HOMOTOPY HYPERBOLIC 3-MANIFOLDS
ARE VIRTUALLY HYPERBOLIC

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The main result of this paper is the only if part of

Theorem 0.1. A closed irreducible 3-manifold \( N \) is homotopy equivalent to a hyperbolic 3-manifold if and only if \( N \) is finitely covered by a hyperbolic 3-manifold.

Remark 0.2. The if direction is a well known, quick consequence of Mostow's Rigidity theorem. Here is the sketch. Let \( p : M \rightarrow N \) be a finite regular covering map. Any covering translation of \( H^3 \) corresponding to an element of \( \pi_1(N) \) is a lift of a covering transformation \( f \) of \( p \), which by Mostow rigidity is homotopic to a unique isometry of \( M \). It follows that \( \pi_1(N) \cong \Gamma \subset \text{Isom}(H^3) \) and \( H^3/\Gamma \) is a hyperbolic 3-manifold \( M' \). Since \( M' \) and \( N \) are \( K(\pi, 1)'s \), they are homotopy equivalent.

The proof of the only if direction is likewise a quick application of well-known results. Here is the sketch. If \( N \) is homotopy equivalent to \( M \), then using the residual finiteness of \( \pi_1(M) \) we can pass to a regular covering space \( M_1 \) of \( M \) which has a closed geodesic \( \gamma \) with an enormously thick embedded regular neighborhood \( U \). Now lift the homotopy equivalence to \( f_1 : M_1 \rightarrow N_1 \) where \( N_1 \) is the corresponding covering of \( N \). Using the fact that the thurston norm equals the singular norm [to replace a singular torus by an embedded one in the same homology class in \( f_1(U) - f_1(\gamma) \)] and the observation that the homotopy equivalence keeps far away points of \( M_1 \) far away, it follows that in \( f_1(U) \) we can find a curve with a thick collar \( W \). The homotopy inverse \( g_1 \) is homotopic to a map which is a homeomorphism on \( W \) and on \( N - W \) restricts to a \( \pi_1 \)-injective degree-1 map. By Waldhausen \( g_1 \) is homotopic to a homeomorphism.

More details are provided in §1. Theorem 0.1 is used in §2 to reduce the general problem of homotopy equivalence implying homeomorphism for hyperbolic 3-manifolds to Conjecture 2.1. Other results related to the proof of Theorem 0.1 are stated in §2.

1. PROOF OF THEOREM 1.1

Notation 1.1. If \( f : M \rightarrow N \) is a homotopy equivalence, let \( g : N \rightarrow M \) be the homotopy inverse and \( F : M \times I \rightarrow M \) be the homotopy of \( g \circ f \) to \( \text{id}_M \). Let
$C > 2 \sup \{ \text{diam } \tilde{F}(m \times I) \mid m \in M \}$, where $\tilde{F}$ is a lift of $F$ to the universal covering of $M$. $l(\gamma)$ denotes length, and $B(n, x) = \{ z \in Z \mid d(x, z) \leq n \}$ where the space $Z$ is clear from context. $N(X)$ denotes (thin) regular neighborhood, and $|E|$ denotes number of components of $E$.

**Lemma 1.2.** If $f: M \to N$ is a homotopy equivalence, then $d(x, y) \geq C$ implies that $f(x) \cap f(y) = \emptyset$. □

**Lemma 1.3.** If $M$ is a closed hyperbolic manifold, $n > 0$, then there exists a regular finite sheeted covering $M_1$ of $M$ with injectivity radius $\geq n$.

**Proof.** Let $p: (H^3, z) \to (M, x)$ the universal covering map. Let $d = \text{diam}(M)$, and assume that $n > d$. Let $V = \{ t \in p^{-1}(x) \mid d(z, t) < 4n \}$. Since $\pi_1(M)$ is residually finite [Ma], there exist regular coverings $q: (H^3, z) \to (M_1, y)$, $\pi: (M_1, y) \to (M, x)$ such that $p = \pi \circ q$ and $V \cap q^{-1}(y) = \emptyset$. To see this let $\{ a_1, \ldots, a_k \} = \{ a \in \pi_1(M, x) \mid$ which lift to paths with the first end point $z$ and the other in $V - z \}$. $M_1$ is a covering corresponding to a finite index normal subgroup which does not contain $\{ a_1, \ldots, a_k \}$. $q|B(2n, z)$ is an embedding, else there exists $w \in B(4n, z)$ such that $q(w) = q(z)$. Since $M_1$ is regular, $q|B(2n, z')$ is an embedding for each $z' \in p^{-1}(x)$. Finally for all $s \in H^3$, there exists $z' \in p^{-1}(x)$ such that $B(n, s) \subset B(2n, z')$. Thus $q|B(n, s)$ is an embedding. □

If $\gamma$ is a closed geodesic in a hyperbolic 3-manifold, then the **tube radius** of $\gamma = \sup \{ \text{radii of embedded hyperbolic tubes about } \gamma \} = \frac{1}{2} \min \{ d(\gamma, \delta) \mid \delta$ is a distinct covering translate of $\gamma \text{ in } H^3 \}$. 

**Lemma 1.4.** If $M_1$ is a closed hyperbolic manifold with injectivity radius $n$, then there exists a geodesic $\gamma$ in $M_1$ with tube radius $> n/2$.

**Proof.** Let $\gamma$ be a shortest geodesic in $M_1$. Let $\gamma_1, \gamma_2$ be distinct lifts of $\gamma$ in $H^3$. If $d(\gamma_1, \gamma_2) \leq n = \frac{1}{3}l(\gamma)$, then there exist $x_i \in \gamma_i$ which are covering translates of each other such that $d(x_1, x_2) < l(\gamma)$, which implies the existence of a geodesic shorter than $\gamma$. □

**Lemma 1.5.** If $M$ is a closed oriented hyperbolic 3-manifold and $f: M \to N$ is a homotopy equivalence such that $N$ is irreducible and $M$ has a geodesic $\gamma$ with tube radius $> 4C$, then $f$ is homotopic to a homeomorphism.

**Proof.** For $0 < i \leq 4$ let $S_i$ be the torus in $M$ at distance $iC$ from $\gamma$, let $V_i$ be the solid torus in $M$ bounded by $S_i$, and let $K = f(S_2)$ and $J = N(K) \cup (\text{components of } N - K \text{ disjoint from } f(S_1 \cup S_3))$. Let $V_0$ also denote $\gamma$.

Claim 1. (0) $f^{-1}(J) \subset V_3 - V_1$ and $g(J) \subset V_4 - V_0$.

(ii) $J$ is irreducible.

(iii) $[K]$ generates $H_2(J) = \mathbb{Z}$. 

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Proof of Claim 1. (0) $K \cap (f(V_1) \cup f(M - V_3)) = \emptyset$ by Lemma 1.2. If $R$ is a component of $\partial N(K)$, then $g(R) \subset V_3$ and, hence, is homologically trivial, so $R$ bounds in $N$ since $f$ is a homotopy equivalence. Each of $f(M - V_3)$, $f(V_1)$ lies in a unique component of $N - K$ and, hence, in a unique component of $N - J$, so $f^{-1}(J) \subset \hat{V}_3 - V_1$, $g(J) \subset \hat{V}_4 - V_0$ now follow from Lemma 1.2.

(i) If $x \in f(\gamma)$, $y \in f(M - V_4)$, and $\alpha \subset N - K$ is a path from $x$ to $y$, then $\deg f = 1$ and choice of $C$ implies that (after possibly a tiny homotopy of $f$) some component $\beta$ of $f^{-1}(\alpha)$ is a path from some element of $f^{-1}(x) \in V_1$ to some element of $f^{-1}(y) \in M - V_3$ disjoint from $S_2$.

(ii) If there exists an essential 2-sphere $P$ in $J$, the irreducibility of $N$ would imply $P$ bounded a ball containing $f(V_1)$ or $f(M - V_3)$. This would contradict the $\pi_1$-injectivity of $f$.

(iii) $g \circ f | S_2$ is homotopic to id in $V_3 - V_1 \subset V_4 - V_0$, and $[S_2]$ generates $H_2(V_4 - V_0)$; therefore, $[f(S_2)] = [K]$ is primitive in $H_2(J)$. Since each closed curve in $J$ can be homotoped out of $J$, $J$ contains no nonseparating surface, so by (i) $H_2(J) = Z$. □

Claim 2. $J$ contains a homologically nontrivial torus $T$ which bounds in $N$ a solid torus $W$ containing $f(\gamma)$. Finally $g : T \to M - V_0$ and $i : T \to N - \hat{W}$ are $\pi_1$-injective.

Proof of Claim 2. Since the thurston norm on $H_2(J)$ equals the singular norm on $H_2(J)$ [G Corollary 6.18] and (iii) there exists an embedded nonbounding torus $T$ in $J$ such that $[T] = [K] \in H_2(J)$. Since $g | T$ is not $\pi_1$-injective as a map into $V_4$, it follows that $T$ is compressible in $N$. A compressible torus in an irreducible 3-manifold bounds either a solid torus or lives in a ball. $\pi_1$-injectivity of $f$ precludes the latter, and $Z \neq \pi_1(M - V_3)$ implies that the solid torus $W$ contains $f(\gamma)$. The $\pi_1$-injectivity of $g | T$ follows from the facts that $g | T$ is $\pi_1$-injective as a map into $V_4 - V_0$ (since each singular sphere in $V_4 - V_0$ is homologically trivial and $[g(T)] = [S_2]$) and $S_4$ is incompressible in $M - \hat{V}_4$. Finally if $T$ is compressed in $N - \hat{W}$, then an application of the loop theorem would imply that either some power of $f(\gamma)$ is homotopically trivial in $N$ or $N = S^2 \times S^1$. □

Claim 3. Let $Q = N - \hat{W}$. $g$ is homotopic to a map $h : N \to M$ such that $h | T$ is a homeomorphism onto $S_2$, $h | W$ is degree-1 onto $V_2$, $h | Q$ is degree-1 onto $M - V_2$, and $h | W$ is $\pi_1$-injective into $V_2$.

Proof of Claim 3. By Claim 1 the map on $T$ obtained by first applying $g$ and then projecting to $S_2$ (in $V_4 - V_0$) is a degree-1 map, so by [K] or [BE] it is homotopic to a homeomorphism. Therefore, to obtain $h$, first homotop $g$ to $g'$ via a homotopy supported in a tiny neighborhood of $T$ so that $g' | T$ is a homeomorphism, $g'(W) \subset V_4$, and $g'(Q) \subset M - V_0$. Applying the natural retractions of $V_4$ to $V_2$ and $M - V_0$ to $M - \hat{V}_2$, to stuff the guts spilling out, we obtain $h$. The degree-1 conclusions follow from the fact that $g$ is degree-1. $h | W$ is obviously $\pi_1$-injective. □
Claim 4. $Q$ is irreducible and $\pi_1$-injects into $f(M - \hat{V}_1)$.

Proof of Claim 4. The irreducibility of $Q$ follows from the irreducibility of $N$ and the fact that $f(\gamma)$ is homotopically nontrivial. If $D$ is a singular disc in $f(M - \hat{V}_1)$ such that $\partial D \subset T$, then $g(D) \subset M - V_0$ and $g | T : T \rightarrow M - V_0$ is $\pi_1$-injective implies that $\partial D$ is homotopically trivial in $T$. Therefore, $T \pi_1$-injects into $f(M - \hat{V}_1)$ and, since $T$ is incompressible in $Q$, Claim 4 follows. □

Claim 5. $h | Q$ is $\pi_1$-injective into $M - V_2$.

Proof of Claim 5. Let $\delta$ be a closed curve in $Q$. Let $\alpha$ (resp. $\beta$) be the curve $h(\delta)$ (resp. $g(\delta)$). By construction $\alpha \subset M - V_2$ and $\beta \subset M - V_0$. Furthermore $\alpha$ is homotopic to $\beta$ in $M - V_0$. $\deg f = 1$ implies that $f^{-1}(\delta)$ contains a curve $\epsilon \in M - V_1$ such that $f | \epsilon$ maps with nonzero degree to $\delta$. $\epsilon$ is homotopic to a nonzero multiple of $\beta$ and, hence, a nonzero multiple of $\alpha$ in $M - V_0$. Therefore, if $h(\delta)$ is homotopically trivial in $M - V_2$, then $\epsilon$ is homotopically trivial in $M - V_1$, so $\delta$ is homotopically trivial in $f(M - V_1)$ [$\pi_1(f(M - \hat{V}_1))$ being torsion free] and so $\delta$ is homotopically trivial in $Q$ by Claim 4. □

Claim 6. $h$ is homotopic to a homeomorphism.

Proof of Claim 6. By Waldhausen [He] $h : (Q, \partial Q) \rightarrow (M - \hat{V}_2, \partial V_2)$ (resp. $h : (W, \partial W) \rightarrow (V_2, \partial V_2)$) is homotopic to a homeomorphism via a homotopy fixed on the boundary. □

Remarks. If $\gamma$ has a larger tube radius, e.g. $12C$, then Claims 4–5 can be replaced by the observation that the homotopy equivalence splits along $S_6$ and $T_6$ to ones on $V_6$ and $W$ and $M - \hat{N}(V_6)$ and $Q$. Hint: there is an embedded torus $T_i$ near $f(S_i)$ for $i = 2, 6, 10$ which bounds a solid torus; furthermore, $T_2, T_{10}$ bound a product homeomorphic to Torus $\times I$. In this setting we now have enough room to homotope $f$ so that $f(S_6) = T_6$. I thank Mike Freedman for suggesting this simplification.

Proof of Theorem 1.1. By Lemmas 1.3, 1.4, $M$ has a finite covering space $M_1$ with a geodesic of tube radius $> 4C$. Let $N_1$ be the associated covering space of $N$. By [MSY] or [D] $N_1$ is irreducible. Now apply Lemma 1.5. □

2. RELATED RESULTS AND A CONJECTURE

Conjecture 2.1. Let $G$ and $H$ be isomorphic finitely generated groups such that $G \subset PSL(2, C) \subset \text{Homeo}(B^3)$ and $H \subset \text{Homeo}(B^3)$. Suppose further:

(a) $G$ and $H$ act freely on $\hat{B}^3$ with closed 3-manifold quotients;
(b) $H | S^2 = G | S^2$; and
(c) there exist subgroups $H', G'$ of finite index in $H$ and $G$ such that $H' | B^3 = G' | B^3$.

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Then $H$ is conjugate to $G$ in $\text{Homeo} B^3$.

**Remark 2.2.** Mostow rigidity and Conjecture 2.1 imply the conjecture "If $f: M \to N$ is a homotopy equivalence where $M$ is hyperbolic and $N$ is irreducible, then $f$ is homotopic to a homeomorphism." For if $N$ is an irreducible homotopy hyperbolic 3-manifold, then by Theorem 0.1 it has a regular finite sheeted covering space $N_1$ which is a hyperbolic 3-manifold. The well-known argument of Remark 0.2 shows that there exists a hyperbolic 3-manifold $M'$ homeomorphically equivalent to $N$ such that $M$ is finitely covered by $N_1$ and if $G$ (resp. $H$) is the group of covering transformations of $M'$ corresponding to $N_1$ (resp. $N$), and extended to act on $B^3$, then $G$ and $H$ satisfy (a), (b), and (c), where $H'$ and $G'$ are the groups associated to $N_1$. The conclusion of Conjecture 2.1 implies that $N$ is homeomorphic to $M'$, and another application of Mostow rigidity shows that $M = M'$ and that the homotopy equivalence $f$ is homotopic to a homeomorphism.

**Theorem 2.3.** Let $f: M \to N$ be a homotopy equivalence between closed irreducible 3-manifolds with residually finite fundamental group. Suppose further that there exists an element $\gamma \in \pi_1(M)$ which generates a maximal abelian subgroup $\langle \gamma \rangle$ whose associated covering space $M_\gamma = D^2 \times S^1$; then $M$ and $N$ have homeomorphic finite sheeted coverings.

**Proof.** Fix any Riemannian metric on $M$. Let $V_i$, $i = 0, 1, \ldots, 4$, be parallel solid tori in $M_\gamma$ containing $\gamma$ as a core with $\partial V_i = S_i$ at least $C$ distance apart. It is well known (to algebraists, see [L]) that maximal abelian subgroups are separable (so given $a_1, \ldots, a_n \in \pi_1(M) - \langle \gamma \rangle$ there exists a subgroup of finite index containing $\langle \gamma \rangle$ but missing $a_1, \ldots, a_n$). An argument related to the one of Lemma 1.3 shows that there exists a finite covering $M_1$ of $M$ such that $M_1$ is covered by $M_\gamma$ and the projection of $V_i$ to $M_1$ is an embedding. We abuse notation by continuing to call the image in $M_1$ of $V_i$ by the same name. Let $N_1$ be the associated finite covering of $N$. Again by [MSY] or [D] $N_1$ is irreducible. The argument of Lemma 1.5 now shows that $M_1$ and $N_1$ are homeomorphic.

Combining Waldhausen [W] with the idea of the proof of Lemma 1.5 we obtain.

**Theorem 2.4.** If $f: M \to M$ is a homeomorphism homotopic to the identity and $M$ is a hyperbolic 3-manifold, then there exists a finite covering space of $M$ such that a lift of $f$ is isotopic to the identity. □

**Remark 2.5.** Actually $M$ need only satisfy the hypothesis of Theorem 2.3.

**References**


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