

ON THE CRITICAL SEMILINEAR WAVE EQUATION OUTSIDE CONVEX OBSTACLES

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1. INTRODUCTION

In this paper we shall show that certain estimates for the Euclidean wave equation also hold on Riemannian manifolds with smooth, strictly geodesically concave boundaries. By the last condition, we understand that the second fundamental form on the boundary of the manifold is strictly positive definite. We shall then give two applications of our estimates. First, we shall show that if Ω is the exterior of a smooth, compact, and strictly convex obstacle $\mathcal{O} \subset \mathbb{R}^3$, then there exists a unique global, smooth solution to the critical wave equation in $\mathbb{R}_+ \times \Omega$:

$$(1.1) \quad \begin{cases} \square u = -u^5, & t > 0, \\ u(0, x) = f(x) \in C^\infty(\Omega), & \partial_t u(0, x) = g(x) \in C^\infty(\Omega), \\ u(t, x) = 0, & x \in \partial\Omega, \end{cases}$$

assuming that the data (f, g) satisfies a necessary compatibility condition arising from the Dirichlet boundary condition. The corresponding result for the critical free wave equation in $\mathbb{R}_+ \times \mathbb{R}^3$ is due to Grillakis [4], following earlier work of Rauch [22] and Struwe [33]. Our other application will be to show that the eigenfunction estimates of the second author [27] carry over to the setting of compact Riemannian manifolds with strictly geodesically concave boundaries. The two-dimensional version of this theorem and also the L^∞ estimates are due to Grieser [3].

Let $\square_g = \partial^2/\partial t^2 - \Delta_g$ be the wave operator associated to our Riemannian manifold (Ω, g) , which we assume has strictly geodesically concave boundary $\partial\Omega$. In what follows, we assume that Ω is either compact, or is the exterior domain in \mathbb{R}^n to a compact obstacle. Our most important estimates concern the Cauchy problem with Dirichlet boundary conditions:

$$(1.2) \quad \begin{cases} \square_g v(t, x) = 0, \\ v(0, x) = f(x), & \partial_t v(0, x) = g(x), \\ v|_{\mathbb{R} \times \partial\Omega} = 0. \end{cases}$$

Received by the editors March 17, 1994 and, in revised form, May 19, 1994.

1991 *Mathematics Subject Classification*. Primary 35L05, 35L70.

The authors were supported in part by the National Science Foundation.

Our most basic inequality is that, if $n = \dim \Omega \geq 2$,

$$(1.3) \quad \|v\|_{L^{\frac{2(n+1)}{n-1}}([0, T] \times \Omega)} \leq C_T \|f\|_{\dot{H}^{\frac{1}{2}}(\Omega)} + C_T \|g\|_{\dot{H}^{-\frac{1}{2}}(\Omega)},$$

where \dot{H}^γ refers to the homogeneous L^2 Sobolev space with γ derivatives. For the Euclidean metric on \mathbb{R}^n , this estimate is due to Strichartz, and is equivalent to his generalization of the Stein-Tomas restriction theorem for the Fourier transform:

$$\left(\int_{\mathbb{R}^n} |\hat{F}(|\xi|, \xi)|^2 \frac{d\xi}{|\xi|} \right)^{1/2} \leq C \|F\|_{L^{\frac{2(n+1)}{n+3}}(\mathbb{R}^{1+n})}.$$

In this case, simple scaling arguments show that if (1.3) holds for a given $T > 0$, then it holds with the same constant for all $T > 0$. These scaling arguments break down when there is an obstacle, and so far we can only show that (1.3) holds with a constant depending on T . Fortunately, these “local” inequalities are sufficient for our applications.

Our proof of (1.3) easily gives two related inequalities. First of all, for our application involving the critical semilinear wave equation, we shall need the estimates

$$(1.4) \quad \|v\|_{L_t^{\frac{2q}{(n-2)q-2n}} L_x^q([0, T] \times \Omega)} \leq C_T \|f\|_{\dot{H}^1(\Omega)} + C_T \|g\|_{L^2(\Omega)},$$

if $6 \leq q < \infty$ when $n = 3$, or if $\frac{2n}{n-2} \leq q \leq \frac{2n}{n-3}$ when $n \geq 4$. We shall also prove that

$$(1.5) \quad \|u\|_{L_x^q L_t^2([0, T] \times \Omega)} \leq C_T \|f\|_{\dot{H}^{\alpha_q}(\Omega)} + C_T \|g\|_{\dot{H}^{\alpha_q-1}(\Omega)},$$

if $\frac{2(n+1)}{n-1} \leq q < \infty$ and $\alpha_q = n(\frac{1}{2} - \frac{1}{q}) - \frac{1}{2}$. The Euclidean version of (1.4) is due to Pecher [21]. The version of (1.5) for Riemannian manifolds without boundary was proved by Mockenhaupt, Seeger and the second author [19]. As we shall see in §4, (1.5) immediately yields sharp L^q estimates for eigenfunctions.

The general strategy behind the proof of (1.3) and the related inequalities (1.4) and (1.5) is to exploit L^2 continuity of certain operators to reduce consideration to operators which are similar to those on a manifold without boundary. For example, we shall see that it suffices to consider the case where the data is supported a fixed distance from the boundary, which will allow us to use the parametrix construction of Melrose and Taylor for the wave group of (1.2).

In order to prove (1.3)–(1.5), we then need show only that the “glancing” part of the parametrix satisfies the same bounds. The estimates for the other terms in the parametrix follow from the smoothing estimates for Fourier integrals in [19]. The glancing term in the parametrix takes the form

$$\int e^{i\theta(x, \xi) + it\xi_1} \left(A_+(\zeta(x, \xi))a(x, \xi) + A'_+(\zeta(x, \xi))b(x, \xi) \right) \times \frac{Ai}{A_+}(\zeta_0(\xi)) \widehat{K}(f, g)(\xi) d\xi,$$

where θ and ζ are solutions of an eikonal equation, a and b are standard symbols of order $1/6$ and $-1/6$, respectively, and K is a standard Fourier integral operator of order zero on f , and order -1 on g . Since the right

sides of the inequalities all involve L^2 , we can exploit the L^2 continuity of standard Fourier integral operators to reduce matters to the case $g = 0$ and $K = \text{Identity}$. Similarly, since the Airy quotient $(Ai/A_+)(\zeta_0)$, which contributes to the scattering relation, is an L^2 multiplier, we can also replace it by 1. Following an idea of Zworski [37], we then split up the resulting operator into two parts: a "main term", which is a Fourier integral operator with symbol of type $(2/3, 1/3)$, and with a canonical relation coming from the free wave equation, plus a "diffractive" term. The arguments in [19] apply to Fourier integrals with type $(2/3, 1/3)$ symbols. Since the diffractive term is small away from a small neighborhood of $\partial\Omega$, it turns out that it too can be handled by our proposition saying that smoothing estimates hold for Fourier integrals of type $(2/3, 1/3)$. This finishes the proof of our estimates for the linear equation (1.2).

The key to establishing global existence for (1.1) is to show that, if the data (f, g) has compact support, and if u is a smooth solution to (1.1) in a half open strip $[0, t_0) \times \Omega$, then u must be uniformly bounded by some constant in that strip. Local existence and regularity theorems then imply global existence and regularity.

To establish the uniform bounds on u , by compactness it suffices to show that u is bounded in a neighborhood of each given point (t_0, x_0) , where $x_0 \in \overline{\Omega}$. We prove the latter using ideas from Grillakis [5], Kapitanski [9], Shatah and Struwe [23], and Struwe [33]. Our approach is somewhat different in that we only use (1.4) and not (1.3). This has two advantages. First, the proof of the key fact that the energy cannot concentrate at (t_0, x_0) is somewhat more direct. Moreover, it is much more convenient to use only (1.4) in our setting since it immediately implies estimates for the inhomogeneous wave equation (see Corollary 2.2), whereas proving the natural Strichartz estimates for the inhomogeneous wave equation outside of an obstacle appears to be considerably more technical than the proof of (1.3), since it is not possible to fully exploit L^2 in that case.

Let us now sketch the proof that u cannot blow up at (t_0, x_0) . As in [5], [23], the first step is to show that the L^6 part of the energy associated to (1.1) cannot concentrate at (t_0, x_0) :

$$(1.6) \quad \lim_{t \nearrow t_0} \int_{\substack{|x-x_0| \leq t_0-t \\ x \in \Omega}} \frac{u^6}{6} dx = 0.$$

The proof of (1.6) follows Struwe [33], exploiting an identity of Morawetz. If $x_0 \in \partial\Omega$, then convexity of the obstacle is required at this step for the flux across the boundary to have the correct sign. A bootstrap argument involving (1.4) and (1.6) then shows that $u \in L_t^4 L_x^{12}(K_0)$, where K_0 is the domain of influence for $\{(t_0, x_0)\}$:

$$K_0 = \{(t, x) : |x - x_0| \leq t_0 - t, (t, x) \in [0, t_0) \times \Omega\}.$$

The next step is to use estimate (1.4) to see that $u \in L_t^4 L_x^{12}(K_0)$ implies $\partial_t u \in L_t^\infty L_x^6(K_0)$. We then use the energy inequality for the linear wave equation to

see that the total energy cannot concentrate at (t_0, x_0) :

$$\lim_{t \nearrow t_0} \int_{\substack{|x-x_0| \leq t_0-t \\ x \in \Omega}} \left(\frac{|\partial_t u|^2 + |\nabla_x u|^2}{2} + \frac{|u|^6}{6} \right) dx = 0.$$

We then repeat these arguments to see that, for some $\delta > 0$,

$$u, \partial_t u \in L_t^\infty L_x^6(K_0^*),$$

if

$$K_0^* = \{ (t, x) \in \mathbb{R}_+ \times \Omega : 0 \leq t < t_0 \text{ and } |x - x_0| < \delta + t_0 - t \}.$$

Energy estimates then imply that $\Delta_x u \in L_t^\infty L_x^2(K_0^*)$. By Sobolev's Theorem, we conclude that u must be bounded in a neighborhood of (t_0, x_0) in $[0, t_0] \times \bar{\Omega}$, finishing the proof that there cannot be blowup.

This paper is organized as follows. In the next section we establish inequalities (1.3)–(1.5). Then, in §3, we turn to global existence for (1.1), while in §4 we present the simple argument showing that (1.5) implies sharp eigenfunction estimates, thus extending results of Grieser [3].

It is a pleasure to thank the people who helped us with this paper. First of all, we are very grateful to H. Lindblad and M. Machedon for helping us to understand some points in [5] and [23]. We are also indebted to R. Melrose for showing us how to prove local existence and regularity for (1.1).

2. SMOOTHING ESTIMATES FOR THE WAVE EQUATION OUTSIDE CONVEX OBSTACLES

Let (Ω, g) be either a smooth n -dimensional compact Riemannian manifold with C^∞ boundary, or $\Omega = \mathbb{R}^n \setminus \mathcal{O}$, where \mathcal{O} is compact with smooth boundary. We shall assume that $n \geq 2$ and that $\partial\Omega$ is strictly geodesically concave throughout.

Let Δ_g be the Laplace-Beltrami operator associated to g . In the case where $\Omega = \mathbb{R}^n \setminus \mathcal{O}$, we assume that we can globally write

$$\Delta_g = \sum_{j,k=1}^n g^{jk}(x) \partial_j \partial_k + \sum_{j=1}^n a_j(x) \partial_j,$$

where the coefficients belong to a bounded subset of C^∞ and the principal part is uniformly elliptic.

We denote the wave operator corresponding to Δ_g by

$$\square_g = \partial^2 / \partial t^2 - \Delta_g.$$

In this section we shall be interested in estimates for the Cauchy problem with Dirichlet boundary conditions:

$$(2.1) \quad \begin{cases} \square_g v(t, x) = 0, \\ v(0, x) = f(x), \quad \partial_t v(0, x) = g(x), \\ v|_{\mathbb{R} \times \partial\Omega} = 0. \end{cases}$$

If $\dot{H}^\gamma(\Omega)$ denotes the homogeneous L^2 -Sobolev space with γ derivatives:

$$\|f\|_{\dot{H}^\gamma(\Omega)} = \|(\sqrt{-\Delta_g})^\gamma f\|_{L^2(\Omega)}, \quad 0 < \gamma < n/2,$$

then we shall prove the following

Theorem 2.1. *Let $S_T = [0, T] \times \Omega$ with $0 < T < \infty$. Then there is a constant $C = C(T)$ such that*

$$(2.2) \quad \|v\|_{L^{\frac{2(n+1)}{n-1}}(S_T)} \leq C\|f\|_{\dot{H}^{\frac{1}{2}}(\Omega)} + C\|g\|_{\dot{H}^{-\frac{1}{2}}(\Omega)}.$$

Also, if $n \geq 3$

$$(2.3) \quad \|v\|_{L_t^{\frac{2q}{(n-2)q-2n}} L_x^q(S_T)} \leq C\|f\|_{\dot{H}^1(\Omega)} + C\|g\|_{L^2(\Omega)}$$

provided that $6 \leq q < \infty$ when $n = 3$, and that $\frac{2n}{n-2} \leq q \leq \frac{2n}{n-3}$ if $n \geq 4$, with C depending on T , and on T and q if $n = 3$. Finally, if $n \geq 2$ and $\frac{2(n+1)}{n-1} \leq q < \infty$, then there is a constant $C = C(T, q)$ so that

$$(2.4) \quad \|v\|_{L_x^q L_t^2(S_T)} \leq C\|f\|_{\dot{H}^{\alpha_q}(\Omega)} + C\|g\|_{\dot{H}^{\alpha_q-1}(\Omega)}, \quad \alpha_q = n(\frac{1}{2} - \frac{1}{q}) - \frac{1}{2}.$$

Estimate (2.2) is a generalization of Strichartz’s estimate in [32] for the free wave equation in $\mathbb{R} \times \mathbb{R}^n$. The fact that (2.2) holds on manifolds without boundary was proved by Kapitanski [10], and, also, independently by Mockenhaupt, Seeger and the second author [19]. Inequality (2.3) generalizes the corresponding estimates for $\mathbb{R} \times \mathbb{R}^n$ which are due to Pecher [21]. To show that there is global existence for (1.1) we shall require the following immediate consequence of (2.3).

Corollary 2.2. *Let u solve the inhomogeneous Cauchy problem with Dirichlet boundary conditions*

$$\begin{cases} \square_g u(t, x) = F(t, x), \\ u(0, x) = f(x), \quad \partial_t u(0, x) = g(x), \\ u|_{\mathbb{R} \times \partial\Omega} = 0. \end{cases}$$

Then

$$(2.5) \quad \|u\|_{L_t^{\frac{2q}{(n-2)q-2n}} L_x^q(S_T)} \leq C(\|f\|_{\dot{H}^1(\Omega)} + \|g\|_{L^2(\Omega)} + \|F\|_{L_t^1 L_x^2(S_T)}),$$

provided that q and C are as in Theorem 2.1.

That (2.3) yields (2.5) follows from the fact that

$$u(t, \cdot) = \cos t\sqrt{-\Delta_g} f + \frac{\sin t\sqrt{-\Delta_g}}{\sqrt{-\Delta_g}} g + \int_0^t \frac{\sin(t-s)\sqrt{-\Delta_g}}{\sqrt{-\Delta_g}} F(s, \cdot) ds,$$

while the solution to the Cauchy problem (2.1), v , is given by

$$v(t, \cdot) = \cos t\sqrt{-\Delta_g} f + \frac{\sin t\sqrt{-\Delta_g}}{\sqrt{-\Delta_g}} g.$$

Hence, one gets (2.5) from (2.3) simply by using Minkowski's integral inequality. Notice that since $\cos t_0\sqrt{-\Delta_g}$ and $\sin t_0\sqrt{-\Delta_g}$ are bounded on \dot{H}^γ , the formula for v also implies that, in proving Theorem 2.1, we may take T to be an arbitrary, fixed small positive number.

Inequality (2.4) generalizes estimates of Mockenhaupt, Seeger and the last author [19]. As we shall see in §4, (2.4) immediately leads to sharp eigenfunction estimates on compact manifolds with concave boundary.

Let us now turn to the proof of our estimates. Notice that the right sides of the inequalities in Theorem 2.1 all involve L^2 Dirichlet-Sobolev spaces. The first step in the proof is to exploit this to reduce matters to proving two types of estimates. The first involves a "main term", which looks like the type of operator arising in the boundaryless case, except that its symbol is of type $(2/3, 1/3)$ rather than $(1, 0)$. The other estimate involves a "diffractive term", which can essentially be handled using the arguments for the main term.

To make this reduction, we may assume that the metric g is extended smoothly across the boundary, so that Ω is a geodesically concave subset of a complete Riemannian manifold $\tilde{\Omega}$, which we assume to be either compact or \mathbb{R}^n . By the free wave equation, we mean the wave equation on $\tilde{\Omega}$, where the data (f, g) has been extended to $\tilde{\Omega}$ by an extension operator preserving the Sobolev spaces. By a broken geodesic in Ω , we mean a geodesic that is allowed to reflect off $\partial\Omega$ according to the reflection law for g .

As we remarked before, it suffices to consider T small. We shall take $T = \delta/2$, where δ is a constant to be fixed below. Let

$$S_T^\delta = \{ (t, x) \in S_T : d(x, \partial\Omega) < \delta \}.$$

On the complement of S_T^δ in S_T , the solution $v(t, x)$ equals the solution to the free wave equation on $\tilde{\Omega}$, for which, as we pointed out before, the estimates are known. It therefore suffices to establish each of the estimates for the norm over S_T^δ .

The wave group

$$U(t) = \begin{pmatrix} \cos t\sqrt{-\Delta_g} & \frac{\sin t\sqrt{-\Delta_g}}{\sqrt{-\Delta_g}} \\ -\sqrt{-\Delta_g} \sin t\sqrt{-\Delta_g} & \cos t\sqrt{-\Delta_g} \end{pmatrix},$$

which maps data at time 0 to data at time t , is an isomorphism on $\dot{H}^\alpha(\Omega) \times \dot{H}^{\alpha-1}(\Omega)$. Consequently, it suffices to establish the estimates over S_T^δ for the function

$$v(f, g) := \cos(t_0 + \cdot)\sqrt{-\Delta_g} f + \frac{\sin(t_0 + \cdot)\sqrt{-\Delta_g}}{\sqrt{-\Delta_g}} g,$$

for some t_0 .

Strict geodesic concavity of Ω implies the existence of $c_0 > 0$, such that if t_0 is less than some fixed constant, then whenever $\gamma(s)$ is a unit speed broken

geodesic in Ω with $d(\gamma(0), \partial\Omega) \leq c_0 t_0^2$, one has:

$$(2.6) \quad d(\gamma(s), \partial\Omega) \geq c_0 t_0^2, \quad s \in [t_0/2, 4t_0].$$

We now define a set $\omega \subset \Omega$. A point p lies in ω if there exists a unit speed broken geodesic with $\gamma(0) = p$, and $d(\gamma(t_0 + t), \partial\Omega) \leq 2\delta$ for some $t \in [-\delta, \delta]$. We assume $2\delta < c_0 t_0^2$, so that ω is a compact subset of Ω with

$$d(\omega, \partial\Omega) \geq c_0 t_0^2.$$

The singularities of the Schwartz kernel of the wave group are contained in the set of (t, x, y) such that there exists a broken geodesic segment of length t from x to y . (See, e.g., [7].) Thus, there is a smooth function $\alpha(x)$ supported in ω , such that the Schwartz kernel of the operator $U(t + t_0) \cdot (1 - \alpha)$ lies in $C^\infty(S_T^\delta \times \Omega)$. Since we are integrating over compact sets, the desired estimates are immediate for this operator, so we are reduced to estimating $v(f, g)$ for data (f, g) supported in ω . We remark that, for functions supported in ω , the norms of $\dot{H}^\alpha(\Omega)$, for $\alpha < n/2$, are comparable to the norms of the nonhomogeneous Sobolev space $H^\alpha(\mathbb{R}^n)$, so we shall henceforth work with the latter norms on the data f and g .

To analyze the wave group, we let Δ_0 denote the Laplacian on $\tilde{\Omega}$. We define the operator \mathcal{R} , which maps a pair of functions on $\tilde{\Omega}$ to a function on $\mathbb{R} \times \partial\Omega$, by

$$\mathcal{R}(f, g) = \left(\cos(t_0 + \cdot) \sqrt{-\Delta_0} f + \frac{\sin(t_0 + \cdot) \sqrt{-\Delta_0}}{\sqrt{-\Delta_0}} g \right) \Big|_{\mathbb{R} \times \partial\Omega}.$$

We also let \mathcal{M} denote the forward solution to the Dirichlet problem for \square_g on $\mathbb{R} \times \Omega$. Thus, if h is a function on $\mathbb{R} \times \partial\Omega$ which vanishes for $t \leq -t_0$, then $\mathcal{M}h$ is the solution on $\mathbb{R} \times \Omega$ to:

$$\begin{cases} \square_g \mathcal{M}h & = 0, \\ \mathcal{M}h & = 0, \quad \text{if } t \leq -t_0, \\ \mathcal{M}h|_{\mathbb{R} \times \partial\Omega} & = h. \end{cases}$$

We then have, for $t \in [-t_0, T]$ and for data (f, g) supported in ω ,

$$v(f, g) = v_0(f, g) - \mathcal{M}\mathcal{R}_+(f, g).$$

Here, \mathcal{R}_+ is the operator \mathcal{R} smoothly cutoff on the right to $t \in [-t_0, t_0]$, and $v_0(f, g)$ is the free solution on $\mathbb{R} \times \tilde{\Omega}$.

The cotangent bundle of $\mathbb{R} \times \partial\Omega$ is divided into three disjoint, time-independent conic sets: the hyperbolic and elliptic regions where the Dirichlet problem is respectively hyperbolic and elliptic, and the glancing region which is the boundary between the two. We write the identity operator as

$$I = \Pi_h + \Pi_e + \Pi_g,$$

where Π_h and Π_e are time-independent conic pseudodifferential cutoffs, essentially supported strictly inside the hyperbolic and elliptic regions, and Π_g is essentially supported in a small conic set about the glancing region, on which the Melrose-Taylor construction of \mathcal{M} is valid.

On S_T^δ , we can write $v(f, g)$ as the sum of four terms:

$$v_0(f, g) - \mathcal{M}\Pi_e\mathcal{R}_+(f, g) - \mathcal{M}\Pi_h\mathcal{R}_+(f, g) - \mathcal{M}\Pi_g\mathcal{R}_+(f, g).$$

For the first term, $v_0(f, g)$, the desired estimates follow as in the boundaryless case.

The projection of any characteristic direction for \square_g onto $T^*(\mathbb{R} \times \partial\Omega)$ is contained in the hyperbolic or glancing regions, so that

$$\mathcal{M}\Pi_e\mathcal{R}_+ \equiv 0,$$

where \equiv denotes equality modulo smoothing operators. The estimates for the second term are then trivial.

On the essential support of Π_h , the forward Dirichlet problem for \square_g can be solved locally, modulo smoothing kernels, on an open set in $\mathbb{R} \times \tilde{\Omega}$ around $\partial\Omega$. Precisely, there is an operator $\tilde{\mathcal{M}}$, from $\mathbb{R} \times \partial\Omega$ to $\mathbb{R} \times \tilde{\Omega}$, and a constant t_1 , so that acting on data supported by $t \in [-t_1, t_1]$:

$$\begin{aligned} \square_0\tilde{\mathcal{M}} &\equiv 0, \quad t \in [-2t_1, 2t_1], \\ (\mathcal{M} - \tilde{\mathcal{M}})\Pi_h &\equiv 0. \end{aligned}$$

We remark that t_1 is comparable to the shortest time it takes a bicharacteristic over the essential support of Π_h , entering $\tilde{\Omega} \setminus \Omega$, to intersect $\partial\Omega$ again.

We assume that $t_0 \leq t_1$. Set

$$\tilde{f} = \tilde{\mathcal{M}}\Pi_h\mathcal{R}_+(f, g)\Big|_{t=-t_0},$$

$$\tilde{g} = \partial_t\tilde{\mathcal{M}}\Pi_h\mathcal{R}_+(f, g)\Big|_{t=-t_0}.$$

The operator $(f, g) \rightarrow (\tilde{f}, \tilde{g})$ is a matrix of classical Fourier integral operators, of the same order as the wave group, associated to the relation of ‘‘reflection about $\partial\Omega$ ’’. We can take the data (\tilde{f}, \tilde{g}) to be compactly supported in $\tilde{\Omega} \setminus \Omega$, and hence

$$\|\tilde{f}\|_{H^\alpha(\mathbb{R}^n)} + \|\tilde{g}\|_{H^{\alpha-1}(\mathbb{R}^n)} \leq C\|f\|_{H^\alpha(\mathbb{R}^n)} + C\|g\|_{H^{\alpha-1}(\mathbb{R}^n)}.$$

On S_T^δ , we then have

$$\mathcal{M}\Pi_h\mathcal{R}_+(f, g) \equiv v_0(\tilde{f}, \tilde{g}).$$

The estimates for the third term then follow as for the first term, and so we are left with estimating the last term.

We decompose this term by writing Π_g as a finite sum of pseudodifferential cutoffs, each essentially supported in a suitably small conic neighborhood of a glancing ray. Each term in the resulting sum can be written, modulo a smoothing operator, in the form

$$\begin{aligned} \int e^{i\theta(x, \xi) + it\xi_1} &\left(A_+(\zeta(x, \xi))a(x, \xi) + A'_+(\zeta(x, \xi))b(x, \xi) \right) \\ &\times \frac{Ai}{A_+}(\zeta_0(\xi))\widehat{K}(f, g)(\xi) d\xi, \end{aligned}$$

where the symbols and phases have the following properties. First of all, a and b are symbols of type $(1, 0)$ and order $1/6$ and $-1/6$, respectively, both of which are supported in a small conic neighborhood of the ξ_1 -axis. With this normalization, the operator K is a classical Fourier integral operator, compactly supported on both sides, of order 0 on f and order -1 on g . The phases θ and ζ are real, smooth and homogeneous in ξ of degree 1 and $2/3$, respectively. If local coordinates are chosen so that Ω is given by $x_n > 0$, the phases satisfy the eikonal equations

$$(2.7) \quad \begin{cases} \xi_1^2 - \langle d_x \theta, d_x \theta \rangle_x + \zeta \langle d_x \zeta, d_x \zeta \rangle_x = 0, \\ \langle d_x \theta, d_x \zeta \rangle_x = 0, \\ \zeta(x', 0, \xi) \equiv \zeta_0(\xi) = -\xi_1^{-1/3} \xi_n, \end{cases}$$

in the region $\zeta(x, \xi) \leq 0$. Here, $x' = (x_1, \dots, x_{n-1})$, and $\langle \cdot, \cdot \rangle_x$ denotes the inner product given by g . The phases also satisfy these equations to infinite order at $x_n = 0$ in the region where $\zeta > 0$. For further discussion and proofs, we refer the reader to the manuscript [18] of Melrose and Taylor.

By the L^2 continuity of K , to finish the proof of Theorem 2.1 we thus need only to show that the operator

$$(2.8) \quad Af(t, x) = \int e^{i\theta(x, \xi) + it\zeta_1} \left(A_+(\zeta(x, \xi)) a(x, \xi) + A'_+(\zeta(x, \xi)) b(x, \xi) \right) \times \frac{Ai}{A_+}(\zeta_0(\xi)) \hat{f}(\xi) d\xi$$

satisfies

$$(2.2') \quad \|Af\|_{L^{\frac{2(n+1)}{n-1}}} \leq C \|f\|_{H^{\frac{1}{2}}(\mathbb{R}^n)},$$

$$(2.3') \quad \|Af\|_{L_t^{\frac{2q}{(n-2)q-2n}} L_x^q} \leq C \|f\|_{H^1(\mathbb{R}^n)},$$

and

$$(2.4') \quad \|Af\|_{L_x^q L_t^2} \leq C \|f\|_{H^{\alpha q}(\mathbb{R}^n)},$$

if the norms on the left side are taken over $\{(t, x) : x_n > 0, |(t, x)| < \delta_0\}$, with $\delta_0 > 0$ small.

To prove these estimates it is convenient to split the operator A into two parts: a main term and a diffractive term. To this end, let $\chi(s)$ be a smooth function satisfying

$$\text{supp } \chi \subseteq (-\infty, -1], \quad \text{supp } (1 - \chi) \subseteq [-2, \infty).$$

We write this operator as the sum of two terms,

$$A = S + T,$$

by decomposing

$$A_+(\zeta(x, \xi)) = ((1 - \chi)A_+)(\zeta(x, \xi)) + (\chi A_+)(\zeta(x, \xi)),$$

and letting the “main term” be defined by

$$Tf(t, x) = \int e^{i\theta(x, \xi) + it\xi_1} \left((\chi_{A_+})(\zeta(x, \xi)) a(x, \xi) + (\chi_{A_+})'(\zeta(x, \xi)) b(x, \xi) \right) \frac{Ai}{A_+}(\zeta_0(\xi)) \hat{f}(\xi) d\xi.$$

The “diffractive term” is then defined by

$$Sf(t, x) = \int e^{i\theta(x, \xi) + it\xi_1} q(x, \xi) \hat{f}(\xi) d\xi,$$

where

$$(2.9) \quad q(x, \xi) = \left(((1 - \chi)A_+)(\zeta(x, \xi)) a(x, \xi) + ((1 - \chi)A_+)'(\zeta(x, \xi)) b(x, \xi) \right) \frac{Ai}{A_+}(\zeta_0(\xi)).$$

We shall postpone the proof of the following, since the arguments for estimating the diffractive term are similar to those that will be used for the main term.

Proposition 2.3. *The operator $f \mapsto Sf(t, x)$ satisfies the analogues of (2.2'), (2.3') and (2.4').*

To estimate the main term, we first use the fact that

$$\left| \frac{Ai(s)}{A_+(s)} \right| \leq 2, \quad s \in \mathbb{R}.$$

Consequently, the estimates for T follow from showing that the operator (2.10)

$$f \mapsto \int e^{i\theta(x, \xi) + it\xi_1} \left((\chi_{A_+})(\zeta(x, \xi)) a(x, \xi) + (\chi_{A_+})'(\zeta(x, \xi)) b(x, \xi) \right) \hat{f}(\xi) d\xi$$

satisfies the same bounds.

Next, following [37, Lemma 4.1], we write (χ_{A_+}) in terms of its Fourier transform to express the Schwartz kernel of this operator in the form

$$\int e^{i\theta(x, \xi) + it\xi_1 + is\xi_1^{-2/3}\zeta(x, \xi) + is^3/3\xi_1^2 - i\langle y, \xi \rangle} \times \xi_1^{-2/3} \Psi_+(\xi_1^{-2/3}s) [a(x, \xi) + s\xi_1^{-2/3}b(x, \xi)] ds d\xi.$$

The function Ψ_+ is a symbol of order 0 on \mathbb{R} , hence the symbol

$$\xi_1^{-2/3} \Psi_+(\xi_1^{-2/3}s) [a(x, \xi) + s\xi_1^{-2/3}b(x, \xi)]$$

belongs to the class $S_{2/3, 1/3}^{-1/2}(\mathbb{R}^n, \mathbb{R}^{n+1})$. The phase function

$$\theta(x, \xi) + t\xi_1 + s\xi_1^{-2/3}\zeta(x, \xi) + s^3/3\xi_1^2 - \langle y, \xi \rangle$$

parameterizes a Lagrangian submanifold Λ of $T^*(\mathbb{R}_{x,t}^{n+1} \times \mathbb{R}_y^n)$, such that the projection on the left factor is contained in the characteristic variety of \square_0 .

The operator (2.10) is then a Fourier integral operator, of order 0 and type $(2/3, 1/3)$, associated to the canonical relation Λ . We cannot directly apply Proposition 2.4 below to this operator, because Λ is not the identity at $t = 0$. However, by [37, Lemma 3.1] (see also [25, Lemma 5.2]), the restriction of this Lagrangian manifold to $t = 0$ is the graph of a canonical transformation (denoted by Λ_0 in the appendix of [25]). The canonical relation $\Lambda \circ \Lambda_0^{-1}$ is thus the flowout of a conical subset of the diagonal at $t = 0$ under the bicharacteristic flow of \square_0 . By the Lax construction, the Lagrangian manifold $\Lambda \circ \Lambda_0^{-1}$ can be parameterized by a phase function

$$\varphi(t, x, \xi) - \langle y, \xi \rangle,$$

where, if

$$p(x, \xi) = \sqrt{\sum g^{jk}(x)\xi_j\xi_k}$$

denotes the principal symbol of Δ_g , then

$$(2.11) \quad \varphi(0, x, \xi) = x \cdot \xi \quad \text{and} \quad \varphi'_t = p(x, \varphi'_x).$$

For the final step in the reduction, we take G and H to be classical Fourier integral operators of order 0, respectively associated to the canonical relations Λ_0^{-1} and Λ_0 , such that $1 - G \circ H$ is smoothing on the appropriate conic set. By continuity of H on $H^\alpha(\mathbb{R}^n)$, we are reduced to considering the mapping properties of the operator

$$f \rightarrow \int e^{i\theta(x, \xi) + it\xi_1} \left((\chi_{A_+})(\zeta(x, \xi)) a(x, \xi) + (\chi_{A_+})'(\zeta(x, \xi)) b(x, \xi) \right) \widehat{Gf}(\xi) d\xi.$$

This is a Fourier integral operator of order 0, of type $(2/3, 1/3)$, associated to the relation $\Lambda \circ \Lambda_0^{-1}$. By the equivalence of phase function theorem, [6, Theorem 3.2.1], and the discussion following Proposition 25.3.3 of [7], it can thus be written in the form (modulo a smoothing operator):

$$(2.12) \quad f \rightarrow \int e^{i\varphi(t, x, \xi)} c(t, x, \xi) \hat{f}(\xi) d\xi,$$

where $c(t, x, \xi)$ is a symbol in $S_{2/3, 1/3}^0(\mathbb{R}^{1+n} \times \mathbb{R}^n)$. The estimates for this term now follow from:

Proposition 2.4. *Suppose that $a(t, x, \xi) \in S_{2/3, 1/3}^{-\mu}(\mathbb{R}^{1+n} \times \mathbb{R}^n)$ is supported in a small conic neighborhood of the ξ_1 axis, and is supported by (t, x) in a small ball, and suppose that $\varphi(t, x, \xi)$ satisfies (2.11). If we define W by*

$$Wf(t, x) = \int e^{i\varphi(t, x, \xi)} a(t, x, \xi) \hat{f}(\xi) d\xi,$$

then the following estimates hold. First,

$$(2.13) \quad \|Wf\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^{1+n})} \leq C \|f\|_{L^2(\mathbb{R}^n)}, \quad \mu = \frac{1}{2}.$$

Also, if α_q is as in (2.3) and $\frac{2(n+1)}{n-1} \leq q < \infty$,

$$(2.14) \quad \|Wf\|_{L_x^q L_t^2(\mathbb{R}^{1+n})} \leq C \|f\|_{L^2(\mathbb{R}^n)}, \quad \mu = \alpha_q.$$

Finally, if $n \geq 3$, and if $6 \leq q < \infty$ for $n = 3$, or $\frac{2n}{n-2} \leq q \leq \frac{2n}{n-3}$ for $n \geq 4$, then

$$(2.15) \quad \|Wf\|_{L_t^{\frac{2q}{(n-2)q-2n}} L_x^q(\mathbb{R}^{1+n})} \leq C \|f\|_{L^2(\mathbb{R}^n)}, \quad \mu = 1.$$

If the phase is replaced by $x \cdot \xi + it|\xi|$, and a is independent of (t, x) , then (2.13) is due to Strichartz [32]. The version with φ as above and a a type $(1, 0)$ symbol is due to Kapitanski [10] and Mockenhaupt, Seeger and the second author [19]. Under these assumptions, (2.14) was also proved in [19]. The Euclidean version of (2.15) is due to Pecher [21], except for the endpoint in higher dimensions which is due to Lindblad and the second author [12].

As in these papers, the first step in the proof of Proposition 2.4 is to reduce matters to proving more symmetric inequalities involving dual spaces. Specifically, if we set

$$(TF)(t, x) = (WW^*F)(t, x) = \iint e^{i\varphi(t, x, \xi) - i\varphi(s, y, \xi)} a(t, x, \xi) \overline{a(s, y, \xi)} F(s, y) d\xi ds dy,$$

then the three inequalities are equivalent to

$$(2.13') \quad \|TF\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^{1+n})} \leq C \|F\|_{L^{\frac{2(n+1)}{n+3}}(\mathbb{R}^{1+n})},$$

$$(2.14') \quad \|TF\|_{L_x^q L_t^2(\mathbb{R}^{1+n})} \leq C \|F\|_{L_x^{\frac{q}{q-1}} L_t^2(\mathbb{R}^{1+n})},$$

$$(2.15') \quad \|TF\|_{L_t^{\frac{2q}{(n-2)q-2n}} L_x^q(\mathbb{R}^{1+n})} \leq C \|F\|_{L_t^{\frac{2q}{2n-(n-4)q}} L_x^{\frac{q}{q-1}}(\mathbb{R}^{1+n})}.$$

To see, for instance, that (2.13') implies (2.13), notice that the dual version of the latter is

$$\|W^*F\|_{L^2(\mathbb{R}^n)} \leq C \|F\|_{L^{\frac{2(n+1)}{n+3}}(\mathbb{R}^{1+n})}.$$

But

$$\int |W^*F|^2 dx = \int WW^*F\overline{F} dt dx \leq \|TF\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^{1+n})} \|F\|_{L^{\frac{2(n+1)}{n+3}}(\mathbb{R}^{1+n})},$$

and hence the estimate for T implies the one for W . Similar reasoning applies to the other two inequalities.

Next, as in [19], we reduce matters to proving uniform dyadic estimates. Since all of the exponents on the left sides of (2.13') – (2.15') are ≥ 2 and all the ones on the right sides are ≤ 2 , one can use Littlewood-Paley theory (see also [28, pp. 221–222]) to see that it suffices to prove estimates for

$$(T_\lambda F)(t, x) = \iint e^{i\varphi(t, x, \xi) - i\varphi(s, y, \xi)} \beta(|\xi|/\lambda) \times a(t, x, \xi) \overline{a(s, y, \xi)} F(s, y) d\xi ds dy,$$

if $\beta \in C_0^\infty(\mathbb{R}_+)$ satisfies

$$\sum_{k=1}^\infty \beta(2^{-k}s) = 1, \quad s > 2.$$

Specifically, (2.13') – (2.15') are equivalent to the uniform dyadic estimates

$$(2.13'') \quad \|T_\lambda F\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^{1+n})} \leq C \|F\|_{L^{\frac{2(n+1)}{n+3}}(\mathbb{R}^{1+n})},$$

$$(2.14'') \quad \|T_\lambda F\|_{L_x^q L_t^2(\mathbb{R}^{1+n})} \leq C \|F\|_{L_x^{\frac{q}{q-1}} L_t^2(\mathbb{R}^{1+n})},$$

$$(2.15'') \quad \|T_\lambda F\|_{L_t^{\frac{2q}{(n-2)q-2n}} L_x^q(\mathbb{R}^{1+n})} \leq C \|F\|_{L_t^{\frac{2q}{2n-(n-4)q}} L_x^{\frac{q}{q-1}}(\mathbb{R}^{1+n})},$$

where $\lambda = 2^k \geq 1$. (The part corresponding to $|\xi| \leq 2$ is trivial to estimate since the symbols vanish outside of a compact subset of \mathbb{R}^{1+n} .)

We have so far proceeded as in [19] and [12]. However, since the symbols here are of type $(2/3, 1/3)$, and not of type $(1, 0)$, we need to make a further decomposition. Fix $\rho \in C_0^\infty(\mathbb{R})$ satisfying $\rho(s) = 1$ near 0, and $\rho(s) = 0$ if $|s| \geq 1$. We then let

$$T_\lambda F = T_\lambda^0 F + T_\lambda^1 F,$$

with

$$(T_\lambda^0 F)(t, x) = \iint e^{i\varphi(t, x, \xi) - i\varphi(s, y, \xi)} \rho(\lambda^{1/3}(t-s)) \beta(|\xi|/\lambda) \\ \times a(t, x, \xi) \overline{a(s, y, \xi)} F(s, y) d\xi ds dy.$$

The two pieces will be handled differently. The kernel of T_λ^0 is supported in a suitably small set, and we shall estimate it by freezing coefficients. To estimate T_λ^1 , we shall be able to use the standard stationary phase arguments behind the proofs of (2.13'') – (2.14'') for type $(1, 0)$ symbols. For type $(2/3, 1/3)$ symbols these stationary phase arguments break down if $|t-s|$ is smaller than $\lambda^{-1/3}$, which motivates the decomposition. We also point out here that the argument we are describing breaks down for symbols of type $(\rho, 1-\rho)$ if $\rho < 2/3$; however, we suspect that the result might hold if $\rho = 1/2$.

Let us first estimate the term T_λ^0 . To establish the estimates (2.13'') – (2.15'') for this piece, it suffices to establish them with both the variables (t, x) and (s, y) restricted to lie in a cube in \mathbb{R}^{n+1} of sidelength comparable to $\lambda^{-1/3}$. To see this for the estimate (2.13''), let $p_n = \frac{2(n+1)}{n-1}$, and decompose S_T into disjoint cubes Q of sidelength $\lambda^{-1/3}$. We then have

$$\|T_\lambda^0 F\|_{p_n}^{p_n} = \sum_Q \|\chi_Q T_\lambda^0 F\|_{p_n}^{p_n},$$

where χ_Q is the characteristic function of Q .

The integral kernel $K_\lambda^0(t, x, s, y)$ of T_λ^0 vanishes if $|s-t| \geq \lambda^{-1/3}$. If $|s-t| \leq \lambda^{1/3}$, and $|x-y| \geq C_0 \lambda^{-1/3}$, then the phase

$$\varphi(t, x, \xi) - \varphi(s, y, \xi)$$

has no critical points with respect to ξ_1 , so that

$$|K_\lambda^0(t, x, s, y)| \leq C_N \lambda^{-N} \quad \forall N, \quad \text{if } |x - y| \geq C_0 \lambda^{-1/3}.$$

It therefore suffices to estimate

$$\sum_Q \|\chi_Q T_\lambda^0 \chi_{Q^*} F\|_{p_n}^{p_n},$$

where Q^* is the dilate of Q by some fixed factor independent of λ . If we establish the estimates (2.13'') uniformly for the $\chi_Q T_\lambda^0 \chi_{Q^*}$, then we can control

$$\sum_Q \|\chi_Q T_\lambda^0 \chi_{Q^*} F\|_{p_n}^{p_n} \leq C_1 \sum_Q \|\chi_{Q^*} F\|_{p_n}^{p_n} \leq C_2 \|F\|_{p_n}^{p_n},$$

where the last inequality holds since $p_n > p'_n$, and the Q^* have fixed finite overlap.

Similar considerations show that for the estimates (2.14'') and (2.15'') also it suffices to restrict both (t, x) and (s, y) to lie in a cube of sidelength comparable to $\lambda^{-1/3}$.

Now let Q be a fixed cube in \mathbb{R}^{n+1} of sidelength $\lambda^{-1/3}$. We label the lower left-hand corner of Q by $(0, 0)$ for convenience, but do not use the special form of the phase at $t = 0$. Let

$$b_\lambda(t, x, s, y, \xi) = \rho(\lambda^{1/3}(t - s)) \beta(|\xi|/\lambda) a(t, x, \xi) \overline{a(s, y, \xi)},$$

and write

$$\begin{aligned} b_\lambda(t, x, s, y, \xi) &= b_\lambda(0, 0, s, y, \xi) + \int_0^t \partial_t b_\lambda(r, 0, s, y, \xi) dr + \dots \\ &\quad + \int_0^t \dots \int_0^{x_n} \partial_t \dots \partial_{x_n} b_\lambda(r, z_1, \dots, z_n, s, y, \xi) dr dz. \end{aligned}$$

The estimates (2.13) – (2.15) each hold if the symbol a of Proposition 2.4 is independent of (t, x) , since $a(\xi)$ then acts as an L^2 multiplier. We use this, for instance, to deduce

$$\begin{aligned} \|T_\lambda^0 F\|_{L^{\frac{2(n+1)}{n-1}}(Q)} &\leq C \left\| \iint e^{ix \cdot \xi - i\varphi(s, y, \xi)} b_\lambda(0, 0, s, y, \xi) F(s, y) d\xi ds dy \right\|_{L^2(\mathbb{R}^n)} + \dots \\ &\quad + C \int_0^{\lambda^{-\frac{1}{3}}} \dots \int_0^{\lambda^{-\frac{1}{3}}} \left\| \iint e^{ix \cdot \xi - i\varphi(s, y, \xi)} \partial_t \dots \partial_{x_n} b_\lambda(r, z, s, y, \xi) \right. \\ &\quad \left. \times F(s, y) d\xi ds dy \right\|_{L^2(\mathbb{R}^n)} dr dz. \end{aligned}$$

Each derivative of $b_\lambda(t, x, s, y, \xi)$ in (t, x) loses a factor of $\lambda^{1/3}$, but this is exactly compensated by the integral over (r, z) , so that it suffices to establish

uniform estimates for fixed (r, z) . By duality, we would be done if we could establish, for instance, the estimate

$$\left\| \iint e^{i\varphi(s, y, \xi)} b_\lambda(0, 0, s, y, \xi) \widehat{f}(\xi) d\xi \right\|_{L^{\frac{2(n+1)}{n+3}}(\mathcal{Q})} \leq C \|f\|_{L^2(\mathbb{R}^n)}.$$

This now follows by using the same argument to freeze the variables (s, y) .

Similar considerations apply to the estimates (2.14'') – (2.15''), which concludes the estimates for the term T_0^λ .

To estimate T_λ^1 and finish the proof of Proposition 2.4 we need the following straightforward stationary phase lemma whose proof we postpone for now.

Lemma 2.5. *Suppose that $a(t, x, \xi) \in S_{2/3, 1/3}^{-\mu}$ is supported in a small conic neighborhood of $(0, 0, (1, \dots, 0)) \in \mathbb{R}^{1+n} \times (\mathbb{R}^n \setminus \{0\})$, and let $K_\lambda^1(t, x, s, y)$ denote the integral kernel of T_λ^1 . Then, there is a constant $1 < C_0 < \infty$ such that*

$$(2.16) \quad |K_\lambda^1(t, x, s, y)| \leq C_N \lambda^{-N} \quad \forall N, \text{ if } \frac{|x-y|}{|t-s|} \notin [C_0^{-1}, C_0].$$

On the other hand, there is a function $\xi_0(t, x, s, y) = (1, \xi_0''(t, x, s, y))$, which is smooth in the variables (s, t) uniformly over (x, y) , so that if $C_0^{-1} \leq \frac{|x-y|}{|t-s|} \leq C_0$, then

$$(2.17) \quad K_\lambda^1 = \lambda^{n-1-2\mu} (\lambda|x-y|)^{-\frac{n-1}{2}} \int e^{i\tau[\varphi(t, x, \xi_0) - \varphi(s, y, \xi_0)]} c_\lambda(t, x, s, y, \tau) d\tau,$$

where c_λ vanishes unless $\tau \approx \lambda$ and

$$(2.18) \quad |\partial_\tau^k \partial_{t,s}^\alpha c_\lambda| \leq C_{k,\alpha} \lambda^{\frac{|\alpha|}{3} - \frac{2k}{3}} \quad \forall k, \alpha,$$

with $C_{k,\alpha}$ uniform over x, y . Furthermore,

$$(2.19) \quad \partial_t(\varphi(t, x, \xi_0(t, x, s, y))) \neq 0, \quad \partial_s(\varphi(s, y, \xi_0(t, x, s, y))) \neq 0.$$

End of proof of Proposition 2.4. We first show that T_λ^1 satisfies the analogues of (2.13'') and (2.15''). To this end, for each t and s let $T_{t,s}$ be the ‘‘frozen’’ operator

$$T_{t,s} f(x) = \int K_\lambda^1(t, x, s, y) f(y) dy.$$

By (2.11), since $a \in S_{2/3, 1/3}^{-\mu}$, $T_{t,s}$ is a Fourier integral operator of order -2μ and type $(2/3, 1/3)$, with a canonical relation which is locally a canonical graph. Since such Fourier integrals of order zero and type $(2/3, 1/3)$ are bounded on $L^2(\mathbb{R}^n)$, we have

$$\|T_{t,s} f\|_{L^2(\mathbb{R}^n)} \leq C \lambda^{-2\mu} \|f\|_{L^2(\mathbb{R}^n)}.$$

On the other hand, (2.17) and (2.18) imply that

$$|K_\lambda^1| \leq C \lambda^{n-2\mu} (1 + \lambda|t-s|)^{-\frac{n-1}{2}},$$

yielding

$$\|T_{t,s} f\|_{L^\infty(\mathbb{R}^n)} \leq C \lambda^{n-2\mu} (1 + \lambda|t-s|)^{-\frac{n-1}{2}} \|f\|_{L^1(\mathbb{R}^n)}.$$

By interpolation, we conclude that for $2 \leq q \leq \infty$

$$(2.20) \quad \|T_{t,s}f\|_{L^q(\mathbb{R}^n)} \leq C \lambda^{-2\mu} \lambda^{2n(\frac{1}{2}-\frac{1}{q})} (1 + \lambda|t-s|)^{-(n-1)(\frac{1}{2}-\frac{1}{q})} \|f\|_{L^{\frac{q}{q-1}}(\mathbb{R}^n)}.$$

If $\mu = 1/2$ and $q = \frac{2(n+1)}{n-1}$, as in (2.13''), this implies

$$\|T_{t,s}f\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^n)} \leq C|t-s|^{-1+(\frac{n+3}{2(n+1)}-\frac{n-1}{2(n+1)})} \|f\|_{L^{\frac{2(n+1)}{n+3}}(\mathbb{R}^n)}.$$

By Minkowski's inequality, we next observe that

$$\begin{aligned} \|(T_\lambda^1 F)(t, \cdot)\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^n)} &\leq \int \|T_{t,s}F(s, \cdot)\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^n)} ds \\ &\leq C \int |t-s|^{-1+(\frac{n+3}{2(n+1)}-\frac{n-1}{2(n+1)})} \|F(s, \cdot)\|_{L^{\frac{2(n+1)}{n+3}}(\mathbb{R}^n)} ds. \end{aligned}$$

The estimate (2.13'') now follows by applying the Hardy-Littlewood inequality for fractional integrals (cf. [28, p. 26]).

The proof of (2.15'') is similar. If $\mu = 1$ as in (2.15''), then for $q \leq \frac{2(n+1)}{n-3}$, inequality (2.20) yields

$$\|T_{t,s}f\|_{L^q(\mathbb{R}^n)} \leq C|t-s|^{-1+(\frac{2n-(n-4)q}{2q}-\frac{(n-2)q-2n}{2q})} \|f\|_{L^{\frac{q}{q-1}}(\mathbb{R}^n)}.$$

If $\frac{2n}{n-2} \leq q < \frac{2n}{n-3}$, then the exponent of $|t-s|$ is in the range $(-1, 0]$, and we obtain (2.15'') for such q by applying the Hardy-Littlewood inequality as above. If $q = \frac{2n}{n-3}$, then (2.20) states

$$\|T_{t,s}f\|_{L^{\frac{2n}{n-3}}(\mathbb{R}^n)} \leq C \lambda (1 + \lambda|t-s|)^{-\frac{3(n-1)}{2n}} \|f\|_{L^{\frac{2n}{n+3}}(\mathbb{R}^n)}.$$

For $n \geq 4$, we use Young's inequality to conclude

$$\|T_\lambda^1 F\|_{L_t^2 L_x^{\frac{2n}{n-3}}(S_T)} \leq C \|F\|_{L_t^2 L_x^{\frac{2n}{n+3}}(S_T)}.$$

The proof of (2.14'') requires an additional step. Condition (2.11) implies that for fixed x_1, y_1 , the phase $\varphi(t, x, \xi) - \varphi(s, y, \xi)$ is nondegenerate in the variables (t, s, x'', y'') . For each x_1 and y_1 , we let T_{x_1, y_1} be the frozen operator

$$(T_{x_1, y_1} f)(t, x'') = \int K_\lambda^1(t, x, s, y) f(s, y'') ds dy''.$$

The L^2 boundedness of Fourier integrals implies

$$\|T_{x_1, y_1} f\|_{L_{t, x''}^2} \leq C \lambda^{-2\mu} \|f\|_{L_{t, x''}^2}.$$

On the other hand, in view of (2.16)–(2.18), for each fixed x and y we have

$$\left\| \int K_\lambda^1(t, x, s, y) h(s) ds \right\|_{L^2(dt)} \leq C \lambda^{n-1-2\mu} (1 + \lambda|x_1 - y_1|)^{-\frac{n-1}{2}} \|h\|_{L^2(dt)}.$$

This implies

$$\|T_{x_1, y_1} f\|_{L_{x''}^\infty L_t^2} \leq C \lambda^{n-1-2\mu} (1 + \lambda|x_1 - y_1|)^{-\frac{n-1}{2}} \|f\|_{L_{x''}^1 L_t^2}.$$

By interpolation, for $2 \leq q \leq \infty$ we have

$$\|T_{x_1, y_1} f\|_{L_{x''}^q L_t^2} \leq C \lambda^{-2\mu} \lambda^{2(n-1)(\frac{1}{2}-\frac{1}{q})} (1 + \lambda|x_1 - y_1|)^{-(n-1)(\frac{1}{2}-\frac{1}{q})} \|f\|_{L_{x''}^{\frac{q}{q-1}} L_t^2}.$$

If, as in (2.14''), $\frac{2(n+1)}{n-1} \leq q < \infty$, and $\mu = n(\frac{1}{2} - \frac{1}{q}) - \frac{1}{2}$, this implies

$$\|T_{x_1, y_1} f\|_{L_{x''}^q L_t^2} \leq C |x_1 - y_1|^{-1+(\frac{q-1}{q}-\frac{1}{q})} \|f\|_{L_{x''}^{\frac{q}{q-1}} L_t^2},$$

which leads to (2.14'') by an application of the Hardy-Littlewood inequality. \square

Proof of Lemma 2.5. We need to use stationary phase to evaluate

$$K_\lambda^1 = \int e^{i\varphi(t, x, \xi) - i\varphi(s, y, \xi)} (1 - \rho(\lambda^{1/3}(t-s))) \beta(|\xi|/\lambda) a(t, x, \xi) \overline{a(s, y, \xi)} d\xi.$$

The critical points of the phase occur where $|x - y| \approx |t - s|$. For some C_0 then,

$$|\nabla_\xi \varphi(t, x, \xi) - \nabla_\xi \varphi(s, y, \xi)| \approx |t - s| + |x - y| \geq \lambda^{-1/3} \quad \text{if } \frac{|x-y|}{|t-s|} \notin [C_0^{-1}, C_0].$$

Since $a \in S_{2/3, 1/3}^{-\mu}$, an easy integration by parts argument leads to (2.16).

To handle the case $|x - y| \approx |t - s|$, we introduce a cutoff function $\psi(|x - y|/|t - s|)$ for an appropriate bump function ψ , and remark that this cutoff satisfies the estimates (2.18) for $|t - s| \geq \lambda^{-1/3}$. We next note that if x is written in geodesic polar coordinates about y , then the function $\xi_0''(t, x, s, y)$ determined by

$$\nabla_{\xi''} \varphi(t, x, (1, \xi'')) - \nabla_{\xi''} \varphi(s, y, (1, \xi'')) = 0,$$

is smooth in the new coordinates. In particular, it is smooth in t and s , with uniform bounds on derivatives as x and y vary. Furthermore,

$$|\partial_{x, y}^\alpha \partial_{s, t}^\gamma \xi_0''(t, x, s, y)| \leq C_{\alpha, \gamma} \lambda^{|\alpha|/3} \quad \text{if } |x - y| \geq \lambda^{-1/3}.$$

Statement (2.19) follows from the eikonal equations (2.11). By equation (2.11), we also see that

$$\det[\partial_t \varphi_{\xi'' \xi''}''(t, x, (1, \xi''))] \approx 1.$$

In the region $|x - y| \approx |t - s|$, it follows that we can write

$$\varphi(t, x, \xi) - \varphi(s, y, \xi) = |x - y| \Phi(t, x, s, y, \xi),$$

where $|\det \Phi_{\xi'' \xi''}''| \approx 1$ if $\xi_1 = 1$. We now introduce new variables

$$\xi_1 = \tau, \quad \xi'' = \tau(\eta + \xi_0''(t, x, s, y)),$$

and for $|x - y| \approx |t - s|$ write

$$K_\lambda^1(t, x, s, y) = \lambda^{n-1-2\mu} \iint e^{i\tau|x-y|\Phi(t, x, s, y, (1, \eta+\xi_0''))} a_\lambda(t, x, s, y, \eta, \tau) d\eta d\tau,$$

where the symbol a_λ is supported in the region $\tau \approx \lambda$, and satisfies

$$|\partial_{t, s, \eta}^\alpha \partial_\tau^k a_\lambda(t, x, s, y, \eta, \tau)| \leq C_{\alpha, k} \lambda^{\frac{|\alpha|}{3} - \frac{2k}{3}}.$$

We want to apply stationary phase to the integral over η , since the phase Φ is nondegenerate in this variable. To do this, we introduce the stretched variable $\omega = \tau |x - y|$. The symbol obeys the following estimates, where $\delta = |x - y|^{3/2}$:

$$|\partial_{t,s,\eta}^\alpha \partial_\omega^k a_\lambda(t, x, s, y, \eta, |x - y|^{-1} \omega)| \leq C_{k,\alpha} \lambda^{|\alpha|/3} (\delta \lambda)^{-2k/3}.$$

By the following Lemma 2.6, we can write

$$\begin{aligned} & \int e^{i\omega \Phi(t,x,s,y,(1,\eta+\xi_0''))} a_\lambda(t, x, s, y, \eta, |x - y|^{-1} \omega) d\eta \\ &= \omega^{-(n-1)/2} e^{i\omega \Phi(t,x,s,y,(1,\xi_0''))} c_\lambda(t, x, s, y, \omega), \end{aligned}$$

where

$$|\partial_{t,s}^\alpha \partial_\omega^k c_\lambda(t, x, s, y, \omega)| \leq C_{k,\alpha} \lambda^{|\alpha|/3} (\delta \lambda)^{-2k/3}.$$

The conclusion of Lemma 2.5 follows by setting $\omega = |x - y| \tau$. \square

Lemma 2.6. *Suppose that $\Phi(z, \theta) \in C^\infty(\mathbb{R}^m \times \mathbb{R}^d)$ is real, $\nabla_\theta \Phi(z, 0) = 0$, $\nabla_\theta \Phi(z, \theta) \neq 0$ if $\theta \neq 0$, and*

$$|\det \Phi''_{\theta\theta}| \geq c_0 > 0 \text{ if } |\theta| \leq 1.$$

Suppose also that

$$|\partial_z^\alpha \partial_\theta^\beta \Phi(z, \theta)| \leq C_{\alpha,\beta} \lambda^{|\alpha|/3} \quad \forall \alpha, \beta.$$

In addition, suppose that the symbol $a(z, \theta, \omega)$ vanishes when $|\theta| \geq 1$, and satisfies

$$|\partial_z^\alpha \partial_\theta^\gamma \partial_\omega^k a(z, \theta, \omega)| \leq C_{k,\alpha,\gamma} \lambda^{(|\alpha|+|\gamma|)/3} (\delta \lambda)^{-2k/3} \quad \forall k, \alpha, \gamma,$$

where on the support of a_λ we have $\omega \geq \lambda^{2/3}$. Then we can write

$$\int_{\mathbb{R}^d} e^{i\omega \Phi(z, \theta)} a(z, \theta, \omega) d\theta = \omega^{-d/2} e^{i\omega \Phi(z, 0)} b(z, \omega),$$

where b satisfies

$$|\partial_\omega^k \partial_z^\alpha b(z, \omega)| \leq C_{k,\alpha} \lambda^{|\alpha|/3} (\delta \lambda)^{-2k/3},$$

with each of the constants depending only on c_0 and the size of finitely many of the constants $C_{\alpha,\beta}$ and $C_{k,\alpha,\gamma}$ above. In particular, the constants are uniform in δ if $1 \geq \delta \geq \lambda^{-1}$.

A variation of this lemma was used by Grieser [3] in his related work. The lemma follows easily from the proof of the standard stationary phase lemma (see, e.g., [28, p. 45]).

Proof of Proposition 2.3: The diffractive term. We begin by showing that, for $x_n \geq 0$, the symbol $q(x, \xi)$ defined by (2.9) can be written in the form

$$q(x, \xi) = p(x, \xi, \zeta(x, \xi)),$$

where

$$\left| \partial_{\xi}^{\alpha} \partial_{\zeta}^j \partial_{x'}^{\beta} \partial_{x_n}^k p(x, \xi, \zeta) \right| \leq C_{\alpha, j, \beta, k} \xi_1^{1/6 - |\alpha| + 2k/3} e^{-c x_n^{3/2} \xi_1 - \frac{1}{2} |\zeta|^{3/2}},$$

for some $c > 0$. Recall that $\xi' = (\xi_1, \dots, \xi_{n-1})$, and that we are working in a small conic neighborhood of the positive ξ_1 axis. Since

$$\left| \partial_{\zeta}^k ((1 - \chi) A_+) (\zeta) \right| \leq C_{k, \varepsilon} e^{(\frac{2}{3} + \varepsilon) |\zeta|^{3/2}} \quad \forall \varepsilon > 0,$$

and the symbols $a(x, \xi)$ and $b(x, \xi)$ belong to $S_{1,0}^6$, the above fact will follow by showing that, in the region $\zeta(x, \xi) \geq -2$,

$$\frac{Ai}{A_+}(\zeta_0(\xi)) = \tilde{p}(x, \xi', \zeta(x, \xi)),$$

where

$$(2.21) \quad \left| \partial_{\xi'}^{\alpha} \partial_{\zeta}^j \partial_{x'}^{\beta} \partial_{x_n}^k \tilde{p}(x, \xi', \zeta) \right| \leq C_{\alpha, j, \beta, k, \varepsilon} \xi_1^{-|\alpha| + 2k/3} e^{-c x_n^{3/2} \xi_1 - (\frac{4}{3} - \varepsilon) |\zeta|^{3/2}}.$$

At $x_n = 0$, one has $\zeta = \zeta_0$. Furthermore, $\partial_{x_n} \zeta < 0$. It follows that

$$\zeta_0(x, \xi) \geq \zeta(x, \xi) + c x_n \xi_1^{2/3},$$

for some positive constant c . In the region $\zeta(x, \xi) \geq 2$, we therefore have by the asymptotics of the Airy functions that

$$(2.22) \quad \left| \left(\frac{Ai}{A_+} \right)^{(m)}(\zeta_0) \right| \leq C_{k, \varepsilon} e^{-c x_n^{3/2} \xi_1 - (\frac{4}{3} - \varepsilon) |\zeta(x, \xi)|^{3/2}}.$$

We introduce a new variable

$$\tau(x, \xi) = \xi_1^{1/3} \zeta(x, \xi).$$

At $x_n = 0$, one has $\tau = -\xi_n$, so that we can write $\xi_n = \sigma(x, \xi', \tau)$, where σ is homogeneous of degree 1 in (ξ', τ) . We thus set

$$\tilde{p}(x, \xi', \zeta) = \frac{Ai}{A_+}(-\xi_1^{-1/3} \sigma(x, \xi', \xi_1^{1/3} \zeta)).$$

The estimates (2.21) will follow by showing that

$$(2.23) \quad \left| \partial_{\xi'}^{\alpha} \partial_{\tau}^j \partial_{x'}^{\beta} \partial_{x_n}^k \frac{Ai}{A_+}(-\xi_1^{-1/3} \sigma(x, \xi', \tau)) \right| \leq C_{\alpha, j, \beta, k, \varepsilon} \xi_1^{-|\alpha| - j + 2k/3} e^{-c x_n^{3/2} \xi_1 - (\frac{4}{3} - \varepsilon) |\tau|^{3/2} \xi_1^{-1/2}}.$$

For $k = 0$, the estimates (2.23) follow from (2.22), together with the fact that

$$\left| \partial_{\xi'}^{\alpha} \partial_{\tau}^j \partial_{x'}^{\beta} (-\xi_1^{-1/3} \sigma(x, \xi', \tau)) \right| \leq C_{\alpha, \beta, j} (x_n \xi_1^{2/3} + \xi^{-1/3} |\tau|) \xi_1^{-|\alpha| - j},$$

which, in turn, holds by homogeneity, together with the fact that $\sigma(x, \xi', \tau) = 0$ if $x_n = \tau = 0$. If $k > 0$, the estimate (2.23) follows by observing that the effect

of differentiating in x_n is similar to multiplying by a symbol of order $\frac{2}{3}$. This concludes the proof of (2.21).

We now take the Fourier transform of the symbol $p(x, \xi, \zeta)$ in ζ to write the Schwartz kernel of the diffractive term S in the form

$$\int e^{i\theta(x, \xi) + it\xi_1 + is\xi_1^{-2/3}\zeta(x, \xi) + is^3/3\xi_1^2 - (v, \xi)} a(x, \xi, \xi_1^{-2/3}s) ds d\xi,$$

where

$$\begin{aligned} & \left| \partial_\zeta^\alpha \partial_s^j \partial_x^\beta \partial_{x_n}^k a(x, \xi, \xi_1^{-2/3}s) \right| \\ & \leq C_{\alpha, j, \beta, k, N} \xi_1^{\alpha - 1/2 - |\alpha| - 2j/3 + 2k/3} e^{-c x_n^{3/2} \xi_1} \langle \xi_1^{-2/3}s \rangle^{-N} \quad \forall N. \end{aligned}$$

Proceeding now as for the main term, we reduce matters to considering the operator

$$\tilde{S}f(t, x) = \int e^{i\varphi(t, x, \xi)} c(t, x, \xi) \hat{f}(\xi) d\xi,$$

where now

$$(2.24) \quad x_n^j \partial_{x_n}^k c(t, x, \xi) \in S_{\frac{2}{3}, \frac{1}{3}}^{2(k-j)/3}(\mathbb{R}_{x', t}^n \times \mathbb{R}_\xi^n)$$

uniformly over x_n .

It suffices to establish estimates (2.2'), (2.3'), and (2.4') uniformly for the operator \tilde{S}_λ , which is defined by dyadically localizing to $\xi_1 \approx \lambda$. If c_λ is the dyadic symbol obtained, then using the freezing arguments behind the proof of the estimates for T_λ^0 , we conclude that

$$\begin{aligned} \|\tilde{S}_\lambda f\| & \leq \left\| \int e^{i\varphi(t, x, \xi)} c_\lambda(t, x', 0, \xi) \hat{f}(\xi) d\xi \right\| \\ & \quad + \int \left\| \int e^{i\varphi(t, x, \xi)} (\partial_{x_n} c_\lambda)(t, x', r, \xi) \hat{f}(\xi) d\xi \right\| dr, \end{aligned}$$

where $\|\cdot\|$ is any of the desired function space norms. Since the symbols

$$c_\lambda(t, x', 0, \xi) \text{ and } \lambda^{-2/3}(1 + \lambda^{4/3}r^2)(\partial_{x_n} c_\lambda)(t, x', r, \xi)$$

are, by (2.24), symbols of order 0 and type (2/3, 1/3), with uniform estimates over r , the estimates for the diffractive term also follow from Proposition 2.4. \square

3. THE CRITICAL SEMILINEAR WAVE EQUATION OUTSIDE CONVEX OBSTACLES

Let $\Omega = \mathbb{R}^3 \setminus \mathcal{O}$, where \mathcal{O} is compact, with smooth, strictly convex boundary (for the Euclidean metric). If Δ here denotes the \mathbb{R}^3 Laplacian and $\square = \partial^2 t / \partial t^2 - \Delta$, we shall be concerned with the critical semilinear wave equation in $\mathbb{R}_+ \times \Omega$:

$$(3.1) \quad \begin{cases} \square u = -u^3, & t > 0, \\ u(0, x) = f(x) \in C^\infty(\Omega), & \partial_t u(0, x) = g(x) \in C^\infty(\Omega), \\ u(t, x) = 0, & x \in \partial\Omega. \end{cases}$$

We assume in what follows, for the sake of simplicity, that the data (f, g) is real. Otherwise, one would have to replace the first part of the equation by $\square u = -|u|^4 u$, and modify the regularity assertions accordingly to take into account the lack of regularity of $u \rightarrow |u|^4 u$ at $u = 0$.

Because of the boundary, one needs to impose certain compatibility conditions on the data for solutions to be smooth. To state these conditions, we define a collection of functions ψ_j on Ω as follows. Set

$$\psi_0 = f, \quad \psi_1 = g.$$

We now recursively define ψ_j such that, if the function \tilde{u} has the following Taylor expansion in t ,

$$\tilde{u}(t, x) \sim \sum_{j=0}^{\infty} \psi_j(x) \frac{t^j}{j!},$$

then $\square \tilde{u} + \tilde{u}^5$ vanishes to infinite order at $t = 0$. This is seen to determine ψ_j uniquely, and ψ_j is then a nonlinear function of the data (f, g) , involving derivatives of order up to j of f , and of order up to $j - 1$ of g .

Definition. We say that the data (f, g) satisfies the compatibility conditions to order k if, for $0 \leq j \leq k$, the functions ψ_j vanish on $\partial\Omega$. We say that the data satisfies the compatibility conditions to infinite order if all the ψ_j vanish on $\partial\Omega$.

In addition to our assumption that the data is smooth and real, we shall assume that it satisfies the compatibility conditions to infinite order. Notice that the last is always satisfied if both f and g vanish near $\partial\Omega$.

Grillakis [4] showed that, for the free wave operator in \mathbb{R}^{1+3} , the associated critical semilinear wave equation has a smooth solution if the data is smooth. Earlier special cases of this result were obtained by Rauch [22] and Struwe [33]. Here we shall show that this result extends naturally to the setting of the wave equation outside of a convex obstacle. Specifically, by using the Pecher-type estimates of Corollary 2.2 and ideas from Grillakis [5] and Shatah and Struwe [23], we shall prove the following

Theorem 3.1. *Assume that the real data $(f, g) \in C^\infty(\Omega)$ satisfies the compatibility conditions to infinite order. Then there is a global C^∞ solution to (3.1).*

Using the finite propagation speed of \square , we can assume that our data (f, g) vanishes when $|x|$ is large. At the end of this section we shall show that, if we fix such data, there must be a local smooth solution in a strip; that is, there must be a $t_0 > 0$ and a solution $u \in C^\infty([0, t_0] \times \Omega)$ to (3.1). Just as in the free case (see, e.g., [34]), we shall also show that, if $|u|$ is uniformly bounded by any finite constant in this strip, then u can be smoothly extended to a larger strip $([0, t'] \times \Omega)$, $t' > t_0$. Just as in [22], [4], [5], [23], therefore, our strategy will be to obtain pointwise bounds for u in any such strip. Specifically, we shall prove the following result, which for the reasons just stated implies Theorem 3.1.

Theorem 3.1'. *Suppose that $u \in C^\infty([0, t_0] \times \Omega)$ solves (3.1), with data (f, g) as above. Then, if $x_0 \in \bar{\Omega}$, u must be bounded in a neighborhood of (t_0, x_0) , and hence $u \in L^\infty([0, t_0] \times \Omega)$.*

To prove this we shall require several lemmas. The first is standard and says that the energy associated with our equation is conserved; furthermore, the energy inside spatial cross-sections of a backwards light cone is monotonic decreasing in time.

Lemma 3.2. *If $u \in C^\infty([0, t_0] \times \Omega)$ is a solution to (3.1), with data (f, g) as above, then*

$$(3.2) \quad \int_{\Omega} \left[\frac{|\partial_t u(t, x)|^2 + |\nabla_x u(t, x)|^2}{2} + \frac{|u|^6}{6} \right] dx$$

is equal to a fixed constant $E_0 < \infty$ for all $0 \leq t < t_0$. Additionally, if $0 \leq s < t < t_0$, $x_0 \in \bar{\Omega}$, and $\delta \geq 0$, then

$$(3.3) \quad \int_{\substack{x \in \Omega \\ |x-x_0| < t_0-t+\delta}} \left[\frac{|\nabla_x u(t, x)|^2 + |\partial_t u(t, x)|^2}{2} + \frac{1}{6} u^6(t, x) \right] dx \\ \leq \int_{\substack{x \in \Omega \\ |x-x_0| < t_0-s+\delta}} \left[\frac{|\nabla_x u(s, x)|^2 + |\partial_t u(s, x)|^2}{2} + \frac{1}{6} u^6(s, x) \right] dx.$$

We shall also require the following nontrivial decay lemma, which generalizes Struwe's [33] Euclidean version, and states that the L^6 part of the energy cannot concentrate at any point (x_0, t_0) .

Lemma 3.3. *If u is as above and $x_0 \in \bar{\Omega}$, then*

$$(3.4) \quad \lim_{t \nearrow t_0} \int_{\substack{x \in \Omega \\ |x-x_0| < t_0-t}} \frac{1}{6} u^6(t, x) dx = 0.$$

We postpone the proofs of Lemmas 3.2 and 3.3 for the moment. The first step in the proof of Theorem 3.1' is the following lemma, which uses Hölder's inequality along with (3.4) and Corollary 2.2, to control certain mixed space-time norms of u over the backwards cone

$$K = \{ (t, x) \in [0, t_0] \times \Omega : |x - x_0| < t_0 - t \}.$$

Lemma 3.4. *Let u be as above. Then*

$$u \in L_t^{\frac{2q}{q-6}} L_x^q(K), \quad \text{if } 6 \leq q < \infty.$$

Proof. Hölder's inequality implies that if $6 < q < q_1$, then $L_t^{\frac{2q}{q-6}} L_x^q \subset L_t^\infty L_x^6 \cap L_t^{\frac{2q_1}{q_1-6}} L_x^{q_1}$. Since $u \in L_t^\infty L_x^6$ by conservation of energy, it therefore suffices to check that

$$(3.5) \quad u \in L_t^{\frac{2q}{q-6}} L_x^q(K) \quad \text{if } 10 \leq q < \infty.$$

If $0 \leq s_1 < s_2 < t_0$, set

$$K_{s_1}^{s_2} = K \cap ([s_1, s_2] \times \Omega),$$

where K is as above. Then, since u is smooth and has relatively compact support in $[0, t_0] \times \Omega$, it suffices to show that for some fixed $0 < s_1 < t_0$, one has

$$\sup_{s_2 \in (s_1, t_0)} \|u\|_{L_t^{\frac{2q}{q-6}} L_x^q(K_{s_1}^{s_2})} < \infty, \quad \text{if } 10 \leq q < \infty.$$

To establish this inequality, we observe that Corollary 2.2 and domain of dependence considerations imply

$$\|u\|_{L_t^{\frac{2q}{q-6}} L_x^q(K_{s_1}^{s_2})} \leq C_q E_0 + C_q \|u^5\|_{L_t^1 L_x^2(K_{s_1}^{s_2})},$$

where E_0 denotes the energy of u . If $q > 10$, another application of Hölder's inequality yields

$$\|u^5\|_{L_t^1 L_x^2(K_{s_1}^{s_2})} \leq \|u\|_{L_t^\infty L_x^6(K_{s_1}^{s_2})}^{5-\frac{2q}{q-6}} \|u\|_{L_t^{\frac{2q}{q-6}} L_x^q(K_{s_1}^{s_2})}^{\frac{2q}{q-6}},$$

and consequently

$$\|u\|_{L_t^{\frac{2q}{q-6}} L_x^q(K_{s_1}^{s_2})} \leq C_q E_0 + C_q \|u\|_{L_t^\infty L_x^6(K_{s_1}^{s_2})}^{5-\frac{2q}{q-6}} \|u\|_{L_t^{\frac{2q}{q-6}} L_x^q(K_{s_1}^{s_2})}^{\frac{2q}{q-6}}.$$

Given $\varepsilon > 0$, (3.4) implies that we can choose s_1 close enough to t_0 so that

$$C_q \|u\|_{L_t^\infty L_x^6(K_{s_1}^{t_0})}^{5-\frac{2q}{q-6}} < \varepsilon.$$

If we choose

$$\varepsilon < 2^{-\frac{2q}{q-6}} (C_q E_0)^{1-\frac{2q}{q-6}},$$

then the following standard lemma implies that

$$\|u\|_{L_t^{\frac{2q}{q-6}} L_x^q(K_{s_1}^{s_2})} < 2C_q E_0,$$

giving us (3.5) and finishing the proof. \square

Lemma 3.5. *Let $0 < C_0 < \infty$ and suppose that $0 \leq y(s) \in C([a, b])$, with $y(a) = 0$, satisfies*

$$y(s) \leq C_0 + \varepsilon y(s)^\gamma.$$

Then if $\varepsilon < 2^{-\gamma} C_0^{1-\gamma}$ it follows that

$$y(s) < 2C_0, \quad s \in [a, b].$$

Proof. Since $C_0 + \varepsilon x^\gamma - x < 0$ if $\varepsilon < 2^{-\gamma} C_0^{1-\gamma}$ and $x = 2C_0$, it follows that

$$0 \leq C_0 + \varepsilon x^\gamma - x \quad \forall x \in [0, x_0] \implies x_0 < 2C_0.$$

Since $y(s)$ must be \leq the supremum of such x_0 , the lemma follows. \square

End of Proof of Theorem 3.1'. We shall use the following special case of Lemma 3.4:

$$u \in L_t^4 L_x^{12}(K).$$

Since $\square(\partial_t u) = -5u^4 \partial_t u$, and $\partial_t u$ vanishes on $\partial\Omega$, estimate (2.5) with $q = 6$ implies that, if $0 \leq s < t < t_0$, then

$$\begin{aligned} \|\partial_t u\|_{L_t^\infty L_x^6(K_s^t)} &\leq C(s) + C\|u^4 \partial_t u\|_{L_t^1 L_x^2(K_s^t)} \\ &\leq C(s) + C\|u\|_{L_t^4 L_x^{12}(K_s^t)} \|\partial_t u\|_{L_t^\infty L_x^6(K_s^t)}. \end{aligned}$$

Hence, if s is close enough to t_0 so that $C\|u\|_{L_t^4 L_x^{12}(K_s^t)}^4 < 1/2$, we conclude that $\partial_t u \in L_t^\infty L_x^6(K_s^t)$ with norm bounded by $2C(s)$ for all $t \in (s, t_0)$, which yields

$$\partial_t u \in L_t^\infty L_x^6(K).$$

If x_0 is interior to Ω , a similar argument can be applied to show that $\nabla_x u \in L_t^\infty L_x^6(K)$, and from this and Hölder's inequality we could conclude that the total energy of u cannot concentrate at (t_0, x_0) :

$$(3.6) \quad \lim_{t \nearrow t_0} \int_{\substack{x \in \Omega \\ |x-x_0| < t_0-t}} \left[\frac{1}{2} |\nabla_{t,x} u(t, x)|^2 + \frac{1}{6} u^6(t, x) \right] dx = 0.$$

If $x_0 \in \partial\Omega$, however, this argument breaks down since $\nabla_x u$ does not vanish on $\partial\Omega$, and we cannot apply (2.5) to estimate it.

Instead, for any $x_0 \in \bar{\Omega}$ we may apply the energy inequality to the equation $\square(\partial_t u) = -5u^4 \partial_t u$ to obtain

$$\begin{aligned} &\left(\int_{\substack{x \in \Omega \\ |x-x_0| < t_0-t}} [|\nabla_x \partial_t u(t, x)|^2 + |\partial_t^2 u(t, x)|^2] dx \right)^{1/2} \\ &\leq \|f\|_{\dot{H}^2} + \|g\|_{\dot{H}^1} + 5 \left(\int_K |u^4 \partial_t u|^2 dt dx \right)^{1/2} \\ &\leq C + C\|u\|_{L_t^4 L_x^{12}(K)}^4 \|\partial_t u\|_{L_t^\infty L_x^6(K)}. \end{aligned}$$

Since the last terms are bounded by previous arguments, we conclude that

$$(3.7) \quad \partial_t \nabla_x u, \partial_t^2 u \in L_t^\infty L_x^2(K).$$

We now write

$$\nabla_{t,x} u(t, x) = \nabla_{t,x} u(s, x) + \int_s^t \partial_t \nabla_{t,x} u(r, x) dr,$$

and conclude that

$$\begin{aligned} & \left(\int_{\substack{x \in \Omega \\ |x-x_0| < t_0-t}} |\nabla_{t,x} u(t, x)|^2 dx \right)^{1/2} \\ & \leq \left(\int_{\substack{x \in \Omega \\ |x-x_0| < t_0-t}} |\nabla_{t,x} u(s, x)|^2 dx \right)^{1/2} + (t_0 - s) \|\partial_t \nabla_{t,x} u\|_{L_t^\infty L_x^2(K)}. \end{aligned}$$

The second term on the right is made small by taking $(t_0 - s)$ small. Since $u(t, x)$ is smooth for $t < t_0$, the first term on the right vanishes as $t \rightarrow t_0$. Thus, the left side goes to 0 as $t \rightarrow t_0$, and together with (3.4) we conclude that (3.6) holds for $x_0 \in \partial\Omega$ as well.

Therefore, given $\varepsilon > 0$, we can find a $0 < t_1 < t_0$ so that

$$\int_{\substack{x \in \Omega \\ |x-x_0| < t_0-t}} \left[\frac{1}{2} |\nabla_{t,x} u(t_1, x)|^2 + \frac{1}{6} u^6(t_1, x) \right] dx < \varepsilon/2.$$

By dominated convergence, there is a $\delta > 0$ so that

$$\int_{\substack{x \in \Omega \\ |x_0-x| < \delta+t_0-t_1}} \left[\frac{1}{2} |\nabla_{t,x} u(t_1, x)|^2 + \frac{1}{6} u^6(t_1, x) \right] dx < \varepsilon,$$

which, by the monotonicity of energy (3.3), yields

$$\int_{\substack{x \in \Omega \\ |x-x_0| < \delta+t_0-t}} \frac{1}{6} u^6(t, x) dx < \varepsilon, \quad t_1 \leq t < t_0.$$

Let

$$K^\delta = \{ (t, x) \in [0, t_0] \times \Omega : |x - x_0| < \delta + t_0 - t \}.$$

For ε sufficiently small, we can repeat the proof of Lemma 3.4 with K replaced by K^δ , to conclude that

$$u \in L_t^4 L_x^{12}(K^\delta).$$

We can now argue as before to conclude that

$$\partial_t u \in L_t^\infty L_x^6(K^\delta).$$

If x_0 is interior to Ω , a similar argument shows that $\nabla_x u \in L_t^\infty L_x^6(K^\delta)$, which implies $u \in L^\infty(K^{\delta/2})$ by Sobolev's theorem.

For $x_0 \in \partial\Omega$, an additional argument is needed since $\nabla_x u$ does not vanish on $\partial\Omega$. First, the proof of (3.7) now yields

$$(3.8) \quad \partial_t^2 u \in L_t^\infty L_x^2(K^\delta).$$

Let $K_{s,t}^\delta = K^\delta \cap ([s, t] \times \Omega)$. The estimates (2.5) imply that

$$\begin{aligned} \|\partial_t u\|_{L_t^5 L_x^{10}(K_{s,t}^\delta)} &\leq CE_0 + C\|u^4 \partial_t u\|_{L_t^1 L_x^2(K_{s,t}^\delta)} \\ &\leq CE_0 + C\|u\|_{L_t^5 L_x^{10}(K_{s,t}^\delta)}^4 \|\partial_t u\|_{L_t^5 L_x^{10}(K_{s,t}^\delta)}. \end{aligned}$$

Since

$$u \in L_t^\infty L_x^6(K^\delta) \cap L_t^4 L_x^{12}(K^\delta) \subseteq L_t^5 L_x^{10}(K^\delta),$$

we see from this that the left side is $\leq 2CE_0$ for s close to t_0 . Consequently, $\partial_t u \in L_t^5 L_x^{10}(K^\delta) \subseteq L_t^1 L_x^{10}(K^\delta)$, which implies that

$$u \in L_t^\infty L_x^{10}(K^\delta).$$

Since $\partial_t^2 u - \Delta_x u + u^5 = 0$, this along with (3.8) yields

$$\Delta_x u \in L_t^\infty L_x^2(K^\delta).$$

Since u satisfies Dirichlet boundary conditions, elliptic regularity implies $u \in L^\infty(K^{\delta/2})$, which completes the proof. \square

We still need to prove Lemmas 3.2 and 3.3. The first one is the easiest and its proof will serve as a model for the proof of Lemma 3.3.

Proof of Lemma 3.2. To prove the conservation of energy one multiplies both sides of the equation $\square u + u^5 = 0$ by $\partial_t u$ to obtain the identity

$$(3.2') \quad \partial_t \left(\frac{|\partial_t u|^2 + |\nabla_x u|^2}{2} + \frac{|u|^6}{6} \right) - \operatorname{div}_x (\partial_t u \nabla_x u) = 0.$$

Thus,

$$0 = \partial_t \int_\Omega \left(\frac{|\partial_t u|^2 + |\nabla_x u|^2}{2} + \frac{|u|^6}{6} \right) dx - \int_\Omega \operatorname{div}_x (\partial_t u \nabla_x u) dx,$$

and since the last term is always zero, by the divergence theorem, due to the fact that $\partial_t u = 0$ on $\partial\Omega$ and $u(t, x) = 0$ for $|x| > C + t$, we see that (3.2') implies that (3.2) must be constant, as desired.

To prove the other half of Lemma 3.2 we need to define the energy flux across part of the domain of dependence of a point.

To do this, we first need to introduce some more notation. First of all, if $0 \leq s < t < t_0$, and if K_s^t is as above, let M_s^t denote the ‘‘mantle’’ associated with K_s^t :

$$M_s^t = \partial K_s^t \cap ((s, t) \times \Omega).$$

Also, let $d\sigma$ denote the induced Lebesgue measure on M_s^t and let $\nu = \nu(r, x)$ denote the outward unit normal through $(r, x) \in M_s^t$. If we then let $e(u)$ be the vector field arising from (3.2'),

$$e(u) = \left(\frac{|\partial_t u|^2 + |\nabla_x u|^2}{2} + \frac{|u|^6}{6}, -\partial_t u \nabla_x u \right),$$

we can define the “energy flux” across M_s^t :

$$\text{Flux} (u, M_s^t) = \int_{M_s^t} \langle e(u), \nu \rangle d\sigma.$$

Since $u = \partial_t u = 0$ on $\partial\Omega$, if we integrate (3.2') over K_s^t we arrive at the “flux identity”:

$$\begin{aligned} (3.3') \quad & \int_{\substack{x \in \Omega \\ |x-x_0| < t_0-t+\delta}} \left[\frac{|\nabla_x u(t, x)|^2 + |\partial_t u(t, x)|^2}{2} + \frac{1}{6} u^6(t, x) \right] dx + \text{Flux} (u, M_s^t) \\ & = \int_{\substack{x \in \Omega \\ |x-x_0| < t_0-s+\delta}} \left[\frac{|\nabla_x u(s, x)|^2 + |\partial_t u(s, x)|^2}{2} + \frac{1}{6} u^6(s, x) \right] dx. \end{aligned}$$

It is easy to check that

$$\langle e(u), \nu \rangle \geq 0.$$

Hence, $\text{Flux} (u, M_s^t) \geq 0$. Therefore (3.3') implies (3.3), which completes the proof. \square

Proof of Lemma 3.3. We follow [5], [23] and [33]. The main difference is that we have to make sure that in the proof the terms arising from the boundary have the correct sign.

For the sake of notation it is convenient to shift $(t_0, x_0) \in \mathbb{R} \times \overline{\Omega}$ to the origin. We then want to use an identity of Morawetz (see [20]). Specifically, by multiplying the equation $\square u + u^5 = 0$ by $tu_t + x \cdot \nabla_x u + u$, one arrives at the identity

$$0 = \text{div}_{t,x} (tQ + u_t u, -tP) + \frac{|u|^6}{3},$$

where

$$\begin{aligned} Q &= \frac{|u_t|^2 + |\nabla_x u|^2}{2} + \frac{|u|^6}{6} + u_t \frac{x}{t} \cdot \nabla_x u, \\ P &= \frac{x}{t} \left(\frac{|u_t|^2 - |\nabla_x u|^2}{2} - \frac{|u|^6}{6} \right) + \nabla_x u \left(u_t + \frac{x}{t} \cdot \nabla_x u + \frac{u}{t} \right). \end{aligned}$$

We want to integrate the Morawetz identity over the truncated cones K_S^T . To apply the divergence theorem we note that

$$\partial K_S^T = (([S, T] \times \partial\Omega) \cap K_S^T) \cup M_S^T \cup D_T \cup D_S,$$

if $D_T = \{x \in \overline{\Omega} : |x| < -T\}$. Thus, if $\nu_{\partial\Omega}$ denotes the outward unit normal

for Ω ,

$$(3.9) \quad 0 = \int_{D_T} (TQ + u_t u) dx - \int_{D_S} (SQ + u_t u) dx + \frac{1}{\sqrt{2}} \int_{M_S^T} (tQ + u_t u + x \cdot P) d\sigma \\ + \int_{([S, T] \times \partial\Omega) \cap K_S^T} \nu_{\partial\Omega} \cdot [-tP] d\sigma + \frac{1}{3} \int_{K_S^T} u^6 dt dx.$$

Clearly the last term is nonnegative. We claim that the same is true for the second to last one. More specifically, we claim that

$$(3.10) \quad -t\nu \cdot P \geq 0,$$

if $\nu = \nu_{\partial\Omega}$. To see this, we note that on $[S, T] \times \partial\Omega$,

$$\nabla_x u = (\partial_\nu u) \nu, \quad u = u_t = 0.$$

Thus,

$$P = \frac{x}{t} \left(-\frac{1}{2} (\partial_\nu u)^2 \right) + \nu (\partial_\nu u)^2 \left(\frac{x}{t} \cdot \nu \right),$$

yielding

$$-t\nu \cdot P = -\frac{1}{2} (\nu \cdot x) (\partial_\nu u)^2,$$

and since ν is the normal pointing into the convex obstacle, $\nu \cdot x \leq 0$ when $x \in \partial\Omega$, giving us (3.10).

Using Hölder's inequality and the conservation of energy, it is not hard to see that the first term in (3.9) tends to zero as $T \nearrow 0$. Thus,

$$0 \geq - \int_{D_S} (SQ + u_t u) dx + \frac{1}{\sqrt{2}} \int_{M_S^0} [tQ + u_t u + x \cdot P] d\sigma \\ = I + II.$$

The next step is to realize that since $t = -|x|$ on M_S^0 , we can rewrite the last term as

$$II = \frac{1}{\sqrt{2}} \int_{M_S^0} \left[-|x| |u_t|^2 + 2u_t x \cdot \nabla_x u - \frac{(x \cdot \nabla_x u)^2}{|x|} - u \frac{x}{|x|} \cdot \nabla_x u + u_t u \right] d\sigma.$$

If we parameterize M_S^0 by

$$\Omega \ni y \rightarrow (-|y|, y), \quad |y| \leq |S|,$$

and let $v(y) = u(-|y|, y)$, then $d\sigma = \sqrt{2} dy$ and

$$y \cdot \frac{\nabla v}{|y|} = \frac{x \cdot \nabla_x u}{|x|} - u_t.$$

Thus, a calculation gives

$$\begin{aligned} II &= - \int_{\{y \in \Omega : |y| \leq |S|\}} \left\{ \frac{|y \cdot \nabla v|^2}{|y|} + v \frac{y \cdot \nabla v}{|y|} \right\} dy \\ &= - \int_{\{y \in \Omega : |y| \leq |S|\}} \frac{1}{|y|} |y \cdot \nabla v + v|^2 dy \\ &\quad + \int_{\{y \in \Omega : |y| \leq |S|\}} \left[\frac{v^2}{|y|} + v \frac{y \cdot \nabla v}{|y|} \right] dy. \end{aligned}$$

But if one integrates by parts and uses the fact that $v = 0$ on $\partial\Omega$, one sees that

$$\int_{\{y \in \Omega : |y| \leq |S|\}} v \frac{y \cdot \nabla v}{|y|} dy = \frac{1}{2} \int_{\{y \in \Omega : |y|=|S|\}} v^2 d\sigma - \int_{\{y \in \Omega : |y| \leq |S|\}} \frac{v^2}{|y|} dy.$$

So if we switch back to the original coordinates, we have the desired conclusion that

$$(3.11) \quad II = \frac{1}{\sqrt{2}} \int_{M_S^0} \frac{1}{t} |tu_t + x \cdot \nabla_x u + u|^2 d\sigma + \frac{1}{2} \int_{\partial D_S} u^2 d\sigma.$$

To handle I , we first rewrite it as

$$\begin{aligned} - \int_{D_S} \left[S \left\{ \frac{1}{2} u_t^2 + \frac{1}{2} \left| \nabla_x u + \frac{x}{|x|^2} u \right|^2 - \frac{x \cdot \nabla_x u}{|x|^2} u - \frac{1}{2} \frac{u^2}{|x|^2} + \frac{|u|^6}{6} \right\} \right. \\ \left. + u_t (x \cdot \nabla_x u + u) \right] dx. \end{aligned}$$

But, as above, integration by parts gives

$$\int_{D_S} u \frac{x}{|x|^2} \cdot \nabla_x u dx + \frac{1}{2} \int_{D_S} \frac{u^2}{|x|^2} dx = \frac{1}{2} \int_{\partial D_S} \frac{1}{|S|} u^2 d\sigma.$$

Thus,

$$\begin{aligned} (3.12) \quad I &= - \frac{1}{2} \int_{\partial D_S} u^2 d\sigma \\ &\quad + \int_{D_S} \left\{ -S \left[\frac{1}{2} u_t^2 + \frac{1}{2} \left| \nabla_x u + \frac{x}{|x|^2} u \right|^2 + \frac{|u|^6}{6} \right] - u_t (x \cdot \nabla_x u + u) \right\} dx \\ &\geq - \frac{1}{2} \int_{\partial D_S} u^2 d\sigma - S \int_{D_S} \frac{|u|^6}{6} dx, \end{aligned}$$

as

$$|u_t (x \cdot \nabla_x u + u)| \leq -S \left[\frac{1}{2} u_t^2 + \frac{1}{2} \left| \nabla_x u + \frac{x}{|x|^2} u \right|^2 \right].$$

Combining (3.9), (3.11), and (3.12) gives

$$-S \int_{D_S} \frac{|u|^6}{6} dx + \frac{1}{\sqrt{2}} \int_{M_S^0} \frac{1}{t} |tu_t + x \cdot \nabla_x u + u|^2 dx \leq 0,$$

and hence

$$\int_{D_S} \frac{|u|^6}{6} dx \leq \frac{1}{\sqrt{2}} \frac{1}{|S|} \int_{M_S^0} \frac{1}{|t|} |tu_t + x \cdot \nabla_x u + u|^2 d\sigma.$$

But, Hölder's inequality implies that the right side is

$$\begin{aligned} &\leq \sqrt{2} \int_{M_S^0} \left| \frac{x}{|x|} u_t - \nabla_x u \right|^2 d\sigma + \sqrt{2} \int_{M_S^0} \frac{|u|^2}{|S||t|} d\sigma \\ &\leq C \text{Flux}(u, M_S^0) + C \text{Flux}(u, M_S^0)^{1/3}. \end{aligned}$$

Therefore,

$$\int_{D_S} \frac{|u|^6}{6} dx \leq C \text{Flux}(u, M_S^0) + C \text{Flux}(u, M_S^0)^{1/3},$$

and since the flux identity, (3.3'), and the monotonicity of local energy, (3.3), imply that the right side goes to zero as $S \nearrow 0$, the proof of (3.4) is complete. \square

Local existence and regularity in (1+3)-dimensions. Here we shall complete the proof of Theorem 3.1. We must show that locally in time there is always a smooth solution to (3.1) if the compatibility conditions are satisfied, and that, if

$$u \in L^\infty([0, t_0] \times \Omega) \cap C^\infty([0, t_0] \times \Omega), \quad t_0 < \infty,$$

then u can be extended to a smooth solution in a larger strip.

If the data is as in Theorem 3.1, by Borel's Lemma, we can find a function $\tilde{u} \in C^\infty(\mathbb{R} \times \Omega)$ vanishing on $\mathbb{R} \times \Omega$ so that, at $t = 0$, \tilde{u} has Taylor expansion

$$\sum_{j=0}^{\infty} \psi_j(x) \frac{t^j}{j!},$$

with the ψ_j being the functions occurring in the compatibility condition for the data. If we then let

$$(3.13) \quad v(t, x) = \begin{cases} -\square \tilde{u} - \tilde{u}^5, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

it follows that

$$(3.14) \quad v \in C^\infty(\mathbb{R} \times \Omega) \quad \text{and} \quad v = 0 \quad \text{on} \quad (-\infty, 0] \times \Omega.$$

Let

$$(3.15) \quad F(t, x, w) = (\tilde{u}(t, x))^5 - (\tilde{u}(t, x) + w)^5.$$

Then F satisfies

$$(3.16) \quad \begin{cases} F \in C^\infty(\mathbb{R} \times \Omega \times \mathbb{R}), \\ F(t, x, 0) = 0, \\ |F^{(\alpha)}(t, x, w)| \leq C_\alpha(R), \quad \text{if } |w| \leq R. \end{cases}$$

For the sake of notation, if w is a function on $\mathbb{R} \times \Omega$ we shall abbreviate $F(t, x, w(t, x))$ by $F(w)$. We shall be interested in local existence and regularity for the inhomogeneous equation with zero data

$$(3.17) \quad \begin{cases} \square w = F(w) + v, \\ w(t, x) = 0, \quad t \leq 0, \\ w(t, x) = 0, \quad x \in \partial\Omega, \end{cases}$$

assuming that v and F satisfy (3.14) and (3.16), respectively. Notice that if v and F are given by (3.13) and (3.15), respectively, then $u = \tilde{u} + w$ solves our original equation (3.1). Thus local existence and regularity for (3.1) is equivalent to local existence and regularity for (3.17).

To proceed we need to make another definition.

Definition. Let

$$V_T^k = V_T^k(\mathbb{R} \times \Omega)$$

be the subspace of $\bigcap_{j=0}^k C^{k-j}((-\infty, T]; \bar{H}^j(\Omega))$ consisting of functions which vanish on $(-\infty, 0] \times \Omega$. For $k \geq 1$, we let $V_{T,d}^k = V_{T,d}^k(\mathbb{R} \times \Omega)$ denote the subspace of functions which in addition vanish on $(-\infty, T] \times \partial\Omega$.

Here $H^j(\Omega)$ is the Sobolev space of restrictions to Ω of elements in $H^j(\mathbb{R}^3)$, where the latter is the inhomogeneous L^2 Sobolev space involving j derivatives. (See [7, Appendix B.2].)

The basic estimate for the linear inhomogeneous wave equation which we shall require is the following

Lemma 3.6. *Suppose that $k \geq 0$, and that $G \in V_T^k$. Set $w = 0$ if $t \leq 0$ and*

$$w(t, \cdot) = \int_0^t \frac{\sin s\sqrt{-\Delta}}{\sqrt{-\Delta}} G(t-s, \cdot) ds, \quad t > 0.$$

Then $w \in V_{T,d}^{k+1}$, and

$$\|w\|_{V_{T,d}^{k+1}} \leq C_0(T) \|G\|_{V_T^k}.$$

Also, for $T \leq 1$, $C_0(T) = O(T)$.

Proof. The operator

$$\frac{\sin s\sqrt{-\Delta}}{\sqrt{-\Delta}}$$

is bounded from $L^2(\Omega)$ to the subspace of $H^1(\Omega)$ of functions which vanish on $\partial\Omega$, with operator norm bounded by a constant times $1+s$. It follows that

$$w \in C^k((-\infty, T]; H^1(\Omega)),$$

and that w vanishes on $((-\infty, 0] \times \Omega) \cup (\mathbb{R} \times \partial\Omega)$. We can integrate by parts to write

$$w(t, \cdot) = \int_0^t \cos s\sqrt{-\Delta} G^{(-1)}(t-s, \cdot) ds,$$

where $G^{(-1)}$ is a primitive of G with respect to t . It follows that

$$w \in C^{k+1}((-\infty, T]; L^2(\Omega)).$$

In each case, the norm is $O(T)$ for $T \leq 1$. If $k = 0$, we are done. For $k \geq 1$, we proceed inductively. Suppose that $j \leq k - 1$, and that we have shown

$$w \in C^{k+1-j}((-\infty, T]; \bar{H}^j(\Omega)) \cap C^{k-j}((-\infty, T]; \bar{H}^{j+1}(\Omega)),$$

with norm $O(T)$ if $T \leq 1$. For each fixed t , $w(t, x)$ is a distributional solution in $\bar{H}^1(\Omega)$ to the equation

$$\Delta w(t, x) = \partial_t^2 w(t, x) - G(t, x),$$

which vanishes on $\partial\Omega$. If $j \leq k - 2$, the right side is in the space

$$C^{k-1-j}((-\infty, T]; \bar{H}^j(\Omega)) \cap C^{k-2-j}((-\infty, T]; \bar{H}^{j+1}(\Omega)),$$

and, if $j = k - 1$, it is in the space

$$C((-\infty, T]; \bar{H}^{k-1}(\Omega)).$$

Furthermore, the norm of the right side is $O(T)$ if $T \leq 1$; this holds by the induction assumption for $\partial_t^2 w$, and holds for G since $\partial_t G$ is bounded in the norm. By elliptic regularity, we conclude that, if $j \leq k - 2$, then

$$w \in C^{k-1-j}((-\infty, T]; \bar{H}^{j+2}(\Omega)) \cap C^{k-2-j}((-\infty, T]; \bar{H}^{j+3}(\Omega)),$$

and if $j = k - 1$, then

$$w \in C((-\infty, T]; \bar{H}^{k+1}(\Omega)),$$

in each case with norm $O(T)$ if $T \leq 1$. \square

Next we shall use Lemma 3.6 to prove the following result which will be used in showing that our local solutions to (3.17) are smooth if they are bounded in a strip.

Lemma 3.7. *Suppose that $w \in V_T^0$ satisfies*

$$w(t, \cdot) = \int_0^t \frac{\sin s\sqrt{-\Delta}}{\sqrt{-\Delta}} (F(w(t-s, \cdot)) + v(t-s, \cdot)) ds,$$

and suppose that there is a constant R such that $|w(t, x)| \leq R$ on $(-\infty, T] \times \Omega$. If $v \in V_T^k$, then $w \in V_{T,d}^{k+1}$, and

$$\|w\|_{V_{T,d}^{k+1}} \leq C(T, R, \|w\|_{V_T^0}, \|v\|_{V_T^k}).$$

Proof. We shall show that, if $w \in V_T^k$, and if $|w(t, x)| \leq R$, then $F(w) \in V_T^k$, with norm bounded by $C(R, \|w\|_{V_T^k})$. The lemma then follows by Lemma 3.6 and induction.

The case $k = 0$ is trivial, and the case $k = 1$ follows easily from the fact that $L^\infty(\Omega) \cap \bar{H}^1(\Omega)$ is an algebra under pointwise multiplication. For $k = 2$, we can deduce

$$F(w) \in C^2((-\infty, T]; L^2(\Omega)) \cap C^1((-\infty, T]; \bar{H}^1(\Omega))$$

from the fact that $\dot{H}^2(\Omega)$ is an algebra, and is also a space of bounded multipliers on $\dot{H}^1(\Omega)$. In showing that $\partial_t^2 F(w) \in C((-\infty, 0]; L^2(\Omega))$, the only nontrivial term to consider is

$$F''(w) (\partial_t w)^2.$$

The term $F''(w)$ is bounded, and the square of a function in $\dot{H}^1(\Omega)$ belongs to L^2 .

Now assume $k \geq 3$. We expand $\partial_t^j F(w)$ as a sum of terms of the form

$$F^{(\alpha)}(w) \prod_{i \leq j} (\partial_t^i w)^{m_i},$$

where $\sum i m_i \leq j$. We need to show that such a product belongs to

$$C((-\infty, T]; \dot{H}^{k-j}(\Omega)).$$

The term $F^{(\alpha)}(w)$ belongs to $C((-\infty, T]; \dot{H}^k(\Omega))$. The product of terms with $k - i \geq 2$ belongs to

$$C((-\infty, T]; \dot{H}^{k-j}(\Omega)) \cap C((-\infty, T]; \dot{H}^2(\Omega)),$$

by the algebra property. Each of these terms is a bounded multiplier on \dot{H}^{k-j} , so it suffices to consider the product over i with $k - i \leq 1$. For $k \geq 3$, this consists of a single term $\partial_t^i u$, which belongs to $C((-\infty, T]; \dot{H}^{k-j}(\Omega))$ since $i \leq j$. \square

Using Lemma 3.6 it is not hard to prove that there is local existence for (3.17):

Lemma 3.8. *Suppose that $v \in V_{T_0}^1$. Then there exist $0 < T \leq T_0$ and a function $w \in V_{T,d}^2$, which satisfies*

$$w(t, \cdot) = \int_0^t \frac{\sin s\sqrt{-\Delta}}{\sqrt{-\Delta}} (F(w(t-s, \cdot)) + v(t-s, \cdot)) ds.$$

Proof. Without loss of generality, we assume $T_0 \leq 1$. We construct w as a fixed point of the map

$$K(w) = \int_0^t \frac{\sin s\sqrt{-\Delta}}{\sqrt{-\Delta}} (F(w)(t-s, \cdot) + v(t-s, \cdot)) ds.$$

Similar arguments to those used in the proof of Lemma 3.7 show that, if

$$\max \left(\|u\|_{V_T^2}, \|w\|_{V_T^2} \right) \leq R,$$

then

$$\|F(u) - F(w)\|_{V_T^2} \leq C(R)\|u - w\|_{V_T^2}.$$

It follows from Lemma 3.6 that

$$\|K(u) - K(w)\|_{V_{T,d}^2} \leq C_1(R) T \|u - w\|_{V_{T,d}^2},$$

uniformly for $T \leq T_0$. Suppose that $v \in V_{T_0}^1$. By Lemma 3.6, $w_0 = K(0)$ satisfies $\|w_0\|_{V_{T_0}^2} \leq R = C_0(T_0)\|v\|_{V_{T_0}^1}$. If $C_1(2R) T \leq 1/2$, then the iterated sequence $w_{j+1} = K(w_j)$ is Cauchy in the $2R$ ball of $V_{T,d}^2$, and converges to a fixed point. \square

End of proof of Theorem 3.1. Since, as we noted before, equations (3.1) and (3.17) are equivalent, if F and v are given by (3.13) and (3.15) and $u = \tilde{u} + w$, we see that Lemmas 3.7 and 3.8 imply that there is a local smooth solution to (3.1). Therefore, we just need to see that Theorem 3.1' implies Theorem 3.1. So let us suppose that the data $(f, g) \in C^\infty(\Omega)$ vanishes for large $|x|$ and satisfies the compatibility conditions to infinite order. Then if $u \in C^\infty([0, t_0] \times \Omega) \cap L^\infty([0, t_0] \times \Omega)$ solves (3.1), we need to show that u can be extended to a smooth solution in a larger strip. To see this we first note that Lemma 3.7 implies that u in fact belongs to $C^\infty([0, t_0] \times \Omega)$. Also, if we let

$$f_0 = u|_{t=t_0}, \quad g_0 = \partial_t u|_{t=t_0},$$

then the smooth data (f_0, g_0) at $t = t_0$ satisfy the compatibility conditions to infinite order. Indeed, we must have

$$\psi_j = \partial_t^j u|_{t=t_0},$$

and this vanishes on $\partial\Omega$ since u vanishes on $[0, t_0] \times \partial\Omega$. If we apply Lemmas 3.7 and 3.8 again we see that we can locally solve (3.1) with data (f_0, g_0) at $t = t_0$ and hence obtain a solution $u \in C^\infty([0, t_1] \times \Omega)$ to (3.1) for some $t_1 > t_0$. \square

4. EIGENFUNCTION ESTIMATES

Let (M^n, g) be an n -dimensional compact manifold with strictly geodesically concave boundary. As before, we assume that $n \geq 2$. The purpose of this section is to find a natural extension of a recent result of Grieser [3], by showing that the sharp L^q estimates of the second author [27] for the spectral projection operators on Riemannian manifolds without boundary hold in the present setting.

As before, let Δ_g be the Dirichlet Laplacian on M^n . Then the eigenvalues, λ_j , of $\sqrt{-\Delta_g}$ are positive and tend to $+\infty$. Here we shall be concerned with estimates for the spectral projection operator

$$\chi_\lambda f = \sum_{\lambda_j \in [\lambda-1, \lambda]} e_j(f),$$

if $e_j(f)$ denotes the projection of f onto the eigenspace of $\sqrt{-\Delta_g}$ with eigenvalue λ_j .

The goal of this section is to prove the following

Theorem 4.1. *Let χ_λ be as above. Then*

$$(4.1) \quad \|\chi_\lambda f\|_{L^q(M^n)} \leq C\lambda^{n(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}} \|f\|_{L^2(M^n)}, \quad \frac{2(n+1)}{n-1} \leq q \leq \infty,$$

and

$$(4.2) \quad \|\chi_\lambda f\|_{L^q(M^n)} \leq C\lambda^{\frac{n-1}{2}(\frac{1}{2}-\frac{1}{q})} \|f\|_{L^2(M^n)}, \quad 2 \leq q \leq \frac{2(n+1)}{n-1}.$$

Furthermore, these estimates are always sharp in the sense that if $\sigma(q)$ denotes the power of λ in (4.1) or (4.2), then

$$(4.3) \quad \limsup_{\lambda \rightarrow +\infty} \lambda^{-\sigma(q)} \|\chi_\lambda\|_{L^2 \rightarrow L^q} > 0.$$

Grieser [3] proved this result for $n = 2$. He also showed that the estimate with $q = \infty$ holds for all n , so in what follows we shall consider only finite q . The sharpness statement, (4.3), follows from arguments for manifolds without boundary in [28, Chapter 5]. Notice that (4.1) and (4.2) also give estimates for individual eigenfunctions, i.e., if $-\Delta_g f_j = \lambda_j^2 f_j$, then

$$(4.4) \quad \|f_j\|_{L^q} / \|f_j\|_{L^2} \leq \lambda_j^{\sigma(q)}.$$

Unlike the estimates for “quasi-modes”, (4.1) and (4.2), the bounds in (4.4) need not be the best possible as $\lambda_j \rightarrow +\infty$.

We first observe that it suffices to prove (4.1), since the estimates of (4.2) follow by interpolating between the trivial $L^2 \rightarrow L^2$ estimates for χ_λ , and the estimate (4.1) with $q = \frac{2(n+1)}{n-1}$.

We now show that the estimates (4.1) follow from the same estimates for certain “approximate spectral projection” operators $\tilde{\chi}_\lambda$, which in turn require only small-time estimates for the Cauchy problem. First notice that (4.1) with $q < \infty$ is equivalent to the dual estimate

$$(4.5) \quad \|\tilde{\chi}_\lambda f\|_{L^2} \leq C\lambda^{n(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}} \|f\|_{L^{\frac{q}{q-1}}}, \quad \frac{2(n+1)}{n-1} \leq q < \infty.$$

Now fix $0 \neq h \in C_0^\infty(\mathbb{R})$ with $\text{supp } h \subset [0, 1]$ and let

$$\tilde{\chi}_\lambda f = \int e^{-it\lambda} h(t) e^{it\sqrt{-\Delta_g}} f dt = \hat{h}(\lambda - \sqrt{-\Delta_g}) f.$$

By Plancherel’s theorem, estimates (4.5) are then a consequence of the estimates

$$\|(\sqrt{-\Delta_g})^{-n(\frac{1}{2}-\frac{1}{q})+\frac{1}{2}} \tilde{\chi}_\lambda f\|_{L^2} \leq C\|f\|_{L^{\frac{q}{q-1}}}, \quad \frac{2(n+1)}{n-1} \leq q < \infty,$$

since the L^2 norm of $(\sqrt{-\Delta_g})^{-\sigma} \chi_\lambda f$ is dominated by the sum of a fixed number of L^2 norms of $(\sqrt{-\Delta_g})^{-\sigma} \tilde{\chi}_\lambda f$, with $|\tilde{\lambda} - \lambda|$ smaller than a fixed constant.

Exploiting duality once more, we conclude that inequality (4.1) for $q < \infty$ would follow from

$$(4.1') \quad \|(\sqrt{-\Delta_g})^{-n(\frac{1}{2}-\frac{1}{q})+\frac{1}{2}} \tilde{\chi}_\lambda f\|_{L^q(M^n)} \leq C_q \|f\|_{L^2(M^n)}, \quad \frac{2(n+1)}{n-1} \leq q < \infty.$$

Theorem 2.1 actually yields the stronger estimate

$$(4.1'') \quad \left\| \sup_{\lambda} |(\sqrt{-\Delta_g})^{-n(\frac{1}{2}-\frac{1}{q})+\frac{1}{2}} \tilde{\chi}_{\lambda} f(x)| \right\|_{L^q(M^n)} \leq C_q \|f\|_{L^2(M^n)}, \quad \frac{2(n+1)}{n-1} \leq q < \infty.$$

To see this, notice that, since $\text{supp } h \subset [0, 1]$, one has

$$\begin{aligned} & |(\sqrt{-\Delta_g})^{-n(\frac{1}{2}-\frac{1}{q})+\frac{1}{2}} \tilde{\chi}_{\lambda} f(x)| \\ &= \left| \int e^{-it\lambda} h(t) (\sqrt{-\Delta_g})^{-n(\frac{1}{2}-\frac{1}{q})+\frac{1}{2}} (e^{it\sqrt{-\Delta_g}} f)(x) dt \right| \\ &\leq \|h\|_{L^2} \cdot \left(\int_0^1 |(\sqrt{-\Delta_g})^{-n(\frac{1}{2}-\frac{1}{q})+\frac{1}{2}} (e^{it\sqrt{-\Delta_g}} f)(x)|^2 dt \right)^{1/2}. \end{aligned}$$

Thus the left side of (4.1'') is controlled by

$$\left\| (\sqrt{-\Delta_g})^{-n(\frac{1}{2}-\frac{1}{q})+\frac{1}{2}} e^{it\sqrt{-\Delta_g}} f \right\|_{L_x^q L_t^2(M^n \times [0, 1])}.$$

Theorem 2.1 implies that this in turn is bounded by a constant times the L^2 norm of f . \square

Final Remarks. It would be interesting to know to what extent the results of this paper hold for domains in \mathbb{R}^n with nonconcave boundaries. A counterexample of Grieser [3] shows that the eigenfunction estimates (4.1), and hence the estimates (2.4), can break down near a strictly convex boundary point if $q < (6n + 4)/(3n - 4)$. Using a related construction it was shown by the authors in [26] that the Strichartz estimates (2.2) can also break down, even on manifolds without boundary, if the coefficients of the operator are not $C^{1,1}$. It is not known, however, whether (2.2) and (2.3) fail, for instance, when $\Omega \subset \mathbb{R}^n$ is a bounded convex domain.

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