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L^2-COHOMOLOGY, NASH BLOWUP, AND SEMISMALL RESOLUTIONS

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1. INTRODUCTION

It was conjectured by Cheeger, Goresky, and MacPherson [CGM] some ten years ago that L^2-cohomology of a complex projective variety is naturally isomorphic to its intersection cohomology; recently, this conjecture has been settled by [Ohsawa].

Most of the interest in this conjecture has been due to its applications to the question of existence of a pure Hodge decomposition in the intersection cohomology; the point of this paper is to give another application to this conjecture. We show that it implies some restrictions on the possible resolutions of singularities.

Our principle result is as follows:

Theorem. (*See Corollary* 2.6.) *Let X be a compact complex analytic space. Then, if a semismall resolution $\pi : \widetilde{X} \to X$ is such that $\widetilde{X}$ is also a modification of the Nash blowup of X, then π is small.*

Recall that a resolution $\pi : \widetilde{X} \to X$ is called *small* if X can be stratified in such a way that the fiber of π has (complex) dimension $< d/2$ for the points of the stratum of codimension d, and π is called *semismall* if the dimension of the fiber is $< d - 1$. (If the resolution is obtained by a sequence of blowups with smooth centers, then the dimensions of the fibers are equal to $d - 1$.) The assertion of the Theorem is that (under the stated assumptions) the situation when the dimensions of the fibers are all $< d - 1$ but not all $< d/2$ is impossible. (See Example 2.8 for a semismall resolution satisfying the assumptions of the Theorem.)

This theorem is a corollary of our main technical result, Theorem 2.1, which describes the sheaves of L^2-forms on X in terms of its resolution $\pi : \widetilde{X} \to X$ provided that $\widetilde{X}$ maps regularly onto the Nash blowup of X.

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2. The main results

Notation. For any (possibly, singular) complex analytic space Y, let $\mathcal{O}_Y$ be the sheaf of analytic functions on Y, and let T^*Y be the sheaf of analytic differentials (1-forms) on Y. The latter may be defined, as usual, as $\mathcal{J}_Y/\mathcal{J}_Y^2$ where $\mathcal{J}_Y$ is the ideal of the diagonal embedding $Y \hookrightarrow Y \times Y$.

Let $\overset{\circ}{Y}$ be the set of nonsingular points of Y. We shall assume that $\overset{\circ}{Y}$ is equipped with a Riemannian metric g_Y which is locally in a neighborhood U of any point $x \in Y$ quasi-isometric to the metric induced by an embedding of U into $\mathbb{C}^N$. Such metric g_Y can be constructed by means of a partition of unity, and any two such metrics are quasi-isometric on all compact subsets of Y.

Denote by Ω_Y^{pq} the sheaf on Y of (p, q)-forms on $\overset{\circ}{Y}$ with measurable coefficients which are locally square-integrable on Y with respect to g_Y. In other words, for an open set $U \subset Y$ the space $\Gamma(U, \Omega_Y^{pq})$ contains all forms on $U \cap \overset{\circ}{Y}$ which have measurable coefficients and are square-integrable in a neighborhood of any point of U.

The notion of square-integrability is the same with respect to all metrics in one quasi-isometry class. Hence, the sheaf Ω_Y^{pq} is independent of the choice of the metric g_Y within the quasi-isometry class described above.

When $p = q = 0$, we get the sheaf $L_Y^2 = \Omega_Y^{0,0}$ of L^2-functions on Y, and the sheaf of all L^2-forms of order k is given by

$$\Omega_Y^k = \bigoplus_{p+q=k} \Omega_Y^{pq}.$$

Clearly, $\mathcal{O}_Y \subset L_Y^2$ and L_Y^2 is an $\mathcal{O}_Y$-module.

Let X be a singular complex-analytic space of pure dimension n, and let $\pi : \widetilde{X} \to X$ be a resolution of singularities which can be mapped onto the Nash blowup X^{Nash} of X so that the diagram

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "\pi"] \arrow[dr] & & X \\ & X^{\text{Nash}} \arrow[ur] & \end{tikzcd} \tag{2.1}$$

is commutative. In this case (see §3) the pullback $\pi^* T^* X$ is locally free modulo torsion. Let

$$\det \pi \overset{\text{def}}{=} \bigwedge^n \left(\frac{\pi^* T^* X}{\text{torsion}} \right) \otimes \left(\bigwedge^n T^* \widetilde{X} \right)^{-1} ;$$

this is an invertible sheaf on $\widetilde{X}$. The homomorphism of sheaves $\pi^* T^* X \to T^* \widetilde{X}$ makes $\det \pi$ an invertible sheaf of ideals.

Let $j : \overset{\circ}{X} \hookrightarrow X$ and $\tilde{\jmath} : \pi^{-1}\overset{\circ}{X} \hookrightarrow \widetilde{X}$ be the embedding maps; then $\Omega_X^{pq} \subset j_*\Omega_{\overset{\circ}{X}}^{pq}$ and we can identify $j_*\Omega_{\overset{\circ}{X}}^{pq} = \pi_*\tilde{\jmath}_*\Omega_{\pi^{-1}\overset{\circ}{X}}^{pq}$; this is the sheaf on X of the forms on $\overset{\circ}{X}$ which are locally L^2 in the neighborhood of any point of $\overset{\circ}{X}$ but may grow without restriction near the singular points of X.

This is our main technical result:

Theorem 2.1. *Let the resolution of singularities π be as in* (2.1). *Then the following two subsheaves of $j_*\Omega_{\overset{\circ}{X}}^{pq} = \pi_*\tilde{\jmath}_*\Omega_{\pi^{-1}\overset{\circ}{X}}^{pq}$ coincide*:

$$\Omega_X^{pq} = \pi_*\left[(\det \pi)^{-1} \underset{\mathcal{O}_{\widetilde{X}}}{\otimes} \bigwedge^p \left(\frac{\pi^* T^* X}{\text{torsion}}\right) \underset{\mathcal{O}_{\widetilde{X}}}{\otimes} \bigwedge^q \left(\overline{\frac{\pi^* T^* X}{\text{torsion}}}\right) \underset{\mathcal{O}_{\widetilde{X}}}{\otimes} L^2_{\widetilde{X}}\right] \tag{2.2}$$

where bar denotes complex conjugation and the right-hand side is considered a subsheaf of $\pi_*\tilde{\jmath}_*\Omega_{\pi^{-1}\overset{\circ}{X}}^{pq}$ in the following way: *the sections of* $(\det \pi)^{-1}$ *are meromorphic functions on $\widetilde{X}$ and the sections of $\pi^*T^*X/$*torsion *are considered as differential forms on $\widetilde{X}$ via the obvious monomorphism of sheaves $\pi^*T^*X/$*torsion $\hookrightarrow T^*\widetilde{X}$.

The proof of this theorem is found in §4.

Remark 2.2. If X is smooth, then this theorem states that (for $\widetilde{X} = X$)

$$\Omega_X^{pq} = \bigwedge^p T^*X \otimes_{\mathcal{O}_X} \bigwedge^q \overline{T^*X} \otimes_{\mathcal{O}_X} L_X^2$$

which is obvious.

Corollary 2.3. *Let π be as in* (2.1). *If, in addition, π is semismall, then $\Omega_X^{pq} = \pi_*\Omega_{\widetilde{X}}^{pq}$.*

Note that here $\Omega_{\widetilde{X}}^{pq}$ is the sheaf of L^2 forms on $\widetilde{X}$ with respect to the "nice" metric $g_{\widetilde{X}}$ on $\widetilde{X}$, as opposed to π^*g_X.

Proof. By definition, π is semismall if it is an isomorphism outside a subvariety $D \subset \widetilde{X}$ of codimension at least two. The invertible sheaf of ideals $\det \pi$ is clearly supported on D; hence, $\det \pi = \mathcal{O}_{\widetilde{X}}$. It follows then that

$$\bigwedge^n \left(\frac{\pi^*T^*X}{\text{torsion}}\right) = \bigwedge^n T^*\widetilde{X} \tag{2.3}$$

where $\pi^*T^*X/\text{torsion} \subset T^*\widetilde{X}$ are two locally free modules of rank n over $\mathcal{O}_{\widetilde{X}}$.

We claim that this implies $\dfrac{\pi^*T^*X}{\text{torsion}} = T^*\widetilde{X}$. Indeed, take any point $P \in \widetilde{X}$, and let $dz_1, dz_2, \dots, dz_n$ be local generators of $T^*\widetilde{X}$ at P, and $\omega_1, \omega_2, \dots, \omega_n$ local generators of $\pi^*T^*X/\text{torsion}$ at P, with $\omega_i = \sum_j f_{ij}dz_j$ where f_{ij} are some germs of analytic functions at P. The equality (2.3) implies that locally $\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n = h\,dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$ with h invertible. As $h = \det(f_{ij})$, the matrix (f_{ij}) is invertible locally, i. e., locally $dz_1, dz_2, \dots, dz_n$ can be expressed in terms of $\omega_1, \omega_2, \dots, \omega_n$ and $\pi^*T^*X/\text{torsion} = T^*\widetilde{X}$.

Substituting all this into formula (2.2), we get

$$\Omega_X^{pq} = \pi_* \left[\bigwedge^p T^* \widetilde{X} \otimes_{\mathscr{O}_{\widetilde{X}}} \bigwedge^q \overline{T^* \widetilde{X}} \otimes_{\mathscr{O}_{\widetilde{X}}} L^2_{\widetilde{X}} \right] = \pi_* \Omega_{\widetilde{X}}^{pq} . \quad \square$$

Let $\Omega_X^\bullet = \bigoplus_{k=0}^{2n} \Omega_X^k = \bigoplus \Omega_X^{pq}$, and let

$$(\operatorname{dom} d_X)^\bullet = \{\omega \in \Omega_X^\bullet \mid d\omega \in \Omega_X^\bullet\} .$$

Then $(\operatorname{dom} d_X)^\bullet$ is a complex of sheaves with differential d, and, if X is compact, the cohomology of the complex of global sections $\Gamma(X, (\operatorname{dom} d_X)^\bullet)$ is the L^2-cohomology of X.

It is known that $(\operatorname{dom} d_X)^\bullet$ is a complex of fine sheaves due to the existence of partitions of unity by functions with bounded differentials. Indeed, for any point $P \in X$ and any chart $U \subset X$, $U \hookrightarrow \mathbb{C}^N$ containing P, we can take a smooth function in $\mathbb{C}^N$ which vanishes outside some neighborhood of P and is identically equal to 1 inside some smaller neighborhood and restrict this function to U. The restriction does not increase the pointwise norm of the differential of this function, which is, consequently, bounded. These functions may be used to construct partitions of unity, so $(\operatorname{dom} d_X)^\bullet$ is fine.

If X is smooth, then $(\operatorname{dom} d_X)^\bullet$ is a resolution of the constant sheaf $\mathbb{C}_X$.

Corollary 2.4. *Let π be as in* (2.1). *If, in addition, π is semismall, then* $(\operatorname{dom} d_X)^\bullet = \pi_*(\operatorname{dom} d_{\widetilde{X}})^\bullet \simeq R\pi_* \mathbb{C}_{\widetilde{X}}$.

(Obvious.)

Denote by $\mathscr{IC}_X^\bullet$ the complex of (middle-perversity) intersection chain sheaves.

Indexing convention: we denote by $\mathscr{IC}_X^k$ what is denoted by $\mathbf{IC}^{k-2n}$ in [GM].

Theorem 2.5 [Ohsawa]. *There is a canonical quasi-isomorphism of complexes of sheaves* $(\operatorname{dom} d_X)^\bullet \simeq \mathscr{IC}_X^\bullet$.

For the proof, see [Ohsawa].

Corollary 2.6. *Let π be as in* (2.1). *If, in addition, π is semismall, then π is small.*

Proof. Indeed, it follows from Corollary 2.4 and Theorem 2.5 that $\mathscr{IC}_X^\bullet \simeq R\pi_* \mathbb{C}_{\widetilde{X}}$, and Lemma 2.7 below shows that this implies that π is small. $\square$

Lemma 2.7. *If π is a resolution of singularities such that $\mathscr{IC}_X^\bullet \simeq R\pi_* \mathbb{C}_{\widetilde{X}}$, then π is small.*

Proof. Indeed, take any point $x \in X$. As $\pi^{-1}x$ is a compact complex-analytic variety, $H^{2k}(\pi^{-1}x, \mathbb{C}) \neq 0$ when $k = \dim_{\mathbb{C}} \pi^{-1} x$.

Denote by $\mathscr{H}^l$ the lth cohomology sheaf of a complex of sheaves. The stalk at x of the sheaf $\mathscr{H}^l R\pi_* \mathbb{C}_{\widetilde{X}}$ is isomorphic to $H^l(\pi^{-1}x, \mathbb{C})$; hence, if $k = \dim_{\mathbb{C}} \pi^{-1} x$, then the sheaf $\mathscr{H}^{2k} \mathscr{IC}_X^\bullet$ is nontrivial at x. We know that $R\pi_* \mathbb{C}_{\widetilde{X}} \simeq \mathscr{IC}_X^\bullet$, and by the definition of $\mathscr{IC}_X^\bullet$, the set of points where

$\mathscr{H}^{2k}\mathscr{I}\mathscr{C}_X^\bullet$ is nontrivial, has real codimension $> k$. It follows that the set of points x such that $\dim_{\mathbb{C}} \pi^{-1}x = k$, has real codimension $> k$. Since this set is complex analytic, it has complex codimension $> k/2$, and this shows that π is small. □

Example 2.8 [Teissier, pp. 370–371]. Let X be the union of of two planes in $\mathbb{C}^4$ intersecting at one point. Then X^{Nash} is the normalization of X (the disjoint union of the same two planes), and X^{Nash} is a small resolution of X.

In this case, indeed, we have a semismall resolution which is mapped onto the Nash blowup of X, and $(\operatorname{dom} d_X)^\bullet \simeq \mathscr{I}\mathscr{C}_X^\bullet$; in agreement with our theorem, the resolution is indeed, small.

More generally, we may consider such X that all its singularities are transverse self-intersections, i. e., X is an image of an immersion which is allowed to have self-intersections, but only transverse ones; then, again, the Nash blowup is a normalization of X, and it is a small resolution of X.

Question 2.9. Does there exist a variety X such that the map $X^{\text{Nash}} \to X$ is semismall but not finite?

3. Nash blowup

Here we recall the definition of the Nash blowup.

First suppose that our singular variety X is embedded as a closed subvariety into a smooth manifold M. As before, let $n = \dim_{\mathbb{C}} X$. Let $\operatorname{Gr}_n TM$ denote the Grassmanian bundle of n-planes in the tangent bundle TM; the point of $\operatorname{Gr}_n TM$ is specified by by indicating a point $x \in M$ and an n-plane in $T_x M$. We have a section $\phi : \overset{\circ}{X} \to \operatorname{Gr}_n TM$ given by $x \mapsto (x, T_x X)$ (a "generalized Gauss map"). The Nash blowup X^{Nash} is defined as the closure of $\phi(\overset{\circ}{X})$ in $\operatorname{Gr}_n TM$. It is easy to see that X^{Nash} is a closed analytic subspace of $\operatorname{Gr}_n TM$, and the projection $\operatorname{Gr}_n TM \to M$ restricts to map $\psi : X^{\text{Nash}} \to X$ which is analytic, proper, and biholomorphic over $\overset{\circ}{X}$. Moreover, it is not hard to see that the construction of X^{Nash} and the map ψ up to natural equivalence is independent of the choice of the embedding $X \hookrightarrow M$ and is local on X (the latter means that the Nash blowup of an open subset of X is an open subset of X^{Nash}, in a natural way). In case the embedding $X \hookrightarrow M$ is not given, we may patch X^{Nash} together from the local embeddings of open subsets of X.

Proposition 3.1. *The coherent sheaf* $\psi^* T^* X/\text{torsion}$ *on* X^{Nash} *is locally free.*

Proof. Consider the tautological vector bundle over $\operatorname{Gr}_n TM$; if (x, L) is a point in $\operatorname{Gr}_n TM$ (x is a point in M and L is an n-dimensional subspace in $T_x M$), then the fiber of the tautological bundle over (x, L) is the space L. Denote by η the sheaf of sections of this tautological vector bundle.

It is easy to see that there is a morphism of coherent sheaves $\psi^* T^* X \to \eta|_{X^{\text{Nash}}}$ which is onto and whose kernel is the torsion of $\psi^* T^* X$. This is the statement of the Proposition. □

4. Proof of Theorem 2.1

After [HP], we consider the pullback $\pi^* g_X$ of our metric to $\widetilde{X}$; it is a nondegenerate metric on $\pi^{-1}\overset{\circ}{X}$. Denote by $\Omega^{pq}_{\widetilde{X}, \pi^* g_X}$ the sheaf on $\widetilde{X}$ of (p, q)-forms on $\pi^{-1}\overset{\circ}{X}$ with measurable coefficients which are square-integrable with respect to $\pi^* g_X$; for an open set $U \subset \widetilde{X}$, $\Gamma(U, \Omega^{pq}_{\widetilde{X}, \pi^* g_X})$ is the space of all forms on $U \cap \pi^{-1}\overset{\circ}{X}$ which have measurable coefficients and are square-integrable with respect to $\pi^* g_X$ in a neighborhood of any point of U.

Lemma 4.1. $\pi_* \Omega^{pq}_{\widetilde{X}, \pi^* g_X} = \Omega^{pq}_X$.

Proof. This follows from the fact that π is proper. □

Lemma 4.2. *The sheaf* $\pi^* T^* X/$torsion *is locally free.*

Proof. It follows immediately from Proposition 3.1 and diagram (2.1), as the pullback of a locally free sheaf is locally free. □

Take any point $x \in \widetilde{X}$ and take a local base $\alpha_1, \alpha_2, \dots, \alpha_n$ of $\pi^* T^* X/$torsion at x. We may consider $\pi^* T^* X/$torsion to be a subsheaf of $T^* \widetilde{X}$, thus making $\alpha_1, \alpha_2, \dots, \alpha_n$ sections of $T^* \widetilde{X}$.

Lemma 4.3. *The metric $\pi^* g_X$ is quasi-isometric to $\sum \alpha_i \overline{\alpha_i}$ in a neighborhood of x.*

Proof. Consider an embedding of a neighborhood of $\pi(x)$ in X into $\mathbb{C}^N$, and let $z_1, z_2, \dots, z_N$ be the coordinate functions in $\mathbb{C}^N$. Then $\pi^* g_X$ is quasi-isometric to $\sum \pi^* dz_j \overline{\pi^* dz_j}$. The sections $\pi^* dz_1, \pi^* dz_2, \dots, \pi^* dz_N$ form another local base of the sheaf $\pi^* T^* X/$torsion at x, hence, they can be expressed in terms of $\alpha_1, \alpha_2, \dots, \alpha_n$ and vice versa:

$$\pi^* dz_j = \sum f_{ji} \alpha_i$$
$$\alpha_i = \sum h_{ij} \pi^* dz_j .$$

Obviously, this implies that $\sum \pi^* dz_j \overline{\pi^* dz_j}$ and $\sum \alpha_i \overline{\alpha_i}$ are quasi-isometric. □

Let $u_1, u_2, \dots, u_n$ be local coordinates on $\widetilde{X}$ at x. We may assume $g_{\widetilde{X}} = \sum du_i \overline{du_i}$.

Lemma 4.4. *If we write at x: $\alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_n = f\, du_1 \wedge du_2 \wedge \dots \wedge du_n$, then f is a local generator at x of the invertible sheaf of ideals* $\det \pi$.

(Obvious.)

Lemma 4.5. *On a small enough neighborhood U of x in $\widetilde{X}$ the sheaf $\Omega^{pq}_{\widetilde{X}, \pi^* g_X}$ can be described in the following way. Let ω be a (p, q)-form with measurable coefficients on $U \cap \pi^{-1}\overset{\circ}{X}$. Then ω is square-integrable with respect to $\pi^* g$ if* *and only if* ω *can be represented in the form*

$$\omega = \sum \frac{1}{|f|} h_{i_1 i_2 \dots i_p \bar{\imath}_1 \bar{\imath}_2 \dots \bar{\imath}_q} \alpha_{i_1} \wedge \alpha_{i_2} \wedge \cdots \wedge \alpha_{i_p} \wedge \overline{\alpha_{\bar{\imath}_1}} \wedge \overline{\alpha_{\bar{\imath}_2}} \wedge \cdots \wedge \overline{\alpha_{\bar{\imath}_q}} \tag{4.1}$$

where $h_{i_1 i_2 \dots i_p \bar{\imath}_1 \bar{\imath}_2 \dots \bar{\imath}_q}$ *are some* L^2*-functions on* U *with respect to the usual metric* $g_{\widetilde{X}} = \sum du_i \overline{du_i}$ *on* U.

Proof. We may choose Usmall enough so that we may assume that $\alpha_1, \alpha_2, \dots, \alpha_n$ generate the sheaf $\pi^* T^* X/\text{torsion}$ on U and $\pi^* g_X = \sum \alpha_i \overline{\alpha_i}$. Then $\alpha_1, \alpha_2, \dots, \alpha_n$ generate the sheaf $T^*\widetilde{X}$ on $U \cap \pi^{-1}\overset{\circ}{X}$, and any (p, q)-form on $U \cap \pi^{-1}\overset{\circ}{X}$ can be represented in the form (4.1) with some coefficients $h_{i_1 i_2 \dots i_p \bar{\imath}_1 \bar{\imath}_2 \dots \bar{\imath}_q}$ (not necessarily square-integrable). A standard calculation shows that

$$\begin{aligned}\omega \wedge *\omega &= \sum \frac{1}{|f|^2} |h_{i_1 i_2 \dots i_p \bar{\imath}_1 \bar{\imath}_2 \dots \bar{\imath}_q}|^2 \alpha_1 \wedge \overline{\alpha_1} \wedge \alpha_2 \wedge \overline{\alpha_2} \wedge \cdots \wedge \alpha_n \wedge \overline{\alpha_n} \\ &= \sum |h_{i_1 i_2 \dots i_p \bar{\imath}_1 \bar{\imath}_2 \dots \bar{\imath}_q}|^2 du_1 \wedge \overline{du_1} \wedge du_2 \wedge \overline{du_2} \wedge \cdots \wedge du_n \wedge \overline{du_n}\end{aligned}$$

(here $*$ is taken with respect to $\pi^* g_X$). This means that ω is square-integrable with respect to $\pi^* g_X$ if and only if each $h_{i_1 i_2 \dots i_p \bar{\imath}_1 \bar{\imath}_2 \dots \bar{\imath}_q}$ is square integable with respect to the usual metric $g_{\widetilde{X}}$ on $\widetilde{X}$ (or U). □

Corollary 4.6.

$$\Omega^{pq}_{\widetilde{X}, \pi^* g_X} = (\det \pi)^{-1} \underset{\mathcal{O}_{\widetilde{X}}}{\otimes} \bigwedge^p \left(\frac{\pi^* T^* X}{\text{torsion}}\right) \underset{\mathcal{O}_{\widetilde{X}}}{\otimes} \bigwedge^q \left(\overline{\frac{\pi^* T^* X}{\text{torsion}}}\right) \underset{\mathcal{O}_{\widetilde{X}}}{\otimes} L^2_{\widetilde{X}}.$$

Proof. Indeed, this is just an invariant reformulation of Lemma 4.5. □

Finally, Theorem 2.1 follows immediately from Lemma 4.1 and Corollary 4.6.

Remark 4.7. It is easy to see that $\Omega^{pq}_{\widetilde{X}, \pi^* g_X} = \pi^{-1}\, \Omega^{pq}_X$.

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Abstract. We study the structure of L^2-forms on singular complex projective varieties with respect to Fubini-Studi metric using resolution of singularities. We show that a semismall resolution of X has to be small if it can be mapped onto the Nash blowup of X.

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