JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 8, Number 4, October 1995

THRESHOLD FUNCTIONS FOR RAMSEY PROPERTIES

VOJTECH RÖDL AND ANDRZEJ RUCIŃSKI

1. INTRODUCTION

Ramsey type problems have played an important role in the development of combinatorial mathematics. Although the classical Ramsey results do not involve explicitly random structures, the proofs are often based on the use of probabilistic techniques. In this paper we study Ramsey type questions from the point of view of random graphs and give a complete solution to a problem initiated in [LRV 92] (see also [FR 86], [RR 94], and [RR 93]).

Our result can be formulated as follows. Let $F \to (G)_r$ mean that for every coloring of the edges of a graph F with r colors there is in F a subgraph isomorphic to G whose edges are all colored by the same color. Call graphs F with the above property (G, r)-*Ramsey*. One could think that there cannot be too many sparse (G, r)-Ramsey graphs. We shall show that there is quite a low threshold on the number of edges, above which almost all graphs are (G, r)-Ramsey.

More precisely, for positive integers n and N, and a graph G, let $\mathscr{F}_{n,N}$ be the family of all $\binom{\binom{n}{2}}{N}$ graphs on the vertex set $[n] = \{1, 2, \dots, n\}$ and with N edges, and let $\mathscr{R}_{G,r}(n, N)$ be the set of those graphs from $\mathscr{F}_{n,N}$ which are (G, r)-Ramsey. Let, for a graph G on at least three vertices,

$$m_G^{(2)} = \max_{H \subseteq G, v_H \geq 3} \frac{e_H - 1}{v_H - 2},$$

where e_G and v_G stand for the number of edges and vertices of a graph G. *A star forest* is a graph every connected component of which is a star.

Theorem 1. *For all integers $r \geq 2$ and for every graph G which is not a star forest there exist constants c and C such that*

$$\lim_{n\to\infty} \frac{|\mathscr{R}_{G,r}(n, N)|}{|\mathscr{F}_{n,N}|} = \begin{cases} 0 & \text{if } N < cn^{2-1/m_G^{(2)}}, \\ 1 & \text{if } N > Cn^{2-1/m_G^{(2)}}. \end{cases}$$

If G is a star forest with maximum degree $d \geq 1$ then the threshold for the above Ramsey property coincides, by The Pigeon-Hole Principle, with that for

Received by the editors September 15, 1993 and, in revised form, May 20, 1994.
1991 *Mathematics Subject Classification*. Primary 05C80; Secondary 05C55.
The first author was supported by NSF grant DUS-9011850. The second author is on leave from the Department of Discrete Mathematics, Adam Mickiewicz University, Poznań, Poland, and was supported by KBN grant 2 1087 9101.

the appearance of vertices of degree $r(d-1)+1$. This threshold was already found in [ER 60] to be $n^{1-\frac{1}{r(d-1)}}$.

Theorem 1 consists, in fact, of two separate statements to which we shall be referring to as the 0-statement and the 1-statement, according to the value of the limit. In case $G = K_3$, $r = 2$, the 1-statement has been already proved by Frankl and Rödl in [FR 86]. It was then reproved, along with the corresponding 0-statement, by Łuczak, Ruciński, and Voigt in [LRV 92]. This case was also considered by Erdős, Sós, and Spencer (personal communication). Recently we have proved that for $G = K_3$ the threshold does not vary with the increase of the number of colors ([RR 94]). As we now see it is true for all graphs except for star forests.

We now state as a corollary two most prominent cases of Theorem 1. Let K_k and C_k denote the complete graph and the cycle on k vertices.

Corollary 1. *For all $r \geq 2$ and all $k \geq 3$ there exist constants c and C such that*

$$\lim_{n\to\infty} \frac{|\mathscr{R}_{K_k,r}(n,N)|}{|\mathscr{F}_{n,N}|} = \begin{cases} 0 & \text{if } N < cn^{\frac{2k}{k+1}}, \\ 1 & \text{if } N > Cn^{\frac{2k}{k+1}}, \end{cases}$$

and

$$\lim_{n\to\infty} \frac{|\mathscr{R}_{C_k,r}(n,N)|}{|\mathscr{F}_{n,N}|} = \begin{cases} 0 & \text{if } N < cn^{\frac{k}{k-1}}, \\ 1 & \text{if } N > Cn^{\frac{k}{k-1}}. \end{cases}$$

The above results have an obvious enumerative flavour. Namely, Theorem 1 says that almost all graphs with n vertices and N edges are (G, r)-Ramsey as soon as N is bigger than $Cn^{2-1/m_G^{(2)}}$, and that almost no graphs are such when the number of edges drops just by a multiplicative constant. This sharp threshold behaviour is a typical feature of random graphs and, indeed, the family $\mathscr{F}_{n,N}$ can be viewed as a uniform space of random graphs, where each graph is assigned the same probability $\binom{\binom{n}{2}}{N}^{-1}$. Since "$F \to (G)_r$" is a monotone graph property, the existence of a threshold follows from [BT 87]. The fact that the threshold occurs at $n^{2-1/m_G^{(2)}}$ is not accidental. This quantity ensures that, for each $H \subseteq G$, $e_H > 0$, the expected number of copies of H in a graph drawn at random from the family $\mathscr{F}_{n,N}$ is at least as big as the number of edges N, or on a more local scale, the expected number of copies of H containing a fixed edge is at least a constant. Intuitively, this seems to be a necessary condition for the property "$F \to (G)_r$" to hold for almost all graphs. Here we have analogy with the result from [LRV 92] establishing $n^{2-1/m_G^{(1)}}$, $m_G^{(1)} = \max_{H\subseteq G, v_H>2} \frac{e_H}{v_H-1}$, as the threshold for the corresponding property when, instead of edges, the vertices are colored. In that case, the expected number of copies of each $H \subseteq G$ is at least of order n, the number of vertices in the random graph.

The 1-statement of Theorem 1, or rather its strengthening proved in Section 3, implies the existence of locally sparse Ramsey graphs. Let, for a family of graphs $\mathscr{H}$, $\mathscr{F}orb(\mathscr{H})$ denote the family of all graphs with no subgraph

isomorphic to a member of $\mathscr{H}$. Let $\mathscr{H}_t^{(2)}(G)$ denote the family of all graphs H with $v_H \leq t$ which satisfy the inequality $m_H^{(2)} > m_G^{(2)}$.

Corollary 2. *For all r, G and t there exist graphs $F = F(r, G, t)$ such that $F \to (G)_r$ and $F \in \mathscr{F}orb(\mathscr{H}_t^{(2)}(G))$.*

In particular, this means that there exist (G, r)-Ramsey graphs F such that the corresponding hypergraph, whose vertices are edges of F and whose edges are copies of G in F, contains no short cycles. A construction of such graphs can be deduced from a more general result of Nešetřil and Rödl [NR 89].

Also, we shed more light on the existence of K_{k+1}-free (K_k, r)-Ramsey graphs. It was Folkman who first constructed such a graph for $r = 2$ [Fo 70] and Nešetřil and Rödl [NR 76] for $r > 2$. The constructions were difficult and of enormous size, and, perhaps, made everyone believe that such graphs are very rare. However, it follows from the stronger version of the 1-statement proved in Section 3 (Theorem 3), that almost all K_{k+1}-free graphs with $Cn^{\frac{2k}{k+1}}$ edges are (K_k, r)-Ramsey. For details on locally sparse Ramsey graphs see Section 4.

Before we turn to the proofs, let us introduce the alternative, and in many respects equivalent, model of a random graph.

Let the random graph $K(n, p)$ be the outcome of the following probability experiment. For every 2-element subset of $[n] = \{1, 2, \dots, n\}$, make it an edge of $K(n, p)$ with probability $p = p(n)$, where all $\binom{n}{2}$ decisions are mutually independent. The main difference between this model of a random graph, called sometimes the binomial model, and the uniform model described before, is that the space $K(n, p)$ consists of all $2^{\binom{n}{2}}$ graphs on n vertices and the number of edges is not fixed but it is, in fact, a random variable with the binomial distribution $Bi(\binom{n}{2}, p)$. As far as the thresholds for Ramsey properties are concerned the two random graph models are equivalent (see [Bo 85] and Section 2 below for the relevant equivalence theorems) and our Theorem 1 can be now restated as follows.

Theorem 1′. *For all integers $r \geq 2$ and for every graph G which is not a star forest there exist constants c and C such that*

$$\lim_{n\to\infty} Prob(K(n, p) \to (G)_r) = \begin{cases} 0 & \text{if } p < cn^{-1/m_G^{(2)}}, \\ 1 & \text{if } p > Cn^{-1/m_G^{(2)}}. \end{cases}$$

In fact, in Section 3 we choose to prove Theorem 1′ rather than Theorem 1. The reason is that the binomial model $K(n, p)$ provides independence of the occurrences of disjoint subsets of edges, a feature we will be frequently relying on in our proof.

It is well known (see for instance [GRS 90]) that for every graph G there is a graph H such that for every r-coloring of the edges of H there is a monochromatic *induced* copy of G in H. Hence the induced version of the 1-statement of Theorem 1′ follows immediately for $p = constant$ from the fact that $K(n, p)$ contains almost surely an induced copy of every graph H. In Section 3 we prove a strengthening of Theorem 1′ which, in particular, claims

that there are not one but many monochromatic copies of G; more precisely, a fraction of the expectation μ of the number of copies of G in $K(n, p)$ become monochromatic in every coloring. As the number of noninduced copies of G is $o(\mu)$ if $p = o(1)$, this allows us to derive the following induced version of our result.

Corollary 3. *For all integers $r \geq 2$ and for every graph G there is a constant C such that if $p > Cn^{-1/m_G^{(2)}}$ then for every r-coloring of the edges of $K(n, p)$ there is a monochromatic induced copy of G.*

The proof of the 0-statement of Theorem $1'$ appeared in [RR 93]. In Section 3 of this paper the 1-statement will be proved. Section 2 contains some preliminary results, whereas in Section 4 we present the corollaries about locally sparse Ramsey graphs. Finally, in Section 5 we indicate how our method of proof can be used to obtain the following threshold result about van der Waerden properties of random subsets of integers. For integers $k \geq 2$ and $r \geq 3$, and for a set $F \subseteq [n]$ we write $F \to (k)_r$ if every r-coloring of F results in at least one monochromatic k-term arithmetic progression. For $0 < N < n$, let $[n]_N$ stand for the random subset of $[n]$ picked from all N-element subsets of $[n]$ with uniform distribution.

Theorem 2. *For all $k \geq 3$ and $r \geq 2$ there exist constants c and C such that*

$$\lim_{n\to\infty} Prob([n]_N \to (k)_r) = \begin{cases} 0 & \text{if } N < cn^{\frac{k-2}{k-1}}, \\ 1 & \text{if } N > Cn^{\frac{k-2}{k-1}}. \end{cases}$$

2. Preliminaries

In this section we collect some probabilistic and graph theoretic tools, and we prove a couple of lemmas to be used in the proof.

Boole's inequality. For any sequence of events $A_1, A_2, \ldots$ defined on the same probability space, $P(\bigcup A_n) \leq \sum P(A_n)$.

Markov's inequality. For a nonnegative random variable X, and a positive constant a, $P(X \geq a) \leq \frac{1}{a} Exp(X)$. In particular, for a nonnegative integer-valued random variable X, $P(X > 0) \leq Exp(X)$.

Chernoff's inequality. For a random variable X with the binomial distribution $Bi(n, p)$, and for every $\varepsilon > 0$,

$$P(|X - np| > \varepsilon np) < e^{-\mathrm{CHERNOFF}(\varepsilon)np},$$

where $\mathrm{CHERNOFF}(\varepsilon) = \min\{-\log(e^{\varepsilon}(1+\varepsilon)^{-(1+\varepsilon)}), \frac{1}{2}\varepsilon^2\}$.

Janson's inequality [Ja 90]. Let $\{J_i\}_{i\in F}$ be a collection of independent 0-1 random variables, $\{F(\alpha)\}_{\alpha\in A}$ – a family of subsets of F, $I_\alpha = \prod_{i\in F(\alpha)} J_i$, and $S = \sum_{\alpha\in A} I_\alpha$. (To better accommodate this stream of definitions, think of F as the edge set of an n-vertex complete graph and of A as an index set of all $\binom{n}{3}$ triangles there. If J_i indicates the presence of the ith edge with probability p, then S is the number of triangles in $K(n, p)$.)

Then, for every $\varepsilon > 0$,

$$P(S \leq (1-\varepsilon)Exp(S)) \leq \exp\left\{-\frac{(\varepsilon Exp(S))^2}{2\sum\sum_{F(\alpha)\cap F(\beta)\neq\varnothing} Exp(I_\alpha I_\beta)}\right\}.$$

FKG inequality. In the same setting as above, if S_1 and S_2 are both nonincreasing or both nondecreasing functions of the random vector $\{J_i\}_{i\in F}$ then $Exp(S_1S_2) \geq Exp(S_1)Exp(S_2)$. In particular,

$$P(S=0) = Exp\left(\prod_{\alpha\in A}(1-I_\alpha)\right) \geq \prod_{\alpha\in A}(1-Exp(I_\alpha))$$
$$\geq \exp\left\{-\frac{Exp(S)}{1-\max_{\alpha\in A} Exp(I_\alpha)}\right\}.$$

Equivalence of random graphs. It costs nothing to have a general setting here. Let F be a set with M elements, $0 < p < 1$, $0 \leq N \leq M$. A random set F_p is obtained from F by independent inclusion of elements, each with probability p. A random set F_N is an N-element subset of F selected uniformly over all $\binom{M}{N}$ N-element subsets of F. If F is the edge set of a complete graph on n vertices, then we adopt the standard notation $K(n, p)$ and $K(n, N)$, respectively. Also, for $A \subset [n]$, $K(A, p)$ is the random graph $K(n, p)$ restricted to A.

For a family $\mathscr{Q}$ of subsets of F, by the law of total probability,

$$P(F_p \in \mathscr{Q}) = \sum_{k=0}^{M} P(F_k \in \mathscr{Q})\binom{M}{k}p^k(1-p)^{M-k}, \tag{1}$$

from which Pittel's inequality follows:

$$P(F_N \in \mathscr{Q}) \leq 3\sqrt{N}P(F_{\frac{N}{M}} \in \mathscr{Q}). \tag{2}$$

We say that family $\mathscr{Q}$ is *increasing* if whenever $A \subseteq B$, $A \in \mathscr{Q}$ implies that $B \in \mathscr{Q}$. It follows from (1) and Chernoff's inequality that if $\mathscr{Q}$ is increasing, $Mp(1-p) \to \infty$, and $P(F_p \in \mathscr{Q}) \to 1$, then $P(F_{\lfloor 2pM\rfloor} \in \mathscr{Q}) \to 1$ while if $P(F_p \in \mathscr{Q}) \to 0$, then $P(F_{\lfloor \frac{1}{2}pM\rfloor} \in \mathscr{Q}) \to 0$.

For the sake of self-containment we now state three celebrated theorems from graph theory in a form suitable for us.

Ramsey's Theorem, 1930. *For all choices of positive integers $r, k_1, \ldots, k_r$, there exists a smallest integer $R = R(k_1, \ldots, k_r)$ such that if the edges of the complete graph K_n with $n \geq R$ are partitioned into r classes, $[n]^2 = E_1 \cup \cdots \cup E_r$, then for some $i \in [r]$, E_i contains a complete subgraph K_{k_i}.* □

Turán's Theorem, 1941. *If a graph on n vertices contains more than $(1-\frac{1}{k})n^2/2$ edges, then it contains a complete subgraph K_{k+1}.* □

For a graph $\Gamma = (V, E)$ and two disjoint subsets of its vertices A, B we set $e(A, B)$ for the number of edges of Γ with one endpoint in A and one in B,

and the density of the pair A, B in Γ is defined as

$$d_\Gamma(A, B) = \frac{e(A, B)}{|A||B|}.$$

Let $\Gamma_1, \dots, \Gamma_r$ be graphs on the same vertex set V. The pair A, B is called *ε-regular* if for all $X \subseteq A$ and $Y \subseteq B$, $|X| \geq \varepsilon|A|$, $|Y| \geq \varepsilon|B|$,

$$|d_{\Gamma_i}(X, Y) - d_{\Gamma_i}(A, B)| < \varepsilon$$

for all $i \in [r]$. A partition $V = C_0 \cup \dots \cup C_t$ is *ε-regular* if $|C_0| \leq \varepsilon|V|$, all $C_1, \dots, C_t$ have the same cardinality, and at least $(1-\varepsilon)\binom{t}{2}$ pairs C_i, C_j, $1 \leq i < j \leq t$, are ε-regular. The following is a modification of the result from [Sz 78], the proof of which follows the lines of the original proof of Szemerédi and therefore is omitted.

Szemerédi's Regularity Lemma, 1978. *For all positive numbers ε, r, and m there exist numbers $N = N(\varepsilon, m)$ and $M = M(\varepsilon, m)$ such that for every choice of r graphs $\Gamma_1, \dots, \Gamma_r$ on the same set V of at least N vertices there is an ε-regular partition $V = C_0 \cup \dots \cup C_t$ with $m < t < M$.* □

Note that the common cardinality x of the sets $C_1, \dots, C_t$ satisfies

$$(1-\varepsilon)\frac{|V|}{M} < x < \frac{|V|}{m}. \tag{3}$$

Now we shall use some of the above collected tools to prove a few lemmas.

For a graph Γ, the *density* of Γ is defined as

$$d(\Gamma) = \frac{|E(\Gamma)|}{\binom{|V(\Gamma)|}{2}}.$$

A graph Γ is called *(ρ, d)-dense* if for every subset $V \subseteq V(\Gamma)$, with $|V| \geq \rho|V(\Gamma)|$, the density of the subgraph of Γ induced by V satisfies

$$d(\Gamma[V]) \geq d. \tag{4}$$

Observe that to claim that Γ is (ρ, d)-dense it is enough to check whether (4) holds for subsets V with $|V| = \lceil \rho|V(\Gamma)| \rceil = v$. Indeed, then for any U with $|U| = u \geq v+1$, we have

$$|E(\Gamma[U])| \geq d\binom{v}{2}\frac{\binom{u}{v}}{\binom{u-2}{v-2}} = d\binom{u}{2}.$$

This is because there are exactly $\binom{u}{v}$ v-element subsets of U, each containing at least $d\binom{v}{2}$ edges, and each edge is counted exactly $\binom{u-2}{v-2}$ times.

Lemma 1. *For all r, ρ', d there exist ρ, n_0, ν such that if Γ is a (ρ, d)-dense graph on $n \geq n_0$ vertices and $E(\Gamma) = E_1 \cup \dots \cup E_r$, then there exists $s \in [r]$ and $V \subseteq V(\Gamma)$, $|V| \geq \nu n$, for which the subgraph of Γ restricted to the vertices of V and to the edges of E_s, $\Gamma_s[V]$, is $(\rho', \frac{d}{20r})$-dense.*

Proof. If $\rho' > 1$, there is nothing to prove. Assume thus that $\rho' \leq 1$. Set

$$k = \left\lceil \frac{2}{d} \right\rceil, \qquad l = \left\lceil \frac{2.05}{\rho'} \right\rceil, \tag{5}$$

$$m = R(k, \underbrace{l, \dots, l}_{r}) , \tag{6}$$

where $R(k, \underbrace{l, \dots, l}_{r})$ is the Ramsey number introduced in the statement of Ramsey's Theorem above,

$$\varepsilon = \min\left\{\frac{1}{20l}, \frac{d}{100r}, \frac{1}{2m}\right\} , \tag{7}$$

$$\rho = \frac{k(1-\varepsilon)}{M} , \qquad \nu = \frac{l(1-\varepsilon)}{M} , \qquad \text{and} \qquad n_0 = \max\left\{N, \frac{20M}{1-\varepsilon}\right\} , \tag{8}$$

where $M = M(\varepsilon, 2m)$ and $N = N(\varepsilon, 2m)$ are the constants from Szemerédi's Regularity Lemma.

Let Γ be a (ρ, d)-dense graph on vertex set V, $|V| = n \geq n_0$, and let $\Gamma_i = \Gamma[E_i]$, $i = 1, \dots, r$, where $E(\Gamma) = E_1 \cup \dots \cup E_r$ is a partition of the edge set of Γ. By Szemerédi's Regularity Lemma there exists an ε-regular partition $V = C_0 \cup C_1 \cup \dots \cup C_t$ with respect to $\Gamma_1, \dots, \Gamma_r$, with $2m < t < M$.

As at least $(1-\varepsilon)\binom{t}{2} \overset{(7)}{>} (1 - \frac{1}{m})\frac{t^2}{2}$ pairs (C_i, C_j) are ε-regular, it follows by Turán's Theorem that there are m sets, $C_1, \dots, C_m$ say, such that all $\binom{m}{2}$ pairs are ε-regular.

Consider the $(r+1)$-coloring of $[m]^2$, $[m]^2 = D_0 \cup D_1 \cup \dots \cup D_r$, where

$$\{i, j\} \in D_0 \qquad \text{if} \qquad d(C_i, C_j) < \frac{d}{2} \tag{9}$$

and

$$\{i, j\} \in D_s \qquad \text{if} \qquad d_{\Gamma_s}(C_i, C_j) \geq \frac{d}{2r} , s \in [r] .$$

By (6) and Ramsey's Theorem there exists either a subset $K \subset [m]$, $|K| = k$, $[K]^2 \subset D_0$, or a subset $L \subset [m]$, $|L| = l$, $[L]^2 \subset D_s$, for some $s \in [r]$. The first option is impossible, since then, setting $x = |C_i|$, the set $C = \bigcup_{j \in K} C_j$ would have density

$$d(C) \overset{(9)}{<} \frac{\binom{k}{2}\frac{d}{2}x^2 + k\binom{x}{2}}{\binom{kx}{2}} < \frac{d}{2} + \frac{1}{k} \overset{(5)}{\leq} d ,$$

while $|C| = kx \overset{(3)}{>} k\frac{(1-\varepsilon)n}{M} \overset{(8)}{=} \rho n$, contradicting the fact that Γ is (ρ, d)-dense. Thus, for some $s \in [r]$, there are l sets, $C_1, \dots, C_l$ say, which satisfy

$$d_{\Gamma_s}(C_i, C_j) \geq \frac{d}{2r} \qquad \text{for all} \qquad \{i, j\} \in [l]^2 .$$

We shall prove that the graph $A = \Gamma_s[V]$, where $V = C_1 \cup \dots \cup C_l$, is $(\rho', \frac{d}{20r})$-dense. (Note that $|V| = lx \overset{(3)}{>} l\frac{(1-\varepsilon)n}{M} \overset{(8)}{=} \nu n$ as required.)

Consider $V' \subset V$, with

$$|V'| = \lceil \rho' |V| \rceil = \lceil \rho' l x \rceil \overset{(5)}{\leq} \left\lceil \left(\frac{2.05}{\rho'} + 1\right)\rho' x \right\rceil .$$

Since by (8) we have $x \geq \frac{(1-\epsilon)n}{M} \geq 20$ and also $\rho' \leq 1$, the R-H-S above can further be bounded from above by $3.1x$ leading to

$$|V'| \leq 3.1x\,. \tag{10}$$

By the remark made after the definiton of (ρ, d)-dense graphs, it is enough to show that $d(A') \geq \frac{d}{20r}$, where $A' = \Gamma_s[V']$. Set $x_i = |V' \cap C_i|$, $i = 1, \ldots, l$. Then, due to (10), we infer that

$$\sum_{i=1}^{l} \binom{x_i}{2} \leq 3\binom{x}{2} + \binom{0.1x}{2} < 3.01\binom{x}{2}\,. \tag{10'}$$

Let $V' = V'' \cup V'''$ be a partition, where $V'' = \bigcup_{i:\, x_i \leq \varepsilon x} V' \cap C_i$. Clearly, $|V''| \leq l\varepsilon x$, and thus

$$|V'''| \geq (\rho' - \varepsilon)lx \overset{(5,7)}{\geq} 2x\,. \tag{11}$$

Let us lowerbound the number of edges in the graph $\Gamma_s[V''']$. By the definition of V''' and by the choice of $C_1, \ldots, C_l$,

$$e(\Gamma_s[V''']) \geq \sum\sum \left(\frac{d}{2r} - \varepsilon\right) x_i x_j\,,$$

where the double summation is taken over all pairs i, j, $1 \leq i < j \leq l$, such that $x_i \geq \varepsilon x$, $x_j \geq \varepsilon x$. But

$$\sum\sum x_i x_j = \binom{\sum x_i}{2} - \sum \binom{x_i}{2} \overset{(10',11)}{\geq} \binom{2x}{2} - 3.01\binom{x}{2} > 0.99\binom{x+1}{2}$$

(the single summations are taken over all i satisfying $x_i \geq \varepsilon x$) and hence

$$e(\Gamma_s[V''']) \geq 0.99\left(\frac{d}{2r} - \varepsilon\right)\binom{x+1}{2}. \tag{12}$$

Finally,

$$\begin{aligned} d(A') &\geq \frac{e(\Gamma_s[V''])}{\binom{|V'|}{2}} \overset{(10,12)}{\geq} \frac{(\frac{d}{2r} - \varepsilon)(0.99)\binom{x+1}{2}}{\binom{3.1x}{2}} \\ &\geq \frac{0.99}{9.7}\left(\frac{d}{2r} - \varepsilon\right) \geq \frac{1}{9.8}\left(\frac{d}{2r} - \varepsilon\right) \overset{(7)}{\geq} \frac{d}{20r}\,. \quad \square \end{aligned}$$

Lemma 2. *For all d and k, there exist ρ, n_0, c_0 such that every (ρ, d)-dense graph on $n \geq n_0$ vertices contains at least $c_0 n^k$ complete subgraphs K_k.*

Proof. The proof is by induction on k. The lemma holds trivially for $k = 1$. For $k \geq 2$, let us assume that the lemma is true for $k-1$, i.e. for all d there exist ρ', n_0', and c_0' such that every (ρ', d)-dense graph on at least $n' \geq n_0'$ vertices contains at least $c_0'(n')^{k-1}$ subgraphs K_{k-1}. We choose $\rho = \frac{d}{2}\rho'$, $n_0 = \frac{2}{d}n_0'$, and $c_0 = \frac{1}{4k}c_0'(\frac{d}{2})^{k-1}$. Let x be the number of vertices of degree less than $\frac{d}{2}n$ in a (ρ, d)-dense graph on $n \geq n_0$ vertices. The subgraph induced by these vertices has less than $\frac{1}{4}dnx$ edges and so its density is smaller than

$\frac{dn}{2(x-1)}$. By the fact that our graph is (ρ, d)-dense, either $x < \rho n$ or $\frac{dn}{2(x-1)} > d$, implying $x \leq \frac{1}{2}(n+1)$. As $\rho < \frac{1}{2}$, we conclude that there are at least $\frac{n}{2}$ vertices of degree at least $\frac{dn}{2}$. Let v be one such vertex and let N_v be its neighborhood. The graph N_v is a (ρ', d)-dense graph on at least n_0' vertices, and by our induction assumption contains at least $c_0'(\frac{d}{2}n)^{k-1}$ subgraphs K_{k-1}. As each of these subgraphs together with vertex v forms a copy of K_k, we conclude that there are at least $\frac{1}{k}\frac{1}{2}(n-1)c_0'(\frac{d}{2}n)^{k-1} = 2c_0(n-1)n^{k-1} \geq c_0 n^k$ subgraphs K_k altogether. (The factor of $\frac{1}{k}$ is due to the fact that for each copy of K_k the vertex v can be chosen in up to k ways.) □

Both Lemmas 3 and 4 below are stated for random subsets rather than graphs. Such a general setting will be appreciated only when we get to the discussion of van der Waerden properties of random sets of integers in Section 5. For a family $\mathscr{Q}$ and an integer k, let $\mathscr{Q}_k$ be the family

$$\{A : \text{ for all } B \subseteq A,\ |B| \leq k,\ A \setminus B \in \mathscr{Q}\}.$$

Note that $\mathscr{Q}_k$ is increasing if $\mathscr{Q}$ is such. Let $\neg\mathscr{Q}$ denote the negation of $\mathscr{Q}$.

Lemma 3. *Let F be a set with M elements, $0 < p < 1$, and c and δ satisfy*

$$\delta(1 + \log_2 e - \log_2 \delta) < c \tag{13}$$

(note that the L-H- S approaches 0 as $\delta \to 0$). Then for any increasing family $\mathscr{Q}$ of subsets of F and for $0 < k \leq \delta Mp/2$, if

$$P(F_{(1-\delta)p} \in \neg\mathscr{Q}) < 2^{-cMp} \tag{14}$$

then, for Mp large enough,

$$P(F_p \in \neg\mathscr{Q}_k) < 2^{-c'Mp}, \tag{15}$$

where $c' = \frac{1}{2}\min(\frac{c}{2}, (\log_2 e)CHERNOFF(\delta/2))$.

Proof. Applying (1), Chernoff's inequality, and the fact that property $\neg\mathscr{Q}_k$ is decreasing, we have

$$\begin{aligned} P(F_p \in \neg\mathscr{Q}_k) &\leq \sum_{|t-Mp|<\frac{\delta}{2}Mp} P(F_t \in \neg\mathscr{Q}_k)\binom{M}{t}p^t(1-p)^{M-t} \\ &\quad + e^{-\mathrm{CHERNOFF}(\delta/2)Mp} \\ &\leq P\left(F_{\lfloor(1-\frac{\delta}{2})Mp\rfloor} \in \neg\mathscr{Q}_k\right) + e^{-\mathrm{CHERNOFF}(\delta/2)Mp}. \end{aligned} \tag{16}$$

Setting $t_0 = \lfloor(1-\delta)Mp\rfloor$ and using Boole's inequality, it is easily seen that

$$\begin{aligned} P(F_{\lfloor(1-\frac{\delta}{2})Mp\rfloor} \in \neg\mathscr{Q}_k) &\leq P\left(F_{t_0+k} \in \neg\mathscr{Q}_k\right) \\ &\leq \binom{k+t_0}{k} P(F_{t_0} \in \neg\mathscr{Q}). \end{aligned} \tag{17}$$

But

$$\binom{k+t_0}{t_0} < \binom{\frac{\delta}{2}Mp + t_0}{\frac{\delta}{2}Mp} < \left[\frac{2e}{\delta}\right]^{\frac{\delta}{2}Mp}. \tag{18}$$

By Pittel's inequality (2),

$$P(F_{t_0} \in \neg\mathscr{C}) \leq 3\sqrt{Mp}\ P(F_{(1-\delta)p} \in \neg\mathscr{C}) ,$$

which together with (13), (14), and (16)–(18) completes the proof. □

The original motivation for Janson's inequality was a search for an exponential bound for the lower tail of the asymptotic distribution of the number X_G of copies of a given graph G in the random graph $K(n, p)$. It follows that

$$P(X_G \leq (1-\varepsilon)Exp(X_G)) \leq \exp\{-c(G)\phi_n(G)\} ,$$

where $\phi_n(G) = \min_{H \subseteq G, e_H > 0} Exp(X_H)$. When p is at least of the order of magnitude of $n^{-1/m_G^{(2)}}$, then $\phi_n(G) = Exp(X_{K_2}) = \binom{n}{2}p$. In general, there is no corresponding upper tail bound of the same magnitude. To avoid this difficulty we shall use the following result which is of some independent interest.

Lemma 4. *Let F be a set with M elements, $0 < p < 1$, k and s are integers, and let $\mathscr{S}$ be a family of s-element subsets of F. Let $\mathscr{A}$ be the event that there exists $E \subseteq F_p$, $|E| \leq k$, such that the set $F_p \setminus E$ contains at most $2\mu = 2|\mathscr{S}|p^s$ sets from $\mathscr{S}$. Then*

$$P(\neg\mathscr{A}) < 2^{-k/s} .$$

Proof. Let Z be the number of $\kappa = \lceil k/s \rceil$-element sequences of disjont sets from $\mathscr{S}$ in F_p. If the event $\mathscr{A}$ does not hold, then $Z \geq (2\mu)^\kappa$. On the other hand, $Exp(Z) \leq |\mathscr{S}|^\kappa p^{s\kappa} = \mu^\kappa$ and by Markov's inequality

$$P(\neg\mathscr{A}) \leq P(Z \geq (2\mu)^\kappa) \leq \frac{Exp(Z)}{(2\mu)^\kappa} < 2^{-\kappa} \leq 2^{-k/s} . \quad \square$$

3. Main proof

In this section a strong generalization of the 1-statement of Theorem $1'$ will be proved. Recall that for a graph G with at least three vertices

$$m_G^{(2)} = \max_{\substack{H \subseteq G \\ v_H \geq 3}} \frac{e_H - 1}{v_H - 2} \qquad \text{and set} \qquad m_{K_2}^{(2)} = 1 .$$

Observe that $m_G^{(2)} > 1$ if and only if G contains a cycle. For G a forest, $m_G^{(2)} = 1$.

Given graphs F and G, integers r, k, and l, and a family $\mathscr{G}$ of copies of G in F, we write $F \xrightarrow[\mathscr{G}]{k,l} (G)_r^2$ if for every $(r+1)$- coloring of the edges of F with color $r+1$ appearing at most k times, there exists a color $s \in [r]$ so that at least l copies of G belonging to the family $\mathscr{G}$ are monochromatic in color s.

A subgraph of F is called *nested* if it is contained in a complete subgraph of F. Let $\mathscr{N}_F$ denote the family of all nested subgraphs of F.

Recall that for $0 < p < 1$, F_p is a random subgraph of F obtained by independent deletion of the edges of F, each with probability $1-p$.

The following theorem is a many-fold strengthening of the 1-statement of Theorem $1'$.

Theorem 3. *For every graph G with at least one edge, for all integers $r \geq 1$ and all numbers $0 < d \leq 1$, there exist positive numbers ε, a, b, C, and n_0 such that if*

(i) $n > n_0$,

(ii) *F is an (ε, d)-dense graph with n vertices, and*

(iii) $p > Cn^{-1/m_G^{(2)}}$,

then, setting $l = an^{v_G}p^{e_G}$,

$$P\left(F_p \xrightarrow[\mathcal{N}_F]{0,l} (G)_r^2\right) > 1 - 2^{-be_F p} .$$

The proof is by double induction on e_G and r. For graphs G with one edge the statement is an immediate consequence of Chernoff's inequality. In the case $r = 1$ we use Janson's inequality. Let S be the number of copies of G in F_p. We shall now apply Janson's inequality to S with $\varepsilon = .1$, say. By Lemma 2 there are at least $c_0 n^{v_G}$ nested copies G' of G in F. Let us associate with each of them a 0-1 random variable $I_{G'}$, equal to 1 if G' is present in F_p and to 0 otherwise. Then $Exp(S) = Exp(\sum I_{G'}) \geq c_0 n^{v_G} p^{e_G}$ and

$$\sum \sum_{G' \cap G'' \neq \varnothing} Exp(I_{G'} I_{G''}) \leq \sum_{K \subseteq G} n^{2v_G - v_K} p^{2e_G - e_K} ,$$

as there are no more than $n^{2v_G - v_K}$ pairs of copies of G in F which intersect on a subgraph isomorphic to K. By the definition of $m_G^{(2)}$, $n^{v_K} p^{e_K} \geq n^2 p$, provided $p \geq n^{-1/m_G^{(2)}}$, giving the bound

$$\sum \sum_{G' \cap G'' \neq \varnothing} Exp(I_{G'} I_{G''}) \leq \sum_{K \subseteq G} n^{2v_G - v_K} p^{2e_G - e_K} \leq 2^{e_G} \frac{n^{2v_G} p^{2e_G}}{n^2 p} .$$

Plugging it all into Janson's inequality and noticing that $e_F \geq d\binom{n}{2}$, we obtain, for some $b > 0$,

$$P(S > .9c_0 n^{v_G} p^{e_G}) > 1 - \exp\{-be_F p\} ,$$

which completes the proof of Theorem 3 in case $r = 1$.

The idea. Before getting overwhelmed by the details let us outline the ideas behind the proof of the induction step.

Assume that $e = e_G \geq 2$ and $r \geq 2$. Let η be an arbitrary edge of G. Set $H = G - \eta = (V(G), E(G) \setminus \{\eta\})$. Our strategy is to generate F_p in two rounds, i.e. to represent it as a union of two independent random graphs F_{p_1} and F_{p_2} with

$$p_2 = \frac{p - p_1}{1 - p_1} ,$$

equivalently $p_1 + p_2 - p_1 p_2 = p$, with p_1 and p_2 suitably chosen. It is planned that p_2 will be sufficiently bigger than p_1, but both of the same order of magnitude. More specifically, we shall set

$$p_1 = \alpha p_2 , \tag{18'}$$

where α will be given by (32). Note that we have

$$(18'') \qquad p < p_1 + p_2 .$$

Throughout the proof we will be assuming that p is smaller than an arbitrarily small constant. We may do so, since for p constant our Theorem 3 follows easily from Chernoff's inequality and Lemmas 1 and 2.

After the first round is completed, we ask the enemy to color the edges of F_{p_1} (coloring h). An edge of $E(F) \setminus E(F_{p_1})$ is said to *close a copy of H* if together they form a copy of G. For $1 \le s \le r$, we call an edge *s-rich* if it closes $\Omega(n^{v_G-2}p_1^{e_G-1})$ nested monochromatic copies of H colored by color s. (All Ω's and Θ's will be replaced by concrete constants later in the proof.) Let Γ_h^s be the graph of all s-rich edges of $E(F) \setminus E(F_{p_1})$ and let $\Gamma = \Gamma_h^1 \cup \cdots \cup \Gamma_h^r$. We want to apply Lemma 1 to graph Γ with parameters r, d_0, and $\rho' = \varepsilon(G, r-1, \frac{d_0}{20r})$, where $d_0 > 0$ is a suitably chosen constant and $\varepsilon(G, r-1, \frac{d_0}{20r})$ is determined by Theorem 3 applied with $r-1$ colors. In order to be able to apply Lemma 1 to Γ, we must show, however, that Γ satisfies the assumptions of Lemma 1, which will be taken care of in the following Claim.

Claim. *For some $\rho = \rho(r, d_0, \rho') > 0$ (very small but independent from n) and for n sufficiently large, the graph Γ has, with high probability, the property that every ρn-element subset of vertices spans at least $d_0\binom{\rho n}{2}$ edges for a suitably chosen d_0, i.e. Γ is (ρ, d_0)-dense.*

Hence the graph Γ satisfies the assumptions of Lemma 1 and consequently there exists a subset $V \subset V(\Gamma) = [n]$, $|V| \ge \nu n$, and $s \in [r]$ such that the graph $B = \Gamma_h^s[V]$ is (ρ', d_0')-dense, where $d_0' = \frac{d_0}{20r}$. (Compare Fact (a) later in the proof.)

Now the second round of generating F_p takes place and we just focus on the random graph $B_{p_2} = \Gamma_h^s[V]_{p_2}$. As $\rho' = \varepsilon(G, r-1, d_0')$, we may apply our induction assumption for $r-1$ colors. Thus, by THEOREM$(G, r-1, d_0')$ combined with Lemma 3, we get that, with high probability, B_{p_2} has the property that if colored by r colors with one of the colors appearing not too extensively, there will be $\Omega((\nu n)^{v_G}p_2^{e_G})$ monochromatic nested copies of G in one of the $r-1$ remaining colors. Finally, we ask the enemy to complete the coloring h by extending it to all edges of F_p (coloring $\bar{h}$). If she (w.l.o.g. we may assume that the enemy is a female) uses color s on a small enough fraction of all $\Theta(n^2p_2)$ edges of B_{p_2}, then at least $\Omega((\nu n)^{v_G}p_2^{e_G})$ nested monochromatic copies of G will appear in one of the remaining $r-1$ colors. If, on the other hand, she uses color s $\Omega(n^2p_2)$ times, then recall that each edge of $\Gamma_h^s \supset B_{p_2}$ closes $\Omega(n^{v_G-2}p_1^{e_G-1})$ nested monochromatic copies of H colored by color s. This way we obtain $\Omega(n^{v_G}p_2p_1^{e_G-1}) = \Omega(n^{v_G}p^{e_G})$ nested s-colored copies of G.

Since the outcome of the second round must be successful no matter how the enemy colored the edges emerging from the first round, the probability of failure in the second round must be much smaller than the reciprocal of the number of all possible r-colorings h of the edges of F_{p_1}, which with very high

probability is less than $r^{n^2 p_1}$. This is taken care of by choosing p_2 sufficiently bigger than p_1.

Details. Let us first recall that we are proving Theorem 3 by double induction on e_G – the number of edges of G, and on r – the number of colors, and that we have already verified both initial cases, i.e. cases $e_G = 1$, r arbitrary, and $r = 1$, e_G arbitrary. Setting $H = G - \{\eta\} = (V(G), E(G) \setminus \{\eta\})$ we will further assume the validity of two instances of Theorem 3: THEOREM(H, r, d) and THEOREM(G, $r-1$, d_0'). We begin with fixing all the constants to come. An unpatient reader is advised to skip this part at the first reading and go directly to the paragraph following formula (35). Let $b(H, r, d)$ be determined through our THEOREM(H, r, d). Setting

$$c = c_1 = b(H, r, d)(1 - \delta_1) \tag{19}$$

we choose δ_1 to satisfy (13). Choose $0 < \gamma < 1$ to satisfy

$$(a(H, r, d)(1 - \delta_1)^{e_H})^2 (1 - \gamma/2)^2 > 3(1 - \gamma) v_G^2 2^{\binom{2v_G - 2}{2}} . \tag{20}$$

We also set

$$d_0 = \frac{2}{3} d(1 - \gamma) \qquad \text{and} \qquad d_0' = \frac{d_0}{20r} , \tag{21}$$

and

$$\rho' = \varepsilon(G, r - 1, d_0') , \tag{22}$$

the latter through our THEOREM(G, $r-1$, d_0'). Furthermore, let

$$\rho = \rho(r, d_0, \rho'), \quad n_0(r, d_0, \rho') \qquad \text{and} \qquad \nu = \nu(r, d_0, \rho') \tag{23}$$

be determined by Lemma 1 and set

$$\varepsilon = \varepsilon(G, r, d) = \rho \varepsilon(H, r, d) . \tag{24}$$

Let

$$b_1 = \frac{1}{2} \min\{b(H, r, d)(1 - \delta_1)/2, (\log_2 e)\text{CHERNOFF}(\delta_1/2)\} , \tag{25}$$

$$b_2 = \frac{\delta_1}{4e_H} 2^{-\binom{2v - 2}{2}} , \tag{26}$$

$$b_3 = \frac{d}{3} \min\{b_1, b_2, \text{CHERNOFF}(1)\}, \tag{27}$$

and

$$b' = \frac{1}{2} b_3 \rho^2 . \tag{28}$$

Furthermore, let δ_2 satisfy (13) with $c = b(G, r - 1, d_0')(1 - \delta_2)$. Set

$$b_4 = \frac{1}{2} \min\{b(G, r - 1, d_0')(1 - \delta_2)/2, (\log_2 e)\text{CHERNOFF}(\delta_2/2)\} \tag{29}$$

and

$$b'' = b_4 \frac{d_0 \nu^2}{80r} = b_4 \frac{d_0'}{4} \nu^2 . \tag{30}$$

Finally, take

$$b = b(G, r, d) = b'''/(\alpha + 1) \quad \text{and} \quad b''' = \min(\alpha b'/2, b''/4) , \tag{31}$$

where

$$\alpha = \frac{b''}{2 \log_2 r} . \tag{32}$$

Let

$$n_0(G, r, d) = \max\left\{ \frac{1}{\rho} n_0(H, r, d), \frac{1}{\nu} n_0(G, r-1, d_0'), n_0(r, d_0, \rho'), n^* \right\} , \tag{33}$$

where n^* guarantees that all not otherwise justified inequalities hold. Let

$$C(G, r, d) = \max\{(1 + 1/\alpha)C_1, (1 + \alpha)C_2\} , \tag{34}$$

where

$$C_1 = \max\left\{ \frac{C(H, r, d)}{(1 - \delta_1)\rho^{1/m_G^{(2)}}}, \frac{1}{\rho^{1/m_G^{(2)}}}, \frac{2}{b_3 \rho^2} \right\}$$

and

$$C_2 = \frac{C(G, r-1, d_0')}{(1 - \delta_2)\nu^{1/m_G^{(2)}}} .$$

Finally, let

$$\begin{aligned} a(G, r, d) = \min\Bigg\{ & a(G, r-1, d_0')(1 - \delta_2)^{e_G} \nu^{v_G} \left(\frac{1}{1+\alpha}\right)^{e_G} , \\ & \frac{\delta_2 d_0' \nu^2}{4 r v_G^2} a(H, r, d)(1 - \delta_1)^{e_G - 1} \rho^{v_G - 2} \frac{\alpha^{e_G - 1}}{(1+\alpha)^{e_G}} \Bigg\} . \end{aligned} \tag{35}$$

Now the conditional argument can be described as follows. Let $\mathscr{A}$ be the event that there is an r-coloring $\bar{h} : E(F_p)) \to [r]$ with less than $an^{v_G} p^{e_G}$ nested monochromatic copies of G in each color.

For a copy H' of H in F, let $cl(H')$ be the set of all edges $\eta \in F$ such that $H' \cup \{\eta\}$ is isomorphic to G. Given an r-coloring $h : E(F_{p_1}) \to [r]$, define the auxiliary graphs

$$\Gamma_h^i = \{\eta \in E(F) \setminus E(F_{p_1}) \ : \ |\{H' \subseteq F_{p_1} \ : \ \eta \in cl(H') \text{ and } h(H') = i\}| > z\} , \tag{36}$$

$i = 1, \dots, r$, where

$$z = a(H, r, d)(1 - \delta_1)^{e_G - 1} (\rho n)^{v_G - 2} p_1^{e_G - 1} . \tag{36'}$$

Let $\mathscr{B}$ be the event that for every $h : E(F_{p_1}) \to [r]$ there exists an $s \in [r]$ and a set $V \subset V(F) = [n]$, $|V| > \nu n$, ν defined by (23), such that the graph $\Gamma_h^s[V]$

is (ρ', d_0')-dense and that $|E(F_{p_1})| < n^2 p_1$. Conditioning on F_{p_1} and fixing h, let $\mathscr{A}_h$ be the event that there is an extension of h, $\bar{h} : E(F_p) \to [r]$, i.e. $\bar{h} = h$ when restricted to $E(F_{p_1})$, such that for each color $s \in [r]$ there are less than $an^{v_G} p^{e_G}$ monochromatic nested copies of G in color s. Then, by the law of total probability,

$$P(\mathscr{A}) \le P(\neg\mathscr{B}) + \sum_{K \in \mathscr{B}} P(\mathscr{A} \mid F_{p_1} = K) P(F_{p_1} = K), \tag{37}$$

and by Boole's inequality

$$P(\mathscr{A} \mid F_{p_1} = K) = P\left(\bigcup_h \mathscr{A}_h \mid F_{p_1} = K\right) \le r^{n^2 p_1} P(\mathscr{A}_{h_0} \mid F_{p_1} = K) \tag{38}$$

(the summation is taken over all r-colorings h of the edges of $F_{p_1} = K$; h_0 maximizes the conditional probability). Hence, all we need to prove are the two following facts:

Fact (a). $P(\mathscr{B}) > 1 - 2^{-b' n^2 p_1}$, *where* b' *is defined by* (28).

Fact (b). *For every* $K \in \mathscr{B}$ *and for every* r*-coloring* h *of the edges of* K, $P(\mathscr{A}_h \mid F_{p_1} = K) \le 2^{-b'' n^2 p_2}$, *where* b'' *is defined by* (30).

Indeed, if both (a) and (b) are true, then, by $(18')$, (32), (37), (38)

$$P(\mathscr{A}) \le 2^{-b' n^2 p_1} + r^{n^2 p_1} 2^{-b'' n^2 p_2} \le 2^{-b' \alpha n^2 p_2} + 2^{-\frac{1}{2} b'' n^2 p_2} \overset{(31)}{\le} 2(2^{-2b''' n^2 p_2}).$$

By (33), $n > n^*$ and by $p \le (\alpha + 1) p_2$ we conclude that

$$P(\mathscr{A}) \le 2(2^{-2b''' n^2 p_2}) \le 2^{-b''' n^2 p_2} \le 2^{-b n^2 p}.$$

Proof of (a). Let us fix a set $A \subset V(F) = [n]$ of size $|A| = \lceil \rho n \rceil$, with ρ defined by (23). Since F is (ε, d)-dense, $F[A]$, the subgraph of F induced by A, is $(\frac{\varepsilon}{\rho}, d)$-dense, and hence, by (24), it is also $(\varepsilon(H, r, d), d)$-dense. Moreover, due to $(18')$, $(18'')$, and (34), we have

$$(1 - \delta_1) p_1 \ge (1 - \delta_1) \frac{\alpha}{\alpha + 1} p \ge C(H, r, d)(\rho n)^{-1/m_G^{(2)}} \ge C(H, r, d)(\rho n)^{-1/m_H^{(2)}}$$

and, by (33),

$$|V(F[A])| = \rho n \ge n_0(H, r, d).$$

Thus we may apply THEOREM(H, r, d), concluding that

$$P\left(F[A]_{(1-\delta_1)p_1} \overset{0,l}{\underset{\mathscr{N}_{F[A]}}{\to}} (H)_r^2\right) > 1 - 2^{-c_1 M p_1},$$

where

$$l = a(H, r, d)(1 - \delta_1)^{e_H} (\rho n)^{v_H} p_1^{e_H}, \tag{39}$$

c_1 is given by (19), and

$$d\binom{\rho n}{2} \le M = |E(F[A])| \le \binom{\rho n}{2}. \tag{39'}$$

Now by Lemma 3 with $c = c_1$ defined in (19) and with $k = \frac{\delta_1}{2} M p_1$, we derive that

$$P\left(F[A]_{p_1} \underset{\mathcal{N}_{F[A]}}{\overset{k,l}{\to}} (H)_r^2\right) > 1 - 2^{-b_1 M p_1}, \tag{40}$$

where b_1 is defined by (25).

Let $\mathcal{T} = \{T_1, \dots, T_t\}$ be the family of all pairwise nonisomorphic graphs which are unions of two copies of H, say $H_1 \cup H_2$, such that for some edge η, $H_1 + \eta$ and $H_2 + \eta$ are isomorphic to G. Note that $t \le 2^{\binom{2v_G - 2}{2}}$. As we are heading toward an application of Lemma 4, let us focus on a particular $T \in \mathcal{T}$ and find an upper bound on the expected number μ_T of the copies of T in $F[A]_{p_1}$. Trivially, the number N_T of copies of T in $F[A]$ is not greater than $(\rho n)^{v_T}$ and consequently, setting $T = H_1 \cup H_2$ and $I = H_1 \cap H_2$,

$$\mu_T = N_T p_1^{e_T} \le (\rho n)^{2v_H - v_I} p_1^{2e_H - e_I}. \tag{41}$$

We claim that

$$(\rho n)^{v_I} p_1^{e_I} \ge (\rho n)^2. \tag{42}$$

To see this, choose $\eta \in cl(H_1) \cap cl(H_2)$ and set $J = I + \eta = (V(I), E(I) \cup \{\eta\})$. (Note that $J \subset G$.) There is nothing to prove when $E(J) = \{\eta\}$, since then $e_I = 0$, $v_I = 2$ and both sides of (42) coincide. Otherwise, i.e. if $v_J \ge 3$, we have $\frac{e_J - 1}{v_J - 2} \le m_G^{(2)}$ and, as by (34) $p \ge (1 + \frac{1}{\alpha})(\rho n)^{-1/m_G^{(2)}}$ and by (18′) $p_1 \ge \frac{\alpha}{\alpha+1} p$, we infer that

$$p_1^{e_I} = p_1^{e_J - 1} \ge (\rho n)^{2 - v_J} = (\rho n)^{2 - v_I},$$

which proves (42). Set

$$v = v_G = v_H \text{ and } e = e_G = e_H + 1, \tag{42'}$$

for convenience. Inequalities (41) and (42) imply together that

$$\mu_T \le (\rho n)^{2v-2} p_1^{2e-2}.$$

On the other hand, setting $k_t = \frac{\delta_1}{2t} M p_1$ (recall that $|\mathcal{T}| = t$), and b_2 as in (26), we have $\frac{k_t}{e_T} \ge b_2 M p_1$, for any $T \in \mathcal{T}$. Hence, by Lemma 4, applied additively to the families $\mathcal{S}_T$ of all copies of $T \in \mathcal{T}$ in F, and with $k = k_t$ and $s = e_T$, we conclude that, with probability at least

$$1 - \sum_{T \in \mathcal{T}} 2^{-\frac{k_t}{e_T}} \ge 1 - t 2^{-b_2 M p_1} \tag{43}$$

there exists a set $E_A \subseteq E(F[A]_{p_1})$, $|E_A| \le k = \frac{1}{2}\delta_1 M p_1$, such that the graph $R = F[A]_{p_1} - E_A$ contains at most $2t(\rho n)^{2v-2} p_1^{2e-2}$ subgraphs isomorphic to members of $\mathcal{T}$, i.e. at most $2t(\rho n)^{2v-2} p_1^{2e-2}$ pairs H_1, H_2 of copies of H in R satisfy $cl(H_1) \cap cl(H_2) \ne \varnothing$.

Let $h : E(F_{p_1}) \to [r]$ be an r-coloring of the edges of F_{p_1}. Still focusing on a particular set A, by (40), with probability at least $1 - 2^{-b_1 M p_1}$ there is a color $s \in [r]$ such that at least l nested copies of H in $F[A]_{p_1} - E_A$ are monochromatic in color s. Let, for every $i \in E(F[A])$, x_i be the number of monochromatic copies H' of H in R colored by color s for which $i \in cl(H')$. Then, combining (40) and (43), using (39′), with probability at least

$$1 - 2^{-b_1 M p_1} - t2^{-b_2 M p_1} \geq 1 - 2^{-b_1 d\binom{\rho n}{2} p_1} - t2^{-b_2 d\binom{\rho n}{2} p_1}, \tag{44}$$

$$x_1 + \cdots + x_M > l \tag{45}$$

and

$$\binom{x_1}{2} + \cdots + \binom{x_M}{2} < 2tv^2(\rho n)^{2v-2} p_1^{2e-2} \tag{46}$$

hold. Note that l was defined by (39) and, as $|cl(H)| \leq \binom{v}{2} - (e-1) < v^2$, the factor v^2 takes care of possible duplications. Inequalities (45) and (46) will soon imply that there should be many terms among x_i's not smaller than, say, half of the average. Let us order them from low to high, $x_1 \leq x_2 \leq \cdots \leq x_M$, and choose γ to satisfy (20). Set $i_0 = \lfloor \gamma M \rfloor$ and suppose that

$$x_{i_0} < \frac{l}{2M}. \tag{47}$$

Then by (45) and (47),

$$\sum_{i=i_0+1}^{M} x_i > l - i_0 \frac{l}{2M} \geq l(1 - \gamma/2). \tag{48}$$

Moreover, by the Schwarz (or Jensen) inequality and by (48),

$$\sum_{i=i_0+1}^{M} \binom{x_i}{2} \geq (M - i_0)\binom{\frac{\sum x_i}{M - i_0}}{2} \geq \frac{(\sum x_i)^2}{3(M - i_0)} > \frac{l^2(1 - \frac{\gamma}{2})^2}{3(M - i_0)},$$

which contradicts (46) by (20). Hence (47) is false and consequently, using (39′),

$$x_M \geq \cdots \geq x_{i_0} \geq \frac{l}{2M},$$

i.e. there are at least

$$M - i_0 \geq d(1 - \gamma)\binom{\rho n}{2} \tag{49}$$

edges $\eta \in E(F[A])$, each belonging to $cl(H')$ for at least

$$\frac{l}{2M} \geq a(H, r, d)(1 - \delta_1)^{e-1}(\rho n)^{v-2} p_1^{e-1} = z \tag{50}$$

(same z as in (36′)) nested monochromatic copies H' of H in $F[A]_{p_1}$ colored by s. Also, by Chernoff's inequality, with probability at least

$$1 - 2^{-\mathrm{CHERNOFF}(1) M p_1} \geq 1 - 2^{-\mathrm{CHERNOFF}(1) d\binom{\rho n}{2} p_1}, \tag{50′}$$

$$|E(F[A]_{p_1})| \le 2Mp_1 \le 2\binom{\rho n}{2}p_1 \tag{51}$$

holds. By (44) and (50′) we infer that inequalities (49), (50), and (51) hold true, with probability at least

$$\begin{aligned} &1 - \binom{n}{\rho n}\left[2^{-db_1\binom{\rho n}{n}p_1} + t2^{-db_2\binom{\rho n}{2}p_1} + 2^{-\mathrm{CHERNOFF}(1)d\binom{\rho n}{2}p_1}\right] \\ &\ge 1 - \binom{n}{\rho n}2^{-b_3(\rho n)^2 p_1}, \end{aligned} \tag{52}$$

for every ρn-vertex subset of $A \subset [n] = V(F)$, where b_3 is defined by (27). Of course, the color s varies from set to set. To single out a "majority" color, we need to apply Lemma 1 to the graph $\Gamma = \bigcup_{s=1}^{r} \Gamma_h^s$, with Γ_h^s defined (cf. (36)) as the graph of all edges of F which were not picked in the first round and belong to $cl(H')$ for at least z monochromatic (in color s) copies H' of H in F_{p_1}. By (49)–(52), with probability at least $1 - \binom{n}{\rho n}2^{-b_3(\rho n)^2 p_1}$, Γ has at least

$$d(1-\gamma)\binom{\rho n}{2} - 2\binom{\rho n}{2}p_1 > d_0\binom{\rho n}{2}$$

edges in each ρn-vertex induced subgraph, where d_0 is defined by (21). This is true by our assumption that p is smaller than any constant. In short, Γ is (ρ, d_0)-dense. We apply Lemma 1 with inputs r, d_0, and ρ' defined by (22) to the partition $E(\Gamma) = E_1 \cup \cdots \cup E_r$, where $E_i = E(\Gamma_h^i)$, and as a result we obtain the existence of an index $s \in [r]$ and a subset V of $V(\Gamma)$, $|V| \ge \nu n$, such that the graph $B = \Gamma_h^s[V] = (V, E_s \cap [V]^2)$ is (ρ', d_0')-dense (here ν is defined by (23) and d_0' by (21)). Summarizing, with probability at least

$$1 - \binom{n}{\rho n}2^{-b_3(\rho n)^2 p_1} > 1 - 2^n 2^{-b_3(\rho n)^2 p_1} > 1 - 2^{-b' n^2 p_1},$$

where b' is defined by (28), the graph B is (ρ', d_0')-dense and, by (51), $|E(F_{p_1})| < n^2 p_1$. (The second bound on probability follows immediately from $n > n^*$ in the case when $m_G^{(2)} > 1$. When $m_G^{(2)} = 1$, i.e. when G is a forest, we use that part of (34) which says that $C_1 \ge \frac{2}{b_3\rho^2}$.) This completes the proof of Fact (a).

Proof of (b). To prove statement (b) we condition on the outcome of round 1, i.e. we fix F_{p_1} satisfying property $\mathscr{B}$, and we also fix a coloring $h : E(F_{p_1}) \to [r]$, which according to property $\mathscr{B}$ yields a color s and a set $V \subset V(F) = [n]$, $|V| \ge \nu n$, such that the graph $B = \Gamma_h^s[V]$ is (ρ', d_0')-dense.

By inequalities (33) and (34), n and p_2 are big enough so that we may apply THEOREM$(G, r-1, d_0')$ to the random graph $B_{(1-\delta_2)p_2}$, where δ_2 satisfies (13) with $c_2 = b(G, r-1, d_0')(1-\delta_2)$. As a result we obtain, setting $l' = a(G, r-1, d_0')(1-\delta_2)^e(\nu n)^v p_2^e$,

$$P\left(B_{(1-\delta_2)p_2} \underset{\mathscr{N}_B}{\overset{0,l'}{\to}} (G)_{r-1}^2\right) > 1 - 2^{-c_2 e_B p_2}.$$

This, by Lemma 3 with k replaced by $k' = \frac{\delta_2}{2} e_B p_2$, implies that

$$P\left(B_{p_2} \underset{\mathcal{N}_B}{\overset{k',l'}{\to}} (G)^2_{r-1}\right) > 1 - 2^{-b_4 e_B p_2} > 1 - 2^{-b'' n^2 p_2} ,$$

where b_4 and b'' are defined by (29) and (30). Here we use the estimate

$$e_B \geq d_0' \binom{\nu n}{2} \geq d_0' \frac{(\nu n)^2}{4} . \tag{53}$$

Now let $\bar{h} : E(F_p) \to [r]$ be any extension of h. Let y be the number of edges of B_{p_2} colored by $\bar{h}$ with color s. If $y < k'$ then there will be at least l' nested monochromatic copies of G in one of the colors different from s. If, on the other hand, $y \geq k'$, then, as each of the edges of B_{p_2} closes at least z nested monochromatic copies of H colored by h with color s, there will be at least $\frac{k'z}{v^2}$ nested monochromatic copies of G colored by color s, where z is as in (50). Note that, by (53), $k' \geq \frac{\delta_2}{2} d_0' \frac{(\nu n)^2}{4} p_2$, and thus, by (35), (18′) and the fact that $p < p_1 + p_2$, there is, one way or another, a color $i \in [r]$, so that there are at least

$$\begin{aligned}
&\min\left\{l', \frac{k'z}{v^2}\right\} \\
&\geq \min\Bigg\{a(G, r-1, d_0')(1-\delta_2)^{e_G} \nu^{v_G} (p_2/p)^{e_G}, \\
&\qquad \frac{\delta_2 d_0' \nu^2}{8 v_G^2} a(H, r, d)(1-\delta_1)^{e_G - 1} \rho^{v_G - 2} (p_1/p)^{e_G - 1} (p_2/p)\Bigg\} n^{v_G} p^{e_G} \\
&\geq \min\Bigg\{a(G, r-1, d_0')(1-\delta_2)^{e_G} \nu^{v_G} \left(\frac{1}{1+\alpha}\right)^{e_G}, \\
&\qquad \frac{\delta_2 d_0' \nu^2}{8 v_G^2} a(H, r, d)(1-\delta_1)^{e_G - 1} \rho^{v_G - 2} \frac{\alpha^{e_G - 1}}{(1+\alpha)^{e_G}}\Bigg\} n^{v_G} p^{e_G} \\
&= a(G, r, d) n^{v_G} p^{e_G}
\end{aligned}$$

monochromatic copies of G colored by color i.

4. Locally sparse Ramsey graphs

By locally sparse Ramsey graphs we mean Ramsey graphs with no small and dense subgraphs. The first question about the existence of such graphs was raised by Erdős and Hajnal [EH 67] when they asked for K_6-free $(K_3, 2)$-Ramsey graphs, i.e. graphs F such that $F \to (K_3)_r$ but $K_6 \not\subseteq F$. Graham [Gr 68] found the smallest graph with this property which has eight vertices and later Irving [Ir 73] constructed a K_5-free $(K_3, 2)$-Ramsey graph on 18 vertices. In 1970 Folkman [Fo 70] constructed for each k an enormous K_{k+1}-free $(K_k, 2)$-Ramsey graph. The existence of such graphs for arbitrary number of colors r was later established by Nešetřil and Rödl [NR 76].

What does our Theorem 3 tell about the existence of sparse Ramsey graphs?

Let $\mathscr{H}$ be a family of graphs. We say that $F \in \mathscr{F}orb(\mathscr{H})$ if no subgraph of F is isomorphic to a member of $\mathscr{H}$.

Let $\mathscr{H}_t(G)$, $\mathscr{H}_t^{(1)}(G)$, and $\mathscr{H}_t^{(2)}(G)$ be the families of all graphs H with $v_H \le t$ and $m_H > m_G^{(2)}$, $m_H^{(1)} \ge m_G^{(2)}$, and $m_H^{(2)} > m_G^{(2)}$, respectively. Part (c) of Corollary 4 below implies Corollary 2 stated in the Introduction.

Corollary 4. *Let $r \ge 2$ and let G be a graph with a cycle, $N = Cn^{2-1/m_G^{(2)}}$, where C is as in Theorem 3. Furthermore let $t > 0$ and let $\mathscr{F}_{n,N}$ be as in the Introduction.*

(a) *almost all $F \in \mathscr{F}_{n,N}$ belong to $\mathscr{F}orb(\mathscr{H}_t(G))$ and are (G, r)-Ramsey,*

(b) *almost all $F \in \mathscr{F}_{n,N}$ which belong to $\mathscr{F}orb\left(\mathscr{H}_t^{(1)}(G)\right)$ are (G, r)-Ramsey,*

(c) *almost all $F \in \mathscr{F}_{n,N}$ contain a subgraph $F' \subset F$ which belongs to $\mathscr{F}orb\left(\mathscr{H}_t^{(2)}(G)\right)$ and is (G, r)-Ramsey.*

Remark. In part (a) of Corollary 4 we can replace the family $\mathscr{H}_t(G)$ by the family of all graphs H, $v_H \ge t$, such that $m_H \ge m_G^{(2)}$, but then it remains true only for a positive fraction of graphs $F \in \mathscr{F}_{n,N}$. This is because, for H with $m_H = m_G^{(2)}$, a positive fraction of all $\binom{\binom{n}{2}}{N}$ graphs are H-free and (G, r)-Ramsey. This follows from the fact that then $0 < \lim_{n\to\infty} P(K(n, N) \not\supset H) < 1$ (see [Ru 90]). Statement (b) implies that there are as many as $(1 + o(1))\binom{\binom{n}{2}}{N} e^{-\Theta(n)}$ (G, r)-Ramsey graphs with N edges and without a subgraph isomorphic to a member of $\mathscr{H}_t^{(1)}$.

Proof. (a) Let X_H denote the number of copies of H in $K(n, N)$ and let $X = \sum_{H\in\mathscr{H}_t} X_H$. Then, by Markov's inequality

$$P(K(n, N) \notin \mathscr{F}orb(\mathscr{H}_t(G))) = P(X > 0) \le Exp(X) = o(1) ,$$

whereas, by Theorem 1, $P(K(n, N) \to (G)_r) \to 1$ as $n \to \infty$.

(b) Setting $p = \frac{N}{\binom{n}{2}}(1 + \varepsilon)$, by the FKG inequality, for every $H \in \mathscr{H}_t^{(1)}(G)$,

$$P(K(n, p) \not\supset H) \ge (1 - p^{e_H})^{\binom{n}{v_H}} \ge \exp\{-O(n^{v_H} p^{e_H})\} \ge \exp\{-O(n)\} .$$

Again by the FKG inequality, using the fact that t and thus also $|\mathscr{H}_t^{(1)}(G)|$ do not depend on n,

(54)
$$P\left(K(n, p) \in \mathscr{F}orb\left(\mathscr{H}_t^{(1)}(G)\right)\right) \ge \prod_{H\in\mathscr{H}_t^{(1)}(G)} P(K(n, p) \not\supset H) \ge \exp\{-O(n)\} ,$$

as $v_H - \frac{e_H}{m_G^{(2)}} \leq 1$. Moreover, by (1) and by Chernoff's inequality,

$$P\left(K(n,p) \in \mathscr{F}orb\left(\mathscr{H}_t^{(1)}(G)\right)\right)$$
$$\leq \sum_{|M-N(1+\varepsilon)|<\varepsilon N} P\left(K(n,M) \in \mathscr{F}orb\left(\mathscr{H}_t^{(1)}(G)\right)\right)\binom{\binom{n}{2}}{M}p^M(1-p)^{\binom{n}{2}-M}$$
$$+ e^{-\mathrm{CHERNOFF}(\frac{\varepsilon}{1+\varepsilon})(1+\varepsilon)N} \leq P\left(K(n,N) \in \mathscr{F}orb\left(\mathscr{H}_t^{(1)}(G)\right)\right) + e^{-\Omega(N)},$$

the last inequality by the monotonicity of subgraph containment. Thus, by (54),

$$P\left(K(n,N) \in \mathscr{F}orb\left(\mathscr{H}_t^{(1)}(G)\right)\right) \geq e^{-O(n)} - e^{-\Omega(N)} = e^{-O(n)}.$$

On the other hand, by Theorem 3 and by Pittel's inequality (2),

$$P(K(n,N) \not\to (G)_r) < 3\sqrt{N}2^{-bn^{2-1/m_G^{(2)}}}. \tag{55}$$

Comparing (54) and (55), and abbreviating $\mathscr{R} = \mathscr{R}_{G,r}(n,N)$ and $\mathscr{F}orb = \mathscr{F}orb(\mathscr{H}_t^{(1)}(G))$, we conclude that

$$\frac{|\neg\mathscr{R} \cap \mathscr{F}orb|}{|\mathscr{F}orb|} \leq \frac{|\neg\mathscr{R}|}{|\mathscr{F}orb|} = \frac{|\mathscr{R}|}{\binom{\binom{n}{2}}{N}}\frac{\binom{\binom{n}{2}}{N}}{|\mathscr{F}orb|} \leq 3\sqrt{N}2^{-bn^{2-1/m_G^{(2)}}}e^{O(n)} = o(1)$$

as $m_G^{(2)} > 1$ for graphs G containing a cycle.

(c) For every $H \in \mathscr{H}_t^{(2)}(G)$, $Exp(X_H) = O(N^{1-\varepsilon})$ for some $\varepsilon > 0$. Thus, by Markov's inequality, almost all $F \in \mathscr{F}_{n,N}$ contain less than $(\log n)^{-2}N$, say, copies of graphs from $\mathscr{H}_t^{(2)}(G)$, which can be destroyed by deleting at most as many edges. Since, by Theorem 3, Lemma 3, and by Pittel's inequality (2), almost all $F \in \mathscr{F}_{n,N}$ are not just (G,r)-Ramsey but they are such even after deleting a fraction of edges, the graph obtained from F after destroying all forbidden subgraphs remains (G,r)-Ramsey and the proof is completed. □

Part (b) of the above corollary is most transparent in the case of such well structured graphs like cliques and cycles.

Corollary 5. *For all $r \geq 2$ and all $k \geq 3$ there exists a constant C such that almost all graphs $F \in \mathscr{F}_{n,N}$ with $N = Cn^{\frac{2k}{k+1}}$ which are K_{k+1}-free are (K_k, r)-Ramsey and almost all graphs $F \in \mathscr{F}_{n,N}$ with $N = Cn^{\frac{k}{k-1}}$ and with girth at least k are (C_k, r)-Ramsey.*

5. Van der Waerden properties of random subsets of integers

The method of proof we developed in this paper can be successfully used for proving Ramsey-type properties of other random combinatorial structures. Below we outline the proof of a threshold result for van der Waerden properties of random subsets of integers.

The question was raised by H.Lefmann in Poznań (1993), and independently, by P.Erdős and V.Sós a month later in Oberwolfach. We would like to thank

all of them for stimulating discussions and, in addition, Paul Erdős for pointing out the paper of Varnavides.

For integers $k \geq 3$ and $r \geq 2$, and for a set $F \subseteq [n]$, we write $F \to (k)_r$ if every r-coloring of F results in at least one monochromatic k-term arithmetic progression (AP_k in short). Recall that for $0 < p < 1$, F_p stands for a random subset of F obtained by independent inclusion of each element with probability p. The theorem below is the binomial version of Theorem 2 stated in the Introduction.

Theorem 2′. *For all $k \geq 3$ and $r \geq 2$ there exist constants c and C such that*

$$\lim_{n\to\infty} P([n]_p \to (k)_r) = \begin{cases} 0 & \text{if } p < cn^{-\frac{1}{k-1}}, \\ 1 & \text{if } p > Cn^{-\frac{1}{k-1}}. \end{cases}$$

In outlining the proof we shall focus on the 1-statement only, which, as in case of graphs, must be first significantly strengthened.

We say that an AP_k is *t-nested in F* if it constitutes the initial segment of an AP_{k+t} belonging to F. The proof is based on double induction on k and r, and the following strengthening is needed for the induction step.

Theorem 4. *For all integers $k \geq 2$, $r \geq 1$, and $t \geq 0$, and for all real numbers $0 < d < 1$, there exist positive numbers a, b, C, and n_0 such that if $n > n_0$, $F \subseteq [n]$, $|F| > dn$, and $p > Cn^{-\frac{1}{k-1}}$ then, with probability at least $1 - 2^{-bnp}$, every r-coloring of F_p results in at least an^2p^k monochromatic t-nested in F AP_k's.*

In the proof of Theorem 3 in Section 3 we needed the notion of (ρ, d)-dense graphs, mainly because for graphs, being just dense does not necessarily imply the existence of many copies of a given graph. Consequently, Lemma 1 was vital to us. Working with integers rather than with graphs puts us in a more comfortable position, since dense subsets of integers contain, roughly, as many AP_k's as the set of all integers does. This follows quite easily from Szemerédi's density version of van der Waerden's Theorem [Sz 75]. For $k = 3$ the next lemma was first proved by Varnavides [Va 59].

Lemma 5. *For every $k \geq 3$ and for every $d > 0$ there exist n_0 and $\delta > 0$ such that every set $F \subseteq [n]$, $|F| > dn$, $n > n_0$, contains at least δn^2 AP_k's.* □

Thus, the quantity an^2p^k appearing in Theorem 4 is a positive fraction of the expected number of AP_k's in F_p.

Now we may start our inductive proof.

Case $r = 1$, k arbitrary. By Lemma 5, there are at least δn^2 AP_{k+t}'s in F, yielding at least δn^2 t-nested AP_k's. By Janson's inequality, with probability at least $1 - 2^{-bnp}$, at least an^2p^k of them will be present in F_p, for some $a, b > 0$. Since only one color is used, they all are bound to be monochromatic.

Case $k = 2$, r arbitrary. Here we run induction on r. Assuming $r \geq 2$, we generate F_p in two rounds, as we did for graphs, i.e. $F_p = F_{p_1} \cup F_{p_2}$,

$p = p_1 + p_2 - p_1 p_2$, p_1, p_2 of the same order of magnitude as p, but p_2 sufficiently larger than p_1.

By Lemma 5, there are at least δn^2 AP_{t+2}'s in F. Denote by $A_s(v, u)$ the AP_s whose first element is v and the difference is u. For $v \in F$ set $N(v) = \{v + u : A_{t+2}(v, u) \subseteq F\}$, i.e. $N(v)$ is the set of all second elements of the AP_{t+2}'s of F originating at v. Now, let $F' = \{v : |N(v)| \geq \frac{1}{2}\delta n\}$. Then

$$\sum_{v \in F \setminus F'} |N(v)| < \frac{1}{2}\delta n |F \setminus F'| \leq \frac{1}{2}\delta n^2$$

and hence $\sum_{v \in F'} |N(v)| > \frac{1}{2}\delta n^2$. As $|N(v)| < n$ for every $v \in F$, we infer that $|F'| \geq \frac{1}{2}\delta n$. By Chernoff's inequality, $|F'_{p_1}| = \Theta(np_1)$ with probability $1 - e^{-\Theta(np_1)}$. (Note that np_1 may be $O(1)$.) For an r-coloring of F'_{p_1}, let B be the largest color class (say, red). Set $N'(v) = N(v) \setminus F'_{p_1}$ and consider the bipartite graph $\mathscr{B}$ with vertex classes B and $\bigcup_{v \in B} N'(v)$ and with edge set $\{\{v, w\} : w \in N'(v)\}$. Again by averaging, there is a set $D \subseteq \bigcup_{v \in B} N'(v)$ such that $|D| = \Theta(n)$ and each $w \in D$ has degree $\deg_{\mathscr{B}}(w) = \Theta(np_1)$.

Now, we expose $F \setminus F_{p_1}$ with probability p_2 (second round) and focus on D_{p_2} only. With probability $1 - e^{-\Theta(np_2)}$, $|D_{p_2}| = \Theta(np_2)$. Let $\varepsilon > 0$ be a sufficiently small constant. If at least $\varepsilon n p_2$ elements of D_{p_2} are colored red then, by the definition of D and $\mathscr{B}$, we obtain $\Theta(n^2 p^2)$ red t-nested AP_2's (pairs). If red is used less than $\varepsilon n p_2$ times then, combining Lemma 3 with our induction assumption for $r - 1$ colors (in the same manner as we did for graphs), gives, with probability at least $1 - e^{-\Theta(np_2)}$, at least $an^2 p_2^2$ t-nested pairs in one of the other colors.

Case $k \geq 3$, r arbitrary. Here we basically follow the steps of case $k = 2$, but instead of the two rather simple averagings, we apply the more sophisticated argument, very similar to that from the proof of Fact (a) in Section 3.

We aim to obtain, with probability $1 - e^{-\Theta(np_1)}$, a set $D \subseteq F \setminus F_{p_1}$, $|D| > \delta n$, such that each $w \in D$ is the kth element of $\Theta(np_1^{k-1})$ t-nested AP_k's, whose first $k - 1$ elements belong to F_{p_1} and are all red.

By the induction assumption we know that with probability at least $1 - e^{-\Theta(np_1)}$, for every r-coloring of $F_{(1-\delta)p_1}$ there are $\Theta(n^2 p_1^{k-1})$ $(t + 1)$-nested monochromatic AP_{k-1}'s (say, red). By Lemma 3, it is also true for F_{p_1}, even after deleting up to $\frac{\delta}{2} n p_1$ elements from F_{p_1}. We do need to delete them in order to make, via Lemma 4, the number of pairs of intersecting $(t + 1)$-nested AP_{k-1}'s in F_{p_1} smaller than their doubled expectation. Here we care for intersections occurring at the nested parts. Now, an averaging argument parallel to that in Section 3 (compare the text between formulas (43)–(50)) gives the existence of the required set D. Finally, we expose D with probability p_2 and complete the proof as in case $k = 2$.

Sparse van der Waerden sets. Theorem 4 has interesting consequences concerning "locally sparse" van der Waerden sets. The following has been constructively proved in [Sp 75] and [NR 76a].

Theorem ([Sp 75],[NR 76a]). *For all k and r there exists a set S of integers such that S contains no AP_{k+1} while $S \to (k)_r$.*

Setting $N = cn^{\frac{k-2}{k-1}}$ and applying the FKG inequality, formula (1), Chernoff's inequality, and monotonicity (as in the proof of Corollary 4(b)) we infer that there are at least $e^{-O\left(n^{\frac{k-3}{k-1}}\right)}\binom{n}{N}$ N-element subsets of $[n]$ containing no AP_{k+1}. Thus, using equivalence between uniform and binomial models, we infer from Theorem 4 the following.

Corollary 6. *For every k and r, there exists a constant C such that almost all $Cn^{\frac{k-2}{k-1}}$-element subsets $S \subseteq [n]$ which contain no AP_{k+1} satisfy $S \to (k)_r$.*

In [Sp 75] it was also conjectured that the above theorem can be further extended as follows: *For all k, l, r there exists a set $S \subseteq [n]$ with the property that the k-uniform hypergraph $\mathcal{G}$ with vertex set S and edge set $\{A : A \subseteq S, A \text{ is } AP_k\}$ has*

(1) *chromatic number $\chi(\mathcal{G}) > r$,*
(2) *no cycle of length $j = 2, 3, \ldots, l$.*

The existence of such sets was established in [Rö 90] and [NR 90] by constructive means. Our Theorem 4 implies the existence of such an S as well. Taking $p = Cn^{-\frac{1}{k-1}}$, the expected number of cycles of length at most l in the hypergraph based on the AP_k's of $[n]_p$ is at most

$$\sum_{j=3}^{l} O(n^j p^{(k-1)j}) + O(n^{\frac{k-3}{k-1}}) = O(n^{\frac{k-3}{k-1}}) .$$

(The sum stands for the cycles of length $j \geq 3$; the second term takes care of all 2-cycles.) But Theorem 4 and Lemma 3 imply jointly that F_p, even after deleting up to $\delta np = \Theta(n^{\frac{k-2}{k-1}})$ elements, contains many monochromatic AP_k's for every r-coloring. To obtain the required set S we destroy all short cycles of $\mathcal{G}$ by deleting no more than $(\log n)n^{\frac{k-3}{k-1}}$ elements from F_p.

Let us note that the existence of a set S with properties (1) and (2) yields also an example of arbitrarily large sets S such that $S \to (k)_r$ but $T \not\to (k)_r$ for any proper subset $T \subseteq S$. The existence of such sets S was established in [GN 86] (cf. [Gr 83]).

Acknowledgments

We would like to thank Joel Spencer for his interest in our work, his support and encouragment. We are very grateful to Alan Frieze for checking our proof in great detail.

REFERENCES

[Bo 81] B.Bollobás, *Threshold functions for small subgraphs*, Math. Proc. Cambridge Philos. Soc. **90** (1981), 197-206.

[Bo 85] ——, *Random graphs*, Academic Press, London, 1985.

[BT 87] B.Bollobás and A.Thomason, *Threshold functions*, Combinatorica **7** (1987), 35–38.

[EH 67] P.Erdős and A.Hajnal, *Research problem* 2-5, J. Combin. Theory **2** (1967), 104.

[ER 60] P. Erdős and A. Rényi, *On the evolution of random graphs*, Publ. Math. Inst. Hungar. Acad. Sci. **5** (1960), 17–61.

[Fo 70] J. Folkman, *Graphs with monochromatic complete subgraphs in every edge coloring*, SIAM J. Appl. Math. **18** (1970), 19–29.

[FR 86] P. Frankl and V. Rödl, *Large triangle-free subgraphs in graphs without* K_4, Graphs Combin. **2** (1986), 135–144.

[Gr 68] R.Graham, *On edgewise 2-colored graphs with monochromatic triangles and containing no complete hexagon*, J. Combin. Theory **4** (1968), 300.

[Gr 83] ——, *Recent developments in Ramsey theory*, Proc. Internat. Congr. Math., PWN, Warszawa, 1983, pp. 1555–1567.

[GN 86] R.Graham and J.Nešetřil, *Large minimal sets which force long arithmetic progressions*, J. Combin. Theory Ser. A **42** (1986), 270–276.

[GRS 90] R.Graham, B.Rothschild, and J.Spencer, *Ramsey theory* (second edition), Wiley, New York, 1990.

[Ir 73] R.Irving, *On a bound of Graham and Spencer for a graph coloring constant*, J. Combin. Theory Ser. B **15** (1973), 200–203.

[Ja 90] S.Janson, *Poisson approximation for large deviations*, Random Structures Algorithms **1** (1990), 221–230.

[ŁRV 92] T.Łuczak, A.Ruciński, and B.Voigt, *Ramsey properties of random graphs*, J. Combin. Theory Ser. B **56** (1992), 55–68.

[NR 76] J.Nešetřil and V.Rödl, *The Ramsey property for graphs with forbidden complete subgraphs*, J. Combin. Theory Ser. B **20** (1976), 243–249.

[NR 76a] ——, *Van der Waerden theorem for sequences of integers not containing an arithmetic progression of* k *terms*, Comment. Math. Univ. Carolin. **17** (1976), 675–681.

[NR 89] ——, *Partite constructions and Ramsey set systems*, Discrete Math. **75** (1989), 149–160.

[NR 90] ——, *Partite constructions and Ramsey space systems*, Mathematics of Ramsey Theory, Springer, Berlin and New York, 1990, pp. 98–108.

[Rö 90] V.Rödl, *On Ramsey families of sets*, Graphs Combin. **6** (1990), 187–195.

[RR 94] V.Rödl and A.Ruciński, *Random graphs with monochromatic triangles in every edge coloring*, Random Struct. Algorithms **5** (1994), 253–270.

[RR 93] ——, *Lower bounds on probability thresholds for Ramsey properties*, Combinatorics, Paul Erdős is Eighty (Volume 1) Keszthely (Hungary), Bolyai Soc. Math. Studies, 1993, pp. 317–346.

[Ru 90] A.Ruciński, *Small subgraphs of random graphs:a survey*, Proceedings of Random Graphs '87, Wiley, Chichester, 1990, pp. 283–303.

[Sp 75] J.Spencer, *Restricted Ramsey configurations*, J. Combin. Theory **19** (1975), 278–286.

[Sz 75] E.Szemerédi, *On sets of integers containing no* k *elements in arithmetic progression*, Acta Arith. **27** (1975), 199–245.

[Sz 78] ——, *Partitions of graphs*, Problems Combin. et Theorie des Graphes, Edition du C.N.R.S. **260** (1978), 399–402, Paris.

[Va 59] P.Varnavides, *On certain sets of positive density*, J. London Math. Soc. **34** (1959), 358–360.

ABSTRACT. Probabilistic methods have been used to approach many problems of Ramsey theory. In this paper we study Ramsey type questions from the point of view of random structures.

Let $K(n, N)$ be the random graph chosen uniformly from among all graphs with n vertices and N edges. For a fixed graph G and an integer r we address the question what is the minimum $N = N(G, r, n)$ such that the random graph $K(n, N)$ contains, almost surely, a monochromatic copy of G in every r-coloring of its edges ($K(n, N) \to (G)_r$, in short).

We find a graph parameter $\gamma = \gamma(G)$ yielding

$$\lim_{n\to\infty} Prob(K(n, N) \to (G)_r) = \begin{cases} 0 & \text{if } N < cn^{\gamma}, \\ 1 & \text{if } N > Cn^{\gamma}, \end{cases}$$

for some $c, C > 0$. We use this to derive a number of consequences that deal with the existence of sparse Ramsey graphs. For example we show that for all $r \geq 2$ and $k \geq 3$ there exists $C > 0$ such that almost all graphs H with n vertices and $Cn^{\frac{2k}{k+1}}$ edges which are K_{k+1}-free, satisfy $H \to (K_k)_r$.

We also apply our method to the problem of finding the smallest $N = N(k, r, n)$ guaranteeing that almost all sequences $1 \leq a_1 < a_2 < \cdots < a_N \leq n$ contain an arithmetic progression of length k in every r-coloring, and show that $N = \Theta(n^{\frac{k-2}{k-1}})$ is the threshold.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, EMORY UNIVERSITY, ATLANTA, GEORGIA 30322

E-mail address: rodl@mathcs.emory.edu

E-mail address: rucinski@vm.amu.edu.pl