converge smoothly to $\partial\omega_1$ away from H. Thus we see that for t sufficiently large, $\partial\omega(t)$ is as in Figure 2. That is, it stays within an ε-neighborhood of $\partial\omega_1$.

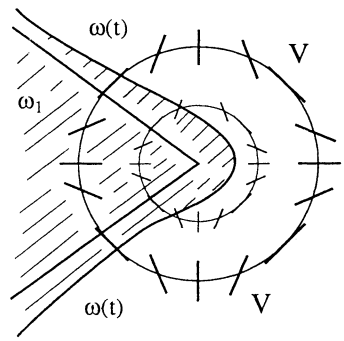

FIGURE 2

Now for $c < 0$, we consider the set

$$\nu_c = \{(z, c) : \Re e z < 0, \quad z\bar{z} + \lambda \Re e z^2 = c_2\}.$$

and the complex disk $\tilde{\nu}_c = \nu_c \times \{c\} \subset \mathbb{C}^2$. By construction, the portion of $\partial \tilde{\nu}_c$ near H lies in Γ. It is clear that for t large, there is a value of c for which $\partial \omega(t)$ and $\partial \nu_c$ intersect in a single point p. Since the disks $\sigma(t)$ converge to τ_1, we see that we must have $\Re e g(z, t) \geq \Re e g_1(z)$ for $z \in \omega_1$. It follows that $\Re e g(z, t) > c$ for $z \in \omega(t)$. We may apply the Uniqueness Theorem to conclude that $\sigma(t) = \tilde{\nu}_c$, which is a contradiction.

Remark. Let E denote the set of elliptic points. When we say that M' may be swept out by boundaries of complex disks, we mean that there are finitely many hyperbolic chains $\mathscr{C}_1, \ldots, \mathscr{C}_r$ such that each hyperbolic point is contained in the closure of a unique chain, and there is a foliation $\mathscr{F}$ of $\Gamma - (\bigcup_j \partial \mathscr{C}_j \cup E)$ such that each leaf of $\mathscr{F}$ is the boundary of a complex disk $\sigma \subset \Omega$.

Lemma 4.5. *Let $M \subset \Gamma$ be an open subset such that $\partial M \subset \Gamma^*$ consists of the boundaries of m complex disks $\{\sigma_1, \ldots, \sigma_m\}$ inside Ω, and assume that for each j there is a one-parameter family of complex disks $\{\sigma_t^{(j)} : t \leq T_j\}$ with boundaries in $\Gamma - \overline{M}$ such that $\sigma_j = \sigma_{T_j}^{(j)}$. Renumbering these families, we may suppose there exists a number e^+ such that the jth family is positive for $1 \leq j \leq e^+$ and negative for $e^+ < j \leq m$. Let $H_1, \ldots, H_n$ denote the hyperbolic points contained in M, and suppose that h^+ is chosen such that H_j is positive for $1 \leq j \leq h^+$ and negative otherwise. If $e^+ > h^+$, then there is an arbitrarily small C^2 perturbation Γ' of Γ with the properties:*

(i) *There is an open set $U \subset X$, $U \supset \Gamma - M$ such that $\Gamma' \cap U = \Gamma \cap U$;*
(ii) *If we set $M' = (\Gamma' - U) \cup (M \cap U)$, then M' may be swept out by the boundaries of complex disks.*

Proof. We proceed by induction on the number of hyperbolic points in M. If $n = 0$, there are no hyperbolic points, and the family $\sigma_1(t)$ may be continued

until it reaches a boundary curve, say σ_2. By the uniqueness theorem, $\sigma_1(\infty) = \sigma_2$. Thus $m = 2$, and the boundaries of $\sigma_1(t)$ sweep out the region M.

Now we suppose $m > 0$. Let us extend the positive families of disks to maximal families, and let us consider the limit chains $\mathscr{C}_1, \ldots, \mathscr{C}_{e^+}$. There are two possibilities to consider.

The first possibility is that the hyperbolic points in $\overline{\sigma}_1(\infty) \cup \cdots \cup \overline{\sigma}_{e^+}(\infty)$ are positive. By Lemma 3.5 there is only one disk that can occupy an approach region to a hyperbolic point. Thus we may group the disks of $\sigma_1(\infty) \cup \cdots \cup \sigma_{e^+}(\infty)$ into a set of chains $\mathscr{C}_1, \ldots, \mathscr{C}_r$ with the property that $\overline{\mathscr{C}}_i \cap \overline{\mathscr{C}}_j = \varnothing$ if $i \neq j$.

Since the hyperbolic points in $\overline{\sigma}_1(\infty) \cup \cdots \cup \overline{\sigma}_{e^+}(\infty)$ are all positive, we conclude from Lemma 4.3 that the approach of any family $\{\sigma_i(t)\}$, $1 \leq i \leq e^+$ to any of these points is from the inside. Thus $\sigma_1(\infty) \cup \cdots \cup \sigma_{e^+}(\infty)$ contains e^+ distinct disks. Since there are fewer positive hyperbolic points than complex disks, we conclude that one of the chains, say $\mathscr{C}_1$, has more disks than hyperbolic points. By Lemmas 2.6 and 2.7, we conclude $\overline{\mathscr{C}}_1$ is saturated and simply connected.

We may apply the Flattening Theorem to conclude that there are a small perturbation M' of M in a neighborhood of $\partial\mathscr{C}_1$ and a chain $\mathscr{C}'$ in M' whose boundary contains hyperbolic points $H_1'', \ldots, H_s''$, which are small perturbations of $H_1', \ldots, H_s'$. In particular, this gives a flattened open set $M_0 \subset \Gamma'$ containing $\mathscr{C}'$ and that is bounded by complex disks $\sigma_1', \ldots, \sigma_r'$.

Now we claim that we may proceed by induction on the region $M - \overline{M}_0$. We observe that each σ_i' is part of a positive family (since it is a continuation of $\{\sigma_1(t)\}$, which was positive). We let $\sigma_1', \ldots, \sigma_a'$ denote the positive families of disks approaching $\mathscr{C}'$, and we let $\sigma_1'', \ldots, \sigma_b''$ denote the positive families continuing on the other side of $\mathscr{C}'$. Since $\mathscr{C}'$ is saturated and simply connected, it follows that $a - b$ is equal to the number of hyperbolic points in $\overline{\mathscr{C}}'$. Thus we have removed as many hyperbolic points as positive families of disks, and so we may continue by induction.

The other possibility to be considered is where there are negative hyperbolic points in $\overline{\sigma}_1(\infty) \cup \cdots \cup \overline{\sigma}_{e^+}(\infty)$. As before, we break $\overline{\sigma}_1(\infty) \cup \cdots \cup \overline{\sigma}_{e^+}(\infty)$ into chains $\mathscr{C}_1, \ldots, \mathscr{C}_r$. Suppose H is a negative hyperbolic point, say $H \in \overline{\mathscr{C}}$ and $H \in \overline{\sigma}_1(\infty)$. By Lemma 4.3, the family $\sigma_1(t)$ must approach H from the outside, and by Lemma 4.4, $\overline{\sigma}_1(\infty)$ must contain both approach regions at H. We conclude, then, that each $\mathscr{C}_i$ is saturated at all of its negative hyperbolic points.

Now let us consider a 2-manifold $\mathscr{D}_i$, which is the same as $\mathscr{C}_i$, except that we fatten it slightly so that it contains a small two-dimensional disk about each negative hyperbolic point in $\overline{\mathscr{C}}_i$. Thus $\mathscr{D}_i$ contains its negative hyperbolic points in its interior and $\mathscr{D}_i$ coincides with $\mathscr{C}_i$ at all of its positive hyperbolic points.

Let $\mathscr{D}_{i,1}, \ldots, \mathscr{D}_{i,s_i}$ denote the connected components of $\mathscr{D}_i$. We may

assume that each $\mathscr{D}_{i,j}$ is a disk. This is seen from the following alternative construction of $\mathscr{D}_{i,j}$.

Let σ denote a disk of $\mathscr{C}_i$, with $\sigma \subset \mathscr{D}_{i,j}$. Let $\{H'\}$ denote the negative hyperbolic points contained in $\overline{\sigma}$. By Lemmas 4.3 and 4.4, σ contains both approach regions at each H'. Since the approach is from outside, we see that by filling in a neighborhood of H', we may obtain something homeomorphic to $\sigma(t)$ for t large. Since this construction gives $\mathscr{D}_{i,j}$, we see that $\mathscr{D}_{i,j}$ is homeomorphic to a disk.

Now we observe that Lemmas 2.6 and 2.7 are essentially combinatorial in nature, and so we may apply them to the disks $\mathscr{D}_{i,j}$ to conclude that some $\mathscr{D}_i$ is simply connected and saturated. In particular we may assume that $\mathscr{C}_1$ is simply connected and saturated.

As in the previous case, we may perturb Γ and flatten in a neighborhood of $\mathscr{C}_1$. Thus we may continue to the other side of $\mathscr{C}_1'$ with new families of disks $\{\sigma_i''(t)\}$. Again we observe that the number of positive hyperbolic points that we have eliminated is the same as the number of positive families of disks. Thus we may proceed by induction, and the proof is complete.

Lemma 4.6. *Suppose that there is a family $\mathscr{F}$ of complex disks whose boundaries are disjoint and sweep out Γ. Let $\sigma \subset \Omega$ be a complex disk with $\partial\sigma \subset \Gamma^*$ and such that $\operatorname{ind}(\sigma, \Gamma) = 0$. Then*

(i) *σ belongs to the family $\mathscr{F}$, and*
(ii) *σ does not intersect any other disk in $\mathscr{F}$; in particular the disks in $\mathscr{F}$ are pairwise disjoint.*

Proof. Let $\mathscr{C}_1, \ldots, \mathscr{C}_r$ be the hyperbolic chains as in the remark above. If $\partial\sigma$ contains a hyperbolic point, then by Lemma 3.5, σ is contained in some $\mathscr{C}_i$, and thus $\sigma \in \mathscr{F}$. Otherwise, $\partial\sigma$ is contained in the totally real points of Γ.

Let us consider the maximal one-parameter family $\Sigma = \{\sigma(t) : -\infty < t < \infty\}$ generated by σ. Let $\mathscr{C}^{\pm}$ denote the forward and backward limits of this maximal family. If $\mathscr{C}^+$ or $\mathscr{C}^-$ is an elliptic point E, then it is clear that Σ must be the family of disks emanating from E. Thus $\sigma \in \mathscr{F}$. Otherwise, $\mathscr{C}^{\pm}$ are hyperbolic chains, and again by Proposition 3.5 $\mathscr{C}^+ \subset \mathscr{C}_i$ for some i. Now let $\sigma_i \in \mathscr{F}$ be a disk which separates σ from $\mathscr{C}_i$. Let $0 < t_i < \infty$ be the first value of t for which

$$\partial\sigma(t_i) \cap \partial\sigma_i \neq \varnothing.$$

By the Uniqueness Theorem, we conclude that $\sigma(t_i) = \sigma_i$. Thus $\Sigma \subset \mathscr{F}$, which completes the proof of (i).

For (ii), we suppose that there exist σ, $\tau \in \mathscr{F}$ such that $\sigma \cap \tau \neq \varnothing$. Now the intersection of two distinct, irreducible analytic varieties is stable, so the set of disks $\tau \neq \sigma$ in $\mathscr{F}$ that intersect σ is open. On the other hand, since the disks of $\mathscr{F}$ have disjoint boundaries, it follows that the points of intersection cannot approach the boundary; thus the set of τ that intersect σ is closed. Finally, the local one-parameter family of disks $\sigma(t)$ near σ does not intersect

σ except in the case $\sigma(t_0) = \sigma$. Thus, since the set of disks is a connected, compact set, we conclude that $\sigma \cap \tau \neq \varnothing$ implies that $\sigma = \tau$.

We will use Lemma 4.5 to sweep out Γ by the boundaries of complex disks. The purpose of Lemma 4.6 is to show that the disks thus obtained actually generate a topological 3-manifold.

Corollary 4.7. *Suppose that there is a one-parameter family $\mathscr{F}$ of complex disks whose boundaries are disjoint and sweep out Γ. Then $B = \bigcup_{\sigma \in \mathscr{F}} \sigma$ is a topological 3-manifold. Further, there are finitely many hyperbolic chains $\mathscr{C}_1, \ldots, \mathscr{C}_r$ whose boundaries are disjoint and contain the hyperbolic points of Γ, and*

$$B - (\mathscr{C}_1 \cup \cdots \cup \mathscr{C}_r) = F_1 \cup \cdots \cup F_s,$$

where each F_i is the union of a maximal one-parameter family of complex disks with boundaries in Γ^*. Each F_i is a smooth 3-manifold with smooth boundary along $\overline{F}_i \cap (\Gamma - (\partial\mathscr{C}_1 \cup \cdots \cup \partial\mathscr{C}_r))$.

We remark that if Γ is a 2-sphere, then it is clear that B is a 3-ball.

Proof. By Lemma 4.6, two disks of $\mathscr{F}$ cannot intersect unless they coincide. Thus the disks sweep out a 3-manifold. The sets F_i are smooth because of the regularity of the construction of a one-parameter family of disks with boundary in Γ^*.

5. Proofs of the theorems

Here we show how the main results of this paper follow from the lemmas of the previous sections.

Lemma 5.1. *If Γ is a smooth 2-sphere in X, then there is a small C^2 perturbation Γ' of Γ such that:*

(i) *All complex tangencies of Γ' are either elliptic or hyperbolic;*
(ii) *Γ' may be flattened in a neighborhood of each elliptic point;*
(iii) *Each hyperbolic point is good.*

Proof. This result is local near the complex tangencies, so we may assume that $\Gamma = \Gamma(\varphi_0)$ is a graph. Let us first consider a complex tangency of $\Gamma(\varphi_0)$, which we may assume occurs at (0,0). We may suppose that at $(0, 0)$ ∂D is given by

$$u = \Re e\alpha_{1,0} z + \alpha_{1,1} z\overline{z} + \Re e\alpha_{2,0} z^2 + o(|z|^2),$$

and that on this tangent plane,

$$\varphi_0 = \Re e\beta_1 z + \beta_{1,1} z\overline{z} + \Re e\beta_{2,0} z^2 + o(|z|^2).$$

The condition that $\Gamma(\varphi_0)$ have a complex tangent is $\alpha_{1,0} = i\beta_{1,0}$. Removing the linear terms by a complex linear change of coordinates, we find that $\Gamma(\varphi_0)$ is given as

$$w = Az\overline{z} + Bz^2 + C\overline{z}^2 + \psi(z),$$

where $A = \alpha_{1,1} + i\beta_{1,1}$, $B = \alpha_{2,0} + i\beta_{2,0}$, $C = \overline{\alpha}_{2,0} + i\overline{\beta}_{2,0}$, and $\psi(z) = o(|z|^2)$. The condition that the complex tangency be elliptic or hyperbolic is $2|A|^{-1}|C| \neq 1$. This may be achieved by arbitrarily small perturbations in either $\beta_{1,1}$ or $\beta_{2,0}$, which gives two real degrees of freedom. It follows from the Thom Transversality Theorem (cf. [GG, p. 54]) that for a generic $\varphi \in C^2(\partial D)$, all complex tangencies of $\Gamma(\varphi)$ are either elliptic or hyperbolic. Clearly there is a dense subset of $\varphi \in C^2(\partial D)$ for which the invariant λ at each hyperbolic point satisfies the conclusions of Lemma 3.1.

Now it remains only to show that at a complex tangency the higher order terms may be flattened by an arbitrary small C^2 perturbation. This is achieved by replacing ψ by $\chi(|z|/\varepsilon)\psi(z)$, where χ is a smooth function with $\chi(t) = 0$ for $t < 1$ and $\chi(t) = 1$ for $t > 2$. It is easily checked that $(1 - \chi(|z|/\varepsilon))\psi$ tends to zero in C^2 as $\varepsilon \to 0$.

Proof of Theorem 1. Let us make a small C^2 perturbation Γ'' of Γ satisfying the conclusions of Lemma 5.1. We may push Γ'' "out" of Ω so that there is a strongly pseudoconvex domain $\Omega'' \supset \overline{\Omega}$ with $\Gamma'' \subset \partial\Omega''$.

Since Γ is a 2-sphere, we have $i^+ + i^- = 2$. Reversing the orientation of Γ, if necessary, we may assume that $i^+ > 0$. Let $E_1, \ldots, E_r$ denote the elliptic points of Γ''. Since Γ'' is flat near E_i, we may build one-parameter families $\{\sigma_i(t) : t \leq T\}$ of complex disks starting at $E_1, \ldots, E_r$. We note that the family $\{\sigma_i(t)\}$ is positive if and only if E_i is a positive elliptic point.

Let M denote the connected region in Γ'' bounded by $\bigcup \sigma_i(T)$. By the condition $i^+ > 0$, we may apply Lemma 4.5, and thus there is a small perturbation M' of M that bounds a one-parameter family of complex disks. Let us denote again by Γ'' the perturbed 2-manifold containing M'. By Corollary 4.7, Γ'' bounds a topological 3-manifold B'', which is locally a one-parameter family of complex disks.

We may make a small perturbation of the disks of B'' so that the 3-manifold B'' is smooth in a neighborhood of $\overline{\Omega}$. Let us set $\Gamma' = \partial\Omega \cap B''$. We note that Γ' is smooth at all points where the complex disk in B'' is not tangent to $\partial\Omega$.

Now the set $\Gamma' = \partial\Omega \cap B''$ is smooth wherever $\partial\Omega$ and B'' intersect transversally. If the intersection is tangential, then it occurs where a complex disk of B'' is tangent to $\partial\Omega$. Generically, there are only two possibilities: an elliptic point (vanishing disk) or a hyperbolic point (splitting disk). At either of these points, a small perturbation of B'' by a holomorphic mapping will guarantee that $\partial\Omega$ and B'' intersect transversally at a complex tangency.

The proof is completed with

Lemma 5.2. *$\overline{B}'$ is the envelope of Γ'.*

Proof. We note that a function holomorphic in a neighborhood of Γ' may be continued to a neighborhood of $\overline{B}'$ by Theorem 3.1 of [BG]. Although that theorem is stated for the case where Γ' is a 2-sphere, the essential ingredient in

the proof is that B' is swept out by a family of complex disks like the family obtained in Lemma 4.5. By moving Γ' "forward" and "backward" inside $\partial\Omega$, we obtain 2-spheres $\Gamma'^{\pm}$ that bound Levi-flat 3-balls $B'^{\pm}$. Let U denote the region of Ω such that $\Omega \cap \partial U = B'^{+} \cup B'^{-}$. By such a construction (bumping out $\partial\Omega$, too) we obtain a fundamental system pseudoconvex neighborhoods of $\overline{B}'$.

Proof of Theorem 2. Theorem 2 is a special case of Theorem 5.3.

Theorem 5.3. *Let X and Γ be as in Theorem* 1. *If all hyperbolic points of Γ are good, then there is a* 3*-dimensional topological manifold B that is the disjoint union of complex disks and $\partial B = \Gamma$. Further, $\overline{B}$ is the envelope of holomorphy of Γ.*

Proof. Let us first show that for any totally real point $p \in \Gamma^*$ there is a complex disk $\sigma_p \subset \Omega$ with $p \in \partial\sigma_p \subset \partial\Omega$. Let $\{\Gamma_j\}$ be a sequence of 2-manifolds in $\partial\Omega$ converging in C^2 to Γ and for which the envelopes B_j satisfy the conclusions of Theorem 1. We let U be a neighborhood of p in Γ on which the Lipschitz estimate holds. We may smoothly identify U with a neighborhood in Γ_j. Then there are disks σ_j with $p \in \partial\sigma_j \subset \Gamma_j$ such that $\partial\sigma_j \cap U$ converges to a smooth curve $\gamma \subset U$ containing p.

By the bound on area in Lemma 4.1, the area of σ_j is uniformly bounded, so the σ_j converge to a subvariety σ_∞ of Ω. By the Lipschitz estimate on $\partial\sigma_j \cap U$ we conclude that σ_∞ is not empty, and $\gamma \subset \partial\sigma_\infty$. Since σ_∞ is the limit of one-parameter families of complex disks, we see that σ_∞ is nonsingular, as in the proof of Lemma 4.1.

By the Uniqueness Theorem in §2, we see that if $\partial\sigma_{p_1} \cap \partial\sigma_{p_2} \cap \Gamma^* \neq 0$, then $\sigma_{p_1} = \sigma_{p_2}$. Further, by Lemma 3.5, if $\partial\sigma_{p_1}$ and $\partial\sigma_{p_2}$ intersect at a hyperbolic point H, then either $\sigma_{p_1} = \sigma_{p_2}$, or σ_{p_1} and σ_{p_2} represent the two approach regions to H. We conclude from this that the family $\mathscr{F} = \{\sigma_p\}$ has disjoint boundaries that sweep out Γ. By Corollary 4.7 $B = \bigcup \sigma_p$ is a topological 3-manifold with $\partial B = \Gamma$. By Lemma 5.2 $\overline{B}$ is the envelope of Γ, which completes the proof.

Proof of Theorem 3. By Theorem 1, we may choose a sequence $\varphi_j \in C^\infty(\partial D)$ that converges to φ in C^2, and so there is a smooth solution Φ_j such that $\hat{\Gamma}(\Phi_j)$ is Levi-flat. We note that $\hat{\Gamma}(\Phi_j - \varepsilon_j)$ is again a Levi-flat surface for any constant ε_j. If we let $\{\varepsilon_j\}$ be a sequence decreasing to zero such that $\varphi_j - \varepsilon_j < \varphi < \varphi_j + \varepsilon_j$, then we conclude that $\{\Phi_j\}$ converges uniformly on $\overline{D}$ to a function Φ. Since $\| \Phi_j \|_{\mathrm{Lip}}$ is uniformly bounded in terms of $\| \varphi_j \|_{C^2}$, we conclude that $\Phi \in \mathrm{Lip}^1(\overline{D})$.

(ii) follows Lemma 5.2; (iii) follows from the proof of Theorem 2 (§9) of [Be]; and (iv) is derived from (iii) as follows. If $p(z, w)$ is any polynomial, then $p(f(\zeta))$ is a bounded analytic function whose boundary values satisfy

$$|p(f(\zeta))| \leq \sup_{\Gamma(\varphi)} |p|$$

for a.e. $\zeta \in \partial\Delta$. It follows then that this inequality holds for any $\zeta \in \Delta$. Since this inequality holds for all polynomials, we conclude that $f(\zeta)$ belongs to the polynomial hull of $\Gamma(\varphi)$. Now by (iii) we conclude that $f(\zeta) \in \hat{\Gamma}(\Phi)$.

To prove (i), we first show that the interior of $\hat{\Gamma}(\Phi)$ is foliated by complex manifolds. Let us fix a point of D, which after a translation of coordinates may be assumed to be (0,0). The interior of each surface $\hat{\Gamma}(\Phi_j)$ is foliated by complex manifolds, and we want to take the limit of the foliations as $j \to \infty$. We can do this because there is a uniform bound $|dw/dz| \leq K$ on the leaves on the foliations. (Here K depends on $\| \varphi_j \|_{C^2}$.) Choosing $\varepsilon > 0$ small enough that $\{|z|, |u| < \varepsilon\} \subset D$, we see that if $|u_0| < \varepsilon/K$, then the leaf of $\hat{\Gamma}(\Phi_j)$ over $(0, u_0)$ is given as a graph $\{w = g_j(z, u_0) : |z| < \varepsilon\}$. Here each g_j is holomorphic in z, $|\partial g/\partial z| \leq K$, $\Re e g_j$ is monotone increasing in u, and $g_j(0, u_0) = u_0$. It follows that $\lim_{j\to\infty} g_j(z, u) = g(z, u)$ exists and again has these properties. Since $\Re e g(z, u)$ is monotone in u, and $g(0, u) = u$, and since the graph of g lies in $\hat{\Gamma}(\Phi)$, it follows that g is continuous in u. This gives the existence of the foliation of the interior of $\hat{\Gamma}(\Phi)$.

If $g : \Delta \to \mathbb{C}^2$ is a holomorphic mapping with $g(\Delta) \subset \operatorname{int} \hat{\Gamma}(\Phi)$, then it follows from Lemma 4.6 of [Be] that $g(\Delta)$ is contained in one of the leaves of M.

Finally, we suppose $\Gamma(\varphi)$ is totally real at the point (z_0, w_0) and assume $(z_0, w_0) = (0, 0)$. By the Lipschitz estimate, we may choose holomorphic coordinates (ζ, η) near (0,0) for which there exists constants $\varepsilon, K > 0$ such that $|d\eta/d\zeta| < K$ holds for any leaf of $M \cap \{|\zeta| + |\eta| < \varepsilon\}$. Further, the directions $\{(1, \alpha) : |\alpha| \leq K\}$ are transverse to the tangent space of $\partial D \times i\mathbb{R}$ at all points of $(\partial D \times i\mathbb{R}) \cap \{|\zeta| + |\eta| < \varepsilon\}$. By the transversality, we may choose ε' small enough that every leaf of M of $M \cap \{|\zeta| < \varepsilon', |\eta| < \varepsilon\}$ may be written as a graph $M = \{\eta = f(\zeta) : \zeta \in \omega_M\}$. By the uniform Lipschitz condition of f, this foliation may be continued from the interior of $\hat{\Gamma}(\Phi)$ to $\hat{\Gamma}(\Phi) \cap \{|\zeta| < \varepsilon', |\eta| < \varepsilon\}$. This completes the proof.

Corollary 5.4. *In Theorems* 1–3, *we may make a small perturbation* Γ' *of* Γ *so that in the corresponding solution* B', *all hyperbolic chains contain only two disks.*

Proof. Let B be the 3-manifold that is swept out by complex disks. A chain may be disconnected at a hyperbolic point $H \in \Gamma$ by pushing Γ "into" B at H. We omit the details.

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