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MANIFOLDS OF POSITIVE RICCI CURVATURE WITH ALMOST MAXIMAL VOLUME

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1°. In this note we consider complete Riemannian manifolds with Ricci curvature bounded from below. The well-known theorems of Myers and Bishop imply that a manifold M^n with $\mathrm{Ric} \geq n-1$ satisfies $\mathrm{diam}(M^n) \leq \mathrm{diam}(S^n(1))$, $\mathrm{Vol}(M^n) \leq \mathrm{Vol}(S^n(1))$. It follows from [Ch] that equality in either of these estimates can be achieved only if M^n is isometric to $S^n(1)$. The natural conjecture is that a manifold M^n with almost maximal diameter or volume must be a topological equivalent to S^n. With respect to diameter this is true only if M^n satisfies some additional assumptions; see [An, O, GP, E]. With respect to volume however no extra restriction is necesary.

Theorem 1. *For any integer $n \geq 2$ there exists $\delta_n > 0$ with the following property. Let M^n be a complete Riemannian manifold with $\mathrm{Ric} \geq n-1$. Suppose that $\mathrm{Vol}(M^n) \geq (1-\delta_n)\mathrm{Vol}(S^n(1))$. Then M^n is homeomorphic to S^n.*

In fact, we prove only that $\pi_i(M^n) = 0$ for all $i < n$ and refer to the work of Hamilton [H] for $n = 3$ and to the solution of generalized Poincaré conjecture (Smale [S], Freedman [F]) for $n \neq 3$.

Vanishing of homotopy groups is a simple consequence of the Main Lemma below. Its further simple corollaries are a noncompact version of Theorem 1 and a corresponding finiteness theorem (cf. [P, Corollary B]).

Let $B^H(R)$ denote a ball of radius R in the simply connected space form of constant curvature H.

Theorem 2. *Let M^n be a complete Riemannian manifold with $\mathrm{Ric} \geq 0$; $p \in M$. Suppose that $\mathrm{Vol}(B_p(R)) \geq (1-\delta_n)\mathrm{Vol}(B^0(R))$ for all $R > 0$. Then M^n is contractible.*

Theorem 3. *For any $n, H, \mathscr{D}, R$ the set $\mathscr{M}_{\delta_n}(n, H, \mathscr{D}, R)$ of all complete Riemannian manifolds M^n with $\mathrm{diam}(M^n) \leq \mathscr{D}$, $\mathrm{Ric} \geq (n-1)H$, and $\mathrm{Vol}(B_p(R)) \geq (1-\delta_n)\mathrm{Vol}(B^H(R))$ for all $p \in M^n$, contains only finitely many homotopy types.*

2°. Henceforward we fix $n \geq 2$ and ignore the dependence on n in our notations. We denote by M an arbitrary compact n-dimensional Riemannian manifold with $\mathrm{Ric} \geq n-1$; all parameters below are supposed to be independent of M.

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Main Lemma. *For any* $c_2 > c_1 > 1$ *and integer* $k \geq 0$ *there exists* $\delta = \delta_k(c_1, c_2) > 0$ *with the following property. Let* $p \in M$, $0 < R < \pi c_2^{-1}$. *Suppose that* $\mathrm{Vol}(B_q(\rho)) \geq (1-\delta)\,\mathrm{Vol}(B^1(\rho))$ *for every ball* $B_q(\rho) \subset B_p(c_2 R)$. *Then*

(A) *Any continuous map* $f\colon S^k = \partial D^{k+1} \to B_p(R)$ *can be continuously extended to a map* $g\colon D^{k+1} \to B_p(c_1 R)$.

(B) *Any continuous map* $f\colon S^k \to M \setminus B_p(R)$ *can be continuously deformed to a map into* $M \setminus B_p(c_1 R)$.

Remark. The Main Lemma can obviously be modified for n-manifolds with $\mathrm{Ric} \geq 0$ or $\mathrm{Ric} \geq -(n-1)$, in the latter case δ may depend on R as $R \to \infty$.

We give below a detailed proof of (A) and outline a similar proof of (B) leaving the details to the reader.

3°. At first we state explicitly all the properties of manifolds with Ricci curvature bounded from below, which are relevant to the proof.

Let $\overline{ab}$ denote a shortest geodesic with endpoints a, b.

(κ) There exists a positive function κ, $\kappa(t) \to 0$ as $t \to 0$, such that

$$|ab| + |ac| - |bc| \leq \kappa\left(|a, \overline{bc}| / \min\{|ab|, |ac|\}\right) \cdot |a, \overline{bc}| \quad \text{for all } a, \overline{bc} \subset M.$$

This is a weakened version of the Abresch-Gromoll inequality [AG].

(γ) For any $c_2 > c_1 > 1$, $\epsilon > 0$ there exists $\gamma = \gamma(c_1, c_2, \epsilon)$ with the following property. Let $p \in M$, $0 < R < \pi c_2^{-1}$. Suppose that $\mathrm{Vol}(B_p(c_2 R)) \geq (1-\gamma)\,\mathrm{Vol}(B^1(c_2 R))$. Then for every $a \in B_p(R)$ there exists $b \in M \setminus B_p(c_1 R)$ such that $|a, \overline{pb}| \leq \epsilon R$.

This is a simple corollary of (the proof of) the Bishop-Gromov volume comparison inequality.

Warning. In the proof of the Main Lemma, we do not use the existence of the injectivity radius and avoid explicit induction on R.

4°. **Outline of the proof of (A).** Assertion (A) is proved by induction on k. The case $k = 0$ is obvious. Assume that (A) holds in dimensions less than k. Fix $c_2 > c_1 > 1$ and let $d_0 > 0$ and $\delta > 0$ be small enough. Now given M, p, R, satisfying the conditions of (A), and a continuous map $f\colon S^k \to B_p(R)$, we can construct another continuous map $\tilde{f}\colon S^k \to B_p((1-d_0)R)$ such that the uniform distance between f and $\tilde{f}$ is small in comparison with R. This is the crucial step, it uses both properties (κ), (γ) and the inductional assumption.

The map $\tilde{f}$ is not known yet to be homotopic to f, and there is no obvious way to construct such a homotopy at once. To go around this difficulty, we take a fine triangulation of S^k and construct a "small" homotopy between f and $\tilde{f}$ on the $(k-1)$-skeleton of this triangulation. In fact, the homotopy is constructed consecutively on i-skeleta, $i = 0, 1, \ldots, k-1$, using the inductional assumption.

The result of previous steps can be interpreted as an extension of f from $S^k = \partial D^{k+1}$ to the k-skeleton of a finite cell decomposition of D^{k+1}. Recall

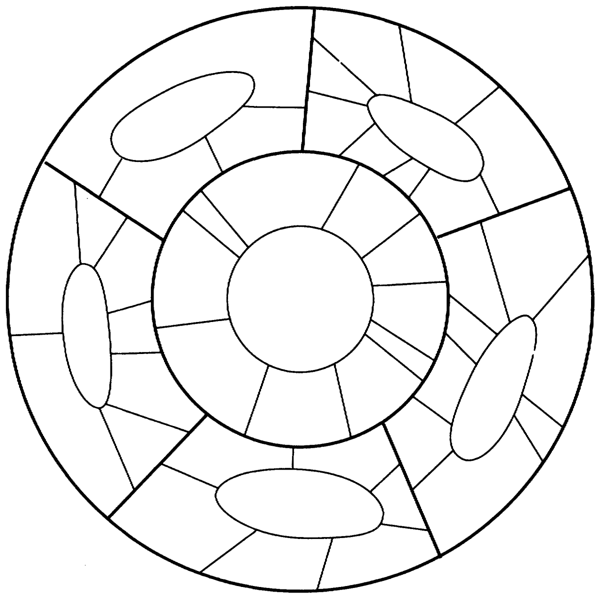

FIGURE 1

that the boundary of the "central" cell is mapped into $B_p((1-d_0)R)$, and the size of the images of the boundaries of all other cells is small in comparison with R. Now we repeat the previous steps for each cell separately and obtain an extension of f to the k-skeleton of a finer cell decomposition (Figure 1), etc. The limit of the infinite repetition of this procedure is the required extension g.

Apparently the argument above cannot be convincing until the choice of "small" parameters is specified. We give a formal exposition below.

5°. **Proof of (A).**

5.1. Consider the following general situation. Let $f\colon S^k \to B_p(R) \subset M$ be a continuous map, and let sequences of finite cell subdivisions K_j of D^{k+1} and continuous maps $f_j\colon \mathrm{skel}_k K_j \to M$ satisfy

(a) K_{j+1} is a cell subdivision of K_j and $f_{j+1} \equiv f_j$ on $\mathrm{skel}_k(K_j)$.

(b) For each $(k+1)$-cell $\sigma \in K_j$ there exist $p_\sigma \in B_p(c_1R)$ and $R_\sigma > 0$ such that $f_j(\partial\sigma) \subset B_{p_\sigma}(R_\sigma)$ and

$$B_{p_{\sigma'}}(c_1R_{\sigma'}) \subset B_{p_\sigma}(c_1R_\sigma)\ , \qquad R_{\sigma'} \le (1-d_0)R_\sigma$$

(for a positive constant d_0), in case $\sigma \in K_j$, $\sigma' \in K_{j+1}$, $\sigma' \subset \sigma$.

(c) $\mathrm{skel}_k(K_0) = S^k = \partial D^{k+1}$, $f_0 \equiv f$, $R_{\sigma_0} = R$ for $\sigma_0 = D^{k+1} \setminus S^k \in K_0$.

Then there exists a continuous map $g\colon D^{k+1} \to B_p(c_1R)$, such that $g \equiv f_j$ on $\mathrm{skel}_k(K_j)$ for all j.

Indeed, let $g(x) = \lim_{j\to\infty} p_{\sigma_j}$ for some sequence of $(k+1)$-cells $\sigma_j \in K_j$, such that $\sigma_{j+1} \subset \sigma_j$ and $x \in \operatorname{clos}(\sigma_j)$ for all j. Obviously p_{σ_j} form a Cauchy sequence for any such $\{\sigma_j\}$, and moreover, $|g(x)p_{\sigma_j}| \le (1-d_0)^j c_1 R$. The sequence σ_j is defined unambiguously if $x \notin \cup_j \operatorname{skel}_k K_j$, and it is clear that $g(x) = f_j(x)$ if $x \in \operatorname{skel}_k K_j$; therefore g is correctly defined and continuous.

5.2. Specify the choice of d_0 and δ in the following way. Let $d_0 > 0$ be so small that for suitably chosen positive numbers $d_1, \dots, d_k$,

$$d_{i+1}/d_i > 100\ , \qquad d_0/d_i > 100k \cdot \kappa(100 d_i/d_{i+1})\ ,$$
$$100 d_k < 10^{-k}(1+d_0/2k)^{-k}(1-c_1^{-1})$$

hold, and let

$$\begin{aligned}\delta &= \delta_k(c_1, c_2) \\ &= \min\{\gamma(c_1, c_2, d_0), \delta_i(1+d_0/2k, c_2), i = 0, 1, \dots, k-1\}.\end{aligned}$$

5.3. Assume that the conditions of (A) are satisfied. Then the extensions f_j from 5.1 can be constructed inductively using the following key assertion (see 6° for the proof).

(C) Given $\rho > 0$, $q \in M$, such that $\operatorname{Vol}(B_q(c_2\rho)) \ge (1-\delta)\operatorname{Vol}(B^1(c_2\rho))$, a continuous map $\phi\colon S^k \to B_q(\rho)$ and a triangulation T of S^k such that $\operatorname{diam}(\phi(\Delta)) \le d_0\rho$ for all $\Delta \in T$, there exists a continuous map $\tilde\phi\colon S^k \to B_q((1-d_0)\rho)$ such that

$$\operatorname{diam}(\phi(\Delta) \cup \tilde\phi(\Delta)) \le 10^{-k-1}(1+d_0/2k)^{-k}(1-c_1^{-1})\rho$$

for all $\Delta \in T$.

Indeed, represent a $(k+1)$-cell $\sigma \in K_j$ as $S^k \times (0, 1] \cup \{0\}$, choose a fine triangulation T of S^k and apply (C) to $f_j\colon S^k \times \{1\} \to B_{p_\sigma}(R_\sigma)$. (The volume condition is satisfied since it follows from 5.1 (b) that $B_{p_\sigma}(c_2 R_\sigma) \subset B_p(c_2 R)$.) Define K_{j+1} by $\sigma \cap \operatorname{skel}_k(K_{j+1}) = S^k \times \{1/2\} \cup S^k \times \{1\} \cup \operatorname{skel}_{k-1}(T) \times [1/2, 1]$, and let $f_{j+1} \equiv f_j$ on $S^k \times \{1\}$ and $f_{j+1} \equiv \tilde f_j$ on $S^k \times \{1/2\}$. Now f_{j+1} can be extended consecutively to $\operatorname{skel}_i(T) \times [1/2, 1]$, $i = 0, 1, \dots, k-1$, in such a way that

$$\operatorname{diam} f_{j+1}(\Delta \times [1/2, 1]) \le 10^{i-k}(1+d_0/2k)^{i+1-k}(1-c_1^{-1})R_\sigma$$

for all $\Delta \in \operatorname{skel}_i(T)$. (Each extension to $\Delta \times [1/2, 1]$ from its boundary, for $\Delta \in \operatorname{skel}_i T$, is ensured by the inductional assumption in dimension i and the inequality $\delta \le \delta_i(1+d_0/2k, c_2)$.) It is easy to check that $f_{j+1}\colon \operatorname{skel}_k K_{j+1} \to M$ satisfies the conditions of 5.1, since the boundary of the "central" cell $S^k \times (0, 1/2] \cup \{0\}$ is mapped into $B_{p_\sigma}((1-d_0)R_\sigma)$, and the images of the boundaries of all other cells have diameters less than $(1/2)(1-c_1^{-1})R_\sigma$.

6°. **Proof of (C).** We construct $\tilde{\phi}$ consecutively on $\mathrm{skel}_i(T)$, $i = 0, \ldots, k$, to satisfy $\tilde{\phi}|_\Delta \equiv \phi|_\Delta$ if $\phi(\Delta) \subset B_q(\rho(1-2d_0))$,

$$\tilde{\phi}(\Delta) \subset B_q((1 - d_0(2 - i/k))\rho) \ , \tag{1}$$

$$\mathrm{diam}(\phi(\Delta) \cup \tilde{\phi}(\Delta)) \le 10 d_i \rho \tag{2}$$

for all $\Delta \in \mathrm{skel}_i(T)$.

To begin with, define $\tilde{\phi}$ on $\mathrm{skel}_0(T)$ by

$$\tilde{\phi}(x) \in \overline{\phi(x)q} \ , \qquad |q\tilde{\phi}(x)| = \rho(1-2d_0) \quad \text{if } |q\phi(x)| > \rho(1-2d_0) .$$

Assume that $\tilde{\phi}$ is defined on $\mathrm{skel}_i(T)$ for some $i < k$ and consider a $(i+1)$-simplex Δ, such that $\phi(\Delta) \not\subset B_q(\rho(1-2d_0))$. Applying (γ) choose a point $r_\Delta \in M \setminus B_q(c_1\rho)$, such that $|\overline{qr_\Delta}, \phi(\Delta)| \le d_0\rho$ and let $q_\Delta \in \overline{qr_\Delta}$ be such that $|qq_\Delta| = \rho(1 - d_{i+1})$; see Figure 2 on the next page. It follows from (2) and the choice of $\{d_i\}$ that for any $x \in \partial\Delta$

$$|\tilde{\phi}(x), \overline{q_\Delta r_\Delta}| < 20 d_i \rho \ , \qquad |\tilde{\phi}(x) q_\Delta| > d_{i+1}\rho/2 \ , \qquad |\tilde{\phi}(x) r_\Delta| > d_{i+1}\rho/2 .$$

Hence we can apply (κ) to $\tilde{\phi}(x)$, $\overline{q_\Delta r_\Delta}$ and obtain

$$|\tilde{\phi}(x) r_\Delta| + |\tilde{\phi}(x) q_\Delta| - |q_\Delta r_\Delta| < 20\kappa(100 d_i/d_{i+1}) d_i \rho .$$

Adding this to the triangle inequality

$$|q_\Delta r_\Delta| + \rho(1 - d_{i+1}) = |qr_\Delta| \le |\tilde{\phi}(x) r_\Delta| + |\tilde{\phi}(x) q|$$

and taking (1) into account we get

$$\begin{aligned} \tilde{\phi}(\partial\Delta) &\subset B_{q_\Delta}(\rho(d_{i+1} - d_0(2 - i/k) + 20\kappa(100 d_i/d_{i+1}) d_i)) \\ &\subset B_{q_\Delta}(\rho(d_{i+1} - d_0(2 - (2i+1)/2k))) \ , \end{aligned}$$

where the last inclusion follows from the choice of $\{d_i\}$.

Since $\dim \Delta = i + 1 \le k$, the inductional assumption can be applied to extend $\tilde{\phi}$ from $\partial\Delta$ to Δ. It follows from the choice of δ that the extension satisfies

$$\tilde{\phi}(\Delta) \subset B_{q_\Delta}(\rho(d_{i+1} - d_0(2 - (i+1)/k))) .$$

It remains to observe that the last inclusion implies (1), (2) with i replaced by $i + 1$.

7°. The proof of (B) can be carried out along the same lines. An argument similar to the proof of (C) shows that a map $f: S^k \to M \setminus B_p(\rho)$, $R \le \rho \le c_1 R$, can be transformed to a map $\bar{f}$ with image outside a markedly larger ball, in such a way that $\mathrm{diam}(f(\Delta) \cup \bar{f}(\Delta))$ is small for every simplex Δ of a fine triangulation T of S^k. A deformation from f to $\bar{f}$ can be constructed consecutively on $\mathrm{skel}_i(T)$, $i = 0, \ldots, k$ making use of the assertion (A). After a bounded number of such deformations we obtain the required map with image outside $B_p(c_1 R)$.

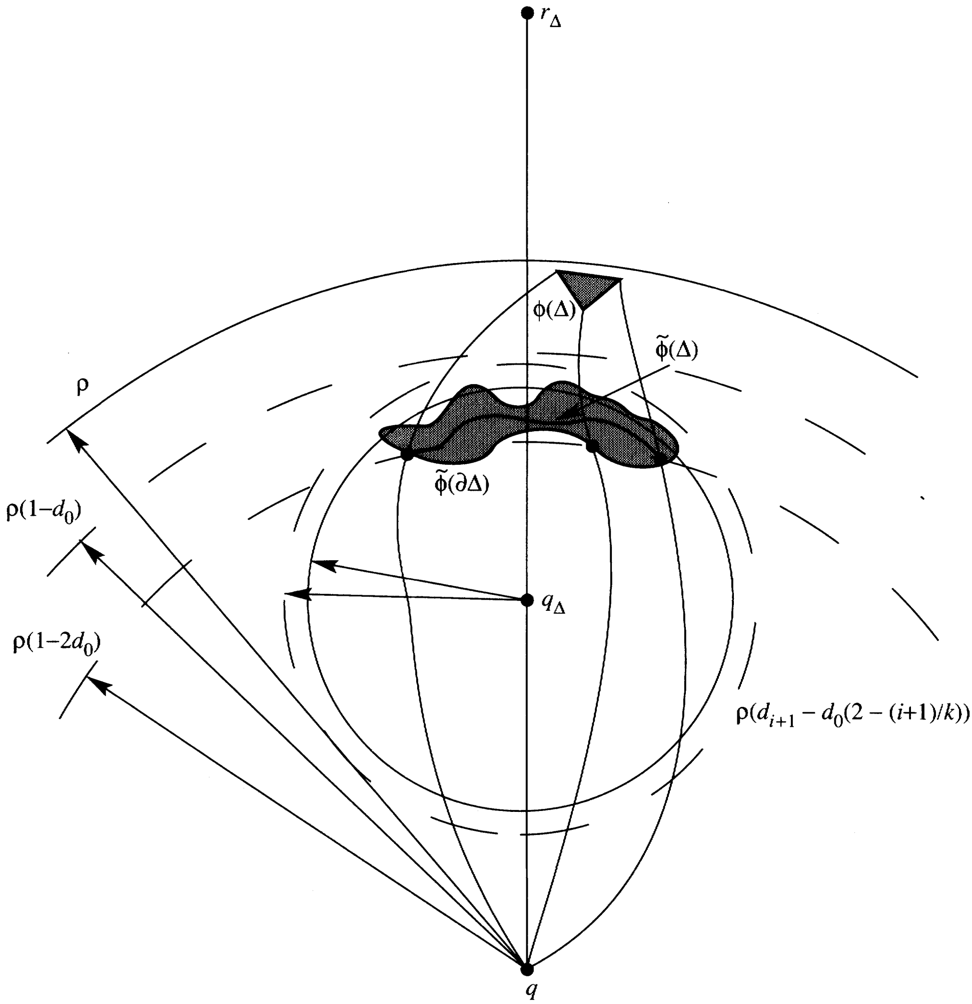

FIGURE 2

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