Manifolds of Positive Ricci Curvature with Almost Maximal Volume

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1°. In this note we consider complete Riemannian manifolds with Ricci curvature bounded from below. The well-known theorems of Myers and Bishop imply that a manifold $M^n$ with $\text{Ric} \geq n - 1$ satisfies $\text{diam}(M^n) \leq \text{diam}(S^n(1))$, $\text{Vol}(M^n) \leq \text{Vol}(S^n(1))$. It follows from [Ch] that equality in either of these estimates can be achieved only if $M^n$ is isometric to $S^n(1)$. The natural conjecture is that a manifold $M^n$ with almost maximal diameter or volume must be a topological equivalent to $S^n$. With respect to diameter this is true only if $M^n$ satisfies some additional assumptions; see [An, O, GP, E]. With respect to volume however no extra restriction is necessary.

**Theorem 1.** For any integer $n \geq 2$ there exists $\delta_n > 0$ with the following property. Let $M^n$ be a complete Riemannian manifold with $\text{Ric} \geq n - 1$. Suppose that $\text{Vol}(M^n) \geq (1 - \delta_n) \text{Vol}(S^n(1))$. Then $M^n$ is homeomorphic to $S^n$.

In fact, we prove only that $\pi_i(M^n) = 0$ for all $i < n$ and refer to the work of Hamilton [H] for $n = 3$ and to the solution of generalized Poincaré conjecture (Smale [S], Freedman [F]) for $n \neq 3$.

Vanishing of homotopy groups is a simple consequence of the Main Lemma below. Its further simple corollaries are a noncompact version of Theorem 1 and a corresponding finiteness theorem (cf. [P, Corollary B]).

Let $B^H(R)$ denote a ball of radius $R$ in the simply connected space form of constant curvature $H$.

**Theorem 2.** Let $M^n$ be a complete Riemannian manifold with $\text{Ric} \geq 0$; $p \in M$. Suppose that $\text{Vol}(B_p(R)) \geq (1 - \delta_n) \text{Vol}(B^0(R))$ for all $R > 0$. Then $M^n$ is contractible.

**Theorem 3.** For any $n$, $H$, $\mathcal{D}$, $R$ the set $\mathcal{M}_{\delta_n}(n, H, \mathcal{D}, R)$ of all complete Riemannian manifolds $M^n$ with $\text{diam}(M^n) \leq \mathcal{D}$, $\text{Ric} \geq (n - 1)H$, and $\text{Vol}(B_p(R)) \geq (1 - \delta_n) \text{Vol}(B^H(R))$ for all $p \in M^n$, contains only finitely many homotopy types.

2°. Henceforward we fix $n \geq 2$ and ignore the dependence on $n$ in our notations. We denote by $M$ an arbitrary compact $n$-dimensional Riemannian manifold with $\text{Ric} \geq n - 1$; all parameters below are supposed to be independent of $M$.

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Main Lemma. For any \( c_2 > c_1 > 1 \) and integer \( k \geq 0 \) there exists \( \delta = \delta_k(c_1, c_2) > 0 \) with the following property. Let \( p \in M, \ 0 < R < \pi c_2^{-1} \). Suppose that \( \text{Vol}(B_q(p)) \geq (1 - \delta) \text{Vol}(B^1(p)) \) for every ball \( B_q(p) \subset B_p(c_2 R) \). Then

(A) Any continuous map \( f: S^k = \partial D^{k+1} \to B_p(R) \) can be continuously extended to a map \( g: D^{k+1} \to B_p(c_1 R) \).

(B) Any continuous map \( f: S^k \to M \setminus B_p(R) \) can be continuously deformed to a map into \( M \setminus B_p(c_1 R) \).

Remark. The Main Lemma can obviously be modified for \( n \)-manifolds with \( \text{Ric} \geq 0 \) or \( \text{Ric} \geq -(n-1) \), in the latter case \( \delta \) may depend on \( R \) as \( R \to \infty \).

We give below a detailed proof of (A) and outline a similar proof of (B) leaving the details to the reader.

3°. At first we state explicitly all the properties of manifolds with Ricci curvature bounded from below, which are relevant to the proof.

Let \( \overline{ab} \) denote a shortest geodesic with endpoints \( a, b \).

(\( \kappa \)) There exists a positive function \( \kappa, \kappa(t) \to 0 \) as \( t \to 0 \), such that
\[
|ab| + |ac| - |bc| \leq \kappa \left( |a, \overline{bc}| / \min \{|ab|, |ac|\} \right) \cdot |a, \overline{bc}| \quad \text{for all} \ a, \overline{bc} \subset M.
\]

This is a weakened version of the Abresch-Gromoll inequality [AG].

(\( \gamma \)) For any \( c_2 > c_1 > 1, \ \epsilon > 0 \) there exists \( \gamma = \gamma(c_1, c_2, \epsilon) \) with the following property. Let \( p \in M, \ 0 < R < \pi c_2^{-1} \). Suppose that \( \text{Vol}(B_p(c_2 R)) \geq (1 - \gamma) \text{Vol}(B^1(c_2 R)) \). Then for every \( a \in B_p(R) \) there exists \( b \in M \setminus B_p(c_1 R) \) such that \( |a, \overline{pb}| \leq \epsilon R \).

This is a simple corollary of (the proof of) the Bishop-Gromov volume comparison inequality.

Warning. In the proof of the Main Lemma, we do not use the existence of the injectivity radius and avoid explicit induction on \( R \).

4°. Outline of the proof of (A). Assertion (A) is proved by induction on \( k \). The case \( k = 0 \) is obvious. Assume that (A) holds in dimensions less than \( k \). Fix \( c_2 > c_1 > 1 \) and let \( d_0 > 0 \) and \( \delta > 0 \) be small enough. Now given \( M, p, R \), satisfying the conditions of (A), and a continuous map \( f: S^k \to B_p(R) \), we can construct another continuous map \( \tilde{f}: S^k \to B_p((1-d_0)R) \) such that the uniform distance between \( f \) and \( \tilde{f} \) is small in comparison with \( R \). This is the crucial step, it uses both properties (\( \kappa \)), (\( \gamma \)) and the inductionsal assumption.

The map \( \tilde{f} \) is not known yet to be homotopic to \( f \), and there is no obvious way to construct such a homotopy at once. To go around this difficulty, we take a fine triangulation of \( S^k \) and construct a “small” homotopy between \( f \) and \( \tilde{f} \) on the \((k-1)\)-skeleton of this triangulation. In fact, the homotopy is constructed consecutively on \( i \)-skeleta, \( i = 0, 1, \ldots, k-1 \), using the inductional assumption.

The result of previous steps can be interpreted as an extension of \( f \) from \( S^k = \partial D^{k+1} \) to the \( k \)-skeleton of a finite cell decomposition of \( D^{k+1} \). Recall
that the boundary of the "central" cell is mapped into $B_p((1 - d_0)R)$, and the size of the images of the boundaries of all other cells is small in comparison with $R$. Now we repeat the previous steps for each cell separately and obtain an extension of $f$ to the $k$-skeleton of a finer cell decomposition (Figure 1), etc. The limit of the infinite repetition of this procedure is the required extension $g$.

Apparently the argument above cannot be convincing until the choice of "small" parameters is specified. We give a formal exposition below.

5°. Proof of (A).

5.1. Consider the following general situation. Let $f: S^k \rightarrow B_p(R) \subset M$ be a continuous map, and let sequences of finite cell subdivisions $K_j$ of $D^{k+1}$ and continuous maps $f_j: \text{skel}_k(M) \rightarrow M$ satisfy

(a) $K_{j+1}$ is a cell subdivision of $K_j$ and $f_{j+1} \equiv f_j$ on $\text{skel}_k(K_j)$.

(b) For each $(k+1)$-cell $\sigma \in K_j$ there exist $p_\sigma \in B_p(c_1R)$ and $R_\sigma > 0$ such that $f_j(\partial \sigma) \subset B_{p_\sigma}(R_\sigma)$ and

$$B_{p_{\sigma'}}(c_1R_{\sigma'}) \subset B_{p_\sigma}(c_1R_\sigma), \quad R_{\sigma'} \leq (1 - d_0)R_\sigma$$

(for a positive constant $d_0$), in case $\sigma \in K_j$, $\sigma' \in K_{j+1}$, $\sigma' \subset \sigma$.

(c) $\text{skel}_k(K_0) = S^k = \partial D^{k+1}$, $f_0 \equiv f$, $R_{\sigma_0} = R$ for $\sigma_0 = D^{k+1} \setminus S^k \subset K_0$.

Then there exists a continuous map $g: D^{k+1} \rightarrow B_p(c_1R)$, such that $g \equiv f_j$ on $\text{skel}_k(K_j)$ for all $j$.

Indeed, let $g(x) = \lim_{j \rightarrow \infty} p_{\sigma_j}$ for some sequence of $(k+1)$-cells $\sigma_j \in K_j$, \ldots
such that \( \sigma_{j+1} \subset \sigma_j \) and \( x \in \text{clos}(\sigma_j) \) for all \( j \). Obviously \( p_{\sigma_j} \) form a Cauchy sequence for any such \( \{\sigma_j\} \), and moreover, \( |g(x)p_{\sigma_j}| \leq (1 - d_0)^j c_1 R \). The sequence \( \sigma_j \) is defined unambiguously if \( x \notin \bigcup_j \text{skel}_k K_j \), and it is clear that \( g(x) = f_j(x) \) if \( x \in \text{skel}_k K_j \); therefore \( g \) is correctly defined and continuous.

5.2. Specify the choice of \( d_0 \) and \( \delta \) in the following way. Let \( d_0 > 0 \) be so small that for suitably chosen positive numbers \( d_1, \ldots, d_k \),

\[
\frac{d_{i+1}}{d_i} > 100, \quad \frac{d_0}{d_i} > 100k \cdot \kappa(100d_i/d_{i+1}),
\]

\[
100d_k < 10^{-k}(1 + d_0/2k)^{-k}(1 - c_1^{-1})
\]

hold, and let

\[
\delta = \delta_k(c_1, c_2) = \min\{\gamma(c_1, c_2, d_0), \delta_i(1 + d_0/2k, c_2), i = 0, 1, \ldots, k - 1\}.
\]

5.3. Assume that the conditions of (A) are satisfied. Then the extensions \( f_j \) from 5.1 can be constructed inductively using the following key assertion (see 6° for the proof).

(C) Given \( \rho > 0 \), \( q \in M \), such that \( \text{Vol}(B_q(c_2 \rho)) \geq (1 - \delta) \text{Vol}(B^1(c_2 \rho)) \), a continuous map \( \phi: S^k \to B_q(\rho) \) and a triangulation \( T \) of \( S^k \) such that \( \text{diam}(\phi(\Delta)) \leq d_0 \rho \) for all \( \Delta \in T \), there exists a continuous map \( \hat{\phi}: S^k \to B_q((1 - d_0) \rho) \) such that

\[
\text{diam}(\phi(\Delta) \cup \hat{\phi}(\Delta)) \leq 10^{-k-1}(1 + d_0/2k)^{-k}(1 - c_1^{-1}) \rho
\]

for all \( \Delta \in T \).

Indeed, represent a \((k + 1)\)-cell \( \sigma \in K_j \) as \( S^k \times (0, 1] \cup \{0\} \), choose a fine triangulation \( T \) of \( S^k \) and apply (C) to \( f_j: S^k \times \{1\} \to B_{p_x}(R_\sigma) \). (The volume condition is satisfied since it follows from 5.1 (b) that \( B_{p_x}(c_2 R_\sigma) \subset B_p(c_2 R) \).) Define \( K_{j+1} \) by \( \sigma \cap \text{skel}_k(K_{j+1}) = S^k \times \{1/2\} \cup S^k \times \{1\} \cup \text{skel}_{k-1}(T) \times [1/2, 1] \), and let \( f_{j+1} = f_j \) on \( S^k \times \{1\} \) and \( f_{j+1} = \hat{f}_j \) on \( S^k \times \{1/2\} \). Now \( f_{j+1} \) can be extended consecutively to \( \text{skel}_i(T) \times [1/2, 1], i = 0, 1, \ldots, k - 1 \), in such a way that

\[
\text{diam} f_{j+1}(\Delta \times [1/2, 1]) \leq 10^{i-k}(1 + d_0/2k)^{i+1-k}(1 - c_1^{-1}) R_\sigma
\]

for all \( \Delta \in \text{skel}_i(T) \). (Each extension to \( \Delta \times [1/2, 1] \) from its boundary, for \( \Delta \in \text{skel}_i T \), is ensured by the inducational assumption in dimension \( i \) and the inequality \( \delta \leq \delta_i(1 + d_0/2k, c_2) \).) It is easy to check that \( f_{j+1}: \text{skel}_k K_{j+1} \to M \) satisfies the conditions of 5.1, since the boundary of the “central” cell \( S^k \times (0, 1/2] \cup \{0\} \) is mapped into \( B_{p_x}((1 - d_0)R_\sigma) \), and the images of the boundaries of all other cells have diameters less than \((1/2)(1 - c_1^{-1}) R_\sigma\).
6°. Proof of (C). We construct \( \tilde{\phi} \) consecutively on \( \text{skel}_i(T) \), \( i = 0, \ldots, k \), to satisfy \( \tilde{\phi}|_\Delta \equiv \phi|_\Delta \) if \( \phi(\Delta) \subset B_q(\rho(1 - 2d_0)) \),

\[
(1) \quad \tilde{\phi}(\Delta) \subset B_q((1 - d_0(2 - i/k))\rho),
\]

\[
(2) \quad \text{diam}(\phi(\Delta) \cup \tilde{\phi}(\Delta) \leq 10d_i\rho
\]

for all \( \Delta \in \text{skel}_i(T) \).

To begin with, define \( \tilde{\phi} \) on \( \text{skel}_0(T) \) by

\[
\tilde{\phi}(x) = q \tilde{\phi}(x)q, \quad |q \tilde{\phi}(x)| = \rho(1 - 2d_0) \quad \text{if} \quad |q \phi(x)| > \rho(1 - 2d_0).
\]

Assume that \( \tilde{\phi} \) is defined on \( \text{skel}_i(T) \) for some \( i < k \) and consider an \( (i + 1) \)-simplex \( \Delta \), such that \( \phi(\Delta) \not\subset B_q(\rho(1 - 2d_0)) \). Applying (\( \gamma \)) choose a point \( r_\Delta \in M \setminus B_q(c_1\rho) \), such that \( |q r_\Delta, \phi(\Delta)| \leq d_0 \rho \) and let \( q_\Delta \in \overline{q r}_\Delta \) be such that

\[
|q_\Delta| = \rho(1 - d_{i+1}); \text{ see Figure 2 on the next page. It follows from (2) and the choice of } \{d_j\} \text{ that for any } x \in \partial \Delta
\]

\[
|\tilde{\phi}(x), \overline{q r}_\Delta| < 20d_i \rho, \quad |\tilde{\phi}(x)q_\Delta| > d_{i+1} \rho/2, \quad |\tilde{\phi}(x)r_\Delta| > d_{i+1} \rho/2.
\]

Hence we can apply (\( \kappa \)) to \( \tilde{\phi}(x), \overline{q r}_\Delta \) and obtain

\[
|\tilde{\phi}(x)r_\Delta| + |\tilde{\phi}(x)q_\Delta| - |q_\Delta r_\Delta| < 20\kappa(100d_i/d_{i+1})d_i \rho.
\]

Adding this to the triangle inequality

\[
|q_\Delta r_\Delta| + \rho(1 - d_{i+1}) = |q r_\Delta| \leq |\tilde{\phi}(x)r_\Delta| + |\tilde{\phi}(x)q|
\]

and taking (1) into account we get

\[
\tilde{\phi}(\partial \Delta) \subset B_{q_\Delta}(\rho(d_{i+1} - d_0(2 - i/k) + 20\kappa(100d_i/d_{i+1})d_i))
\]

\[
\subset B_{q_\Delta}(\rho(d_{i+1} - d_0(2 - (2i + 1)/2k)))
\]

where the last inclusion follows from the choice of \( \{d_j\} \).

Since \( \dim \Delta = i + 1 \leq k \), the inductional assumption can be applied to extend \( \tilde{\phi} \) from \( \partial \Delta \) to \( \Delta \). It follows from the choice of \( \delta \) that the extension satisfies

\[
\tilde{\phi}(\Delta) \subset B_{q_\Delta}(\rho(d_{i+1} - d_0(2 - (i + 1)/k))).
\]

It remains to observe that the last inclusion implies (1), (2) with \( i \) replaced by \( i + 1 \).

7°. The proof of (B) can be carried out along the same lines. An argument similar to the proof of (C) shows that a map \( f: S^k \to M \setminus B_p(\rho), \ R \leq \rho \leq c_1 R \), can be transformed to a map \( \bar{f} \) with image outside a markedly larger ball, in such a way that \( \text{diam}(f(\Delta) \cup \bar{f}(\Delta)) \) is small for every simplex \( \Delta \) of a fine triangulation \( T \) of \( S^k \). A deformation from \( f \) to \( \bar{f} \) can be constructed consecutively on \( \text{skel}_i(T) \), \( i = 0, \ldots, k \) making use of the assertion (A). After a bounded number of such deformations we obtain the required map with image outside \( B_p(c_1 R) \).
FIGURE 2

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REFERENCES


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