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NILPOTENT ORBITS, NORMALITY, AND HAMILTONIAN GROUP ACTIONS

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The theory of coadjoint orbits of Lie groups is central to a number of areas in mathematics. A list of such areas would include (1) group representation theory, (2) symmetry-related Hamiltonian mechanics and attendant physical theories, (3) symplectic geometry, (4) moment maps, and (5) geometric quantization. From many points of view the most interesting cases arise when the group G in question is semisimple. For semisimple G the most familiar of the orbits are orbits of semisimple elements. In that case the associated representation theory is pretty much understood (the Borel-Weil-Bott Theorem and noncompact analogs, e.g., Zuckerman functors). The isotropy subgroups are reductive and the orbits are in one form or another related to flag manifolds and their natural generalizations.

A totally different experience is encountered with nilpotent orbits of semisimple groups. Here the associated representation theory (unipotent representations) is poorly understood and there is a loss of reductivity of isotropy subgroups. To make matters worse (or really more interesting) orbits are no longer closed and there can be a failure of normality for orbit closures. In addition simple connectivity is generally gone but more seriously there may exist no invariant polarizations.

The interest in nilpotent orbits of semisimple Lie groups has increased sharply over the last two decades. This perhaps may be attributed to the recurring experience that sophisticated aspects of semisimple group theory often lead one to these orbits (e.g., the Springer correspondence with representations of the Weyl group).

This paper presents new results concerning the symplectic and algebraic geometry of the nilpotent orbits O and the symmetry groups of that geometry. The starting point is the recognition (made also by others) that the ring R of regular functions on any G-homogeneous cover M of O is not only a Poisson algebra (the case for any coadjoint orbit) but that R is also naturally graded. The key theme is that the same nilpotent orbit may be "shared" by more than one simple group. The key result is the determination of all pairs of simple Lie groups having a shared nilpotent orbit. Furthermore there is then a unique maximal such group and this group is encoded in the symplectic and algebraic

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geometry of the orbit. Remarkably a covering of a nilpotent orbit of a classical group may "see" an exceptional Lie group as the maximal symmetry group of this symplectic manifold. A beautiful instance of this is that the complex simple Lie group G_2 of exceptional type G_2 is the symmetry group of the simply-connected covering of the maximal nilpotent orbit of $SL(3, \mathbb{C})$ and that this 6-dimensional space "becomes" (after adding a boundary of codimension 2) the minimal nilpotent orbit of G_2.

Our work began with a desire to thoroughly investigate a striking discovery of Levasseur, Smith, and Vogan. They found that the failure of the closure of the 8-dimensional nilpotent orbit of G_2 to be a normal variety may be "remedied" by refinding this orbit (with a codimension 2 boundary added) as the minimal nilpotent orbit of $SO(7, \mathbb{C})$. The failure has a lot to do with the 7-dimensional representation of G_2. In general given M we have found that there exists a unique minimal representation π (containing the adjoint) wherein M may be embedded with normal closure. It was the study of π which led to the discovery of the maximal symmetry group. Using a new general transitivity result for coadjoint orbits we prove that, modulo a possible normal Heisenberg subgroup (and that occurs in only one case), the maximal symmetry group is semisimple.

Past experience has shown that the action of a subgroup H on a coadjoint orbit of G is a strong prognosticator as to how the corresponding representation L of G decomposes under H. If this continues to hold for unipotent representations our classification result should yield all cases where L remains irreducible (or decomposes finitely) under a semisimple subgroup.

Our results here were announced in [B-K2].

1. Normal closure of orbits and the Poisson algebra of functions

For any algebraic complex quasi-projective variety Z, let $R(Z)$ be the algebra of regular functions on Z and let $K(Z)$ be the field of rational functions on Z. Let Spec A be the affine model (i.e., the maximal ideal spectrum) of a finitely generated commutative algebra A. We note

Lemma 1.1. *If Z is a normal algebraic variety then the ring $R(Z)$ is integrally closed in the field $K(Z)$.*

Let G be a simply-connected semisimple complex Lie group with Lie algebra $\mathfrak{g}$. Let O be the adjoint orbit of a nilpotent element $e \in \mathfrak{g}$. Let

$$\nu : M \to O \tag{1-1}$$

be a G-homogeneous covering and choose $\varepsilon \in M$ such that $\nu(\varepsilon) = e$. Then $G^e_o \subset G^\varepsilon \subset G^e$ where G^e and G^ε are the isotropy groups and G^e_o is the identity component of G^e. There is G-equivariant isomorphism $M \simeq G/G^\varepsilon$ of complex manifolds and of complex algebraic varieties. The fundamental group $\pi_1(M)$ is finite and $\pi_1(M) \simeq G^\varepsilon/G^e_o$.

Let $Q = Q(M)$ be the group of all maps $\alpha: M \to M$ which commute with the action of G. Then Q identifies with the quotient $N^\varepsilon/G^\varepsilon$ where N^ε is the normalizer of G^ε in G and if $g \in G$, $n \in N^\varepsilon$, and $\overline{n}$ is the image of n in

$N^\varepsilon/G^\varepsilon$ then $\overline{n}$ acts by $\overline{n} \bullet (g \cdot \varepsilon) = gn^{-1} \cdot \varepsilon$. Then $G \times Q$ acts rationally by algebra automorphisms on $R = R(M)$.

We assume that the nilpotent e has nonzero projection to every simple component of $\mathfrak{g}$; i.e., O spans $\mathfrak{g}$.

Now let $(\, , \,)$ be a fixed $\mathfrak{g}$-invariant nonsingular symmetric bilinear form on $\mathfrak{g}$ which is negative definite on some compact form of $\mathfrak{g}$. Define $\phi^x \in R$ for $x \in \mathfrak{g}$ by

$$\phi^x(p) = (\nu(p), x) \tag{1-2}$$

where $p \in M$. Let $R[\mathfrak{g}] = R(M)[\mathfrak{g}] \subset R$ be the linear span of the functions ϕ^x where $x \in \mathfrak{g}$. Clearly $R[\mathfrak{g}] \simeq \mathfrak{g}$. Furthermore if $\overline{O}$ is the closure of O in $\mathfrak{g}$ then the subalgebra S of R generated by $R[\mathfrak{g}]$ may be identified with $R(\overline{O})$.

Proposition 1.2. *There exists a unique affine variety X containing M as a Zariski open subset such that all regular functions on M extend to X. The ring $R = R(M) = R(X)$ is a finitely generated $\mathbb{C}$-algebra and*

$$X = \operatorname{Spec} R. \tag{1-3}$$

The commuting actions of G and Q on M extend uniquely to commuting (algebraic) actions of G and Q on X. The covering map ν extends uniquely to a finite surjective G-equivariant morphism

$$\overline{\nu}: X \to \overline{O}. \tag{1-4}$$

Moreover X is a normal variety and in fact X is the normalization of $\overline{O}$ in the function field of M. G has finitely many orbits on X and each is even dimensional. M is the unique Zariski open orbit of G on X and its boundary has codimension at least 2.

Proof. It is easy to show using Lemma 1.1 and the fact that the boundary of O in $\overline{O}$ has codimension at least 2 that R is the integral closure of S in $K(M)$. Then it follows from the corresponding facts about S that R is a finitely generated algebra and furthermore $K(M)$ is the fraction field of R. Now the commuting actions of G and Q on R define commuting actions on X. But then it follows that the natural $(G \times Q)$-map $M \to X$ must be an embedding. Finally the inclusion $S \subset R$ defines the finite morphism (1-4). Then since $\overline{\nu}$ has finite fibers the proposition now follows from the corresponding statements about the G-action on $\overline{O}$.

We call X the *normal closure* of M.

Now O and $\overline{O}$ are both stable under the scaling action of $\mathbb{C}^*$ on $\mathfrak{g}$. In particular the Euler vector field ξ on $\mathfrak{g}$ is tangent to O and so lifts to a vector field ξ_M on M where $\xi = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$ and $\{x_i\}_{i=1}^n$ is a basis of $\mathfrak{g}^*$.

The Jacobson-Morosov Theorem says that there are $h, f \in \mathfrak{g}$ such that the triple (h, e, f) spans an $\mathfrak{sl}(2, \mathbb{C})$-subalgebra of $\mathfrak{g}$, with

$$[h, e] = 2e, \tag{1-5}$$

$[h, f] = -2f$, and $[e, f] = h$. Then $\exp(\mathbb{C}h) \subset N^\varepsilon$ so that $\exp(\mathbb{C}h)$ defines a subgroup C of Q and hence C acts on X. Associated to this C-action is the *right scaling action* of $\mathbb{C}^*$ on M defined by the homomorphism $\mathbb{C}^* \to C$,

$s \to \tilde{s}$ where $\tilde{s} = \overline{\exp th}$ if $s = \exp t$. Now if η_h is the vector field on M corresponding to the (right) infinitesimal action of h then by (1-5) $\eta_h = 2\xi_M$. This proves

Lemma 1.3. *The right scaling action of $\mathbb{C}^*$ on X lifts the square of the Euler action of $\mathbb{C}^*$ on $\overline{O}$ so that for $s \in \mathbb{C}^*$ and $x \in X$ we have $\overline{\nu}(\tilde{s} \bullet x) = s^2 \overline{\nu}(x)$.*

Notice that Lemma 1.3 implies that the right scaling action of $\mathbb{C}^*$ is independent of the choices of e, ε, and h.

Now for $k \in \mathbb{Z}$ let $R[k] = R(M)[k] = \{\phi \in R | \tilde{s} \bullet \phi = s^k \phi \text{ for all } s \in \mathbb{C}^*\}$. Then $\phi^x \in R[2]$ for all $x \in \mathfrak{g}$ by Lemma 1.3, and hence

$$R[\mathfrak{g}] \subset R[2]. \tag{1-6}$$

Proposition 1.4. *$R[k] = 0$ for $k < 0$ so that there is a G-invariant algebra grading*

$$R = \bigoplus_{k=0}^{\infty} R[k]. \tag{1-7}$$

Each $R[k]$ is a finite-dimensional G-stable subspace of R and $R[0] = \mathbb{C} \cdot 1$ where 1 is the constant function on X. Moreover there is a unique point $o \in X$ such that $\overline{\nu}(o) = 0$. This o is the unique G-fixed point in X and also the unique C-fixed point in X. Let $\mathfrak{m} \subset R$ be the maximal ideal at o. Then

$$\mathfrak{m} = \bigoplus_{k=1}^{\infty} R[k] \tag{1-8}$$

and also $\mathfrak{m}$ is equal to the sum in R of all nontrivial G-modules. Finally if the degree of the cover ν is odd (e.g., if $M = O$) then $R[k] = 0$ if k is odd.

Proof. The completely reducible C-action on R defines an algebra grading. But then S is a graded subalgebra by Lemma 1.3. Since R is integral over S, the statements about R then follow from the corresponding statements about S. It is easy to check that (1-8) defines the unique C-stable and also the unique G-stable maximal ideal of R. But then the point $o \in X$ defined by (1-8) is the unique C-fixed and the unique G-fixed point of X. Clearly $\overline{\nu}(o) = 0$. But then o must be the unique point lying above 0 since each point of the finite set $\overline{\nu}^{-1}(0)$ is necessarily C-fixed. Finally by Lemma 1.3 the subgroup $\{\widetilde{\pm 1}\}$ of C leaves stable each fiber of ν. So if the cardinality of $\nu^{-1}(e)$ is odd then $\widetilde{-1}$ fixes some point of $\nu^{-1}(e)$. But then $\widetilde{-1}$ must fix every point of M and it follows that $R[k] = 0$ if k is odd.

Remark 1.5. It is easy to now show that (1-7) is the unique G-invariant algebra grading such that $R[\mathfrak{g}]$ lies in homogeneous degree 2.

We now introduce the machinery of symplectic geometry. A smooth complex variety admits a unique complex analytic manifold structure extending the algebraic structure (so that all regular functions are holomorphic). In this paper, a symplectic structure on a smooth complex variety P is a complex-valued algebraic symplectic 2-form on P. In particular the Poisson bracket of functions defines a Poisson algebra structure on $R(P)$. The moment maps for Hamiltonian G-spaces are algebraic morphisms and take their values in the complex dual space $\mathfrak{g}^*$ to $\mathfrak{g}$. But we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ using $(\, , \,)$ so that the moment map takes values in $\mathfrak{g}$.

The adjoint orbit O admits a canonical G-invariant symplectic structure $\omega(O)$ which is defined at e by $\omega(O)_e(x \cdot e, y \cdot e) = (e, [y, x])$ where $x, y \in \mathfrak{g}$. But then the pullback of $\omega(O)$ to M defines a symplectic form $\omega = \omega(M)$ on M. Each $\phi \in R$ defines a Hamiltonian vector field ξ_ϕ on M by $\xi_\phi \rfloor \omega = d\phi$. Then R is a Poisson algebra with Poisson bracket given by $[\phi, \psi] = \xi_\phi \psi$, where $\phi, \psi \in R$, or equivalently $[\phi, \psi] = \omega(\xi_\psi, \xi_\phi)$. M is a Hamiltonian G-space with corresponding Lie algebra homomorphism

$$\rho : \mathfrak{g} \to R, \qquad \rho(x) = \phi^x. \tag{1-9}$$

Then $\mathfrak{g} \simeq \rho(\mathfrak{g}) = R[\mathfrak{g}]$ and $x \in \mathfrak{g}$ defines a Hamiltonian vector field $\xi^x = \xi_{\phi^x}$.

Now the C-action on M scales ω and then there is a corresponding degree shift in the Poisson bracket of functions. Indeed, let $\phi \in R[k]$ and $\psi \in R[l]$. Using (1-5) we have $\mathscr{L}_{\eta_h}(\omega) = 2\omega$ where $\mathscr{L}$ denotes Lie differentiation. Then $\tilde{s} \bullet \omega = s^2 \omega$ where $s \in \mathbb{C}^*$ so that the definition of ξ_ϕ implies $\tilde{s} \bullet \xi_\phi = s^{k-2} \xi_\phi$. But then $\tilde{s} \bullet [\phi, \psi] = \tilde{s} \bullet (\xi_\phi \psi) = (\tilde{s} \bullet \xi_\phi)(\tilde{s} \bullet \psi) = s^{k-2+l} \xi_\phi \psi$. This proves

Proposition 1.6. *If $k, l \in \mathbb{Z}$ then*

$$[R[k], R[l]] \subset R[k + l - 2]. \tag{1-10}$$

2. Semisimple and Heisenberg Lie algebras inside R

We have $[R[2], R[2]] \subset R[2]$ by (1-10) and thus $R[2]$ is a finite-dimensional Lie subalgebra of R containing $R[\mathfrak{g}]$. Furthermore then $[R[1], R[1]] \subset \mathbb{C} \cdot 1$ so that $R[1] + R[0]$ is a 2-step nilpotent Lie algebra if $R[1] \neq 0$ and

$$[\phi, \psi] = \omega_1(\phi, \psi) \cdot 1 \tag{2-1}$$

defines an alternating bilinear form ω_1 on $R[1]$. Let $\mathfrak{sp}(\omega_1)$ be the Lie algebra of linear transformations of $R[1]$ infinitesimally preserving ω_1.

Theorem 2.1. (i) *$R[2]$ is a semisimple Lie algebra. Moreover any Lie subalgebra of $R[2]$ containing $R[\mathfrak{g}]$ is semisimple.*

(ii) *$R[2]+R[1]+R[0]$ is the unique maximal finite-dimensional Lie subalgebra of R containing $R[\mathfrak{g}]$.*

(iii) *If $R[1] \neq 0$ then ω_1 is a G-invariant symplectic form on $R[1]$. Hence $R[1] + R[0]$ is a Heisenberg Lie algebra and $\mathfrak{sp}(\omega_1) \simeq \mathfrak{sp}(2n, \mathbb{C})$ where $2n = \dim R[1]$. The bracket action of $R[2]$ on $R[1]$ defines a Lie algebra surjection*

$$\delta : R[2] \to \mathfrak{sp}(\omega_1). \tag{2-2}$$

The proof of Theorem 2.1 rests on the following general transitivity theorem on coadjoint orbits.

Theorem 2.2. *Assume that $\mathfrak{s}$ is a finite-dimensional complex Lie algebra and that $\mathfrak{s}$ is a semidirect sum*

$$\mathfrak{s} = \mathfrak{g} + \mathfrak{u} \tag{2-3}$$

where $\mathfrak{g}$ *is a semisimple subalgebra of* $\mathfrak{s}$, $\mathfrak{u}$ *is an abelian ideal of* $\mathfrak{s}$, *and* $\mathfrak{u}^{\mathfrak{g}} = 0$. *Regard* $\mathfrak{s}^* = \mathfrak{g}^* + \mathfrak{u}^*$ *in the obvious way. Let* $\gamma \in \mathfrak{s}^*$ *and write* $\gamma = \mu + \lambda$ *with* $\mu \in \mathfrak{g}^*$ *and* $\lambda \in \mathfrak{u}^*$. *Then*

$$\mathfrak{s} \cdot \gamma = \mathfrak{g} \cdot \gamma \tag{2-4}$$

if and only if $\lambda = 0$.

Proof. The coadjoint action of $\mathfrak{s}$ satisfies the relations $\mathfrak{g} \cdot \mathfrak{g}^* \subset \mathfrak{g}^*$ and $\mathfrak{g} \cdot \mathfrak{u}^* \subset \mathfrak{u}^*$ whereas $\mathfrak{u} \cdot \mathfrak{g}^* = 0$ and $\mathfrak{u} \cdot \mathfrak{u}^* \subset \mathfrak{g}^*$. Clearly then $\lambda = 0$ implies (2-4).

Now assume (2-4). Then the above relations easily imply $\mathfrak{u} \cdot \lambda \subset \mathfrak{g}^{\lambda} \cdot \mu$. But also the space $\mathfrak{u} \cdot \lambda$ identifies with the subspace $(\mathfrak{g}/\mathfrak{g}^{\lambda})^*$ of $\mathfrak{g}^*$. Indeed clearly $\mathfrak{u} \cdot \lambda \subset (\mathfrak{g}/\mathfrak{g}^{\lambda})^*$. But it is immediate (since $\mathfrak{u}$ is abelian) that $\mathfrak{s} \cdot \lambda = \mathfrak{g} \cdot \lambda \oplus \mathfrak{u} \cdot \lambda$ is a Lagrangian decomposition of the tangent space $T_{\lambda}(S \cdot \lambda)$ at λ of the coadjoint orbit $S \cdot \lambda$ where $S = G \ltimes \mathfrak{u}$. Hence by dimension we have $\mathfrak{u} \cdot \lambda = (\mathfrak{g}/\mathfrak{g}^{\lambda})^*$. Thus we conclude $(\mathfrak{g}/\mathfrak{g}^{\lambda})^* \subset \mathfrak{g}^{\lambda} \cdot \mu$. Equivalently, we have $(\mathfrak{g}^{\lambda})^{\perp} \subset [\mathfrak{g}^{\lambda}, x_{\mu}]$ where $(\mathfrak{g}^{\lambda})^{\perp} \subset \mathfrak{g}$ is the the orthogonal subspace to $\mathfrak{g}^{\lambda}$ and $x_{\mu} \in \mathfrak{g}$ corresponds to μ with respect to $(\ ,\ )$.

In fact $x_{\mu} \in \mathfrak{g}^{\lambda}$. Indeed easily $x_{\mu} \cdot \gamma$ is ω_{γ}-orthogonal to $\mathfrak{g} \cdot \gamma$ and hence by (2-4) to all of $\mathfrak{s} \cdot \gamma$ so that necessarily $x_{\mu} \cdot \lambda = 0$. We conclude $(\mathfrak{g}^{\lambda})^{\perp} \subset \mathfrak{g}^{\lambda}$. But then, since $\mathfrak{g}^{\lambda}$ is an algebraic Lie subalgebra of $\mathfrak{g}$, $\mathfrak{g}^{\lambda}$ must be parabolic and it follows that $G \cdot \lambda$ is a point. Then by hypothesis $\lambda = 0$.

There is a notion of an *algebraic vector field* ξ on a complex variety Z. In particular ξ is a regular section of the tangent bundle if Z is smooth, and ξ is a derivation of $R(Z)$ if Z is quasi-affine. An algebraic *infinitesimal action* of a Lie algebra $\mathfrak{a}$ on Z is a Lie algebra homomorphism from $\mathfrak{a}$ to the algebraic vector fields.

Suppose P is a smooth complex symplectic variety. Then $R(P)$ is a central extension by the constants of the Lie algebra $\mathrm{Ham}(P)$ of algebraic Hamiltonian vector fields on P. Let S be an algebraic group with Lie algebra $\mathfrak{s}$. A symplectic S-action on P is *strongly symplectic* if the corresponding vector fields are Hamiltonian. A *Hamiltonian* S-action on P is a strongly symplectic algebraic S-action together with a Lie algebra homomorphism $j \colon \mathfrak{s} \to R(P)$ which lifts the corresponding infinitesimal action $\mathfrak{s} \to \mathrm{Ham}(P)$; i.e., the following diagram is commutative:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C} \arrow[r] & R(P) \arrow[r] & \mathrm{Ham}(P) \arrow[r] & 0 \\ & & & \mathfrak{s} \arrow[ul, "j"] \arrow[u] & \end{tikzcd} \tag{2-5}$$

Then j defines the S-equivariant *moment map*

$$\pi \colon P \to \mathfrak{s}^*, \qquad \langle \pi(p), z \rangle = j_z(p) \tag{2-6}$$

where $p \in P$ and $z \in \mathfrak{s}$. The Covariant Covering Theorem in the algebraic setting yields finite covers:

Theorem 2.3 ([K3]). *Assume as above that P is a Hamiltonian S-space where S is an algebraic group. If S acts transitively on P then the map $P \to \pi(P)$ defined by the moment map (2-6) is a finite S-equivariant cover of a coadjoint orbit of S.*

Theorem 2.2 implies

Corollary 2.4. *Assume G has a dense orbit on a Hamiltonian G-space P. Then*

(i) *there exists no nonzero finite-dimensional G-stable space $\mathfrak{u}$ of Poisson commuting holomorphic functions on P, except the constants.*

(ii) *$\mathfrak{f}$ is semisimple if $\mathfrak{f}$ is a finite-dimensional Lie algebra of holomorphic functions on P such that the corresponding Hamiltonian vector fields contain the infinitesimal $\mathfrak{g}$-action and $\mathfrak{f}$ does not contain the constants.*

Proof. (i) The G-finite holomorphic functions on P are algebraic and the G-action on $R(P)$ is completely reducible. Let $\mathfrak{u} \subset R(P)$ be a nonzero simple G-submodule with $\mathfrak{u} \neq \mathbb{C} \cdot 1$. Clearly $\mathfrak{u}^{\mathfrak{g}} = 0$.

Let $k: \mathfrak{g} \to R(P)$ be the Lie algebra map defining the Hamiltonian action. Then $\mathfrak{s} = k(\mathfrak{g}) + \mathfrak{u}$ is a semidirect sum Lie subalgebra of $R(P)$ and $\mathfrak{s}$ acts infinitesimally on P by the corresponding Hamiltonian vector fields. If $j: \mathfrak{s} \to R(P)$ is the inclusion then j defines by (2-6) an infinitesimally $\mathfrak{s}$-equivariant map $\pi: P \to \mathfrak{s}^*$. But then $\mathfrak{g} \cdot \pi(p) = \mathfrak{s} \cdot \pi(p)$ at each point p in the dense G-orbit P' on P. But then Theorem 2.2 implies that $\pi(p) \in \mathfrak{g}^*$ for all $p \in P'$. Consequently $j(\mathfrak{u}) = 0$ and so $\mathfrak{u} = 0$.

(ii) If $\mathfrak{f}$ is not semisimple then $\mathfrak{f}$ has some nonzero abelian ideal $\mathfrak{u}$. But this is impossible by (i).

Let $S^d(V)$ be the dth symmetric power of a vector space V for $d \geq 0$.

Proof of Theorem 2.1. Corollary 2.4(ii) and Proposition 1.4 give (i).

Next let $\mathfrak{r}$ be a finite-dimensional Lie subalgebra of R containing $R[\mathfrak{g}]$. Let k be the largest integer such that the projection $p_k: R \to R[k]$ defined by the grading (1-7) is nonzero. By (1-6) $k \geq 2$. But then $k = 2$ since otherwise $p_k(\mathfrak{r})$ is abelian by (1-10) and then $p_k(\mathfrak{r}) = 0$ by Corollary 2.4(i). This proves (ii) since $R[2] + R[1] + R[0]$ is closed under Poisson bracket.

Now the radical $\mathfrak{u}$ of ω_1 is G-stable and abelian by (2-1) so that consequently $\mathfrak{u} = 0$ by Corollary 2.4(i). Therefore ω_1 is a symplectic form on $R[1]$. Furthermore ω_1 is infinitesimally $R[2]$-invariant because of (1-10) and so (2-2) is defined. Clearly (2-2) is a Lie algebra homomorphism and using the standard isomorphism $S^2(R[1]) \simeq \mathfrak{sp}(\omega_1)$ we find that the multiplication map $S^2(R[1]) \to R[2]$ is injective and (2-2) is surjective.

Let $Z(G)$ be the center of G so that $G_{\mathrm{ad}} = G/Z(G)$ is the adjoint group.

Proposition 2.5. *Assume G_{ad} operates on M (e.g., $M = O$). If $\mathfrak{r}$ is a Lie subalgebra of* $R[2]$ *containing* $R[\mathfrak{g}]$ *with $\mathfrak{r} \neq R[\mathfrak{g}]$ then rank $\mathfrak{g} <$ rank $\mathfrak{r}$.*

Proof. Suppose $\mathfrak{g}$ and $\mathfrak{r}$ have the same rank. Let $\mathfrak{r} = R[\mathfrak{g}] + \mathfrak{p}$ be a $\mathfrak{g}$-stable direct sum decomposition. Then equality of ranks implies that as a G-representation $\mathfrak{p}$ has a trivial zero-weight space. But then the weights of $\mathfrak{p}$ are not in the root lattice of $\mathfrak{g}$ and it follows that $Z(G)$ operates nontrivially on $\mathfrak{p}$. But then $Z(G)$ acts nontrivially on R and it follows that G_{ad} does not operate on M.

We conclude this section with an example of Theorem 2.1. First we need a result computing $R[2]$ as a G-module.

Algebraic Frobenius reciprocity implies as in [K2] that for each G-module V there is a G-linear isomorphism $t\colon V^{G^\varepsilon} \to \mathrm{Hom}_G(V^*, R)$ defined by $t(v)(\gamma)(g\cdot\varepsilon) = \langle g\cdot v, \gamma\rangle$ where $g \in G$, $v \in V^{G^\varepsilon}$, and $\gamma \in V^*$. If h acts on a space U then let $U[k]$ be the k-eigenspace of h. This is consistent as $R[k]$ is the k-eigenspace with respect to the right action of h. It is routine to show

Lemma 2.6. *For every G-module V and $k \in \mathbb{Z}$, t defines by restriction to $V^{G^\varepsilon}[k]$ a linear isomorphism $t_k\colon V^{G^\varepsilon}[k] \to \mathrm{Hom}_G(V^*, R[k])$*

Example 2.7. Let $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ and let M be the simply-connected (3-fold) cover of the principal nilpotent orbit $O \subset \mathfrak{g}$. Let $V \simeq \mathbb{C}^3$ be the standard representation of $\mathfrak{g}$ so that h has eigenvalues $2, 0, -2$ on V. We find immediately that the spaces $V^{\mathfrak{g}^e}[2]$, $(\bigwedge^2 V)^{\mathfrak{g}^e}[2]$, and $\mathfrak{g}^{\mathfrak{g}^e}[2]$ are all 1-dimensional so that by Lemma 2.6 the simple $\mathfrak{g}$-modules $\mathfrak{g}$, $\mathbb{C}^3$, and $\bigwedge^2\mathbb{C}^3$ occur exactly once in $R[2]$. Furthermore it is easy to check (using Corollary 2.4(i) and the $\mathfrak{g}$-module structure) that if $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are the $\mathfrak{g}$-submodules of $R[2]$ carrying V and $\bigwedge^2 V$ then $\mathfrak{r} = R[\mathfrak{g}] + \mathfrak{q}_1 + \mathfrak{q}_2$ is a 14-dimensional Lie subalgebra of $R[2]$. Furthermore $\mathfrak{r}$ is semisimple by Theorem 2.1 and easily $\mathfrak{r}$ has rank 2. But then $\mathfrak{r}$ must be simple of type G_2. In fact $\mathfrak{r} = R[2]$—see Theorem 6.1(i) and also Remark 6.3. This example was found in collaboration with D. Vogan.

3. Shared orbit pairs

We next consider the problem of integrating the infinitesimal action of $R[2]$ on M. We find that such a group action exists on a larger symplectic manifold M' lying in X.

In analogy with (1-9) we introduce an abstract Lie algebra $\mathfrak{g}'$ where $\mathfrak{g} \subset \mathfrak{g}'$ and a Lie algebra isomorphism

$$\rho'\colon \mathfrak{g}' \to R[2] \tag{3-1}$$

extending ρ. We put $\phi^z = \rho'(z)$ for $z \in \mathfrak{g}'$. Then $\mathfrak{g}'$ acts infinitesimally on M by the Hamiltonian vector fields ξ_{ϕ^z} where $z \in \mathfrak{g}'$. We identify $\mathfrak{g}'^* \simeq \mathfrak{g}'$ using a fixed $\mathfrak{g}'$-invariant nonsingular symmetric bilinear form $(\ ,\ )'$ on $\mathfrak{g}'$ which is negative definite on some compact form.

Let G' be the simply-connected complex Lie group with Lie algebra $\mathfrak{g}'$. We will say an action of G' extends an action of G if it is compatible with the Lie group homomorphism $\iota\colon G \to G'$ corresponding to the inclusion $\mathfrak{g} \subset \mathfrak{g}'$.

The constructions in the next theorem and proof are central throughout the rest of the paper. A key point is that we have a $\mathfrak{g}'$-module structure on R defined by

$$z \cdot \psi = [\phi^z, \psi] \tag{3-2}$$

where $z \in \mathfrak{g}'$ and $\psi \in R$. Clearly then $\mathfrak{g}'$ acts on R by derivations.

Theorem 3.1. *The G-action on X extends uniquely to an algebraic G'-action on X which integrates the infinitesimal action of $\mathfrak{g}'$ on X. Furthermore the G'-action on X preserves the Poisson bracket on R. G' has finitely many orbits on X and each is even-dimensional. G' has a unique Zariski open orbit M' on X. We have*

$$M \subset M' \subset X \tag{3-3}$$

and M is Zariski open in M' with boundary of codimension at least 2. All regular functions on M extend to M' so that

$$R = R(M) = R(M') = R(X). \tag{3-4}$$

The symplectic form ω on M extends uniquely to a G'-invariant symplectic form ω' on M' so that M' is a smooth complex symplectic variety. M' is a homogeneous Hamiltonian G'-space and the corresponding moment map is a finite G'-covering

$$\nu' : M' \to O' \tag{3-5}$$

onto a nilpotent orbit $O' \subset \mathfrak{g}'$ where O' spans $\mathfrak{g}'$. X is the normal closure of both M and M'.

Proof. Since each space $R[k]$ is finite-dimensional by Proposition 1.4 and $\mathfrak{g}'$-stable by (1-10) it follows that the derivation action of $\mathfrak{g}'$ on R integrates uniquely to an automorphism action of G' on R. But this defines an action of G' on X as a group of algebraic automorphisms. Clearly the $\mathfrak{g}'$-action on R extends the $\mathfrak{g}$-action and preserves Poisson brackets so that therefore the corresponding statements hold for the G'-action. This proves the first two statements of the theorem.

Now we can verify all the statements up to (3-4) by using the corresponding properties of the G-action given in Proposition 1.2. Indeed G' has finitely many orbits on X since this is already true for G. But then each G'-orbit is a finite union of G-orbits and so is even-dimensional. Obviously M lies in the G'-orbit $M' = G' \cdot \varepsilon$ so that clearly M' is the unique Zariski open orbit of G'. The boundary of M in M' lies in the boundary of M in X and hence has codimension at least 2. All regular functions on M extend to M' since they extend even to X.

Now M' is a smooth complex variety since it is an orbit. It is easy to check that the relation $\omega'(\xi_y, \xi_z) = [z, y]$ where $y, z \in R[2]$ defines a G'-invariant symplectic form ω' on M extending ω. Furthermore by (3-2) the G'-action on M' is Hamiltonian so that Theorem 2.3 implies that the moment map is the covering (3-5) onto an adjoint orbit O'.

As ν' is a moment map we have $\phi^z(p) = (\nu'(p), z)'$ by (2-6). So if $z \in \mathfrak{g}'$ satisfies $(z, y)' = 0$ for all $y \in O'$ then ϕ^z vanishes on M' and hence $z = 0$. This proves O' spans $\mathfrak{g}'$. But also $\phi^z \in R[2]$ where $z \in \mathfrak{g}'$ and it follows that $\nu'(\tilde{s} \bullet \varepsilon) = s^2 \nu'(\varepsilon)$ where $s \in \mathbb{C}^*$. Thus $\mathbb{C}^* e' \in O'$ where

$$e' = \nu'(\varepsilon) \in O' \tag{3-6}$$

and so e' is nilpotent.

Corollary 3.2. *By adding a boundary, we can uniquely complete M to a symplectic manifold M' such that the infinitesimal $\mathfrak{g}'$-action on M extends to M' and then integrates to a homogeneous Hamiltonian G'-action on M'.*

We call the pair (M, M') arising in Theorem 3.1 a *shared orbit pair with maximal symmetry* of (G, G') if $\mathfrak{g} \neq \mathfrak{g}'$.

Proposition 3.3. *The Euler grading and Poisson bracket on R arising from M and M' are the same.*

Proof. The two Poisson brackets on R coincide since they are completely determined by the symplectic forms ω and ω'. Now since $\nu'(\tilde{s} \bullet \varepsilon) = s^2 e'$ where $s \in \mathbb{C}^*$ (see the last proof) we find

$$[h, e'] = 2e'. \tag{3-7}$$

Consequently if (h', e', f') is an S-triple in $\mathfrak{g}'$ then $h - h' \in \mathfrak{g}'^{e'}$. But then $\exp th$ and $\exp th'$ define the same class in the quotient group $N_{G'}(G'^{\varepsilon})/G'^{\varepsilon}$. It follows that the right action of $\exp(\mathbb{C}h')$ on M' stabilizes M and coincides on M with the right action of $\exp(\mathbb{C}h)$. In particular then the two gradings on R coincide.

Proposition 3.4. *The inclusion $M \subset M'$ induces an isomorphism of fundamental groups $\pi_1(M) \simeq \pi_1(M')$.*

Proof. We will prove that restriction of covers defines a bijection between connected covers of M and connected covers of M'. First let $\sigma: P \to M'$ be a G'-homogeneous cover of M'. Then by dimension and the connectedness of P it follows that $\sigma^{-1}(M)$ is a G-orbit and hence a connected cover of M.

Next let $\mu: N \to M$ be a G-homogeneous covering of M. Clearly then μ is equivariant with respect to the right scaling actions of $\mathbb{C}^*$. Consequently $R[2]$ defines a Lie subalgebra $\mathfrak{r}$ of $R(N)[2]$ and the infinitesimal $\mathfrak{g}'$-action on M lifts to an infinitesimal action on N corresponding to $\mathfrak{r}$. Let $Y = \operatorname{Spec} R(N)$. Then the map $\overline{\mu}: Y \to X$ defined by the algebra homomorphism $R \to R(N)$ corresponding to μ is a finite infinitesimally $\mathfrak{g}'$-equivariant morphism extending μ. We can now conclude by arguing as in the proof of Theorem 3.1 that there is a G'-action on Y corresponding to the infinitesimal $\mathfrak{g}'$-action and a unique open dense G'-orbit $N' \subset Y$ such that $N \subset N'$ and the restriction of $\overline{\mu}$ to N' is a G'-homogeneous cover $\mu': N' \to M'$ extending μ.

We remark that Proposition 3.4 implies that the kernel of $\iota: G \to G'$ lies in G_o^e.

Example 3.5. A beautiful example due to Levasseur and Smith [L-S] and Vogan [V1] provides an instance of shared orbit pairs. Let O_8 be the (unique) 8-dimensional nilpotent orbit in the simple Lie algebra $\mathfrak{g}(2, \mathbb{C})$ of type G_2. Let $O_{\min}$ be the minimal nonzero nilpotent orbit of $\mathfrak{so}(7, \mathbb{C})$ so that $O_{\min}$ is also 8-dimensional. Let $\pi: \mathfrak{so}(7, \mathbb{C}) \to \mathfrak{g}(2, \mathbb{C})$ be the projection corresponding to the decomposition $\mathfrak{so}(7, \mathbb{C}) = \mathfrak{g}(2, \mathbb{C}) \oplus \mathbb{C}^7$ where $\mathbb{C}^7$ carries the 7-dimensional fundamental representation of $\mathfrak{g}(2, \mathbb{C})$. Then Levasseur and Smith proved that the restriction of π to $\overline{O}_{\min}$ is a birational finite morphism $\overline{\pi}: \overline{O}_{\min} \to \overline{O_8}$ which is then the normalization map. Hence $(O_8, O_{\min})$ is a shared orbit pair. We will return to this example below in Theorem 6.4.

We will use the following results about O' and e' later in the classification of shared orbit pairs. Notice that (by a routine argument) O' depends only on the choice of M and is independent of the choices of the invariant nonsingular bilinear forms on $\mathfrak{g}$ and $\mathfrak{g}'$. Let $\tau': \mathfrak{g}' \to \mathfrak{g}$ be the G-linear map defined by $(\tau'(z), x) = (z, x)'$ where $x \in \mathfrak{g}$ and $z \in \mathfrak{g}'$. It is easy to verify that

Proposition 3.6. *By restriction τ' defines two G-equivariant morphisms $\tau: O' \to \overline{O}$ and $\overline{\tau}: \overline{O'} \to \overline{O}$. We have $\tau(e') = e$ and $\overline{\nu} = \overline{\tau} \circ \overline{\nu'}$. Moreover $\overline{\tau}$ is a finite morphism so that in particular $\overline{\tau}$ is surjective and has finite fibers.*

Now let $\mathfrak{p} \subset \mathfrak{g}'$ be the subspace orthogonal to $\mathfrak{g}$ with respect to $(\ ,\ )'$. Then we have a $\mathfrak{g}$-stable direct sum

$$\mathfrak{g}' = \mathfrak{g} \oplus \mathfrak{p}. \tag{3-8}$$

Proposition 3.7. *Assume $\mathfrak{g} \neq \mathfrak{g}'$. After changing if necessary the choice of $(\ ,\ )$ we have $e' = e + e_{\mathfrak{p}}$ where $0 \neq e_{\mathfrak{p}} \in \mathfrak{p}$. Furthermore e, e', and $e_{\mathfrak{p}}$ are commuting nilpotent elements of $\mathfrak{g}'$. We have $\mathfrak{g}'^{e'} + \mathfrak{g} = \mathfrak{g}'$. Regarded as nilpotents in $\mathfrak{g}'$, the element e' is strictly more degenerate than e; i.e., e' lies on the boundary of $G' \cdot e$.*

Proof. The first statement is routine, noting that $e_{\mathfrak{p}} \neq 0$ by the definition of $\mathfrak{g}'$. Since $[h, e_{\mathfrak{p}}] = 2e_{\mathfrak{p}}$ by (1-5) and (3-7) it follows that $e_{\mathfrak{p}}$ is nilpotent. Clearly $G^e \cdot e'$ is a finite set and so $\mathfrak{g}^e \cdot e' = 0$. Consequently $[e, e'] = 0$ and so e', e, and $e' - e$ all commute. Since M is dense in M' it follows that $\mathfrak{g} \cdot e' = \mathfrak{g}' \cdot e'$ so that equivalently $\mathfrak{g}'^{e'} + \mathfrak{g} = \mathfrak{g}'$.

Next notice that $\mathfrak{g}' \cdot e = [\mathfrak{g}, e] \oplus [\mathfrak{p}, e]$. Now $[\mathfrak{p}, e] \neq 0$ follows easily. Thus $\mathfrak{g}' \cdot e$ is strictly larger than $\mathfrak{g} \cdot e$ and hence $\dim G' \cdot e > \dim G \cdot e = \dim G' \cdot e'$. Furthermore, since (h, e, f) is an S-triple in $\mathfrak{g}'$, we know from [K1] that (3-7) implies that e' lies in the closure of $G'^h \cdot e$.

Remark 3.8. Suppose now that $\mathfrak{r}$ is any Lie subalgebra of $R[2]$ containing $R[\mathfrak{g}]$ and let $\mathfrak{g}^+ = \rho'^{-1}(\mathfrak{r})$. Then we have Lie algebra inclusions

$$\mathfrak{g} \subset \mathfrak{g}^+ \subset \mathfrak{g}' \tag{3-9}$$

and $\rho^+: \mathfrak{g}^+ \to \mathfrak{r}$ is a Lie algebra isomorphism where $\rho^+ = \rho'|_{\mathfrak{g}^+}$. Let G^+ be the simply-connected group corresponding to $\mathfrak{g}^+$ and let $G \to G^+$ be the group homomorphism corresponding to the inclusion $\mathfrak{g} \subset \mathfrak{g}^+$.

Not only is $\mathfrak{g}^+$ semisimple by Theorem 2.1(i) but the results in this section involving G' and $\mathfrak{g}'$ apply equally well to G^+ and $\mathfrak{g}^+$. (Indeed none of the proofs used the maximality of $\mathfrak{g}'$.) In particular G^+ has a unique open orbit M^+ on X and then

$$M \subset M^+ \subset M'. \tag{3-10}$$

Then $R(M) = R(M^+) = R(M')$ as graded Poisson algebras. The moment map defines a G^+-homogeneous cover $\nu^+ : M^+ \to O^+$ where $O^+ \subset \mathfrak{g}^+$ is a nilpotent orbit and then ν^+ extends to a G^+-equivariant finite morphism $\overline{\nu^+} : X \to \overline{O^+}$. Let $e^+ = \nu^+(\varepsilon) \in O^+$.

We call such a pair (M, M^+) a *shared orbit pair* of (G, G^+) if $\mathfrak{g} \neq \mathfrak{g}^+$. If $\mathfrak{g}, \mathfrak{g}^+$, and $\mathfrak{g}'$ are all three different then we call the triple (M, M^+, M') in (3-10) a *shared orbit triple* of (G, G^+, G').

Shared orbit pairs are characterized by the following result.

Proposition 3.9. *Let $\mathfrak{f}$ be a semisimple Lie subalgebra of $\mathfrak{g}$ with $\mathfrak{f} \neq \mathfrak{g}$ and let F be the simply-connected group corresponding to $\mathfrak{f}$. Assume F has a Zariski open orbit N on M. Then (N, M) is a shared orbit pair of (F, G). In particular, N is a covering of a nilpotent orbit of F and all regular functions on N extend to M.*

Proof. Since M is a Hamiltonian G-space clearly N is a symplectic submanifold and a Hamiltonian F-space. But then by Theorem 2.3 the moment map defines a finite cover of N onto an adjoint orbit $F \cdot x$ where $x \in \mathfrak{f}$. Now the C-action on M commutes with the F-action and hence preserves N. It follows that $F \cdot x$ is stable under scaling. Thus x is nilpotent.

Restriction of functions from M to N defines an inclusion of Poisson algebras $R(M) \subset R(N)$. Furthermore $R(N)$ is integral over $R(M)$ since $R(N)$ is integral over the subring generated by $R(N)[\mathfrak{f}]$ (by Proposition 1.2) and clearly $R(N)[\mathfrak{f}] \subset R(M)[\mathfrak{g}]$. But then $R(N) = R(M)$ by Lemma 1.1. Thus X is the normal closure of N and consequently by Theorem 3.1 (N, M) is a shared orbit pair.

4. Simple symmetry, $R[1]$, and the geometry of X

The results of this section reduce the problem of determining all shared orbit pairs to the case where both $\mathfrak{g}$ and $\mathfrak{g}'$ are simple Lie algebras and also $R[1] = 0$.

From now on we adopt the following notations. Choose a Cartan subalgebra $\mathfrak{h}$ inside a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. Let T and B be the corresponding subgroups of G. Let Δ be the set of roots of $(\mathfrak{h}, \mathfrak{g})$ and let Δ_+ be the set of roots of $(\mathfrak{h}, \mathfrak{b})$. Let $\Lambda \subset \mathfrak{h}^*$ be the associated set of dominant integral weights so that for $\lambda \in \Lambda$ we have the corresponding finite-dimensional simple G-module V_λ. Let $\psi \in \Delta_+$ be the highest root.

If $\mathfrak{g}$ is simple then let $O_{\min} \subset \mathfrak{g}$ be the orbit of the highest root vector so that it is the minimal (nonzero) nilpotent orbit. Similarly we define $O^+_{\min} \subset \mathfrak{g}^+$ and $O'_{\min} \subset \mathfrak{g}'$.

Remark 4.1. Suppose G is not simple. Then we have a decomposition $G = G_1 \times \cdots \times G_k$ of G into a direct product of simply-connected simple Lie groups and a corresponding Lie algebra decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$. Then in turn we have $O = O_1 \times \cdots \times O_k$ and $M = M_1 \times \cdots \times M_k$ where M_i is a G_i-homogeneous cover of a nilpotent orbit $O_i \subset \mathfrak{g}_i$ for each i. Furthermore there is a corresponding identification $R = R(M_1) \otimes \cdots \otimes R(M_k)$ of graded Poisson algebras.

Theorem 4.2. *If* $\mathfrak{g}$ *is a simple Lie algebra then any Lie subalgebra* $\mathfrak{r}$ *of* $R[2]$ *containing* $R[\mathfrak{g}]$ *is also a simple Lie algebra.*

Proof. Assume $\mathfrak{r}$ is not simple. Let $\mathfrak{r} = \mathfrak{r}_1 + \cdots + \mathfrak{r}_m$ be a direct sum decomposition of $\mathfrak{r}$ into nonzero simple ideals where $m \geq 2$. Let $\pi_i : \mathfrak{r} \to \mathfrak{r}_i$, $i = 1, \ldots, m$, be the corresponding projections. Clearly then $\mathfrak{s} = \pi_1(R[\mathfrak{g}]) + \cdots + \pi_m(R[\mathfrak{g}])$ is a Lie subalgebra of $R[2]$ containing $R[\mathfrak{g}]$. Now $\pi_i(R[\mathfrak{g}]) \neq 0$ for each i since otherwise $[R[\mathfrak{g}], \mathfrak{r}_i] = 0$ so that G acts trivially on $\mathfrak{r}_i$ which is false. But then $\pi_i(R[\mathfrak{g}]) \simeq \mathfrak{g}$ since $\mathfrak{g}$ is simple. Hence we can identify the Lie algebra $\mathfrak{g}^+ = \rho'^{-1}(\mathfrak{s})$ with $\mathfrak{g}^{\oplus m}$ and then the inclusion $\mathfrak{g} \subset \mathfrak{g}^+$ is the m-fold diagonal embedding.

Next define M^+, ν^+, O^+, and e^+ as in Remark 3.8. Then $e^+ = (e_1, \ldots, e_m)$ and each $e_i \in \mathfrak{g}$ is necessarily nonzero since $\pi_i(R[\mathfrak{g}]) \neq 0$. Now G has only finitely many orbits on X (Proposition 1.2) and hence on its the image $\overline{O^+} = \overline{\nu^+(X)}$. But clearly there is a G-embedding $O_{\min} \times O_{\min} \to \overline{O^+}$ and thus G has only finitely many orbits on $O_{\min} \times O_{\min}$. This is false. Indeed let $(t_\psi, x_\psi, x_{-\psi})$ be an S-triple spanning the root $\mathfrak{sl}(2)$-subalgebra defined by ψ such that $t_\psi \in \mathfrak{h}$. Then the points $(sx_{-\psi}, x_\psi)$, where $s \in \mathbb{C}^*$, all lie in different G-orbits. This is clear since no nontrivial element of G^{x_ψ} scales $x_{-\psi}$. Indeed $G^{x_\psi} = L_\psi \ltimes U_\psi$ where L_ψ is kernel of the character χ by which G^{t_ψ} acts on $\mathbb{C}x_\psi$, U is the unipotent radical of B, and $U_\psi = G^{x_\psi} \cap U$. But clearly L_ψ fixes $x_{-\psi}$ and no nontrivial element of U_ψ scales $x_{-\psi}$.

Corollary 4.3. *Assume* $R[1] = 0$. *If* $\mathfrak{r}$ *is any Lie subalgebra of* $R[2]$ *containing* $R[\mathfrak{g}]$ *then* $\mathfrak{g}$ *and* $\mathfrak{r}$ *have the same number of simple components. In particular* $\mathfrak{g}$ *is simple if and only if* $\mathfrak{r}$ *is simple.*

Proof. Suppose $\mathfrak{g}$ has exactly k simple components $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$. Then since $R[1] = 0$ we have by Remark 4.1 that $R(M)[2] = \bigoplus_{i=1}^k R(M_i)[2]$ where the summands Poisson commute. But then there is a corresponding direct sum of Lie algebras $\mathfrak{r} = \mathfrak{r}_1 \oplus \cdots \oplus \mathfrak{r}_k$ so that for each i we have $R(M_i)[\mathfrak{g}_i] \subset \mathfrak{r}_i \subset R(M_i)[2]$. Then by Theorem 4.2, $\mathfrak{r}_1, \ldots, \mathfrak{r}_k$ are the simple components of $\mathfrak{r}$.

Next we analyze the case $R[1] \neq 0$. First we need

Proposition 4.4. *Suppose that* $\mathfrak{g}$ *is simple and* $O = O_{\min}$. *Then the closure* $\overline{O}$ *is a normal variety and* $R(O) = R(\overline{O})$. $R(O)$ *is a multiplicity-free* G*-module and furthermore as* G*-modules we have* $R(O)[2d] = V_{d\psi}$ *for* $d \geq 0$ *so that* $R(O) = \bigoplus_{d \geq 0} V_{d\psi}$. *If* $\mathfrak{g}$ *is of type* C_n, *i.e., if* $\mathfrak{g} \simeq \mathfrak{sp}(2n, \mathbb{C})$, *then* $\pi_1(O) = \mathbb{Z}_2$. *In every other case* O *is simply connected.*

Proof. The module decomposition and the normality are a special case of the corresponding result (see, e.g., [V-P]) about the variety of the highest weight orbit in any simple finite-dimensional G-module. Then $\overline{O}$ is the normal closure of O and $R(O) = R(\overline{O})$.

Let $(t_\psi, x_\psi, x_{-\psi})$ and L_ψ be as in the last part of the proof of Theorem 4.2. Then $\pi_1(O)$ identifies with the component group of $L_\psi = \chi^{-1}(1)$ and also with the component group of $\chi_T{}^{-1}(1)$ where $\chi_T = \chi|_T$ is the restricted character (since O is the orbit of a highest weight vector). Now $\chi_T{}^{-1}(1)$ is connected if $\mathfrak{g}$ is not of type C_n since then χ_T is a primitive character. If $\mathfrak{g}$ has type C_n then χ_T is the square of a primitive character and hence ${\chi_T}^{-1}(1)$ has two components.

Example 4.5. Let ω_0 be the standard symplectic form on $\mathbb{C}^{2n}$ so that $\omega_0 = \sum_{i=1}^n dp_i \wedge dq_i$ where $\{p_1, \dots, p_n, q_1, \dots, q_n\}$ is a linear coordinate system on $\mathbb{C}^{2n}$ with linear span V. Then the polynomial algebra $S(V)$ is a Poisson algebra. The homogeneous degree-2 polynomials form a Lie subalgebra of type C_n, i.e., $S^2(V) = \mathfrak{sp}(2n, \mathbb{C})$. We can identify $S^2(V) = S^2(\mathbb{C}^{2n})$ using the Killing form.

Let $M = \mathbb{C}^{2n} - \{0\}$. Then M, with its symplectic structure defined by ω_0, is the simply-connected 2-fold covering of the minimal nilpotent orbit $O \subset \mathfrak{sp}(2n, \mathbb{C})$ where the (moment map) covering $\sigma: M \to O$ is given by $\sigma(v) = v^2$. Then $X = \mathbb{C}^{2n}$, $o = 0$, and $R = S(V)$ as graded Poisson algebras. So $R[d] = S^d(V)$ where $d \geq 0$ and in particular $R[1] = V$ and $R[2] = R[\mathfrak{g}]$. Easily then $R(O) = R(\overline{O}) = S^{\text{even}}(V)$ where $S^{\text{even}}(V)$ is the subalgebra of $S(V)$ spanned by all homogeneous even-degree polynomials. We call M the $\mathfrak{sp}(\cdot, \mathbb{C})$ *double-covering minimal nilpotent orbit* case.

Theorem 4.6. *Assume $\mathfrak{g}$ is simple. Then $R[1] \neq 0$ in only one case: the $\mathfrak{sp}(\cdot, \mathbb{C})$ double-covering minimal nilpotent orbit case in Example 4.5. The normal variety X is nonsingular in this one and only one case.*

Proof. Suppose $2n = \dim R[1] \neq 0$. Since $R[2]$ is simple (Theorem 4.2) the surjection $R[2] \to \mathfrak{sp}(2n, \mathbb{C})$ in (2-2) must be an isomorphism. So $\mathfrak{g}' = \mathfrak{sp}(2n, \mathbb{C})$ and then by dimension the injective map $p: S^2(R[1]) \to R[2]$ defined by multiplication (see the proof of Theorem 2.1(iii)) is an isomorphism. Then p defines a surjective algebra homomorphism $S^{\text{even}}(R[1]) \to S'$, where S' is the subalgebra of R generated by $R[2]$. But $S^{\text{even}}(R[1]) = R(\overline{O'_{\text{min}}})$ (by Example 4.5) and clearly $S' = R(\overline{O'})$. So we have a surjective algebra homomorphism $R(\overline{O'_{\text{min}}}) \to R(\overline{O'})$ and by minimality we conclude $O' = O'_{\text{min}}$.

Now $M' \neq O'$ since otherwise $R[1] = 0$ by Proposition 1.4. But then the only possibility by Proposition 4.4 is that M' is the simply-connected double cover of O'. Thus $X \simeq \mathbb{C}^{2n}$ by Example 4.5 so that certainly X is nonsingular.

Next we can determine $\mathfrak{g}$. Indeed G is a subgroup of $G' = SP(2n, \mathbb{C})$ acting on $X = \mathbb{C}^{2n}$ with a dense orbit and hence $G = SP(2n, \mathbb{C})$ by [K-S, Proposition 1, §6]. Thus $O = O'$ and $M = M'$. This proves the first statement.

Finally suppose $R[1] = 0$. Then (1-8) implies immediately that $R[2] \subset \mathfrak{m}$ whereas $R[2] \cap \mathfrak{m}^2 = 0$. Hence $\dim \mathfrak{m}/\mathfrak{m}^2 \geq \dim R[2] \geq \dim \mathfrak{g}$ and consequently $\dim \mathfrak{m}/\mathfrak{m}^2 > \dim X$. But $\mathfrak{m}/\mathfrak{m}^2 = T_o^*(X)$ (Zariski cotangent space) and so X is singular at o.

In the general case where $\mathfrak{g}$ is semisimple, Theorem 4.6 and Remark 4.1 give

Corollary 4.7. $R[1] \neq 0$ *if and only if the $\mathfrak{sp}(\cdot, \mathbb{C})$ double-covering minimal nilpotent orbit example occurs in some component of M corresponding to a simple summand of $\mathfrak{g}$.*

Furthermore using Theorem 2.1 it now follows easily that

Corollary 4.8. *$R[2]$ is the unique maximal finite-dimensional semisimple Lie subalgebra of R containing $R[\mathfrak{g}]$.*

Example 4.9. Suppose $\mathfrak{g} = \mathfrak{sp}(2n_1, \mathbb{C}) \oplus \cdots \oplus \mathfrak{sp}(2n_k, \mathbb{C})$ where $n = n_1 + \cdots + n_k$ with $k \geq 2$ and all $n_i > 0$. Let M_i be the double cover of the minimal (nonzero) nilpotent orbit in $\mathfrak{sp}(2n_i, \mathbb{C})$ for $i = 1, \dots, k$ and let $M = M_1 \times \cdots \times M_k$. Then it follows immediately from Example 4.5 and Remark 4.1 that $R[1]$ is $2n$-dimensional and $\mathfrak{g}' \simeq R[2] \simeq \mathfrak{sp}(2n, \mathbb{C})$. In particular M is an example where $\mathfrak{g}'$ is simple even though $\mathfrak{g}$ is not simple. Moreover it follows from Remark 4.1 and Theorem 4.6 that every such example arises from this case.

In Example 4.5 we found $R[1]$ generates R as an algebra. In the general case it follows easily using Proposition 1.4 that

Lemma 4.10. *Let $\mathfrak{v} \subset \mathfrak{m}$ be a graded subspace. Then $\mathfrak{v}$ generates R as an algebra if and only if $\mathfrak{m} = \mathfrak{v} + \mathfrak{m}^2$.*

Now the Zariski tangent space $T_o(X) = (\mathfrak{m}/\mathfrak{m}^2)^*$ of X at o is a canonical finite-dimensional graded G-representation attached to M. Indeed since o is $G \times C$-fixed (Proposition 1.4) $T_o(X)$ carries commuting actions of $G \times C$. Lemma 4.10 implies that $(\mathfrak{m}/\mathfrak{m}^2)^*$ is the unique minimal embedding space of X. In fact we have the stronger result

Proposition 4.11. *There exists a closed G-embedding $X \to (\mathfrak{m}/\mathfrak{m}^2)^*$. Furthermore if $\mu: X \to V$ is any closed G-embedding into a G-module V then there exists a G-linear surjection $p: V \to (\mathfrak{m}/\mathfrak{m}^2)^*$.*

Proof. We can choose a $G \times C$-stable subspace $\mathfrak{v} \subset \mathfrak{m}$ such that $\mathfrak{m} = \mathfrak{v} \oplus \mathfrak{m}^2$. Then by Lemma 4.10 we have a surjective G-algebra homomorphism $S(\mathfrak{v}) \to R$ and hence a closed G-embedding $X \to \mathfrak{v}^* \simeq (\mathfrak{m}/\mathfrak{m}^2)^*$. On the other hand, if μ is given then clearly $\mu(o)$ is G-fixed so that therefore the differential $d\mu : T_o(X) \to T_{\mu(o)}(V)$ is an G-linear injection. As $T_{\mu(o)}(V) \simeq V$ as G-modules it follows by complete reducibility that p exists.

We remark that a closed G-embedding $X \to V$ into a G-module V is equivalent to an algebraic G-embedding $M \to V$ such that all regular functions on M arise as pullbacks of polynomial functions on V.

Remark 4.12. We have found that for the simply-connected (2-fold) covers of the three classical orbits associated to the three (classical) rank 4 Jordan algebras, the dimensions of the corresponding spaces $\mathfrak{m}/\mathfrak{m}^2$ are 78, 133, and 248. The connections of $\mathfrak{m}/\mathfrak{m}^2$ with the exceptional Lie groups of type E_6, E_7, and E_8 will appear in our next paper. In particular we will relate our results to the Joseph ideal in the universal enveloping algebra.

Remark 4.13. Suppose M is the simply-connected cover of the principal nilpotent orbit in $\mathfrak{g}$ where $\mathfrak{g}$ is simple and $\mathfrak{g} \neq \mathfrak{sl}(2)$. Then W. Graham [Gr] proved our conjecture that as a G-module $\mathfrak{m}/\mathfrak{m}^2$ is equal to the multiplicity-free sum of $\mathfrak{g}$ and its minuscule representations. This shows that $T_o(X)$ can be "large" compared to $\mathfrak{g}$, e.g., for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. Furthermore we can show using [B, Theorem 8.1] that the morphism $\overline{\nu}: X \to \overline{O}$ is bijective over the boundary of O.

Now regarding the normality of $\overline{O}$ we have the following. Notice that Proposition 1.2 implies: $X \simeq \overline{O}$ if and only if $M = O$ and $\overline{O}$ is normal.

Proposition 4.14. *Assume $M = O$ so that $R = R(O)$. Then the following three conditions are equivalent:*

(i) *$\overline{O}$ is a normal variety.*
(ii) *$R[\mathfrak{g}]$ is a complement to $\mathfrak{m}^2$ in $\mathfrak{m}$.*
(iii) *As $\mathfrak{g}$-modules we have $T_o(X) \simeq \mathfrak{g}$.*

Furthermore if (i), (ii), (iii) *are satisfied then*

$$R[\mathfrak{g}] = R[2] \tag{4-1}$$

so that $\mathfrak{g}' = \mathfrak{g}$ and $O' = O$.

Proof. (iii) implies (by the results about $R[1]$ in this section) that $R[1] = 0$ so that $R[2]$, and hence $R[\mathfrak{g}]$, lies in a complement to $\mathfrak{m}^2$ in $\mathfrak{m}$. The equivalence of (ii) and (iii) follows immediately and moreover we see (ii) implies (4-1). Now (i) implies that $X \simeq \overline{O}$ and clearly $T_0(\overline{O}) \simeq \mathfrak{g}$ so that we obtain (iii). Finally (ii) implies by Lemma 4.10 that $R[\mathfrak{g}]$ generates R. But then $R = S$ and hence $\overline{O} \simeq X$ which gives (i).

Remark 4.15. We can in fact recover O' from just the geometry of the singularity of X at o. Indeed recall (see, e.g., [M]) that the reduced variety $TC_o(X)_{\text{red}}$ of the Zariski tangent cone $TC_o(X)$ of X at o is a closed subvariety of the Zariski tangent space $T_o(X)$. If $R[1] = 0$ (so that o is a singular point of X) then as $(G \times C)$-varieties $TC_o(X)_{\text{red}} \simeq \overline{O'}$ and furthermore the linear span of $TC_o(X)_{\text{red}}$ in $(\mathfrak{m}/\mathfrak{m}^2)^*$ is the space $(\mathfrak{m}/\mathfrak{m}^2)^*[2] \simeq \mathfrak{g}'$. (We omit the proof.)

5. Root system structure of $\mathfrak{g}'$

In this section we determine all possible pairs $(\mathfrak{g}, \mathfrak{g}^+)$ corresponding to shared orbit pairs. Throughout this section we use the setup of Remark 3.8. Furthermore we assume that $\mathfrak{g}$, and hence also $\mathfrak{g}'$ and $\mathfrak{g}^+$ by Theorem 4.2, is simple.

The main idea is to analyze how the corresponding root systems are related to each other. Before we begin this however we have a result that reduces the problem of classifying the pairs $(\mathfrak{g}, \mathfrak{g}^+)$ to the case where $\mathfrak{g}^+ = \mathfrak{g}'$ and $M' \simeq O' = O'_{\min}$. This was essentially pointed out to us by D. Vogan.

Proposition 5.1. *Given a shared orbit pair (M, M') and an intermediate Lie algebra $\mathfrak{g}^+$, there exist a nilpotent orbit $O_\alpha \subset \mathfrak{g}$ and a G-homogeneous cover M_α of O_α such that $(M_\alpha, O^+_{\min})$ is a shared orbit pair of (G, G^+). Furthermore $(M_\alpha, O^+_{\min})$ has maximal symmetry; i.e., $\mathfrak{g}^+ \simeq R(M_\alpha)[2]$.*

Proof. Since G has finitely many orbits on X, it follows (cf. the proof of Theorem 4.2) that G has a unique open dense orbit M_α on $O^+_{\min}$. Then by Proposition 3.9 $(M_\alpha, O^+_{\min})$ is a shared orbit pair of (G, G^+). But clearly we have $R(M_\alpha)[2] = R(O^+_{\min})[2] = R(O^+_{\min})[\mathfrak{g}^+] \simeq \mathfrak{g}^+$ where the middle equality follows by Proposition 4.14, (4-1) since $\overline{O^+_{\min}}$ is normal.

Corollary 5.2 (of proof). *If $M^+ = O^+_{\min}$ then $\mathfrak{g}^+ = \mathfrak{g}'$ so that $M' \simeq O' = O'_{\min}$ and furthermore $X \simeq \overline{O'}$.*

Let Δ' be the set of roots of $(\mathfrak{h}', \mathfrak{g}')$ where $\mathfrak{h}'$ is a Cartan subalgebra (CSA) of $\mathfrak{g}'$ containing $\mathfrak{h}$. Let $\Delta_{\mathscr{L}} \subset \Delta$ and $\Delta'_{\mathscr{L}} \subset \Delta'$ be the sets of long roots (where we call all roots long in the simply-laced case).

Let V^μ be the μ-weight space in a $\mathfrak{g}$-module V for $\mu \in \mathfrak{h}^*$.

Theorem 5.3. *We have*

$$\mathfrak{h}' = \mathfrak{g}'^{\mathfrak{h}} \tag{5-1}$$

so that $\mathfrak{h}'$ is the unique CSA of $\mathfrak{g}'$ containing $\mathfrak{h}$. Furthermore if $\beta \in \Delta'$ is a long root of $\mathfrak{g}'$ then $\beta|_{\mathfrak{h}}$ is a root of $\mathfrak{g}$ so that we have a mapping

$$\Delta'_{\mathscr{L}} \to \Delta, \qquad \beta \mapsto \beta|_{\mathfrak{h}}. \tag{5-2}$$

Proof. By Proposition 5.1 we can assume without loss that $M' \simeq O' = O'_{\min}$ so that then $X \simeq \overline{O'}$.

Let $\beta \in \Delta'_{\mathscr{L}}$ and let $x'_\beta \in O'$ be a corresponding root vector. Then (since τ' is $\mathfrak{g}$-linear) $x_\phi = \tau(x'_\beta)$ lies in $\mathfrak{g}^\phi$ where $\phi = \beta|_{\mathfrak{h}}$. Now $x_\phi \neq 0$ since Propositions 1.4 and 3.6 imply that τ is never zero on O'. But also x_ϕ is nilpotent since $x_\phi \in \overline{O}$. Hence $\phi \neq 0$ so that $\phi \in \Delta$. This proves (5-2).

Now to show that the reductive algebra $\mathfrak{g}'^{\mathfrak{h}}$ is a CSA it suffices to show that it is abelian. Suppose not. Then $\mathfrak{k} = [\mathfrak{g}'^{\mathfrak{h}}, \mathfrak{g}'^{\mathfrak{h}}]$ is a nonzero semisimple Lie algebra. Then $\mathfrak{h}_{\mathfrak{k}} = \mathfrak{h}' \cap \mathfrak{k}$ is a CSA of $\mathfrak{k}$. Since $\Delta'_{\mathscr{L}}$ spans $\mathfrak{h}'^*$ we can now choose β so that $\beta|_{\mathfrak{h}_{\mathfrak{k}}} \neq 0$. Then $[\mathfrak{k}, x'_\beta] \neq 0$ and so the $\mathfrak{k}$-module $V \subset \mathfrak{g}'$ generated by x'_β is a nontrivial $\mathfrak{k}$-module. Consequently $\dim V > 1$. We next produce a contradiction to this.

Let $K \subset G'$ be the subgroup corresponding to $\mathfrak{k}$ and put $Z = K \cdot x'_\beta \subset O'$. Then Z spans V and also Z is stable under scaling. Now clearly $\tau(Z) = \mathbb{C}^* x_\phi$ since K centralizes $\mathfrak{h}$. It follows since τ has finite fibers that Z is equal to a finite union of root spaces in $\mathfrak{g}'$ with the origin deleted. But Z is connected. Thus $Z = \mathbb{C}^* x'_\beta$ and so $\dim V = 1$. Contradiction.

Let $\mathfrak{g}'_{\mathscr{L}}$ be the linear span in $\mathfrak{g}'$ of $\mathfrak{h}'$ and the root vectors corresponding to $\Delta'_{\mathscr{L}}$. Let $\mathfrak{e}(6, \mathbb{C}), \mathfrak{e}(7, \mathbb{C}), \mathfrak{e}(8, \mathbb{C}), \mathfrak{f}(4, \mathbb{C})$, and $\mathfrak{g}(2, \mathbb{C})$ denote the simple complex Lie algebras of types E_6, E_7, E_8, F_4, and G_2 respectively.

Corollary 5.4. *We have* $\operatorname{rank}\mathfrak{g} = \operatorname{rank}\mathfrak{g}'$ *if and only if*

$$\mathfrak{g}'_{\mathscr{L}} \subset \mathfrak{g} \subset \mathfrak{g}'. \tag{5-3}$$

In this case if $\mathfrak{g} \neq \mathfrak{g}'$ *then the pair type of* $(\mathfrak{g}, \mathfrak{g}')$ *must be one of*

$$\begin{array}{llll} & \text{(i)} & (A_2, G_2), & \\ \text{(5-4)} & \text{(ii)} & (D_n, B_n), & n \geq 3, \\ & \text{(iii)} & (D_4, F_4), & \\ & \text{(iv)} & (B_4, F_4). & \end{array}$$

Proof. If $\mathfrak{g}$ and $\mathfrak{g}'$ have the same rank then by Theorem 5.3 the long root spaces of $\mathfrak{g}'$ are root spaces of $\mathfrak{g}$ and so $\mathfrak{g}'_{\mathscr{L}} \subset \mathfrak{g}$. The converse is clear.

Now let $\mathfrak{s}$ be any simple Lie algebra and let $\mathfrak{s}_{\mathscr{L}}$ be the subspace spanned by a CSA $\mathfrak{h}_{\mathfrak{s}}$ of $\mathfrak{s}$ and the corresponding long root spaces. Since the long roots span $\mathfrak{h}_{\mathfrak{s}}^*$, $\mathfrak{s}_{\mathscr{L}}$ is a semisimple Lie subalgebra of $\mathfrak{s}$ and furthermore a list of all such pairs $(\mathfrak{s}_{\mathscr{L}}, \mathfrak{s})$ up to type is: (a) (A_2, G_2), (b) (D_n, B_n) for $n \geq 2$, (c) $(A_1^{\times n}, C_n)$ for $n \geq 3$, and (d) (D_4, F_4). We can find all strictly intermediate subalgebras $\mathfrak{s}_0$ such that $\mathfrak{s}_{\mathscr{L}} \subset \mathfrak{s}_0 \subset \mathfrak{s}$. In cases (a) and (b) easily there are none. In case (c) it is clear that $\mathfrak{s}_0$ is semisimple of type $C_{n_1} \times \cdots \times C_{n_k}$ where $n = n_1 + \cdots + n_k$. In particular if $\mathfrak{s}_0 \neq \mathfrak{s}$ then $\mathfrak{s}_0$ is not simple. Thus (a) and (b) give (i) and (ii) whereas (c) gives nothing.

Case (d) however is more interesting and gives cases (iii) and (iv). Indeed the 24 short roots of $(\mathfrak{h}_{\mathfrak{s}}, \mathfrak{s})$ are exactly the weights of the three irreducible 8-dimensional representations of $\mathfrak{so}(8, \mathbb{C})$, i.e., the standard and the two half-spin representations. Moreover the sum of $\mathfrak{s}_{\mathscr{L}}$ and the weight spaces corresponding to any one of these 8-dimensional spaces is a Lie subalgebra $\mathfrak{s}_0$ and $\mathfrak{s}_0 \simeq \mathfrak{so}(9, \mathbb{C})$. These are the only choices for $\mathfrak{s}_0$ since in each case the representation of $\mathfrak{so}(9, \mathbb{C})$ on $\mathfrak{s}/\mathfrak{s}_0$ is the irreducible 16-dimensional spin representation.

Remark 5.5. In the proof of Corollary 5.4 the pairs $(\mathfrak{s}_0, \mathfrak{s})$ where $\mathfrak{s}_0$ is semisimple but not simple coincides with the pairs $(\mathfrak{g}, \mathfrak{g}')$ in Example 4.9.

Now recall $\mathfrak{p}$ from (3-8) and let $\mathfrak{h}_{\mathfrak{p}} = \mathfrak{h}' \cap \mathfrak{p}$ so that then

$$\mathfrak{h}' = \mathfrak{h} \oplus \mathfrak{h}_{\mathfrak{p}} \tag{5-5}$$

and we have a corresponding inclusion $\mathfrak{h}^* \subset \mathfrak{h}'^*$.

Theorem 5.6. *There exists an extension of* $\mathfrak{b}$ *to a Borel subalgebra* $\mathfrak{b}'$ *of* $\mathfrak{g}'$ *containing* $\mathfrak{h}'$. *Choose any such* $\mathfrak{b}'$. *Let* Δ'_+ *be the set of roots of* $(\mathfrak{h}', \mathfrak{b}')$ *and let* $\psi' \in \Delta'_+$ *be the highest root. Then* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *have the same highest root*; *i.e.,*

$$\psi' = \psi. \tag{5-6}$$

In particular $\mathfrak{g}$ *and* $\mathfrak{g}'$ *have the same highest root space. That is, the highest root vector* $x'_{\psi'}$ *of* $\mathfrak{g}'$ *lies in* $\mathfrak{g}$ *and is a highest root vector* x_ψ *of* $\mathfrak{g}$ *so that*

$$x_\psi = x'_{\psi'}. \tag{5-7}$$

Proof. We can choose a real semisimple element $t \in \mathfrak{h}$ such that $\mathfrak{g}'^{\mathfrak{h}} = \mathfrak{g}'^t$ and t is in the fundamental chamber of $\mathfrak{h}$, i.e., $\phi(t) > 0$ for all $\phi \in \Delta^+$. Then clearly $\mathfrak{g}'^t = \mathfrak{h}'$ and the linear span of $\mathfrak{h}'$ and the root vectors corresponding to $\Delta'_+ = \{\beta \in \Delta' | \beta(t) > 0\}$ is a Borel subalgebra $\mathfrak{b}' \subset \mathfrak{g}'$ containing $\mathfrak{b}$. Corresponding to this $\mathfrak{b}'$ we now have ψ' and $x'_{\psi'}$.

Next let $\phi = \psi'|_{\mathfrak{h}}$ and $x_\phi = \tau(x'_{\psi'})$. Then (see the proof of Theorem 5.3) $\phi \in \Delta$ and x_ϕ is a corresponding nonzero root vector in $\mathfrak{g}$. In fact $\phi = \psi$ since clearly $x'_{\psi'}$, and so also x_ϕ (since τ is $\mathfrak{g}$-linear), is a highest weight vector for the $\mathfrak{g}$-action. Furthermore $\dim \mathfrak{g}'^{\psi} = 1$. Indeed if $\alpha|_{\mathfrak{h}} = \psi$ where $\alpha \in \Delta'$, then it follows from the maximality of $\psi'(t)$ that $\alpha = \psi'$. But then $\mathfrak{g}'^{\psi} = \mathbb{C}x'_{\psi'} = \mathfrak{g}^{\psi}$.

Next we show that $\psi' = \psi$. We need to check that $\psi'|_{\mathfrak{h}_\mathfrak{p}} = 0$. This is true since $[\mathfrak{h}_\mathfrak{p}, x'_{\psi'}] \subset \mathbb{C}x'_{\psi'} \subset \mathfrak{g}$ and $[\mathfrak{h}_\mathfrak{p}, x'_{\psi'}] \subset [\mathfrak{p}, \mathfrak{g}] \subset \mathfrak{p}$ so that $[\mathfrak{h}_\mathfrak{p}, x'_{\psi'}] = 0$.

Finally notice that if $\mathfrak{c}$ is some other choice of $\mathfrak{b}'$ and $\gamma \in \Delta'$ is the corresponding highest root then by the same argument as above we have $\gamma|_{\mathfrak{h}} = \psi$. But then necessarily $\gamma = \psi'$ since $\mathfrak{g}'^{\psi} = \mathbb{C}x'_{\psi'}$. This proves (5-6).

Assume now that $(\, , \,)$ and $(\, , \,)'$ are the Killing forms on $\mathfrak{g}$ and $\mathfrak{g}'$ respectively. Let $| \ |$ and $| \ |'$ be the corresponding norms on the real vector spaces spanned by the root lattices. An immediate consequence of (5-6) is

Corollary 5.7. *If $\beta \in \Delta'$ then $|\beta|' \le |\psi|'$.*

We can now bound the $\mathfrak{g}$-modules appearing in $\mathfrak{p}$. This is a key result.

Corollary 5.8. *Let μ be a weight of $\mathfrak{h}$ on the $\mathfrak{g}$-module $\mathfrak{p} \simeq \mathfrak{g}'/\mathfrak{g}$. Then*

$$|\mu| < |\psi|. \tag{5-8}$$

In particular the adjoint representation of $\mathfrak{g}$ does not appear in $\mathfrak{g}'/\mathfrak{g}$.

Proof. It suffices to show that $|\mu|' < |\psi|'$. Assume not. Suppose $\beta \in \Delta'$ with $\beta|_{\mathfrak{h}} = \mu$. Then $|\mu|' \le |\beta|' \le |\psi|'$ and hence by assumption the three norms are equal. It follows that $\beta = \mu$ and furthermore (because of (5-6) and Theorem 5.3) β is a long root of Δ' and therefore $\mu \in \Delta$. This proves $\mathfrak{g}'^{\mu} = \mathfrak{g}^{\mu}$. But $\mathfrak{g}'^{\mu}$ contains a nonzero vector of $\mathfrak{p}$ by hypothesis. Contradiction.

We can now complete the determination of possible pairs $(\mathfrak{g}, \mathfrak{g}')$.

Theorem 5.9. *Assume* $\operatorname{rank}\mathfrak{g} < \operatorname{rank}\mathfrak{g}'$. *Then $\mathfrak{g}$ is doubly laced. Furthermore the pair type of $(\mathfrak{g}, \mathfrak{g}')$ must be one of*

$$\begin{array}{lll} \text{(i)} & (G_2, B_3), & \\ \text{(ii)} & (B_n, D_{n+1}), & n \ge 2, \\ \text{(iii)} & (C_n, A_{2n-1}), & n \ge 2, \\ \text{(iv)} & (F_4, E_6), & \\ \text{(v)} & (G_2, D_4). & \end{array} \tag{5-9}$$

In cases (i)–(iv) *we have $\mathfrak{g}'/\mathfrak{g} \simeq V_\alpha$ as $\mathfrak{g}$-modules and in case* (v) *$\mathfrak{g}'/\mathfrak{g} \simeq V_\alpha \oplus V_\alpha$ where $\alpha \in \Delta$ is the highest short root.*

Proof. Since $\mathfrak{p}^0 \neq 0$ it follows that $\mathfrak{p}^{\mu} \neq 0$ for some $\mu \in \Delta$. Then by (5-8) μ is short and hence $\mathfrak{g}$ is doubly laced.

Choose $\mathfrak{g}^+ \subset \mathfrak{g}'$ to be the subalgebra of invariants under $Z(G)$. Clearly $\mathfrak{h}'$ is a CSA of $\mathfrak{g}^+$. So $\operatorname{rank} \mathfrak{g}^+ = \operatorname{rank} \mathfrak{g}'$ and in particular $\mathfrak{g} \neq \mathfrak{g}^+$. We conclude from (5-8) that $\mathfrak{g}^+ = \mathfrak{g} \oplus \mathfrak{p}^+$ where $\mathfrak{p}^+$ is the V_α-primary component in $\mathfrak{g}^+$.

In fact we can choose a *simple* $\mathfrak{g}$-submodule $\mathfrak{q} \subset \mathfrak{p}^+$ such that $\mathfrak{a} = \mathfrak{g} \oplus \mathfrak{q}$ is a Lie subalgebra of $\mathfrak{g}^+$. Indeed, clearly there exists a root γ of $(\mathfrak{h}', \mathfrak{g}^+)$ with a corresponding root vector $x_\gamma^+ \in \mathfrak{g}^+$ such that $\gamma|_\mathfrak{h} = \alpha$ and $x_\gamma^+ \notin \mathfrak{g}$. Then $x_\gamma^+ = x_\alpha + v$ where $x_\alpha \in \mathfrak{g}^\alpha$ and $0 \neq v \in \mathfrak{p}^{+\alpha}$. But then v is a highest weight vector of a $\mathfrak{g}$-submodule $\mathfrak{q} \subset \mathfrak{p}^+$ such that $\mathfrak{q} \simeq V_\alpha$. We claim that $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{g}+\mathfrak{q}$. We know by representation theory that the $\mathfrak{g}$-submodule of $[\mathfrak{q}, \mathfrak{q}]$ generated by $[\mathfrak{q}^\alpha, \mathfrak{q}^0]$ contains the whole V_α-primary component in $[\mathfrak{q}, \mathfrak{q}]$. So it suffices to show that $[\mathfrak{q}^\alpha, \mathfrak{q}^0] \subset \mathfrak{g} + \mathfrak{q}$. Now this is clear since $[\mathfrak{q}^\alpha, \mathfrak{q}^0] = [x_\gamma^+ - x_\alpha, \mathfrak{q}^0]$ while $[x_\gamma^+, \mathfrak{q}^0] \subset [x_\gamma^+, \mathfrak{h}'] = \mathbb{C}x_\gamma^+$ and $[x_\alpha, \mathfrak{q}^0] \subset \mathfrak{q}^\alpha$. This proves the claim. Furthermore by Theorem 4.2 again $\mathfrak{a}$ is simple.

Next we analyze the pairs $(\mathfrak{g}, \mathfrak{a})$. As $\mathfrak{g}$ is simple and doubly laced there are four possible cases.

(a) $\mathfrak{g}$ has type G_2. Then $\dim V_\alpha = 7$ so that $\dim \mathfrak{a} = 14 + 7 = 21$. The only simple Lie algebras of dimension 21 are those of types B_3 and C_3, i.e., $\mathfrak{a} = \mathfrak{so}(7, \mathbb{C})$ or $\mathfrak{a} = \mathfrak{sp}(6, \mathbb{C})$. Since $\mathfrak{g}(2, \mathbb{C})$ has no nontrivial 6-dimensional representation we conclude $\mathfrak{a} = \mathfrak{so}(7, \mathbb{C})$.

(b) $\mathfrak{g}$ has type B_n, $n \geq 2$. Then V_α is the standard representation of dimension $2n + 1$ and as a $\mathfrak{g}$-module $\bigwedge^2 V_\alpha = \mathfrak{g}$. It follows that $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{g}$ so that therefore $\mathfrak{a} = \mathfrak{g} + \mathfrak{q}$ is a complexified Cartan decomposition of $\mathfrak{a}$. But then $\mathfrak{g}$ and the "tangential representation" of $\mathfrak{g}$ on $\mathfrak{q}$ determine $\mathfrak{a}$ up to isomorphism. We can now conclude $\mathfrak{a} = \mathfrak{so}(2n + 2, \mathbb{C})$.

(c) $\mathfrak{g}$ has type C_n, $n \geq 3$. Then V_α is the (fundamental) $\mathfrak{g}$-representation defined by $V_\alpha \oplus \mathbb{C} = \bigwedge^2 \mathbb{C}^{2n}$. So $\dim \mathfrak{a} = \dim S^2(\mathbb{C}^{2n}) + \dim \bigwedge^2(\mathbb{C}^{2n}) - 1 = 4n^2 - 1$. Since $\dim V_\alpha^0 = n - 1$ it follows using (5-1) that $\operatorname{rank} \mathfrak{a} = 2n - 1$. Now because of the rank and dimension it follows that $\mathfrak{a} = \mathfrak{sl}(2n, \mathbb{C})$.

(d) $\mathfrak{g}$ has type F_4. Then $\dim V_\alpha = 26$ so that $\dim \mathfrak{a} = 52 + 26 = 78$. The only simple algebras of dimension 78 are those of types B_6, C_6, and E_6. But $\mathfrak{a}$ cannot be $\mathfrak{so}(13, \mathbb{C})$ or $\mathfrak{sp}(12, \mathbb{C})$ since $\mathfrak{f}(4, \mathbb{C})$ has no nontrivial representation of dimension 13 or 12. Thus $\mathfrak{a}$ has type E_6.

We can now determine all possible pairs $(\mathfrak{g}, \mathfrak{g}')$. A key point is that the pair $(\mathfrak{a}, \mathfrak{g}')$ itself corresponds to a shared orbit pair with maximal symmetry.

Suppose $\mathfrak{a} = \mathfrak{g}^+$. Then $\operatorname{rank} \mathfrak{a} = \operatorname{rank} \mathfrak{g}'$ and so either (I) $\mathfrak{a} = \mathfrak{g}'$ or (II) the type of $(\mathfrak{a}, \mathfrak{g}')$ appears in (5-4). If (I) holds then (a), (b), (c), (d) give four possible pairs $(\mathfrak{g}, \mathfrak{g}')$ and their types are listed in (5-9)(i)–(iv). If (II) holds then comparing (a), (b), (c), (d) with (5-4) the only possibilities are case (b) together with (5-4)(ii)(iii). Then $\mathfrak{g} = \mathfrak{so}(2n + 1, \mathbb{C})$, $\mathfrak{a} = \mathfrak{so}(2n + 2, \mathbb{C})$, and either $\mathfrak{g}' = \mathfrak{so}(2n + 3, \mathbb{C})$ or, if $n = 3$, $\mathfrak{g}' = \mathfrak{f}(4, \mathbb{C})$. However none of these cases can occur since in each case $\mathfrak{g}'$ would contain a nonzero $\mathfrak{g}$-invariant vector and this is impossible. So (II) is ruled out.

On the other hand suppose $\mathfrak{a} \neq \mathfrak{g}^+$. Then clearly $\operatorname{rank} \mathfrak{a} < \operatorname{rank} \mathfrak{g}'$ so that (by the argument above) $\mathfrak{a}$ is doubly laced. But then case (a) is the only possibility, i.e., $\mathfrak{g} = \mathfrak{g}(2, \mathbb{C})$ and $\mathfrak{a} = \mathfrak{so}(7, \mathbb{C})$. This proves already that (5-9)(i)–(iv) exhaust all pair types of unequal rank pairs $(\mathfrak{g}, \mathfrak{g}')$ where $\mathfrak{g}$ is not of type G_2. But then the pair type of $(\mathfrak{a}, \mathfrak{g}')$ itself must appear in in (5-9)(i)–(iv). We conclude $(\mathfrak{a}, \mathfrak{g}')$ has type (B_3, D_4). Therefore $\mathfrak{g} = \mathfrak{g}(2, \mathbb{C})$ and $\mathfrak{g}' = \mathfrak{so}(8, \mathbb{C})$ is the final possibility and this is listed in (5-9)(v).

In the next section we prove all pair types in (5-4) and (5-9) do in fact arise from shared orbit pairs.

6. Construction of shared orbit pairs

In this section we construct shared orbit pairs and triples. If (M, M') is a shared orbit pair of (G, G') then recall $R[\mathfrak{g}] \simeq \mathfrak{g}$ and $R[2] \simeq \mathfrak{g}'$.

Theorem 6.1. *Each of the pairs* $(\mathfrak{g}, \mathfrak{g}')$ *given in* (5-4) *arises from a shared orbit pair* (M, M'). *In particular*

(i) *Let* $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ *and let* M *be the simply-connected* (3*-fold*) *cover of the principal nilpotent orbit* O *in* $\mathfrak{g}$. *Then* $\mathfrak{g}' = \mathfrak{g}(2, \mathbb{C})$ *and as* $\mathfrak{g}$*-modules*

$$R[2] \simeq R[\mathfrak{g}] \oplus \mathbb{C}^3 \oplus \bigwedge^2 \mathbb{C}^3. \tag{6-1}$$

(ii) *Let* $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ $(n \geq 3)$ *and let* M *be the simply-connected* (2*-fold*) *cover of the nilpotent orbit* O *in* $\mathfrak{g}$ *of Jordan type* $(3, 1^{2n-3})$. *Then* $\mathfrak{g}' = \mathfrak{so}(2n+1, \mathbb{C})$ *and as* $\mathfrak{g}$*-modules*

$$R[2] \simeq R[\mathfrak{g}] \oplus \mathbb{C}^{2n} \tag{6-2}$$

where $\mathbb{C}^{2n}$ *is the standard representation of* $\mathfrak{g}$.

(iii) *Let* $\mathfrak{g} = \mathfrak{so}(8, \mathbb{C})$ *and let* M *be the simply-connected* (4*-fold*) *cover of the nilpotent orbit* O *in* $\mathfrak{g}$ *of Jordan type* $(3, 2, 2, 1)$. *Then* $\mathfrak{g}' = \mathfrak{f}(4, \mathbb{C})$ *and as* $\mathfrak{g}$*-modules*

$$R[2] \simeq R[\mathfrak{g}] \oplus \mathbb{C}^8 \oplus \mathbb{C}^8_+ \oplus \mathbb{C}^8_- \tag{6-3}$$

where $\mathbb{C}^8$ *is the standard representation and* $\mathbb{C}^8_+$ *and* $\mathbb{C}^8_-$ *are the two* (*inequivalent*) *half-spin representations of* $\mathfrak{g}$.

(iv) *Let* $\mathfrak{g} = \mathfrak{so}(9, \mathbb{C})$ *and let* M *be the simply-connected* (2*-fold*) *cover of the nilpotent orbit* O *in* $\mathfrak{g}$ *of Jordan type* $(2, 2, 2, 2, 1)$. *Then* $\mathfrak{g}' = \mathfrak{f}(4, \mathbb{C})$ *and as* $\mathfrak{g}$*-modules*

$$R[2] \simeq R[\mathfrak{g}] \oplus \mathbb{C}^{16} \tag{6-4}$$

where $\mathbb{C}^{16}$ *is the spin representation of* $\mathfrak{g}$.

In every case $M' \simeq O'$ *where* O' *is the orbit of the highest root vector in* $\mathfrak{g}'$ *and furthermore* $X \simeq \overline{O'}$.

Proof. Assume $\mathfrak{g}$ is simple and let $\mathfrak{a} \subset \mathfrak{g}$ be the $\mathfrak{sl}(2, \mathbb{C})$-subalgebra containing e and h. Let $\mathfrak{g}^+$, O^+, and M^+ be as in Remark 3.8.

(i) We return to the setup Example 2.7. Corresponding to $\mathfrak{r}$ we have $\mathfrak{g}^+ = \mathfrak{g}(2, \mathbb{C})$. Since both O and $O^+_{\min}$ are 6-dimensional we conclude $O^+ = O^+_{\min}$.

But then $M^+ \simeq O^+_{\min}$ by Proposition 4.4 since $\mathfrak{g}^+$ is not of type C_n. Now all the statements concerning (i) follow by Corollary 5.2.

(ii) Clearly we have an $\mathfrak{a}$-stable decomposition $\mathbb{C}^{2n} = V \oplus (\mathbb{C}^{2n})^{\mathfrak{a}}$ where $V \simeq \mathfrak{a}$ as $\mathfrak{a}$-modules. Then the eigenvalues of h on $\mathbb{C}^{2n}$ are ± 2 and 0 and $\mathbb{C}^{2n}[2] = \mathbb{C}v$ where $0 \neq v \in V^e$. Furthermore using the Levi decomposition $\mathfrak{g}^e = \mathfrak{g}^{\mathfrak{a}} \oplus \mathfrak{g}^e_{\mathrm{nil}}$ it follows that $\mathfrak{g}^e \cdot v = 0$. Indeed $\mathfrak{g}^e_{\mathrm{nil}}$ is the sum of the positive eigenspaces of ad h on $\mathfrak{g}^e$ and hence $\mathfrak{g}^e_{\mathrm{nil}}$ kills the highest eigenspace in $\mathbb{C}^{2n}$. On the other hand $\mathfrak{g}^{\mathfrak{a}}$ acts by scalars on V as V is a simple $\mathfrak{a}$-module. But also $\mathfrak{g}^{\mathfrak{a}}$ preserves a nonsingular bilinear form on V (namely the form obtained by restricting the standard form on $\mathbb{C}^{2n}$) and so these scalars are all zero.

We have shown that $(\mathbb{C}^{2n})^{\mathfrak{g}^e}[2] = \mathbb{C}v$. Thus (by Lemma 2.6) $\mathfrak{g}'$ contains a unique $\mathfrak{g}$-submodule $\mathfrak{q}$ equivalent to $\mathbb{C}^{2n}$. But $\bigwedge^2 \mathfrak{q} \simeq \mathfrak{g}$ as $\mathfrak{g}$-modules and so we conclude using Proposition 2.4(i) and Corollary 5.8 that $[\mathfrak{q}, \mathfrak{q}] = \mathfrak{g}$. Therefore $\mathfrak{g}^+ = \mathfrak{g} + \mathfrak{q}$ is a Lie subalgebra of $\mathfrak{g}'$ and it follows (see (b) in the proof of Theorem 5.9) that $\mathfrak{g}^+ = \mathfrak{so}(2n+1, \mathbb{C})$. Now both O and $O^+_{\min}$ have dimension equal to $4n - 4$ (see, e.g., [S-S] or [Kr-P]). Therefore as in (i) we have $O^+ = O^+_{\min}$ and everything else follows (again using Corollary 5.2).

(iii) Since e has Jordan type $(3, 2, 2, 1)$ we have an $\mathfrak{a}$-stable decomposition $\mathbb{C}^8 = V \oplus U_1 \oplus U_2 \oplus \mathbb{C}$ where as $\mathfrak{a}$-modules $\mathbb{C}$ is trivial, $U_1 \simeq U_2 \simeq \mathbb{C}^2$ (the standard representation), and again $V \simeq \mathfrak{a}$. Then the eigenvalues of h on $\mathbb{C}^8$ are (counting multiplicities) $\pm 2, \pm 1, \pm 1, \pm 0$. Furthermore by the same argument as in (ii) we conclude that $\mathbb{C}^8[2] = \mathbb{C}v$ and $\mathfrak{g}^e \cdot v = 0$.

Now e acts on $\mathbb{C}^8_+$ and $\mathbb{C}^8_-$ by nilpotent endomorphisms. We claim that on each space e again has Jordan type $(3, 2, 2, 1)$! Indeed the Jordan type is completely determined by the eigenvalues of h (with multiplicities counted). Recall that if $\pm\mu_1, \pm\mu_2, \pm\mu_3, \pm\mu_4$ are the weights of $\mathfrak{h}$ on $\mathbb{C}^8$ then the weights on $\mathbb{C}^8_+$ and $\mathbb{C}^8_-$ are those of the form $\frac{1}{2}(\pm\mu_1 \pm \mu_2 \pm \mu_3 \pm \mu_4)$ where we allow for (say) $\mathbb{C}^8_+$ only those sums with an even number of minus signs and for $\mathbb{C}^8_-$ only those with an odd number of minus signs. Thus (remarkably) the h-eigenvalues on both $\mathbb{C}^8_+$ and $\mathbb{C}^8_-$ are (counting multiplicities) $\pm 2, \pm 1, \pm 1, \pm 0$. This proves the claim. Furthermore by the same argument as for $\mathbb{C}^8$ we find that $\mathbb{C}^8_+[2]$ and $\mathbb{C}^8_-[2]$ are both 1-dimensional and $\mathfrak{g}^e$-invariant. Thus (by Lemma 2.6) $\mathfrak{g}'$ contains a unique copy of each of $\mathbb{C}^8, \mathbb{C}^8_+$, and $\mathbb{C}^8_-$.

Now by Theorem 5.9 rank $\mathfrak{g}$ = rank $\mathfrak{g}'$ since $\mathfrak{g}$ is simply laced. Therefore the type of $(\mathfrak{g}, \mathfrak{g}')$ must appear in (5-4). Hence this is case (5-4)(iii) so that $\mathfrak{g}' = \mathfrak{f}(4, \mathbb{C})$ and (6-3) holds. We conclude $O' = O'_{\min}$ since both orbits have dimension 16. This proves (iii).

Moreover we can now prove (iv) too. Indeed let $\mathfrak{q} \subset \mathfrak{g}'$ be the $\mathfrak{g}$-submodule equivalent to $\mathbb{C}^8$. Then (see (d) in the proof of Corollary 5.4) $\mathfrak{g}^+ = \mathfrak{g} + \mathfrak{q}$ is a Lie subalgebra of $\mathfrak{g}'$, $\mathfrak{g}^+ \simeq \mathfrak{so}(9, \mathbb{C})$, and $\mathfrak{g}'/\mathfrak{g}^+$ carries the spin representation of $\mathfrak{so}(9, \mathbb{C})$. Now there is only one nilpotent orbit in $\mathfrak{so}(9, \mathbb{C})$ of the same dimension (16) as O', and hence O has Jordan type $(2, 2, 2, 2, 1)$. By

Proposition 3.4, M^+ is simply connected because M' is simply connected.

Corollary 6.2 (of proof). *In* (iii) *of Theorem* 6.1 *there is a subalgebra* $\mathfrak{r} \simeq \mathfrak{so}(9, \mathbb{C})$ *between* $R[\mathfrak{g}]$ *and* $R[2]$ *and therefore we have a shared orbit triple* (M, M^+, M') *for the Lie algebras*

$$\mathfrak{so}(8, \mathbb{C}) \subset \mathfrak{so}(9, \mathbb{C}) \subset \mathfrak{f}(4, \mathbb{C}). \tag{6-5}$$

Furthermore M^+ *is equal to the space* M *in Theorem* 6.1(iv).

Remark 6.3. Let O' and M be as in (i) of Theorem 6.1. A noncommutative analog of this example in was given by Zahid in [Z]. In [Ge] S. Gelfand quantized the ring of functions on the 6-dimensional orbit O' by a ring $\mathscr{A}$ of differential operators on 3-space (with rational coefficients). Furthermore [Ge, Theorem 4(ii)] and [Gr, Theorem 1.2] imply that $\mathscr{A}$ and $R(M)$ have the same $\mathfrak{sl}(3, \mathbb{C})$-module structure.

Next we recover the Levasseur-Smith result (see Example 3.5) and explain how it extends to the other cases of short root vector orbits.

Theorem 6.4. *Assume that* $\mathfrak{g}$ *is a simple doubly-laced Lie algebra,* $O \subset \mathfrak{g}$ *is the orbit of a short root vector, and* M *is the simply-connected cover of* O. *Then* $\mathfrak{g} \neq \mathfrak{g}'$ *and in fact* $\mathfrak{g}'$ *is a simple simply-laced Lie algebra. The pairs* $(\mathfrak{g}, \mathfrak{g}')$ *in these four cases are*

$$\begin{aligned} &\text{(i)} && (\mathfrak{g}(2, \mathbb{C}), \mathfrak{so}(7, \mathbb{C})), \\ &\text{(ii)} && (\mathfrak{so}(2n+1, \mathbb{C}), \mathfrak{so}(2n+2, \mathbb{C})), \qquad n \geq 2, \\ &\text{(iii)} && (\mathfrak{sp}(2n, \mathbb{C}), \mathfrak{sl}(2n, \mathbb{C})), \qquad n \geq 2, \\ &\text{(iv)} && (\mathfrak{f}(4, \mathbb{C}), \mathfrak{e}(6, \mathbb{C})). \end{aligned} \tag{6-6}$$

These pairs coincide with those in (5-9)(i)–(iv). *Let* $\alpha \in \Delta$ *be the the highest short root. Then as* $\mathfrak{g}$*-modules*

$$R[2] \simeq R[\mathfrak{g}] \oplus V_\alpha. \tag{6-7}$$

In (i) $M = O$ *and* X *is the normalization of (the nonnormal variety)* $\overline{O}$. *In cases* (ii)–(iv) M *is a double cover of* O. *In every case* $M' \simeq O'$ *where* O' *is the orbit of the highest root vector in* $\mathfrak{g}'$ *and furthermore* $X \simeq \overline{O'}$

Proof. Let $(t_\alpha, x_\alpha, x_{-\alpha})$ be the standard basis of the root $\mathfrak{sl}(2, \mathbb{C})$-subalgebra $\mathfrak{a} \subset \mathfrak{g}$ be corresponding to α. We may assume $e = x_\alpha$ and $h = t_\alpha$.

Now the weights in V_α are the short roots (each with multiplicity 1) and zero (with positive multiplicity). It follows easily that the eigenvalues of h on V_α are ± 2, ± 1, and 0 and furthermore the eigenvalues ± 2 have multiplicity 1 so that $V_\alpha[2] = V_\alpha^\alpha$.

Next we prove $\mathfrak{g}^e \cdot V_\alpha^\alpha = 0$. By the argument given in the proof of Theorem 6.1(ii) it suffices to show that $\mathfrak{g}^\mathfrak{a}$ annihilates V_α^α. But $\mathfrak{g}^\mathfrak{a}$ is a reductive algebra and so $\mathfrak{g}^\mathfrak{a} = \mathfrak{c} + [\mathfrak{g}^\mathfrak{a}, \mathfrak{g}^\mathfrak{a}]$ where $\mathfrak{c}$ is the center of $\mathfrak{g}^\mathfrak{a}$. Now $[\mathfrak{g}^\mathfrak{a}, \mathfrak{g}^\mathfrak{a}]$ is semisimple and hence acts trivially on the 1-dimensional space V_α^α. On the other hand $\mathfrak{c}$ is an algebra of commuting semisimple elements of $\mathfrak{g}$ and also $\mathfrak{c}$ is $\mathfrak{h}$-stable (since $\mathfrak{a}$ is $\mathfrak{h}$-stable). It follows that $\mathfrak{c} \subset \mathfrak{h}$ and therefore $\mathfrak{c}$ acts on both $\mathbb{C}e$

$(=\mathfrak{g}^{\alpha})$ and V_{α}^{α} by the weight α. But $\mathfrak{c}$ acts trivially on $\mathbb{C}e$ so therefore $\mathfrak{c}$ acts trivially on V_{α}^{α}.

This proves $\dim V_{\alpha}^{\mathfrak{g}^e}[2]=1$ and so (as in the proof of Theorem 6.1(ii)) we conclude that $\mathfrak{g}'$ contains a unique $\mathfrak{g}$-submodule $\mathfrak{q}$ equivalent to V_{α}. But also $\operatorname{rank}\mathfrak{g}<\operatorname{rank}\mathfrak{g}'$ by (5-1) since $V_{\alpha}^{0}\neq 0$. So the pair type of $(\mathfrak{g},\mathfrak{g}')$ must occur in (5-9) and consequently the four cases here are the four cases 186(i)–(iv). (Notice that 186(v) is excluded because it requires that V_{α} occur twice in $\mathfrak{g}'$.) In all cases then $\mathfrak{g}'=\mathfrak{g}+\mathfrak{q}$. This proves (6-7) and also that $\mathfrak{q}=\mathfrak{p}$.

Now by Proposition 3.7 we have (after rescaling e if necessary) $e'=e+v\in\mathfrak{g}'$ where $0\neq v\in\mathfrak{q}$. But $[h,v]=2v$ by (3-7) and so $v\in\mathfrak{q}[2]=\mathfrak{q}^{\alpha}$. Thus $e'\in\mathfrak{g}'^{\alpha}$ and so $\mathbb{C}e'$ is $\mathfrak{h}$-stable. But then Proposition 3.7 and (5-1) imply that $\mathfrak{h}'=\mathfrak{h}'^{e'}+\mathfrak{h}$. Therefore $\mathbb{C}e'$ is $\mathfrak{h}'$-stable and so e' is a root vector corresponding to a root $\beta\in\Delta'$ such that $\beta|_{\mathfrak{h}}=\alpha$. Thus $O'=O'_{\min}$ in the three cases (5-9)(ii),(iii)(iv) where $\mathfrak{g}'$ is simply laced. But also $O'=O'_{\min}$ in case (5-9)(i) by dimension (both orbits are 8-dimensional). Thus β is long in all cases so that (by Proposition 4.4 and Corollary 5.2) $M'\simeq O'_{\min}$ and $X\simeq\overline{O'_{\min}}$.

Finally we determine the degree of the cover $\nu\colon M\to O$. This will also provide some insight into why the Levasseur-Smith case (i) is different from (ii), (iii), and (iv).

Notice that Propositions 1.2 and 3.6 imply that the map $\tau^{-1}(O)\to O$ defined by τ is a G-covering which is G-isomorphic (via ν') to the covering ν. We claim that the degree of ν is equal to the number of *long* roots in Δ' which restrict to α and furthermore the fiber $Z=\tau^{-1}(e)$ is a corresponding set of root vectors in $\mathfrak{g}'$. Indeed clearly $Z=G^e\cdot e'$. But T acts trivially on G^e/G^e_o and so Z lies in $\mathfrak{g}'^{\alpha}$. Furthermore for each $z\in Z$ we find (using Proposition 3.7) that $\mathfrak{g}'=\mathfrak{g}+\mathfrak{g}'^{z}$ and so (as for e') it follows that z is a root vector corresponding to a root $\gamma\in\Delta'$. Then γ is long since $z\in O'_{\min}$. On the other hand if $\gamma\in\Delta'_{\mathscr{L}}$ with a corresponding root vector x'_{γ} and $\gamma|_{\mathfrak{h}}=\alpha$ then (see the proof of Theorem 5.3) $\tau(\mathbb{C}x'_{\gamma})=\mathbb{C}e$ and there exists a a unique scalar $s\in\mathbb{C}^*$ such that $\tau(sx'_{\gamma})=e$. This proves the claim.

Now using (6-7) we find $\mathfrak{g}'^{\alpha}=\mathbb{C}e+\mathbb{C}v$. So there are exactly two roots in Δ' restricting to α. Therefore ν is a double cover in the cases where $\mathfrak{g}'$ is simply laced. In the case where $\mathfrak{g}=\mathfrak{g}(2,\mathbb{C})$ we know (see, e.g., [L-S,2.3] or [H,19.3]) that both a long and a short root restrict to α and so therefore the degree of ν is equal to 1, i.e., $M=O$. But then by Proposition 4.14, (4-1) $\overline{O}$ is not normal.

Remark 6.5. In Theorem 6.4 if $\mathfrak{g}$ is of type B_n then O has Jordan type $(3,1^{2n-3})$ just as in (ii) of Theorem 6.1. These two cases really represent a single example. In [So2] and [So3] there is a real analog of this example for the compact Lie groups $SO(m,1)$ and $SO(m,2)$.

Now $\mathfrak{g}(2,\mathbb{C})$ has exactly four (nonzero) nilpotent orbits and their dimensions are 6, 8, 10, and 12. The orbits of dimension 6 and 8 occurred in shared orbit pairs in the last two theorems. Finally the 10-dimensional orbit appears.

Theorem 6.6. *Let $\mathfrak{g} = \mathfrak{g}(2, \mathbb{C})$ and let M be the simply-connected (6-fold) covering of the 10-dimensional nilpotent orbit O in $\mathfrak{g}$. Then $\mathfrak{g}' = \mathfrak{so}(8, \mathbb{C})$. If $\alpha \in \Delta$ is the highest short root then as $\mathfrak{g}$-modules*

$$R[2] \simeq R[\mathfrak{g}] \oplus V_\alpha \oplus V_\alpha. \tag{6-8}$$

Then $M' \simeq O'$ where O' is the orbit of the highest root vector in $\mathfrak{g}'$ and $X \simeq \overline{O'}$. Furthermore there is a subalgebra $\mathfrak{r} \simeq \mathfrak{so}(7, \mathbb{C})$ between $R[\mathfrak{g}]$ and $R[2]$ and we have a shared orbit triple (M, M^+, M') for the Lie algebras

$$\mathfrak{g}(2, \mathbb{C}) \subset \mathfrak{so}(7, \mathbb{C}) \subset \mathfrak{so}(8, \mathbb{C}) \tag{6-9}$$

where M^+ is the simply-connected (double cover) of the orbit of the short root vector in $\mathfrak{so}(7, \mathbb{C})$.

Proof. Let e be a principal nilpotent element in the subalgebra $\mathfrak{g}_{\mathscr{L}} \simeq \mathfrak{sl}(3, \mathbb{C})$ spanned by $\mathfrak{h}$ and the long root vectors. Then $e \in O$. Indeed $\mathfrak{g}^e = \mathfrak{g}^e_{\mathscr{L}} + \mathfrak{q}^e_1 + \mathfrak{q}^e_2$ where $\mathfrak{q}_1, \mathfrak{q}_2 \subset \mathfrak{g}$ are the $\mathfrak{g}_{\mathscr{L}}$-submodules equivalent to $\mathbb{C}^3$ and $\bigwedge^2 \mathbb{C}^3$ and so clearly $\mathfrak{g}^e$ is 4-dimensional. Hence we can take (h, e, f) to be a principal S-triple in $\mathfrak{g}_{\mathscr{L}}$.

Since the six short roots in Δ are just the weights of $\mathfrak{h}$ on $\mathfrak{q}_1$ and $\mathfrak{q}_2$ we find the eigenvalues of h on the 7-dimensional space V_α are (counting multiplicity) $\pm 2, \pm 2, 0, 0, 0$. But then $V_\alpha[2]$ is 2-dimensional and furthermore $\mathfrak{g}^e$ kills $V_\alpha[2]$ since clearly the eigenvalues of h on $\mathfrak{g}^e$ are all positive. Consequently (by Lemma 2.6) $\mathfrak{g}'$ contains two distinct $\mathfrak{g}$-submodules $\mathfrak{p}_1$ and $\mathfrak{p}_2$ isomorphic to V_α. But then by (5-1) $\operatorname{rank} \mathfrak{g} < \operatorname{rank} \mathfrak{g}'$ and it follows from Theorem 5.9 that $(\mathfrak{g}, \mathfrak{g}')$ has type (5-9)(v) so that $\mathfrak{g}' = \mathfrak{so}(8, \mathbb{C})$. Then $O' = O'_{\min}$ by dimension and so as usual $M' \simeq O'$ and $X \simeq \overline{O'}$.

Furthermore the argument in the proof of Theorem 5.9 shows that there is a Lie subalgebra $\mathfrak{g}^+ = \mathfrak{so}(7, \mathbb{C})$ between $\mathfrak{g}$ and $\mathfrak{g}'$. Then the corresponding orbit $O^+ \subset \mathfrak{g}^+$ is 10-dimensional so that (by dimension) O^+ is the orbit of the short root vector. Also M^+ is simply connected by Proposition 3.4 because M' is simply connected.

Remark 6.7. The fundamental groups of all nilpotent orbits are nearly known; at least tables of the component group of G^e_{ad} have been computed where G_{ad} is the adjoint group—see, e.g., [C]. On the other hand in Theorems 6.1 and 6.6 we can directly determine the fundamental group of O in each case from our construction of M'. In particular in Theorem 6.6 $\pi_1(O) \simeq S_3$ where S_3 is the symmetric group on three letters.

Remark 6.8. Levasseur and Smith already showed in [L-S] that the group G of type G_2 has an open dense orbit on the orbit of the highest root vector in $\mathfrak{so}(8, \mathbb{C})$. In addition Kraft showed in [Kr] that there is a finite surjective map $\overline{O^+} \to \overline{O}$ where $O^+ \subset \mathfrak{so}(7, \mathbb{C})$ and $O \subset \mathfrak{g}(2, \mathbb{C})$ are the 10-dimensional nilpotent orbits. McGovern in [McG, Theorem 4.1] constructed a Dixmier algebra analog of the shared orbit pair of G and $SO(8, \mathbb{C})$. Furthermore Vogan has constructed an analog for unipotent representations. If π denotes the minimal unitary representation of $SO(4, 4)$ (see, e.g., [K4]) then π extends to the outer automorphism group A of $\mathfrak{g}$ and in particular to a group $S \simeq S_3$ which induces A. Vogan shows that a split form G_o of G and S_3 behave like a Howe pair with respect to π and that the restriction of π to G_o decomposes into six irreducible components.

Remark 6.9. By Proposition 3.6 we have in general that $O \subset \tau(O') \subset \overline{O}$. Now in the cases where $\mathfrak{g}'$ is simple and $X \simeq \overline{O'_{\min}}$ then clearly $\tau(O')$ is the complement of the origin in $\overline{O}$. In this way we obtain examples where $\tau(O')$ contains *singular* points of $\overline{O}$.

The results of this section and the previous one give immediately

Theorem 6.10. *The examples in Theorems* 6.1, 6.4, and 6.6 *exhaust all Lie algebra pairs* $(\mathfrak{g}, \mathfrak{g}^+)$ *arising from shared orbit pairs where* $\mathfrak{g}$ *is simple. Every doubly laced simple Lie algebra occurs as* $\mathfrak{g}$. *Furthermore the only cases where* $\mathfrak{g}$ *is simply laced are those where* $\mathfrak{g}^+$ *is doubly laced and then* $\mathfrak{g}$ *is isomorphic to the subalgebra* $\mathfrak{g}^+_{\mathscr{L}}$ *spanned by a CSA of* $\mathfrak{g}^+$ *and the corresponding long root spaces, so that in particular* $\mathfrak{g}$ *and* $\mathfrak{g}^+$ *have the same rank.*

Remark 6.11. In Theorem 6.10 if we include the cases in Example 4.9 then we exhaust all pairs $(\mathfrak{g}, \mathfrak{g}^+)$ arising from shared orbit pairs where $\mathfrak{g}^+$ is simple. Furthermore among these pairs the set where $\operatorname{rank} \mathfrak{g} = \operatorname{rank} \mathfrak{g}^+$ coincides with the set of pairs $(\mathfrak{s}_0, \mathfrak{s})$ where $\mathfrak{s}$ is simple and $\mathfrak{s}_0$ is a proper Lie subalgebra of $\mathfrak{s}$ containing a CSA and all long root vectors.

7. Geometry of automorphism actions

In this section we give some applications to the geometry of M and the symmetry of compact homogeneous spaces of G.

First we resolve a question left over from [B-K1]—we find the rank of the toroidal part of the identity component Q_o of the group of G-automorphisms of M. Recall from [B-K1, Theorem 23] that Q_o is a solvable algebraic group with Levi decomposition $Q_o = T^d \ltimes Q_+$ where T^d is a d-dimensional algebraic torus and Q_+ is the unipotent radical of Q_o.

Theorem 7.1. *The dimension* d *of the toroidal factor of* Q_o *is equal to the number of simple components of* $\mathfrak{g}$. *In particular if* $\mathfrak{g}$ *is simple then* $d = 1$.

Proof. We have $d = \dim \operatorname{Hom}_G(\mathfrak{g}, R[2])$ because of Lemma 2.6 and the fact $d = \dim \mathfrak{g}^{G^\varepsilon}[2]$ from [B-K1, Theorem 23 and Proposition 4(27)]. But then by Corollary 5.8 we have $d = 1$ if $\mathfrak{g}$ is simple. We now get the result for general $\mathfrak{g}$ using Remark 4.1.

Next we consider the "symplectic" automorphisms of X. Even though X is in general singular we will say an automorphism of X is symplectic if the corresponding automorphism of R preserves the Poisson bracket structure. The action of G' on X is symplectic in this sense (by Theorem 3.1). As such G' is maximal—except if $R[1] \neq 0$ and then there is also a translation action of $R[1]$ arising from the Heisenberg Lie algebra $R[1] + \mathbb{C} \cdot 1$. In fact we have a stronger result.

Theorem 7.2. *Assume $R[1] = 0$. Suppose we have an infinitesimal action of a finite-dimensional Lie algebra $\mathfrak{f}$ of holomorphic symplectic vector fields on M which extends the infinitesimal action of $\mathfrak{g}$. Then this extends uniquely to an infinitesimal action of $\mathfrak{f}$ on M' which then integrates to a Hamiltonian algebraic group action on M' given by a semisimple subgroup of G'.*

Proof. First observe that every symplectic holomorphic vector field ξ on M is Hamiltonian. Indeed ξ lifts to a symplectic holomorphic vector field η on the simple-connected cover $\widetilde{M}$ of M. Simple-connectivity insures that η is Hamiltonian with a corresponding Hamiltonian holomorphic function ϕ on $\widetilde{M}$. But then since η descends to M it follows that the functions $H \cdot \phi$ are the same up to addition of constants where H is the group of deck transformations of the cover $\widetilde{M} \to M$. However these constants must then be zero since here H is finite. Therefore ϕ is H-fixed so that ϕ defines a holomorphic function $\overline{\phi}$ on M and then $\xi_{\overline{\phi}} = \xi$.

Now the infinitesimal $\mathfrak{f}$-action defines a Lie algebra $\mathfrak{r}$ of all corresponding holomorphic Hamiltonian functions on M including the constant functions. Then $\mathfrak{r}$ is the central extension of $\mathfrak{f}$ by a 1-dimensional space constructed in [K3] and [So1]. Since the infinitesimal action of $\mathfrak{f}$ contains the infinitesimal action of $\mathfrak{g}$ we have $\mathbb{C} \cdot 1 + R[\mathfrak{g}] \subset \mathfrak{r}$. But then $\mathfrak{r}$ is $\mathfrak{g}$-finite and hence $\mathfrak{r} \subset R$ so that in fact $\mathfrak{r} \subset \mathbb{C} \cdot 1 + R[2]$ by Theorem 2.1(ii). It follows now that the central extension splits and there is a Lie algebra isomorphism $\mathfrak{r}[2] \to \mathfrak{f}$ where $\phi \mapsto \xi_\phi$. Furthermore, by Theorem 2.1(i), $\mathfrak{r}[2]$ is semisimple.

Let F be the subgroup of G' corresponding to $\mathfrak{r}[2]$. The derivation action of $\mathfrak{r}[2]$ on R defines an algebraic infinitesimal action of $\mathfrak{f}$ on X and a corresponding algebraic action of F on X (see the proof of Theorem 3.1). These actions restrict to the Zariski open dense G'-orbit M'.

Corollary 7.3. *The action of any connected Lie group of holomorphic symplectic automorphisms of X that extends the action of G is given by a semisimple subgroup of G'*

Finally we make an application to a classical problem in geometry. If P is a parabolic Lie subgroup of a simple Lie group G it is a solved problem (see [T, D]) to determine the connected component of the full group F of holomorphic automorphisms of the projective variety G/P. It is precisely for the groups G appearing in (i), (ii), and (iii) of Theorem 6.4 that P exists so that F is larger than that given by the action of G. Furthermore in those cases F is given by the action of G'. This statement and an even stronger statement (Corollary 7.6) can be obtained from the classification of pairs $(\mathfrak{g}, \mathfrak{g}')$ and Theorem 7.5 below—basically because any holomorphic automorphism of G/P operates as a symplectic automorphism of the cotangent bundle $T^*(G/P)$.

Proposition 7.4. *Let P be a parabolic subgroup of G. Choose M to be the unique open G-orbit on the cotangent bundle $T^*(G/P)$ so that $O \subset \mathfrak{g}$ is the orbit of a Richardson element in the nilradical of the Lie algebra of P. Then there is a G-equivariant desingularization map*

$$\zeta \colon T^*(G/P) \to X. \tag{7-1}$$

In particular ζ is birational and furthermore ζ is an isomorphism over M. All regular functions on M extend to $T^*(G/P)$, i.e.,

$$R(M) = R(T^*(G/P)). \tag{7-2}$$

Proof. Regard $A = R(T^*(G/P))$ as a subring of $R = R(M)$ by restriction of functions. Then R is integral over A. Indeed the moment map $\mu\colon T^*(G/P) \to \overline{O}$ defines an inclusion of algebras $R(\overline{O}) \subset A$ and R is integral over $R(\overline{O})$ (see the proof of Proposition 1.2). But also R and A are both integrally closed in the function field $K(M)$ by Lemma 1.1 since X is normal and $T^*(G/P)$ is even smooth. Consequently $R = A$ and this defines (7-1). The rest of the result is now immediate.

Theorem 7.5. *Keep the setup of Proposition 7.4. Then the action of G' on X lifts uniquely to an action on $T^*(G/P)$ as a group of symplectic holomorphic automorphisms. In fact as such G' is maximal—any connected Lie group of symplectic holomorphic automorphisms of $T^*(G/P)$ containing the action of G is given by a subgroup of G'. Also $R[1] = 0$. Furthermore G' preserves the cotangent polarization of $T^*(G/P)$ so that G' acts on G/P and consequently there exists a parabolic subgroup $P' \subset G'$ such that $G/P = G'/P'$. Finally the action of G' on G/P is the connected component $(\operatorname{Aut} G/P)_o$ of the group of all holomorphic automorphisms of G/P.*

Proof. $T^*(G/P)$ has a natural $\mathbb{C}^*$-action defined by the Euler action on the cotangent polarization, i.e., the fibers of the natural projection $\pi\colon T^*(G/P) \to G/P$. This $\mathbb{C}^*$-action is the lifting with respect to the moment map of the scaling action on $\overline{O}$. It follows using (7-2) and Lemma 1.3 that $R[2]$ is the space of regular functions on $T^*(G/P)$ which are linear on the cotangent polarization. Consequently the Hamiltonian vector fields ξ_ϕ, $\phi \in R[2]$, define an infinitesimal $\mathfrak{g}'$-action on $T^*(G/P)$ which is then the canonical lift of an infinitesimal $\mathfrak{g}'$-action on G/P.

Now the infinitesimal $\mathfrak{g}'$-action on G/P defines a linear isomorphism $\mathfrak{g}/\mathfrak{p} \to \mathfrak{g}'/\mathfrak{p}'$ where $\mathfrak{p}' \subset \mathfrak{g}'$ is the isotropy algebra of $\mathfrak{g}'$ at the point P/P. It follows that if P' is a connected algebraic subgroup of G' corresponding to $\mathfrak{p}'$ then the natural map $G/P \to G'/P'$ is a Zariski open dense embedding of a projective variety and hence is an isomorphism. Now in fact P' exists. Indeed it is easy to check that $\mathfrak{p}' = \mathfrak{p} + \mathfrak{g}'^{\varepsilon}$ where $\mathfrak{g}'^{\varepsilon}$ is the isotropy algebra of a point $\varepsilon \in M$ such that $\pi(\varepsilon) = P/P$. But then $\mathfrak{g}'^{\varepsilon}$ is the Lie algebra of the isotropy group $G'^{\zeta(\varepsilon)}$ (since G' acts on X and (7-1) is infinitesimally $\mathfrak{g}'$-equivariant). Therefore $\mathfrak{p}'$ is a sum of algebraic Lie subalgebras of $\mathfrak{g}'$ and hence is itself algebraic.

This proves $G/P = G'/P'$ and so P' must be parabolic since it gives a projective quotient. Moreover then G' operates on G/P integrating the infinitesimal $\mathfrak{g}'$-action and hence G' operates on $T^*(G/P)$ integrating the infinitesimal $\mathfrak{g}'$-action.

The maximality of G' follows from Theorem 7.2 as soon we check that $R[1] = 0$. This is clear since Corollary 4.7 and Example 3.5 imply that $Z(G)$ acts nontrivially on $R[1]$ when $R[1] \neq 0$ whereas in our case $Z(G)$ acts trivially on G/P and hence also on M.

Finally $(\mathrm{Aut}\ G/P)_o$ acts by symplectic holomorphic automorphisms on $T^*(G/P)$ so that it must act by a subgroup of G' because of the maximality. Hence $G' = (\mathrm{Aut}\ G/P)_o$.

Corollary 7.6. *Any connected Lie group of symplectic holomorphic automorphisms of $T^*(G/P)$ containing the action of G automatically preserves the cotangent space polarization of $T^*(G/P)$ and consequently acts as a group of holomorphic automorphisms of G/P.*

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References

[B] R. K. Brylinski, *Twisted ideals of the nullcone*, Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory (A. Connes et al., eds.), Birkhäuser, Boston, 1990, pp. 289–316.

[B-K1] R. Brylinski and B. Kostant, *The variety of all invariant symplectic structures on a homogeneous space and normalizers of isotropy groups*, Symplectic Geometry and Mathematical Physics (P. Donato et al., eds.), Birkhäuser, Boston, 1991, pp. 80–113.

[B-K2] ——, *Nilpotent orbits, normality and Hamiltonian group actions*, Bull. Amer. Math. Soc. (N.S.) **26** (1992), 269–275.

[C] R. W. Carter, *Finite groups of Lie type: Conjugacy classes and complex characters*, Wiley-Interscience, Chichester and New York, 1985.

[D] M. Demazure, *Automorphismes et déformations des variétés de Borel*, Invent. Math. **39** (1977), 179–186.

[Ge] S. Gelfand, *The Weil representation of a Lie algebra of type G_2 and related representations of SL_3*, Funktsional. Anal. i Prilozhen. **14** (1980), no. 1, 51–52. (Russian)

[Gr] W. A. Graham, *Functions on the universal cover of the principal nilpotent orbit*, Invent. Math **108** (1992), 15–27.

[H] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, Berlin and New York, 1972.

[K-S] J. Kimura and M. Sato, *A classification of irreducible prehomogeneous spaces and their relative invariants*, Nagoya Math. J. **65** (1977), 1–155.

[K1] B. Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex semisimple Lie group*, Amer. J. Math. **81** (1959), 973–1032.

[K2] ——, *Lie group representations on polynomial rings*, Amer J. Math. **85** (1963), 327–404.

[K3] ——, *Quantization and unitary representations*, Lectures in Modern Analysis and Applications III, Lecture Notes in Math., vol. 170, Springer-Verlag, Berlin and New York, 1970, pp. 87–208.

[K4] ——, *The vanishing of scalar curvature and the minimal representation of $SO(4, 4)$*, Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory (A. Connes et al., eds.), Birkhäuser, Boston, 1990, pp. 85–124.

[Kr] H. Kraft, *Closures of conjugacy classes in G_2*, J. Algebra **126** (1989), 454–465.

[Kr-P] H. Kraft and C. Procesi, *On the geometry of conjugacy classes in classical groups*, Comment. Math. Helv. **57** (1982), 539–602.

[L-S] T. Levasseur and S. P. Smith, *Primitive ideals and nilpotent orbits in type G_2*, J. Algebra **1 14** (1988), 81–105.

[McG] W. M. McGovern, *Dixmier algebras and the orbit method*, Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory (A. Connes et al., eds.), Birkhäuser, Boston, 1990, pp. 397–416.

[M] D. Mumford, *The red book of varieties and schemes*, Lecture Notes in Math., vol. 1358, Springer-Verlag, Berlin and New York, 1988.

[So1] J.-M. Souriau, *Structure des systèmes dynamiques*, Dunod, Paris, 1970.

[So2] ——, *Sur la variété de Kepler*, Sympos. Math., vol. 14, Academic Press, London and New York, 1974, pp. 343–360.

[So3] ——, *Sur la variété de Kepler*, Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. (Proc. IUTAM-ISIMM Modern Developments in Analytical Mechanics) **117(suppl.)** (1983), 369–418.

[S-S] T. A. Springer and R. Steinberg, *Conjugacy classes*, Seminar on Algebraic Groups and Related Finite Groups (A. Borel et al., eds.), Lecture Notes in Math., vol. 131, Springer-Verlag, New York, 1970, pp. 167–266.

[T] J. Tits, *Espaces homogènes complexes compacts*, Comment. Math. Helv. **37** (1962), 111–120.

[V-P] E. B. Vinberg and V. L. Popov, *On a class of quasihomogeneous affine varieties*, Math. USSR-Izv. **6** (1972), no. 4, 743–758.

[V1] D. A. Vogan, *The orbit method and primitive ideals for semisimple Lie algebras*, Lie Algebras and Related Topics, CMS Conf. Proc., vol. 5, Amer. Math. Soc., Providence, RI, 1986, pp. 281–316.

[V2] ——, *Noncommutative algebras and unitary representations*, Proc. Sympos. Pure Math., vol. 48, Amer. Math. Soc., Providence, RI, 1988, pp. 35–60.

[Z] A. Zahid, *Les endomorphismes $\mathfrak{k}$-finis des modules de Whittaker*, Bull. Soc. Math. France **117** (1989), 451–477.

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