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THE FUNDAMENTAL GROUP OF THE VON NEUMANN ALGEBRA OF A FREE GROUP WITH INFINITELY MANY GENERATORS IS $\mathbb{R}_+\backslash\{0\}$

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In this paper we show that the fundamental group $\mathcal{F}$ of the von Neumann algebra $\mathcal{L}(F_\infty)$ of a free (noncommutative) group with infinitely many generators is $\mathbb{R}_+ \setminus \{0\}$. This extends the result of Voiculescu who previously proved [26, 27] that $\mathbb{Q}_+ \setminus \{0\}$ is contained in $\mathcal{F}(\mathcal{L}(F_\infty))$. This solves a classical problem in the harmonic analysis of the free group F_∞. In particular, it follows that there exists subfactors of $\mathcal{L}(F_\infty)$ with index s for every $s \in [4, \infty)$. We will use the noncommutative (quantum) probabilistic approach introduced in Voiculescu's paper.

Von Neumann algebras were introduced by Murray and von Neumann in the early thirties to provide a framework for quantum physics. As formulated by Heisenberg, quantization amounts to replacing the algebra of "observable" functions on the phase space of a classical system by a noncommuting algebra of infinite matrices, or more precisely, operators on a Hilbert space. By definition, a von Neumann algebra is a weakly closed, selfadjoint algebra of bounded operators on a Hilbert space that contains the identity operator.

Where as any commutative von Neumann algebra is *-isomorphic to the algebra of bounded functions on a measure space, the structure of the noncommutative algebras is much more subtle. The simplest noncommutative von Neumann algebras are the $n \times n$ matrix algebras and the "hyperfinite algebras," i.e., the inductive limits of matrix algebras. Von Neumann algebras have provided a very powerful tool for studying noncommutative ergodic theory. More recently, Vaughan Jones has shown [13, 14] that the study of inclusions of von Neumann algebras naturally leads to new polynomial invariants for knot theory and to solutions of the Yang Baxter equation.

Murray and von Neumann began by distinguishing three "type" classes for von Neumann algebras. Restricting to factors (von Neumann algebras with trivial center) on a separable Hilbert space, the type I_n factors have a dimension function on the projections (i.e., selfadjoint idempotents) which assumes the values $\{0, 1, \dots, n\}$ $(n \leq \infty)$. Type II_1 (resp., II_∞) algebras also have a dimension function, which assumes the values in $[0,1]$ (resp., $[0, \infty]$). Such

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continuous dimensions arise quite naturually in the theory of transversally elliptic operators for a foliation and in the theory of Wiener-Hopf operators with almost periodic symbols [22, 2, 3]. Finally, the type III factors are characterized by the fact that they have no dimension function [9, 1, 20].

Any algebra of type I_n is isomorphic to the algebra of operators on a Hilbert space of dimension n. The simplest examples of II_1 factors are provided by infinite tensor products of matrix algebras and by the regular von Neumann algebra completions of the group algebras of suitable discrete groups G. The algebra of the ergodic relation associated with the holonomy group of a foliation is often of type III. Owing to the work of Connes [7] and Haagerup [12], the hyperfinite factors have been essentially classified, and, in particular, there is only one hyperfinite III_1 factor.

Murray and von Neumann primarily focused their attention on type II_1 factors. They proved that if $G = F_N$, the free group on $2 \leq N \leq \infty$ generators, then the factor $\mathscr{L}(F_N)$ is not hyperfinite. In order to distinguish other examples, they introduced an invariant they called the fundamental group $\mathscr{F}(M)$ of a II_1 factor M. Letting M have the trace τ (this is the linear extension of the dimension function), they defined

$$\mathscr{F}(M) = \{t > 0 \mid \exists \text{ projection } p \in M, \tau(p) = t, pMp \cong M\}.$$

We have, for example, that $n \in \mathbb{N}$ belongs to $\mathscr{F}(M)$ if and only if $M_n(\mathbb{C}) \otimes M$ is isomorphic to M, and thus for dimension reasons the fundamental group of $M = M_n$ is trivial. On the other hand, for the hyperfinite II_1 factor M, $\mathscr{F}(M) = \mathbb{R}_+$. It remained open whether or not $\mathscr{F}(\mathscr{L}(F_N))$ contained values other 1 itself. Connes proved [5] that for rigid groups like $\mathrm{SL}(3, \mathbb{Z})$ the fundamental group is countable (see also [18]).

Later it appeared that the fundamental group is also important for other parts of the theory of Neumann algebras. Connes's analysis [6, 4] of the type III factors showed that one can build a specific type III_t factor out of a number $t \in (0, 1)$ which sits in the fundamental group $\mathscr{F}(M)$ of a type II_1 factor M (see also [25]). For pairs of subfactors $N \subset M$ Jones proved [13] the striking result that the "dimension" of M as projective left module over N (the index of N into M) takes a discrete and a continuous series of values: $\{4\cos^2 \frac{\pi}{n}, n = 3, 4, \dots\} \cup [4, \infty)$, each value leading to specific representations of the braid group (with associated Markov traces). He also proved that given any number $t/(1-t) \in \mathscr{F}(M)$ one constructs a subfactor of index $t^{-1} + (1-t)^{-1}$, for any II_1 factor M (see also [18, 10, 11, 17, 29]).

The first breakthrough in the analysis of von Neumann algebras associated with free groups was recently achieved by Voiculescu, who showed the occurrence of free algebras in concrete probabilistic situations. The starting point was Wigner's idea (which, in fact, is much closer to Heisenberg's matricial mechanics) to view hamiltonians as matrices of very large size rather than infinite ones as in von Neumann's approach [28]. Based on experimental data he observed that the density of eigenvalues of random matrices tends to obey a semicircular law (rather then a Gaussian law as expected).

Voiculescu introduced [26, 27] a noncommutative (quantum) probability theory whose basic objects inherit the asymptotic properties of families of random matrices and showed that these objects are naturally isomorphic to the von Neumann algebras of free groups. Hence these last algebras are the natural frame for noncommutative probability, playing the same role as commutative algebras play in usual probability theory (i.e., bosonic case) and hyperfinite factors in the anticommutative case (i.e., fermionic case). This led him to conclude [27] that $\mathbb{Q}_+$ is contained in the fundamental group $\mathscr{F}(\mathscr{L}(F_\infty))$ of a free group with infinitely many generators.

In this setting we proved that a noncommutative (quantum) probabilistic free infinite family $(X^s)_{s\in S}$ is well behaved by reduction by p where p is idempotent, i.e., that the reduced family $(pX^sp)_{s\in S}$ can be continued to a larger noncommutative probabilistic free family. This was in a certain sense a missing step in the harmonic analysis of a free group F_∞ and when translated into von Neumann algebra terms this gives

Theorem. *The fundamental group $\mathscr{F}(\mathscr{L}(F_\infty))$ of the type* II_1 *factor $\mathscr{L}(F_\infty)$ of a free group with infinitely many generators is exactly the the nonnegative real numbers* $\mathbb{R}_+\backslash\{0\}$.

This shows that $\mathscr{L}(F_\infty)$, which is a universal model for noncommutative relations and which, as shown before, is the best type II_1 factor after the hyperfinite one, has the same behaviour as the hyperfinite one with respect to the reduction operation. This suggests that the analysis and his consequences carried out for the latter factor could also be done for $\mathscr{L}(F_\infty)$. As shown at the beginning, the theorem provides new numbers as indices for subfactors in $\mathscr{L}(F_\infty)$.

Corollary. *For any $s \in [4, \infty)$ there is a (nonirreducible) subfactor of $\mathscr{L}(F_\infty)$ of index s.*

Moreover, using the theorem and by Connes's classification, one can construct "wild" examples of type III factors with specific noncommutative properties:

Corollary. *For every* tin(0, 1) *there exists a type* III_t *factor with a core isomorphic to $\mathscr{L}(F_\infty)\overline{\otimes}B(H)$.*

Remark. If we replace in the construction before $\mathscr{L}(F_\infty)\overline{\otimes}B(H)$ by

$$N = (\mathscr{L}(F_\infty)\overline{\otimes}B(H))\overline{\otimes}R$$

where R is the hyperfinite factor and take $Q_{\lambda,\gamma,p}$ be the type III_λ factor obtained as the cross product $N \rtimes_{\theta_\lambda\overline{\otimes}\alpha_p} \mathbb{Z}$, then we obtain by [8] an infinite family of non-anti-isomorphic type III_λ factors, which in addition have the core isomorphic to $\mathscr{L}(F_\infty)\overline{\otimes}B(H)\overline{\otimes}R$. Here H is an infinite-dimensional space, $\gamma^2 \neq 1$, and α_p is the cyclic (of order p) outer automorphism of R that has invariants p and γ [4]. Moreover, θ_λ, $\lambda \in (0, 1)$, is the automorphism of $\mathscr{L}(F_\infty) \otimes B(H)$, scaling trace by λ, given by our theorem.

1. Definitions

Let H be a Hilbert space, $B(H)$ the space of all bounded linear operators acting on H. A weakly closed unital subalgebra M of $B(H)$ is called a von Neumann algebra. When we do not mention the underlying Hilbert space on which M acts, we will also say that M is a W^*-algebra [21]. In this case the weak topology on M comes from its predual M_* [21]. An abelian von Neumann subalgebra of M is called diffuse if it does not contain minimal projections. As usual, if B is a W^*-algebra and Σ is a subset of B, Σ'' will be the von Neumann subalgebra of B generated by Σ and $1 \in B$.

By $\mathscr{P}(M)$ we denote the set of all projections $p = p^2 = p^*$ in M, while by $\mathscr{Z}(M)$ we denote the center of M. For a projection e in M we denote by eMe or M_e the reduced algebra which, regarded as acting on eH, is in fact a von Neumann subalgebra of $B(eH)$. If M is a finite factor (i.e., $\mathscr{Z}(M) = \mathbb{C}$ and there exists a normal finite faithful trace τ on M with $\tau(1) = 1$) then we denote by M_t the isomorphism class of eMe, when e is any projection in M of trace t.

If $t \geq 1$ then we consider an infinite dimensional separable space H, and endow $B(H)$ with the usual faithful semifinite normal trace that takes value 1 when evaluated on projections of dimension 1. Let e be any projection in $M \otimes B(H)$ (equipped with the tensorial product traces τ') and, in this case, let M_t be the isomorphism class of $(M \otimes B(H))_e$. It is well known [9, 15, 24, 23] that this isomorphism class does not depend on the choices made in the selection of the projection e. In particular, we obtain in this way an equivalent definition for the fundamental group of M:

$$\mathscr{F}(M) = \{t > 0 \mid (\exists)\theta \in \operatorname{Aut}(M\overline{\otimes}B(H)),\ \tau'(\theta(x) = t\tau'(x),\ x \in M\overline{\otimes}B(H))\}.$$

Here $\operatorname{Aut}(M)$ stands for the automorphism group of a W^*- algebra M, while the property of the automorphism θ, in the above formula, is called "scaling trace by t" [6, 25].

If $\mathscr{A}$ is any W^*- algebra then a matrix unit $(w_{ij})_{i,j=1}^n$ in $\mathscr{A}$ is simply a copy of $M_n(\mathbb{C})$ inside $\mathscr{A}$ with the same unit, i.e., $(w_{ij})_{i,j=1}^n$ are partial isometries with $w_{ij}w_{kl} = \delta_{jk}w_{il}$ (where δ_{kl} denotes the Kronecker symbol). Usually $M_n(\mathbb{C})$ comes with a canonical matrix unit $(e_{ij})_{i,j=1}^n$ where e_{ij} is the matrix whose only nonnull entry is the entry (i, j), which has value 1. Moreover, $M_n(\mathbb{C})$ has a canonical trace τ_n which is normalized by $\tau_n(1) = 1$.

Recall some definitions and facts from [26]. Let (A, φ) be a C^*-algebra with unit 1 equipped with a trace φ. A family of subalgebras $1 \in A_i \subseteq A$ $(i \in I)$ is called a free family of subalgebras if $\varphi(a_1a_2\cdots a_n) = 0$ whenever $a_j \in A_{i_j}$ with $i_j \neq i_{j+1}$, $1 \leq j \leq n-1$, $\varphi(a_j) = 0$, $j = 1, 2, \dots, n$. A family of subsets $(\Omega_i)_{i\in I}$ is called free if the family of the subalgebras generated by the Ω_i and 1 is free, and a family $(f_i)_{i\in I}$ is free if the family $\{f_i\}_i$ is free. Moreover, a free family $(f_i)_i$ is called semicircular if f_i are selfadjoint and if the distributions of the f_i, $i \in I$, are given by the semicircle law

$$\varphi(f_i^k) = 2/\pi \int_{-1}^{1} t^k(1-t^2)^{1/2}\,dt, \qquad k \in \mathbb{Z},\ k \geq 0.$$

It is called circular if the family $\{x_i\}_i \cup \{y_i\}_i$ is semicircular where $f_j = x_j + iy_j$, $j \in I$, is the decomposition of f_j into real and imaginary parts.

A natural example of a free semicircular family is provided [26] by $(\lambda_{g_i} + (\lambda_{g_i})^{-1})_{i\in S}$, where F_S is a nonabelian free group with generators $(g_s)_{s\in S}$.

Moreover, the family $(\lambda_{g_i})_{i\in S}$ is a free family (with respect to the usual trace on $\mathscr{L}(F_\infty)$).

For any discrete group G, we denote by $\lambda = (\lambda_g)_{g\in G} : G \to B(l^2(G))$ the regular left representation of G, i.e., $(\lambda_g f)(h) = f(g^{-1}h)$, $g, h \in G$, $f \in L^2(G)$. Moreover, $\mathscr{L}(G)$ is the von Neumann algebra generated in $B(L^2(G))$ by $(\lambda_g)_{g\in G}$, and if G has infinite conjugacy classes then $\mathscr{L}(G)$ is a factor with trace τ given by $\tau(\lambda_g) = 0$ for $g \neq e$ and by $\tau(\lambda_e) = 1$, e being the neutral element in G. In particular, it follows that the von Neumann algebra generated by an infinite free semicircular family $(f_i)_{i\in S}$ is isomorphic to $\mathscr{L}(F_\infty)$. To observe that, one has to consider unitaries F_i that generate the same abelian (diffuse) von Neumann algebra as f_i, for each $i \in S$. It follows that the family $(F_i)_{i\in S}$ is also free and hence that the von Neumann algebra generated by them is isomorphic to $\mathscr{L}(F_\infty)$ (see [26]). By abuse of language we will continue to call a family semicircular (and consequently circular) even if we assume that $2/\pi$ is replaced in the defining formula by powers of a certain nonnull constant.

2. Outline of the proof

The main device in Voiculescu's (noncommutative) approach to the structure of the von Neumann algebras of free groups was to obtain a different view of a free semicircular family $(X_s)_{s\in S}$ by considering it as subset of $M_n(\mathbb{C}) \otimes B$, where B is a W^*-algebra containing an infinite free family. Namely [27, Proposition 2.6], if $\omega_1 = \{g(i, j, s) \mid 1 \le i, j \le n, s \in S\}$ is a free circular family in b, $\omega_2 = \{f(i, s) \mid 1 \le n, s \in S\}$ is a free semicircular family in B, $\omega_1 \cup \omega_2$ is free, and $1 \in D \subseteq B$ is a commutative subalgebra, then letting

$$X_s = \sum_{1\le i,j\le n} (g(i, j, s)\otimes e_{ij}) + (g(i, j, s)^* \otimes e_{ji}) + \sum_{1\le i\le n} f(i, s)\otimes e_{ii}, \qquad s \in S$$

we obtain a free semicircular family $(X_s)_{s\in S} \subseteq M_n(\mathbb{C})\otimes B$ which is also free with respect to the algebra $D_n \otimes D$. Here $(e_{ij})_{i,j=1}^n$ is the canonical matrix unit in $M_n(\mathbb{C})$ and D_n is the diagonal algebra generated by $\{e_{11}, \dots, e_{nn}\}$. Using this, Voiculescu was able to prove that in the reduced algebra $(1\otimes e_{11})\{X_s\}''(1\otimes e_{11})$ one can find another free semicircular family. Suitably rephrased, this shows that $\mathscr{L}(F_k)_{(1/N)} \cong \mathscr{L}(F_{kN^2-N^2+1})$, and hence that $\mathbb{Q} \subseteq \mathscr{L}(F_\infty)$ [27, Theorem 3.2].

The first step towards the proof of this result was the remark [27, Lemma 3.1] that if $(w_{ij})_{i,j=1}^n$ is a matrix unit in a W^* algebra $\mathscr{A}$ and Σ is a system of generators for $\mathscr{A}$ (i.e., $\Sigma'' = \mathscr{A}$) then a system of generators for the reduced algebra $w_{11}\mathscr{A}w_{11}$ is

$$\bigcup_{1\le i,j\le n} w_{1i}\sum w_{j1}.$$

If $\mathscr{A} = \{X_s\}'' \subseteq M_n(\mathbb{C}) \otimes B$ then fixing $\sigma \in S$, one can build a matrix unit in $\mathscr{A}$ by taking the unitaries coming out of the polar decomposition of the elements $g(1, j, \sigma)$, $2 \le j \le n$. Since all the entries of the X_s's were free, it follows that by bringing them back on the (1,1) entry (in order to get a system of generators for the reduced algebra), one obtains another free family of generators (with more generators). This last assertion follows from Lemma 2.5 in [27], which says that in a free family, replacing one element by the other two elements coming out of its polar decomposition yields a free family.

As a first step towards passing from rational to real numbers, we first describe explicitly the free generators of the reduced algebra $\mathscr{L}(F_\infty)_{(k/n)}$ (which by Voiculescu's theorem is isomorphic to $\mathscr{L}(F_\infty)$), where $1 \leq k < n$, $n \in \mathbb{N}$. We prove in this way that if $e = 1 \otimes e_{11} + \cdots + 1 \otimes e_{kk}$, then (with the notation as before) the family $(e(X_s)e)_{s\in S}$ can be continued to a larger free semicircular family $(Y_t)_{t\in T} \subseteq e\mathscr{A}e$ that generates the reduced algebra $e\mathscr{A}e$, where $\mathscr{A} = \{(X_s)_{s\in S}\}''$.

To prove that we use an obvious extension of the earlier mentioned lemma [27, Lemma 3.1]: if $(w_{ij})_{i,j=1}^n \subseteq \mathscr{A}$ is a matrix unit and Σ a system of generators, then to obtain a system of generators for the reduced algebra $e\mathscr{A}e$, one has to include in a system of generators for $e\mathscr{A}e$ (in addition to $(w_{ij})_{i,j=1}^k$ and $e\Sigma e$), an element of the form $w_{pi}\sigma w_{jq}$, for each i, j with $i > k$ or $j > k$ and for each $\sigma \in \Sigma$, where $p, q \leq k$ and p, q depend on i, j, and σ. Using this trick and by recombining the entries of the X^s that were brought back on the lower indexed entries (i, j), $i, j \leq k$, we get a family in $M_k(\mathbb{C}) \otimes B$ containing the reduced elements $e(X_s)e$, $s \in S$, and whose entries obey the freeness conditions that were assumed for the entries of the initial family (again we have to use Lemma 2.5 in [27], which shows that by replacing, in a circular family, an element by the two other elements coming from its polar decomposition, one obtains another free family). Moreover, by construction $\{(Y_t)_{t\in T}, eA\}'' = e\mathscr{A}e$.

Assume we want to show (by repeated use of the above procedure) that

$$e\mathscr{L}(F_\infty)e \cong \mathscr{L}(F_\infty)$$

whenever e is a projection in the algebra, with $\tau(e) \in \mathbb{R} \setminus \mathbb{Q}_+$. To do this, we choose one of the generators and construct in the abelian von Neumann algebra A generated by it, a decreasing family of projections $(e_n)_{n\in\mathbb{N}}$ of rational trace and with $e_n \to e$.

As in Voiculescu's paper, the algebra A will be represented by the inductive limit of the diagonal algebras of the matrix algebras that appear in the iterated construction (this algebra will be free with respect to all others elements because it may, at each step, be identified with the subalgebra $D \otimes D_n$ that appears in the construction of the X^s's as elements of $M_n(\mathbb{C}) \otimes B$.

We make repeated use of the procedure described above to obtain recursively families $(Y_t^{(n+1)})_{t\in T_{n+1}}$ in $e_{n+1}\mathscr{A}e_{n+1} = e_{n+1}(e_n\mathscr{A}e_n)e_{n+1}$ that together with the corresponding reduced algebra Ae_{n+1} generate the algebra $e_{n+1}\mathscr{A}e_{n+1}$. Moreover, we may assume that this new family also contains all the $e_{n+1}Y_s^n e_{n+1}$, $s \in T_{n+1}$.

A natural guess would be that the free semicircular family obtained as the increasing union of $(eY_t^n e)_{t\in T_n}$ is (together with eAe) a system of generators for $e\mathscr{A}e$. Unfortunately this is not always true. We can make it true by a careful choice of the new family $(Y_t^{n+1})_{t\in T_{n+1}}$ at each step; namely, by imposing that for large values of i or j, the functions p, q take small values.

This means that at each step n, the family $(Y_t^n)_{t\in T}$ contains more and more elements from the reduced algebra. Hence we have to modify the induction step (the procedure of obtaining generators for the reduced algebra) by requiring that given two other projections $e_+, e_- \in A$, with $e_+ \geq e \geq e_-$, almost all pieces (entries) of the $(X_s)_{s\in S} \subseteq M_n(\mathbb{C}) \otimes B$ that have the initial or the final space under the projection $1 - e_+$ are moved in the procedure of construction of generators of the reduced algebra (by a suitable choice of the functions p, q) into entries sitting under the projection e_-.

In this way one can pass to the limit to obtain, given any free semicircular family $(X_s)_{s\in S}$ and any projection e sitting in an algebra A, which is free with respect to $\{(X_s)_{s\in S}\}''$, another free semicircular family $(Y_t)_{t\in T}$ that contains the reduced pieces $(eX_se)_{s\in S}$ and is sufficiently big, i.e., it generates (together with eAe) the reduced algebra $e\{(X_s)_{s\in S}, A\}''e$.

This last fact concerns, in fact, the harmonic analysis of a free group, and by Voiculescu's theorem that describes the von Neumann algebra of a free group with infinitely many generators as the von Neumann algebra generated by an infinite semicircular family, we obtain that the reduced algebra (by e) is also isomorphic to $\mathscr{L}(F_\infty)$. Since the trace of e was arbitrary and since the isomorphism class does not depend on the choice of the projection e (as long as it has the same trace), it follows that the fundamental group of $\mathscr{L}(F_\infty)$ is indeed $\mathbb{R}_+ \setminus \{0\}$.

3. Proof of the results

The first lemma we prove is an extension of Lemma 3.1 in [27]. It shows how to choose a suitable family of generators for a reduced algebra, given a family of generators (containing a matrix unit) in the initial algebra.

Lemma 1. *Let B be a (unital) W^*-algebra, let Σ be a system of elements in B, and let $(w_{ij})_{i,j=1}^n$ be a matrix unit in B. Assume that B is generated by Σ and $(w_{ij})_{i,j=1}^n$, as a von Neumann algebra. Let $1 \leq r < n$ be an integer and let e_1 be the projection $w_{11} + \cdots + w_{rr}$.*

Then for each choice of two functions p,q defined on $\{1, 2, \ldots, n\}^2 \times \Sigma$ with values into $\{1, 2, \ldots, r\}$, a possible system of generators for the reduced W^*-algebra e_1Be_1 is: $(w_{ij})_{i,j=1}^r \cup e_1\Sigma e_1 \cup \Sigma_1$ where $e_1\Sigma e_1 = \{e_1\alpha e_1 \mid \alpha \in \Sigma\}$ and

$$\Sigma_1 = \{w_{p(k,l,\alpha),k}\,\alpha\, w_{l,q(k,l,\alpha)} \mid \alpha \in \Sigma,\ 1 \leq k, l \leq n,\ k \text{ or } l > r\}.$$

Moreover whenever α is selfadjoint it is sufficient to consider in the set Σ_1 only those k, l such that $1 \leq k \leq l \leq n$, $l > r$.

Proof. By density arguments (and since $e_1\Sigma e_1$ is contained between the specified generators of e_1Be_1) it is sufficient to prove that all products P of the form

$$P = w_{i_0,i_1}\alpha_1 w_{i_1,i_2}\alpha_2 \cdots w_{i_{2n-2},i_{2n-1}}\alpha_n w_{i_{2n},i_{2n+1}},$$

with $i_1, i_2, \dots, i_{2n} \in \{r+1, \dots, n\}$, $i_0, i_{2n+1} \in \{1, 2, \dots, r\}$, and $\alpha_1, \dots, \alpha_n \in \Sigma$, are contained in the von Neumann algebra generated by the elements specified in the statement. Denoting

$$p_j = p(i_{2j-1}, i_{2j}, \alpha_j) \quad q_j = q(i_{2j-1}, i_{2j}, \alpha_j), \qquad j = 1, \dots, n$$

we have the following expression for P

$$P = w_{i_0, p_1}(w_{p_1, i_1}\alpha_1 w_{i_2, q_1})w_{q_1, p_2}(w_{p_2, i_3}\alpha_2 w_{i_4, q_2}) \cdots \\ \times w_{q_{n-1}, p_n}(w_{p_n, i_{2n-1}}\alpha_n w_{i_{2n}, q_n})w_{q_n, i_{2n+1}},$$

which shows that P is indeed a product of elements from the set $(w_{ij})_{i,j=1}^r \cup \Sigma_1$. This completes the proof.

Let M be a type II_1 factor and $A \subseteq M$ a diffuse abelian von Neumann subalgebra. Let $(X^s)_{s\in S}$ be an infinite semicircular family, and assume that the family $(X^s)_{s\in S}$ is free with respect to A and that $\{(X^s)_{s\in S}, A\}'' = M$. Let e be any projection in A. In the next lemma we construct a free semicircular family, that includes the reduced part of the initial family $(eX^se)_{s\in S}$, and which does for eMe and eAe the same task as does $(X^s)_{s\in S}$ for M and A (i.e., $(Y^t)_{t\in T}$ is free with respect to eA and $\{(Y^t)_{t\in T}, eA\}'' = eMe$). In particular, M and eMe are both isomorphic to $\mathscr{L}(F_\infty)$.

When extending this result to the case when e has irational trace, we will also need that $(Y^t)_t$ has a supplementary property (property (iv) in the next statement).

Lemma 2. *Let (M, τ) be a type II_1 factor endowed with the normal trace τ and A an abelian diffuse von Neumann subalgebra. Assume that M is generated (as a von Neumann algebra) by the infinite semicircular family $(X^s)_{s\in S}$ and by A and that $(X^s)_{s\in S}$ is free with respect to A.*

For any three distinct projections with rational trace e_+, e_-, e_1 in A that are nontrivial (i.e., different from 0, 1) and satisfy $e_+ \geq e_1 \geq e_-$, there exists a family $(Y_t)_{t\in T}$ in e_1Me_1 (with $S \subseteq T$) such that the following properties hold true

(i) $(Y_t)_{t\in T}$ *is a semicircular family that is free with respect the reduced algebra $e_1Ae_1 = e_1A$.*

(ii) *The von Neumann algebra generated in e_1Me_1 by $(Y_t)_{t\in T}$ and Ae_1 is e_1Me_1 itself.*

(iii) $Y_s = e_1X^se_1$ *for s in S.*

(iv) *If $f = (1-e_+)+e_-$ and B is the von Neumann algebra generated in fMf by $(fX^sf)_{s\in S}$ and fA, then e_-Be_- is contained in the von Neumann algebra generated by $(e_-Y_te_-)_{t\in T}$ and $(e_-)A$.*

Proof. Assume that the traces of $e_1, e_\pm$ have a common denominator $n \in \mathbb{N}$, and let $(g_i)_{i=1}^n$ be a partition of the unity in A (i.e., $(g_i)_{i=1}^n$ are orthogonal projections of equal trace with sum 1), such that $e_1 = g_1 + \cdots + g_r$, $e_\pm = g_1 + \cdots + g_{r_\pm}$, where $r, r_\pm$ are integers such that $r_+ > r > r_-$ (which by taking n large enough, we may assume also satisfy $1 < r_- < r_+ < n$).

We will use the unicity, up to trace preserving isomorphisms, of a free family [27, Proposition 2.9] to give the following realization of $(X^s)_{s\in S}$ and A.

Let (N, φ) be a W^*-algebra endowed with the normal finite trace φ and containing an infinite semicircular family, let $M_n(\mathbb{C})$ be the matrix algebra with the usual normalized trace, and let $(e_{ij})_{i,j=1}^n$ be the canonical matrix unit. Let $N \otimes M_n(\mathbb{C})$ be endowed with the canonical tensorial product trace. We will identify M with a subalgebra of $N \otimes M_n(\mathbb{C})$ by taking

$$X^s = \sum_{i<j}(g(i, j, s) \otimes e_{ij} + g(i, j, s)^* \otimes e_{ji}) + \sum_{i=1}^n f(i, s) \otimes e_{ii}, \qquad s \in S$$

where

$$\omega_1 = \{g(i, j, s) \mid 1 \le i < j \le n, s \in S\}$$

is a circular family in N and

$$\omega_2 = \{f(i, s) \mid 1 \le i \le n, s \in S\}$$

is a semicircular family in N. Moreover, we assume that $\omega_1 \cup \omega_2$ is a free family.

Finally we realize A as the von Neumann subalgebra of $N \otimes M_n(\mathbb{C})$, generated by the selfadjoint Z given by the expression

$$Z = \sum_k (a + k) \otimes e_{kk}.$$

Here $0 \le a \le 1$ is a positive element in N generating a diffuse abelian von Neumann algebra and such that the family $\omega_1 \cup \omega_2 \cup \{a\}$ is free.

In this way (using as we said before Proposition 2.9 of [27]) we realize M as a subalgebra of $N \otimes M_n(\mathbb{C})$. Moreover, we may perform this identification such that $g_i = e_{ii}$ for $1 \le i \le n$. As in the proof of Theorem 3.2 in [27], we construct a matrix unit $(w_{i,j})_{i,j=1}^n$ in M as follows: let σ in S be fixed and let

$$g(1, p, \sigma) \otimes e_{1p} = (v(1, p, \sigma) \otimes e_{1p})(b(1, p, \sigma) \otimes e_{pp}), \qquad 2 \le p \le n,$$

be the polar decomposition of $(1 \otimes e_{11})X^s(1 \otimes e_{pp})$.

By Proposition 2.5 of [27], replacing in ω_1 the set $\{g(1, p, \sigma) \mid 2 \le p \le n\}$ by

$$\{v(1, p, \sigma); b(1, p, \sigma) \mid 2 \le p \le n\},$$

we obtain another free family , where $v(1, p, \sigma)$ are unitaries in N. Let $v(1, 1, \sigma) = 1$, $v_{ij} = (v(1, i, \sigma))^*(v(1, j, \sigma))$, $w_{ij} = v_{ij} \otimes e_{ij}$, $1 \le i, j \le n$. In this way we obtain a specific matrix unit $(w_{ij})_{i,j=1}^n \subseteq M$.

We will apply Lemma 1 to M, the set of generators for M being the matrix unit $(w_{ij})_{i,j=1}^n$ and the set

$$\begin{aligned}\Sigma = (X^s)_{s\in S\setminus\sigma\}} \cup \{Z\} \cup \{b(1, p, \sigma) \otimes e_{pp} \mid p \ge 2\} \\ \cup \{g(i, j, \sigma) \otimes e_{ij} \mid 2 \le i < j \le n\} \\ \cup \{f(i, \sigma) \otimes e_{ii} \mid 1 \le i \le n\}.\end{aligned}$$

Lemma 1 will allow us to set up a system of generators Σ_1 for e_1Me_1. Moreover, by imposing certain conditions on the functions p, q (in the statement of that lemma), we will be able to select these generators so that condition (iv) is also fulfilled.

By the construction of $(w_{ij})_{i,j=1}^n$, we have that $(w_{ij})_{i,j=1}^n$ is already contained in $\{(e_1X^\sigma e_1)\cup Ae_1\}''$. Hence, by the Lemma 1, it remains to include in Σ_1, for each α in Σ and for each entry $w_{ii}\alpha w_{jj}$ of α, with $r<i$ or $r<j$, an element of the form $w_{pi}\alpha w_{jq}$, where p, q are functions of (i, j, α) and $p, q \leq r$.

This means that in order to get a system of generators for the reduced algebra, we are bringing back (by means of the unitaries in the matrix unit) all others entries of the generators in Σ, onto those entries that are indexed by numbers within the set $\{1, 2, \dots, r\}$. Moreover, whenever α is selfadjoint, we may restrict ourselves to those entries of α with $i \leq j$. Hence Σ_1 must contain $(e_1X^se_1)_{s\in S}\cup e_1A$ and the following sets of elements (in order to obtain a set of generators for the von Neumann algebra e_1Me_1)

(1) $\{v_{pk}g(k, l, s)v_{lq}\otimes e_{pq} \mid s\in S\setminus\{\sigma\},\ 1\leq k<l\leq n,\ l>r\}$,
(1′) $\{v_{pk}g(k, l, \sigma)v_{lq}\otimes e_{pq} \mid 2\leq k<l\leq n,\ l>r\}$,
(2) $\{v_{p,l}f(l, s)v_{l,q}\otimes e_{pq} \mid 1\leq l\leq n,\ s\in S\}$,
(3) $\{v(1, p, \sigma)B(1, p, \sigma)v(1, p, \sigma)^*\otimes e_{11} \mid r<p\leq n\}$,
(4) $\{v(1, p, \sigma)D(v(1, p, \sigma))^*\otimes e_{11}, \mid r<p\}$.

The first set (1) comes (via the procedure in Lemma 1) from the entries of $(X^s)_{s\in S\setminus\{\sigma\}}$ while (1′) comes by the same procedure, from the remaining entries of X^σ, i.e., $\{g(i, j, \sigma)\otimes e_{jj} \mid 2\leq i\leq j\leq n\}$. Here we use the symbols p, q instead of $p(k, l, s)$, $q(k, l, s)$.

The third set is collecting the entries of

$$\{f(i, s)\otimes e_{ii} \mid r<i\leq n,\ s\in S\}$$

(moved back by the isometries in the matrix unit onto the entries indexed with numbers in $\{1, \dots, r\}$. Again we use the abreviation p, q for $p(l, l, s)$, $q(l, l, s)$.

We impose the following condition on the functions p, q before (assumed to take values within the set $\{1, 2, \dots, r\}$)

(x) $p(k, l, s) = q(k, l, s)$ iff $k=l$, $s\in S$;
(y) $r_+\leq l\leq n$ implies $p(k, l, s)$, $q(k, l, s)\leq r_-$, $s\in S$;
(z) the function

$$(p, q): \{(k, l) \mid 1\leq k\leq l\leq n,\ l>r\}\times(S\setminus\{\sigma\})\to\{1\leq p\leq q\leq r\}$$

takes each value infinitely many times.

The first two conditions will be important when putting all the entries together to obtain a family $(Y_t)_{t\in T}$, whose entries behave like those of $(X^s)_{s\in S}$. The last condition will be necessary when proving condition (iv) in the statement.

Out of the initial set Σ, we have still to cover the elements of the form $b(1, p, \sigma)\otimes e_{pp}$, $p>r$. This means that we also have to include in Σ_1 elements that generate the same von Neumann algebra as

$$\{v(1,p,\sigma)b(1,p,\sigma)v(1,p,\sigma)^* \otimes e_{11} \mid p = r+1, \dots, n\}.$$

Letting $B(1,p,\sigma)$, $p > r$, be a semicircular element generating (in N) the same algebra as $b(1,p,\sigma)$ for each $p > r$, it follows that we have to include in Σ_1 the elements in the set (3).

Finally we have to describe how to include, between the generators of the reduced algebra, the elements corresponding to Z. We may do this by including, between the generators in Σ_1, the elements in the set

$$\{v(1,p,\sigma)(a+p)v(1,p,\sigma)^* \otimes e_{11} \mid r < p \leq n\}.$$

When we are interested only in the von Neumann algebra that is generated by these elements, we may replace this set by the set in formula (4), where D is a semicircular element that generates the same von Neumann algebra as a (recall that by definition we had $Z = \sum(a+p) \otimes e_{pp}$).

Since we applied the construction from Lemma 1 to all elements from Σ, it follows that the sets (1), (2), (3), (4), $(e_1 X^s e_1)_{s\in S}$, and $e_1 A$ form a family of generators for $e_1 M e_1$ (by Lemma 1).

The construction of the family $(Y_t)_{t\in T}$ runs as follows: for t in S we take $Y_t = e_1 X^s e_1$, while for $t \in T \setminus S$ we choose $(Y_t)_{t\in T\setminus S} \subseteq N \times M_n(\mathbb{C})$ so that the set of the diagonal entries of $(Y_t)_{t\in T\setminus S}$ is exactly the union of the sets (2), (3), and (4). Moreover, the set of the elements beyond the diagonal in $(Y_t)_{t\in T\setminus S}$ should coincide with set in (1). Finally we will impose the condition that $(Y_t)_t$ are all selfadjoint for all t in T. It is clear that the conditions (x), (y) that we imposed on p, q allow us to carry out such a construction.

Since we included all elements from the sets (1), (2), (3), and (4) in the $(Y_t)_t$, and by the construction, it follows that conditions (ii) and (iii) are satisfied (using Lemma 1).

By Lemma 3 it follows that the entries of $(Y_t)_{t\in T}$, below or on the diagonal, form a free family that is also free with respect to a; moreover, the entries on the diagonal are semicircular and those below the diagonal are circular. Hence by Proposition 2.8 of [27], it follows that condition (i) also holds true.

Finally condition (z) and the fact that in the sets (3), (4) we chose the partial isometries so that the corresponding elements are mapped onto the (1,1) entry, it follows by the same arguments as before that condition (iv) also holds. (We have to use that $(w_{ij})_{i,j=1}^r$ is already contained in the von Neumann algebra generated by $e_1 X^\sigma e_1$ and $e_1 A$.)

The following technical lemma was used in the proof of Lemma 2.

Lemma 3. *Let (A, φ) be a W^*-algebra with normal faithful finite trace φ. Let $\omega = (Y^s)_{s\in S} \subseteq A$ be an infinite free family composed from circular elements for $s \in S_c$ and semicircular for S_c (where $S = S_c \cup S_s$ is a partion of S). Let $v_1, \dots, v_N$, $N \geq 1$, be a free family of unitaries such that $\omega \cup \{v_1, \dots, v_N\}$ is free and $\varphi(v_i^k) = 0$ for all nonnull integers k. Let σ in S_s be fixed and for each s in $S\setminus\{\sigma\}$ choose $v_{p_s}, v_{q_s}, v_{r_s}, v_{t_s}$ in $\{v_1, v_2, \dots, v_N\} \cup \{1\}$ and $w_{i_s} = v_{p_s} v_{q_s}^*$, $w_{j_s} = v_{r_s} v_{t_s}^*$, so that if s is in S_s then $w_{j_s} = w_{i_s}^*$.*

Then the family

$$\{Y^{\sigma}\} \cup \{w_{i_s} Y^s w_{j_s}\}_{s \in S \setminus \{\sigma\}} \cup \{v_i Y^{\sigma} v_i^*\}_{i=1}^n$$

is still free and is composed from circular and semicircular elements ($w_{i_s} Y^s w_{j_s}$ is semicircular if and only if Y^s is for each $s \neq \sigma$).

Proof. We will use the fact that in an arbitrary von Neumann algebra M with finite normal faithful trace τ, and for any y in M and any v, w unitaries in M, the polar decomposition of vyw is $(vuw)|vyw|$ with $|vyw| = w^*|y|w$, if the polar decomposition of y is $u|y|$. Recall that by Proposition 2.6 in [6], if y is circular then $\{u, y\}$ is free and u is a unitary such that $\tau(u^k) = 0$ whenever $k \neq 0$, while $b = |y|$ has the distribution given by formula (5) below.

For s in S let D_s be a unitary in A that generates the same algebra as $|Y_s|$ for s in S_c or, respectively, as Y_s for s in S_s. Let u_s denote the unitary from the polar decomposition of Y_s for s in S_c. By the proposition quoted before the family

$$\{u_s\}_{s \in S_c} \cup \{D_s\} \cup \{v_1, \dots, v_n\}$$

is free and hence so is the family

$$\{D_{\sigma}\} \cup \{w_{i_s} u_s w_{j_s}\}_{s \in S_c} \cup \{w_{j_s}^* D_s w_{j_s}\}_{s \in S \setminus \{\sigma\}} \cup \{v_i^* D_{\sigma} v_i\}_{i=1}^n.$$

Consequently the family in the statment is free. To conclude the proof we have to use the fact that if u is a unitary with $\varphi(u^k) = 0$ for $k \in \mathbb{Z} \setminus \{0\}$ and $\{u, b\}$ is free, where b is positive with

$$\varphi(b^k) = (\text{const})^k \int_0^1 t^k (1 - t^2)^{1/2} dt, \tag{5}$$

then ub is circular (if the constant is nonzero). This ends the proof.

The next lemma extends, in a certain sense, the results from Lemma 2 to the case of projections with irrational trace.

Lemma 4. *Let (M, τ) be a von Neumann algebra with a finite faithful trace τ and let $(X^s)_{s \in S} \subseteq M$ be a free semicircular family, which is also free with respect to the abelian diffuse von Neumann subalgebra $A \subseteq M$. Assume also that $\{(X^s)_{s \in S} \cup A\}'' = M$. Let e be a projection in A of irrational trace, ε a strictly positive real number, and F a fixed finite set in eMe.*

Then there exists a projection e_1 in M, $e_1 \geq e$, $\tau(e_1 - e) \leq \varepsilon$, and a family $(Y_t)_{t \in T}$ in $e_1 M e_1$, with $S \subseteq T$, such that:

(i) *$(Y_t)_{t \in T}$ is semicircular and free with respect $e_1 A$.*
(ii) *The von Neumann algebra generated in $e_1 M e_1$ by $(Y_t)_{t \in T}$ and $e_1 A$ is $e_1 M e_1$ itself.*
(iii) *$Y^s = e_1 X^s e_1$ for s in S.*
(iv) *F is ε-contained in the von Neumann algebra generated in eMe by $(eY^t e)_{t \in T}$ and eA.*

(The assertion about F means that for every f in F there is f' in the algebra eMe such that $\|f'-f\|_{2,\tau}=\tau((f'-f)(f'-f))^{1/2}$ is less than ε.)

Proof. By usual continuity arguments, taking the linear span we may assume that all elements of F are of the form

$$e_{i_0}X^{s_1}e_{i_1}X^{s_2}\cdots e_{i_{n-1}}X^{s_n}e_{i_n},$$

where $s_1,\dots,s_n\in S$ and $e_{i_0},\dots,e_{i_n}$ are projections in A with $e_{i_0},e_{i_n}\le e$.

Since F is finite, it follows that there is a real number $\delta>0$ such that whenever e', f are projections in A with $e'\le e$, $\tau(e-e')<\delta$, $\tau(1-f)<\delta$, the elements of the set $F_{f,e'}$ (obtained by replacing everywhere in F the $(X^s)_{s\in S}$ by $(fX^sf)_{s\in S}$ and replacing, in a product defining an element in F, the first and the last projection e_{i_0}, e_{i_n} by $e'e_{i_0}$, $e'e_{i_n}$) are at distance less than ε with respect to the corresponding elements in F (in the norm $\|\ \|_{2,\tau}$ given by $\|X\|_{2,\tau}^2=\tau(X^*X)$, $X\in M$).

Let e_+, e_-, e_1 be three distinct projections in A of rational trace, different from 0, 1 such that $e_+\ge e_1\ge e\ge e_-$ and $\tau(e_+-e_-)<\min(\delta,\varepsilon)$ (in particular $\tau(e_1-e)<\varepsilon$). By Lemma 2 we obtain a semicircular family $(Y^t)_{t\in T}$ which already has properties (i), (ii), (iii).

Taking $f=(1-e_+)+e_-$ we have (by Lemma 2(iv)) that F_{f,e_-} is contained in the von Neumann algebra generated in e_-Me_- by $(e_-Y^te_-)_{t\in T}$ and $(e_-)A$. Hence $F_{f,e}$ is also contained in the von Neumann algebra generated by $(e_-Y^te_-)_{t\in T}$ and eA. Taking into account the way we chose δ, we also obtain that $(Y^t)_{t\in T}$ has property (iv). This completes the proof of the lemma.

We are now able to prove our theorem, which states that $\mathbb{R}_+\setminus\{0\}$ is the fundamental group of $\mathscr{L}(F_\infty)$, the von Neumann algebra of a free group with infinitely many generators. In the course of the proof we will prove the behaviour of a semicircular family in a reduced algebra that we announced in the introductory section.

Theorem. *Let $M=\mathscr{L}(F_\infty)$ be the von Neumann algebra of a free group with infinitely many generators and let e be a nonnull projection in M. Then $e\mathscr{L}(F_\infty)e$ is isomorphic to $\mathscr{L}(F_\infty)$. In particular, $\mathscr{F}(\mathscr{L}(F_\infty))=\mathbb{R}_+\setminus\{0\}$.*

To prove the theorem by [15] it is sufficient to consider, for each value of the trace, a single projection. The unicity of the von Neumann algebra generated by a free family [26] then shows that the Theorem is a consequence of the

Proposition. *Let $M=\mathscr{L}(F_\infty)$ be the von Neumann algebra of the free group with infinitely many generators endowed with the canonical trace τ. Assume that M is generated by a free semicircular family $(X^s)_{s\in S}$ and by the diffuse abelian von Neumann algebra A. Moreover, assume that A is free with respect to the family $(X^s)_{s\in S}$.*

Let e be a nonnull projection in M. Then there is a free family $(Y^t)_{t\in T}$ in eMe such that:

(a) *$(Y^t)_{t\in T}$ is semicircular and free with respect eA.*

(b) *The von Neumann algebra generated in* eMe *by* $(Y^t)_{t\in T}$ *and* eA *is* eMe *itself.*

(c) T *contains* S *and* $Y^s = eX^s e$ *for* s *in* S.

In particular, eMe *is also isomorphic to the von Neumann algebra of a free group with infinitely many generators.*

Proof. If e has rational trace then we are already done by Lemma 2. If not, let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers, decreasing to zero, such that $\tau(e)+\varepsilon_n < 1$ for $n \in \mathbb{N}$ where $\varepsilon_0 = 1$. Let $(F_n)_{n\in\mathbb{N}}$ be an increasing family of finite subsets of eMe such that $(\bigcup_{n\geq 0} F_n)'' = eMe$.

By induction we construct a sequence of projections $(e_n)_{n\in\mathbb{N}}$ in A of rational trace, decreasing to e, such that $\tau(e_n - e) < \varepsilon_n$ (where $e_0 = 1$) and a free family $(Y_n^t)_{t\in T_n}$ in $e_n M e_n$, $n \in \mathbb{N}\cup\{0\}$, where $S = T_0 \subseteq T_1 \cdots \subseteq T_n \cdots$. Moreover, we also have that $Y_0^t = eX^t e$ for t in S and for each $n \in \mathbb{N}$

(a′) $(Y_n^t)_{t\in T}$ is semicircular and free with respect $e_n A$.

(b′) The von Neumann algebra generated in $e_n M e_n$ by $(Y_n^t)_{t\in T_n}$ and $e_n A$, is $e_n M e_n$ itself.

(c′) $e_n Y_{n-1}^t e_n = Y_n^t$ for t in T_{n-1}.

(d′) F_n is ε_n -contained into the von Neumann algebra generated in eMe by $(eY^t e)_{t\in T_n}$ and eA.

Assume that we succeded in constructing such a family, we may take $T = \bigcup_{n\geq 0} T_n$ and $(Y_t)_{t\in T} = \bigcup_{n\geq 0}(eY_n^t e)_{t\in T_n}$. It is clear by property (c′) that this family is well defined and moreover, since $(Y_0^t)_{t\in T_0} = (X^t)_{t\in T_0 = S}$, that property (c) holds.

Property (b) is a consequence of (d′) and of the fact that $(\bigcup_{n\geq 0} F_n)'' = eMe$ and $\varepsilon_n \downarrow 0$.

Property (a) follows from (a′) by taking the limit (we also have to use that the second moment of the variables $(Y_n^t)_{t\in T}$ is constantly equal to the second moment of $eX^s e$ for any s in S, which is nonvanishing, since e is nonnull).

It remains to carry out the construction. Assume that we have realised the construction up to stage n. We construct e_{n+1} and $(Y_{n+1}^t)_{t\in T_{n+1}}$.

To do this we simply apply Lemma 4 to the von Neumann algebra $e_n M e_n$ that is generated (as a consequence of (b′)) by $(Y_n^t)_{t\in T_n}$ and $e_n A$ (taking $\varepsilon = \varepsilon_{n+1}$ and $F = F_{n+1}$, the last one being regarded as a subset of $eMe = e(e_{n+1} M e_{n+1})e$).

In this way we obtain a projection e_{n+1} (which is the e_1 given by the Lemma 4) and a family $(Y_{n+1}^t)_{t\in T_{n+1}}$ that satisfies (a′), (b′), (c′), (d′). This completes the proof of the proposition and hence we also proved the theorem.

As we already mentioned in the introduction (by [13]), if M is an arbitrary type II_1 factor with trace τ and if t is a real number in $[0, 1]$ with the property that there is a projection $p \in M$ with $\tau(p) = t$ and a von Neumann algebras isomorphism $\theta\colon pMp \to (1-p)M(1-p)$, then there exists a subfactor N of

M, of index $t^{-1} + (1-t)^{-1}$, given by $N = \{x + \theta(x) \mid x \in M\}$. Moreover, N has nontrivial relative commutant. In particular, if $t/(1-t) \in \mathscr{F}(M)$, then such an automorphism always exists by the definition of the fundamental group. Hence our result implies the following

Corollary. *For any $s \in [4, \infty)$ there exists a subfactor (with nontrivial relative commutant) of $\mathscr{L}(F_\infty)$ of index s.*

In [3] Connes proved that any type III_λ factor, with $\lambda \in (0, 1)$, admits a discrete decomposition as the cross product of a semifinite factor (N with semifinite faithful normal trace τ), by an automorphism θ that scales the trace τ by λ (i.e., that $\tau(\theta(x)) = \lambda\tau(x)$, $x \in M$. Moreover, in this case N, θ are unique (up to conjugacy) and N is called the core of M. Hence we have

Corollary. *For any λ in $(0, 1)$ there exists a type III_λ factor having a core isomorphic to $\mathscr{L}(F_\infty)\overline{\otimes}B(H)$.*

As suggested to us by M. Takesaki, this last result could be used to improve our main theorem if one could find an abstract characterization of a type III factor having a core that is isomorphic to $\mathscr{L}(F_\infty)\overline{\otimes}B(H)$. One should then construct, by taking the tensor product of the already constructed type III factors, another type III_1 with a similar core. S. Popa and E. Effros also suggested to us that such an abstract characterization of $L(F_N)$ should be possible, along these lines.

Note added in proof

In a recent paper (to appear in C. R. Acad. Sci. Paris) we prove the existence of a type III_1 factor with a core isomorphic to $\mathscr{L}(F_\infty) \otimes B(H)$.

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References

1. H. Araki and E. J. Woods, *A classification of factors,* Publ. Res. Inst. Math. Sci. **4** (1968), 51–130.
2. M. F. Atiyah and I. M. Singer, *The index of elliptic operators* I.V, Ann. of Math. **93** (1971), 119–138.
3. A. Connes, *A survey of foliations and operator algebras,* Proc. Sympos. Pure Math., vol. 38, Amer. Math. Soc., Providence, RI, 1982, pp. 521–628.
4. ——, *Classification of injective factors,* Ann. of Math. **104** (1976), 73–115.
5. ——, *Un facteur du type* II_1 *avec le groupe fondamentale denombrable,* J. Operator Theory **4** (1980), 151–153.
6. ——, *Une classification des facteurs de type* III, Ann. Sci. École. Nom. Sup. (4) **6** (1973), 133–252.
7. ——, *Property* L'_λ *and automorphisms of factors,* J. Operator Theory **14** (1985), 189–211.
8. ——, *A factor non anti-isomorphic to itself,* Ann. of Math. **101** (1974), 536–554.

9. J. Diximier, *Les algèbres des operateurs dans l'espaces hilbertien,* Gauthier-Villard, Paris, 1969.

10. S. Doplicher, R. Haag, and J. E. Robertson, *Local observables and particle statistics* II, Comm. Math. Phys. **35** (1974), 49–85.

11. J. Frolich, *Statistics of fields*, The Yang Baxter equation and the theory of knots, Proceedings of the 1987 Carése School, World Scientific, London, 1991.

12. U. Haagerup, *Connes' bicentralizer problem and uniqueness of the injective factor of type* III_1, Acta Math. **158** (1987), 95–147.

13. V. F. R. Jones, *Index of subfactors*, Invent. Math. **72** (1983), 1–25.

14. ——, *A polynomial for knots via von Neumann Algebras*, Bull. Amer. Math. Soc. **12** (1985), 103–112.

15. R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras,* vols. I, II, Academic Press, New York, 1983.

16. F. J. Murray and J. von Neumann, *On ring of operators* IV, Ann. of Math. **44** (1943), 716–808.

17. A. Ocneanu, *Quantized groups, string algebras and Galois theory for algebras*, Pitman Lecture Notes, vol. 123, 1990.

18. S. Popa, *Some rigidity results in type* II_1 *factors*, C. R. Acad. Sci. Paris Sér. I Math. **311** (1990), 535–538.

19. ——, *Classification of subfactors; reduction to commuting squares,* Invent. Math. **101** (1990), 19–43.

20. R. T. Powers, *Representations of uniformly hyperfinite algebras, and their associated rings*, Ann. of Math. **86** (1967), 138–171.

21. S. Sakai, C^**-algebras and* W^**-algebras*, Springer-Verlag, New York, 1966.

22. I. M. Singer, *Some remarks on operator algebras and index theory*, Lecture Notes in Math., vol. 575, Springer, New York, 1977, pp. 128–138.

23. Ş. Strătilă and L. Zsido, *Lectures or von Neumann algebras*, Editura Academiei, Bucharest, 1980.

24. M. Takesaki, *Theory of operators algebras* I, II, Springer, New York, 1979.

25. ——, *Duality for crossed products and the structure of von Neumann algebras of type* III, Acta Math. **131** (1973), 249–310.

26. D. Voiculescu, *Limit laws for random matrices and free products*, Operator Algebras, Unitary Representations and Invariant Theory, Progress in Math., vol. 92, Birkhäuser, 1990, pp. 45–60.

27. ——, Circular and semicircular systems and free product factors, Inventiones Math. **104** (1991), 201–220.

28. E. Wigner, *On the distribution of the roots of certain symmetric matrices,* Ann. of Math. **67** (1958), 325–327.

29. R. Longo, *Index of subfactors and statistics of quantum fields*, Comm. Math. Phys. **126**, (1989), 217–247.

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