

CONVOLUTION SINGULAR INTEGRALS ON LIPSCHITZ SURFACES

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1. INTRODUCTION

Let us recall some results about convolution singular integrals on curves.

There have been many proofs of the L_p -boundedness of the singular Cauchy integral operator on a Lipschitz curve γ in the complex plane \mathbb{C} since this was first proved by Calderón [2] (for curves with small Lipschitz constants) and by Coifman, McIntosh, and Meyer [3] in the general case. In particular two straightforward proofs have recently been given in the paper [5] by Coifman, Jones, and Semmes. In the first of these it is shown that bounds on the Cauchy operator can be obtained directly from quadratic estimates proved by Kenig [11] for those L_2 functions on γ which are in the Hardy spaces $H_\pm^2(\gamma)$ consisting of boundary values of holomorphic functions defined on one side of the curve γ or the other. Actually simple proofs are also given for those results of Kenig which are needed.

An alternative presentation of this, more in the spirit of functional calculi of operators, is presented in [15], where the following theorem is proved. In it, γ denotes a curve of the form $\gamma = \{x + ig(x) \in \mathbb{C} : x \in \mathbb{R}\}$, where g is a Lipschitz function which satisfies $\|g'\|_\infty \leq \tan \omega < \infty$, and S_μ^o denotes the

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open double sector which consists of those complex numbers z which satisfy $|\arg(\pm z)| < \mu$, with $\omega < \mu < \pi/2$, while $S_{\mu+}^o$ denotes the component in which $|\arg z| < \mu$.

Theorem 1.1. *Suppose that ϕ and ϕ_1 are holomorphic functions on S_{μ}^o and $S_{\mu+}^o$, that $z\phi(z)$ and $\phi_1(z)$ are bounded in z , that $\phi_1'(z) = \phi(z) + \phi(-z)$ for all $z \in S_{\mu+}^o$, and that $1 < p < \infty$. Then there is a bounded linear operator $T_{(\phi, \phi_1)}$ on $L_p(\gamma)$ defined for all $u \in L_p(\gamma)$ and almost all $z \in \gamma$ by*

$$(T_{(\phi, \phi_1)}u)(z) = \lim_{\varepsilon \rightarrow 0+} \left\{ \int_{\substack{\zeta \in \gamma \\ |z-\zeta| \geq \varepsilon}} \phi(z-\zeta)u(\zeta)d\zeta + \phi_1(\varepsilon \underline{t}(z))u(z) \right\}$$

where $\underline{t}(z)$ is the unit tangent vector to γ which is defined at almost all $z \in \gamma$. Moreover

$$\|T_{(\phi, \phi_1)}u\|_p \leq C_{\omega, \mu, p} \left[\sup \left\{ |z\phi(z)| : z \in S_{\mu}^o \right\} + \sup \left\{ |\phi_1(z)| : z \in S_{\mu+}^o \right\} \right] \|u\|_p$$

for some constants $C_{\omega, \mu, p}$ which depend only on ω, μ and p .

Of course the singular Cauchy integral operator on γ occurs when $\phi(z) = i/\pi z$ and $\phi_1 = 0$. We remark that Coifman and Meyer first showed in [4] how the L_p -boundedness of the singular Cauchy integral operator on γ could be used to prove the L_p -boundedness of other convolution singular integral operators on γ .

In this paper we present a higher-dimensional analogue of this result, concerning the L_p -boundedness of convolution singular integral operators on a Lipschitz surface

$$\Sigma = \{g(\mathbf{x})e_0 + \mathbf{x} \in \mathbb{R}^{n+1} : \mathbf{x} \in \mathbb{R}^n\}$$

where g is a Lipschitz function which satisfies $\|\nabla g\|_{\infty} \leq \tan \omega < \infty$. Here we have embedded \mathbb{R}^{n+1} in the Clifford algebra $\mathbb{R}_{(n)}$ with identity e_0 , and are considering convolution with right-monogenic (right Clifford-regular) functions ϕ which satisfy $|\phi(x)| \leq C|x|^{-n}$ on a sector

$$S_{\mu}^o = \{x = x_0 + \mathbf{x} \in \mathbb{R}^{n+1} : |x_0| < |\mathbf{x}| \tan \mu\}$$

where $\mu > \omega$. Provided there exists an L_{∞} function $\underline{\phi}$ satisfying

$$\underline{\phi}(R) - \underline{\phi}(r) = \int_{\substack{r < |x| < R \\ x \in \mathbb{R}^n}} \phi(x) dx,$$

then the related convolution singular integral operator $T_{(\phi, \underline{\phi})}$ is bounded on $L_p(\Sigma)$ for $1 < p < \infty$. A detailed statement is given in §3, while the proof occupies §§4–6.

The operators $T_{(\phi, \underline{\phi})}$ act on functions defined on the Lipschitz surface Σ taking values in the Clifford algebra $\mathbb{R}_{(n)}$, its complexification $\mathbb{C}_{(n)}$, or indeed in any finite-dimensional left-Clifford module, such as those which are considered in the papers of Gilbert and Murray, e.g., [9].

As in the holomorphic case, we first derive quadratic estimates for those L_2 functions on Σ which are in the Hardy spaces $H_{\pm}^2(\Sigma)$ consisting of boundary

values of left-monogenic functions defined on one side of the surface Σ or the other. We then use the idea in [5] to find L_p bounds for convolution with functions ϕ_{\pm} which are right-monogenic, not just on S_{μ}^o , but also above or below S_{μ}^o . Finally, we obtain the L_p bounds for convolution with ϕ by decomposing ϕ as a sum of such functions ϕ_+ and ϕ_- . As in the one-dimensional case, the surprising thing is that quadratic estimates in the Hardy spaces $H_{\pm}^2(\Sigma)$ lead to bounds on all of $L_2(\Sigma)$.

It should be possible to prove our theorem by other means, as for example by using the Clifford Tb theorem [7]. We remark that Clifford martingales have recently been developed by Gaudry, Long, and Qian and used to give a new proof of the Clifford Tb theorem [8].

The most important special case occurs when $\phi(x) = \frac{2}{\sigma_n} \frac{\bar{x}}{|x|^{n+1}}$, $\underline{\phi} = 0$, and $T_{(\phi, \phi)} = C_{\Sigma}$, the singular Cauchy integral operator on Σ . This operator appears to have been first studied by Iftimie who proved in 1965 that it is a bounded operator on Hölder continuous functions on Lyapunov surfaces [10]. Although the L_p -boundedness of this operator on Lipschitz surfaces follows from the results in [3] using the Calderón rotation method, it is of interest to obtain such bounds directly using Clifford generalizations of the one-dimensional proofs. This was done by Murray for surfaces with small Lipschitz constants [17], though this restriction was removed in [13].

Observe that the singular double-layer potential operator on Σ is the scalar part of the singular Cauchy integral operator on Σ .

Although the operators $T_{(\phi, \phi)}$ of Theorem 1.1 form an algebra, this is not true in higher dimensions for two reasons. First, it is not natural to define the convolution product $\phi_1 * \phi_2$ of two right-monogenic functions ϕ_1 and ϕ_2 on S_{μ}^o unless ϕ_2 is also left-monogenic. So we need to restrict our attention to functions which are two-sided monogenic. Second, in order to decompose such ϕ as $\phi = \phi_+ + \phi_-$ with ϕ_+ and ϕ_- two-sided monogenic, we need the extra condition (*) stated in §3.

We show that the operators $T_{(\phi, \phi)}$ corresponding to two-sided monogenic functions ϕ satisfying condition (*) do form an algebra. Moreover $T_{(\phi, \phi)} = T_{\phi_+} + T_{\phi_-}$ where T_{ϕ_+} and T_{ϕ_-} are bounded operators from $L_p(\Sigma)$ to $H_{\pm}^p(\Sigma)$.

That these operators are precisely those arising from an H_{∞} -functional calculus of the operator \mathbf{D}_{Σ} introduced by Coifman and used in [17] and [13] will be shown in [12]. Indeed $T_{(\phi, \phi)} = b(\mathbf{D}_{\Sigma})$ where b is the Fourier transform of (ϕ, ϕ) , extended to a sector in \mathbb{C}^n .

We remark that if we decompose \mathbf{D}_{Σ} as a sum of operators \mathbf{D}_{Σ}^+ and \mathbf{D}_{Σ}^- acting in the Hardy spaces $H_{\pm}^2(\Sigma)$, then $\mp \mathbf{D}_{\Sigma}^{\pm}$ are the infinitesimal generators of the semigroups $u(x) \rightarrow u(x \pm te_0)$.

This paper is actually closely related to the paper [12] being written jointly with Tao Qian by two of the authors. Although in the current paper we have avoided using the characterisations by Fourier transforms just mentioned, many of the statements and results in this paper were actually found by using those

ideas. So we would especially like to express our appreciation to Qian with whom we have been developing this topic.

We are also indebted to the other mathematicians with whom we have discussed this material, in particular to John Ryan.

2. CLIFFORD ANALYSIS

We embed the vector space \mathbb{R}^{n+1} in the real 2^n -dimensional Clifford algebra $\mathbb{R}_{(n)}$ or in the complex 2^n -dimensional Clifford algebra $\mathbb{C}_{(n)}$ in the usual way.

Let $e_0, e_1, e_2, \dots, e_n$ be the standard basis of \mathbb{R}^{n+1} , and denote the basis vectors of $\mathbb{R}_{(n)}$ or $\mathbb{C}_{(n)}$ by e_S , where S is any subset of $\{1, 2, \dots, n\}$. Make the identifications $e_0 = e_\emptyset$ and $e_j = e_{\{j\}}$ for $1 \leq j \leq n$, and define the associative multiplication of basis vectors by

$$\begin{aligned} e_0 &= 1, \\ e_j^2 &= -e_0 = -1 \quad \text{for } 1 \leq j \leq n, \\ e_j e_k &= -e_k e_j = e_{\{j,k\}} \quad \text{for } 1 \leq j < k \leq n, \end{aligned}$$

and

$$e_{j_1} e_{j_2} \cdots e_{j_s} = e_S \quad \text{if } 1 \leq j_1 < j_2 < \cdots < j_s \leq n \text{ and } S = \{j_1, j_2, \dots, j_s\}.$$

The product of two elements $u = \sum_S u_S e_S$ and $v = \sum_S v_S e_S$ in $\mathbb{R}_{(n)}$ (or $\mathbb{C}_{(n)}$) is $uv = \sum_{S,T} u_S v_T e_S e_T$. The term $u_\emptyset e_\emptyset$ is usually written as $u_0 e_0$ or just u_0 , and is called the scalar part of u .

We use the Euclidean norm $|u|$ in $\mathbb{R}_{(n)}$ (or $\mathbb{C}_{(n)}$), defined by $|u|^2 = \sum_S |u_S|^2 = (u, u)$, and the inner product $(u, v) = \sum_S u_S \bar{v}_S = (\bar{v}u)_0$ where \bar{v}_S is the complex conjugate of v_S and $\bar{v} = \sum_S \bar{v}_S \bar{e}_S$ with $\bar{e}_S = \pm e_S$, the sign chosen so that $\bar{e}_S e_S = e_S \bar{e}_S = 1$.

Let us now outline some analysis. The results we quote here can be found in [1]. The differential operator $D = \frac{\partial}{\partial x_0} + \mathbf{D}$ where $\mathbf{D} = \sum_{k=1}^n \frac{\partial}{\partial x_k} e_k$ acts on C^1 -functions f to give

$$Df = \sum_{k=0}^n \frac{\partial f_S}{\partial x_k} e_k e_S$$

and also

$$fD = \sum_{k=0}^n \frac{\partial f_S}{\partial x_k} e_S e_k.$$

A C^1 -function defined on an open subset of \mathbb{R}^{n+1} with values in $\mathbb{R}_{(n)}$ or $\mathbb{C}_{(n)}$ is called *left-monogenic* if $Df = 0$ and *right-monogenic* if $fD = 0$. All analytic functions F on domains in \mathbb{R}^n have both left- and right-monogenic extensions to domains in \mathbb{R}^{n+1} . These two extensions coincide if and only if $\mathbf{D}F = F\mathbf{D}$, so in particular they coincide if F is scalar valued.

The most important functions which are both left- and right-monogenic are the Cauchy kernels

$$\frac{1}{\sigma_n} \frac{\overline{y-x}}{|y-x|^{n+1}} \quad \text{for } x \in \mathbb{R}^{n+1} \setminus \{y\}$$

where σ_n is the volume of the unit n -sphere in \mathbb{R}^{n+1} .

The following version of Cauchy's theorem and the Cauchy integral formula holds.

Theorem 2.1. *Let Ω be a bounded open subset of \mathbb{R}^{n+1} with Lipschitz boundary $\partial\Omega$ and exterior unit normal $n(y)$ defined for almost all $y \in \partial\Omega$. Suppose f is left-monogenic and g is right-monogenic on a neighbourhood of $\Omega^- = \Omega \cup \partial\Omega$. Then*

- (i)
$$\int_{\partial\Omega} g(y)n(y)f(y) dS_y = 0,$$
- (ii)
$$\frac{1}{\sigma_n} \int_{\partial\Omega} g(y)n(y) \frac{\overline{y-x}}{|y-x|^{n+1}} dS_y = \begin{cases} g(x), & x \in \Omega \\ 0, & x \notin \Omega^-, \end{cases}$$
- (iii)
$$\frac{1}{\sigma_n} \int_{\partial\Omega} \frac{\overline{y-x}}{|y-x|^{n+1}} n(y)f(y) dS_y = \begin{cases} f(x), & x \in \Omega, \\ 0, & x \notin \Omega^-. \end{cases}$$

A result we need is the following.

Theorem 2.2. *Suppose f is right-monogenic on $\mathbb{R}^{n+1} \setminus \{0\}$ and satisfies the bound $|f(x)| \leq C/|x|^n$ for $x \in \mathbb{R}^{n+1} \setminus \{0\}$. Then $f(x) = c\bar{x}/|x|^{n+1}$ for some constant $c \in \mathbb{C}_{(n)}$.*

The proof can be found in §12, Chapter 2 of [1].

We note that parts (i) and (iii) of Theorem 2.1 remain valid when f is a left-monogenic function taking its values in \mathcal{L} , where \mathcal{L} is a finite-dimensional left Clifford module. That is, \mathcal{L} is a finite-dimensional real (or complex) linear space together with a representation of $\mathbb{R}_{(n)}$ (or $\mathbb{C}_{(n)}$) as linear operators on \mathcal{L} . If $u \in \mathbb{R}_{(n)}$ (or $\mathbb{C}_{(n)}$) and $v \in \mathcal{L}$, we denote the action of u on v by uv .

We consider \mathcal{L} together with a norm $\|\cdot\|$ and note that there exists a constant C such that $\|uv\| \leq C\|u\|\|v\|$ for all $u \in \mathbb{R}_{(n)}$ (or $\mathbb{C}_{(n)}$) and $v \in \mathcal{L}$. However we do not equip \mathcal{L} with an inner product, and in particular do not require that the basis vectors e_j be represented by skew-adjoint operators.

3. CONVOLUTION SINGULAR INTEGRALS ON LIPSCHITZ SURFACES

Throughout this paper Σ denotes the Lipschitz surface consisting of points $x = g(\mathbf{x}) + \mathbf{x} = g(\mathbf{x})e_0 + \mathbf{x} \in \mathbb{R}^{n+1}$, where $\mathbf{x} \in \mathbb{R}^n$, and g is a real-valued Lipschitz function which satisfies $\|\nabla g\|_\infty \leq \tan \omega < \infty$ where $0 \leq \omega < \pi/2$.

Moreover \mathcal{L} denotes a finite-dimensional left module over the Clifford algebra $\mathbb{R}_{(n)}$ (or $\mathbb{C}_{(n)}$). If $1 \leq p < \infty$ then $L_p(\Sigma) = L_p(\Sigma, \mathcal{L})$ is the space of

equivalence classes of functions $u: \Sigma \rightarrow \mathcal{R}$ which are measurable with respect to $dS_x = \sqrt{1 + |\nabla g(x)|^2} dx$, and for which

$$\|u\|_p = \left\{ \int_{\Sigma} \|u(x)\|^p dS_x \right\}^{1/p} < \infty, \quad 1 \leq p < \infty.$$

Our aim is to prove the L_p -boundedness of convolution singular integral operators associated with right-monogenic functions on open sectors S_{μ}^o in \mathbb{R}^{n+1} defined by

$$S_{\mu}^o = \{x = x_0 + \mathbf{x} \in \mathbb{R}^{n+1} : |x_0| < |\mathbf{x}| \tan \mu\}$$

when $\omega < \mu < \pi/2$.

Define $K(S_{\mu}^o)$ to be the Banach space of pairs $(\phi, \underline{\phi})$ of functions with ϕ right-monogenic from S_{μ}^o to $\mathbb{R}_{(n)}$ (or $\mathbb{C}_{(n)}$) and with $\underline{\phi}$ a differentiable function on $(0, \infty)$ such that $\underline{\phi}'(r) = \int_{|x|=r} \phi(x) dx$ (where the integral is over the $(n-1)$ -sphere $S(0, r)$ in \mathbb{R}^n) and

$$\|(\phi, \underline{\phi})\|_{K(S_{\mu}^o)} = \frac{\sigma_n}{2} \sup \{ |x|^n |\phi(x)| : x \in S_{\mu}^o \} + \sup \{ |\underline{\phi}(r)| : r > 0 \} < \infty$$

where σ_n is the volume of the unit n -sphere in \mathbb{R}^{n+1} .

Note that $\underline{\phi}$ is determined by ϕ up to an additive constant, and that

$$\underline{\phi}(R) - \underline{\phi}(r) = \int_{\substack{r < |x| < R \\ x \in \mathbb{R}^n}} \phi(x) dx$$

when $0 < r < R$.

Given $(\phi, \underline{\phi}) \in K(S_{\mu}^o)$ we extend $\underline{\phi}$ to the open cone

$$T_{\mu}^o = \{y = y_0 + \mathbf{y} \in \mathbb{R}^{n+1} : y_0 > |\mathbf{y}| \cot \mu\}$$

by

$$\underline{\phi}(rt) = \underline{\phi}(r) - \int_{A(r,t)} (\phi(x) + \phi(-x))n(x) dS_x, \quad t \in T_{\mu}^o, |t| = 1, r > 0,$$

where $A(r, t) = \{x \in \mathbb{R}^{n+1} : |x| = r, x_0 > 0, \langle x, t \rangle < 0\}$ and $n(x)$ is the exterior normal to $A(r, t)$ at the point x . It is easy to see that

$$\sup\{|\underline{\phi}(x)| : x \in T_{\mu}^o\} \leq \|(\phi, \underline{\phi})\|_{K(S_{\mu}^o)}.$$

Theorem 3.1. *Suppose that $\omega < \mu < \pi/2$ and $1 < p < \infty$. Given $(\phi, \underline{\phi}) \in K(S_{\mu}^o)$, there is a bounded linear operator $T_{(\phi, \underline{\phi})}$ on $L_p(\Sigma)$ defined for all $u \in L_p(\Sigma)$ and almost all $x \in \Sigma$ by*

$$(T_{(\phi, \underline{\phi})}u)(x) = \lim_{\varepsilon \rightarrow 0^+} \left\{ \int_{\substack{y \in \Sigma \\ |x-y| \geq \varepsilon}} \phi(x-y)n(y)u(y) dS_y + \underline{\phi}(\varepsilon n(x))u(x) \right\}.$$

Moreover

$$\|T_{(\phi, \underline{\phi})}u\|_p \leq C_{\omega, \mu, p} \|(\phi, \underline{\phi})\|_{K(S_{\mu}^o)} \|u\|_p$$

for some constant $C_{\omega, \mu, p}$ which depends only on ω, μ and p (and dimension n).

In particular, when $(\phi, \underline{\phi}) = (2k, 0)$ where $k(x) = \frac{1}{\sigma_n} \frac{\bar{x}}{|x|^{n+1}}$, then $T_{(\phi, \underline{\phi})} = C_{\Sigma}$, the Cauchy singular integral operator on Σ given by

$$C_{\Sigma}u(x) = \text{p.v.} \frac{2}{\sigma_n} \int_{\Sigma} \frac{\overline{x-y}}{|x-y|^{n+1}} n(y)u(y) dS_y$$

and we have the known result that C_{Σ} is bounded on $L_p(\Sigma)$ for $1 < p < \infty$.

More generally, if $\phi(x)$ is an odd function which is right-monogenic and satisfies $|\phi(x)| \leq C|x|^{-n}$ for all $x \in S_{\mu}^o$, then the principal value convolution operator $\phi * u$ equals $T_{(\phi, 0)}$ and is therefore bounded on $L_p(\Sigma)$.

We postpone the proof until §6, though a special case, which implies the boundedness of the Cauchy singular integral operator, will be proved in §5.

The space $K(S_{\mu}^o)$ is not a convolution algebra. But if we define

$$M(S_{\mu}^o) = \{(\phi, \underline{\phi}) \in K(S_{\mu}^o) : \phi \text{ is left-monogenic and } (*) \text{ holds}\}$$

where

$$(*) \quad \int_{y \in \mathbb{R}^n} \langle y, x \rangle (\phi(y)n(y) - n(y)\phi(y)) dS_y - (\underline{\phi}(r)x - x\underline{\phi}(r)) = 0,$$

then $M(S_{\omega}) = \bigcup_{\mu > \omega} M(S_{\mu}^o)$ is a convolution algebra. Details will be given in §6, where we prove the following result.

Theorem 3.2. *Suppose $1 < p < \infty$. If $(\phi, \underline{\phi}), (\psi, \underline{\psi}) \in M(S_{\omega})$, then*

$$T_{(\phi, \underline{\phi})} \cdot T_{(\psi, \underline{\psi})} = T_{(\phi, \underline{\phi}) * (\psi, \underline{\psi})}.$$

The algebra of operators $\{T_{(\phi, \underline{\phi})} : (\phi, \underline{\phi}) \in M(S_{\omega})\}$ is precisely that corresponding to the H_{∞} -functional calculus of \mathbf{D}_{Σ} developed in [12].

4. REVERSE SQUARE FUNCTION ESTIMATES

Let Ω_+ be the domain in \mathbb{R}^{n+1} above the surface Σ and Ω_- the one below. Suppose $\sigma(x) = \text{dist}(x, \Sigma)$. Then $\mathcal{H}_+ = L_2(\Omega_+, \mathbb{C}_{(n)}, \sigma(x) dx)$ is a Hilbert space with norm

$$\begin{aligned} \|f\|_{\mathcal{H}_+} &= \left(\iint_{\Omega_+} |f(x)|^2 \sigma(x) dx \right)^{1/2} \\ &= \sup_{g \in B} \left| \iint_{\Omega_+} (f(x), g(x)) \sigma(x) dx \right| \end{aligned}$$

where

$$B = \{f \in \mathcal{H}_+ : \|f\|_{\mathcal{H}_+} \leq 1, f \text{ compactly supported in } \Omega_+\}.$$

Similarly we define \mathcal{H}_- .

Theorem 4.1 (reverse square function estimate). *Suppose that $f : \Omega_+ \rightarrow \mathbb{C}_{(n)}$ is right-monogenic and continuous to the boundary Σ . If $|f(x)| \leq c/(1 + |x|)^n$ and $|\nabla f(x)| \leq c/(1 + |x|)^{n+1}$ for all $x \in \Omega_+ \cup \Sigma$, then*

$$\left\{ \int_{\Sigma} |f(x)|^2 dS_x \right\}^{1/2} \leq C \|\nabla f\|_{\mathcal{H}_+}$$

where $\|\nabla f\|_{\mathcal{H}_+}^2 = \sum_{j=0}^n \left\| \frac{\partial}{\partial x_j} f \right\|_{\mathcal{H}_+}^2$.

It is easy to see that f is the Cauchy integral of its restriction to Σ and $\|f(x + te_0)\|_{L_2(\Sigma, dx)} \rightarrow 0$ as $t \rightarrow \infty$. So

$$\begin{aligned} \int_{\Sigma} |f(x)|^2 dS_x &= \int_0^\infty \int_{\Sigma} \frac{\partial}{\partial t} |f(x + te_0)|^2 dS_x dt \\ &= \int_0^\infty \int_{\Sigma} \frac{\partial}{\partial t} (f(x + te_0), f(x + te_0)) dS_x dt \\ &= 2 \Re e \iint_{\Omega_+} \left(\left(\frac{\partial}{\partial x_0} f \right) (x), f(x) \right) dx. \end{aligned}$$

For functions f and g defined on Ω_+ , set the bilinear form

$$B(f, g) = \iint_{\Omega_+} \left(\left(\frac{\partial}{\partial x_0} f \right) (x), g(x) \right) dx.$$

The following result was first proved by Dahlberg, who only required that f be harmonic. The extra assumption of monogenicity will allow us to give a simpler proof, although some of the ingredients are the same as in Dahlberg’s proof [6].

For a function g on Ω_+ , the nontangential maximal function is

$$N(g)(x) = \sup_{y \in x + \Gamma(\theta)} |g(y)|, \quad x \in \Sigma,$$

where $\Gamma(\theta) = \{x = x_0 + \mathbf{x} : |\mathbf{x}| < x_0 \tan \theta\}$ for some $\theta \in (0, \frac{\pi}{2} - \omega)$.

Lemma 4.2. *Suppose f satisfies the conditions in Theorem 4.1, g is smooth on Ω_+ , $|g(x)| \leq c/(1 + |x|)^n$, $|\nabla g(x)| \leq c/(1 + |x|)^{n+1}$. Then*

$$|B(f, g)| \leq C \left(\iint_{\Omega_+} |\nabla f|^2 \sigma(x) dx \right)^{1/2} \|g\|$$

where

$$\|g\| = \left(\iint_{\Omega_+} |\nabla g|^2 \sigma(x) dx \right)^{1/2} + \left(\int_{\Sigma} N(g)^2 dS_x \right)^{1/2}$$

and $N(g)$ is the nontangential maximal function of g .

Before proving Lemma 4.2, let us indicate how it implies Theorem 4.1. We have already got

$$\int_{\Sigma} |f(x)|^2 dS_x = 2 \Re e B(f, f) \leq C \left(\iint_{\Omega_+} |\nabla f|^2 \sigma(x) dx \right)^{1/2} \|f\|.$$

We claim that

$$\int_{\Sigma} |N(f)(x)|^2 dS_x \leq C \int_{\Sigma} |f(x)|^2 dS_x .$$

Indeed, let $\mathcal{E}_{\Sigma}(f)$ denote the Cauchy integral of the boundary values of f . That is,

$$\mathcal{E}_{\Sigma}(f)(x) = \pm \int_{\Sigma} k(x - y)n(y)f(y) dS_y, \quad x \in \Omega_{\pm} .$$

Then by using Cauchy’s integral formula and the assumption that f decays at ∞ we get $\mathcal{E}_{\Sigma}(f) = f$ on Ω_+ , $\mathcal{E}_{\Sigma}(f) = 0$ on Ω_- . If $x \in \Omega_+$, we can find $y(x) \in \Omega_-$ such that $|x - y(x)| \approx \text{dist}(x, \Sigma) \approx \text{dist}(y(x), \Sigma)$. By direct computation on the Cauchy kernel, the nontangential maximal function of $f(x) = \mathcal{E}_{\Sigma}(f)(x) - \mathcal{E}_{\Sigma}(f)(y(x))$ is controlled by the Hardy-Littlewood maximal function $M(f)$ of f pointwise on Σ . So

$$\int_{\Sigma} N(f)(x)^2 dS_x \leq C \int_{\Sigma} M(f)(x)^2 dS_x \leq C \int_{\Sigma} |f(x)|^2 dS_x .$$

Thus we get

$$\begin{aligned} \int_{\Sigma} |f(x)|^2 dS_x &\leq C \left(\iint_{\Omega_+} |\nabla f|^2 \sigma(x) dx \right)^{1/2} \\ &\quad \times \left\{ C \left(\int_{\Sigma} |f(x)|^2 dS_x \right)^{1/2} + \left(\iint_{\Omega_+} |\nabla f|^2 \sigma(x) dx \right)^{1/2} \right\} . \end{aligned}$$

Using $A^2 \leq C^2 B^2 + CAB \Rightarrow A \leq 2CB$, we get

$$\int_{\Sigma} |f(x)|^2 dS_x \leq C \iint_{\Omega_+} |\nabla f|^2 \sigma(x) dx$$

as desired.

Now we go back to prove Lemma 4.2. Following Dahlberg, we use a good bilipschitz change of variables. Choose ϕ to be a smooth function on \mathbb{R}^n , compactly supported, $\int_{\mathbb{R}^n} \phi(x) dx = 1$, and define $\phi_{\delta}(x) = \delta^{-n} \phi(\frac{x}{\delta})$,

$$\rho(x) = x_0 + \mathbf{x} + \phi_{x_0} * g(\mathbf{x}), \quad x = \mathbf{x} + x_0 \in \mathbb{R}_+^{n+1} .$$

For properly chosen ϕ , ρ defines a bilipschitz map from \mathbb{R}_+^{n+1} onto Ω_+ such that $|\nabla \rho|^2 x_0 dx$ is a Carleson measure. (The existence of such a change of variables is due to Dahlberg, but this particularly simple mapping was found by Kenig and Stein.)

After making the change of variables, we can rewrite $B(f, g)$ as

$$\iint_{\mathbb{R}_+^{n+1}} \left(\sum_{i=1}^n a_i(x) \frac{\partial}{\partial x_i} F(x), G(x) \right) dx \quad F = f \cdot \rho, G = g \cdot \rho .$$

The $a_i \in L_{\infty}(\mathbb{R}_+^{n+1}, \mathbb{R}_{(n)})$ and $|\nabla a_i(x)|^2 x_0 dx$ are Carleson measures on \mathbb{R}_+^{n+1} . We have used the monogenicity of f to replace the $\partial/\partial x_0$ term by terms of $\partial/\partial x_j$, $j = 1, 2, \dots, n$, with coefficients satisfying the above conditions.

Let R_j denote the Riesz transforms and Δ the negative of the Laplacian on \mathbb{R}^n . Then the above becomes

$$\begin{aligned} B(f, g) &= \iint_{\mathbb{R}^{n+1}} \left(\sum_{i=1}^n (\Delta^{1/4} R_i F)(x), \Delta^{1/4} (\bar{a}_i G)(x) \right) dx \\ &\leq C \left(\iint_{\mathbb{R}^{n+1}} |\Delta^{1/4} F(x)|^2 dx \right)^{1/2} \left(\sum_{i=1}^n \iint_{\mathbb{R}^{n+1}} |\Delta^{1/4} (\bar{a}_i G)(x)|^2 dx \right)^{1/2}. \end{aligned}$$

Lemma 4.3. *If H is a smooth function defined on \mathbb{R}_+^{n+1} and $H, \nabla H \in L_2(\mathbb{R}_+^{n+1})$, then*

$$\int_{\mathbb{R}^n} \left| \Delta^{1/4} H \right|^2 dx \leq C \iint_{\mathbb{R}_+^{n+1}} \left| \nabla H \right|^2 dx.$$

This well-known result can be derived by applying the even better known trace estimate

$$\int_{\mathbb{R}^n} |\Delta^{1/4} H_t|^2 dx \leq C \iint_{\mathbb{R}_+^{n+1}} (|\nabla H_t|^2 + |H_t|^2) dx$$

to the function $H_t(x) = H(tx)$, and letting $t \rightarrow \infty$.

From the lemma we get that

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^n} |\Delta^{1/4} F(x_0 + \mathbf{x})|^2 d\mathbf{x} dx_0 &\leq C \int_0^\infty \int_{\mathbb{R}^n} \int_0^\infty |\nabla F(x_0 + \mathbf{x} + te_0)|^2 dt d\mathbf{x} dx_0 \\ &= C \int_0^\infty \int_{\mathbb{R}^n} |\nabla F(\mathbf{x} + te_0)|^2 t dt d\mathbf{x} \\ &\leq C \iint_{\Omega_+} |\nabla f|^2 \sigma(x) dx. \end{aligned}$$

Similarly

$$\begin{aligned} \iint_{\mathbb{R}_+^{n+1}} |\Delta^{1/4} (\bar{a}_i G)(x)|^2 dx &\leq C \int_0^\infty \int_{\mathbb{R}^n} |\nabla (\bar{a}_i G)(\mathbf{x} + te_0)|^2 t d\mathbf{x} dt \\ &\leq C \|G\|^2 \leq C \|g\|^2 \end{aligned}$$

because $\bar{a}_i \in L_\infty(\mathbb{R}_+^{n+1}, \mathbb{R}_{(n)})$ and $|\nabla \bar{a}_i(x)|^2 x_0 dx$ are Carleson measures. This proves Lemma 4.2 and therefore Theorem 4.1.

5. OPERATOR BOUNDS

Let us first prove the special case of Theorem 3.1 where ϕ can be extended monogenically to \mathbb{R}_+^{n+1} or \mathbb{R}_-^{n+1} . More precisely, for $0 < \mu < \pi/2$, define the open cones $C_{\mu+}^o, C_{\mu-}^o$ in \mathbb{R}^{n+1} by

$$C_{\mu+}^o = \{x = x_0 + \mathbf{x} \in \mathbb{R}^{n+1} : x_0 > -|\mathbf{x}| \tan \mu\}, \quad C_{\mu-}^o = -C_{\mu+}^o.$$

Then $S_\mu^o = C_{\mu+}^o \cap C_{\mu-}^o$.

Define the Banach spaces $K(C_{\mu\pm}^o)$ to be the spaces of right-monogenic functions ϕ on $C_{\mu\pm}^o$ for which

$$\|\phi\|_{K(C_{\mu\pm}^o)} = \frac{\sigma_n}{2} \sup \left\{ |x|^n |\phi(x)| : x \in C_{\mu\pm}^o \right\} < \infty.$$

Given $\phi \in K(C_{\mu\pm}^o)$ and $r > 0$, define

$$\phi_{\pm}(r) = \int_{\pm x_0 > 0}^{|x|=r} \phi(x)n(x) dS_x$$

where $n(x)$ is the upward pointing normal to the sphere $\{x \in \mathbb{R}^{n+1} : |x| = r\}$.

In §6 we will show that $K(S_{\mu}^o) \cong K(C_{\mu+}^o) \oplus K(C_{\mu-}^o)$ and use this decomposition to prove that Theorem 3.1 is a consequence of the following result.

Theorem 5.1. *Suppose that $\omega < \mu < \pi/2$ and $1 < p < \infty$. Given $\phi \in K(C_{\mu+}^o)$ or $K(C_{\mu-}^o)$, there is a bounded linear operator T_{ϕ} on $L_p(\Sigma)$ defined for all $u \in L_p(\Sigma)$ and almost all $x \in \Sigma$ by*

$$\begin{aligned} (T_{\phi}u)(x) &= \lim_{\delta \rightarrow 0^+} \left\{ \int_{\Sigma} \phi(x \pm \delta - y)n(y)u(y) dS_y \right\} \\ &= \lim_{\varepsilon \rightarrow 0^+} \left\{ \int_{\substack{y \in \Sigma \\ |x-y| \geq \varepsilon}} \phi(x-y)n(y)u(y) dS_y + \phi_{\pm}(\varepsilon n(x))u(x) \right\} \end{aligned}$$

where $n(x)$ is the upward pointing unit normal vector to Σ which is defined at almost all $x \in \Sigma$. Moreover

$$\|T_{\phi}u\|_p \leq C_{\omega, \mu, p} \|\phi\|_{K(C_{\mu\pm}^o)} \|u\|_p$$

for some constants $C_{\omega, \mu, p}$ which depend only on ω, μ and p .

The following known results follow directly from this theorem.

Corollary 5.2. *Let $k(x) = \frac{1}{\sigma_n} \frac{\bar{x}}{|x|^{n+1}}$ when $x \neq 0$. Then there are bounded linear Cauchy operators P_+, P_- and C_{Σ} on $L_p(\Sigma)$ defined for all $u \in L_p(\Sigma)$ and almost all $x \in \Sigma$ by*

$$(P_{\pm}u)(x) = \pm \lim_{\delta \rightarrow 0^+} \int_{\Sigma} k(x \pm \delta - y)n(y)u(y) dS_y$$

and

$$(C_{\Sigma}u)(x) = 2 \text{ p.v. } \int_{\Sigma} k(x-y)n(y)u(y) dS_y.$$

Moreover, $P_{\pm} = \frac{1}{2}(\pm C_{\Sigma} + I) = \pm T_{k_{\pm}}$ where $k_{\pm} = k|_{C_{\mu\pm}^o}$. (Note that $\underline{k}_{\pm} = \pm \frac{1}{2}$.) So $P_+ + P_- = I$ and $P_+ - P_- = C_{\Sigma}$.

These are Plemelj formulae [10].

Proof of Theorem 5.1. We prove the case when $\phi \in K(C_{\mu+}^o)$. Our proof is an adaptation of the first proof in the paper [5] by Coifman, Jones, and Semmes on the L_p -boundedness of the Cauchy singular integral operator on a Lipschitz curve. It depends on the reverse square function estimates presented in Theorem 4.1 and the following lemma.

Lemma 5.3. *Suppose f is a function in \mathcal{H}_+ with compact support in Ω_+ , $\phi \in K(C_{\mu+}^o)$, $j = 0, 1, \dots, n$, and $\delta > 0$. Define*

$$S_{\phi, \delta, j} f(y) = \iint_{\Omega_+} f(x) \frac{\partial \phi}{\partial x_j}(x - y + \delta) \sigma(x) dx, \quad y \in \overline{\Omega}_-$$

Then

$$\|S_{\phi, \delta, j} f\|_{L_2(\Sigma)} \leq C \|\phi\|_{K(C_{\mu+}^o)} \|f\|_{\mathcal{H}_+}$$

where C depends only on μ and ω .

Proof. Since f is compactly supported we have $|S_{\phi, \delta, j} f(x)| \leq c/(1 + |x|)^n$ and $|\nabla S_{\phi, \delta, j} f(x)| \leq c/(1 + |x|)^{n+1}$. Since ϕ is right-monogenic on $C_{\mu+}^o$, $S_{\phi, \delta, j} f$ is right-monogenic on $\overline{\Omega}_-$. By Theorem 4.1, we have

$$\|S_{\phi, \delta, j} f\|_{L_2(\Sigma)} \leq C \|\nabla(S_{\phi, \delta, j} f)\|_{\mathcal{H}_-}$$

Since

$$\left| \nabla_y \frac{\partial \phi}{\partial x_j}(x - y) \right| \leq \frac{C \|\phi\|_{K(C_{\mu+}^o)}}{|x - y|^{n+2}} \Big/ \sin^{n+2} \left(\frac{\mu - \omega}{2} \right) \quad \forall x - y \in C_{(\mu+\omega)/2}^o,$$

we have

$$\begin{aligned} |\nabla(S_{\phi, \delta, j} f)(y)| &= \left| \iint_{\Omega_+} f(x) \nabla_y \frac{\partial \phi}{\partial x_j}(x - y + \delta) \sigma(x) dx \right| \\ &\leq C \|\phi\|_{K(C_{\mu+}^o)} \iint_{\Omega_+} \frac{|f(x)| \sigma(x)}{|x - y + \delta|^{n+2}} dx \end{aligned}$$

where C depends on μ and ω .

Since

$$\sup_{y \in \Omega_-} \iint_{\Omega_+} \frac{\sigma(y)^{1/2} \sigma(x)^{1/2}}{|x - y + \delta|^{n+2}} dx \leq C$$

and

$$\sup_{x \in \Omega_+} \iint_{\Omega_-} \frac{\sigma(y)^{1/2} \sigma(x)^{1/2}}{|x - y + \delta|^{n+2}} dy \leq C$$

with C independent of δ (notice that $\sigma(x) \leq |x - y|$, $\sigma(y) \leq |x - y|$ for $x \in \Omega_+$, $y \in \Omega_-$), by Schur's lemma,

$$\left\| \iint_{\Omega_+} \frac{|f(x)| \sigma(x)}{|x - y + \delta|^{n+2}} dx \right\|_{\mathcal{H}_-} \leq C \|f\|_{\mathcal{H}_+}.$$

Therefore

$$\|S_{\phi, \delta, j} f\|_{L_2(\Sigma)} \leq C \|\nabla(S_{\phi, \delta, j} f)\|_{\mathcal{H}_-} \leq C \|\phi\|_{K(C_{\mu+}^o)} \|f\|_{\mathcal{H}_+}.$$

This proves the lemma.

Now we are almost ready to prove Theorem 5.1. Consider first scalar-valued functions $u \in L_p(\Sigma)$, $1 < p < \infty$. Then, for $\delta > 0$, $W_{\phi_\delta} u$ is right-monogenic on $\bar{\Omega}_+$ where

$$(W_{\phi_\delta} u)(x) = \int_{\Sigma} \phi(x + \delta - y)u(y) dS_y, \quad x \in \bar{\Omega}_+.$$

Let us first consider $u \in L_2(\Sigma)$ which is compactly supported on Σ , in which case it is clear that $W_{\phi_\delta} u$ satisfies Theorem 4.1. Thus by applying it, we have

$$\begin{aligned} \|W_{\phi_\delta} u\|_{L_2(\Sigma)} &\leq C \|\nabla(W_{\phi_\delta} u)\|_{\mathcal{L}_+} \\ &\leq C \sum_{j=0}^n \sup_{f_j \in B} \left| \iint_{\Omega_+} \left(\frac{\partial}{\partial x_j} (W_{\phi_\delta} u)(x), \bar{f}_j(x) \right) \sigma(x) dx \right|. \end{aligned}$$

But

$$\begin{aligned} &\left| \iint_{\Omega_+} \left(\frac{\partial}{\partial x_j} (W_{\phi_\delta} u)(x), \bar{f}_j(x) \right) \sigma(x) dx \right| \\ &= \left| \iint_{\Omega_+} \int_{\Sigma} \left(\frac{\partial \phi}{\partial x_j} (x + \delta - y)u(y), \bar{f}_j(x) \right) dS_y \sigma(x) dx \right| \\ &= \left| \int_{\Sigma} \iint_{\Omega_+} \left(f_j(x) \frac{\partial \phi}{\partial x_j} (x + \delta - y) \right)_0 \sigma(x) dx u(y) dS_y \right| \\ &= \left| \int_{\Sigma} (S_{\phi, \delta, j} f_j)_0(y) u(y) dS_y \right| \\ &\leq \|u\|_{L_2(\Sigma)} \|S_{\phi, \delta, j} f_j\|_{L_2(\Sigma)} \\ &\leq C \|\phi\|_{K(C_{\mu^+}^o)} \|u\|_{L_2(\Sigma)} \|f_j\|_{\mathcal{L}_+} \quad (\text{by Lemma 5.3}) \\ &\leq C \|\phi\|_{K(C_{\mu^+}^o)} \|u\|_{L_2(\Sigma)}. \end{aligned}$$

Thus we get

$$\|W_{\phi_\delta} u\|_{L_2(\Sigma)} \leq C \|\phi\|_{K(C_{\mu^+}^o)} \|u\|_{L_2(\Sigma)}$$

for all $u \in L_2(\Sigma)$ with compact support. Therefore the operators W_{ϕ_δ} are Calderón-Zygmund operators, so

$$\|W_{\phi_\delta} u\|_{L_p(\Sigma)} \leq C_p \|\phi\|_{K(C_{\mu^+}^o)} \|u\|_{L_p(\Sigma)}$$

for $u \in L_p(\Sigma)$.

We now prove the same estimates for $T_{\phi_\delta} u$, $u \in L_p(\Sigma, \mathcal{L})$, where

$$(T_{\phi_\delta} u)(x) = \int_{\Sigma} \phi(x + \delta - y)n(y)u(y) dS_y.$$

(We could not do this directly because $T_{\phi_\delta} u$ is not monogenic on Ω_+ .)

Let $(v_k)_{k=1, \dots, M}$ be a basis in \mathcal{L} . Then $nu = \sum_{k=1}^M (nu)_k v_k$ where $(nu)_k$ are scalar valued and $\sum_{k=1}^M \|(nu)_k\|_p \leq c\|nu\|_p \leq c\|u\|_p$. So

$$\begin{aligned} \|T_{\phi_\delta} u\|_{L_p(\Sigma)} &\leq \sum_{k=1}^M \|W_{\phi_\delta} (nu)_k\|_{L_p(\Sigma)} \leq C\|\phi\|_{K(C_{\mu^+}^o)} \sum_{k=1}^M \|(nu)_k\|_{L_p(\Sigma)} \\ &\leq C\|\phi\|_{K(C_{\mu^+}^o)} \|u\|_{L_p(\Sigma)}. \end{aligned}$$

We next prove that for $u \in L_p(\Sigma)$, $T_{\phi_\delta} u$ converges in $L_p(\Sigma)$ as $\delta \rightarrow 0$.

Lemma 5.4. *Let*

$$\Phi_t(x) = \frac{1}{\sigma_n} \left(\frac{\overline{x+t}}{|x+t|^{n+1}} - \frac{\overline{x-t}}{|x-t|^{n+1}} \right), \quad t > 0, x \in S_{\pi/2}^o.$$

Then for $u \in L_p(\Sigma)$, $\Phi_t * u \rightarrow u$ in $L_p(\Sigma)$ as $t \rightarrow 0$.

This is proved by noticing that

$$|\Phi_t(x)| \leq C \frac{t}{(|x|^2 + t^2)^{(n+1)/2}} \quad \text{and} \quad \int_{\mathbb{R}^n} \Phi_t(x) dx = 1.$$

According to this lemma, we only have to prove $T_{\phi_\delta}(\Phi_t * u)$ converges in $L_p(\Sigma)$ as $\delta \rightarrow 0$ for fixed $t > 0$ and u compactly supported, Lipschitz continuous (for the set of such functions $\Phi_t * u$ is dense in $L_p(\Sigma)$, and the operators T_{ϕ_δ} are uniformly bounded in δ).

Notice that $T_{\phi_\delta}(\Phi_t * u) = \phi_{\delta+t} * u$ for $\delta < t$ and

$$|\phi_{\delta+t}(x) - \phi_{\delta'+t}(x)| \leq C \frac{|\delta - \delta'|}{(|x|^2 + t^2)^{(n+1)/2}} \quad \text{for } x \in S_\mu^o.$$

We get $\|\phi_{\delta+t} * u - \phi_{\delta'+t} * u\|_{L_p(\Sigma)} \rightarrow 0$ as $\delta, \delta' \rightarrow 0$ which means $T_{\phi_\delta}(\Phi_t * u)$ converges in $L_p(\Sigma)$ as $\delta \rightarrow 0$. Hence there exists a bounded operator T_ϕ on $L_p(\Sigma)$ given by

$$T_\phi u = \lim_{\delta \rightarrow 0} T_{\phi_\delta} u \quad \text{for } u \in L_p(\Sigma)$$

which satisfies the required estimate

$$\|T_\phi u\|_{L_p(\Sigma)} \leq C\|\phi\|_{K(C_{\mu^+}^o)} \|u\|_{L_p(\Sigma)}.$$

Actually $T_\phi u = \sum_{k=1}^M (T_{\phi, k} u) v_k$ where

$$(T_{\phi, k} u)(x) = \lim_{\delta \rightarrow 0} \int_\Sigma \phi(x - y + \delta)(nu)_k(y) dS_y = T_\phi(n^{-1}(nu)_k)(x)$$

are right-monogenic functions, so we also get

$$\begin{aligned} \|N(T_\phi u)\|_{L_p(\Sigma)} &\leq C \sum_{k=1}^M \|N(T_{\phi, k} u)\|_{L_p(\Sigma)} \leq C \sum_{k=1}^M \|T_\phi(n^{-1}(nu)_k)\|_{L_p(\Sigma)} \\ &\leq C \sum_{k=1}^M \|n^{-1}(nu)_k\|_{L_p(\Sigma)} \leq C\|u\|_{L_p(\Sigma)} \end{aligned}$$

where N is the nontangential maximal function. So the convergence is also almost everywhere. That is,

$$T_\phi u(x) = \lim_{\delta \rightarrow 0} T_{\phi_\delta} u(x)$$

for almost all $x \in \Sigma$, which is the first formula for T_ϕ given in Theorem 5.1.

To derive the second formula, we first consider functions u which are compactly supported and Lipschitz continuous. Then for any $\varepsilon > 0$

$$\begin{aligned} T_{\phi_\delta} u(x) &= \int_\Sigma \phi(x - y + \delta)n(y)u(y) dS_y \\ &= \int_{\substack{|x-y| \geq \varepsilon \\ y \in \Sigma}} \phi(x - y + \delta)n(y)u(y) dS_y + \int_{\substack{|x-y| \leq \varepsilon \\ y \in \Sigma}} \phi(x - y + \delta)n(y)u(y) dS_y. \end{aligned}$$

Since

$$\begin{aligned} &\left| \int_{\substack{|x-y| \leq \varepsilon \\ y \in \Sigma}} \phi(x - y + \delta)n(y)u(y) dS_y - \int_{\substack{|x-y| \leq \varepsilon \\ y \in \Sigma}} \phi(x - y + \delta)n(y) dS_y u(x) \right| \\ (1) \quad &= \left| \int_{\substack{|x-y| \leq \varepsilon \\ y \in \Sigma}} \phi(x - y + \delta)n(y)(u(y) - u(x)) dS_y \right| \\ &\leq C \int_{\substack{|x-y| \leq \varepsilon \\ y \in \Sigma}} \frac{1}{|x - y + \delta|^n} |u(y) - u(x)| dS_y \\ &\leq C \|u'\|_\infty \int_{\substack{|x-y| \leq \varepsilon \\ y \in \Sigma}} \frac{dS_y}{|x - y|^{n-1}} \leq C\varepsilon \|u'\|_\infty, \end{aligned}$$

$$(2) \quad \int_{\substack{|x-y| \leq \varepsilon \\ y \in \Sigma}} \phi(x - y + \delta)n(y) dS_y = \int_{\substack{|x-y| = \varepsilon \\ y \in \Omega_+}} \phi(x - y + \delta)n(y) dS_y,$$

(by the right-monogenicity of ϕ) and

$$(3) \quad \lim_{\varepsilon \rightarrow 0} \left\{ \int_{\substack{|x-y| = \varepsilon \\ y \in \Omega_+}} \phi(x - y)n(y) dS_y - \underline{\phi}_+(\varepsilon n(x)) \right\} = 0 \quad \text{a.e.}$$

we get

$$T_\phi u(x) = \lim_{\varepsilon \rightarrow 0} \left\{ \int_{\substack{|x-y| > \varepsilon \\ y \in \Sigma}} \phi(x - y)n(y)u(y) dS_y + \underline{\phi}_+(\varepsilon n(x))u(x) \right\} \quad \text{a.e.}$$

This gives that T_ϕ is a Calderón-Zygmund operator and so we can apply Theorem 5 in Chapter VII of [16] to conclude that, for all $u \in L_p(\Sigma)$, we have

$$T_\phi u(x) = \lim_{\varepsilon \rightarrow 0} \left\{ \int_{\substack{|x-y| > \varepsilon \\ y \in \Sigma}} \phi(x - y)n(y)u(y) dS_y + \underline{\phi}_+(\varepsilon n(x))u(x) \right\} \quad \text{a.e.}$$

A similar proof applies to $\phi \in K(C_{\mu-}^o)$. This completes the proof of Theorem 5.1.

Let us turn our attention to convolutions. Define $M(C_{\mu\pm}^o) = \{\phi \in K(C_{\mu\pm}^o) : \phi \text{ is left-monogenic as well}\}$.

Theorem 5.5. *Let $\phi \in K(C_{\mu+}^o)$ and $\psi \in M(C_{\mu+}^o)$.*

(1) *For $x \in C_{\mu+}^o$ define*

$$\begin{aligned} (\phi * \psi)(x) &= \int_{\partial C_{\mu+}} \phi(x - y - \delta)n(y)\psi(y + \delta) dS_y \\ &\quad \text{where } 0 < \delta < \text{dist}(x, \partial C_{\mu+}) \\ &= \lim_{\varepsilon \rightarrow 0} \left\{ \int_{\substack{H^+(\varepsilon, \mu) \\ |y| > \varepsilon}} \phi(x - y)n(y)\psi(y) dS_y + \phi_+(\varepsilon)\psi(x) \right\} \end{aligned}$$

where $H^+(\varepsilon, \mu) = \{x + \varepsilon \tan \mu : x = x_0 + \mathbf{x} \in \mathbb{R}^{n+1}, x_0 = -|\mathbf{x}| \tan \mu\}$.

Then $\phi * \psi \in K(C_{\nu+}^o)$ for all $\nu < \mu$ and

$$\|\phi * \psi\|_{K(C_{\nu+}^o)} \leq C \|\phi\|_{K(C_{\mu+}^o)} \|\psi\|_{K(C_{\mu+}^o)}.$$

(2) $T_\phi \cdot T_\psi = T_{\phi * \psi}$.

(3) *If $\psi_1 \in M(C_{\nu+}^o)$, then $\psi * \psi_1 \in M(C_{\nu+}^o)$ for all $\nu < \mu$, and $\phi * (\psi * \psi_1) = (\phi * \psi) * \psi_1$.*

Proof. (1) and (3) are easy to check.

To prove (2), we have

$$\begin{aligned} T_\phi \cdot T_\psi u(x) &= \lim_{\delta \rightarrow 0+} \int_\Sigma \phi(x + \delta - y)n(y) \lim_{\varepsilon \rightarrow 0+} \int_\Sigma \psi(y + \varepsilon - z)n(z)u(z) dS_z dS_y \\ &= \lim_{\delta \rightarrow 0+} \lim_{\varepsilon \rightarrow 0+} \int_\Sigma \left(\int_\Sigma \phi(x + \delta - y)n(y)\psi(y + \varepsilon - z) dS_y \right) n(z)u(z) dS_z. \end{aligned}$$

But

$$\begin{aligned} \int_\Sigma \phi(x + \delta - y)n(y)\psi(y + \varepsilon - z) dS_y &= \int_{\Sigma - z} \phi(x - z + \delta - y)n(y)\psi(y + \varepsilon) dS_y \\ &= \int_{\partial C_{\mu+}} \phi(x - z + \delta - y)n(y)\psi(y + \varepsilon) dS_y \quad (\text{by using Cauchy's theorem}) \\ &= (\phi * \psi)(x - z + \delta + \varepsilon). \end{aligned}$$

Therefore

$$T_\phi \cdot T_\psi u(x) = \lim_{\delta \rightarrow 0+} \int_\Sigma (\phi * \psi)(x + \delta - z)n(z)u(z) dS_z = T_{\phi * \psi} u(x).$$

Similarly we define $\phi * \psi$ for $\phi, \psi \in M(C_{\mu-}^o)$ and the corresponding results are true.

This theorem implies the following well-known results about the Cauchy operators P_\pm and C_Σ defined in Corollary 5.2.

Corollary 5.6. *Let $\phi_\pm \in M(C_{\mu\pm}^o)$. Then*

$$(1) \quad \begin{aligned} P_+ T_{\phi_+} &= T_{\phi_+} P_+ = T_{\phi_+}, & P_- T_{\phi_+} &= T_{\phi_+} P_- = 0, \\ P_- T_{\phi_-} &= T_{\phi_-} P_- = T_{\phi_-}, & P_+ T_{\phi_-} &= T_{\phi_-} P_+ = 0. \end{aligned}$$

$$(2) \quad P_+^2 = P_+, \quad P_-^2 = P_-, \quad P_+P_- = P_-P_+ = 0, \quad C_\Sigma^2 = I.$$

$$(3) \quad T_{\phi_+}T_{\phi_-} = T_{\phi_-}T_{\phi_+} = 0.$$

Proof. This follows immediately from Theorem 5.5 and the identities $k_+ * \phi_+ = \phi_+ * k_+ = \phi_+$ and $k_- * \phi_- = \phi_- * k_- = -\phi_-$.

6. HARDY DECOMPOSITION OF $K(S_\mu^o)$

Our proofs of Theorem 3.1 and Theorem 3.2 depend on the following decomposition results.

Theorem 6.1.

(1) Given $\phi_\pm \in K(C_{\mu\pm}^o)$, let $\phi = \phi_+ + \phi_-$, $\underline{\phi} = \underline{\phi}_+ + \underline{\phi}_-$ where $\underline{\phi}_\pm$ were defined in §5. Then $(\phi, \underline{\phi}) \in K(S_\mu^o)$ and

$$\|(\phi, \underline{\phi})\|_{K(S_\mu^o)} \leq (\|\phi_+\|_{K(C_{\mu+}^o)} + \|\phi_-\|_{K(C_{\mu-}^o)}).$$

(2) Conversely, given $(\phi, \underline{\phi}) \in K(S_\mu^o)$ there exist unique functions $\phi_+ \in K(C_{\mu+}^o)$ and $\phi_- \in K(C_{\mu-}^o)$ such that $\phi = \phi_+ + \phi_-$, $\underline{\phi} = \underline{\phi}_+ + \underline{\phi}_-$. Moreover

$$\|\phi_\pm\|_{K(C_{\mu\pm}^o)} \leq c\|(\phi, \underline{\phi})\|_{K(S_\mu^o)}$$

where c depends on the dimension n and μ . Indeed, for $x \in C_{\mu\pm}^o$

$$\phi_\pm(x) = \pm \lim_{\varepsilon \rightarrow 0+} \frac{1}{\sigma_n} \left\{ \int_{\substack{H^\pm(\varepsilon, \mu) \\ |y| > \varepsilon}} \phi(y)n(y) \frac{\overline{x-y}}{|x-y|^{n+1}} dS_y + \underline{\phi}(\varepsilon) \frac{\overline{x}}{|x|^{n+1}} \right\}$$

where $H^\pm(\varepsilon, \mu)$ are the n -surfaces defined by

$$H^\pm(\varepsilon, \mu) = \{x \pm \varepsilon \tan \mu : x = x_0 + \mathbf{x} \in \mathbb{R}^{n+1}, x_0 = \mp |\mathbf{x}| \tan \mu\}$$

and $n(y)$ is the normal to the surface at y which is directed upwards.

(3) $(\phi, \underline{\phi}) \in M(S_\mu^o)$ if and only if $\phi_\pm \in M(C_{\mu\pm}^o)$.

Proof of Theorem 3.1. Define $T_{(\phi, \underline{\phi})} = T_{\phi_+} + T_{\phi_-}$ by using the decomposition of Theorem 6.1 (2). Then $T_{(\phi, \underline{\phi})}$ is a bounded linear operator on $L_p(\Sigma)$ which satisfies the formula and the bound given in Theorem 3.1.

Proof of Theorem 6.1.

(1) To see that

$$\underline{\phi}_\pm(R) - \underline{\phi}_\pm(r) = \int_{r < |x| < R} \phi_\pm(x) dS_x,$$

apply Cauchy's theorem to the monogenic functions ϕ_\pm . The bounds are straightforward.

(2) To prove uniqueness, suppose that $\phi_+ + \phi_- = 0$, $\underline{\phi}_+ + \underline{\phi}_- = 0$. Then $\phi_+ = -\phi_-$ can be extended monogenically to $\mathbb{R}^{n+1} \setminus \{0\}$. Thus by Theorem 2.2,

$$\phi_+(x) = c \frac{\bar{x}}{|x|^{n+1}}, \quad \phi_- = -c \frac{\bar{x}}{|x|^{n+1}}$$

for some $c \in \mathbb{C}_{(n)}$. From the definition of $\underline{\phi}_\pm$ we obtain $\underline{\phi}_+ = \underline{\phi}_- = \frac{c}{2}$, so $c = 0$, and therefore $\phi_+ = \phi_- = 0$.

Let

$$\phi_\pm^\varepsilon(x) = \pm \frac{1}{\sigma_n} \left\{ \int_{\substack{H^\pm(\varepsilon, \mu) \\ |y| > \varepsilon}} \phi(y) n(y) \frac{\overline{x-y}}{|x-y|^{n+1}} dS_y + \underline{\phi}(\varepsilon) \frac{\bar{x}}{|x|^{n+1}} \right\}.$$

We prove that $\lim_{\varepsilon \rightarrow 0} \phi_\pm^\varepsilon(x)$ exist when $\varepsilon \rightarrow 0$, then define $\phi_\pm(x) = \lim_{\varepsilon \rightarrow 0} \phi_\pm^\varepsilon(x)$. We only look at $\phi_+^\varepsilon(x)$. For fixed

$$x \in C_{\mu+}^o \quad \text{and} \quad 0 < \varepsilon_2 < \varepsilon_1 < \min \left(\frac{1}{2}|x|, \text{dist}(x, \partial C_{\mu+}^o) \right),$$

$$\begin{aligned} \phi_+^{\varepsilon_1}(x) - \phi_+^{\varepsilon_2}(x) &= \frac{1}{\sigma_n} \left(\int_{\substack{H^+(\varepsilon_1, \mu) \\ |y| > \varepsilon_1}} - \int_{\substack{H^+(\varepsilon_2, \mu) \\ |y| > \varepsilon_2}} \right) \phi(y) n(y) \frac{\overline{x-y}}{|x-y|^{n+1}} dS_y \\ &\quad + (\underline{\phi}(\varepsilon_1) - \underline{\phi}(\varepsilon_2)) \frac{\bar{x}}{|x|^{n+1}} \\ &= -\frac{1}{\sigma_n} \int_{\substack{\varepsilon_2 < |y| < \varepsilon_1 \\ y \in \mathbb{R}^n}} \phi(y) \left(\frac{\overline{x-y}}{|x-y|^{n+1}} - \frac{\bar{x}}{|x|^{n+1}} \right) dy. \end{aligned}$$

Hence

$$\begin{aligned} |\phi_+^{\varepsilon_1}(x) - \phi_+^{\varepsilon_2}(x)| &\leq \frac{C}{\sigma_n} \int_{\substack{\varepsilon_2 < |y| < \varepsilon_1 \\ y \in \mathbb{R}^n}} \frac{1}{|y|^n} \left| \frac{\overline{x-y}}{|x-y|^{n+1}} - \frac{\bar{x}}{|x|^{n+1}} \right| dy \\ &\leq \frac{C}{\sigma_n} \int_{\substack{\varepsilon_2 < |y| < \varepsilon_1 \\ y \in \mathbb{R}^n}} \frac{1}{|y|^n} \frac{1}{|x|^{n+1}} |y| dy \\ &= \frac{C}{\sigma_n} \frac{1}{|x|^{n+1}} (\varepsilon_1 - \varepsilon_2) \rightarrow 0 \quad \text{as } \varepsilon_1 \rightarrow 0, \varepsilon_2 \rightarrow 0. \end{aligned}$$

Thus $\{\phi_+^\varepsilon(x)\}_\varepsilon$ is a Cauchy net for $x \in C_{\mu+}^o$ and therefore $\phi_+(x) = \lim_{\varepsilon \rightarrow 0} \phi_+^\varepsilon(x)$ exists pointwise on $C_{\mu+}^o$. Furthermore from the proof we can see that the convergence is uniform on any compact subsets of $C_{\mu+}^o$. So ϕ_+ is right-monogenic because it can be uniformly approximated by the right-monogenic functions ϕ_+^ε on every compact subset of $C_{\mu+}^o$. Similarly ϕ_- is right-monogenic.

Let $E(\varepsilon, \mu)$ be the set of points x between $H^+(\varepsilon, \mu)$ and $H^-(\varepsilon, \mu)$ with $|x| > \varepsilon$. From the Cauchy integral formula, $\phi_+^\varepsilon(x) + \phi_-^\varepsilon(x) = \phi(x)$. On letting $\varepsilon \rightarrow 0$ we get

$$\phi_+(x) + \phi_-(x) = \phi(x), \quad x \in S_\mu^o.$$

To prove $\underline{\phi}_+(r) + \underline{\phi}_-(r) = \underline{\phi}(r)$, we look at $\phi_\pm^\varepsilon(r)$ when $\varepsilon < r$. By Cauchy's theorem, we have

$$\begin{aligned} \phi_+^\varepsilon(x) &= \frac{1}{\sigma_n} \left\{ \int_{\substack{y \in \mathbb{R}^n \\ |y| > \varepsilon}} \phi(y) \frac{\overline{x-y}}{|x-y|^{n+1}} dy + \underline{\phi}(\varepsilon) \frac{\overline{x}}{|x|^{n+1}} \right\}, & x \in \mathbb{R}_+^{n+1}, \\ \phi_-^\varepsilon(x) &= -\frac{1}{\sigma_n} \left\{ \int_{\substack{y \in \mathbb{R}^n \\ |y| > \varepsilon}} \phi(y) \frac{\overline{x-y}}{|x-y|^{n+1}} dy + \underline{\phi}(\varepsilon) \frac{\overline{x}}{|x|^{n+1}} \right\}, & x \in \mathbb{R}_-^{n+1}. \end{aligned}$$

Then

$$\begin{aligned} \underline{\phi}_+^\varepsilon(r) &= \int_{\substack{|x|=r \\ x_0 > 0}} \left(\frac{1}{\sigma_n} \int_{\substack{y \in \mathbb{R}^n \\ |y| > \varepsilon}} \phi(y) \frac{\overline{x-y}}{|x-y|^{n+1}} dy \right) n(x) dS_x + \frac{1}{2} \underline{\phi}(\varepsilon), \\ \underline{\phi}_-^\varepsilon(r) &= \int_{\substack{|x|=r \\ x_0 < 0}} \left(\frac{1}{\sigma_n} \int_{\substack{y \in \mathbb{R}^n \\ |y| > \varepsilon}} \phi(y) \frac{\overline{x-y}}{|x-y|^{n+1}} dy \right) n(x) dS_x + \frac{1}{2} \underline{\phi}(\varepsilon), \end{aligned}$$

where $n(x)$ is the outward normal to the n -sphere $\{x \in \mathbb{R}^{n+1} : |x| = r\}$. Thus

$$\begin{aligned} \underline{\phi}_+^\varepsilon(r) + \underline{\phi}_-^\varepsilon(r) &= \int_{|x|=r} \left(\frac{1}{\sigma_n} \int_{\substack{y \in \mathbb{R}^n \\ |y| > \varepsilon}} \phi(y) \frac{\overline{x-y}}{|x-y|^{n+1}} dy \right) n(x) dS_x + \underline{\phi}(\varepsilon) \\ &= \int_{\substack{y \in \mathbb{R}^n \\ |y| > \varepsilon}} \phi(y) \left(\frac{1}{\sigma_n} \int_{|x|=r} \frac{\overline{x-y}}{|x-y|^{n+1}} n(x) dS_x \right) dy + \underline{\phi}(\varepsilon). \end{aligned}$$

By Cauchy's integral formula,

$$\frac{1}{\sigma_n} \int_{|x|=r} \frac{\overline{x-y}}{|x-y|^{n+1}} n(x) dS_x = \begin{cases} 1, & |y| < r, \\ 0, & |y| > r. \end{cases}$$

Hence

$$\underline{\phi}_+^\varepsilon(r) + \underline{\phi}_-^\varepsilon(r) = \int_{\varepsilon < |y| < r} \phi(y) dy + \underline{\phi}(\varepsilon) = \underline{\phi}(r).$$

On letting $\varepsilon \rightarrow 0$ we get $\underline{\phi}_+(r) + \underline{\phi}_-(r) = \underline{\phi}(r)$.

To prove the estimates, for $x \in \mathbb{R}_+^{n+1}$, $|x| > 2\varepsilon$, we have

$$\begin{aligned} &\int_{\substack{H^+(\varepsilon, \mu) \\ |y| > \varepsilon}} \phi(y) n(y) \frac{\overline{x-y}}{|x-y|^{n+1}} dS_y \\ &= \int_{\substack{H^+(\varepsilon, \mu) \\ \varepsilon < |y| < \frac{1}{2}|x|}} \phi(y) n(y) \frac{\overline{x-y}}{|x-y|^{n+1}} dS_y + \int_{\substack{H^+(\varepsilon, \mu) \\ \frac{1}{2}|x| < |y|}} \phi(y) n(y) \frac{\overline{x-y}}{|x-y|^{n+1}} dS_y \\ &= I_1 + I_2. \end{aligned}$$

For I_2 , notice that when $y \in H^+(\varepsilon, \mu)$, and $\frac{1}{2}|x| < |y|$, then $|x-y| > |y| \sin \mu$. Therefore

$$\begin{aligned} |I_2| &\leq \sigma_n^{-1} \|(\phi, \underline{\phi})\|_{K(S_\mu^0)} \int_{\substack{H^+(\varepsilon, \mu) \\ \frac{1}{2}|x| < |y|}} \frac{1}{|x-y|^n} \frac{1}{|y|^n} dS_y \\ &\leq \sigma_n^{-1} \|(\phi, \underline{\phi})\|_{K(S_\mu^0)} \int_{\substack{H^+(\varepsilon, \mu) \\ \frac{1}{2}|x| < |y|}} \frac{1}{(\sin \mu)^n} \frac{1}{|y|^{2n}} dS_y \\ &\leq c \|(\phi, \underline{\phi})\|_{K(S_\mu^0)} / |x|^n. \end{aligned}$$

For I_1 , we want to prove that

$$\left| I_1 + \underline{\phi}(\varepsilon) \frac{\bar{x}}{|x|^{n+1}} \right| \leq c \|(\phi, \underline{\phi})\|_{K(S_\mu^o)} / |x|^n.$$

Since

$$\begin{aligned} \underline{\phi}(\varepsilon) &= \underline{\phi} \left(\frac{1}{2}|x| \right) - \int_{\varepsilon < |y| < \frac{1}{2}|x|} \phi(y) dy, \\ I_1 &= \int_{\varepsilon < |y| < \frac{1}{2}|x|} \phi(y) \frac{\overline{x-y}}{|x-y|^{n+1}} dy + \int_{S_\mu(\frac{1}{2}|x|)} \phi(y)n(y) \frac{\overline{x-y}}{|x-y|^{n+1}} dS_y \end{aligned}$$

where $S_\mu(r) = \{x = x_0 + \mathbf{x} \in \mathbb{R}^{n+1}, |x| = r, 0 < -x_0/|\mathbf{x}| < \tan \mu\}$ and

$$\begin{aligned} &\left| \int_{S_\mu(\frac{1}{2}|x|)} \phi(y)n(y) \frac{\overline{x-y}}{|x-y|^{n+1}} dS_y \right| \\ &\leq \sigma_n^{-1} \|(\phi, \underline{\phi})\|_{K(S_\mu^o)} \int_{S_\mu(\frac{1}{2}|x|)} \frac{1}{|x-y|^n} \frac{1}{|y|^n} dS_y \\ &\leq c \|(\phi, \underline{\phi})\|_{K(S_\mu^o)} / |x|^n. \end{aligned}$$

Therefore

$$\begin{aligned} &\left| I_1 + \underline{\phi}(\varepsilon) \frac{\bar{x}}{|x|^{n+1}} \right| \\ &\leq \left| \int_{\varepsilon < |y| < \frac{1}{2}|x|} \phi(y) \left(\frac{\overline{x-y}}{|x-y|^{n+1}} - \frac{\bar{x}}{|x|^{n+1}} \right) dy \right| + c \|(\phi, \underline{\phi})\|_{K(S_\mu^o)} / |x|^n \\ &\quad + \left| \underline{\phi} \left(\frac{1}{2}|x| \right) \frac{\bar{x}}{|x|^{n+1}} \right| \\ &\leq c \sigma_n^{-1} \|(\phi, \underline{\phi})\|_{K(S_\mu^o)} \int_{\varepsilon < |y| < \frac{1}{2}|x|} \frac{1}{|x|^{n+1}} \frac{1}{|y|^{n-1}} dy + c \|(\phi, \underline{\phi})\|_{K(S_\mu^o)} / |x|^n \\ &\leq c \|(\phi, \underline{\phi})\|_{K(S_\mu^o)} / |x|^n. \end{aligned}$$

Thus we have proved that giving any $\varepsilon > 0$, for $x \in \mathbb{R}_+^{n+1}$, $|x| > 2\varepsilon$,

$$\begin{aligned} &\left| \int_{\substack{H^+(\varepsilon, \mu) \\ |y| > \varepsilon}} \phi(y)n(y) \frac{\overline{x-y}}{|x-y|^{n+1}} dS_y + \underline{\phi}(\varepsilon) \frac{\bar{x}}{|x|^{n+1}} \right| \\ &\leq c \|(\phi, \underline{\phi})\|_{K(S_\mu^o)} / |x|^n \end{aligned}$$

with c a constant depending on n and μ . On letting $\varepsilon \rightarrow 0$, we get

$$|\phi_+(x)| \leq \frac{c}{\sigma_n} \|(\phi, \underline{\phi})\|_{K(S_\mu^o)} / |x|^n, \quad x \in \mathbb{R}_+^{n+1}.$$

Similarly we get

$$|\phi_-(x)| \leq \frac{c}{\sigma_n} \|(\phi, \underline{\phi})\|_{K(S_\mu^o)} / |x|^n, \quad x \in \mathbb{R}_-^{n+1}.$$

These estimates can easily be extended from the half-spaces \mathbb{R}_\pm^{n+1} to the cones $C_{\mu\pm}^o$ by using the identity $\phi_+ + \phi_- = \phi$. For, when $x \in C_{\mu+}^o \setminus \mathbb{R}_+^{n+1}$, then

$$|\phi_+(x)| \leq |\phi_-(x)| + |\phi(x)| \leq \frac{c}{\sigma_n} \|(\phi, \underline{\phi})\|_{K(S_\mu^o)} / |x|^n,$$

and when $x \in C_{\mu-}^o \setminus \mathbb{R}_-^{n+1}$, then

$$|\phi_-(x)| \leq |\phi_+(x)| + |\phi(x)| \leq \frac{c}{\sigma_n} \|(\phi, \underline{\phi})\|_{K(S_\mu^o)} / |x|^n.$$

Therefore

$$\|\phi_\pm\|_{K(C_{\mu\pm}^o)} \leq c \|(\phi, \underline{\phi})\|_{K(S_\mu^o)}.$$

This completes the proof of (2).

(3) Let $(\phi, \underline{\phi}) \in M(S_\mu^o)$. We have to prove that ϕ_\pm are left-monogenic as well as right-monogenic. Just as we defined the right-monogenic functions ϕ_\pm on $C_{\mu\pm}^o$, we can also define left-monogenic functions ${}^L\phi_\pm(x) = \lim_{\varepsilon \rightarrow 0} {}^L\phi_\pm^\varepsilon(x)$ by

$${}^L\phi_\pm^\varepsilon(x) = \pm \frac{1}{\sigma_n} \left\{ \int_{\substack{H_\pm^\pm(\varepsilon, \mu) \\ |y| > \varepsilon}} \frac{\overline{x-y}}{|x-y|^{n+1}} n(y) \phi(y) dS_y + \frac{\bar{x}}{|x|^{n+1}} \underline{\phi}(\varepsilon) \right\}$$

for $x \in C_{\mu\pm}^o$. So what we need to prove is that ${}^L\phi_\pm(x) = \phi_\pm(x)$. Let us consider ϕ_+ . If $x \in \mathbb{R}_+^{n+1}$ and $\varepsilon < \frac{1}{2}|x|$, then

$$\begin{aligned} \phi_+^\varepsilon(x) - {}^L\phi_+^\varepsilon(x) &= \frac{1}{\sigma_n} \int_{\substack{y \in \mathbb{R}^n \\ |y| > \varepsilon}} \left(\phi(y) \frac{\overline{x-y}}{|x-y|^{n+1}} - \frac{\overline{x-y}}{|x-y|^{n+1}} \phi(y) \right) dy \\ &\quad + \frac{1}{\sigma_n} \left(\underline{\phi}(\varepsilon) \frac{\bar{x}}{|x|^{n+1}} - \frac{\bar{x}}{|x|^{n+1}} \underline{\phi}(\varepsilon) \right). \end{aligned}$$

It follows from condition (*) in the definition of $M(S_\mu^o)$ that

$$\underline{\phi}(\varepsilon) \bar{x} - \bar{x} \underline{\phi}(\varepsilon) = - \int_{\substack{|y|=\varepsilon \\ y \in \mathbb{R}^n}} \langle y, x \rangle (\phi(y) n(y) - n(y) \phi(y)) dy.$$

Then $F(r, x)$ is actually independent of $r > 0$.

Proof. For $0 < r < R$,

$$\begin{aligned} & F(R, x) - F(r, x) \\ &= \int_{\substack{|y|=R \\ y \in \mathbb{R}^n}} \langle y, x \rangle (\phi(y)n(y) - n(y)\phi(y)) dy \\ &\quad - \int_{\substack{|y|=r \\ y \in \mathbb{R}^n}} \langle y, x \rangle (\phi(y)n(y) - n(y)\phi(y)) dy - (\underline{\phi}(R) - \underline{\phi}(r))x \\ &\quad + x(\underline{\phi}(R) - \underline{\phi}(r)) \\ &= \int_{\substack{r < |y| < R \\ y \in \mathbb{R}^n}} \left\{ (\langle y, x \rangle \phi(y)) \mathbf{D}_y - \mathbf{D}_y (\langle y, x \rangle \phi(y)) \right\} dy \\ &\quad - (\underline{\phi}(R) - \underline{\phi}(r))x + x(\underline{\phi}(R) - \underline{\phi}(r)) \\ &= \int_{\substack{r < |y| < R \\ y \in \mathbb{R}^n}} (\phi(y)x - x\phi(y)) dy - (\underline{\phi}(R) - \underline{\phi}(r))x + x(\underline{\phi}(R) - \underline{\phi}(r)) = 0 \end{aligned}$$

(as $\mathbf{D}_y \phi = -\partial \phi / \partial y_0 = \phi \mathbf{D}_y$). So $F(r, x)$ is independent of r .

What remains is to show that if $\phi_{\pm} \in M(C_{\mu_{\pm}}^o)$, then $(\phi, \phi) \in M(S_{\mu}^o)$. Clearly ϕ is left-monogenic. Let us check that (ϕ_+, ϕ_+) satisfies (*). Since ϕ_+ equals its own left- and right-monogenic extensions, we see that $(\phi_+)_+^{\varepsilon}(x) - {}^L(\phi_+)_+^{\varepsilon}(x) \rightarrow 0$ for $x \in C_{\mu_+}^o$. Especially for $x \in \mathbb{R}_+^{n+1}$,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{1}{\sigma_n} \left\{ \int_{\substack{y \in \mathbb{R}^n \\ |y| > \varepsilon}} \left(\phi_+(y) \frac{\overline{x-y}}{|x-y|^{n+1}} - \frac{\overline{x-y}}{|x-y|^{n+1}} \phi_+(y) \right) dS_y \right. \\ \left. + \phi_+(\varepsilon) \frac{\overline{x}}{|x|^{n+1}} - \frac{\overline{x}}{|x|^{n+1}} \phi_+(\varepsilon) \right\} = 0 \end{aligned}$$

Notice that by Gauss's integral formula and the identity $\mathbf{D}_y \phi_+ = \phi_+ \mathbf{D}_y$, it follows as before that

$$\begin{aligned} & \frac{1}{\sigma_n} \int_{\substack{y \in \mathbb{R}^n \\ |y| > \varepsilon}} \left(\phi_+(y) \frac{\overline{x-y}}{|x-y|^{n+1}} - \frac{\overline{x-y}}{|x-y|^{n+1}} \phi_+(y) \right) dS_y \\ &= \frac{1}{(n-1)\sigma_n} \int_{\substack{|y|=\varepsilon \\ y \in \mathbb{R}^n}} (\phi_+(y)n(y) - n(y)\phi_+(y)) \frac{1}{|x-y|^{n-1}} dS_y. \end{aligned}$$

Since for fixed x ,

$$\frac{1}{|x-y|^{n-1}} = \frac{1}{|x|^{n-1}} - (n-1) \frac{\langle x, y \rangle}{|x|^n} + \frac{o(\varepsilon)}{|x|^{n+1}}$$

we have

$$\int_{\substack{|y|=\varepsilon \\ y \in \mathbb{R}^n}} \langle y, x \rangle (\phi_+(y)n(y) - n(y)\phi_+(y)) dS_y - (\phi_+(\varepsilon)x - x\phi_+(\varepsilon)) \rightarrow 0$$

as $\varepsilon \rightarrow 0$. By Lemma 6.2 it equals zero and therefore (ϕ_+, ϕ_+) satisfies (*). Similarly (ϕ_-, ϕ_-) satisfies (*).

This finishes the proof of Theorem 6.1.

We finish this section with a proof of Theorem 3.2. Given $(\phi, \underline{\phi}), (\psi, \underline{\psi}) \in M(S_\mu^o)$, define $(\phi, \underline{\phi}) * (\psi, \underline{\psi}) = (\phi_+ * \psi_+ + \phi_- * \psi_-, (\phi_+ * \psi_+)_+ + (\phi_- * \psi_-)_-)$. Then from Theorem 5.4 and Theorem 6.1, $(\phi, \underline{\phi}) * (\psi, \underline{\psi}) \in M(S_\nu^o)$ for all $\nu < \mu$ and

$$\|(\phi, \underline{\phi}) * (\psi, \underline{\psi})\|_{K(S_\nu^o)} \leq C \|(\phi, \underline{\phi})\|_{K(S_\mu^o)} \|(\psi, \underline{\psi})\|_{K(S_\mu^o)}.$$

(In particular $M(S_\omega)$ is an algebra under the $*$ -product.) Moreover,

$$\begin{aligned} T_{(\phi, \underline{\phi})} \cdot T_{(\psi, \underline{\psi})} &= (T_{\phi_+} + T_{\phi_-})(T_{\psi_+} + T_{\psi_-}) \\ &= T_{\phi_+} \cdot T_{\psi_+} + T_{\phi_-} \cdot T_{\psi_-} \quad (\text{from Corollary 5.6}) \\ &= T_{\phi_+ * \psi_+} + T_{\phi_- * \psi_-} \quad (\text{from Theorem 5.5}) \\ &= T_{(\phi, \underline{\phi}) * (\psi, \underline{\psi})}. \end{aligned}$$

7. CONDITION (*)

Our aim in this final section is to show that condition (*) is not very restrictive.

Theorem 7.1. *Given $(\phi, \underline{\phi}) \in K(S_\mu^o)$ with ϕ right-monogenic as well, there exists a constant $v \in \mathbb{R}_{(n)}$ (or $\mathbb{C}_{(n)}$) such that $(\phi, \underline{\phi} - v) \in M(S_\mu^o)$ and $|v| \leq C \|(\phi, \underline{\phi})\|_{K(S_\mu^o)}$, where C is a constant which depends only on n .*

Consequently there exist $\phi_\pm \in M(C_{\mu\pm}^o)$ such that $\|\phi_\pm\|_{K(C_{\mu\pm}^o)} \leq C \|(\phi, \underline{\phi})\|_{K(S_\mu^o)}$ and $T_{(\phi, \underline{\phi})} = T_{\phi_+} + T_{\phi_-} - vI$.

To prove this, we need the following results about Clifford algebras.

Lemma 7.2. *Let $u_k \in \mathbb{R}_{(n)}$ (or $\mathbb{C}_{(n)}$) for $1 \leq k \leq n$. Then there exists $v \in \mathbb{R}_{(n)}$ (or $\mathbb{C}_{(n)}$) such that $ve_k - e_k v = u_k$ for all k with $1 \leq k \leq n$ if and only if*

$$(7.1) \quad e_k u_j + e_j u_k + u_j e_k + u_k e_j = 0 \quad \text{whenever } 1 \leq j \leq n \text{ and } 1 \leq k \leq n.$$

One solution is given by

$$v = \sum_{j=1}^n \sum_{N=0}^n \alpha_N^{-1} e_j u_{j,N}$$

where $u_{j,N} = \sum_{|S|=N} u_{j,S} e_S$, and $\alpha_N = 2(n+1-N)$ if N is even and $\alpha_N = 2(N+1)$ if N is odd.

Proof. It is easy to check that (7.1) holds if $u_k = ve_k - e_k v$, whether or not $j = k$.

For the converse, suppose that the u_k satisfies (7.1), in which case the $u_{k,N}$ do too. This is because $e_k u_{j,N} + u_{j,N} e_k$ is of degree $N + (-1)^N$, so there is no mixing of degrees involved in (7.1).

Let $w_N = \sum_{j=1}^n e_j u_{j,N}$. Then, for $1 \leq k \leq n$,

$$\begin{aligned} w_N e_k - e_k w_N &= \sum_{j=1}^n (e_j u_{j,N} e_k - e_k e_j u_{j,N}) \\ &= \sum_{j=1}^n e_j (u_{j,N} e_k + e_k u_{j,N}) + 2u_{k,N} \\ &= - \sum_{j=1}^n e_j (e_j u_{k,N} + u_{k,N} e_j) + 2u_{k,N} \\ &= (n+2)u_{k,N} - \sum_{|S|=N} u_{k,S} \left\{ \sum_{j \in S} e_j e_S e_j + \sum_{j \notin S} e_j e_S e_j \right\} \\ &= (n+2)u_{k,N} - \sum_{|S|=N} u_{k,S} \{ (-1)^N N + (-1)^{N+1} (n-N) \} e_S \\ &= \alpha_N u_{k,N} \end{aligned}$$

and so $v = \sum_{N=0}^n \alpha_N^{-1} w_N$ as required.

Lemma 7.3. Given a C^1 -function F on $\mathbb{R}^n \setminus \{0\}$ which satisfies $\mathbf{D}F - F\mathbf{D} = 0$, define

$$u_k = \int_{|y|=1} y_k (F(y)y - yF(y)) dS_y, \quad 1 \leq k \leq n.$$

Then the u_k satisfy (7.1).

Proof. First consider (7.1) with $j = k$. (For ease of notation, take $j = k = 1$.) Then

$$\begin{aligned} e_1 u_1 + u_1 e_1 &= \int_{|y|=1} y_1 \{ e_1 (F(y)y - yF(y)) + (F(y)y - yF(y)) e_1 \} dS_y \\ &= \int_{|y|=1} y_1 \{ e_1 (F(y)y^+ - y^+ F(y)) + (F(y)y^+ - y^+ F(y)) e_1 \} dS_y \\ &\qquad \text{where } y^+ = y_2 e_2 + y_3 e_3 + \dots + y_n e_n \\ &= \int_{\substack{|y| \leq 1 \\ y_1 \neq 0}} y_1 \{ e_1 (F\mathbf{D}^+(y) - \mathbf{D}^+ F(y)) + (F\mathbf{D}^+(y) - \mathbf{D}^+ F(y)) e_1 \} dS_y \\ &\qquad \qquad \qquad \left(\text{where } \mathbf{D}^+ = \sum_{k=2}^n \frac{\partial}{\partial y_k} e_k \right) \end{aligned}$$

(using Gauss's theorem on the $(n-1)$ -balls $y_1 = \alpha$, $|y^+| \leq \sqrt{1-\alpha^2}$)

$$\begin{aligned} &= - \int_{\substack{|y| \leq 1 \\ y_1 \neq 0}} y_1 \left\{ e_1 \left(\frac{\partial F}{\partial x_1}(y) e_1 - e_1 \frac{\partial F}{\partial x_1}(y) \right) \right. \\ &\qquad \qquad \qquad \left. + \left(\frac{\partial F}{\partial x_1}(y) e_1 - e_1 \frac{\partial F}{\partial x_1}(y) \right) e_1 \right\} dS_y \\ &\qquad \qquad \qquad \text{(by the hypothesis } \mathbf{D}F - F\mathbf{D} = 0) \\ &= 0 \quad \text{(as the integrand is zero).} \end{aligned}$$

The proof of (7.1) with $j \neq k$ is similar (though a little more involved), requiring the use of Gauss's theorem on $(n-2)$ -balls.

To prove Theorem 7.1, let

$$u_k = \int_{\substack{|y|=r \\ y \in \mathbb{R}^n}} y_k (\phi(y)n(y) - n(y)\phi(y)) dS_y - (\underline{\phi}(r)e_k - e_k \underline{\phi}(r))$$

(which by Lemma 6.2 are independent of r). By Lemma 7.3 the u_k satisfy (7.1). So according to Lemma 7.2, there is a solution $v \in \mathbb{R}_{(n)}$ (or $\mathbb{C}_{(n)}$) satisfying $ve_k - e_kv = u_k$ and $|v| \leq C \sum_{k=1}^n |u_k| \leq C \|(\phi, \underline{\phi})\|_{X(S_\mu^n)}$. It is easy to check that, for this v ,

$$\int_{\substack{|y|=r \\ y \in \mathbb{R}^n}} \langle y, x \rangle (\phi(y)n(y) - n(y)\phi(y)) dy - ((\underline{\phi}(r) + v)x - x(\underline{\phi}(r) + v)) = 0$$

which means that $(\phi, \underline{\phi} + v) \in M(S_\mu^n)$.

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ABSTRACT. We prove the L_p -boundedness of convolution singular integral operators on a Lipschitz surface

$$\Sigma = \{g(\mathbf{x})e_0 + \mathbf{x} \in \mathbb{R}^{n+1} : \mathbf{x} \in \mathbb{R}^n\}$$

where g is a Lipschitz function which satisfies $\|\nabla g\|_\infty \leq \tan \omega < \infty$. Here we have embedded \mathbb{R}^{n+1} in the Clifford algebra $\mathbb{R}_{(n)}$ with identity e_0 , and are considering convolution with right-monogenic functions ϕ which satisfy $|\phi(x)| \leq C|x|^{-n}$ on a sector

$$S_\mu^o = \{x = x_0 + \mathbf{x} \in \mathbb{R}^{n+1} : |x_0| < |\mathbf{x}| \tan \mu\}$$

where $\mu > \omega$. Provided there exists an L_∞ function $\underline{\phi}$ satisfying

$$\underline{\phi}(R) - \underline{\phi}(r) = \int_{\substack{r < |x| < R \\ x \in \mathbb{R}^n}} \phi(x) dx,$$

then the related convolution singular integral operator

$$(T_{(\phi, \underline{\phi})}u)(x) = \lim_{\varepsilon \rightarrow 0^+} \left\{ \int_{\substack{y \in \Sigma \\ |x-y| \geq \varepsilon}} \phi(x-y)n(y)u(y) dS_y + \underline{\phi}(\varepsilon n(x))u(x) \right\}$$

is bounded on $L_p(\Sigma)$ for $1 < p < \infty$.

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