JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 7, Number 3, July 1994

THE INVERSE EIGENVALUE PROBLEM FOR REAL SYMMETRIC TOEPLITZ MATRICES

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INTRODUCTION

A *Toeplitz* matrix is one for which the entries are constant on diagonals: $c_{i,j} = c_{i-j}$. Thus Toeplitz matrices are discrete analogues of convolution operators. They are connected to analysis by the trigonometric moment problem and its many ramifications, and to applications by discrete time-invariant linear systems and stationary stochastic processes, which they represent. They have therefore been extensively studied, and far-ranging theory has grown from the problem of inverting them, and from questions about the quadratic forms which they define. Nevertheless, relatively little is known about their spectral properties. In particular, the fundamental question of whether or not a real symmetric Toeplitz matrix can have arbitrary real eigenvalues, which was posed by P. Delsarte and Y. Genin and solved by them for $n \leq 4$ [DG], has been open. The problem is challenging because of its analytic intractability and because the few available examples exhibit multiple solutions that form no apparent pattern [F]. Here we use a nonconstructive method to give an affirmative answer, within a class of Toeplitz matrices having a certain additional regularity. These matrices therefore constitute another canonical form for Hermitian matrices under unitary transformation.

To describe the result more precisely, let a vector $(v_1, \ldots, v_k)$ be called *even* if $v_{1+i} = v_{k-i}$ and *odd* if $v_{1+i} = -v_{k-i}$, $0 \leq i \leq k-1$. Delsarte and Genin showed that each eigenvector of a real symmetric Toeplitz matrix can be taken to be either even or odd, and that the numbers of even and odd eigenvectors differ by one at most, not only for the matrix as a whole, but also for each subspace of eigenvectors corresponding to a multiple eigenvalue. They pointed out that, in consequence, for the inverse eigenvalue problem to have a continuous solution, the association of eigenvectors to eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ must assign eigenvectors of opposite parity to adjacent eigenvalues. We extend this crucial observation by focusing on matrices, termed *regular*, for which all the principal submatrices also have this property. Following S. Friedland [F], we

Received by the editors July 13, 1993.
1991 *Mathematics Subject Classification*. Primary 15A18, 15A21; Secondary 55M25.
Key words and phrases. Toeplitz matrices, inverse eigenvalue problem, regular Toeplitz matrix.

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then consider the topological degree of the map which takes such a matrix to its eigenvalues. Using little more than a prior characterization [SL] of matrices with multiple eigenvalues, we succeed in showing that the degree does not vanish; it follows that the eigenvalues of matrices in this class already attain all possible n-tuples of real numbers. It is remarkable that a conclusion so inaccessible analytically nevertheless can be derived from such simple information.

Notation and preliminaries

A symmetric Toeplitz matrix is determined by its top row, $(t_1, \dots, t_n)$; accordingly, we denote such a matrix by $M_n(t_1, \dots, t_n)$. Limiting consideration to real matrices in which $t_1 = 0$ and $t_2 = 1$, we associate each $M_n(0, 1, t_3, \dots, t_n)$ with the point $(t_3, \dots, t_n)$ of R^{n-2}. Such a matrix has n real eigenvalues, which we label so that $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$; here

$$\sum \lambda_i = \operatorname{trace}(M_n) = 0 ,$$

hence $\lambda_1 < 0$, else all the eigenvalues vanish, and $\lambda_n = -(\lambda_1 + \cdots + \lambda_{n-1})$. We can rescale these so that $\lambda_1 = -1$, and so associate with M_n the normalized eigenvalues $y_i \equiv \lambda_i/|\lambda_1| = \lambda_i/(-\lambda_1)$, satisfying

$$(1) \qquad -1 \le y_2 \le \cdots \le y_{n-1} \le y_n = -(-1 + y_2 + \cdots + y_{n-1}) ,$$

which we view as a point $(y_2, \dots, y_{n-1}) \in R^{n-2}$ lying in the simplex L_n defined by the $n-1$ linear inequalities

$$(2) \qquad \begin{aligned} &-1 \le y_2, \\ &y_i \le y_{i+1}, \qquad 2 \le i \le n-2, \\ &y_2 + y_3 + \cdots + y_{n-2} + 2y_{n-1} \le 1 \end{aligned}$$

that correspond to (1). Let $\Lambda = \Lambda(M_n)$ denote the function which maps a point $(t_3, \dots, t_n)$ onto these normalized eigenvalues $(\lambda_2/|\lambda_1|, \dots, \lambda_{n-1}/|\lambda_1|)$ of $M_n(0, 1, t_3, \dots, t_n)$. Λ is then a map of R^{n-2} into $L_n \subset R^{n-2}$.

We want to show that every set of n real numbers $(y_1, \dots, y_n)$, which we can label in increasing order, is the set of eigenvalues of some real symmetric $n \times n$ Toeplitz matrix. Since the eigenvalues of $M_n + \alpha I$ are a uniform translation by α of those of M_n, we may apply such a shift to the y_i to ensure $\sum y_i = 0$, and again scale by $|y_1| \ne 0$ to produce a point in L_n. Our object therefore is to prove that the map Λ covers all of L_n. We will show that this is so already when Λ is restricted to the subset of regular matrices M_n described earlier. The argument will be based on the topological degree of Λ.

More specifically, on the strength of certain basic facts we will show that the set $\mathcal{T}_n$ of normalized $n \times n$ regular matrices is bounded, with boundary made up entirely of matrices having multiple eigenvalues. The complement of the image of this boundary under the eigenvalue map Λ therefore includes the interior of L_n in a connected component. To determine the topological degree of Λ, as a map of $\mathcal{T}_n$, at some point inside L_n we begin with the boundary point of L_n where the lowest $n-1$ eigenvalues coincide; this corresponds to a unique matrix. Following the action of Λ in a neighborhood of that matrix, we show that it is locally one-to-one along a succession of boundary facets of $\mathcal{T}_n$ of increasing dimension, made up of matrices for which the lowest k eigenvalues coincide, $k = n-2, \ldots, 1$. Ultimately this enables us to show that Λ is one-to-one also at some point interior to L_n; the degree of Λ is then not zero in L_n. Since the degree vanishes at a point outside $\Lambda(\mathcal{T}_n)$, it follows that $\Lambda(\mathcal{T}_n)$ covers L_n.

We begin with some basic properties of Toeplitz matrices. Throughout, M_n and M_k will denote, respectively, $M_n(t_1, t_2, \ldots, t_n)$ without the preceding normalizations, and $M_k(t_1, \ldots, t_k)$, its principal submatrix, $k \leq n$. We will always number the eigenvalues in nondecreasing order. Propositions 1 and 3(d) are known; we include brief proofs for completeness. Proposition 2 represents an elaboration of results in [SL]. In addition, we will frequently invoke the fact that the eigenvalues of M_k interlace those of M_{k+1}. The motion of eigenvalues was studied also in [T1, T2].

Definition. For a vector $\mathbf{v} = (v_1, \ldots, v_k)$, let $\mathbf{v}_{\text{rev}} = (v_k, \ldots, v_1)$ denote the vector obtained by writing the coordinates of $\mathbf{v}$ in *reverse* order; $\mathbf{v}$ is *even* or *odd* as $\mathbf{v} = \pm\mathbf{v}_{\text{rev}}$. We will refer to the property of being even or odd as the *parity* of a vector. When there is no confusion, we will also use terms defined for eigenvalues to apply to the corresponding eigenvectors, and conversely, as for example by speaking of the smallest eigenvector or of an even eigenvalue. We will call an eigenvalue *simple* or *single* if its multiplicity is 1, and *multiple* if it exceeds 1.

Proposition 1. *If $\mathbf{v}$ is an eigenvector of M_n, so is $\mathbf{v}_{\text{rev}}$. If λ is a simple eigenvalue of M_n, then the corresponding eigenvector is either even or odd.*

Proof. It is easy to see that, since M_n is Toeplitz, $(M_n\mathbf{v}_{\text{rev}}) = (M_n\mathbf{v})_{\text{rev}}$; hence if $M_n\mathbf{v} = \lambda\mathbf{v}$ then also $M_n\mathbf{v}_{\text{rev}} = \lambda\mathbf{v}_{\text{rev}}$. If the eigenspace is one-dimensional, $\mathbf{v}_{\text{rev}}$ must be a constant multiple of $\mathbf{v}$, and the constant can be only ± 1; hence $\mathbf{v}$ is even or odd.

Definition. For a vector $\mathbf{v} = (v_1, \ldots, v_k)$, let $\mathbf{v}_{\text{ext}} \equiv (v_1, \ldots, v_k, 0)$ denote $\mathbf{v}$ *extended* by a last component of zero.

Proposition 2. *Let $n_k(\lambda) > 0$ denote the multiplicity of λ as an eigenvalue of M_k, and $E_k(\lambda)$ the corresponding $n_k(\lambda)$-dimensional subspace of eigenvectors. Then*

(a) $n_{k+1}(\lambda) - n_k(\lambda) = \pm 1$;
(b) *the following conditions are equivalent*:
 (i) *for every* $\mathbf{v} \in E_k(\lambda)$, $\mathbf{v}_{\text{ext}} \in E_{k+1}(\lambda)$,
 (ii) $n_{k+1}(\lambda) = n_k(\lambda) + 1$,

(iii) $E_{k+1}(\lambda)$ *contains a vector with nonzero first* (*equivalently, last*) *component*;

(c) *property* (b)(iii) *implies the same for* $E_k(\lambda)$.

Proof. The equivalence asserted in (b)(iii) derives from the reversibility of eigenvectors.

The fact that

$$n_k(\lambda) - 1 \le n_{k+1}(\lambda) \le n_k(\lambda) + 1 \tag{3}$$

is a consequence of the interlacing of eigenvalues of M_k and M_{k+1}, but it also has the following suggestive direct proof; for brevity, we use E_k and n_k to denote $E_k(\lambda)$ and $n_k(\lambda)$. Choose a basis $\{\mathbf{w}^{(i)}\}$, $1 \le i \le n_{k+1}$, of E_{k+1}. If the last component of one of these vectors, say of $\mathbf{w}^{(1)}$, does not vanish (case (b)(iii)), let us replace $\mathbf{w}^{(i)}$ by $\mathbf{w}^{(i)} - \alpha_i \mathbf{w}^{(1)}$, $i \ge 2$, with α_i chosen so that the last component of this linear combination vanishes; these $(n_{k+1} - 1)$ vectors (n_{k+1} if all the last components of the basis vanish) remain linearly independent since $\{\mathbf{w}^{(i)}\}$ are. As their last component vanishes, truncation to size k preserves linear independence and produces eigenvectors of M_k. Thus $n_k \ge n_{k+1} - 1$, and equality, which is (b)(ii), implies (b)(iii). Similarly, let $\{\mathbf{v}^{(i)}\}$, $1 \le i \le n_k$, be a basis for E_k, and extend each $\mathbf{v}^{(i)}$ to $\mathbf{v}^{(i)}_{\text{ext}}$. Then for the first k components, $M_{k+1}\mathbf{v}^{(i)}_{\text{ext}} = \lambda \mathbf{v}^{(i)}_{\text{ext}}$, so $\mathbf{v}^{(i)}_{\text{ext}} \in E_{k+1}$ if and only if the last component of $M_{k+1}\mathbf{v}^{(i)}_{\text{ext}}$ vanishes. Either this happens for every $\mathbf{v}^{(i)}_{\text{ext}}$ or there is $\mathbf{v}^{(1)}$ for which it does not. In the former case, E_{k+1} contains all $\{\mathbf{v}^{(i)}_{\text{ext}}\}$ and so we have (b)(i). In the latter, again on replacing $\mathbf{v}^{(i)}_{\text{ext}}$ by $\mathbf{v}^{(i)}_{\text{ext}} - \beta_i \mathbf{v}^{(1)}_{\text{ext}}$, $i \ge 2$, with β_i chosen so that the last component of $M_{k+1}(\mathbf{v}^{(i)}_{\text{ext}} - \beta_i \mathbf{v}^{(1)}_{\text{ext}})$ vanishes, we obtain $n_k - 1$ linearly independent vectors (n_k for case (b)(i) in E_{k+1}). Thus for either situation $n_{k+1} \ge n_k - 1$, completing the proof of (3). We have also shown that (b)(ii) implies (b)(iii).

We return to (b)(i). Select $\mathbf{v}^{(1)}$ to have the earliest nonzero component among $\mathbf{v} \in E_k$; say it is the jth. Since $\mathbf{v}_{\text{rev}} \in E_k$ for each $\mathbf{v} \in E_k$, all the $\mathbf{v}^{(i)}$ and $\mathbf{v}^{(i)}_{\text{rev}}$ compete for the role of $\mathbf{v}^{(1)}$, hence must all have their first and last $j-1$ components vanish. The extensions $\mathbf{v}^{(i)}_{\text{ext}}$, $1 \le i \le n_k$, remain linearly independent and, by the assumption of (b)(i), all are eigenvectors of M_{k+1}; their last j components vanish. As $\mathbf{v}^{(1)}_{\text{ext}}$ is in E_{k+1}, so is $(\mathbf{v}^{(1)}_{\text{ext}})_{\text{rev}}$, which places the nonvanishing jth component of $\mathbf{v}^{(1)}_{\text{ext}}$ in position $(k+2-j)$, where the $\mathbf{v}^{(i)}_{\text{ext}}$ all vanish. Thus $(\mathbf{v}^{(1)}_{\text{ext}})_{\text{rev}} \in E_{k+1}$ is linearly independent of the $\mathbf{v}^{(i)}_{\text{ext}}$, $1 \le i \le n_k$, and so $n_{k+1} \ge n_k + 1$. By (3), this shows that $n_{k+1} = n_k + 1$. Thus (b)(i) implies (b)(ii), which implies (b)(iii). Moreover, E_{k+1} is spanned by $(\mathbf{v}^{(i)})_{\text{ext}}$ and $(\mathbf{v}^{(1)}_{\text{ext}})_{\text{rev}}$, and as the former all have vanishing last component, (b)(iii) implies that $(\mathbf{v}^{(1)}_{\text{ext}})_{\text{rev}}$ does not. Hence $\mathbf{v}^{(1)}_{\text{ext}}$ has nonzero first component, as therefore does $\mathbf{v}^{(1)} \in E_k$. Consequently (b)(i) also implies the conclusion of (c).

Next we show that (b)(iii) implies (b)(i). For suppose, by (b)(iii), that $\mathbf{w} \in E_{k+1}$ has nonzero last component w_{k+1}. Let $\mathbf{v}$ be any eigenvector in E_k, and denote by μ_{k+1} the last component of $M_{k+1}\mathbf{v}_{\text{ext}}$. Since M_{k+1} is symmetric,

$$(M_{k+1}\mathbf{w}, \mathbf{v}_{\text{ext}}) = (\mathbf{w}, M_{k+1}\mathbf{v}_{\text{ext}}) .$$

By definition of $\mathbf{w}$,

$$(M_{k+1}\mathbf{w}, \mathbf{v}_{\text{ext}}) = \lambda(\mathbf{w}, \mathbf{v}_{\text{ext}}) ,$$

and as the first k components of $M_{k+1}\mathbf{v}_{\text{ext}}$ coincide with those of $M_k\mathbf{v}$, we find that

$$(\mathbf{w}, M_{k+1}\mathbf{v}_{\text{ext}}) = \lambda(\mathbf{w}, \mathbf{v}_{\text{ext}}) + w_{k+1}\mu_{k+1} .$$

Thus $\mu_{k+1} = 0$, whence $\mathbf{v}_{\text{ext}} \in E_{k+1}$, establishing (b)(i). This cycle of implications proves (b) and (c).

Finally, we consider (a). If (b)(ii), which is a subcase of (a), is not satisfied, neither is (b)(iii). Thus the last component of all the vectors of E_{k+1} must vanish. Thereupon, the vectors of any basis of E_{k+1}, when truncated to size k, remain linearly independent, lie in E_k, and are extendible by $(\cdot)_{\text{ext}}$ to E_{k+1}. Since (b)(i) is also false, they cannot include all of E_k; hence $n_{k+1} \le n_k - 1$. By (3), $n_{k+1} = n_k - 1$, completing the proof of (a).

Corollary 1 [I]. *In any block of consecutive values of k for which $n_k(\lambda) > 0$, the sequence of numbers $n_k(\lambda)$ can have at most one relative maximum.*

Proof. It follows from Proposition 2(b) that, if $n_{k+1}(\lambda) \le n_k(\lambda)$, all the vectors of E_{k+1} have vanishing first and last components. By Proposition 2(c), the same must be true for E_{k+2}, and so this condition propagates. Thus once the sequence $\{n_k(\lambda)\}$ starts to decrease, it continues to do so.

Proposition 3. *The following statements describe the behavior of multiple eigenvectors:*

(a) *Suppose $n_{k-1}(\lambda) = 0$ and $n_k(\lambda) = 1$. Then for the eigenvector $\mathbf{v} = (v_1, \dots, v_k)$ of M_k corresponding to λ the component $v_1 \ne 0$, and $n_{k+1}(\lambda) = 2$ if and only if*

$$t_{k+1} = -v_1^{-1}(t_k v_2 + \cdots + t_2 v_k) .$$

With this value of t_{k+1}, the eigenspaces $E_{k+1}(\lambda)$ is spanned by $(v_1, \dots, v_k, 0)$ and $(0, v_1, \dots, v_k)$.

(b) *If $n_j(\lambda)$ increases from $n_k(\lambda) = 1$ to $n_{k+m}(\lambda) = m + 1$, $m \ge 2$, each matrix entry t_{k+q}, $1 \le q \le m$, is determined from the preceding $\{t_i\}$ by the convolution*

$$t_{k+q} = -v_1^{-1}(t_{k+q-1}v_2 + t_{k+q-2}v_3 + \cdots + t_{q+1}v_k) , \tag{4}$$

defined by the components of the eigenvector $\mathbf{v}$ of M_k. The corresponding eigenspace E_{k+q} is spanned by the vectors of length $k + q$ formed by preceding the block $(v_1, \dots, v_k)$ by i zeros, and following it by $q - i$ zeros, $0 \le i \le q$.

In the convolution (4), *when* $l+1 \le q \le m$, *the value of* t_{k+q} *is unchanged when* $\mathbf{v}$ *is replaced by any eigenvector* $\mathbf{w} = (w_1, \dots, w_{k+l}) \in E_{k+l}(\lambda)$ *having nonzero first component, viz.*

$$t_{k+q} = -w_1^{-1}(t_{k+q-1}w_2 + \cdots + t_{q-l+1}w_{k+l}) .$$

(c) *If* t_{k+m+1} *differs from the value given by* (4), *then* $n_{k+m+1}(\lambda) = m$, *and* $n_{k+m+j}(\lambda)$ *decreases monotonically to* $n_{k+2m+1}(\lambda) = 0$. *The eigenspace* E_{k+m+j} *is spanned by the vectors of length* $k+m+j$ *formed by preceding the block* $(v_1, \dots, v_k)$ *by* i *zeros, and following it by* $m+j-i$ *zeros,* $j \le i \le m$.

(d) [DG] $E_{k+q}(\lambda)$ *can be decomposed into the sum of subspaces of even and odd vectors, the dimensions of which differ by* 1 *at most. Moreover, when* $n_{k+q}(\lambda)$ *is odd, the subspace of larger dimension corresponds to vectors having the same parity as* $\mathbf{v}$, *the single eigenvector in* $E_k(\lambda)$.

(e) *If* λ *is the largest eigenvalue of* M_k *then its multiplicity as an eigenvalue of* M_{k-1} *is* $n_{k-1}(\lambda) = n_k(\lambda) - 1$, *and if* $n_k(\lambda) \ge 2$ *then* λ *is also the largest eigenvalue of* M_{k-1}. *The analogous assertion holds true for the smallest eigenvalue.*

Proof. If $n_{k-1}(\lambda) = 0$ and $n_k(\lambda) = 1$, then λ is an eigenvalue of $M_k(t_1, \dots, t_k)$ with eigenvector $\mathbf{v} = (v_1, \dots, v_k)$, but λ is not an eigenvalue of $M_{k-1}(t_1, \dots, t_{k-1})$. Consequently $v_1 \ne 0$, else $(v_2, \dots, v_k)$ is an eigenvector of M_{k-1} with eigenvalue λ, contrary to hypothesis. By Proposition 2(b)(i), $n_{k+1}(\lambda) = 2$ if and only if $\mathbf{v}_{\text{ext}}$ remains an eigenvector of M_{k+1}, that is, providing the last component of $M_{k+1}(\mathbf{v}_{\text{ext}})$ vanishes; this means that

$$t_{k+1} = -v_1^{-1}(t_k v_2 + \cdots + t_2 v_k) .$$

With t_{k+1} so determined, the eigenspace $E_{k+1}(\lambda)$ is spanned by $\mathbf{v}_{\text{ext}} = (v_1, \dots, v_k, 0)$ and $(\mathbf{v}_{\text{ext}})_{\text{rev}} = (0, v_k, \dots, v_1)$; since $\mathbf{v}$ is even or odd, the second of these is equivalent to $(0, v_1, \dots, v_k)$. This establishes (a).

To maintain the increase of $n_k(\lambda)$, we must ensure that each of these basis vectors, when extended by $(\cdot)_{\text{ext}}$, remains an eigenvector of M_{k+2}, but by Proposition 2(b)(iii) it is sufficient to verify this for any vector of $E_{k+1}(\lambda)$ with nonzero first component, in particular for $(v_1, \dots, v_k, 0)$. Once more, this is equivalent to

$$t_{k+2} = -v_1^{-1}(t_{k+1}v_2 + \cdots + t_3 v_k) .$$

Thereupon, E_{k+1} is spanned by $(v_1, \dots, v_k, 0, 0)$, $(0, v_1, \dots, v_k, 0)$, and their reversed versions, equivalently by $(v_1, \dots, v_k, 0, 0)$, $(0, v_1, \dots, v_k, 0)$, and $(0, 0, v_1, \dots, v_k)$, of which only the first has nonzero initial component. Continuing in this way, we see that if $n_j(\lambda)$ increases from 1 to $m+1$ as j increases from $j = k$ to $j = k+m$, then each t_{k+q}, for $1 \le q \le m$, is determined from the preceding t_i by the same convolution

$$t_{k+q} = -v_1^{-1}(t_{k+q-1}v_2 + t_{k+q-2}v_3 + \cdots + t_{q+1}v_k) ,$$

defined by the components of the eigenvector $\mathbf{v}$ of M_k. This expresses the requirement that every eigenvector of $E_{k+q-1}(\lambda)$ extend to one of $E_{k+q}(\lambda)$; as we have seen from Proposition 2(b)(iii), any eigenvector $\mathbf{w}$ of $E_{k+q-1}(\lambda)$ having a nonzero initial component can replace $\mathbf{v}$ for this purpose. The corresponding eigenspace E_{k+q} is spanned as described in (b).

Thereafter, if t_{k+m+1} is not given by the convolution (4), the eigenvalue multiplicity $n_{k+m+1}(\lambda) = n_{k+m}(\lambda) - 1 = m$, and by Corollary 1 it continues to decrease for succeeding indices, regardless of the value of the corresponding t_{k+m+j}, $2 \leq j \leq m$; each of the eigenspaces E_{k+m+j}, $1 \leq j \leq m$, of dimension $m + 1 - j$, is formed from the preceding one by removing the basis vector having the fewest leading zeros and extending the remaining vectors by an appended component of zero, so is spanned as described in (c).

We can generate the even and odd subspaces of $E_{k+q}(\lambda)$ by forming pairs $\{\mathbf{v}^{(i)} \pm \mathbf{v}^{(i)}_{\text{rev}}\}$, with $\{\mathbf{v}^{(i)}\}$ the basis vectors for $E_{k+q}(\lambda)$ described in (a), (b), or (c), consisting of $\mathbf{v}$ preceded and followed by suitable blocks of zeros; $\{\mathbf{v}^{(i)}_{\text{rev}}\}$ are also in this collection. For each i, the span of these even and odd vectors includes $\mathbf{v}^{(i)}$ and $\mathbf{v}^{(i)}_{\text{rev}}$. If the numbers of leading and trailing zero components for $\mathbf{v}^{(i)}$ differ, $\mathbf{v}^{(i)}$ and $\mathbf{v}^{(i)}_{\text{rev}}$ are linearly independent, so account for distinct members of the basis. Thus there are $[n_{k+q}(\lambda)/2]$ such pairs ([ ] denoting the integer part) and the dimension of each of the subspaces constructed from them can be no greater than this number. However, the span of these subspaces includes all of the $\mathbf{v}^{(i)}$ used, hence has dimension no smaller than $2[n_{k+q}(\lambda)/2]$. Consequently each subspace must have dimension $[n_{k+q}(\lambda)/2]$. When $n_{k+q}(\lambda)$ is odd, there remains the vector $\mathbf{v}^{(i)}$, consisting of $\mathbf{v}$ preceded and followed by the same number of zeros, for which $\mathbf{v}^{(i)} \pm \mathbf{v}^{(i)}_{\text{rev}}$ generate only a single nonzero vector, having the same parity as $\mathbf{v}$. Because the span now includes the additional $\mathbf{v}^{(i)}$, this vector must increase the dimension of its subspace. This establishes (d).

Finally, let λ be the largest eigenvalue of M_k and have multiplicity exceeding 1. By Proposition 2(a), λ is also an eigenvalue of M_{k-1}, and, from the interlacing of eigenvalues of these matrices, λ is the largest eigenvalue of M_{k-1}. The interlacing shows also that $n_{k-1}(\lambda)$ cannot exceed $n_k(\lambda)$ even when $n_k(\lambda) = 1$. The same considerations apply to the smallest eigenvalue of M_k. This completes the proof of Proposition 3.

Regular Toeplitz matrices

From Proposition 3(d) [DG, p. 204] the authors drew the important conclusion that for the existence of a continuous map from ordered eigenvalues $(\lambda_1, \ldots, \lambda_n)$ to Toeplitz matrices it is necessary that the eigenvectors corresponding to successive eigenvalues alternate in parity. For otherwise, if both λ_j and λ_{j+1}, say, correspond to even eigenvectors, a smooth deformation of the eigenvalue sequence which makes λ_j and λ_{j+1} coincide could not be accompanied by a similarly smooth deformation of the matrices, since one of the eigenvectors corresponding to the multiple eigenvalue is necessarily odd. We amplify this crucial observation by restricting attention to matrices $M_n(t_1, \ldots, t_n)$ in which not only M_n, but also each principal submatrix $M_k(t_1, \ldots, t_k)$, has this property.

Definition. A simple eigenvalue is termed *even* or *odd* according to the parity of its eigenvector. A matrix $M_n(t_1, \ldots, t_n)$ is *regular* provided it, and every submatrix $M_k(t_1, \ldots, t_k)$, $1 \le k \le n$, has the property that its eigenvalues are distinct and alternate being even and odd, with the largest one even. Let $\mathcal{T}_n$ denote the set of regular matrices $M_n(0, 1, t_3, \ldots, t_n)$.

D. Slepian has proved that $A_n \equiv M_n(1, \rho, \ldots, \rho^{n-1})$ has even eigenvectors with components $v_j = \cos(n+1-2j)\theta$, $1 \le j \le n$, for θ a solution of $\tan n\theta = [(1-\rho)/(1+\rho)]\cot\theta$, $0 < \theta < \pi/2$, and odd eigenvectors with components $v_j = \sin(n+1-2j)\theta$, for θ a solution of $\tan n\theta = -[(1+\rho)/(1-\rho)]\tan\theta$, $0 < \theta < \pi/2$. The corresponding eigenvalues are $\lambda = (1-\rho^2)/(1-\rho\cos 2\theta + \rho^2)$. This shows explicitly that A_n is regular. We can also verify this without calculation by considering T_n, the inverse of $-A_n$, which is tridiagonal and extendible to a Jacobi matrix. Viewing this matrix as a three-term recursion which defines a sequence $\{P_i(x)\}$ of orthogonal polynomials [A, p. 5], the eigenvalues λ_i of T_n are the zeros of $P_n(x)$, with eigenvectors $(1, P_1(\lambda_i), \ldots P_{n-1}(\lambda_i))$; by Proposition 1, these are even or odd according to the sign of $P_{n-1}(\lambda_i)$. Since the zeros of orthogonal polynomials are distinct and interlace, the eigenvectors of T_n, equivalently those of A_n, alternate parity in the way required for A_n to be regular. Thus the set of regular matrices is not empty.

We henceforth return to the normalization $t_1 = 0$, $t_2 = 1$. Since single eigenvectors and eigenvalues deform smoothly under variations of the matrix, the set of regular matrices corresponds to an open set of points $(t_3, \ldots, t_n) \in R^{n-2}$. To simplify notation, we will identify points $(t_3, \ldots, t_n)$ with the corresponding matrices $M_n(0, 1, t_3, \ldots, t_n)$ and refer to them interchangeably. We now describe the geometry of $\mathcal{T}_n$.

Lemma 1. (a) *For $k > 2$, $\mathcal{T}_k$ corresponds to a set in R^{k-2} coordinatized by $(t_3, \ldots, t_k)$ for which $(t_3, \ldots, t_{k-1}) \in \mathcal{T}_{k-1}$ and whose boundary consists entirely of the closure of (portions of) the $(k-1)$ surfaces B_i, defined as functions over $\mathcal{T}_{k-1}$ by*

$$B_i:\ t_k = -(v_1^{(i)})^{-1}(t_{k-1}v_2^{(i)} + t_{k-2}v_3^{(i)} + \cdots + t_3 v_{k-2}^{(i)} + v_{k-1}^{(i)}),$$

with $\mathbf{v}^{(i)} = (v_1^{(i)}, v_2^{(i)}, \ldots, v_{k-1}^{(i)})$ the ith eigenvector of the matrix $M_{k-1}(0, 1, t_3, \ldots, t_{k-1})$, $1 \le i \le k-1$. The points of $\overline{B}_i$ represent those matrices for which $\lambda_i = \lambda_{i+1}$.

(b) *The set $\mathcal{T}_k$ lies above B_i if $\mathbf{v}^{(i)}$ is even, and below B_i if $\mathbf{v}^{(i)}$ is odd.*

(c) *If* M_k *corresponds to a boundary point of* $\mathcal{T}_k$, *then, for each of its eigenvalues,* $n_k(\lambda) = n_{k-1}(\lambda) + 1$.

(d) *The closure* $\overline{\mathcal{T}}_k$ *is compact.*

Proof. We proceed by induction. $M_2(0, 1)$ has eigenvectors $(1, 1)$ and $(1, -1)$ corresponding to eigenvalues ± 1. $M_3(0, 1, t_3)$ has odd eigenvector $(1, 0, -1)$ with eigenvalue $-t_3$; its even eigenvalues are $(t_3 \pm \sqrt{t_3^2 + 8})/2$. Thus $M_3(0, 1, t_3)$ is regular provided

$$t_3 - \sqrt{t_3^2 + 8} < -2t_3 < t_3 + \sqrt{t_3^2 + 8} ,$$

that is, for $-1 < t_3 < 1$. The statement concerning the boundary behavior is immediate. This verifies the lemma for $\mathcal{T}_3$.

If $(t_3, \dots, t_k) \in \mathcal{T}_k$, then $(t_3, \dots, t_{k-1}) \in \mathcal{T}_{k-1}$ by definition. $\mathcal{T}_k$ can therefore be pictured as a subset of the cylinder in R^{k-2} obtained by restricting $(t_3, \dots, t_{k-1})$ to the base $\mathcal{T}_{k-1}$, and letting t_k be unconstrained. We now consider the $(k-3)$-dimensional boundary of the $(k-2)$-dimensional set $\mathcal{T}_k$. One part of this is accounted for by the matrices $M_k(0, 1, t_3, \dots, t_k)$ having a double eigenvalue, but other possible boundaries are in the cylindrical wall, that is, consist of matrices for which $M_{k-1}(0, 1, t_3, \dots, t_{k-1})$ lies in the $(k-4)$-dimensional boundary of $\mathcal{T}_{k-1}$, with arbitrary t_k. We will eliminate the latter possibility.

Starting with $M_k \in \mathcal{T}_k$, we examine first the effect on a matrix $M_k \in \mathcal{T}_k$ of varying t_k, while keeping $M_{k-1} \in \mathcal{T}_{k-1}$ fixed. Suppose $\mathbf{w} = (w_1, \dots, w_k)$ is an eigenvector of M_k with eigenvalue λ. If $0 = w_1 = \pm w_k$, then $(w_2, \dots, w_{k-1}, 0)$ and $(0, w_2, \dots, w_{k-1})$ form linearly independent eigenvectors of M_{k-1}, a contradiction since $M_{k-1} \in \mathcal{T}_{k-1}$ has only simple eigenvalues. Thus $w_1 \neq 0$. On replacing t_k by $t_k \pm \epsilon$, with $\epsilon > 0$, we obtain

$$M_k^\epsilon = M_k(0, 1, t_3, \dots, t_k) \pm \epsilon M_k(0, \dots, 0, 1), \qquad \epsilon > 0. \tag{5}$$

We know that M_k^ϵ has an eigenvector $\mathbf{w}(\epsilon)$ near $\mathbf{w}$, of the same parity, and corresponding neighboring eigenvalue $\lambda(\epsilon)$. More precisely, by the theory of analytic (in ϵ) perturbations of symmetric matrices [K, p. 120; RN, p. 376], there exists a vector $\mathbf{u}$ and a constant μ such that

$$\mathbf{w}(\epsilon) = \mathbf{w} + \epsilon \mathbf{u} + O(\epsilon^2), \qquad \lambda(\epsilon) = \lambda + \epsilon\mu + O(\epsilon^2). \tag{6}$$

By substituting (6) into (5), and collecting terms of order ϵ, we find

$$\mu = \left.\frac{d\lambda(\epsilon)}{d\epsilon}\right|_{\epsilon=0+} = (\pm M_k(0, \dots, 0, 1)\mathbf{w}, \mathbf{w})/\|\mathbf{w}\|^2. \tag{7}$$

Since

$$(M_k(0, \dots, 0, 1)\mathbf{w}, \mathbf{w}) = \pm 2w_1^2/\|\mathbf{w}\|^2 , \tag{8}$$

according as $\mathbf{w}$ is even or odd, we see that, as t_k increases (that is, when $+$ is chosen in (5) and (7)), the even eigenvalues of M_k increase monotonically and the odd ones decrease monotonically. M_k reaches the boundary of $\mathscr{T}_k$ when, in this process, two eigenvalues coincide; since $M_{k-1} \in \mathscr{T}_{k-1}$, the eigenvectors of M_{k-1} are simple, their first components do not vanish, and the explicit formula of (a) comes from Proposition 3(a). The value of restricting M_{k-1} to $\mathscr{T}_{k-1}$ lies in the fact that the eigenvectors $\mathbf{v}^{(i)}$ of M_{k-1} are distinct, smooth, unambiguously determined functions of $(t_3, \ldots, t_{k-1})$, so also this formula, having the form $t_k = f(t_3, \ldots, t_{k-1})$, defines a smooth surface B_i over $\mathscr{T}_{k-1}$. This establishes (a) when $M_{k-1} \in \mathscr{T}_{k-1}$.

If $M_k \in B_i$ with $M_{k-1} \in \mathscr{T}_{k-1}$, it has a double eigenvalue $\lambda_i = \lambda_{i+1}$, with a corresponding even and odd eigenvector, having nonzero first components, and the perturbed matrix $M_k(\epsilon)$ has a neighboring even and odd eigenvalue $\lambda_e(\epsilon)$ and $\lambda_o(\epsilon)$. On comparing the parity of eigenvalues of M_k and M_{k-1}, we see that for $M_k^{(\epsilon)}$ to lie in $\mathscr{T}_k$ we require $\lambda_o(\epsilon) < \lambda_e(\epsilon)$, hence an increase in t_k when $\mathbf{v}^{(i)}$ is even, and the reverse inequality, hence a decrease in t_k when $\mathbf{v}^{(i)}$ is odd. Once t_k crosses the boundary surface B_i, the alternation of eigenvalues required for M_k cannot be restored without another exchange of even and odd eigenvalues between λ_i and λ_{i+1}, but this coincidence occurs only on B_i. Thus $\mathscr{T}_k$ lies entirely on one side of each B_i, in the manner described by (b).

We now consider a sequence of matrices $M_k^{(n)} \in \mathscr{T}_k$ for which $M_{k-1}^{(n)}$ approaches a matrix M_{k-1}^* in the boundary of $\mathscr{T}_{k-1}$. By choosing a subsequence of $M_{k-1}^{(n)}$ if necessary, we may suppose that each eigenvector of $M_{k-1}^{(n)}$, when normalized to unit length, also converges. By the induction hypothesis (a), M_{k-1}^* has a multiple eigenvalue λ, and $n_{k-1}(\lambda) > n_{k-2}(\lambda)$ by (c). It follows from Proposition 2(b)(iii) that the eigenspace $E_{k-1}^*(\lambda)$ has a vector with nonzero first component, and from the construction proving Proposition 3(d) that both the even and the odd subspaces of $E_{k-1}^*(\lambda)$ do also. The eigenvectors of $M_{k-1}^{(n)}$ corresponding to the eigenvalues which approach λ are mutually orthogonal, include even and odd vectors, and converge to some orthonormal basis for $E_{k-1}^*(\lambda)$. At least one of these even eigenvectors must have its first component converge to a nonzero value, since the even subspace of $E_{k-1}^*(\lambda)$ contains such a vector; say it is the ith eigenvector of $M_{k-1}^{(n)}$, and denote its limit by $\mathbf{v}_e$. It follows from (a) that the corresponding lower bounding surface B_i of $\mathscr{T}_k$ converges along the sequence $M_{k-1}^{(n)}$, and its height at M_{k-1}^* is given by (a), with $\mathbf{v}^{(i)} = \mathbf{v}_e \in E_{k-1}^*(\lambda)$, having nonzero first component. By the identical argument, some jth, odd, eigenvector of $M_{k-1}^{(n)}$ converges to $\mathbf{v}_o \in E_{k-1}^*(\lambda)$, having nonzero first component, and the corresponding upper bounding surface B_j of $\mathscr{T}_k$ has height at M_{k-1}^* given by (a) with $\mathbf{v}^{(j)} = \mathbf{v}_o$. However, by Proposition 3(b), these two values coincide; hence t_k is determined as the intersection of B_i and B_j at M_{k-1}^*. As t_k depends only on $E_{k-1}^*(\lambda)$, the choice of sequence $M_{k-1}^{(n)} \to M_{k-1}^*$ plays no role, and so the surfaces B_i and B_j are continuous at M_{k-1}^*, with limit t_k there. This shows that they account for the boundary of $\mathscr{T}_k$ also over $\overline{\mathscr{T}_{k-1}}$, and completes the proof of (a).

Let λ be an eigenvalue of $M_k \in \overline{\mathscr{T}_k}$. If $n_{k-1}(\lambda) = 0$, then (c) is automatically true. Otherwise, by Proposition 2(a), $n_{k-1}(\lambda) \geq 2$; hence M_{k-1} lies on the boundary of $\mathscr{T}_{k-1}$, and the argument just given shows that t_k is determined to satisfy (a). By Proposition 3(b), this choice of t_k produces an increase in the multiplicity, and so establishes (c).

Finally, since $\overline{\mathscr{T}_{k-1}}$ is compact by the induction hypothesis, and t_k is bounded there, $\overline{\mathscr{T}_k}$ remains compact. This concludes the proof of Lemma 1.

To illustrate for $n = 4$, $\mathscr{T}_3$ is the interval $|t_3| < 1$, and $\mathscr{T}_4$ is bounded by the curves

$$B_1 : \ t_4 = -1 + \left(t_3^2 + t_3\sqrt{t_3^2 + 8}\right)/2, \qquad 0 \leq t_3 < 1 \ ,$$

$$B_2 : \ t_4 = 1 \ ,$$

$$B_3 : \ t_4 = -1 + \left(t_3^2 - t_3\sqrt{t_3^2 + 8}\right)/2, \qquad -1 < t_3 \leq 0 \, .$$

We remark that this example is atypical, in that, for $k \geq 5$, the projection of $\mathscr{T}_k$ onto $t_k = 0$ is strictly smaller than $\mathscr{T}_{k-1}$; that is, not all matrices of $\mathscr{T}_{k-1}$ are extendible to $\mathscr{T}_k$. In particular, if $M_{k-1} \in \overline{\mathscr{T}_{k-1}}$ has two different multiple eigenvalues, the heights t_k determined in the preceding construction by the corresponding eigenspaces of M_{k-1} are not likely to coincide, and if they differ there can be no matrix in $\mathscr{T}_k$ above M_{k-1}.

The eigenvalue map

Following S. Friedland [F], we next focus on the topological degree [S, B] of the normalized eigenvalue map Λ. This map takes $\mathscr{T}_k \subset R^{k-2}$ into $L_k \subset R^{k-2}$, and, by Lemma 1(a), the boundary $\partial\mathscr{T}_k$ into the hyperplane boundaries of the simplex L_k. The topological degree, defined for a point disjoint from $\Lambda(\partial\mathscr{T}_k)$, the image of the boundary, measures the number of times the point is covered by the map Λ acting on $\mathscr{T}_k$, counted with orientation. Like the winding number in two dimensions, it is known to remain constant in each connected component of the complement of $\Lambda(\partial\mathscr{T}_k)$, hence in particular in L_k. At a point not covered by $\Lambda(\mathscr{T}_k)$, the degree is zero. Therefore to prove that all of L_k is covered, it is sufficient to exhibit a single point in L_k where the degree is not zero. We do not do this explicitly, not knowing enough about Λ to invert it anywhere in L_k. Instead, we proceed indirectly, beginning with a special boundary point of L_k.

Let $M_k(0, 1, t_3, \ldots, t_k)$ have eigenvalues $\lambda_1 = \lambda_2 = \cdots = \lambda_{k-1} = -1$, $\lambda_k = k - 1$; thus $n_k(-1) = k - 1$. By Proposition 3(e), (b) we then find that $n_i(-1) = i - 1$, for $2 \leq i \leq k$; hence every t_i, $3 \leq i \leq k$, is given by (4), with $\mathbf{v}$ the eigenvector $(1, -1)$ of $M_2(0, 1)$ corresponding to $\lambda = -1$. We conclude that M_k is unique, with $t_3 = \cdots = t_k = 1$; it lies in the boundary of $\mathcal{T}_k$, being the limit of $M_k(0, 1, \rho, \dots, \rho^{k-2})$ as $\rho \to 1$. This is not yet useful for our purpose, because it is only a boundary point of $\mathcal{T}_k$, and one at which Λ is not differentiable. However, starting with this matrix, whose first $k-1$ eigenvalues coincide, we will perturb it, freeing one eigenvalue at a time. We will be able to parametrize this sequence of boundary facets of $\mathcal{T}_k$ in a way which will allow us to follow Λ from the boundary of $\mathcal{T}_k$ to its interior, and to show that in this progression Λ is locally one-to-one. (For example, for $n = 4$, we find how Λ extends from the point $t_3 = t_4 = 1$, first along a segment of B_1 which is mapped onto a segment of $\lambda_2 = -1$ near $(-1, -1)$, and thence to the interior of $\mathcal{T}_4$.) Finally, the uniqueness of the starting point will show that Λ, as a map of $\mathcal{T}_k$, must also be globally one-to-one at some nearby point inside L_k, and at such a point the degree does not vanish.

Definition. A matrix $M_k(0, 1, t_3, \dots, t_k)$ is *boundary-regular* if, for each $j \le k$, the eigenvalues of $M_j(0, 1, t_3, \dots, t_j)$ are single except for a possibly multiple lowest eigenvalue, and eigenvectors can be chosen in that eigenspace so that all the eigenvectors of M_j alternate in parity in the manner required of $\mathcal{T}_k$.

The definition of a boundary-regular matrix differs from that of a regular one only in allowing a multiple lowest eigenvalue.

Lemma 2. *Let M_k be a boundary-regular matrix with lowest eigenvalue of multiplicity m. If $m = 1$, M_k is regular. If $m \ge 2$, then M_k is the endpoint of an arc $M_k^{(\epsilon)}$, $\epsilon \to 0+$, of boundary-regular matrices with lowest eigenvalue of multiplicity $m - 1$.*

Proof. Suppose M_k is boundary-regular with single lowest eigenvalue λ. Then by Proposition 3(e), the lowest eigenvalue μ of M_{k-1} must be larger than λ. Since M_k is boundary-regular, $n_k(\mu) \le 1$; hence $n_{k-1}(\mu) \le 2$ by Proposition 2(a). However, if $n_{k-1}(\mu) = 2$, then $n_k(\mu) = 1$, and $n_{k-2}(\mu) = 1$ by Proposition 3(e). If $(v_1, \dots, v_{k-2})$ is the eigenvector of M_{k-2} corresponding to μ, then by Proposition 3(a), (c) that of M_k is $(0, v_1, \dots, v_{k-2}, 0)$, hence has the same parity. However, the alternation of parity required of the eigenvalues of M_k and M_{k-2}, which are single, shows that the lowest eigenvalue of M_{k-2} and the second-lowest of M_k must have opposite parities. This contradiction shows that $n_{k-1}(\mu) = 1$, so that M_{k-1} also has a single lowest eigenvalue. By iterating this argument, the same is true of each M_j, $j \le k$, hence $M_k \in \mathcal{T}_k$.

Suppose $n_k(\lambda) = m \ge 2$. By Proposition 3(e), M_{k-m+1} has λ as a single lowest eigenvalue with corresponding eigenvector $\mathbf{v}$, and by Proposition 3(b) each t_j, $k - m + 2 \le j \le k$, is determined by (4) from $\mathbf{v}$. As we have seen, $M_{k-m+1} \in \mathcal{T}_{k-m+1}$; hence M_{k-m+2} lies in the boundary surface of $\mathcal{T}_{k-m+2}$ defined by $\mathbf{v}$ in Lemma 1(a). It follows from Lemma 1(b) that the perturbation

$$t_{k-m+2}^{(\epsilon)} = t_{k-m+2} \pm \epsilon ,$$

$\epsilon > 0$, with the sign positive or negative as $\mathbf{v}$ is even or odd, respectively, will move M_{k-m+2} to $M_{k-m+2}^{(\epsilon)} \in \mathcal{T}_{k-m+2}$, thereby separating the double lowest eigenvalue λ of M_{k-m+2} into $\lambda^-(\epsilon) < \lambda^+(\epsilon)$; by definition of $\mathcal{T}_{k-m+2}$ and the interlacing of eigenvalues with those of M_{k-m+1}, the parity of $\lambda^-(\epsilon)$ is opposite to that of $\mathbf{v}$. We now define $t_j(\epsilon)$, $k-m+3 \le j \le k$, by (4), using the eigenvector $\mathbf{w}(\epsilon)$ of $M_{k-m+2}^{(\epsilon)}$ corresponding to $\lambda^-(\epsilon)$. This produces a matrix $M_k^{(\epsilon)}$ with lowest eigenvalue $\lambda^-(\epsilon)$ of multiplicity $(m-1)$, and an eigenvalue $\lambda_m(\epsilon)$ for which, by the interlacing property,

$$\lambda_m(\epsilon) \ge \lambda^+(\epsilon) . \tag{9}$$

We now show that $M_k^{(\epsilon)}$ approaches M_k as $\epsilon \to 0^+$, and that for all ϵ sufficiently small, $M_k^{(\epsilon)}$ is boundary-regular.

The lowest eigenvector $\mathbf{w}(\epsilon)$ of $M_{k-m+2}^{(\epsilon)}$ has for its limit as $\epsilon \to 0+$ that one of the two lowest eigenvectors of M_{k-m+2} which has the parity of $\mathbf{w}(\epsilon)$, specifically, the one of opposite parity to $\mathbf{v}$; by Proposition 3(b), the entries t_j of M_k, for $k-m+3 \le j \le k$, are also definable by (4) using this eigenvector. Thus $M_k^{(\epsilon)}$ approaches M_k as $\epsilon \to 0+$. Now for ϵ sufficiently small, eigenvectors of M_j are perturbed to corresponding eigenvectors of $M_j^{(\epsilon)}$ without a change of parity or increase of multiplicity; hence the single eigenvalues of M_j yield neighboring single eigenvalues of the same parity for $M_j^{(\epsilon)}$, while the single eigenvalue split from the block of lowest eigenvalues of M_j, $k-m+2 \le j \le k$, is the next-to-smallest in $M_j^{(\epsilon)}$. We now determine the parity of this eigenvalue; in order to maintain the alternation required for boundary-regularity of $M_k^{(\epsilon)}$, it must be that of $\mathbf{v}$ for each j.

The multiple block of eigenvectors of M_j is generated, as in Proposition 3(b), by $\mathbf{v}$, while that of $M_j^{(\epsilon)}$ comes from $\mathbf{w}(\epsilon)$, of opposite parity. By Proposition 3(d), when the block of M_j has odd multiplicity, its subspace of vectors with the parity of $\mathbf{v}$ has the larger dimension, whereas in the block remaining in $M_j^{(\epsilon)}$ even and odd vectors have equal dimension; consequently, the eigenvector removed from the block by $M_j^{(\epsilon)}$ has the parity of $\mathbf{v}$. Similarly, if the block of M_j has even multiplicity, then that remaining in $M_j^{(\epsilon)}$ has an excess of vectors with the parity of $\mathbf{w}(\epsilon)$, opposite to $\mathbf{v}$; hence again the eigenvector removed has the parity of $\mathbf{v}$. This shows that $M_k^{(\epsilon)}$ is boundary-regular, completing the proof of Lemma 2.

Corollary 2. *A boundary-regular matrix* M_k *lies in* $\overline{\mathcal{T}}_k$.

Proof. By Lemma 2, we can approximate M_k arbitrarily closely by boundary-regular matrices for which the smallest eigenvalue has lower multiplicity. On iterating this, we can reduce the multiplicity to 1, obtaining approximants in $\mathcal{T}_k$. Thus $M_k \in \overline{\mathcal{T}}_k$.

Definition. Let l_j be the j-dimensional boundary face

$$-1 = \lambda_1 = \lambda_2 = \cdots = \lambda_{k-j-1}$$

of L_k. With k fixed, denote by S_j, $0 \le j \le k-3$, the subset of $\overline{\mathcal{T}}_k$ sent by Λ to points of l_j. Let Λ_j represent the restriction of Λ to S_j.

Identifying S_{k-2} with $\overline{\mathcal{T}}_k$ for consistency, we see that $S_j \subset S_{j+1}$, and that S_j, for $j < k-2$, lies in the boundary of $\mathcal{T}_k$. In terms of matrices, S_j corresponds to those for which the lowest $k-j-1$ eigenvalues coincide; hence

$$S_j = \overline{\mathcal{T}}_k \bigcap_{i=1}^{k-j-2} \overline{B}_i.$$

Lemma 3. *For each j, $1 \le j \le k-2$, S_j contains a subset O_j of boundary-regular matrices, which is open in S_j and includes a neighborhood of O_{j-1} in S_j, such that Λ_j, construed as a map of S_j into l_j, is locally one-to-one in $\overline{O}_j$, with nonvanishing Jacobian.*

Proof. S_{j-1} serves as a boundary of S_j. We will proceed inductively, by extending Λ_{j-1} locally from S_{j-1} into S_j. Accordingly, we start with $O_0 = S_0$ which, as we have seen, is the single point $(1, 1, \ldots, 1)$ corresponding to the boundary-regular $M_k(0, 1, \ldots, 1)$, and we consider S_1 in the neighborhood of its boundary point O_0. Since for a matrix in S_1 the multiplicity of the smallest eigenvalue is $(k-2)$, it follows as before from Proposition 3(e), (b) that the matrix consists of the extension (4) of some matrix of $\mathcal{T}_3$ near $M_3(0, 1, 1)$, using for $\mathbf{v}$ the smallest eigenvector. This is exactly the construction of Lemma 2, applied to the boundary-regular $M_k(0, 1, \ldots, 1)$. We conclude that, sufficiently near S_0, the set S_1 is an arc, parametrized by the boundary-regular $M_k^{(\epsilon)}$, $0 < \epsilon < \epsilon_0$, formed by (4) from the smallest eigenvector $\mathbf{w}(\epsilon)$ of

$$M_3^{(\epsilon)} = M_3(0, 1, 1) - \epsilon M_3(0, 0, 1) ,$$

which is even since $M_3^{(\epsilon)} \in \mathcal{T}_3$; the negative sign in the perturbation is dictated by Lemma 1(b), and implies that S_1 departs from S_0 in one direction only. To describe the resulting behavior of eigenvalues, we again appeal to the basic properties of perturbations. Let $\mathbf{v}_e$ and $\mathbf{v}_o$, with first component $v_1 \neq 0$, be the even and odd vectors in the lowest eigenspace of $M_3(0, 1, 1)$. The perturbed $\mathbf{w}(\epsilon) = \mathbf{w}_e(\epsilon)$ is differentiable in ϵ, and we know its first component to be bounded away from 0. Consequently the entries $t_j(\epsilon)$ of $M_k^{(\epsilon)}$, defined successively by (4) from $\mathbf{w}(\epsilon)$, are likewise differentiable in ϵ. Since both sides of (9) approach the same limit as $\epsilon \to 0+$, for the eigenvalue $\lambda_{k-1}(\epsilon)$ of $M_k^{(\epsilon)}$ we find, by (9) and (7),

$$\left.\frac{d\lambda_{k-1}(\epsilon)}{d\epsilon}\right|_{\epsilon=0+} \ge \left.\frac{d\lambda^+(\epsilon)}{d\epsilon}\right|_{\epsilon=0+} = \frac{2v_1^2}{\|\mathbf{v}_o\|^2}.$$

The normalized map Λ takes $M_k^{(\epsilon)}$ into $(-1, \ldots, -1, \lambda_{k-1}(\epsilon)/(-\lambda^-(\epsilon)))$. Since $\lambda_{k-1}(\epsilon)$ likewise approaches -1 as $\epsilon \to 0+$, we find

$$\frac{d}{d\epsilon}\left(\frac{\lambda_{k-1}(\epsilon)}{-\lambda^-(\epsilon)}\right)\bigg|_{\epsilon=0+} \geq \frac{2v_1^2}{\|\mathbf{v}_o\|^2} + \frac{2v_1^2}{\|\mathbf{v}_e\|^2}.$$

Being continuous, this expression remains positive in an open interval of the arc S_1, having O_0 as boundary point, on which $M_k^{(\epsilon)}$ remains boundary-regular; we take this to be O_1. In turn, the positivity of the derivative means that the map Λ_1 is one-to-one in the neighborhood of every point of the closed arc $\overline{O}_1$, that is, on O_1 and its endpoint O_0. This proves the lemma for $j = 1$.

Now suppose $O_j \subset S_j$ has been determined to satisfy the requirements of the lemma. We argue as earlier. Let $M_k(0, 1, t_3, \ldots, t_k) \in O_j$ be boundary-regular. One way of moving M_k into S_{j+1} is to apply the construction of Lemma 2. It splits the double lowest eigenvalue of M_{j+3} into $\lambda^-(\epsilon) < \lambda^+(\epsilon)$ and yields a boundary-regular

$$M_k^{(\epsilon)} = M_k(0, 1, t_3, \ldots, t_{j+2}, t_{j+3} \pm \epsilon, t_{j+4}(\epsilon), \ldots, t_k(\epsilon)), \tag{10}$$

in which $t_i(\epsilon)$, $j+4 \leq i \leq k$, are determined as in (4) to produce a block of $k-j-2$ coincident eigenvalues at $\lambda^-(\epsilon)$; for the next eigenvalue we have, as in (9), $\lambda_{k-j-1}(\epsilon) \geq \lambda^+(\epsilon)$. By the same argument as for S_1, $M_k^{(\epsilon)}$ is differentiable in ϵ. The normalized eigenvalue map Λ_{j+1} takes $M_k^{(\epsilon)} \in S_{j+1}$ onto $(-1, \ldots, -1, (\lambda_{k-j-1}(\epsilon)/-\lambda^-(\epsilon)), \ldots, (\lambda_k(\epsilon)/-\lambda^-(\epsilon)))$, and we find, just as earlier for S_0,

$$\frac{d}{d\epsilon}\left(\frac{\lambda_{k-j-1}(\epsilon)}{-\lambda^-(\epsilon)}\right)\bigg|_{\epsilon=0+} \geq \frac{v_1^2}{|\lambda|}\left(\frac{2}{\|\mathbf{v}_e\|^2} + \frac{2}{\|\mathbf{v}_o\|^2}\right) > 0, \tag{11}$$

with $\lambda \neq 0$ the lowest eigenvalue of M_k, and $\mathbf{v}_e$ and $\mathbf{v}_o$ the lowest eigenvectors of M_{j+3}. This shows that Λ_j can be extended from S_j to S_{j+1}, with $M_k^{(\epsilon)}$ boundary-regular.

Now consider an arbitrary $M_k^{(\epsilon)}(0, 1, t_3(\epsilon), \ldots, t_k(\epsilon)) \in S_{j+1}$ that approaches a boundary-regular $M_k(0, 1, t_3, \ldots, t_k) \in S_j$. As before, by Proposition 3(e), (b), its lowest eigenvalue $\lambda(\epsilon)$ is a single eigenvalue of $M_{j+3}^{(\epsilon)}$, $t_i(\epsilon)$ in the range $j+4 \leq i \leq k$ is defined successively by (4), with $\mathbf{v} = \mathbf{w}(\epsilon)$ the lowest eigenvector of $M_{j+3}^{(\epsilon)}$, and the $(k-j-2)$-dimensional eigenspace $E_k^{(\epsilon)}$ of $M_k^{(\epsilon)}$ corresponding to $\lambda(\epsilon)$ is generated from this $\mathbf{w}(\epsilon)$ as in Proposition 2(b). As $\epsilon \to 0+$, $\mathbf{w}(\epsilon)$ approaches that of the two lowest eigenvectors of M_{j+3} which has the same parity as $\mathbf{w}(\epsilon)$; since $M_{j+3}^{(\epsilon)} \in \mathscr{T}_{j+3}$, we know this to be even or odd, according as the integer j is. Consequently this limit, $\mathbf{w}(0)$, is determined independently of the perturbation, and the eigenspace $E_k^{(\epsilon)}$ generated by $\mathbf{w}(\epsilon)$ has as its limit the eigenspace $E_k^{(0)}$ of M_k analogously generated by $\mathbf{w}(0)$. The eigenvector $\mathbf{v}_{k-j-1}(\epsilon)$ of $M_k^{(\epsilon)}$ corresponding to the eigenvalue

$\lambda^{(\epsilon)}_{k-j-1}$, which has been split from the block of common eigenvalues of M_k, being orthogonal to $E^{(\epsilon)}_k$, then approaches the orthogonal complement in the $(k-j-1)$-dimensional lowest eigenspace of M_k of the $(k-j-2)$-dimensional subspace $E^{(0)}_k$. This is a vector $\mathbf{w}$ which depends only on M_k. We conclude that for any local variation of $M_k \in S_j$ into S_{j+1} the eigenvectors beyond the $(k-j-2)$th are perturbations of fixed vectors that do not depend on the variation. We point out that this is not generally true of all variations of S_j into $\mathcal{T}_k$—a fact which explains why Λ is not differentiable at low-dimensional boundary facets of L_k.

The following argument for the differentiability of the perturbed eigenvalues was suggested by P. Deift. Let $M^{(\epsilon)}_{j+3} = M_{j+3}(0, 1, t_3 \pm \epsilon_3, \dots, t_{j+3} \pm \epsilon_{j+3})$, the signs chosen so that $M^{(\epsilon)}_{j+3} \subset \mathcal{T}_{j+3}$ for $\epsilon \to 0+$. By perturbation theory for single eigenvectors [K, p. 119], each single eigenvector of M_{j+3} is perturbed analytically in all the components $\epsilon_3, \dots, \epsilon_{j+3}$ of ϵ. The same is true of the two lowest (multiple) even and odd eigenvectors of M_{j+3}, since the perturbation of each is definable by orthogonality to the remaining eigenvectors of $M^{(\epsilon)}_{j+3}$ of the same parity. In turn, the analyticity of $\mathbf{w}(\epsilon)$ propagates to $t_i(\epsilon)$, $j+4 \le i \le k$, by (4), and to the j multiple lowest eigenvectors of $M^{(\epsilon)}_k$ constructed from $\mathbf{w}(\epsilon)$ as in the proof of Proposition 3(d). Since $M^{(\epsilon)}_k$ is now an analytic perturbation of M_k, so again is that of each single eigenvector of M_k, and the remaining $(j+1)$st eigenvector of $M^{(\epsilon)}_k$ which has been split from the multiple eigenspace of M_k is analytic as well, being definable by orthogonality to all the others. Thereupon the eigenvalues are also analytic in $\epsilon_3, \dots, \epsilon_{j+3}$, and by differentiating the formula $\lambda_i(\epsilon) = (M^{(\epsilon)}_k \mathbf{v}_i(\epsilon), \mathbf{v}_i(\epsilon))/\|\mathbf{v}_i(\epsilon)\|^2$ we obtain

$$\left.\frac{d\lambda_i(\epsilon)}{d\epsilon}\right|_{\epsilon=0+} = \frac{(T_k \mathbf{v}_i, \mathbf{v}_i)}{\|\mathbf{v}_i\|^2}, \qquad k-j-1 \le i \le k, \tag{12}$$

where $\mathbf{v}_{k-j-1} = \mathbf{w}$, the vector just defined, $\mathbf{v}_i$ is the ith eigenvector of M_k, $k-j \le i \le k$, and $T_k = dM^{(\epsilon)}_k/d\epsilon|_{\epsilon=0+}$; for $i \le k-j-2$, the eigenvalues are equal and

$$\left.\frac{d\lambda_i(\epsilon)}{d\epsilon}\right|_{\epsilon=0+} = \frac{(T_k \mathbf{u}, \mathbf{u})}{\|\mathbf{u}\|^2}, \tag{13}$$

for any eigenvector $\mathbf{u}$ of M_k in the subspace $E^{(0)}_k$, that is, any eigenvector corresponding to the lowest eigenvalue, but orthogonal to $\mathbf{w}$. We conclude that the extension Λ_{j+1} of Λ_j is smooth and locally linear in the perturbation; indeed, (12) and (13) exhibit the tangent space to Λ_{j+1}. Consequently, the variation of M_k to $M^{(\epsilon)}_k$ can be decomposed into one within S_j, followed by one of the form (10). This means that (10) gives the general parametrization of S_{j+1} near S_j; as with S_1, since $\epsilon > 0$, S_{j+1} lies locally on only one side of S_j.

Finally, we show that if Λ_j is locally one-to-one on O_j at a point M_k, its extension Λ_{j+1} remains so. Intuitively, this happens because Λ_j maps S_j into l_j and, by (11), a direction transverse to S_j into one transverse to l_j. More precisely, we recall that the independent variables of a matrix $M_k \in S_j$ are $t_3, \ldots, t_{j+2}$, since the remaining t_i are determined by (4) to produce the required multiplicity of the lowest eigenvalue. If Λ_j is locally one-to-one at M_k, then, by the local linearity, the $j \times j$ Jacobian matrix J of $(\lambda_{k-j}/-\lambda_1), \ldots, (\lambda_{k-1}/-\lambda_1)$ with respect to these independent variables $t_3, \ldots, t_{j+2}$ is not singular at M_k. In extending the map to Λ_{j+1}, we increase its domain and range by one dimension, corresponding to the variables t_{j+3} and $\lambda_{k-j-1}/(-\lambda_1)$, respectively. Let K be the enlarged $(j+1) \times (j+1)$ Jacobian matrix of $(\lambda_{k-j-1}/-\lambda_1)$, $(\lambda_{k-j}/-\lambda_1), \ldots, (\lambda_{k-1}/-\lambda_1)$ with respect to $t_3, \ldots, t_{j+2}, t_{j+3}$.

When viewed as a subset of $t_3, \ldots, t_{j+2}, t_{j+3}$, the independent variables of S_{j+1}, the matrices of S_j correspond to values $t_3, \ldots, t_{j+2}, f(t_3, \ldots, t_{j+2})$, with f determined, as usual, by (4), using for $\mathbf{v}$ the lowest eigenvector of M_{j+2}. Therefore the tangent space $\mathscr{U}$ to S_j at M_k is generated by the vectors

$$\left(\delta t_3, \ldots, \delta t_{j+2}, \sum_{i=3}^{j+2} \frac{\partial f}{\partial t_i} \delta t_i\right) .$$

The last entry being linear in $\{\delta t_i\}$, $\mathscr{U}$ forms a subspace of dimension j in the $(j+1)$-dimensional space of $(\delta t_3, \ldots, \delta t_{j+2}, \delta t_{j+3})$, and is the domain of J in that larger space; that is, K coincides with J on $\mathscr{U}$. Because J is nonsingular, the image $K\mathscr{U}$ is likewise j-dimensional, and hence there exists a unique vector $p \neq 0$ with

$$(K\mathscr{U}, p) = 0 \; ; \tag{14}$$

indeed, p is the vector in l_{j+1} orthogonal to l_j, that is, $(1, 0, \ldots, 0)$ in the present coordinates. By definition of K, we see from (11) that

$$(K\delta t_{j+3}, p) \neq 0 .$$

It follows from (14) that the range of K, which contains $K\mathscr{U}$, is strictly larger than $K\mathscr{U}$, hence $(j+1)$-dimensional. K is therefore nonsingular, so that Λ_{j+1} is one-to-one locally at M_k, or, equivalently, $\det K$, the Jacobian of Λ_{j+1}, does not vanish at M_k. By continuity, it is also bounded away from zero in a neighborhood of M_k in S_{j+1}. On defining O_{j+1} to be the union of such neighborhoods for $M_k \in O_j$, we find that $\det K \neq 0$ on $\overline{O}_{j+1}$; hence O_{j+1} satisfies the lemma. This completes the induction and so proves Lemma 3.

Lemma 4. *There exists a point y interior to the simplex L_k which is covered only once by the map Λ acting on $\mathscr{T}_k$, and the Jacobian of Λ does not vanish at $\Lambda^{-1}(y) \in \mathscr{T}_k$.*

Proof. By Lemma 3, the Jacobian of Λ does not vanish on the open subset O_{k-2} of $\mathcal{T}_k$. We consider the points in L_k onto which Λ takes O_{k-2}. If the assertion of the lemma is false, each of these is covered more than once by $\Lambda(\mathcal{T}_k)$. Thus to each point $\tau \in O_{k-2}$ there corresponds $\tau' \in \mathcal{T}_k$ with $\tau' \neq \tau$ but

$$\Lambda(\tau') = \Lambda(\tau) . \tag{15}$$

Let τ approach a point σ of the boundary O_{k-3} of O_{k-2}. On taking a subsequence if necessary, τ' likewise converges to σ' and, by (15), $\Lambda(\sigma') = \Lambda(\sigma)$. But since $\sigma \in O_{k-3} \subset S_{k-3}$, the image $\Lambda(\sigma)$ lies in the boundary face l_{k-3}; hence $\sigma' \in S_{k-3}$. The points σ and σ' cannot coincide, since by Lemma 3 the map Λ is one-to-one in an O_{k-2}-neighborhood of σ, so once this neighborhood contains an approximant τ_k of σ, it cannot contain the corresponding τ'_k. We thus find that O_{k-3} inherits from O_{k-2} the property that to each of its points there exists a different point in S_{k-3} with the same image under Λ. Continuing the argument for O_i with decreasing i, we derive the same conclusion for $O_0 = S_0$. But as S_0 is a single point, it cannot accommodate the distinct σ and σ'. This contradiction proves Lemma 4.

We now have the ingredients for a proof of the main result.

Theorem 1. *For every* k, $\mathcal{T}_k \in R^{k-2}$ *is bounded by portions of all the surfaces* B_i, $1 \leq i \leq k-1$, *and the image of* $\mathcal{T}_k$ *under* Λ *covers all of* L_k.

Proof. The image of the boundary of $\mathcal{T}_k$ lies exclusively in the bounding hyperplanes of L_k. By Lemma 4, there exists a point τ interior to $\mathcal{T}_k$ at which the Jacobian of Λ does not vanish, and which is the only solution of $\Lambda(x) = \Lambda(\tau)$, for $x \in \mathcal{T}_k$. Set $y = \Lambda(\tau)$, and let $\mathcal{A}_y$ denote the connected component of the complement of $\Lambda(\partial\mathcal{T}_k)$ in R^{k-2} which contains y; clearly $L_k \subset \mathcal{A}_y$.

For the smooth map Λ of $\mathcal{T}_k$ into R^{k-2}, the degree at $y \notin \Lambda(\partial\mathcal{T}_k)$ is definable as [S, B]

$$\deg(y, \Lambda, \mathcal{T}_k) = \sum_{\substack{x \in \mathcal{T}_k \\ \Lambda(x)=y}} \operatorname{sign}(\text{Jacobian of } \Lambda \text{ at } x) . \tag{16}$$

This is an integer, known to be constant on all of $\mathcal{A}_y$. Since $x = \tau$ is unique, $\deg(y, \Lambda, \mathcal{T}_k) = \pm 1 \neq 0$. This implies that $\Lambda(\mathcal{T}_k)$ covers all of $\mathcal{A}_y$, for if there were a point $y' \in \mathcal{A}_y$ with no solution to the equation $\Lambda(x) = y'$, $x \in \mathcal{T}_k$, then by (16)

$$\deg(y', \Lambda, \mathcal{T}_k) = 0 \neq \deg(y, \Lambda, \mathcal{T}_k) ,$$

a contradiction. Moreover, if B_i is not in the boundary of $\mathcal{T}_k$, then $\Lambda(\partial\mathcal{T}_k)$ does not include the boundary hyperplane $\lambda_i = \lambda_{i+1}$ of L_k; hence $\mathcal{A}_y$ also contains that boundary. Since $\Lambda(\mathcal{T}_k)$ covers $\mathcal{A}_y$, this implies in turn that

$\mathcal{T}_k$ contains points in its interior for which $\lambda_i = \lambda_{i+1}$, a contradiction. This completes the proof of the theorem.

REMARKS

It is tempting to conjecture that the above correspondence between eigenvalues and regular Toeplitz matrices is one-to-one. Of course, it would also be very interesting to find an algorithm for carrying it out.

ACKNOWLEDGMENT

I am grateful to S. Friedland for a preprint of his paper, which proposed the topological degree as an approach to problems of this type; to R. Osserman for essential clarifying conversations; to P. Deift for questions and comments that vastly improved the argument; and above all to D. Slepian for unceasing interest and encouragement.

REFERENCES

[A] N. I. Akhiezer, *The classical moment problem*, Hafner, New York, 1965.

[B] M. Berger and M. Berger, *Perspectives in nonlinearity*, Benjamin, New York, 1968.

[DG] P. Delsarte and Y. Genin, *Spectral properties of finite Toeplitz matrices*, Lecture Notes in Control and Inform. Sci., vol. 58, Springer, New York, 1984, pp. 194–213

[F] S. Friedland, *Inverse eigenvalue problems for symmetric Toeplitz matrices*, SIAM J. Matrix Anal. Appl. **13** (1992), 1142–1153.

[I] I. S. Iohvidov, *Hankel and Toeplitz matrices and forms*, Birkhaüser, Boston, 1982.

[K] T. Kato, *Perturbation theory for linear operators*, Springer, New York, 1984.

[RN] F. Riesz and B. Sz-Nagy, *Functional analysis*, Ungar, New York, 1971.

[S] J. T. Schwartz, *Nonlinear functional analysis*, Gordon and Breach, New York, 1969.

[SL] D. Slepian and H. J. Landau, *A note on the eigenvalues of Hermitian matrices*, SIAM J. Math. Anal. **9** (1978), 291–297.

[T1] W. F. Trench, *Spectral evolution of a one-parameter extension of a real symmetric Toeplitz matrix*, SIAM J. Matrix Anal. Appl. **11** (1990), 601–611.

[T2] ———, *Interlacement of the even and odd spectra of real symmetric Toeplitz matrices*, Linear Algebra Appl. **195** (1993), 59–69.

ABSTRACT. We show that every set of n real numbers is the set of eigenvalues of an $n \times n$ real symmetric Toeplitz matrix; the matrix has a certain additional regularity. The argument—based on the topological degree—is nonconstructive.

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