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THE Δ^0_3-AUTOMORPHISM METHOD AND NONINVARIANT CLASSES OF DEGREES

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1. Introduction

A set A of nonnegative integers is *computably enumerable (c.e.)*, also called *recursively enumerable (r.e.)*, if there is a computable method to list its elements. Let $\mathcal{E}$ denote the structure of the computably enumerable sets under inclusion, $\mathcal{E} = (\{W_e\}_{e\in\omega}, \subseteq)$. Most previously known automorphisms Φ of the structure $\mathcal{E}$ of sets were effective (computable) in the sense that Φ has an effective presentation. We introduce here a new method for generating noneffective automorphisms whose presentation is Δ^0_3, and we apply the method to answer a number of long open questions about the orbits of c.e. sets under automorphisms of $\mathcal{E}$. For example, we show that the orbit of every noncomputable (*i.e.*, nonrecursive) c.e. set contains a set of high degree, and hence that for all $n > 0$ the well-known degree classes $\mathbf{L}_n$ (the low$_n$ c.e. degrees) and $\overline{\mathbf{H}}_n = \mathbf{R} - \mathbf{H}_n$ (the complement of the high$_n$ c.e. degrees) are noninvariant classes.

Let $\{W_e\}_{e\in\omega}$ be a standard indexing of the c.e. sets, let $\mathcal{E}$ denote the structure of the computably enumerable sets under inclusion, $\mathcal{E} = (\{W_e\}_{e\in\omega}, \subseteq)$, and let $\mathrm{Aut}(\mathcal{E})$ denote the group of automorphisms of $\mathcal{E}$. An automorphism $\Phi \in \mathrm{Aut}(\mathcal{E})$ is *effective* if there is a recursive function h (called a *presentation* of Φ) such that for all $n \in \omega$, $\Phi(W_n) =^* W_{h(n)}$. Soare [26] introduced a method for generating effective automorphisms of $\mathcal{E}$ and proved that any two maximal sets are automorphic. This effective automorphism method has been substantially developed and applied to study $\mathcal{E}$ and the relationship between the algebraic structure of $A \in \mathcal{E}$ and $\deg(A)$, the Turing degree of A. (See [28], Chapters XV and XVI, for a presentation of this method, the maximal set result, references to later results, and for any unspecified notation or definitions below.) Let $A \simeq B$ ($A \simeq_{eff} B$, $A \simeq_{\Delta^0_3} B$) denote that A is automorphic (effectively automorphic, Δ^0_3-automorphic) to B. The *orbit* of A, written $[A]$, is $\{B : A \simeq B\}$. The orbit of A is *nontrivial* if A is noncomputable.

Recently there have been two important new developments concerning automorphisms of $\mathcal{E}$. First, new $\mathcal{E}$-definable properties have been discovered which demonstrate that certain automorphisms cannot exist ([8] and [12]). Second, a new method has been developed for generating certain automorphisms Φ whose presentation h is a Δ^0_3 function and which will therefore be called Δ^0_3-*automorphisms*. The purpose of the present paper is to present this method and to apply it to study

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the possible Turing degree of sets B in an arbitrary nontrivial orbit. Before doing this we summarize some results on the first topic.

A property of c.e. sets is *invariant* if it is invariant under $\mathrm{Aut}(\mathcal{E})$, and *$\mathcal{E}$-definable* if there is a first order property in the language $L(\subset)$ which defines it over $\mathcal{E}$. In 1984 Harrington [28, page 339] proved that Post's property [24] of being a creative set is $\mathcal{E}$-definable and hence the creative sets form an orbit. In 1991, Harrington and Soare [8] positively answered a question arising from Post's 1944 program [24] which was to find an easily definable property on a noncomputable c.e. set A which guarantees that A is Turing incomplete, *i.e.*, $K \not\leq_{\mathrm{T}} A$, where K is the complete set.

Theorem 1.1 ([8])**.** *There is a nonempty $\mathcal{E}$-definable property $Q(A)$ such that every c.e. set A satisfying $Q(A)$ is noncomputable and Turing incomplete.*

The discovery of these properties was not accidental, but arose in studying the dynamic obstacle to producing the required automorphism, and converting that obstacle to an $\mathcal{E}$-definable property. In a forthcoming paper [12], Harrington and Soare continue this approach by producing several other $\mathcal{E}$-definable properties which prevent the existence of certain automorphisms for A which one might expect.

Although not *every* nontrivial orbit contains a complete set, a large class of orbits do.

Theorem 1.2 (Harrington-Soare)**.** *If A is any c.e. set of promptly simple degree, then A is automorphic (indeed, effectively automorphic) to a complete set.*

Harrington and Soare have also strengthened Theorem 1.2 by proving that it holds for sets A in a strictly larger class of c.e. degrees called *almost prompt* (*a.p.*) degrees, and hence the class of non-promptly-simple (*i.e.*, *tardy*) degrees $\mathbf{M} = \mathbf{R} - \mathbf{PS}$ (*i.e.*, the degrees of halves of minimal pairs) is not *invariant* as defined below. The following theorem asserts that a version of Theorem 1.2 holds for *every* noncomputable c.e. set A if we enlarge the class of target sets for $\Phi(A)$ from the complete sets to the *high* sets.

Theorem 1.3. *For every noncomputable c.e. set A there is a c.e. set B which is high* (i.e., $\deg(B') = \mathbf{0}''$) *such that A is Δ^0_3-automorphic to B.*

Theorem 1.3 asserts that every nontrivial orbit contains a high set. (This has been independently proven by P. Cholak as discussed in §12.) The next theorem, which is the main result of the present paper, considerably strengthens this by showing that every nontrivial orbit intersects *every* upper cone $\{B : B \geq_{\mathrm{T}} D\}$ such that $Q(D)$ holds. (Note that Theorem 1.4 implies Theorem 1.3 because every such D is a major subset and hence is high, so B is also.)

Theorem 1.4. *For every noncomputable c.e. set A and every c.e. set D which satisfies the property Q of Theorem 1.1 there is a c.e. set $B \geq_{\mathrm{T}} D$ such that A is Δ^0_3-automorphic to B.*

Theorem 1.4 gives an unexpected connection between the Q property and the coding of information into a set B in the orbit of A. Its proof uses the fact that by Harrington and Soare [11] the property $Q(D)$ corresponds to a certain computational complexity property on D which forces elements x to be enumerated into D slowly and thus gives time for the corresponding coding markers $\gamma(x)$ (required for $D = \Gamma^B$) to be moved into B slowly enough to respect the automorphism machinery needed to guarantee $A \simeq B$.

A major open problem has been to determine which subclasses of the c.e. degrees $\mathbf{R}$ (particularly which jump classes $\mathbf{H}_n$ and $\mathbf{L}_n$ and their complements) are *invariant*. A class $\mathbf{C}$ of c.e. degrees is *invariant* if it is the set of degrees of sets in some class $\mathcal{C} \subseteq \mathcal{E}$ which is invariant under automorphisms of $\mathcal{E}$ (e.g. if $\mathcal{C}$ is $\mathcal{E}$-definable). Define $\mathbf{H}_n = \{\mathbf{a} \in \mathbf{R} : \mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}\}$, $\mathbf{L}_n = \{\mathbf{a} \in \mathbf{R} : \mathbf{a}^{(n)} = \mathbf{0}^{(n)}\}$, $\mathbf{L}_0 = \{\mathbf{0}\}$, $\mathbf{H}_0 = \{\mathbf{0}'\}$, and $\overline{\mathbf{C}} = \mathbf{R} - \mathbf{C}$. The degrees in $\mathbf{H}_n$ ($\mathbf{L}_n$) are called *high*$_n$ (*low*$_n$) and the high$_1$ (low$_1$) degrees are called *high* (*low*).

Martin [23] showed that the degrees of maximal c.e. sets are exactly $\mathbf{H}_1$. Lachlan [14] and Shoenfield [25] showed that the degrees of coinfinite c.e. sets with no maximal supersets are exactly the nonlow$_2$ c.e. degrees $\overline{\mathbf{L}}_2$. Thus, $\mathbf{H}_1$ and $\overline{\mathbf{L}}_2$ are invariant. For the trivial jump classes corresponding to $n = 0$, $\mathbf{L}_0$, $\overline{\mathbf{L}}_0$, and $\mathbf{H}_0$ are invariant, while $\overline{\mathbf{H}}_0$ is noninvariant by Theorem 1.2. The following immediate corollary of Theorem 1.3 answers the invariance question for the *downward* closed jump classes for $n > 0$.

Corollary 1.5. *For all $n > 0$ the downward closed jump classes of c.e. degrees $\mathbf{L}_n$ and $\overline{\mathbf{H}}_n$ are noninvariant.*

For the *upward* closed classes $\mathbf{H}_n$ and $\overline{\mathbf{L}}_n$, $n > 0$, after the discovery of invariance of $\mathbf{H}_1$ and $\overline{\mathbf{L}}_2$, attention has been focused on $\overline{\mathbf{L}}_1$ because of the important role played by the low c.e. sets, and researchers had tried unsuccessfully for over 15 years to find a property defining $\overline{\mathbf{L}}_1$ analogous to the property for $\overline{\mathbf{L}}_2$. However, Harrington and Soare recently proved the noninvariance of $\overline{\mathbf{L}}_1$ as an immediate corollary of the following result.

Theorem 1.6 ([13])**.** *There is a nonlow c.e. set D such that every c.e. set $A \leq_{\mathrm{T}} D$ is Δ^0_3-automorphic to a low set B.*

Corollary 1.7 ([13])**.** *The upward closed jump class $\overline{\mathbf{L}}_1$ is noninvariant.*

As in [28, p. 167] let $\mathbf{M}$ denote the ideal of c.e. degrees $\mathbf{a}$ such that $\mathbf{a} = \mathbf{0}$ or $\mathbf{a}$ is half of a minimal pair. In Corollary 11.9 we shall prove that $\mathbf{M}$ is not invariant.

Researchers have tried to classify not only the orbit of a noncomputable c.e. set A but also its lattice of supersets, denoted by $\mathcal{L}(A) = \{W : A \subseteq W\}$, or equivalently $\mathcal{L}^*(A)$, the quotient lattice of $\mathcal{L}(A)$ modulo the ideal $\mathcal{F}$ of finite sets. Soare [27] proved that if A is a coinfinite low$_1$ c.e. set, then $\mathcal{L}^*(A) \cong_{eff} \mathcal{E}^*$. This can be extended from low$_1$ to low$_2$ if we replace effective isomorphisms by Δ^0_3-isomorphisms.

Theorem 1.8 (Harrington, Lachlan, Maass, and Soare [7])**.** *If A is a coinfinite low$_2$ c.e. set, then $\mathcal{L}^*(A) \cong_{\Delta^0_3} \mathcal{E}^*$.*

Theorem 1.8 cannot be extended from $\mathbf{L}_2$ to any strictly larger class of c.e. degrees because Shoenfield proved [25] that every degree $\mathbf{a} \in \mathbf{R} - \mathbf{L}_2$ contains an atomless c.e. set A and hence $\mathcal{L}(A) \ncong \mathcal{E}$. Also one cannot improve Theorem 1.8 to prove that every low$_2$ c.e. set is automorphic to some low$_1$ set.

Theorem 1.9 ([12])**.** *There is an $\mathcal{E}$-definable property $P(X)$ satisfied by a c.e. set A which is promptly simple, low$_2$, and such that $\overline{A}$ is semilow$_{1.5}$, but such that $P(B)$ is satisfied by no low$_1$ set B.*

It follows from Theorem 1.9 by Maass [19, Theorem 1.2] that if A is as in Theorem 1.9 and B is any promptly simple low$_1$ set, then $\mathcal{L}^*(A) \cong_{eff} \mathcal{L}^*(B)$, but A and

B are not automorphic. The significance is that in order to prove that A and B are automorphic even if they are promptly simple it does not suffice to prove that $\mathcal{L}^*(A) \cong_{eff} \mathcal{L}^*(B)$ and then to use the prompt simplicity of A and B to satisfy the hypotheses of the extension theorem apparatus of [28, Chapter XV]. (This stands in contrast to other results of Maass [18, page 821] that if A and B are both promptly simple and low, then they are effectively automorphic, or Maass [20] that if A and B are both hyper-hypersimple and also $\mathcal{L}^*(A) \cong_{\Delta^0_3} \mathcal{L}^*(B)$, then A is automorphic to B.) In contrast to Theorem 1.1 which produced an incomplete orbit, the next theorem produces a complete orbit different from that of the creative sets.

Theorem 1.10 ([12]). *There is an $\mathcal{E}$-definable property T satisfied by a promptly simple set A such that for all W, $T(W)$ implies that $K \leq_{\mathrm{T}} W$.*

Theorems 1.3, 1.4, 1.6, and 1.8 all use the Δ^0_3-automorphism method. This method was conceived in 1984 in unpublished work by Harrington who used it to show that for every c.e. set A, $\emptyset <_{\mathrm{T}} A <_{\mathrm{T}} \emptyset'$, there is an r.e. set B automorphic to A such that $B \not\leq_{\mathrm{T}} A$, as announced in [28, p. 379]. The method was further modified in 1988 to prove Theorem 1.8, and finally developed to the form presented here by Harrington and Soare in order to prove Theorems 1.3, 1.4, and 1.6 in 1990.

The purpose of this paper is to introduce the Δ^0_3-automorphism method in as general a form as possible, and to use it to prove Theorems 1.2, 1.3, and 9.1. In §2 and §3 we present the properties required for an automorphism and the construction necessary to achieve the properties. The Automorphism Theorem 4.2 in §4 states that additional steps may be added to the basic construction (for a variety of applications in this and subsequent papers) and if they satisfy certain basic conditions, then the construction will still produce an automorphism. The Automorphism Theorem is proved in §5 and §6. It will be applied in subsequent papers [13] and [7] to prove Theorems 1.6 and 1.8.

We assume that $A = U_0$ is a noncomputable c.e. set. In §6 we exploit this hypothesis by adding Step 6 to the construction in §3. In §7 we add additional steps to code information into B where $B = \widehat{U}_0$ is the intended image of A under the automorphism being constructed. This allows us to state and prove a general Coding Theorem 7.5 which gives a method for coding information into B while maintaining B automorphic to A using the Automorphism Theorem 4.2. We use the Coding Theorem 7.5 to prove Theorem 1.2 (which is Theorem 10.2) in §10 and Theorem 11.5 in §11.

Of particular interest is the Refined Coding Theorem 7.6 in §7.4 which is a slight simplification and restatement of the Coding Theorem 7.5 in a form which is self-contained and can be read and cited in this and subsequent papers without reading any other section here except §7.4. Here we use it to give short easy proofs of Theorem 1.3 in §8, and Theorem 9.1 in §9. In our subsequent paper [10] we use it to prove Theorem 1.4. Thus, it is possible (and perhaps even desirable) to read this paper by reading §7.4 first followed by §8 and §9, then reading §7.3 for a statement of the Coding Theorem 7.5 followed by §10 and §11 on prompt and almost prompt sets first taking the coding theorems on faith and suppressing the automorphism machinery, and later reading the automorphism part. We use the terms "computably enumerable (c.e.)" and "recursively enumerable (r.e.)" interchangeably, and likewise "computable" and "recursive".

2. The intuition and definitions

2.1. **Background.** By [28, page 343] building an automorphism of $\mathcal{E}$ is equivalent to building one of $\mathcal{E}^*$, the quotient lattice of $\mathcal{E}$ modulo the ideal $\mathcal{F}$ of finite sets. To do this we fix two copies of the natural numbers ω and $\widehat{\omega}$. We let variables $x, y, \dots$ ($\hat{x}, \hat{y}, \dots$) range over ω ($\widehat{\omega}$). Normally, we shall specify the definitions and action for only one side (usually the ω-side) since those for the opposite side will be entirely dual.

We view the construction of the automorphism Φ as a game between two players in the sense of Lachlan [15]. Player 1 (whom we call RED) produces two standard indexings $\{U_n\}_{n\in\omega}$ and $\{V_n\}_{n\in\omega}$ of the r.e. sets, where we view U_n as being on the ω-side and V_n on the $\widehat{\omega}$-side. Player 2 (whom we call BLUE) responds by building r.e. sets $\{\widehat{U}_n\}_{n\in\omega}$ on the $\widehat{\omega}$-side and $\{\widehat{V}_n\}_{n\in\omega}$ on the ω-side. The condition necessary to show that this correspondence $\Phi(U_n) = \widehat{U}_n$ and $\widehat{V}_n = \Phi^{-1}(V_n)$ is an automorphism is best stated in terms of the following notion of full e-state.

Definition 2.1. Given two sequences of r.e. sets $\{X_n\}_{n\in\omega}$ and $\{Y_n\}_{n\in\omega}$, define $\nu(e,x)$, the *full e-state* of x with respect to (w.r.t.) $\{X_n\}_{n\in\omega}$ and $\{Y_n\}_{n\in\omega}$, to be the triple $\langle e, \sigma(e,x), \tau(e,x)\rangle$, where

$$\sigma(e,x) = \{i : i \le e \ \ \& \ \ x \in X_i\}, \text{ and}$$

$$\tau(e,x) = \{i : i \le e \ \ \& \ \ x \in Y_i\}.$$

To see that Φ is an automorphism it suffices to satisfy the requirement

$$\begin{aligned}(1)\qquad & (\forall\nu)(\exists^\infty x \in \omega)[\nu(e,x) = \nu \text{ w.r.t. } \{U_n\}_{n\in\omega} \text{ and } \{\widehat{V}_n\}_{n\in\omega}]\\ & \iff (\exists^\infty \hat{y} \in \hat{\omega})[\nu(e,\hat{y}) = \nu \text{ w.r.t. } \{\widehat{U}_n\}_{n\in\omega} \text{ and } \{V_n\}_{n\in\omega}].\end{aligned}$$

Definition 2.2. Given recursive enumerations $\{X_s\}_{s\in\omega}$ and $\{Y_s\}_{s\in\omega}$ of r.e. sets X and Y, define

(i) $X \setminus Y = \{z : (\exists s)[z \in X_s - Y_s]\}$,
(ii) $X \searrow Y = (X \setminus Y) \cap Y$.

2.2. **Using a tree T to define the automorphism Φ.** In the effective automorphism method $\{\widehat{U}_n\}_{n\in\omega}$ is a *recursive* sequence of r.e. sets so that Φ has an effective presentation. For the Δ^0_3-automorphism method we combine the ideas of the effective automorphism method with the tree method of Lachlan [16] as explained in [28, Chapter XIV]. We shall define in §2.9 a recursive tree T with *true path* f. For each $n \in \omega$ there is some $m_n \in \omega$ such that for every $\alpha \in T$ of length m_n, $\widehat{U}_\alpha$ will be a potential candidate for $\widehat{U}_n$ and if $\alpha \subset f$, then $U_\alpha =^* U_n$ and $\widehat{U}_\alpha$ will be the correct candidate for $\widehat{U}_n$. Thus, f will specify the sequence $\{\widehat{U}_{f\restriction m_n}\}_{n\in\omega}$ which will be the desired sequence $\{\widehat{U}_n\}_{n\in\omega}$. In a tree construction f is not in general recursive but only $\emptyset''$-recursive, so the sequence $\{\widehat{U}_n\}_{n\in\omega}$ will only have a $\emptyset''$-recursive (*i.e.*, Δ^0_3) presentation.

We use the usual notation for trees as in [28, page 301]. By coding the intended nodes we may regard the tree T as a subset of $\omega^{<\omega}$. Let $[T]$ be the set of infinite paths through T, where h is an infinite path through T if $h\restriction n \in T$ for all n. Let $\alpha, \beta, \gamma, \delta, \dots$ range over T. Let $|\alpha|$ denote the length of α. Let $\alpha \subseteq \beta$ ($\alpha \subset \beta$) denote that string β extends (properly extends) α. Let λ denote the empty string,

and α^- the predecessor of α if $\alpha \neq \lambda$. Let $\langle a \rangle$ denote the string consisting of element a alone. Let $\alpha^\frown\beta$ denote the concatenation of string α followed by string β.

Definition 2.3. Let $\alpha, \beta \in T$.

(i) α is to the *left* of β ($\alpha <_L \beta$) if

$$(\exists a, b \in \omega)\ (\exists \gamma \in T)\ [\gamma^\frown\langle a \rangle \subseteq \alpha\ \ \&\ \ \gamma^\frown\langle b \rangle \subseteq \beta\ \ \&\ \ a < b].$$

(ii) $\alpha \leq \beta$ if $\alpha <_L \beta$ or $\alpha \subseteq \beta$.

(iii) $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$.

(iv) If $h \in [T]$, we say $\alpha <_L h$ ($h <_L \alpha, \alpha < h, h < \alpha$) if there exists $\beta \subset h$ such that $\alpha <_L \beta$ ($\beta <_L \alpha, \alpha < \beta, \beta < \alpha$, respectively).

Note that $\alpha \leq \beta$ is a kind of modified Kleene-Brouwer ordering. If $\alpha \subset \beta$, then α is a *predecessor* of β and β is a *successor* of α. (Thus, we view the tree T as growing downward with λ as the top node.)

2.3. **The α-section S_α, α-region R_α, and r.e. set Y_α.** We divide the ω-side into disjoint *α-sections*, S_α, for $\alpha \in T$. We shall define during the construction in §3 a function $\alpha(x, s)$ with range T which indicates that x is in section $S_{\alpha(x,s)}$ at the end of stage s, and we shall guarantee that $\alpha(x) = \lim_s \alpha(s, x)$ exists. The *α-region* R_α consists of all S_γ such that $\alpha \subseteq \gamma$. For each stage s we define,

$$S_{\alpha,s} = \{x : \alpha(x, s) = \alpha\},$$

$$R_{\alpha,s} = \{x : \alpha(x, s) \supseteq \alpha\}, \text{ and}$$

$$Y_{\alpha,s} = \bigcup \{R_{\alpha,t} : t \leq s\}.$$

Define $S_{\alpha,\infty} = \{x : \alpha(x) = \alpha\}$, and $R_{\alpha,\infty} = \{x : \alpha(x) \supseteq \alpha\}$. An element x will enter R_α at most once, but x may later leave R_α. Thus, $R_{\alpha,\infty}$ is a d.r.e. set, but the sets Y_α are r.e. with simultaneous recursive enumeration $\{Y_{\alpha,s}\}_{\alpha \in T, s \in \omega}$ and Y_α consists of those x which enter R_α at some stage. If $\alpha \subset f$, then we shall ensure that $Y_\alpha =^* R_{\alpha,\infty}$ so $R_{\alpha,\infty}$ is r.e. It will follow by §3 (1.2) and (2.2) that if $\alpha \neq \lambda$, then for all $x \in Y_\alpha$, $x > |\alpha|$.

We shall guarantee that for all $\alpha \in T$, $\alpha \neq \lambda$,

$$Y_\alpha \setminus Y_{\alpha^-} = \emptyset, \text{ and} \tag{2}$$

$$\alpha \subset f \implies R_{\alpha,\infty} =^* Y_\alpha =^* \omega. \tag{3}$$

We shall ensure (2) by making x enter S_{α^-} before x enters R_α. Also x will enter R_α at most once (although x may later leave R_α). During the construction in §3 we shall define a recursive sequence $\{f_s\}_{s \in \omega}$ such that $f = \liminf_s f_s$.

If $f_s <_L \alpha$ for some $s \geq x$ we say x is *α-ineligible* at all stages $t \geq s$, and we insist that $x \notin S_{\alpha,t}$. Hence, $R_{\alpha,\infty} = \emptyset$ for all α with $f <_L \alpha$. Secondly, Y_α will be finite for all $\alpha <_L f$. Finally, $S_{\alpha,\infty}$ will be finite for all α. These three facts imply (3).

2.4. **The α-states $\nu(\alpha, x, s)$, and lists $\mathcal{E}_\alpha$, $\mathcal{F}_\alpha$, $\mathcal{M}_\alpha$.** For conceptual simplicity we do as little action as possible at each node $\alpha \in T$. If $|\alpha| \equiv 1 \bmod 5$ ($|\alpha| \equiv 2 \bmod 5$), we consider one new U set (V set). If $|\alpha| \equiv 3 \bmod 5$ ($|\alpha| \equiv 4 \bmod 5$), we consider new α-states ν ($\hat{\nu}$) which may be non-well-resided on Y_α ($\widehat{Y}_\alpha$). If $\alpha \equiv 0 \bmod 5$, we make no new commitments for the automorphism machinery but we may perform action for some additional requirement (such as coding information into B for Theorem 1.3). We shall arrange for all $n \in \omega$ that for $\alpha \subset f$,

$$|\alpha| = 5n + 1 \implies U_\alpha =^* U_n, \text{ and} \tag{4}$$

$$|\alpha| = 5n + 2 \implies V_\alpha =^* V_n. \tag{5}$$

We let U_α and $\widehat{U}_\alpha$ (V_α and $\widehat{V}_\alpha$) be undefined if $|\alpha| \not\equiv 1 \bmod 5$ ($|\alpha| \not\equiv 2 \bmod 5$). We let e_α ($\hat{e}_\alpha$) correspond to n in (4) (respectively (5)). Namely, define $e_\lambda = \hat{e}_\lambda = -1$ and if $|\alpha| \equiv 1 \bmod 5$, then let $e_\alpha = e_{\alpha^-} + 1$, and otherwise let $e_\alpha = e_{\alpha^-}$. Define $\hat{e}_\alpha$ similarly with $|\alpha| \equiv 2 \bmod 5$ in place of $|\alpha| \equiv 1 \bmod 5$. Hence, $e_\alpha > e_{\alpha^-}$ ($\hat{e}_\alpha > \hat{e}_{\alpha^-}$) iff $|\alpha| \equiv 1 \bmod 5$ ($|\alpha| \equiv 2 \bmod 5$).

Definition 2.4. An *α-state* is a triple $\langle \alpha, \sigma, \tau \rangle$ where $\sigma \subseteq \{0, \ldots, e_\alpha\}$ and $\tau \subseteq \{0, \ldots, \hat{e}_\alpha\}$. The only λ-state is $\nu_{-1} = \langle \lambda, \emptyset, \emptyset \rangle$.

The construction in §3 will produce a simultaneous recursive enumeration $U_{\alpha,s}$, $V_{\alpha,s}$, $\widehat{U}_{\alpha,s}$, $\widehat{V}_{\alpha,s}$, for $\alpha \in T$ and $s \in \omega$, of these r.e. sets which we use in the following definition.

Definition 2.5. (i) The *α-state of x at stage s*, $\nu(\alpha, x, s)$, is the triple

$$\langle \alpha, \sigma(\alpha, x, s), \tau(\alpha, x, s) \rangle$$

where

$$\sigma(\alpha, x, s) = \{e_\beta : \beta \subseteq \alpha \ \ \& \ \ e_\beta > e_{\beta^-} \ \ \& \ \ x \in U_{\beta,s}\},$$

$$\tau(\alpha, x, s) = \{\hat{e}_\beta : \beta \subseteq \alpha \ \ \& \ \ \hat{e}_\beta > \hat{e}_{\beta^-} \ \ \& \ \ x \in \widehat{V}_{\beta,s}\}.$$

(ii) The *final α-state of x* is $\nu(\alpha, x) = \langle \alpha, \sigma(\alpha, x), \tau(\alpha, x) \rangle$ where $\sigma(\alpha, x) = \lim_s \sigma(\alpha, x, s)$ and $\tau(\alpha, x) = \lim_s \tau(\alpha, x, s)$.

For each $\alpha \in T$ we define the following sets of α-states called *lists*,

$$\mathcal{E}_\alpha = \{\nu : (\exists^\infty x)(\exists s)[x \in S_{\alpha,s} - \bigcup \{S_{\alpha,t} : t < s\} \ \ \& \ \ \nu(\alpha, x, s) = \nu]\}, \text{ and}$$

$$\mathcal{F}_\alpha = \{\nu : (\exists^\infty x)(\exists s)[x \in R_{\alpha,s} \ \ \& \ \ \nu(\alpha, x, s) = \nu]\}.$$

Note that $\mathcal{E}_\alpha$ consists of states well visited by elements x when they first *enter* R_α, and $\mathcal{F}_\alpha$ consists of those states well visited while they *remain* in Y_α so $\mathcal{E}_\alpha \subseteq \mathcal{F}_\alpha$. Each $\alpha \in T$ will have an associated list $\mathcal{M}_\alpha$ (to be defined in §2.8) which is roughly α's "guess" at the true $\mathcal{F}_\alpha$ such that if $\alpha \subset f$, then $\mathcal{M}_\alpha = \mathcal{F}_\alpha$. For $\alpha \subset f$ we shall achieve $\mathcal{M}_\alpha = \mathcal{F}_\alpha$ by ensuring the following properties of $\mathcal{M}_\alpha$,

$$\mathcal{E}_\alpha \subseteq \mathcal{M}_\alpha, \tag{6}$$

$$\begin{gathered}\text{(a.e. } x)[\text{if } x \in Y_{\alpha,s},\ \nu_0 = \nu(\alpha, x, s) \in \mathcal{M}_\alpha, \\ \text{and RED causes enumeration of } x \text{ so that} \\ \nu_1 = \nu(\alpha, x, s+1), \text{ then } \nu_1 \in \mathcal{M}_\alpha],\end{gathered} \tag{7}$$

$$\text{(8)} \qquad \begin{gathered}(\text{a.e. } x)[\text{if } x \in Y_{\alpha,s},\ \nu_0 = \nu(\alpha, x, s) \in \mathcal{M}_\alpha \\ \text{and BLUE causes enumeration of } x \text{ so that} \\ \nu_1 = \nu(\alpha, x, s+1), \text{ then } \nu_1 \in \mathcal{M}_\alpha].\end{gathered}$$

(Here (a.e. x) denotes "for almost every x".) Blue enumeration which satisfies (8) is called α-*legal*. Two main constraints on BLUE's moves will be (6) and (8). Clearly, (6), (7), and (8) guarantee

$$\text{(9)} \qquad \mathcal{F}_\alpha \subseteq \mathcal{M}_\alpha.$$

During Step 1 of the construction in §3 we shall promptly pull elements $x \in Y_{\alpha^-,s}$ into $S_{\alpha,s+1}$ in order to ensure

$$\text{(10)} \qquad \mathcal{M}_\alpha \subseteq \mathcal{E}_\alpha.$$

Hence, by (9), (10), and $\mathcal{E}_\alpha \subseteq \mathcal{F}_\alpha$ we have

$$\text{(11)} \qquad \mathcal{M}_\alpha = \mathcal{F}_\alpha = \mathcal{E}_\alpha.$$

On the $\widehat{\omega}$-side we have dual definitions for the above items by replacing ω, x, U_α, $\widehat{V}_\alpha$ by $\widehat{\omega}, \hat{x}, \widehat{U}_\alpha, V_\alpha$ respectively. These dual items will be denoted by $\hat{\nu}(\alpha, \hat{x}, s)$, $\widehat{S}_\alpha$, $\widehat{R}_\alpha, \widehat{Y}_\alpha, \widehat{\mathcal{E}}_\alpha, \widehat{\mathcal{F}}_\alpha$, and $\widehat{\mathcal{M}}_\alpha$. We write hats over the α-states, e.g. $\hat{\nu}_1 = \nu(\alpha, \hat{x}, s)$, to indicate α-states for elements $\hat{x} \in \widehat{\omega}$. We shall ensure

$$\text{(12)} \qquad \widehat{\mathcal{M}}_\alpha = \{\hat{\nu} : \nu \in \mathcal{M}_\alpha\},$$

which implies by (11) that the well-visited α-states on both sides coincide.

Definition 2.6. Given α-states $\nu_0 = \langle \alpha, \sigma_0, \tau_0 \rangle$ and $\nu_1 = \langle \alpha, \sigma_1, \tau_1 \rangle$:
(i) $\nu_0 \leq_R \nu_1$ if $\sigma_0 \subseteq \sigma_1$ and $\tau_0 = \tau_1$.
(ii) $\nu_0 \leq_B \nu_1$ if $\tau_0 \subseteq \tau_1$ and $\sigma_0 = \sigma_1$.
(iii) $\hat{\nu}_0 \leq_R \hat{\nu}_1$ if $\hat{\sigma}_0 = \hat{\sigma}_1$ and $\hat{\tau}_0 \subseteq \hat{\tau}_1$.
(iv) $\hat{\nu}_0 \leq_B \hat{\nu}_1$ if $\hat{\sigma}_0 \subseteq \hat{\sigma}_1$ and $\hat{\tau}_0 = \hat{\tau}_1$.
(v) $\nu_0 <_R \nu_1$ ($\nu_0 <_B \nu_1$) if $\nu_0 \leq_R \nu_1$ ($\nu_0 \leq_B \nu_1$) and $\nu_0 \neq \nu_1$, and similarly for $\hat{\nu}_0 <_R \hat{\nu}_1$ and $\hat{\nu}_0 <_B \hat{\nu}_1$.

The intuition is that if $\nu_0 = \nu(\alpha, x, s)$ and $\nu_0 <_R \nu_1$ ($\nu_0 <_B \nu_1$), then RED (BLUE) can enumerate x in the necessary U sets ($\widehat{V}$ sets) causing $\nu_1 = \nu(\alpha, x, s{+}1)$. For $\hat{\nu}_0$ and $\hat{\nu}_1$ the role of σ and τ is reversed because on the $\widehat{\omega}$-side BLUE (RED) plays the $\widehat{U}$ sets (V sets), and hence

$$\text{(13)} \qquad [\nu_0 <_R \nu_1 \iff \hat{\nu}_0 <_B \hat{\nu}_1] \ \&\ [\nu_0 <_B \nu_1 \iff \hat{\nu}_0 <_R \hat{\nu}_1].$$

Definition 2.7. Given $\beta \subseteq \alpha \in T$ and an α-state $\nu_0 = \langle \alpha, \sigma_0, \tau_0 \rangle$ or a set $\mathcal{C}_\alpha$ of α-states:
(i) $\nu_0 \restriction \beta = \langle \beta, \sigma_1, \tau_1 \rangle$ where we define $\sigma_1 = \sigma_0 \cap \{0, \ldots, e_\beta\}$ and we define $\tau_1 = \tau_0 \cap \{0, \ldots, \hat{e}_\beta\}$.
(ii) $\nu_1 \preceq \nu_0$ (read " ν_0 extends ν_1") if $\nu_0 \restriction \beta = \nu_1$.
(iii) $\mathcal{C}_\alpha \restriction \beta = \{\nu \restriction \beta : \nu \in \mathcal{C}_\alpha\}$.
(iv) Given a finite set of α-states, $\{\nu(\alpha, \sigma_i, \tau_i) : i \in I\}$, we then define

$$\bigcup \{\nu(\alpha, \sigma_i, \tau_i) : i \in I\} =_{\text{dfn}} \langle \alpha, \sigma, \tau \rangle,$$

where $\sigma = \bigcup \{\sigma_i : i \in I\}$, and where we define $\tau = \bigcup \{\tau_i : i \in I\}$.

The combination of (6)–(11) and their duals together with (12) may cause additional upward closure of $\mathcal{M}_\alpha$ under $\leq_B$. For example, if $e_\alpha > e_{\alpha^-}$ (so α builds U_α and $\widehat{U}_\alpha$), suppose $\nu_0 \in \mathcal{M}_\alpha$ for some $\nu_0 = \langle \alpha, \sigma_0, \tau_0 \rangle$ with $e_\alpha \in \sigma_0$. Hence, $\hat{\nu}_0 \in \widehat{\mathcal{M}}_\alpha$ by (12). But if for infinitely many $\hat{y}$, $\nu(\alpha, \hat{y}, s) = \hat{\nu}_0$ and for some s, RED causes $\nu(\alpha, \hat{y}, s+1) = \hat{\nu}_1 >_R \hat{\nu}_0$ (say by enumerating $\hat{y}$ in V_β for some $\beta \subset \alpha$), then $\hat{\nu}_1 \in \widehat{\mathcal{M}}_\alpha$ by the dual of (7) and hence $\nu_1 \in \mathcal{M}_\alpha$ by (12), and $\nu_0 <_B \nu_1$ by (13) because $\hat{\nu}_0 <_R \hat{\nu}_1$. We do not wait for RED to cause $\hat{\nu}_1 \in \widehat{\mathcal{M}}_\alpha$. Rather in the following definition we anticipate by *now* putting all such $\nu_1 \in \mathcal{M}_\alpha$ and many more as well. Indeed for each α which is $\mathcal{M}$-*consistent* in the following definition (which includes all $\alpha \subseteq f$) we put every ν_1 in $\mathcal{M}_\alpha$ if ν_1 is a blue move away from some $\nu_0 \in \mathcal{M}_\alpha$ (i.e., $\nu_0 <_B \nu_1$), so long as the blue move from ν_0 to ν_1 is β-legal, *i.e.*, $\nu_1 \restriction \beta \in \mathcal{M}_\beta$. But by (11) this means we must make all such ν_1 well visited on R_α. Since there is no evidence that RED will actually make the proposed move, this extreme closure of $\mathcal{M}_\alpha$ seems unwarranted and outrageously bold. Step 3 and Lemma 5.6 prove that it is not.

Definition 2.8. A node $\alpha \in T$ is $\mathcal{M}$-*inconsistent* if $e_\alpha > e_\beta$, where $\beta = \alpha^-$, and there are α-states $\nu_0 <_B \nu_1$ such that $\nu_0 \in \mathcal{M}_\alpha$ and $\nu_1 \restriction \beta \in \mathcal{M}_\beta$ but $\nu_1 \notin \mathcal{M}_\alpha$. Otherwise α is $\mathcal{M}$-*consistent*.

We shall take action in Step 3 of the construction in §3 to ensure that α is $\mathcal{M}$-consistent if $\alpha \subset f$.

2.5. **Non-well-resided α-states and the lists $\mathcal{R}_\alpha$ and $\mathcal{B}_\alpha$.** Define the set of non-well-resided α-states,

$$\mathcal{K}_\alpha = \{\nu_1 : \neg(\exists^\infty x)[x \in Y_\alpha \ \ \& \ \ \nu(\alpha, x) = \nu_1]\}. \tag{14}$$

Likewise define $\widehat{\mathcal{K}}_\alpha$ for the $\widehat{\omega}$-side. To satisfy the automorphism requirement (1) we must show for $\alpha \subset f$ that

$$\widehat{\mathcal{K}}_\alpha = \{\hat{\nu} : \nu \in \mathcal{K}_\alpha\}. \tag{15}$$

To achieve (15) note that unlike $\mathcal{E}_\alpha$ and $\mathcal{F}_\alpha$ of §2.4 $\mathcal{K}_\alpha$ is Σ^0_3 not Π^0_2 so α cannot guess at $\mathcal{K}_\alpha$ directly but only at a certain Π^0_2 approximation $\mathcal{N}_\alpha \subseteq \mathcal{K}_\alpha$. We divide $\mathcal{N}_\alpha$ into the disjoint union of sets $\mathcal{R}_\alpha$ and $\mathcal{B}_\alpha$ which correspond to those $\nu \in \mathcal{N}_\alpha$ which α believes are being emptied by RED and BLUE respectively.

To define $\mathcal{R}_\alpha$ and $\mathcal{B}_\alpha$ fix $\alpha \in T$, let $\beta = \alpha^-$, and assume that $\mathcal{R}_\gamma$, $\mathcal{B}_\gamma$ and their duals $\widehat{\mathcal{R}}_\gamma$, $\widehat{\mathcal{B}}_\gamma$ have been defined for all $\gamma \subset \alpha$. We decompose $\mathcal{R}_\alpha$ into the disjoint union,

$$\mathcal{R}_\alpha = \mathcal{R}^\alpha_\alpha \sqcup \mathcal{R}^{<\alpha}_\alpha, \text{ where} \tag{16}$$

$$\mathcal{R}^{<\alpha}_\alpha =_{\text{dfn}} \{\nu : \nu \in \mathcal{M}_\alpha \ \ \& \ \ \nu \restriction \beta \in \mathcal{R}_\beta\}, \text{ and} \tag{17}$$

$$\mathcal{R}^\alpha_\alpha =_{\text{dfn}} \mathcal{R}_\alpha - \mathcal{R}^{<\alpha}_\alpha. \tag{18}$$

Note that $\mathcal{R}^{<\alpha}_\alpha$ is determined by $\mathcal{R}_\beta$, $\beta \subset \alpha$, but $\mathcal{R}^\alpha_\alpha$ may contain new elements and for $\alpha \subset f$ it has the meaning described below in (20). Likewise, let $\mathcal{B}_\alpha = \mathcal{B}^\alpha_\alpha \sqcup \mathcal{B}^{<\alpha}_\alpha$, where $\mathcal{B}^{<\alpha}_\alpha$ is defined as in (17) but with $\mathcal{B}_\beta$ in place of $\mathcal{R}_\beta$.

If $|\alpha| \not\equiv 3 \bmod 5$, define $\mathcal{R}^\alpha_\alpha = \widehat{\mathcal{B}}^\alpha_\alpha = \emptyset$. If $|\alpha| \equiv 3 \bmod 5$, we let $\mathcal{M}_\alpha = \mathcal{M}_\beta$ (since α-states are β-states because $e_\alpha = e_\beta$ and $\hat{e}_\alpha = \hat{e}_\beta$), we define the Π^0_2 predicate,

$$F(\beta, \nu) \equiv (\forall x)[[x > |\beta| \ \ \& \ \ x \in Y_\beta] \implies \nu(\alpha, x) \neq \nu], \tag{19}$$

and we allow $\mathcal{R}^\alpha_\alpha \neq \emptyset$ with the intention that for $\alpha \subset f$,

$$\mathcal{R}^\alpha_\alpha = \{\nu : \nu \in \mathcal{M}_\alpha - (\mathcal{R}^{<\alpha}_\alpha \cup \mathcal{B}^{<\alpha}_\alpha) \ \ \& \ \ F(\beta, \nu)\}. \tag{20}$$

We define

$$\widehat{\mathcal{B}}^\alpha_\alpha =_{\text{dfn}} \{\hat{\nu} : \nu \in \mathcal{R}^\alpha_\alpha\}. \tag{21}$$

If $|\alpha| \not\equiv 4 \bmod 5$, define $\widehat{\mathcal{R}}^\alpha_\alpha = \mathcal{B}^\alpha_\alpha = \emptyset$. If $\alpha \equiv 4 \bmod 5$, we allow $\widehat{\mathcal{R}}^\alpha_\alpha \neq \emptyset$ (using the duals of (16)–(20) where e.g. in the dual of (19) we use $\widehat{Y_\beta}$ in place of Y_β), and we define

$$\mathcal{B}^\alpha_\alpha =_{\text{dfn}} \{\nu : \widehat{\nu} \in \widehat{\mathcal{R}}^\alpha_\alpha\}. \tag{22}$$

At most one of $\mathcal{R}^\alpha_\alpha$ and $\widehat{\mathcal{R}}^\alpha_\alpha$ is nonempty so by (22), (21), and (20),

$$\mathcal{R}^\alpha_\alpha \cap \mathcal{B}^\alpha_\alpha = \emptyset \ \ \& \ \ ((\mathcal{R}^\alpha_\alpha \cup \mathcal{B}^\alpha_\alpha) \cap (\mathcal{R}^{<\alpha}_\alpha \cup \mathcal{B}^{<\alpha}_\alpha) = \emptyset), \tag{23}$$

and hence

$$\mathcal{R}_\alpha \cap \mathcal{B}_\alpha = \emptyset. \tag{24}$$

If $\alpha \subset f$, then $\nu \in \mathcal{R}_\alpha$ implies $F(\alpha^-, \nu)$ and hence

$$(\forall \nu \in \mathcal{R}_\alpha)(\forall x \in Y_\alpha)(\forall s)[\nu(\alpha, x, s) = \nu \implies (\exists t > s)[\nu(\alpha, x, t) \neq \nu]]. \tag{25}$$

It will be BLUE's responsibility to change the α-state of x if $\nu(\alpha, x, s) \in \mathcal{B}_\alpha$, and $x \in R_\alpha$. However, $\mathcal{B}_\alpha \cap \mathcal{R}_\alpha = \emptyset$ so if $\nu(\alpha, x, s) = \nu \in \mathcal{R}_\alpha$, then BLUE can wait for RED to change the α-state of x to meet (25), namely

$$\begin{gathered}(\forall \nu \in \mathcal{R}_\alpha)(\forall x \in R_\alpha)(\forall s)[\text{if } \nu(\alpha, x, s) = \nu, \text{ then} \\ \text{it is an } \alpha\text{-admissible move for BLUE to restrain} \\ x \text{ from further BLUE enumeration until} \\ (\exists t > s)[\nu(\alpha, x, s) <_R \nu(\alpha, x, t)]].\end{gathered} \tag{26}$$

Definition 2.9. A node $\alpha \in T$ is $\mathcal{R}$-*consistent* if

$$(\forall \nu_0 \in \mathcal{R}_\alpha)(\exists \nu_1)[\nu_0 <_R \nu_1 \ \ \& \ \ \nu_1 \in \mathcal{M}_\alpha], \tag{27}$$

and $\mathcal{R}$-*inconsistent* otherwise.

By applying (26) BLUE will ensure that α is $\mathcal{R}$-consistent for $\alpha \subset f$. Now (27) (21), and (13) imply for $\alpha \subset f$ that

$$(\forall \hat{\nu}_0 \in \widehat{\mathcal{B}}_\alpha)(\exists \hat{\nu}_1)[\hat{\nu}_0 <_B \hat{\nu}_1 \ \ \& \ \ \hat{\nu}_1 \in \widehat{\mathcal{M}}_\alpha]. \tag{28}$$

By repeatedly applying (28) BLUE can achieve $\hat{\nu}_1 \in \widehat{\mathcal{M}}_\alpha - \widehat{\mathcal{B}}_\alpha$, namely

$$(\exists \text{ function } \hat{h}_\alpha)[\hat{h}_\alpha : \widehat{\mathcal{B}}_\alpha \to (\widehat{\mathcal{M}}_\alpha - \widehat{\mathcal{B}}_\alpha) \ \ \& \ \ (\forall \hat{\nu} \in \widehat{\mathcal{B}}_\alpha)[\hat{\nu} <_B \hat{h}_\alpha(\hat{\nu})]]. \tag{29}$$

It is BLUE's responsibility to move any $\hat{x} \in \widehat{R}_\alpha$ for which $\nu(\alpha, \hat{x}, s) = \hat{\nu}_0 \in \widehat{\mathcal{B}}_\alpha$ to the *target state* $\hat{\nu}_1 = \hat{h}_\alpha(\hat{\nu}_0)$ (and where $\hat{h}$ is called the *target function*) so that BLUE can achieve

$$(\forall \hat{x} \in \hat{R}_\alpha)(\forall s)[\nu(\alpha, \hat{x}, s) \in \widehat{\mathcal{B}}_\alpha \implies (\exists t > s)[\nu(\alpha, \hat{x}, t) \in \widehat{\mathcal{M}}_\alpha - \widehat{\mathcal{B}}_\alpha]], \tag{30}$$

and hence BLUE will cause every state $\hat{\nu}_0 \in \widehat{\mathcal{B}}_\alpha$ to be emptied. To achieve (30) on $\widehat{R}_\alpha$ it suffices to achieve the following on $\widehat{S}_\gamma$ for each $\gamma \supseteq \alpha$,

$$(\forall \hat{x} \in \hat{S}_\gamma)(\forall s)[\nu(\gamma, \hat{x}, s) \in \widehat{\mathcal{B}}_\gamma \implies (\exists t > s)[\nu(\gamma, \hat{x}, t) \in \widehat{\mathcal{M}}_\gamma - \widehat{\mathcal{B}}_\gamma]]. \tag{31}$$

(For BLUE to achieve (31) from the hypothesis of (30) there is a subtle but crucial point. Suppose $\nu_0 \in \mathcal{R}_\alpha$ so $\hat{\nu}_0 \in \widehat{\mathcal{B}}_\alpha$. Hence, $\hat{\nu}'_0 \in \widehat{\mathcal{B}}_\gamma$ for all $\gamma \supset \alpha$ such that $\hat{\nu}'_0 \restriction \alpha = \hat{\nu}_0$. Now by (30) BLUE is required for every $\hat{x}$ in region $\widehat{R}_\alpha$ such that $\nu(\alpha, \hat{x}, s) = \hat{\nu}_0 \in \widehat{\mathcal{B}}_\alpha$ to enumerate $\hat{x}$ in blue sets to achieve $\nu(\alpha, \hat{x}, t) = \hat{\nu}_1 >_B \hat{\nu}_0$ for some $t > s$. However, if $\hat{x} \in \hat{S}_{\gamma,s}$ for some $\gamma \supset \alpha$, then BLUE can only make γ-*legal* moves, namely BLUE must ensure that $\nu(\gamma, \hat{x}, s) \in \widehat{\mathcal{M}}_\gamma$. Hence, on the γ-level if $\hat{\nu}'_0 = \nu(\gamma, \hat{x}, s)$ and $\hat{\nu}'_0 \restriction \alpha = \hat{\nu}_0 \in \hat{\mathcal{B}}_\alpha$, then $\nu_0 \in \mathcal{R}_\alpha$ so $\nu'_0 \in \mathcal{R}_\gamma$ and BLUE needs a γ-*target* $\hat{\nu}'_1 >_B \hat{\nu}'_0$ for $\hat{x}$ not merely an α-target $\hat{\nu}_1 >_B \hat{\nu}_0$. To obtain this γ-target ν'_1, BLUE can hold some $y \in S_\gamma$ in γ-state ν'_0 until, by $(26)_\gamma$, RED is forced to cause $\nu(\alpha, y, t) = \nu_1 >_R \nu_0$, for some $t > s$, and hence $\nu(\gamma, y, t) = \nu'_1 >_R \nu'_0$, thus ensuring that γ is $\mathcal{R}$-consistent and giving a target γ-state $\hat{\nu}'_1$ for $\hat{x}$. This action may have to be repeated for each of the infinitely many $\gamma \supseteq \alpha$, even for those $\gamma <_L f$. Hence, (30) constitutes a very strong BLUE constraint on the entire downward cone $\widehat{R}_\alpha$. This procedure for producing an appropriate target j-state ν'_1 for $j > e$ when an e-state ν_0 is emptied is taken from the effective automorphism machinery in [28, Chapter XV], and [26], where it also plays a central role.)

We often refer to the dual of (29) which asserts

$$(\exists \text{ function } h_\alpha)[h_\alpha : \mathcal{B}_\alpha \to (\mathcal{M}_\alpha - \mathcal{B}_\alpha) \ \ \& \ \ (\forall \nu \in \mathcal{B}_\alpha)[\nu <_B h_\alpha(\nu)]], \tag{32}$$

and which enables us to achieve the dual of (31), namely

$$(\forall x \in S_\gamma)(\forall s)[\nu(\gamma, x, s) \in \mathcal{B}_\gamma \implies (\exists t > s)[\nu(\gamma, x, t) \in \mathcal{M}_\gamma - \mathcal{B}_\gamma]]. \tag{33}$$

Finally, we have ensured

$$\begin{aligned} &(\forall \gamma \subset f)(\forall \nu_0 \in \mathcal{M}_\gamma)[(\exists^{<\infty} x)[x \in Y_\gamma \ \ \& \ \ \nu(\gamma, x) = \nu_0] \\ &\implies (\exists \alpha)_{\gamma \subset \alpha \subset f}[\{\nu_1 \in \mathcal{M}_\alpha : \nu_1 \restriction \gamma = \nu_0\} \subseteq \mathcal{R}_\alpha \cup \mathcal{B}_\alpha]]. \end{aligned} \tag{34}$$

To check (34) fix $\gamma \subset f$ and $\nu_0 \in \mathcal{M}_\gamma$. By (3) $Y_\gamma =^* \omega$ so if the hypothesis of (34) holds, then we can choose b such that

$$(\forall x \in \omega)[x > b \implies \nu(\gamma, x) \neq \nu_0].$$

Choose $\alpha \subset f$ such that $\alpha \supset \gamma$, $|\alpha| > b$ and $|\alpha| \equiv 3 \bmod 5$. Consider any $\nu_1 \in \mathcal{M}_\alpha$ such that $\nu_1 \restriction \gamma = \nu_0$. If $\nu_1 \notin \mathcal{R}^{<\alpha}_\alpha \cup \mathcal{B}^{<\alpha}_\alpha$, then $F(\alpha^-, \nu_1)$ holds so $\nu_1 \in \mathcal{R}^\alpha_\alpha$ by (20), and hence $\nu_1 \in \mathcal{R}_\alpha$ by (16).

Equations (21), (25), (30), (34) and their duals guarantee (15).

2.6. **Verifying the automorphism requirement (1).** We shall arrange that $\lim_{\alpha \subset f} e_\alpha = \infty$. By (4) and (5) the sets $\{U_\alpha\}_{\alpha \subset f}$ and $\{V_\alpha\}_{\alpha \subset f}$ constitute skeletons for $\{W_n\}_{n \in \omega}$. By (11), its dual, and (12) we know that the well-visited α-states on the ω-side and $\widehat{\omega}$-side coincide. By (15) the non-well-resided α-states also coincide so (1) is satisfied. The construction in §3 and verification in §5 will demonstrate that the equations of §2.3, §2.4, and §2.5 are satisfied. First we need a few more definitions in §2.8 and §2.9.

2.7. **Splitting S_α into S^0_α and S^1_α.** We divide the α-section S_α into two subsections S^0_α and S^1_α. For $k \in \{0, 1\}$ let $S^k_{\alpha,s}$ denote the set of elements $x \in S_{\alpha,s}$ which lie in S^k_α at the end of stage s. The elements $x \in S^0_\alpha$ may be appointed as α-witnesses (e.g. the position of an α-coding marker), and may require special enumeration into or restraint from certain blue sets to meet certain additional requirements (such as making B high) beyond the automorphism requirements. The other elements of

S_α, namely $x \in S^1_\alpha$, are available to be moved into S_γ for any $\gamma \supset \alpha$ as in §3. For $k \in \{0,1\}$, define

$$\mathcal{E}^k_\alpha = \{\nu : (\exists^\infty x)(\exists s)[x \in S^k_{\alpha,s} - \bigcup \{S^k_{\alpha,t} : t < s\} \ \ \& \ \ \nu(\alpha,x,s) = \nu]\}. \tag{35}$$

We shall arrange that the stream of elements entering S_α is split into two equivalent streams entering S^0_α and S^1_α so that $\mathcal{E}_\alpha = \mathcal{E}^0_\alpha = \mathcal{E}^1_\alpha$. Similarly, we define

$$R^1_{\alpha,s} = \{x : x \in S^1_{\alpha,s} \text{ or } (\exists \gamma \supset \alpha)[x \in S_{\gamma,s}]\}, \text{ and} \tag{36}$$

$$Y^1_{\alpha,s} = \bigcup \{R^1_{\alpha,t} : t \le s\}. \tag{37}$$

We shall arrange that $S^0_{\alpha,\infty}$, the set of permanent residents of S^0_α, is finite. Thus, it will suffice to use $R^1_{\alpha,s}$ and $Y^1_{\alpha,s}$ (rather than the slightly larger $R_{\alpha,s}$ and $Y_{\alpha,s}$) in §2.8 and Steps 1 and 2 of the construction in §3, since the former are the elements truly available to those $\gamma \supset \alpha$.

2.8. **The set $\mathcal{F}^+_\beta$ and the definition of $\mathcal{M}_\alpha$.** In §2.4 we said that every $\alpha \in T$ would have an associated set $\mathcal{M}_\alpha$ such that $\mathcal{M}_\alpha = \mathcal{F}_\alpha$ if $\alpha \subset f$. However, although this is the *property* we want $\mathcal{M}_\alpha$ to have, we cannot simply *define* $\mathcal{M}_\alpha$ to be α's guess at $\mathcal{F}_\alpha$ because that definition would be circular. Rather we must define here a certain set $\mathcal{F}^+_\beta$ which depends only on β, and then let $\mathcal{M}_\alpha$ be α's guess at $\mathcal{F}^+_\beta$ so that $\mathcal{M}_\alpha = \mathcal{F}^+_\beta$ $(= \mathcal{F}_\alpha)$ for $\alpha \subset f$.

Fix $\alpha \in T$ such that $e_\alpha > e_\beta$ for $\beta = \alpha^-$. Define the r.e. set $Z_{e_\alpha} = \bigcup_s Z_{e_\alpha,s}$ where

$$Z_{e_\alpha,s+1} =_{\text{dfn}} \{x : x \in U_{e_\alpha,s+1} \ \ \& \ \ x \in Y^1_{\beta,s}\}. \tag{38}$$

Define the α-state function $\nu^+(\alpha,x,s)$ exactly as for $\nu(\alpha,x,s)$ in Definition 2.5 but with $Z_{e_\alpha,s}$ in place of $U_{\alpha,s}$.

Define

$$\mathcal{F}^+_\beta = \{\nu : (\exists^\infty x)(\exists s)[x \in Y^1_{\beta,s} \ \ \& \ \ \nu^+(\alpha,x,s) = \nu]\}, \tag{39}$$

$$\begin{aligned} k^+_\beta = \min\{y : (\forall x > y)(\forall s) \\ [[x \in Y^1_{\beta,s} \ \ \& \ \ \nu^+(\alpha,x,s) = \nu_1] \implies \nu_1 \in \mathcal{F}^+_\beta]\}. \end{aligned} \tag{40}$$

If $e_\alpha > e_\beta$, we also define $\widehat{\mathcal{F}}^+_\beta = \{\hat\nu : \nu \in \mathcal{F}^+_\beta\}$. (Note that Z_{e_α} and hence $\mathcal{F}^+_\beta$ and k^+_β depend only upon β not α and thus α can make guesses $\mathcal{M}_\alpha$ and k_α for $\mathcal{F}^+_\beta$ and k^+_β.)

If $\hat e_\alpha > \hat e_\beta$, we first define $\widehat{\mathcal{F}}^+_\beta$ and k^+_β using the duals of (39) and (40) (with $\hat Y_{\beta,s}$, $V_{\hat e_\alpha}$, $\hat Z_{\hat e_\alpha}$, and $\nu^+(\alpha,\hat x,s)$ in place of $Y_{\beta,s}$, U_{e_α}, Z_{e_α}, and $\nu^+(\alpha,x,s)$, respectively), and then we define $\mathcal{F}^+_\beta = \{\nu : \hat\nu \in \mathcal{F}^+_\beta\}$. (Note that there is no $\hat k^+_\beta$, only k^+_β.)

Every $\alpha \in T$ will have associated items $\mathcal{M}_\alpha$ and k_α such that $\mathcal{M}_\alpha = \mathcal{F}^+_\beta$ and $k_\alpha = k^+_\beta$ for $\alpha \subset f$. We allow x to enter Y_α only if $x > k_\alpha$. If $e_\alpha = e_\beta$ and $\hat e_\alpha = \hat e_\beta$, we define $\mathcal{F}^+_\beta = \mathcal{F}_\beta$, $\widehat{\mathcal{F}}^+_\beta = \widehat{\mathcal{F}}_\beta$, and $k^+_\beta = k_\beta$. If

$$(\exists x)(\exists s)[x \in Y_{\alpha,s} \ \ \& \ \ \nu(\alpha,x,s) \notin \mathcal{M}_\alpha], \tag{41}$$

then we say that α is *provably incorrect* at all stages $t \ge s$ and we ensure that $\alpha \not\subset f$.

2.9. **The definition of the tree T.**

Definition 2.10. We say that $\alpha \in T$ is *consistent* if α is $\mathcal{M}$-consistent (Definition 2.8), $\mathcal{R}$-consistent (Definition 2.9), and also $\mathcal{C}$-consistent (Definition 6.3).

Note that, by clause (i) in the following Definition 2.11 of T,

$$\beta \in T \implies [\beta \text{ inconsistent} \iff \beta \text{ is a terminal node on } T]. \tag{42}$$

We shall show that if $\alpha \subset f$, then α is consistent, and therefore $\lim_{\alpha \subset f} e_\alpha = \infty$, so the argument of §2.6 applies.

Definition 2.11. Put $\lambda \in T$ and define $\mathcal{M}_\lambda = \mathcal{R}_\lambda = \mathcal{B}_\lambda = \emptyset$, and $k_\lambda = e_\lambda = \hat{e}_\lambda = -1$. If $\beta \in T$, we put $\alpha = \beta^\frown\langle \mathcal{M}_\alpha, \mathcal{R}_\alpha, \mathcal{B}_\alpha, k_\alpha\rangle$ in T providing the following conditions hold:

(i) β is consistent (as defined in Definition 2.10),
(ii) $\mathcal{M}_\alpha$ is a set of α-states, $\mathcal{R}_\alpha \subseteq \mathcal{M}_\alpha$, $\mathcal{B}_\alpha \subseteq \mathcal{M}_\alpha$, and $\mathcal{R}_\alpha \cap \mathcal{B}_\alpha = \emptyset$,
(iii) $\mathcal{M}_\alpha \restriction \beta \subseteq \mathcal{M}_\beta$,
(iv) $[e_\alpha = e_\beta \ \&\ \hat{e}_\alpha = \hat{e}_\beta] \implies \mathcal{M}_\alpha = \mathcal{M}_\beta$,
(v) $\mathcal{R}^{<\alpha}_\alpha =_{\text{dfn}} \{\nu \in \mathcal{M}_\alpha : \nu \restriction \beta \in \mathcal{R}_\beta\} \subseteq \mathcal{R}_\alpha$,
(vi) $\mathcal{B}^{<\alpha}_\alpha =_{\text{dfn}} \{\nu \in \mathcal{M}_\alpha : \nu \restriction \beta \in \mathcal{B}_\beta\} \subseteq \mathcal{B}_\alpha$,
(vii) $\mathcal{R}^{\alpha}_\alpha =_{\text{dfn}} \mathcal{R}_\alpha - \mathcal{R}^{<\alpha}_\alpha \neq \emptyset \implies |\alpha| \equiv 3 \bmod 5$,
(viii) $\mathcal{B}^{\alpha}_\alpha =_{\text{dfn}} \mathcal{B}_\alpha - \mathcal{B}^{<\alpha}_\alpha \neq \emptyset \implies |\alpha| \equiv 4 \bmod 5$.

In addition, each $\alpha \in T$ has associated dual sets $\widehat{\mathcal{M}}_\alpha$, $\widehat{\mathcal{R}}_\alpha$, and $\widehat{\mathcal{B}}_\alpha$ which are determined from $\mathcal{M}_\alpha$, $\mathcal{B}_\alpha$, and $\mathcal{R}_\alpha$ by (12), (22), and (21), respectively. Also α has associated integers e_α and $\hat{e}_\alpha$ (depending only on $|\alpha|$) as defined at the beginning of §2.4. (We identify the finite object $\langle \mathcal{M}_\alpha, \mathcal{R}_\alpha, \mathcal{B}_\alpha, k_\alpha\rangle$ with an integer under some effective coding so we may regard $T \subseteq \omega^{<\omega}$.)

Definition 2.12. The *true path* $f \in [T]$ is defined by induction on n. Let $\beta = f \restriction n$ be consistent. Then $f \restriction (n+1)$ is the $<_L$-least $\alpha \in T$, $\alpha \supset \beta$, of length $m = n+1$ such that:

(i) $m \equiv 1 \bmod 5 \implies \mathcal{M}_\alpha = \mathcal{F}^+_\beta \ \&\ k_\alpha = k^+_\beta$,
(ii) $m \equiv 2 \bmod 5 \implies \widehat{\mathcal{M}}_\alpha = \widehat{\mathcal{F}}^+_\beta \ \&\ k_\alpha = k^+_\beta$,
(iii)
$$\begin{aligned} &m \equiv 3 \bmod 5 \implies \\ &[\mathcal{R}^\alpha_\alpha = \{\nu : \nu \in \mathcal{M}_\alpha - (\mathcal{R}^{<\alpha}_\alpha \cap \mathcal{B}^{<\alpha}_\alpha) \ \&\ F(\beta,\nu)\} \\ &\ \&\ \widehat{\mathcal{B}}^\alpha_\alpha = \{\hat{\nu} : \nu \in \mathcal{R}^\alpha_\alpha\}], \end{aligned}$$
(iv)
$$\begin{aligned} &m \equiv 4 \bmod 5 \implies \\ &[\widehat{\mathcal{R}}^\alpha_\alpha = \{\hat{\nu} : \hat{\nu} \in \widehat{\mathcal{M}}_\alpha - (\widehat{\mathcal{R}}^{<\alpha}_\alpha \cup \widehat{\mathcal{B}}^{<\alpha}_\alpha) \ \&\ \widehat{F}(\beta,\nu)\} \\ &\ \&\ \mathcal{B}^\alpha_\alpha = \{\nu : \hat{\nu} \in \widehat{\mathcal{R}}^\alpha_\alpha\}], \end{aligned}$$
(v) unless otherwise specified in (i)–(iv), $\mathcal{M}_\alpha$, $\mathcal{R}_\alpha$, $\mathcal{B}_\alpha$, and k_α take the values $\mathcal{M}_\beta$, $\mathcal{R}_\beta$, $\mathcal{B}_\beta$, and k_β, respectively.

For a consistent $\beta = f \restriction n$, note that $\mathcal{F}^+_\beta$ is just a finite set of states and k^+_β is an integer, so clearly α exists. We shall prove that if $\alpha \subset f$, then α is consistent, so the true path f exists and is infinite. Note that each of the conditions in Definition 2.12 is Π^0_2. Hence, there is a recursive collection of r.e. sets $\{C_\alpha\}_{\alpha \in T}$ such that $\alpha \subset f$

iff $|C_\alpha| = \infty$. Fix a simultaneous recursive enumeration $\{C_{\alpha,s}\}_{\alpha\in T, s\in\omega}$ which will be used in §3 to define a recursive sequence $\{f_s\}_{s\in\omega}$ such that $f = \liminf_s f_s$.

Remark 2.13. It does not hurt the present construction if we expand the tree T to include other components for action which will not interfere with the automorphism construction. For example, in [10] we modify the tree T by putting $\alpha = \beta^\frown\langle \mathcal{M}_\alpha, \mathcal{R}_\alpha, \mathcal{B}_\alpha, k_\alpha, n_\alpha\rangle$ in T providing $\beta \in T$, $n_\alpha \in \omega$, and conditions (i)–(viii) of Definition 2.11 hold as before. The Definition 2.12 is the same but with a new clause (vi) which asserts that n_α must have a certain property depending on β.

To ensure that $\mathcal{M}_\alpha \subseteq \mathcal{E}_\alpha$ for (10) we have a list $\mathcal{L}$ to be defined in §3. Very roughly when $\alpha \subset f_s$ we add to the bottom of $\mathcal{L}$ an (unmarked) α-entry of the form $\langle \alpha, \nu_1\rangle$ for each $\nu_1 \in \mathcal{M}_\alpha$. At some later stage $t+1 > s$ if we see some $x \in Y_{\beta,t} - Y_{\alpha,t}$ such that $\nu^+(\alpha, x, t) = \nu_1$ and $\nu(\alpha, x, t)\restriction\beta = \nu_1\restriction\beta$, then (under Step 1 of §3) we move x to S_α, enumerate x in $U_{\alpha,t+1}$ if necessary so that $\nu(\alpha, x, t+1) = \nu_1$, and we *mark* the α-entry $\langle \alpha, \nu_1\rangle$ on $\mathcal{L}$. When each α-entry $\langle \alpha, \nu_1\rangle$ on $\mathcal{L}$ has been marked we say that $\mathcal{L}$ has been α-*marked*, and we repeat the process by adding new (unmarked) entries $\langle \alpha, \nu_1\rangle$ to $\mathcal{L}$ when next $\alpha \subset f_v$. We define $m(\alpha, s)$ to be the number of times $\mathcal{L}$ has been α-marked at stages $\leq s$, and we prove that $\lim_s m(\alpha, s) = \infty$ for $\alpha \subset f$. Let $\mathcal{L}_s$ denote that portion of $\mathcal{L}$ defined by the end of stage s.

3. The construction

To *initialize* node α means: to remove every $x \in S_{\alpha,s}$ ($\hat{x} \in \hat{S}_{\alpha,s}$), and put x in S^1_β ($\hat{x}$ in $\hat{S}^1_\beta$) for $\beta = \alpha \cap f_{s+1}$ (where $\alpha \cap \delta$ denotes the longest γ such that $\gamma \subseteq \alpha$ and $\gamma \subseteq \delta$); and if x ($\hat{x}$) is an α-*witness* as explained in §7, then cancel it as an α-witness.

We present in this section Steps 1–5 for the construction and a final Step 11 at which we define f_{s+1}. (Steps $\hat{1}$–$\hat{5}$ are the obvious duals to Steps 1–5, and will not be stated. There is no dual of Step 11.) These properties will produce the automorphism. In later sections we may add additional Steps n ($\hat{n}$), $5 < n < 11$, to achieve additional properties.

Stage $s = 0$. For all $\alpha \in T$ define $U_{\alpha,0} = V_{\alpha,0} = \widehat{U}_{\alpha,0} = \widehat{V}_{\alpha,0} = \emptyset$, and define $m(\alpha, 0) = 0$. Define $Y_{\lambda,0} = \widehat{Y}_{\lambda,0} = \emptyset$, and $f_0 = \lambda$.

Stage $s + 1$. Find the least $n < 11$ such that Step n applies to some $x \in Y_{\alpha,s}$, and perform the indicated action. If there is no such n, then likewise find the least $n < 11$ such that Step $\hat{n}$ applies to some $\hat{x} \in \widehat{Y}_{\alpha,s}$, and perform the indicated action. If none of these steps applies, then apply Step 11, and go to stage $s+1$. (It is important that these steps be performed in the indicated order.)

In the following Steps 1–5 (Steps $\hat{1}$–$\hat{5}$) we let $\alpha \in T$, $\alpha \neq \lambda$, be arbitrary, let $\beta = \alpha^-$, and let $x \in Y_{\lambda,s}$ ($\hat{x} \in \widehat{Y}_{\lambda,s}$) be arbitrary.

Step 1. (Prompt pulling of x from R^1_β to S_α to ensure $\mathcal{M}_\alpha \subseteq \mathcal{E}_\alpha$.) Suppose $\langle \alpha, \nu_1\rangle$ is the first unmarked entry on the list $\mathcal{L}_s$ such that the following conditions hold for some x, where $\nu_1 = \langle \alpha, \sigma_1, \tau_1\rangle$,

(1.1) $x \in R^1_{\beta,s} - Y_{\alpha,s}$,

(1.2) $x > k_\alpha$ and $x > |\alpha|$,

(1.3) x is α-eligible (*i.e.*, $\neg(\exists t)[x \leq t \leq s \ \ \& \ \ f_t < \alpha]$),

(1.4) $\neg[\alpha(x, s) <_L \alpha]$,

(1.5) $x > m(\alpha, s)$,

(1.6) $\nu(\beta, x, s) = \nu_1 \restriction \beta$,

(1.7) $e_\alpha > e_\beta \implies \nu^+(\alpha, x, s) = \nu_1$.

Action. Choose the least x corresponding to $\langle \alpha, \nu_1 \rangle$, and do the following.

(1.8) Mark the α-entry $\langle \alpha, \nu_1 \rangle$ on $\mathcal{L}_s$, and suppose this is the kth occurrence of $\langle \alpha, \nu_1 \rangle$ on $\mathcal{L}_s$.

(1.9) Move x to S^i_α, where $k \equiv i \bmod 2$.

(1.10) If $e_\alpha > e_\beta$ and $e_\alpha \in \sigma_1$, then enumerate x in $U_{\alpha,s+1}$.

(1.11) If $\hat{e}_\alpha > \hat{e}_\beta$ and $\hat{e}_\alpha \in \tau_1$, then enumerate x in $\widehat{V}_{\alpha,s+1}$. (Hence, $\nu(\alpha, x, s{+}1) = \nu_1$. Also $\nu_1 \in \mathcal{M}_\alpha$ because $\langle \alpha, \nu_1 \rangle \in \mathcal{L}$ implies $\nu_1 \in \mathcal{M}_\alpha$.)

(1.12) If $\alpha <_L \alpha(x, s)$, then for every γ such that $\alpha <_L \gamma$, cancel all γ-witnesses if any exist, where the latter are defined in §7.

Step 2. (Move x from S^1_β to S^1_α so $Y_\alpha =^* \omega$.) Suppose there is an x such that,

(2.1) $x \in S^1_{\beta,s}$,

(2.2) $x > |\alpha|$ and $x > k_\alpha$,

(2.3) x is α-eligible,

(2.4) $x < m(\alpha, s)$,

(2.5) α is the $<_L$-least $\gamma \in T$ with $\gamma^- = \beta$ satisfying (2.1)–(2.4).

Action. Choose the least pair $\langle \alpha, x \rangle$ and

(2.6) move x from S^1_β to S^1_α.

(In Step 2 we need (2.4) so Y_α will not grow while α is waiting for another prompt pulling under Step 1.)

Step 3. (For α $\mathcal{M}$-inconsistent to ensure $\alpha \not\subset f$.) Suppose for $\alpha \in T$ there exists x such that,

(3.1) $e_\alpha > e_\beta$,

(3.2) $x \in S_{\alpha,s}$,

(3.3) $\nu(\alpha, x, s) = \nu_0 \in \mathcal{M}_\alpha$,

(3.4) $(\exists \nu_1)[\nu_0 <_B \nu_1 \ \ \& \ \ \nu_1 \restriction \beta \in \mathcal{M}_\beta \ \ \& \ \ \nu_1 \notin \mathcal{M}_\alpha]$.

Action. Choose the least such pair $\langle \alpha, x \rangle$ and,

(3.5) enumerate x in $\widehat{V}_{\delta,s+1}$ for all $\delta \subset \alpha$ such that $e_\delta \in \tau_1$. (This action causes $\nu(\alpha, x, s+1) = \nu_1$. Hence, α is provably incorrect at all stages $t \geq s+1$ so $\alpha \not\subset f$.)

Step 4. (Delayed RED enumeration into U_α.) Suppose $x \in R_{\alpha,s}$ and

(4.1) $e_\alpha > e_\beta$,

(4.2) $x \notin U_{\alpha,s}$,

(4.3) $x \in Z_{e_\alpha,s} =_{\text{dfn}} U_{e_\alpha,s} \cap Y_{\beta,s-1}$.

Action. Choose the least such pair $\langle \alpha, x \rangle$ and,

(4.4) enumerate x in $U_{\alpha,s+1}$.

Step 5. (BLUE emptying of state $\nu \in \mathcal{B}_\alpha$.) Suppose for $\alpha \in T$ there exists x such that either Case 1 or Case 2 holds.

Case 1. Suppose

(5.1) $\nu(\alpha, x, s) = \nu_0 \in \mathcal{B}_\alpha$, say $\nu_0 = \langle \alpha, \sigma_0, \tau_0 \rangle$,

(5.2) $x \in S_{\alpha,s}$,

(5.3) α is $\mathcal{M}$-consistent and $\mathcal{R}$-consistent.

Action. Choose the least such pair $\langle \alpha, x \rangle$. Let $\nu_1 = h_\alpha(\nu_0) >_B \nu_0$, where h_α is a target function satisfying (32). In §6 and thereafter we shall assume that h_α also satisfies (46). In §7 and thereafter we shall assume that $\hat{h}_\alpha$ also satisfies (54). Let $\nu_1 = \langle \alpha, \sigma_1, \tau_1 \rangle$.

(5.4) Enumerate $x \in \widehat{V}_\delta$ for all $\delta \subseteq \alpha$ such that $\hat{e}_\delta > \hat{e}_{\delta^-}$ and also $e_\delta \in \tau_1 - \tau_0$. (Hence, $\nu(\alpha, x, s+1) = \nu_1$.)

Case 2. Suppose that (5.1) holds and

(5.5) $x \in S_{\gamma,s}$ where $\gamma^- = \alpha$, and

(5.6) γ is either $\mathcal{M}$-inconsistent or $\mathcal{R}$-inconsistent.

Action. Perform the same action as in Case 1 to achieve $\nu(\alpha, x, s+1) = \nu_1$.

(In (5.6) note that by (42) $\gamma \in T$ implies (5.3) for $\alpha = \gamma^-$, so h_α exists in Case 2. Note in Step 5 Case 2 that the enumeration may not be γ-legal, *i.e.*, perhaps $\nu(\gamma, x, s+1) \notin \mathcal{M}_\gamma$, but this will not matter because we shall prove that $\gamma \not\subset f$ if γ is inconsistent. Hence, it only matters that the enumeration is α-legal, *i.e.*, $\nu(\alpha, x, s) \in \mathcal{M}_\alpha$.)

Step 11. (Defining f_{s+1}, $m(\alpha, s+1)$, $\mathcal{L}_{s+1}$ and $Y_{\lambda,s+1}$.)

Substep 11A. (Defining f_{s+1}.) First we define δ_t by induction on t for $t \leq s+1$. Let $\delta_0 = \lambda$. Given δ_t let $v \leq s$ be maximal such that $\delta_t \subseteq f_v$ if v exists and let $v = 0$ otherwise. (Let $\{C_{\gamma,v}\}_{\gamma \in T, v \in \omega}$ be the simultaneous recursive enumeration specified at the end of §2.9.) Choose the $\leq_L$-least $\alpha \in T$ such that $\alpha^- = \delta_t$ and $C_{\alpha,s} \neq C_{\alpha,v}$ if α exists and define $\delta_{t+1} = \alpha$. If α does not exist, define $\delta_{t+1} = \delta_t$. Finally, define $f_{s+1} = \delta_{s+1}$.

Substep 11B. (Defining $m(\alpha, s+1)$, $\mathcal{L}_{s+1}$, and their duals.) For every $\alpha \subseteq f_{s+1}$ if every α-entry $\langle \alpha, \nu \rangle$ on $\mathcal{L}_s$ and every α-entry $\langle \alpha, \hat{\nu} \rangle$ on $\widehat{\mathcal{L}}_s$ is marked, we say that the lists are α-*marked* and we

(11.1) define $m(\alpha, s+1) = m(\alpha, s) + 1$, and

(11.2) add to the bottom of list $\mathcal{L}_s$ ($\widehat{\mathcal{L}}_s$) a new (unmarked) α-entry $\langle \alpha, \nu \rangle$ ($\langle \alpha, \hat{\nu} \rangle$) for every such α and every $\nu \in \mathcal{M}_\alpha$. Let the resulting list be $\mathcal{L}_{s+1}(\widehat{\mathcal{L}}_{s+1})$.

If the lists are not both α-marked, then let $m(\alpha, s+1) = m(\alpha, s)$, $\mathcal{L}_{s+1} = \mathcal{L}_s$ and $\widehat{\mathcal{L}}_{s+1} = \widehat{\mathcal{L}}_s$.

Substep 11C. (Emptying R_α to the right of f_{s+1}.) For every α such that $f_{s+1} <_L \alpha$, *initialize* α.

Substep 11D. (Moving from S^0_α to S^1_α.) If currently x is in S^0_α but x is not an α-witness, then move x to S^1_α.

(Steps n ($\hat{n}$), $5 < n < 11$, to be defined in later sections, will determine when $x \in S^0_\alpha$ starts and stops being an α-witness. Up through the present section there are no α-witnesses so every $x \in S^0_\alpha$ is eventually moved to S^1_α under Substep 11D, unless x is first removed from S_α by some other step such as Step 11C or Step 1_β for $\beta <_L \alpha$.)

Substep 11E. (Filling Y_λ and $\hat{Y}_\lambda$.) Choose the least $x \notin Y_{\lambda,s}$ ($\hat{x} \notin \hat{Y}_{\lambda,s}$) and $x < s$. Put x in S_λ ($\hat{x}$ in $\hat{S}_\lambda$).

For each $x \in Y_{\lambda,s+1}$ ($\hat{x} \in \hat{Y}_{\lambda,s+1}$) let $\alpha(x, s+1)$ ($\alpha(\hat{x}, s+1)$) denote the unique γ such that $x \in S_{\gamma,s+1}$. This completes stage $s+1$ and the construction.

(Note that after each application of Step 11, the other Steps 1–5 and Steps $\hat{1} - \hat{5}$ can apply only finitely often until the next application of Step 11 as we prove in Lemma 5.6.)

4. The Automorphism Theorem

From now on we assume that $A = U_0$ is a nonrecursive r.e. set. In §6 we introduce Step 6 to exploit this hypothesis. Step 6 together with Steps 1–5, Steps $\hat{1}$–$\hat{5}$, and Step 11 of §3 constitute the *basic construction* designed to ensure that we achieve an automorphism. We may also want to add in later sections of this paper (and in subsequent papers) certain additional Steps n ($\hat{n}$), $6 < n < 11$, to ensure special properties about $B = \widehat{U}_0$, such as B is high or $D \leq_{\mathrm{T}} B$, for a given set D. We

now wish to isolate certain minimal conditions which these additional steps must satisfy so that the resulting construction will still produce an automorphism.

Convention 4.1. *From now on Step n ($\hat{n}$) denotes one of these new steps for $6 \leq n < 11$. In addition we assume that given finitely many elements in $Y_{\lambda,s}$, Step n can apply for at most finitely many stages until another element is put in Y_λ, and similarly for Step $\hat{n}$.*

Theorem 4.2 (Automorphism Theorem)**.** *Assume that $A = U_0$ is a nonrecursive r.e. set. Suppose r.e. sets $\{U_\alpha\}_{\alpha\in T}$, $\{V_\alpha\}_{\alpha\in T}$, $\{\widehat{U}_\alpha\}_{\alpha\in T}$, and $\{\widehat{V}_\alpha\}_{\alpha\in T}$ are enumerated by the construction in §3 using Steps 1–5, Steps $\hat{1}$–$\hat{5}$, and Step 11 of §3, Step 6 of §6, and possibly also some additional Steps n ($\hat{n}$), $6 < n < 11$, such that for all n, $6 \leq n < 11$, Steps n ($\hat{n}$) satisfy the following conditions* P1–P4 *(and their duals* $\widehat{\text{P1}}$–$\widehat{\text{P4}}$ *for $\widehat{S}_\alpha$). Then the correspondence $U_\alpha \leftrightarrow \widehat{U}_\alpha$ and $\widehat{V}_\alpha \leftrightarrow V_\alpha$, $\alpha \subset f$, defines an automorphism of $\mathcal{E}$.*

(P1) *If α is $\mathcal{R}$-inconsistent or $\mathcal{M}$-inconsistent, then Step n does not apply to α. If α is $\mathcal{C}$-inconsistent, then Step n applies to α only if $n = 6$. (Step 6 and $\mathcal{C}$-inconsistent are defined in §6.)*

(P2) *Step n cannot enumerate x in any red set U_α. If Step n at stage $s+1$ enumerates x in a blue set $\widehat{V_\alpha}$, then $x \in R_{\alpha,s}$, and this enumeration must be α-legal,* i.e.*, must satisfy (8), so that $\nu(\alpha, x, s+1) \in \mathcal{M}_\alpha$.*

(P3) *Step n cannot move x from S_α to S_γ for $\alpha \neq \gamma$, or from S^1_α to S^0_α, but can only appoint some x already in S^0_α as an α-witness, and can later cancel x as an α-witness and simultaneously move x from S^0_α to S^1_α.*

(P4) *For all α, $S^0_{\alpha,\infty} =^* \emptyset$.*

The importance of the Automorphism Theorem 4.2 is that from now on we need only verify that the new Steps n ($\hat{n}$), $6 \leq n < 11$, satisfy conditions (P1)–(P4) (and their duals) and we need not mention anything about automorphisms explicitly. For our purposes in this paper conditions (P1)–(P3) for some new Step n will be immediately verifiable by inspection, and (P4) will be true by Lemma 7.2. On the other hand the new Steps n ($\hat{n}$) have great latitude to enumerate and restrain elements, subject primarily to (P2), Step 5, (P4), and their duals. Namely, suppose that Step n operates on S_α, where $\alpha \subset f$, and that after some stage v_α, α is not initialized, and no $\beta <_L \alpha$ acts.

First, Step n may cause certain elements $x \in S_\alpha$ (not just $x \in S^0_\alpha$) to be enumerated in various blue sets, so long as this enumeration is α-legal by (P2). Second, Step n may cause certain elements $x \in S^0_\alpha$ to become α-witnesses, *i.e.*, the positions of α-markers, whereupon by holding x as an α-witness Step n may restrain x from leaving S^0_α, and hence restrain x from being enumerated in $\widehat{V}_\gamma$ for any $\gamma \supset \alpha$, and may also restrain x from entering any further blue sets $\widehat{V}_\gamma$, $\gamma \subseteq \alpha$, subject only to Step 5. Note that Steps 1 and 2 cannot apply to $x \in S^0_\alpha$ after stage v_α, and Step 3 only applies to an α which is $\mathcal{M}$-inconsistent but such $\alpha \not\subset f$ by Lemma 5.9. Hence, only Steps 4 and 5 from the basic Steps 1–5, and 11, can apply to $x \in S^0_\alpha$ after stage v_α. The latter will still hold after we add to the basic construction Step 6 in §6, because Step 6 only applies to an α which is $\mathcal{C}$-inconsistent and such $\alpha \not\subset f$ by Lemma 6.4.

An element x enters S^0_α at most once (when it is first pulled to S_α by Step 1), x becomes an α-witness at most once, and if x ceases to be an α-witness, then x moves from S^0_α to S^1_α. Finally, the new steps must satisfy (P4), that $S^0_{\alpha,\infty} =^* \emptyset$,

so that at most finitely many elements are permanently restrained in S_α and thus almost every $x \in S_\alpha$ is available to be passed to S_γ for $\gamma \supset \alpha$. Hence, the new steps will not interfere with the basic construction which produces an automorphism.

We shall prove the Automorphism Theorem 4.2 in §5 and §6.

5. The verification

All the lemmas of §5 have obvious duals established by the analogous proofs except for Lemmas 5.2, 5.6, 5.7, and 5.10, which either do not require duals, or in which the dual case is explicitly mentioned already. The construction and (P2) clearly establish the following two lemmas.

Lemma 5.1. *At stage $s+1$,*

(i) *if x enters R_α, $\alpha \neq \lambda$, then Step 1 or Step 2 applies to α and x;*

(ii) *if x moves from S_α to S_δ, then one of the following steps must apply to x: Step 1_δ for $\delta <_L \alpha$ or $\delta^- = \alpha$; Step 2_δ for δ such that $\delta^- = \alpha$; or Step 11_α Substep C applying to α, so $f_{s+1} <_L \alpha$; and in the second two cases x enters S^1_δ;*

(iii) *if $x \in S_{\alpha,s}$ is enumerated in a red set U_α at stage $s+1$, then Step 1 or Step 4 must apply to x;*

(iv) *if $x \in S_{\alpha,s}$ is enumerated in a blue set $\widehat{V}_\alpha$, then Step 1, Step 3, Step 5, or Step n must apply to x.*

Lemma 5.2 (True Path Lemma)**.** $f = \liminf_s f_s$.

Proof. This is immediate from the definitions of f_s in Step 11A, of f in Definition 2.12, and of C_α in §2.9. □

We now verify the properties we stated in the three subsections §2.3, §2.4, and §2.5, and we divide the lemmas here into three corresponding subsections. For each lemma there are obvious dual lemmas with similar proofs unless we state and prove the dual explicitly.

5.1. The lemmas of motion, Y_α, and $\alpha(x,s)$.

Lemma 5.3. *For all $\alpha \in T$,*

(i) $f <_L \alpha \implies R_{\alpha,\infty} = \emptyset$,

(ii) $\alpha <_L f \implies Y_\alpha =^* \emptyset$,

(iii) $\alpha \subset f \implies Y_{<\alpha} =_{\mathrm{dfn}} \bigcup \{Y_\delta : \delta <_L \alpha\} =^* \emptyset$.

Proof. (i) Given x choose $s > x$ such that $f_s <_L \alpha$. By Step 11C $R_{\alpha,s} = \emptyset$. Now x is γ-ineligible for all $t \geq s$ and all $\gamma \supseteq \alpha$ so $x \notin S_{\gamma,t}$ and hence $x \notin R_{\alpha,t}$ by (1.3) and (2.3).

(ii) Assume $\alpha <_L f$. Hence, $|C_\alpha| < \infty$, so $\alpha \subset f_s$ for finitely many s and there are only finitely many α-entries $\langle \alpha, \nu \rangle$ on the list $\mathcal{L}$ under (11.2). Hence, finitely many x enter S_α under Step 1 because every such x must mark some unmarked α-entry on $\mathcal{L}$. Thus, $m(\alpha) =_{\mathrm{dfn}} \lim_s m(\alpha, s) < \infty$ since $\mathcal{L}$ will be α-marked at most finitely often. Hence, by (2.4) Step 2 moves only finitely many x into R_α. But each x enters R_α only under Step 1 or Step 2 so $Y_\alpha =^* \emptyset$.

(iii) Immediate by (ii) since there are finitely many $\delta <_L \alpha$ such that $\delta^- = \alpha^-$. □

Lemma 5.4. *For every $\alpha \in T$ if $\alpha \neq \lambda$ and $\beta = \alpha^-$, then*

(i) *$Y_\alpha \setminus Y_\beta = \emptyset$ and $Y_\alpha \subseteq Y_\beta$,*

(ii) $(\forall x)(\exists^{\leq 1} s)[x \in R_{\alpha,s+1} - R_{\alpha,s}]$,
(iii) $(\forall x)(\exists^{\leq 1} s)[x \in S^0_{\alpha,s+1} - S^0_{\alpha,s}]$,
(iv) $U_\alpha \setminus Y_\alpha = \widehat{V}_\alpha \setminus Y_\alpha = \emptyset$,
(v) $\alpha \subset f \implies (\exists v_\alpha)(\forall x)(\forall s \geq v_\alpha)[x \in R_{\alpha,s} \implies (\forall t \geq s)[x \in R_{\alpha,t}]]$.

Proof. (i) Suppose $x \in Y_{\alpha,s+1} - Y_{\alpha,s}$. Then at stage $s+1$ either Step 1 or Step 2 applies to x and α so $x \in Y_{\beta,s}$ by (1.1) and (2.1).

(ii) Suppose $x \in R_{\alpha,s+1} - R_{\alpha,s}$ and $x \in R_{\alpha,t} - R_{\alpha,t+1}$ for some $t > s$. Then $x < s$ by Step 11E. Hence, by Lemma 5.1(ii) at stage $t+1$ either: (1) Step 11C applies to α and x; or (2) Step 1 applies to δ and x for some $\delta <_L \alpha$, $\delta = \alpha(x, t+1)$. If (1), then $f_{s+1} <_L \alpha$ so x is γ-ineligible at all stages $v \geq t+1$ and all $\gamma \supseteq \alpha$, and x can never reenter R_α because of (1.3) and (2.3). If (2), then by Lemma 5.1(ii), (1.4), and induction on $v \geq t$, either for all $v \geq t$, $\alpha(x, v) <_L \alpha$ so $x \notin R_{\alpha,v}$, or else Step 11C applies at stage $v+1$ to x and some $\eta <_L \alpha$, $\eta = \alpha(x, v)$, in which case the argument for (1) shows that $x \notin R_{\alpha,w}$, for all $w \geq v$.

(iii) By (ii), Lemma 5.1, and Step 1 (1.1) and (1.9), x can enter S^0_α only when x first enters S_α and if x ever leaves S^0_α, then it can never reenter.

(iv) Enumeration of x in $U_{\alpha,s+1}$ ($\widehat{V}_{\alpha,s+1}$) takes place only under Step 1, in which case $x \in Y_{\alpha,s+1}$, or under Step 4 (respectively, Step 3, Step 5, or Step n), in which case $x \in Y_{\alpha,s}$ already by (P2).

(v) Assume $\alpha \subset f$. Choose v_α such that for $s \geq v_\alpha$, $f_s \not<_L \alpha$, and no $\beta <_L \alpha$ acts at stage s, and hence $Y_{<\alpha,s} = Y_{<\alpha}$. Thus, if $x \in R_{\alpha,s}$ for $s \geq v_\alpha$, then x cannot be pulled to S_γ for $\gamma <_L \alpha$ by Step 1_γ and x cannot be removed from R_α by Step 11C so x must remain in $R_{\gamma,t}$ for all $t \geq s$. □

Lemma 5.5. *For all x,*

(i) $\alpha(x) =_{\mathrm{dfn}} \lim_s \alpha(x, s)$ *exists, and*

(ii) *x is enumerated in at most finitely many r.e. sets U_γ, $\widehat{V}_\gamma$, and hence for $\alpha = \alpha(x)$,*

$$\nu(\alpha, x) =_{\mathrm{dfn}} \lim_s \nu(\alpha, x, s) \text{ exists.}$$

Proof. (i) By (1.2), (2.2), and Lemma 5.1(i), $x \in S_{\alpha,s}$ implies $x > |\alpha|$. Fix x, let $\gamma = f \restriction x$ and choose $s > v_\gamma$ (as defined in Lemma 5.4(v)) such that $\gamma \subset f_s$. Let $\delta_0 = \alpha(x, s)$. Clearly, $\delta_0 <_L \gamma$ or $\delta_0 \subseteq \gamma$ by Step 11C. Also by induction on $t \geq s$, if $\delta_1 = \alpha(x, t)$ and $\delta_2 = \alpha(x, t+1)$, then $\delta_2 <_L \delta_1$ or $\delta_2 \supset \delta_1$ because Step 1 or Step 2 must have applied to δ_1 and x at stage $t+1$ since Step 11C cannot apply to x after stage v_γ. But there is no infinite sequence $\{\delta_0, \delta_1, \dots\}$ such that for all k, $\delta_{k+1} <_L \delta_k$ or $\delta_{k+1} \supset \delta_k$.

(ii) By (i) choose $t_x \geq v_\gamma$ such that $\alpha(x, s) = \alpha$ for all $s \geq t_x$. Then $\nu(\alpha, x, s) \subseteq \nu(\alpha, x, s+1)$ for all $s \geq t_x$. Hence,

$$\nu(\alpha, x) = \bigcup \{\nu(\alpha, x, s+1) : s \geq t_x\},$$

where this union is defined as in Definition 2.7(iv). □

Lemma 5.6. (i) *Step* 11 *applies infinitely often.*

(ii) *If the hypotheses of some Step* 1–5, *n (Step* $\hat{1}$–$\hat{5}$, *$\hat{n}$) remain satisfied, then that step eventually applies.*

Proof. (i) If Step 11 applies at stage s, then the finitely many $x \in Y_{\lambda,s}$ ($\hat{x} \in \widehat{Y}_{\lambda,s}$) remain the same until the next application of Step 11. Each later application of

Step 1–5 (Step $\hat{1}$–$\hat{5}$) chooses some x ($\hat{x}$) to change position or to be enumerated in some set U_γ or $\hat{V}_\gamma$ ($\hat{U}_\gamma$ or V_γ). By Lemma 5.5, this can happen at most finitely often for each $x \in Y_{\lambda,s}$ ($\hat{x} \in \widehat{Y}_{\lambda,s}$). By Convention 4.1 Steps n or $\hat{n}$, $n \geq 6$, can apply at most finitely often until the next application of Step 11. Hence, Step 11 applies at some stage $t > s$.

(ii) Step 11 cannot apply at stage t if the hypotheses for some Step 1–5, n (Step $\hat{1}$–$\hat{5}$, $\hat{n}$) are satisfied because the latter steps are performed before Step 11 by the basic construction in §3. □

5.2. **Exact covering and the lemmas for $\mathcal{E}_\alpha$, $\mathcal{F}_\alpha$ and $\mathcal{M}_\alpha$.**

Lemma 5.7. *If $\alpha \subset f$, $\alpha \neq \lambda$, and $\beta = \alpha^-$, then*
(i) $(\forall \gamma <_L f)[m(\gamma) =_{\text{dfn}} \lim_s m(\gamma, s) < \infty]$,
(ii) $m(\alpha) =_{\text{dfn}} \lim_s m(\alpha, s) = \infty$,
(iii) $\mathcal{E}_\alpha \supseteq \mathcal{M}_\alpha = \mathcal{F}^+_\beta$,
(iv) $\widehat{\mathcal{E}}_\alpha \supseteq \widehat{\mathcal{M}}_\alpha = \widehat{\mathcal{F}}^+_\beta$, *and*
(v) $\mathcal{E}_\alpha = \mathcal{E}^0_\alpha = \mathcal{E}^1_\alpha$ *and* $\widehat{\mathcal{E}}_\alpha = \widehat{\mathcal{E}}^0_\alpha = \widehat{\mathcal{E}}^1_\alpha$.

Proof. (i) If $\gamma <_L f$, then $\gamma \subset f_s$ for finitely many s, so finitely many γ-entries are ever added to $\mathcal{L}$ and hence $\mathcal{L}$ is γ-marked finitely often and $m(\gamma) < \infty$.

(ii) Fix $\alpha \subset f$, $\alpha \neq \lambda$, and let $\beta = \alpha^-$. Now $\alpha \subset f$ implies $\mathcal{M}_\alpha = \mathcal{F}^+_\beta$ and $\widehat{\mathcal{M}}_\alpha = \widehat{\mathcal{F}}^+_\beta$. Suppose for a contradiction that $m(\alpha) < \infty$, say $m(\alpha, s) = m_0$ for all $s \geq s_0$.

Claim 1. Every α-entry $\langle \alpha, \nu_1 \rangle$ on $\mathcal{L}$ ($\langle \alpha, \widehat{\nu}_1 \rangle$ on $\widehat{\mathcal{L}}$) is eventually marked.

Proof. Suppose that some α-entry $\langle \alpha, \nu_1 \rangle$ on $\mathcal{L}$ is never marked. Hence, by Step 11B there are only finitely many α-entries on $\mathcal{L}$. Choose $s_1 \geq s_0$ such that every α-entry on $\mathcal{L}$ and every entry on $\mathcal{L}$ preceding $\langle \alpha, \nu_1 \rangle$ which will ever be marked is marked by stage s_1, $Y_{<\alpha,s_1} = Y_{<\alpha}$, and for all $x \leq m_0$, $x \in Y_{\alpha,s_1}$ iff $x \in Y_\alpha$. Hence, $Y_\alpha = Y_{\alpha,s_1}$ because no $x > m_0$ can later enter R_α under Step 2 because of (2.4) and no x can later enter R_α under Step 1 because by (1.8) such an x must cause an (unmarked) α-entry on $\mathcal{L}$ to be marked.

Now $\nu_1 \in \mathcal{M}_\alpha$ since $\langle \alpha, \nu_1 \rangle \in \mathcal{L}$. Also $\mathcal{M}_\alpha = \mathcal{F}^+_\beta$ since $\alpha \subset f$. Hence, by the definition of $\mathcal{F}^+_\beta$ in (39) of §2.8,

$$(\exists^\infty x)(\exists s > s_1)[x \in Y^1_{\beta,s} \ \ \& \ \ \nu^+(\alpha, x, s) = \nu_1].$$

By the choice of s_1 almost every such x also satisfies (1.1)–(1.7). Thus, some such x is moved to S_α under Step 1 at some stage $s + 1 > s_1$ and the entry $\langle \alpha, \nu_1 \rangle$ is then marked, contrary to hypothesis. This establishes the claim for $\mathcal{L}$, and the same proof also establishes it for $\widehat{\mathcal{L}}$. □

To complete the proof of (ii) use the claim to find $s > s_0$ such that $\alpha \subset f_{s+1}$ and every α-entry on $\mathcal{L}_s$ and $\widehat{\mathcal{L}}_s$ is marked. Now by Step 11B, $m(\alpha, s+1) > m(\alpha, s) = m_0$ contrary to the choice of s_0.

(iii) By (ii) and (11.2) for every $\nu_1 \in \mathcal{M}_\alpha$, infinitely often an entry $\langle \alpha, \nu_1 \rangle$ is added to $\mathcal{L}$ and later marked when some $x \in S_{\alpha,s} - \bigcup \{S_{\alpha,t} : t < s\}$ such that $\nu(\alpha, x, s) = \nu_1$. Hence, $\nu_1 \in \mathcal{E}_\alpha$.

(iv) Likewise $\widehat{\mathcal{E}}_\alpha \supseteq \widehat{\mathcal{M}}_\alpha = \widehat{\mathcal{F}}^+_\beta$ by the same proof as in (iii).

(v) By (iii), for every $\nu_1 \in \mathcal{M}_\alpha = \mathcal{E}_\alpha$, infinitely often an entry $\langle \alpha, \nu_1 \rangle$ is added to $\mathcal{L}$ and later marked when some $x \in S_{\alpha,s} - \bigcup \{S_{\alpha,t} : t < s\}$ such that $\nu(\alpha, x, s) = \nu_1$. By Step 1 (1.8) and (1.9), infinitely many such x enter S^i_α, $i = 0, 1$, and hence $\mathcal{E}_\alpha = \mathcal{E}^0_\alpha = \mathcal{E}^1_\alpha$. □

Lemma 5.8. $\alpha \subset f \implies R_{\alpha,\infty} =^* Y_\alpha =^* Y_\lambda = \omega$.

Proof. By Lemma 5.6(i) Step11E must eventually put every element $x \in \omega$ into Y_λ. By induction we may assume $R_{\beta,\infty} =^* Y_\beta =^* \omega$, for $\beta^- = \alpha$. By Lemma 5.7 $m(\alpha) = \infty$ and $m(\gamma) < \infty$ for all $\gamma <_L \alpha$ with $\gamma^- = \beta$.

By Lemma 5.3, $Y_{<\alpha} =^* \emptyset$ and almost every $x \in R_\beta$ not yet in R_α must eventually lie in S_β, and hence in S^1_β, because $S^0_{\beta,\infty} =^* \emptyset$ by (P4). Hence, almost every $x \in R_\beta$ not yet in R_α must eventually satisfy (2.1)–(2.5) and must eventually move to S_α by (2.6). By Lemma 5.4(v) almost every such x will remain in R_α forever. □

Lemma 5.9. $\alpha \subset f \implies \alpha$ *is* $\mathcal{M}$*-consistent.*

Proof. Let $\alpha \subset f$ and $\beta = \alpha^-$. Assume for a contradiction that α is not $\mathcal{M}$-consistent. Then $e_\alpha > e_\beta$ and there exist $\nu_0 \in \mathcal{M}_\alpha$, $\nu_1 \notin \mathcal{M}_\alpha$, $\nu_0 <_B \nu_1$ and $\nu_1 \restriction \beta \in \mathcal{M}_\beta$. By (42) α is a terminal mode on T so $S_\alpha = R_\alpha$. By Lemmas 5.8 and 5.4(v), $S_{\alpha,\infty} =^* \omega$ and no $x \in S_{\alpha,s}$, $s > v_\alpha$, later leaves S_α. By Lemma 5.7, $\mathcal{E}_\alpha \supseteq \mathcal{M}_\alpha$ so

$$(\exists^\infty x)(\exists s)[x \in S_{\alpha,s+1} - S_{\alpha,s} \ \ \& \ \ \nu(\alpha, x, s+1) = \nu_0].$$

Choose any such x and $s > v_\alpha$. Now neither Step 1_γ nor Step 2_γ can apply to x at any stage $t > s$. Hence, by the ordering of the steps, Step 3_α must apply to some such x' at some stage $t + 1 > s + 1$ with $\nu(\alpha, x', t) = \nu_0$ and must cause $\nu(\alpha, x', t+1) = \nu_1$. Thus, α is provably incorrect at all stages $v \geq t+1$ so $\alpha \not\subset f$. □

Lemma 5.10. *If* $\alpha \subset f$*, then*
(i) $\widehat{\mathcal{M}}_\alpha = \{\hat{\nu} : \nu \in \mathcal{M}_\alpha\}$,
(ii) $\mathcal{M}_\alpha = \mathcal{F}_\alpha = \mathcal{E}_\alpha$*, and*
(iii) $\widehat{\mathcal{M}}_\alpha = \widehat{\mathcal{F}}_\alpha = \widehat{\mathcal{E}}_\alpha$.

Proof. Fix $\alpha \subset f$, and let $\beta = \alpha^-$. Now (i) holds by the definitions of $\mathcal{M}_\alpha$ and $\widehat{\mathcal{M}}_\alpha$. Assume (ii) and (iii) for β. We know $\mathcal{E}_\alpha \subseteq \mathcal{F}_\alpha$ by their definitions, and $\mathcal{M}_\alpha \subseteq \mathcal{E}_\alpha$ by Lemma 5.7. Thus, to prove (ii) (and (iii)) it suffices to prove $\mathcal{F}_\alpha \subseteq \mathcal{M}_\alpha$, (and $\widehat{\mathcal{F}}_\alpha \subseteq \widehat{\mathcal{M}}_\alpha$). By (P2) it suffices to consider Steps 1–5 ($\hat{1}$–$\hat{5}$).

Case 1. $e_\alpha = e_\beta$ and $\hat{e}_\alpha = \hat{e}_\beta$.

Then $\mathcal{M}_\alpha = \mathcal{M}_\beta$. Also $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$ since $Y_\alpha \subseteq Y_\beta$. Finally, $\mathcal{M}_\beta = \mathcal{F}_\beta$ by the inductive hypothesis (ii) for β. Hence,

$$\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta = \mathcal{M}_\beta = \mathcal{M}_\alpha,$$

so (ii) holds for α. Likewise, $\widehat{\mathcal{F}}_\alpha \subseteq \widehat{\mathcal{M}}_\alpha$ so (iii) holds for α.

Before considering Case 2 we need a technical sublemma.

Sublemma. *If* $e_\alpha > e_\beta$*,* $\nu_2 = \langle \alpha, \sigma_2, \tau_2 \rangle \in \mathcal{F}^+_\beta$*, and* $\nu_1 = \langle \alpha, \sigma_1, \tau_2 \rangle$*, where* $\sigma_1 = \sigma_2 - \{e_\alpha\}$*, then* $\nu_1 \in \mathcal{F}^+_\beta$ *also.*

Proof. Suppose $\nu_2 \in \mathcal{F}^+_\beta$. Then $\nu_3 = \nu_2 \restriction \beta \in \mathcal{F}_\beta$, and $\mathcal{F}_\beta = \mathcal{E}_\beta = \mathcal{E}^1_\beta$ by the inductive hypothesis (ii) for β and Lemma 5.7(v). Hence, by the definition of $\mathcal{E}^1_\beta$,

$$(\exists^\infty x)(\exists s)[x \in Y^1_{\beta,s} - Y^1_{\beta,s-1} \ \ \& \ \ \nu(\beta, x, s) = \nu_3].$$

But for each such x and s, $x \notin Z_{e_\alpha,s}$ (by the definition of $Z_{e_\alpha,s}$ in §2.8) so $\nu^+(\alpha,x,s) = \nu_1$. Hence, $\nu_1 \in \mathcal{F}^+_\beta$ by the definition of $\mathcal{F}^+_\beta$ in (39). □

Case 2. $e_\alpha > e_\beta$.

We prove $\mathcal{F}_\alpha \subseteq \mathcal{M}_\alpha$ and its dual $\widehat{\mathcal{F}}_\alpha \subseteq \widehat{\mathcal{M}}_\alpha$ in the next five claims. (The proof of Case 3, $\hat{e}_\alpha > \hat{e}_\beta$, is entirely dual and will be omitted.)

Claim 1. $\mathcal{F}_\alpha \subseteq \mathcal{M}_\alpha$.

Proof. Suppose $\nu_1 \in \mathcal{F}_\alpha$. Let $\nu_1 = \langle \alpha, \sigma_1, \tau_1 \rangle$. Then

$$(\exists^\infty x)(\exists s)[x \in Y_{\alpha,s} \ \ \& \ \ \nu(\alpha,x,s) = \nu_1]. \tag{43}$$

Note that $Y_{\alpha,s} \subseteq Y^1_{\beta,s}$ and $\nu(\alpha,x,s) \leq_R \nu^+(\alpha,x,s)$ because $U_{\alpha,s} \subseteq Z_{e_\alpha,s}$. First suppose

$$(\exists^\infty x)(\exists s)[x \in Y_{\alpha,s} \ \ \& \ \ \nu^+(\alpha,x,s) = \nu_1]. \tag{44}$$

Then $\nu_1 \in \mathcal{F}^+_\beta$ by definition of $\mathcal{F}^+_\beta$ because $Y_{\alpha,s} \subseteq Y^1_{\beta,s}$, and $\mathcal{F}^+_\beta = \mathcal{M}_\alpha$ since $\alpha \subset f$.

If (44) fails, then for almost every x in (43), $\nu^+(\alpha,x,s) = \nu_2 >_R \nu_1$ so $\nu_2 = \langle \alpha, \sigma_2, \tau_1 \rangle$ where $e_\alpha \notin \sigma_1$ and $\sigma_2 = \sigma_1 \cup \{e_\alpha\}$. Now $\nu_2 \in \mathcal{F}^+_\beta$ since $Y_{\alpha,s} \subseteq Y^1_{\beta,s}$, so $\nu_1 \in \mathcal{F}^+_\beta = \mathcal{M}_\alpha$ by the Sublemma. □

Claim 2. $\widehat{\mathcal{F}}_\alpha \subseteq \widehat{\mathcal{M}}_\alpha$.

Proof. We establish Claim 2 by the next three claims which are the duals of (6), (7), and (8).

Claim 3. $\widehat{\mathcal{E}}_\alpha \subseteq \widehat{\mathcal{M}}_\alpha$.

Proof. Assume $\hat{\nu}_1 \in \widehat{\mathcal{E}}_\alpha$. Hence,

$$(\exists^\infty \hat{x})(\exists s)[\hat{x} \in \widehat{S}_{\alpha,s+1} - \widehat{Y}_{\alpha,s} \ \ \& \ \ \nu(\alpha,\hat{x},s+1) = \hat{\nu}_1].$$

For every such x and s, x must have entered $\widehat{S}_{\alpha,s+1}$ under Step $\hat{1}$ or Step $\hat{2}$ by ($\widehat{\mathrm{P}}$3). If Step $\hat{1}$ applied, then we marked an entry $\langle \alpha, \hat{\nu}_1 \rangle$ on $\widehat{\mathcal{L}}_s$ so $\hat{\nu}_1 \in \widehat{\mathcal{M}}_\alpha$ by the definition of $\widehat{\mathcal{L}}$ in Step 11. If Step $\hat{2}$ applied, then $\hat{x} \notin \widehat{U}_{\alpha,s+1}$ because $\hat{x} \notin \widehat{U}_{\alpha,s}$ by Lemma 5.4(iv) and no enumeration takes place at stage $s+1$ under Step $\hat{2}$. Hence, $e_\alpha \notin \sigma_1$ where $\nu_1 = \langle \alpha, \sigma_1, \tau_1 \rangle$.

Let $\nu_3 = \nu_1 \restriction \beta$. Now $\hat{\nu}_3 \in \widehat{\mathcal{F}}_\beta = \widehat{\mathcal{M}}_\beta$ so $\nu_3 \in \mathcal{M}_\beta = \mathcal{F}_\beta$ and thus either $\nu_1 \in \mathcal{F}^+_\beta$ or $\nu_2 \in \mathcal{F}^+_\beta$ where $\nu_2 = \langle \alpha, \sigma_1 \cup \{e_\alpha\}, \tau_1 \rangle$. But if $\nu_2 \in \mathcal{F}^+_\beta$, then $\nu_1 \in \mathcal{F}^+_\beta$ by the Sublemma. In either case $\nu_1 \in \mathcal{F}^+_\beta = \mathcal{M}_\alpha$, so $\hat{\nu}_1 \in \widehat{\mathcal{M}}_\alpha$. □

Claim 4. If $\hat{x} \in \widehat{Y}_{\alpha,s}$, $\hat{\nu}_1 = \nu(\alpha,\hat{x},s) \in \widehat{\mathcal{M}}_\alpha$, $s > v_\alpha$ of Lemma 5.4(v), and RED causes enumeration of $\hat{x}$ so that $\hat{\nu}_2 = \nu(\alpha,\hat{x},s+1)$, then $\hat{\nu}_2 \in \widehat{\mathcal{M}}_\alpha$.

Proof. Suppose this enumeration occurs. Then $\hat{\nu}_1 <_R \hat{\nu}_2$ so $\nu_1 <_B \nu_2$ by (13). Now $\nu_1 \in \mathcal{M}_\alpha$ since $\hat{\nu}_1 \in \widehat{\mathcal{M}}_\alpha$. But α is $\mathcal{M}$-consistent by Lemma 5.9 so $\nu_2 \in \mathcal{M}_\alpha$, and hence $\hat{\nu}_2 \in \widehat{\mathcal{M}}_\alpha$. □

Claim 5. If $\hat{x} \in \widehat{Y}_{\alpha,s}$, $\hat{\nu}_1 = \nu(\alpha,\hat{x},s) \in \widehat{\mathcal{M}}_\alpha$, $s > v_\alpha$ of Lemma 5.4(v), and BLUE causes enumeration of $\hat{x}$ so that $\hat{\nu}_2 = \nu(\alpha,\hat{x},s+1)$, then $\hat{\nu}_2 \in \widehat{\mathcal{M}}_\alpha$.

Proof. Suppose $\hat{x} \in \widehat{Y}_{\alpha,s}$ and BLUE causes this enumeration at stage $s+1$, so $\hat{\nu}_1 <_B \hat{\nu}_2$. Since $s > v_\alpha$, $\hat{x} \in \widehat{R}_{\alpha,s} \cap \widehat{R}_{\alpha,s+1}$. Hence, Step $\hat{1}$, Step $\hat{3}$, Step $\hat{5}$, or Step $\hat{n}$ applies to $\hat{x}$ at stage $s+1$ for some $\gamma \supseteq \alpha$. If Step $\hat{n}$ applies, then $\hat{\nu}_2 \in \widehat{\mathcal{M}}_\alpha$ by ($\widehat{\text{P}}$2). If Step $\hat{1}_\gamma$ or Step $\hat{5}_\gamma$ applies, then $\hat{\nu}_3 = \nu(\gamma, \hat{x}, s+1) \in \widehat{\mathcal{M}}_\gamma$ so $\hat{\nu}_2 = \hat{\nu}_3 \restriction \alpha \in \widehat{\mathcal{M}}_\alpha$. (Here Step $\hat{5}_\gamma$ means Step $\hat{5}$ Case 1 for $\hat{x} \in \widehat{Y}_{\gamma,s}$ or Step $\hat{5}$ Case 2 for $\hat{x} \in \widehat{Y}_{\delta,s}$ where $\gamma = \delta^-$.) If Step $\hat{3}_\gamma$ applies, then $\gamma \supset \alpha$ (since α is $\mathcal{M}$-consistent and γ is not) and $\hat{\nu}_3 = \nu(\gamma^-, \hat{x}, s+1) \in \widehat{\mathcal{M}}_{\gamma^-}$ by (3.4) so $\hat{\nu}_2 = \hat{\nu}_3 \restriction \alpha \in \widehat{\mathcal{M}}_\alpha$. This completes the proof of Claim 5, Claim 2 Case 2, and Lemma 5.10. □

5.3. Emptying α-states and the lemmas for $\mathcal{R}_\alpha$ and $\mathcal{B}_\alpha$.

Lemma 5.11. *$\alpha \subset f \implies \alpha$ is $\mathcal{R}$-consistent.*

Proof. Assume for a contradiction that $\alpha \subset f$ and α is not $\mathcal{R}$-consistent. Choose $\nu_1 \in \mathcal{R}_\alpha$ such that for all $\nu_2 \in \mathcal{M}_\alpha$, $\nu_1 \not<_R \nu_2$. By (42) α is a terminal node on T so $S_\alpha = R_\alpha$. By Lemmas 5.8 and 5.4(v), $S_{\alpha,\infty} =^* \omega$ and no $x \in S_{\alpha,s}$, $s > v_\alpha$, later leaves S_α. Now $\nu_1 \in \mathcal{R}_\alpha \subseteq \mathcal{M}_\alpha = \mathcal{E}_\alpha$ by Lemma 5.10 so

$$(\exists^\infty x)(\exists s > v_\alpha)[x \in S_{\alpha,s+1} - Y_{\alpha,s} \ \ \& \ \ \nu(\alpha, x, s) = \nu_1].$$

For each such x and s, $x \in S_{\alpha,t}$ for all $t > s+1$ so neither Step 1 nor Step 2 can apply to x at any stage $t > s+1$. Now Step 3 cannot apply to $x \in S_{\alpha,t}$ because α is $\mathcal{M}$-consistent by Lemma 5.9. Furthermore, Step 5 cannot apply to $x \in S_{\alpha,t}$ while $\nu(\alpha, x, t) = \nu_1$ because $\nu_1 \in \mathcal{R}_\alpha$ and $\mathcal{R}_\alpha \cap \mathcal{B}_\alpha = \emptyset$. But if $\nu(\alpha, x, t) = \nu_1$ for all $t \geq s$, then x witnesses that $F(\alpha^-, \nu_1)$ fails so $\nu_1 \in \mathcal{R}_\alpha$ contradicts $\alpha \subset f$. By (P1) Step n cannot apply to α. Hence, Step 4 applies to $x \in S_{\alpha,t}$ at some stage $t+1 > s+1$ such that $\nu_1 = \nu(\alpha, x, s) = \nu(\alpha, x, t)$, $\nu_2 = \nu(\alpha, x, t+1)$, and $\nu_1 <_R \nu_2$. Choose ν_2 such that this happens for infinitely many $x \in S_\alpha$. Now $\nu_2 \in \mathcal{F}_\alpha$ so $\nu_2 \in \mathcal{M}_\alpha$ by Lemma 5.10. □

Lemma 5.12. *If $\alpha \subset f$ and $\nu_1 \in \mathcal{B}_\alpha$, then $\{x : x \in Y_\alpha \ \ \& \ \ \nu(\alpha, x) = \nu_1\} =^* \emptyset$.*

Proof. Fix $\alpha \subset f$ and $\nu_1 \in \mathcal{B}_\alpha$. Let v_α be as in Lemma 5.4(v). Assume for a contradiction that $x \in R_{\alpha,s}$ for some $s > v_\alpha$ and that for all $t \geq s$, $\gamma = \alpha(x, t)$, and $\nu_1 = \nu(\alpha, x, t)$. Now $\gamma \supseteq \alpha$ and $\alpha \in T$ so by the Definition 2.11 (vi) of T we have $\nu_1' \in \mathcal{B}_\gamma$ for all $\nu_1' \in \mathcal{M}_\gamma$ such that $\nu_1' \restriction \alpha = \nu_1$.

Case 1. γ is $\mathcal{R}$-consistent. Then Step 5 Case 1 applies to x and γ at some stage $t+1 > s$ so $\nu_1' = \nu(\gamma, x, t)$, $\nu_2' = \nu(\gamma, x, t+1)$, $\nu_1' <_B \nu_2'$, and $\nu_2' \in \mathcal{M}_\gamma - \mathcal{B}_\gamma$. Hence, $\nu_2 = \nu_2' \restriction \alpha \in \mathcal{M}_\alpha - \mathcal{B}_\alpha$, and $\nu(\alpha, x, t+1) = \nu_2 >_B \nu_1$.

Case 2. Otherwise. Then at some stage $t+1 > s$, Step 5 Case 2 applies to x and $\delta = \gamma^- \supseteq \alpha$ so $\nu(\alpha, x, t+1) = \nu_2 >_B \nu_1$ as in Case 1 but with δ in place of γ. □

Lemma 5.13. *If every $\alpha \subset f$ is $\mathcal{C}$-consistent, then the correspondence $U_\alpha \leftrightarrow \widehat{U}_\alpha$ and $\widehat{V}_\alpha \leftrightarrow V_\alpha$, $\alpha \subset f$, defines an automorphism of $\mathcal{E}$.*

Proof. Choose $\alpha \subset f$. By Lemmas 5.9 and 5.11, α is $\mathcal{M}$-consistent and α is also $\mathcal{R}$-consistent. By our hypothesis, which will be discharged in Lemma 6.4, α is also $\mathcal{C}$-consistent. Hence, α is consistent by Definition 2.10. Thus, by Definition 2.12, f is infinite, and hence $\lim_{\alpha \subset f} e_\alpha = \infty$.

By Lemma 5.8, $Y_\alpha =^* \omega$; by Lemma 5.10, we have (11), its dual, and (12) (so the well-visited α-states on ω coincide with those on $\widehat{\omega}$); and by Lemma 5.12 and its dual, we have (15) (so the well-resided α-states also coincide). It immediately follows that the automorphism requirement (1) is satisfied as remarked in §2.6. □

6. Using that A is nonrecursive to obtain the set $\widehat{\mathcal{C}}_\alpha$ of coding states

For the rest of this paper we assume that RED specifies a nonrecursive r.e. set A and BLUE replies by constructing an r.e. set B automorphic to A such that B also codes certain additional information (such as B is high as in the conclusion of Theorem 1.3). We let $U_0 = A$ and $B = \widehat{U}_\rho$, where $\rho = f \restriction 1$. Define $B_s = \widehat{U}_{\rho,s}$. (From now on we consider only nodes $\alpha \in T$ such that $\rho \subset \alpha$.) To code this information into B BLUE will choose an α-state $\hat{\nu}_1$ with certain properties, choose a witness $\hat{y} \in \widehat{S}^0_\alpha$ in α-state $\hat{\nu}_1$, begin by holding $\hat{y}$ in $\overline{B}$ and in α-state $\hat{\nu}_1$, and perhaps later attempt to move $\hat{y}$ into B. To see that $\widehat{\mathcal{C}}_\alpha \neq \emptyset$ where $\widehat{\mathcal{C}}_\alpha$ is the set of α-states $\hat{\nu}_1$ (called *coding* states) with the necessary properties we now use the nonrecursiveness of A to verify that the dual set $\mathcal{C}_\alpha \neq \emptyset$, where $\widehat{\mathcal{C}}_\alpha = \{\hat{\nu} : \nu \in \mathcal{C}_\alpha\}$.

Definition 6.1. (i) Let $\mathcal{W}_\alpha$ be that subset of $\mathcal{M}_\alpha$ which is generated by the following three clauses:

1. $[\nu_1 = \langle \alpha, \sigma_1, \tau_1 \rangle \ \&\ 0 \in \sigma_1] \implies \nu_1 \in \mathcal{W}_\alpha$,
2. $(\exists \nu_2)[\nu_1 <_R \nu_2 \ \&\ \nu_2 \in \mathcal{W}_\alpha] \implies \nu_1 \in \mathcal{W}_\alpha$,
3. $[\nu_1 \in \mathcal{B}_\alpha \ \&\ (\forall \nu_2 \in \mathcal{M}_\alpha)[\nu_1 <_B \nu_2 \implies \nu_2 \in \mathcal{W}_\alpha]] \implies \nu_1 \in \mathcal{W}_\alpha$.

(ii) Define $\mathcal{W}^\#_\alpha = \{\nu_1 : \nu_1 = \langle \alpha, \sigma_1, \tau_1 \rangle \in \mathcal{W}_\alpha \ \&\ 0 \notin \sigma_1\}$.
(iii) Define $\mathcal{V}_\alpha = \mathcal{M}_\alpha - \mathcal{W}_\alpha$.

Note that $\mathcal{W}_\alpha$ consists of the α-states $\nu_1 \in \mathcal{M}_\alpha$ for which RED has a *winning strategy* F_α to force any x in α-state ν_1 into A. Namely, if $\nu(\alpha, x, s) = \nu_1$ and (1) holds, then $x \in U_0 = A$ already; if (2), then RED can change the α-state of x from ν_1 to $F_\alpha(\nu_1) = \nu_2 >_R \nu_1$; and if (3), then by (22) and (28) RED can wait for BLUE to change x from α-state ν_1 to some $\nu_2 >_B \nu_1$ and then RED can apply F_α to ν_2. Hence, this winning strategy can be identified with a function,

$$F_\alpha : (\mathcal{W}^\#_\alpha - \mathcal{B}_\alpha) \to \mathcal{W}_\alpha \ \&\ (\forall \nu_1 \in (\mathcal{W}^\#_\alpha - \mathcal{B}_\alpha))[\nu_1 <_R F_\alpha(\nu_1)]. \tag{45}$$

Similarly, if $\nu(\alpha, x, s) = \nu_1 \in \mathcal{V}_\alpha$, then BLUE has a winning strategy G_α to keep x out of A. Namely, BLUE keeps x in α-state ν_1 unless $\nu_1 \in \mathcal{B}_\alpha$ in which case by the negation of (3), BLUE can change x to some α-state $G_\alpha(\nu_1) = \nu_2 >_B \nu_1$ such that $\nu_2 \in \mathcal{V}_\alpha$. Meanwhile if RED causes $\nu(\alpha, x, t) = \nu_3 >_R \nu_1$ at some $t > s$, then by the negation of (2), $\nu_3 \in \mathcal{V}_\alpha$ so BLUE continues to play as for ν_1. By repeatedly applying G_α if necessary we may assume that $\text{range}(G_\alpha) \cap \mathcal{B}_\alpha = \emptyset$. Hence, from now on we may assume that BLUE's target function h_α of (32) agrees with the function G_α on their common domain, namely h_α also satisfies

$$(\forall \nu \in \mathcal{V}_\alpha \cap \mathcal{B}_\alpha)[\nu_1 <_B h_\alpha(\nu_1) = G_\alpha(\nu_1) \in \mathcal{V}_\alpha - \mathcal{B}_\alpha]. \tag{46}$$

(Thus, by using this h_α any BLUE enumeration under Step 5 of §3 is automatically following BLUE's winning strategy G_α for all $\nu_1 \in \mathcal{V}_\alpha$.)

Definition 6.2. If $\alpha \neq \lambda$, let $\mathcal{C}_\alpha$ be the set of $\nu_1 \in \mathcal{M}_\alpha$ such that
(i) $\nu_1 \in \mathcal{W}^\#_\alpha$,
(ii) $\neg(\exists \nu_2 \in \mathcal{M}_\alpha)[\nu_1 <_B \nu_2]$,
(iii) $\nu_1 \notin \mathcal{N}_\alpha =_{\text{dfn}} \mathcal{R}_\alpha \cup \mathcal{B}_\alpha$.

Property (ii) asserts that ν_1 is maximal with respect to α-legal enumeration in blue sets, and the import of (iii) is that $\nu_1 \notin \mathcal{R}_\alpha$. Hence, if x is in state ν_1, then RED can hold x forever in that state (and hence in $\overline{A}$), or by (i) RED can later force x to eventually enter A.

Definition 6.3. (i) A node $\alpha \in T$ is *$\mathcal{C}$-consistent* if $\alpha = \lambda$ or $\mathcal{C}_\alpha \neq \emptyset$, and α is *$\mathcal{C}$-inconsistent* otherwise.

(ii) A node $\alpha \in T$ is *consistent* if α is $\mathcal{M}$-consistent (Definition 2.8), $\mathcal{R}$-consistent (Definition 2.9), and also $\mathcal{C}$-consistent.

Any inconsistent α is terminal by (42). From now on we assume that the following Step 6 has been added to the construction in §3. (Step 6 will ensure that α is $\mathcal{C}$-consistent for $\alpha \subset f$. There is no dual Step $\hat{6}$.)

Step 6. Suppose $\alpha \in T$, α is $\mathcal{C}$-inconsistent, but $\mathcal{M}$-consistent and $\mathcal{R}$-consistent, $x \in S_{\alpha,s}$, $\nu(\alpha, x, s) = \nu_1$, and

$$(\exists \nu_2 \in \mathcal{M}_\alpha)[\nu_1 <_B \nu_2].$$

Action. Choose the least such pair $\langle \alpha, x\rangle$, and the first such $\nu_2 \in \mathcal{M}_\alpha$. Let $\nu_2 = \langle \alpha, \sigma_2, \tau_2\rangle$. Enumerate x in $\widehat{V}_{\beta,s+1}$ for all $\beta \subseteq \alpha$ such that $\hat{e}_\beta > \hat{e}_{\beta^-}$ and $\hat{e}_\beta \in \tau_2$. (Hence, $\nu(\alpha, x, s+1) = \nu_2$.)

Clearly, Step 6 satisfies conditions (P1)–(P4) of the Automorphism Theorem 4.2, because $\nu_2 \in \mathcal{M}_\alpha$.

Lemma 6.4. *If $\alpha \subset f$, then α is $\mathcal{C}$-consistent.*

Proof. Assume for a contradiction that $\alpha \subset f$ and α is $\mathcal{C}$-inconsistent. As in Lemmas 5.8 and 5.11, α is terminal on T, $S_\alpha = R_\alpha$, $S_{\alpha,\infty} =^* \omega$, and no $x \in S_{\alpha,s}$, $s > v_\alpha$, later leaves S_α. Thus, neither Step 1 nor Step 2 can apply to any $x \in S_{\alpha,s}$ after stage v_α, and neither Step 3 nor Step 5 Case 2 can ever apply because α is $\mathcal{M}$-consistent and $\mathcal{R}$-consistent by Lemmas 5.8 and 5.11. For each $\nu_1 \in \mathcal{M}_\alpha$ define the r.e. set

$$D_{\nu_1} = \{x : (\exists s > v_\alpha)[x \in S_{\alpha,s} \ \ \& \ \ \nu(\alpha, x, s) = \nu_1]\}.$$

Now $D_{\nu_1} \subseteq \overline{A}$ for every $\nu_1 \in \mathcal{V}_\alpha$ because by (P1) the only red (blue) enumeration of x after $x \in D_{\nu_1,s}$ comes from Step 4 (Step 5), but in Step 5 the target function h_α now satisfies (46) so $\nu(\alpha, x, t) \in \mathcal{V}_\alpha$ for all $t \geq s$.

Let $\mathcal{K}_\alpha$ be as in (14). For each $\nu_1 \in \mathcal{M}_\alpha - \mathcal{K}_\alpha$ such that $0 \notin \sigma_1$ (*i.e.*, each ν_1 well-resided on $\overline{A}$) let $E_{\nu_1} = \{x : \nu(\alpha, x) = \nu_1\}$. If $\nu_1 \in \mathcal{V}_\alpha$, then $E_{\nu_1} \subseteq D_{\nu_1} \subseteq \overline{A}$. Since A is nonrecursive, there must exist $\nu_1 \in (\mathcal{M}_\alpha - \mathcal{K}_\alpha) \cap \mathcal{W}^\#_\alpha$. Hence, $\nu_1 \notin \mathcal{N}_\alpha = \mathcal{R}_\alpha \cup \mathcal{B}_\alpha$ because $\mathcal{N}_\alpha \subseteq \mathcal{K}_\alpha$. By Step 6, every such ν_1 must satisfy Definition 6.2(ii), and hence $\nu_1 \in \mathcal{C}_\alpha$. Thus, α is $\mathcal{C}$-consistent. □

Lemmas 6.4 and 5.13 complete the proof of the Automorphism Theorem 4.2. □

7. Moving α-witnesses into B

Let A and B be as at the beginning of §6. Let the set of coding states $\widehat{\mathcal{C}}_\alpha$ be the dual of $\mathcal{C}_\alpha$ of §6, namely $\widehat{\mathcal{C}}_\alpha = \{\hat{\nu} : \nu \in \mathcal{C}_\alpha\}$. To code information into B we define Step $\hat{7}$ in §7.1, which determines when an element $\hat{x} \in \hat{S}^0_\alpha$ in some state $\hat{\nu}_1 \in \widehat{\mathcal{C}}_\alpha$ becomes an α-witness; various versions of Steps $\hat{n}$, $9 \leq n < 11$, defined in later sections (to prove one of several different theorems about B) will determine when an α-witness $\hat{x}$ later becomes *activated* indicating that $\hat{x}$ wants to enter B; Step $\hat{8}$ defined in §7.2 processes an activated witness until it enters B; and finally the Coding Theorem 7.5 in §7.3 proves that this coding procedure succeeds. (There are no dual steps 7, or 8.) Let $\widehat{L}_\alpha$ ($\widehat{J}_\alpha$) denote the d.r.e. set of α-witnesses (activated α-witnesses) and $\widehat{L}_{\alpha,s}$ ($\widehat{J}_{\alpha,s}$) the set of elements in $\widehat{L}_\alpha$ (respectively $\widehat{J}_\alpha$) at the end

of stage s. We shall assume from now on that any additional Steps $\hat{n}$, $9 \leq n < 11$, cannot add elements to or remove elements from $\widehat{L}_\alpha$.

7.1. **Appointing α-witnesses and Step $\hat{7}$.** As input to Step $\hat{7}$ we have a recursive function $g(\alpha, s)$. The choice of g will depend on the theorem being proved. For example, in Theorem 1.3, $g(\alpha, s) = 1$ for all α and s. Furthermore, g may even be defined *during* the construction (as in Theorem 9.1) providing that for all α and s, $g(\alpha, s)$ is defined by the end of stage s.

We order the set $\widehat{L}_\alpha$ of α-witnesses so that for all α and every i, $1 \leq i \leq g(\alpha, s)$, we attempt to define a *primary* witness $\hat{y}_{\alpha,i,s}$ and a *backup* witness $\hat{y}'_{\alpha,i,s}$. (The intuition is explained in the example in §8.1.) First we divide the witness set $\widehat{L}_\alpha$ into the disjoint union of subsets $\widehat{L}'_{\alpha,i}$, $1 \leq i$, such that $\widehat{L}'_{\alpha,i,s}$ is the set of elements in $\widehat{L}_{\alpha,i}$ at the end of stage s, and $\widehat{L}'_{\alpha,i,s} = \{\hat{y}_{\alpha,i,s}, \hat{y}'_{\alpha,i,s}\}$ if these are defined. We define $\widehat{L}'_{\alpha,i,s}$, $\hat{y}_{\alpha,i,s}$, and $\hat{y}'_{\alpha,i,s}$, by induction on s as follows.

Definition 7.1. (i) If $\hat{x} \in \widehat{L}_{\alpha,s+1} - \widehat{L}_{\alpha,s}$ (necessarily because Step $\widehat{7}$ Case 1 applies at stage $s+1$ so there will be at most one such $\hat{x}$), then let i be the least $j \geq 1$ such that $|\widehat{L}'_{\alpha,j,s}| < 2$. Put $\hat{x}$ in $\widehat{L}'_{\alpha,i,s+1}$.

(ii) $\hat{x}$ remains in $\widehat{L}'_{\alpha,i}$ until, if ever, $\hat{x}$ is removed from $\widehat{L}_\alpha$, at which time $\hat{x}$ is also removed from $\widehat{L}'_{\alpha,i}$.

(iii) Define $\hat{y}_{\alpha,i,s}$ and $\hat{y}'_{\alpha,i,s}$ by

$$\hat{y}_{\alpha,i,s} = (\mu\hat{x})[\hat{x} \in \widehat{L}'_{\alpha,i,s}], \text{ and } \hat{y}'_{\alpha,i,s} = (\mu\hat{x})[\hat{x} > \hat{y}_{\alpha,i,s} \ \ \& \ \ \hat{x} \in \widehat{L}'_{\alpha,i,s}], \tag{47}$$

if these elements exist.

Step $\hat{7}$. (Putting $\hat{x}$ into $\widehat{L}_\alpha$.) Assume α is consistent as defined in Definition 6.3.
Case 1. If $1 \leq i \leq g(\alpha, s)$, $|\widehat{L}'_{\alpha,i,s}| < 2$, and there exists $\hat{x} \in \widehat{S}^0_{\alpha,s}$ such that
(7.1) $\nu(\alpha, \hat{x}, s) \in \widehat{\mathcal{C}}_\alpha$, and
(7.2) $\hat{x} > \max(\bigcup_{t \leq s} \widehat{L}_{\alpha,t})$,
then put the least such $\hat{x}$ into $\widehat{L}'_{\alpha,i,s+1}$, for the least such i.
Case 2. For all $i > g(\alpha, s)$, remove $\hat{y}_{\alpha,i,s}$ and $\hat{y}'_{\alpha,i,s}$ from $\widehat{L}_\alpha$ and from $\widehat{S}^0_\alpha$, and put them in $\widehat{S}^1_\alpha$.

Lemma 7.2. *Assume that the construction of* §3 *is performed but also with Step* 6 *and Step* $\hat{7}$ *and perhaps with additional Steps* n $(\hat{n})$, $8 \leq n < 11$. *Let* $g(\alpha, s)$ *be the function for Step* $\hat{7}$. *Suppose* $(\forall\gamma \subset f)[\liminf_s \ g(\gamma, s) < \infty]$. *Then for all* $\alpha \in T$, $S^0_{\alpha,\infty} = \emptyset$, *and* $\widehat{S}^0_{\alpha,\infty} =^* \emptyset$, *so conditions* (P4) *and* ($\widehat{\text{P}}$4) *of the Automorphism Theorem* 4.2 *are satisfied.*

Proof. There is no Step 7 so $L_{\alpha,s} = \emptyset$ for all s, and every element $x \in S^0_{\alpha,s}$ is eventually removed from S^0_α by Step 11D, so $S^0_{\alpha,\infty} = \emptyset$. Hence, condition (P4) is satisfied. If $\alpha \not\subset f$, then $\widehat{S}_{\alpha,\infty} =^* \emptyset$ by Lemma 5.3. Now consider $\alpha \subset f$. Let $g(\alpha) = \liminf_s g(\alpha, s)$. By Lemma 5.6, Step $\hat{7}$ infinitely often has an opportunity to act. By Step $\hat{7}$ Case 2, $|\widehat{L}_{\alpha,\infty}| \leq 2g(\alpha)$, and hence by Step 11D, $|\widehat{S}^0_{\alpha,\infty}| \leq 2g(\alpha)$. Thus, $\widehat{S}^0_{\alpha,\infty} =^* \emptyset$, and condition ($\widehat{\text{P}}$4) is satisfied. □

Note that if $g(\alpha, s) = m$, for all α and s (for example, $m = 1$ in Theorem 1.3 of §8), then $|\widehat{L}_{\alpha,s}| \leq 2m$ for all s and hence Case 2 of Step $\hat{7}$ will never apply.

7.2. **Coding states $\widehat{\mathcal{C}}_\alpha$ and Step $\hat{8}$.** In this section we use the coding states $\widehat{\mathcal{C}}_\alpha$ to produce a strategy formalized in Step $\hat{8}$ below for moving $\hat{x} \in \widehat{J}_\alpha$ into B. Assume $\alpha \subset f$. Since $\mathcal{C}_\alpha \neq \emptyset$ by Lemma 6.4, we have by (12), (21), and (22) that $\widehat{\mathcal{C}}_\alpha \neq \emptyset$ where $\widehat{\mathcal{C}}_\alpha = \{\hat{\nu} : \nu \in \mathcal{C}_\alpha\}$. Choose any $\hat{\nu}_1 \in \widehat{\mathcal{C}}_\alpha$. By the dual of Definition 6.2 and (13) we have,

$$\hat{\nu}_1 \in \widehat{\mathcal{W}}^{\#}_\alpha, \tag{48}$$

$$\neg(\exists \hat{\nu}_2 \in \widehat{\mathcal{M}}_\alpha)[\hat{\nu}_1 <_R \hat{\nu}_2], \tag{49}$$

$$\hat{\nu}_1 \notin \widehat{\mathcal{N}}_\alpha =_{\mathrm{dfn}} \widehat{\mathcal{R}}_\alpha \cup \widehat{\mathcal{B}}_\alpha. \tag{50}$$

If $\nu(\alpha, \hat{x}, s) = \hat{\nu}_1 \in \widehat{\mathcal{C}}_\alpha$, then by (49) $\hat{\nu}_1$ is maximal with respect to α-legal red moves so RED cannot change the α-state of $\hat{x}$, and by (50), $\hat{\nu}_1 \notin \widehat{\mathcal{B}}_\alpha$ so BLUE does not have to change the state; and hence BLUE can hold $\hat{x}$ in α-state $\hat{\nu}_1$ forever if he chooses. However, BLUE can later force $\hat{x}$ into B as follows. By the dual of Definition 6.1 and the remarks following it, if $\hat{\nu}_2 = \langle \alpha, \sigma_2, \tau_2 \rangle \in \widehat{\mathcal{W}}^{\#}_\alpha$, then

$$0 \notin \sigma_2 \ \ (\text{so if } \nu(\alpha, \hat{x}, s) = \hat{\nu}_2, \text{ then } \hat{x} \notin B_s), \ \text{ and} \tag{51}$$

$$\begin{gathered}\text{BLUE has a winning strategy, } \widehat{F}_\alpha, \text{ to force} \\ \text{any element } \hat{x} \text{ in } \alpha\text{-state } \hat{\nu}_2 \text{ into } B.\end{gathered} \tag{52}$$

Namely, by the dual of (45), we have

$$(\forall \hat{\nu}_2 \in (\widehat{\mathcal{W}}^{\#}_\alpha - \widehat{\mathcal{R}}_\alpha))[\hat{\nu}_2 <_B \widehat{F}_\alpha(\hat{\nu}_2) \in \widehat{\mathcal{W}}_\alpha]. \tag{53}$$

By repeatedly applying $\widehat{F}_\alpha$ if necessary we may assume in (53) that $\widehat{F}_\alpha(\hat{\nu}_3) \notin \widehat{\mathcal{B}}_\alpha$. Hence, from now on we may assume that the target function $\hat{h}_\alpha$ for (29) used in Step $\hat{5}_\alpha$ agrees with $\widehat{F}_\alpha$ on their common domain, namely,

$$(\forall \hat{\nu}_2 \in (\widehat{\mathcal{W}}^{\#}_\alpha \cap \widehat{\mathcal{B}}_\alpha)\,)\,[\hat{\nu}_2 <_B \hat{h}_\alpha(\hat{\nu}_2) = \widehat{F}_\alpha(\hat{\nu}_2) \in \widehat{\mathcal{W}}_\alpha - \widehat{\mathcal{B}}_\alpha], \tag{54}$$

so that while $\hat{x} \in \widehat{S}_\alpha$ any BLUE enumeration under Step $\hat{5}_\alpha$ Case 1 automatically follows strategy $\widehat{F}_\alpha$.

Step $\hat{8}$. (To move $\hat{x} \in \widehat{J}_\alpha$ toward B.) Suppose

(8.1) $\hat{x} \in \widehat{J}_{\alpha,s} - B_s$, and

(8.2) $\nu(\alpha, \hat{x}, s) = \hat{\nu}_1 \in \widehat{\mathcal{W}}^{\#}_\alpha - \widehat{\mathcal{R}}_\alpha$.

Action. Choose the least such pair $\langle \alpha, s \rangle$. Let $\widehat{F}_\alpha(\hat{\nu}_1) = \hat{\nu}_2 = \langle \alpha, \widehat{\sigma}_2, \widehat{\tau}_2 \rangle$. (Necessarily $\widehat{F}_\alpha(\hat{\nu}_1)\downarrow$ because $\hat{\nu}_1 \in \widehat{\mathcal{W}}^{\#}_\alpha$.)

(8.3) Enumerate $\hat{x}$ in $\widehat{U}_{\delta,s+1}$ for all $\delta \subseteq \alpha$ such that $\hat{e}_\delta \in \widehat{\sigma}_2$.

(8.4) If $\hat{x} \in B_{s+1} - B_s$, then move $\hat{x}$ from $\widehat{S}^0_\alpha$ to $\widehat{S}^1_\alpha$, and let $\hat{x}$ be cancelled as an α-witness (*i.e.*, remove $\hat{x}$ from $\widehat{L}_\alpha$, and hence from $\widehat{J}_\alpha$).

Clearly, Step $\hat{8}$ satisfies ($\widehat{\mathrm{P}}$1)–($\widehat{\mathrm{P}}$4) of the Automorphism Theorem 4.2 because $\widehat{\mathcal{W}}_\alpha \subseteq \widehat{\mathcal{M}}_\alpha$ so ($\widehat{\mathrm{P}}$2) is satisfied; the others are obvious.

7.3. **The Coding Theorem.**

Definition 7.3. Let the *basic coding construction* denote the construction in §3 (consisting of Steps 1–5, $\hat{1}$–$\hat{5}$, and 11) but also with Step 6, Step $\hat{7}$, and Step $\hat{8}$ (as defined in §6, §7.1, and §7.2, respectively).

Definition 7.4. For the following theorem define $t(\alpha, i)$ by

$$t(\alpha, i) = \begin{cases} (\mu t)(\forall s \geq t)[i \leq g(\alpha, s)] & \text{if } t \text{ exists,} \\ \infty & \text{otherwise.} \end{cases}$$

Theorem 7.5 (Coding Theorem)**.** *Let $A = U_0$ be a given nonrecursive r.e. set, and let $B = \widehat{U}_\rho$ where $\rho = f \restriction 1$. Let $g(\alpha, s)$ be a recursive function (to be used in Step $\hat{7}$). Perform the basic coding construction consisting of Steps* 1–6, 11, $\hat{1}$–$\hat{5}$, $\hat{7}$, $\hat{8}$, *and possibly with additional Steps $\hat{n}$, $9 \leq n < 11$, defined later which satisfy conditions* ($\widehat{\mathrm{P}}$1)–($\widehat{\mathrm{P}}$3) *from the Automorphism Theorem* 4.2.

(i) $(\forall \gamma \subseteq f)[\liminf_s g(\gamma, s) < \infty] \implies A$ *is Δ^0_3-automorphic to B.*

In addition, if the Steps $\hat{n}$, $9 \leq n < 11$, satisfy the following conditions ($\widehat{\mathrm{Q}}$1)–($\widehat{\mathrm{Q}}$4)*, then conclusions* (ii)–(viii) *hold.*

($\widehat{\mathrm{Q}}$1) *Step $\hat{n}$ may not put any element $\hat{x}$ into the witness set $\widehat{L}_\alpha$.*

($\widehat{\mathrm{Q}}$2) *Step $\hat{n}$ may not remove any element $\hat{x}$ from the witness set $\widehat{L}_\alpha$.*

($\widehat{\mathrm{Q}}$3) *If Step $\hat{n}$ puts $\hat{x}$ into $\widehat{J}_{\alpha,s+1} - \widehat{J}_{\alpha,s}$, then $\hat{x} \in \widehat{L}_{\alpha,s}$, and Step $\hat{n}$ may not remove any element $\hat{x}$ from $\widehat{J}_\alpha$.*

($\widehat{\mathrm{Q}}$4) *If $\hat{x} \in \widehat{S}^0_{\alpha,s}$, then Step $\hat{n}$ may not enumerate $\hat{x} \in \widehat{U}_{\alpha,s+1} - \widehat{U}_{\alpha,s}$ for any blue set $\widehat{U}_\alpha$.*

Assume $\alpha \subset f$, $\alpha \neq \lambda$. Choose v_α such that for all $s \geq v_\alpha$, α is not initialized and no $\beta <_L \alpha$ acts at stage s. Then for all $\hat{x}$ and s and all $i \geq 1$,

(ii) $\widehat{J}_{\alpha,s} \subseteq \widehat{L}_{\alpha,s} \subseteq \widehat{S}^0_{\alpha,s}$, *and $\widehat{L}_\alpha$ and $\widehat{J}_\alpha$ are d.r.e. sets;*

(iii) $[\hat{x} \in (\widehat{S}^0_{\alpha,s} - \widehat{J}_{\alpha,s})\ \&\ \nu(\alpha, \hat{x}, s) = \hat{\nu}_1 \in \widehat{\mathcal{C}}_\alpha] \implies \nu(\alpha, \hat{x}, s+1) = \hat{\nu}_1$;

(iv) $\hat{x} \in \widehat{L}_{\alpha,s} - \widehat{J}_{\alpha,s} \implies [\nu(\alpha, \hat{x}, s) \in \widehat{\mathcal{C}}_\alpha\ \&\ \hat{x} \in \overline{B}_s]$;

(v) $\hat{x} \in \widehat{J}_{\alpha,s} - B_s \implies \nu(\alpha, \hat{x}, s) \in \widehat{\mathcal{W}}^\#_\alpha$;

(vi) $[s \geq \max\{v_\alpha, t(\alpha, i)\}\ \&\ \hat{y}_{\alpha,i,s} \in \widehat{J}_{\alpha,s}] \implies \hat{y}_{\alpha,i,s} \in B$;

(vii) $i \leq \liminf_s g(\alpha, s) \implies (\exists^\infty s)[\hat{y}_{\alpha,i,s}\downarrow\ \&\ \nu(\alpha, \hat{y}_{\alpha,i,s}, s) \in \widehat{\mathcal{C}}_\alpha]$;

(viii) $[i \leq \liminf_s g(\alpha, s)\ \&\ (\exists^{<\infty} s)[\hat{y}_{\alpha,i,s} \in B_{s+1} - B_s]] \implies [\lim_s \hat{y}_{\alpha,i,s} < \infty]$.

In addition, if Steps $\hat{n}$, $9 \leq n < 11$, satisfy the following condition ($\widehat{\mathrm{Q}}$5)*, then conclusion* (ix) *holds for α and i as above.*

($\widehat{\mathrm{Q}}$5) *Step $\hat{n}$ may not put $\hat{y}'_{\alpha,i,s}$ into $\widehat{J}_{\alpha,s+1} - \widehat{J}_{\alpha,s}$, and may put $\hat{y}_{\alpha,i,s}$ into $\widehat{J}_{\alpha,s+1} - \widehat{J}_{\alpha,s}$ only if $\hat{y}'_{\alpha,i,s}$ is defined.*

(ix) $i \leq \liminf_s g(\alpha, s) \implies [\,(a.e.\ s)[\hat{y}_{\alpha,i,s}\downarrow]\ \&\ (\exists^\infty s)[\hat{y}_{\alpha,i,s}\downarrow\ \&\ \hat{y}'_{\alpha,i,s}\downarrow]]$.

Proof. (i) Clearly, Step 6, Step $\hat{7}$, and Step $\hat{8}$ satisfy conditions (P1)–(P3) and ($\widehat{\mathrm{P}}$1)–($\widehat{\mathrm{P}}$3) of the Automorphism Theorem 4.2. By hypothesis Steps $\hat{n}$, $9 \leq n < 11$, satisfy conditions ($\widehat{\mathrm{P}}$1)–($\widehat{\mathrm{P}}$3) also. By the hypothesis in (i), $\liminf_s g(\alpha, s) < \infty$, so by Lemma 7.2, conditions (P4) and ($\widehat{\mathrm{P}}$4) are satisfied also. Hence, A is automorphic to B by the Automorphism Theorem 4.2.

(ii) By ($\widehat{\mathrm{Q}}$1) $\hat{x}$ can only enter $\widehat{L}_\alpha$ under Step $\hat{7}$, and hence only while $\hat{x} \in \widehat{S}^0_\alpha$. If $\hat{x}$ leaves $\widehat{L}_\alpha$, then $\hat{x}$ can never reenter $\widehat{L}_\alpha$ by (7.2) of Step $\hat{7}$, and by Step 11D, $\hat{x}$ eventually leaves $\widehat{S}^0_\alpha$ and never reenters. By ($\widehat{\mathrm{Q}}$3) $\hat{x}$ can enter $\widehat{J}_\alpha$ only while $\hat{x} \in \widehat{L}_\alpha$,

and $\hat{x}$ will later leave $\widehat{J}_\alpha$ exactly when $\hat{x}$ leaves $\widehat{L}_\alpha$. Hence, $\widehat{J}_{\alpha,s} \subseteq \widehat{L}_{\alpha,s} \subseteq \widehat{S}^0_{\alpha,s}$, and $\widehat{L}_\alpha$ and $\widehat{J}_\alpha$ are d.r.e.

(iii) Suppose $\hat{x} \in (\widehat{S}^0_{\alpha,s} - \widehat{J}_{\alpha,s})$ and $\nu(\alpha, \hat{x}, s) = \hat{\nu}_1 \in \widehat{\mathcal{C}}_\alpha$. Steps $\hat{1}$ and $\hat{2}$ cannot apply to $\hat{x}$ at stage $s+1$ because $\hat{x} \in \widehat{S}^0_{\alpha,s}$; Step $\hat{3}$ cannot apply because $\alpha \subset f$; Step $\hat{4}$ cannot apply because $\hat{\nu}_1 \in \widehat{\mathcal{C}}_\alpha$ so (49) asserts that $\hat{\nu}$ is maximal with respect to α-legal red enumeration; Step $\hat{5}$ cannot apply because $\hat{\nu}_1 \notin \widehat{\mathcal{B}}_\alpha$ by (50); and Step $\hat{8}$ cannot apply because $\hat{x} \notin \widehat{J}_{\alpha,s}$. Hence, only some Step $\hat{n}$, $9 \leq n < 11$, or Step 11 can apply to $\hat{x}$. By hypothesis ($\widehat{\mathrm{Q}}4$), Step $\hat{n}$ cannot enumerate $\hat{x}$ in any blue set $\widehat{U}_\beta$, and by condition ($\widehat{\mathrm{P}}2$), Step $\hat{n}$ cannot enumerate x in any red set V_β, and Step 11 does not cause any enumeration of x, so $\nu(\alpha, \hat{x}, s+1) = \hat{\nu}_1$. By hypothesis ($\widehat{\mathrm{Q}}2$), Step $\hat{n}$ cannot remove $\hat{x}$ from $\widehat{L}_\alpha$, so $\hat{x} \in \widehat{S}^0_{\alpha,s+1}$ unless Step 11D removes $\hat{x}$ from $\widehat{S}^0_\alpha$ because $\hat{x} \notin \widehat{L}_\alpha$.

(iv) If $\hat{x}$ enters $\widehat{L}_\alpha$ at stage $s+1$, then $\hat{x} \in \widehat{S}^0_{\alpha,s} \cap \widehat{S}^0_{\alpha,s+1}$, and $\nu(\alpha, \hat{x}, s) = \hat{\nu}_1 \in \widehat{\mathcal{C}}_\alpha$ by Step $\hat{7}$, which must apply to $\hat{x}$ by ($\widehat{\mathrm{Q}}1$). Hence, by the argument in (iii), while $\hat{x}$ remains in $\widehat{L}_{\alpha,t} - \widehat{J}_{\alpha,t}$, it remains in α-state $\hat{\nu}_1$, and in $\widehat{S}^0_{\alpha,t+1}$ because Step 11D cannot apply to $\hat{x}$. If $\nu(\alpha, \hat{x}, t) = \hat{\nu}_1$, then $\hat{x} \notin B_t$ by the duals of the Definitions 6.2 and 6.1 which define $\widehat{\mathcal{C}}_\alpha$ and $\widehat{\mathcal{W}}^\#_\alpha$ respectively.

(v) If $\hat{x}$ enters $\widehat{J}_\alpha$ at stage s, then $\hat{x} \in \widehat{L}_{\alpha,s}$ by (ii), and $\nu(\alpha, \hat{x}, s) = \hat{\nu}_1 \in \widehat{\mathcal{C}}_\alpha \subseteq \widehat{\mathcal{W}}^\#_\alpha$ by (iii) and (iv). Fix some $t \geq s$ and assume by induction that $\hat{x} \in \widehat{J}_{\alpha,t} \cap \widehat{J}_{\alpha,t+1}$, and $\nu(\alpha, \hat{x}, t) = \hat{\nu}_1 \in \widehat{\mathcal{W}}^\#_\alpha$. Suppose $\hat{\nu}_1 \neq \hat{\nu}_2 = \nu(\alpha, \hat{x}, t+1)$. Then by ($\widehat{\mathrm{Q}}4$) and the remarks in the proof of (iii), either Step $\hat{4}$, $\hat{5}$, or $\hat{8}$ must have applied to $\hat{x}$ at stage $t+1$. If Step $\hat{4}$ applied, then $\hat{\nu}_2 \in \widehat{\mathcal{W}}^\#_\alpha$ because $\widehat{\mathcal{W}}_\alpha$ must be closed under α-legal red moves by the dual of Definition 6.1(i)(3). If Step $\hat{5}$ or Step $\hat{8}$ applied, then $\hat{\nu}_1 \notin \widehat{\mathcal{R}}_\alpha$ so $\hat{\nu}_2 = \widehat{F}_\alpha(\hat{\nu}_1) \in \widehat{\mathcal{W}}_\alpha$ by (53) and (54). If $\hat{x} \notin B_{t+1}$, then $\hat{\nu}_2 \in \widehat{\mathcal{W}}^\#_\alpha$.

(vi) Assume $\hat{y}_{\alpha,i,s} = \hat{x} \in \widehat{J}_{\alpha,s}$ for $s \geq \max\{v_\alpha, t(\alpha, i)\}$. Now $\hat{x}$ cannot be removed from $\widehat{L}_\alpha$ by Step $\hat{1}$ (1.12) or Step 11C because $s \geq v_\alpha$, cannot be removed by Step $\hat{7}$ Case 2, because $s \geq t(\alpha, i)$, and cannot be removed by Step $\hat{n}$ by ($\widehat{\mathrm{Q}}2$). Hence, $\hat{x}$ remains in $\widehat{J}_\alpha$, $\widehat{L}_\alpha$, and $\widehat{S}^0_\alpha$, until $\hat{x}$ enters B. While $\hat{x} \in \widehat{J}_{\alpha,t} - B_t$, we have $\nu(\alpha, \hat{x}, t) = \hat{\nu}_1 \in \widehat{\mathcal{W}}^\#_\alpha$ by (v). If $\hat{\nu}_1 \in \widehat{\mathcal{R}}_\alpha$, then there exists $v > t$ such that $\hat{\nu}_1 <_R \nu(\alpha, \hat{x}, v) = \hat{\nu}_2$ because Steps $\hat{5}$ and $\hat{8}$ cannot apply while $\hat{x}$ remains in state $\hat{\nu}_1$, so Step $\hat{4}$ must cause $\hat{x}$ to be enumerated in a red set (which must occur by (25) and (26) since $\alpha \subset f$). If $\hat{\nu}_1 \in \widehat{\mathcal{W}}^\#_\alpha - \widehat{\mathcal{R}}_\alpha$, then eventually Step $\hat{5}$ or $\hat{8}$ applies to $\hat{x}$ at some stage $v > t$ and causes $\hat{\nu}_1 <_B \nu(\alpha, \hat{x}, v) = \hat{\nu}_3 = \widehat{F}_\alpha(\hat{\nu}_1) \in \widehat{\mathcal{W}}_\alpha$. Since $\hat{x}$ can change α-state at most finitely often, eventually $\nu(\alpha, \hat{x}, v) = \langle \alpha, \sigma, \tau \rangle \in \widehat{\mathcal{W}}_\alpha - \widehat{\mathcal{W}}^\#_\alpha$ where $0 \in \sigma$ so $\hat{x} \in B_v$.

(vii) Choose $u \geq \max\{v_\alpha, t(\alpha, i)\}$. Now α is $\mathcal{C}$-consistent by Lemma 6.4. Also $\widehat{\mathcal{E}}^0_\alpha = \widehat{\mathcal{E}}_\alpha$. Hence, for some $\hat{\nu}_1 \in \widehat{\mathcal{C}}_\alpha$ infinitely many elements $\hat{x}$ enter $\widehat{S}^0_\alpha$ in α-state $\hat{\nu}_1$ and remain in $\widehat{S}^0_\alpha$ and in α-state $\hat{\nu}_1$ until either they enter $\widehat{L}_\alpha$ or are removed from $\widehat{S}^0_\alpha$ by Step 11D. But Step $\hat{7}$ is performed before Step 11. Hence, Step $\hat{7}$ Case 1 ensures that $\hat{y}_{\alpha,i,w} = \hat{x}$ is defined for some $w \geq u$, and some $\hat{x}$ such that $\nu(\alpha, \hat{x}, w) = \hat{\nu}_1$.

(viii) Assume $i \leq \liminf g(\alpha, s)$. By (iv) and (vi), the hypothesis $(\exists^{<\infty} s)[\hat{y}_{\alpha,i,s} \in B_{s+1} - B_s]$ is equivalent to $(\exists^{<\infty} s)[\hat{y}_{\alpha,i,s} \in \widehat{J}_{\alpha,s+1} - \widehat{J}_{\alpha,s}]$. By (vi) and (vii) choose $w > \max\{v_\alpha, t(\alpha, i)\}$ such that $\hat{y}_{\alpha,i,w} \downarrow = \hat{x}$, $\hat{\nu}(\alpha, \hat{x}, w) = \hat{\nu}_1 \in \widehat{\mathcal{C}}$, and for all $s \geq w$,

$\hat{y}_{\alpha,i,s} \notin \widehat{J}_{\alpha,s}$. Hence, by (iv) for all $s \geq w$, $\hat{y}_{\alpha,i,s} \notin B_s$. Now by the same argument as in (iii) and (iv), $\hat{x}$ remains in $\widehat{L}_\alpha - \widehat{J}_\alpha$ and in α-state $\hat{\nu}_1$ forever after stage w. Hence, $\lim_s \hat{y}_{\alpha,i,s} = \hat{x}$. If in addition we assume $(\exists^{<\infty} s)[\hat{y}'_{\alpha,i,s} \in \widehat{J}_{\alpha,s+1} - \widehat{J}_{\alpha,s}]$, then by the same proof $\lim_s \hat{y}'_{\alpha,i,s}$ exists.

(ix) Choose $u \geq \max\{v_\alpha, t(\alpha,i)\}$. By (vii), $\hat{y}_{\alpha,i,w} \downarrow = \hat{x}$ for some $w \geq u$. Either $\hat{y}_{\alpha,i,s} \downarrow = \hat{x}$ for all $s \geq w$ or else for some $v > w$, $\hat{x} \in \widehat{L}_{\alpha,v} - \widehat{L}_{\alpha,v+1}$ in which case $\hat{x} \in B_{v+1} - B_v$ by (8.4). But then $\hat{x} \in \widehat{J}_{\alpha,t+1} - \widehat{J}_{\alpha,t}$, for some t, $w < t < v$, by (iv). Hence, $\hat{y}'_{\alpha,i,t} \downarrow = \hat{x}'$ by condition ($\widehat{\mathrm{Q}}5$), and necessarily $\hat{y}'_{\alpha,i,v} \downarrow = \hat{x}'$ as well because by ($\widehat{\mathrm{Q}}1$), ($\widehat{\mathrm{Q}}2$), and the choice of w, neither $\hat{y}_{\alpha,i,s}$ nor $\hat{y}'_{\alpha,i,s}$ can change in value at any stage s, $t < s \leq v$. Thus, $\hat{y}_{\alpha,i,v+1} \downarrow = \hat{x}'$ by the definition of $\hat{y}_{\alpha,i,s+1}$ in Definition 7.1 . Hence, $\hat{y}_{\alpha,i,s} \downarrow$ for all $s \geq w$. Also $\hat{y}_{\alpha,i,t} \downarrow = \hat{x}$ and $\hat{y}'_{\alpha,i,t} \downarrow = \hat{x}'$ so the second conjunct in the conclusion of (ix) is also satisfied. □

7.4. **The Refined Coding Theorem.** The main point of the Coding Theorem 7.5 is that for applications in this and subsequent papers we may view it as a kind of "black box" with inputs $g(\alpha,s)$ and $\widehat{J}_\alpha$ and output $\widehat{L}_\alpha$, which we can apply without knowing anything about the internal workings of the basic coding machinery from §1–§6, §7.1, §7.2 (such as Steps 1-6, $\hat{1}$–$\hat{8}$, and 11, $\widehat{S}^0_\alpha$, etc.), but only the material from §7.3. The construction can thus be split into two parts performed simultaneously, the first (the basic coding construction) done by the "automorphism builder" and the second done by the "coder". The coder gives up direct control over enumerating elements into B but can enumerate into B indirectly by putting elements of $\widehat{L}_\alpha$ into $\widehat{J}_\alpha$.

For an intended application (such as Theorem 1.3 in §8 or Theorem 9.1 in §9) the coder specifies, as additional input to the basic coding construction, additional Steps $\hat{n}$, $9 \leq n < 11$, satisfying conditions ($\widehat{\mathrm{P}}1$)–($\widehat{\mathrm{P}}3$) and ($\widehat{\mathrm{Q}}1$)–($\widehat{\mathrm{Q}}5$), which are easy to verify, and a recursive function $g(\alpha,s)$. By Theorem 7.5 the basic coding construction will produce a set $\widehat{L}_\alpha$ of α-witnesses labeled as $\hat{y}_{\alpha,i,s}$ and $\hat{y}'_{\alpha,i,s}$ (according to Definition 7.1) such that by Theorem 7.5(ix) and (viii) if $1 \leq i \leq \liminf_s g(\alpha,s)$, then $\hat{y}_{\alpha,i,s} \downarrow$ for almost every s, and if the coder puts $\hat{y}_{\alpha,i,s} \in \widehat{J}_{\alpha,s+1} - \widehat{J}_{\alpha,s}$ for at most finitely many s, then $\lim_s \hat{y}_{\alpha,i,s}$ and $\lim_s \hat{y}'_{\alpha,i,s}$ exist.

Assume $\alpha \subset f$. Choose v_α such that for all $s \geq v_\alpha$, α is not initialized and no $\beta <_L \alpha$ acts at stage s. Let $\hat{y}_{\alpha,i,s} = \hat{x}$, $s > v_\alpha$, for $1 \leq i \leq \liminf_s g(\alpha,s)$ and $s > \max\{v_\alpha, t(\alpha,i)\}$. By withholding $\hat{x}$ from the set $\widehat{J}_\alpha$ of *activated* witnesses, the coder can ensure by Theorem 7.5(iv) that $\hat{x}$ will not enter B. If the coder later changes his mind and inserts $\hat{x}$ in $\widehat{J}_\alpha$, then $\hat{x}$ must eventually enter B by Theorem 7.5(vi).

Finally, the coder must ensure that $\liminf_s g(\alpha,s) < \infty$, which implies that B is automorphic to A by Theorem 7.5(i). This is a significant restriction. For example, one cannot code K into B by putting $\hat{y}_{\alpha,i,s}$ into $\widehat{J}_\alpha$ exactly if $i \in K_s$ because for each $i \in \overline{K}$ one must keep $\hat{y}_{\alpha,i,s} \in \widehat{L}_\alpha - \widehat{J}_\alpha$, which would cause $\liminf_s g(\alpha,s) = \infty$. (By the main result of Harrington and Soare [8] we know that we cannot always achieve $K \leq_{\mathrm{T}} B$.) Nevertheless, the restriction $\liminf_s g(\alpha,s) < \infty$ still allows a lot of information to be coded into B as we shall see in Theorems 1.3, 9.1, and 1.4.

We now wish to reformulate the Coding Theorem 7.5 so that it formally expresses this intuition but in such a way that it is self-contained and can be cited in subsequent papers without knowledge of the rest of this paper analogously as the

Extension Theorem in [26] was cited in subsequent papers on effective automorphisms. (We still need the full Coding Theorem 7.5 in §10 and §11.)

Theorem 7.6 (Refined Coding Theorem). *Let $A = U_0$ be a given nonrecursive r.e. set. Perform the basic coding construction (consisting of Steps 1–6, 11, $\hat{1}$–$\hat{5}$, $\hat{7}$, $\hat{8}$), and possibly with additional Steps $\hat{n}$, $9 \le n < 11$, which we may specify later, and which may be performed at any stage during the construction, but which must satisfy condition ($\widehat{\mathrm{R}}1$) below. Let T be the priority tree of the construction, f the true path through T, and $\{f_s\}_{s\in\omega}$ the recursive approximation to f so that $f = \liminf_s f_s$. Let $B = \widehat{U}_\rho$ where $\rho = f \restriction 1$, and $B_s = \widehat{U}_{\rho,s}$. Let $g(\alpha, s)$ be a recursive function which we may define during the construction but such that for all $\alpha \in T$, $g(\alpha, s)$ is defined by the end of stage s. For $\alpha \in T$ and $i \in \omega$, define $t(\alpha, i)$ by*

$$t(\alpha,i) = \begin{cases} (\mu t)(\forall s \ge t)[i \le g(\alpha,s)] & \text{if } t \text{ exists,} \\ \infty & \text{otherwise.} \end{cases}$$

For every $\alpha \in T$, $\alpha \ne \lambda$ (the empty node on T), the construction will produce a d.r.e. set of α-witnesses, $\widehat{L}_\alpha = \lim_s \widehat{L}_{\alpha,s}$, and pairwise disjoint subsets $\widehat{L}'_{\alpha,i,s} \subseteq \widehat{L}_{\alpha,s}$ such that $|\widehat{L}'_{\alpha,i,s}| \le 2$ and $\widehat{L}_{\alpha,s} = \bigcup \{\widehat{L}'_{\alpha,i,s} : i \le g(\alpha,s)\}$, and from which $\hat{y}_{\alpha,i,s}$ and $\hat{y}'_{\alpha,i,s}$ are defined by

$$\hat{y}_{\alpha,i,s} = (\mu\hat{x})[\hat{x} \in \widehat{L}'_{\alpha,i,s}], \text{ and } \hat{y}'_{\alpha,i,s} = (\mu\hat{x})[\hat{x} > \hat{y}_{\alpha,i,s} \ \ \& \ \ \hat{x} \in \widehat{L}'_{\alpha,i,s}],$$

if these elements exist. From $\widehat{L}_\alpha$ we may select a subset $\widehat{J}_\alpha = \lim_s \widehat{J}_{\alpha,s}$ of activated *α-witnesses using the additional Steps $\hat{n}$, $9 \le n < 11$, providing that these steps satisfy the following property ($\widehat{\mathrm{R}}1$).*

($\widehat{\mathrm{R}}1$) *If $\hat{x} \in \widehat{L}_{\alpha,s}$, then Step $\hat{n}$ may put $\hat{x}$ in $\widehat{J}_{\alpha,s+1} - \widehat{J}_{\alpha,s}$. Step $\hat{n}$ may not remove $\hat{x}$ from $\widehat{L}_\alpha$ or $\widehat{J}_\alpha$, or add $\hat{x}$ to $\widehat{L}_\alpha$. (It is understood that Step $\hat{n}$ may not perform any other action which would affect the automorphism machinery but Step $\hat{n}$ may perform additional external action, such as defining a use function $\psi^{B_s}(j)$.)*

Assume $\alpha \subset f$, $\alpha \ne \lambda$. Choose v_α such that for all $s \ge v_\alpha$, α is not initialized and no $\beta <_L \alpha$ acts at stage s. Then for all $\hat{x}$ and s and all $i \ge 1$,

(i) $(\forall\gamma \subseteq f)[\liminf_s\ g(\gamma,s) < \infty] \implies$ *A is Δ^0_3-automorphic to B;*
(ii) $\hat{x} \in \widehat{L}_{\alpha,s} - \widehat{J}_{\alpha,s} \implies \hat{x} \in \overline{B}_s$;
(iii) $[s \ge \max\{v_\alpha, t(\alpha,i)\} \ \ \& \ \ \hat{y}_{\alpha,i,s} \in \widehat{J}_{\alpha,s}] \implies (\exists t > s)[\hat{y}_{\alpha,i,s} \in B_t]$;
(iv) $i \le \liminf_s g(\alpha,s) \implies (\exists^\infty s)[\hat{y}_{\alpha,i,s}\downarrow]$;
(v) $[i \le \liminf_s g(\alpha,s) \ \ \& \ \ (\exists^{<\infty} s)[\hat{y}_{\alpha,i,s} \in B_{s+1} - B_s]] \implies [\lim_s \hat{y}_{\alpha,i,s} < \infty]$.

In addition, if Steps $\hat{n}$, $9 \le n < 11$, satisfy the following condition ($\widehat{\mathrm{R}}2$), then conclusion (vi) *holds for α and i as above.*

($\widehat{\mathrm{R}}2$) *Step $\hat{n}$ may not put $\hat{y}'_{\alpha,i,s}$ into $\widehat{J}_{\alpha,s+1} - \widehat{J}_{\alpha,s}$, and may put $\hat{y}_{\alpha,i,s}$ into $\widehat{J}_{\alpha,s+1} - \widehat{J}_{\alpha,s}$ only if $\hat{y}'_{\alpha,i,s}$ is defined.*

(vi) $i \le \liminf_s g(\alpha,s) \implies [\ (a.e.\ s)[\hat{y}_{\alpha,i,s}\downarrow] \ \ \& \ \ (\exists^\infty s)[\hat{y}_{\alpha,i,s}\downarrow \ \ \& \ \ \hat{y}'_{\alpha,i,s}\downarrow]]$.

Proof. If Steps $\hat{n}$, $9 \le n < 11$, satisfy condition ($\widehat{\mathrm{R}}1$), then they satisfy conditions ($\widehat{\mathrm{P}}1$)–($\widehat{\mathrm{P}}3$) and conditions ($\widehat{\mathrm{Q}}1$)–($\widehat{\mathrm{Q}}4$). Note that condition ($\widehat{\mathrm{Q}}5$) is ($\widehat{\mathrm{R}}2$). Apply the Coding Theorem 7.5.

Note that Step $\hat{n}$, $9 \le n < 11$, may be performed at *any* stage, unlike the other Steps m ($\hat{m}$), $m \le 8$ or $m = 11$, which must be performed in the order specified in §3, *i.e.*, are performed at stage $s+1$ only if no Step k ($\hat{k}$), $k < m$, wants to act.

The reason here is that the action of Step $\hat{n}$, $9 \leq n < 11$, is entirely *external* to the automorphism construction, since by condition ($\widehat{\mathrm{R}}1$) the Step $\hat{n}$ can merely put some element $\hat{x}$ of $\widehat{L}_\alpha$ into $\widehat{J}_\alpha$, indicating a desire that $\hat{x}$ begin its journey toward B. However, $\widehat{J}_\alpha$ is *not* a set internal to the automorphism machinery, so the journey of $\hat{x}$ does not actually begin until some later stage $t+1$ when Step $\hat{8}$ recognizes that $\hat{x}$ is in $\widehat{J}_{\alpha,t} - B_t$, and changes the α-state of $\hat{x}$ from $\hat{\nu}_1 \in \widehat{\mathcal{C}}_\alpha$ to $\hat{\nu}_2 = \widehat{F}_\alpha(\hat{\nu}_1)$.

It will be crucial in later applications such as Theorem 1.4 proved in [10] that we allow these new Steps $\hat{n}$, $9 \leq n < 11$, to be performed at *any* stage in the construction because timing is crucial for them. However, notice that Step $\hat{n}$ can perform at most finitely much action on the finitely many elements $\hat{x} \in \widehat{Y}_{\alpha,s}$ so Lemma 5.6 still applies even if we still insist that Step 11 apply only when no other step (including Step $\hat{n}$) wants to act. □

Remark 7.7. For the special case of $\alpha = \rho =_{\mathrm{dfn}} f \restriction 1$ in the Refined Coding Theorem 7.6(iii) if $\hat{y}_\rho$ enters $\widehat{J}_\rho$, then $\hat{y}_\rho$ definitely enters B unless $\hat{y}_\rho$ is removed from $\widehat{L}_\rho$ because either $f_s <_L \rho$ (which happens at most finitely often) or $g(\rho, s)$ decreases sufficiently to cancel $\hat{y}_\rho$ under Step $\hat{7}$ Case 2. Here $U_0 = A$ and $\widehat{U}_\rho = B$. For $\alpha = \rho$ there are only two ρ-states $\nu_0 = \langle \rho, \sigma_0, \emptyset \rangle$ and $\nu_1 = \langle \rho, \sigma_1, \emptyset \rangle$, where $\sigma_0 = \emptyset$ and $\sigma_1 = \{0\}$ represent the states $x \notin U_0$ and $x \in U_0$ respectively. Both ν_0 and ν_1 are in $\mathcal{M}_\rho$ because A is nonrecursive, and hence both $\hat{\nu}_0$ and $\hat{\nu}_1$ are in $\widehat{\mathcal{M}}_\rho$, by (12). While $\hat{y}_\rho \in \widehat{L}_\rho - \widehat{J}_\rho$ necessarily $\hat{y}_\rho$ lies in ρ-state $\hat{\nu}_0$, the only ρ-state in $\overline{B}$. When $\hat{y}_\rho$ enters $\widehat{J}_\rho$, then at the next application of Step $\widehat{8}$ we move $\hat{y}_\rho$ into ρ-state $\hat{\nu}_1$, and thus into B. (The second action does not necessarily happen exactly simultaneously unless we make a slight change in our construction for the special case of ρ by performing any Step 8 ρ-action before action for any $\gamma \neq \rho$, but for later coding applications [10] it is enough to know that $\hat{y}_\rho$ enters B after at most a small delay.)

7.5. **The Second Refined Coding Theorem.** In the Refined Coding Theorem 7.6 if the function g satisfies $g(\alpha, s) = g(\alpha)$ for all s, then the statement can be simplified further as we now state for easy reference in later papers.

Theorem 7.8 (Second Refined Coding Theorem)**.** *Let $A = U_0$ be a given nonrecursive r.e. set, and g a recursive function. Perform the basic coding construction (consisting of Steps* 1–6, 11, $\hat{1}$–$\hat{5}$, $\hat{7}$, $\hat{8}$*), and possibly with additional Steps $\hat{n}$, $9 \leq n < 11$, which we may specify later, and which may be performed at any stage during the construction, but which must satisfy condition* ($\widehat{\mathrm{R}}1$) *below. Let T be the priority tree of the construction, f the true path through T, and $\{f_s\}_{s\in\omega}$ the recursive approximation to f so that $f = \liminf_s f_s$. Let $B = \widehat{U}_\rho$ where $\rho = f \restriction 1$, and $B_s = \widehat{U}_{\rho,s}$. For every $\alpha \in T$, $\alpha \neq \lambda$ (the empty node on T), the construction will produce a d.r.e. set of* α-witnesses, $\widehat{L}_\alpha = \lim_s \widehat{L}_{\alpha,s}$, *and pairwise disjoint subsets $\widehat{L}'_{\alpha,i,s} \subseteq \widehat{L}_{\alpha,s}$ such that $|\widehat{L}'_{\alpha,i,s}| \leq 2$ and $\widehat{L}_{\alpha,s} = \bigcup \{\widehat{L}'_{\alpha,i,s} : i \leq g(\alpha)\}$, and from which $\hat{y}_{\alpha,i,s}$ and $\hat{y}'_{\alpha,i,s}$ are defined by*

$$\hat{y}_{\alpha,i,s} = (\mu\hat{x})[\hat{x} \in \widehat{L}'_{\alpha,i,s}], \text{ and } \hat{y}'_{\alpha,i,s} = (\mu\hat{x})[\hat{x} > \hat{y}_{\alpha,i,s} \ \ \& \ \ \hat{x} \in \widehat{L}'_{\alpha,i,s}],$$

if these elements exist. From $\widehat{L}_\alpha$ we may select a subset $\widehat{J}_\alpha = \lim_s \widehat{J}_{\alpha,s}$ of activated *α-witnesses using the additional Steps $\hat{n}$, $9 \leq n < 11$, providing that these steps satisfy the following property* ($\widehat{\mathrm{R}}1$).

($\widehat{\mathrm{R}}$1) *If $\hat{x} \in \widehat{L}_{\alpha,s}$, then Step $\hat{n}$ may put $\hat{x}$ in $\widehat{J}_{\alpha,s+1} - \widehat{J}_{\alpha,s}$. (Step $\hat{n}$ may not remove $\hat{x}$ from $\widehat{L}_\alpha$ or $\widehat{J}_\alpha$, or add $\hat{x}$ to $\widehat{L}_\alpha$.)*

Assume $\alpha \subset f$, $\alpha \neq \lambda$. Choose v_α such that for all $s \geq v_\alpha$, α is not initialized and no $\beta <_L \alpha$ acts at stage s. Then for all $\hat{x}$ and s and all $i \geq 1$,

(i) *A is Δ^0_3-automorphic to B;*

(ii) $\hat{x} \in \widehat{L}_{\alpha,s} - \widehat{J}_{\alpha,s} \implies \hat{x} \in \overline{B}_s$;

(iii) $[s \geq v_\alpha \ \&\ \hat{y}_{\alpha,i,s} \in \widehat{J}_{\alpha,s}] \implies (\exists t > s)[\hat{y}_{\alpha,i,s} \in B_t]$;

(iv) $i \leq g(\alpha) \implies (\exists^\infty s)[\hat{y}_{\alpha,i,s}\downarrow]$;

(v) $[i \leq g(\alpha) \ \&\ (\exists^{<\infty} s)[\hat{y}_{\alpha,i,s} \in B_{s+1} - B_s]] \implies [\lim_s \hat{y}_{\alpha,i,s} < \infty]$.

In addition if Steps $\hat{n}$, $9 \leq n < 11$, satisfy the following condition ($\widehat{\mathrm{R}}$2)*, then conclusion* (vi) *holds for α and i as above.*

($\widehat{\mathrm{R}}$2) *Step $\hat{n}$ may not put $\hat{y}'_{\alpha,i,s}$ into $\widehat{J}_{\alpha,s+1} - \widehat{J}_{\alpha,s}$, and may put $\hat{y}_{\alpha,i,s}$ into $\widehat{J}_{\alpha,s+1} - \widehat{J}_{\alpha,s}$ only if $\hat{y}'_{\alpha,i,s}$ is defined.*

(vi) $i \leq \liminf_s g(\alpha, s) \implies [\ (a.e.\ s)[\hat{y}_{\alpha,i,s}\downarrow] \ \&\ (\exists^\infty s)[\hat{y}_{\alpha,i,s}\downarrow \ \&\ \hat{y}'_{\alpha,i,s}\downarrow]]$.

8. The proof of Theorem 1.3

In this section we add the necessary extra steps and lemmas to the construction and verification of the Refined Coding Theorem 7.6 to prove Theorem 1.3. Specifically, we add new Steps $\hat{9}$ and $\widehat{10}$ so that $B = \widehat{U}_\rho$ is high, where $\rho = f{\restriction}1$.

8.1. **The function Ψ to make B high.** To ensure that B is high it suffices to construct a B-recursive functional $\Psi^B(i,j)$ with use function $\psi(i,j)$ such that for all i,

$$\mathrm{Inf}(i) = \lim_j \Psi^B(i,j), \tag{55}$$

where $\mathrm{Inf} = \{i : W_i \text{ is infinite}\}$. If $\alpha \subset f$ and $\alpha \equiv 0 \bmod 5$, then α will achieve (55) for $i = |\alpha|/5$. The following α-module to accomplish this has two α witnesses, the *primary* witness $\hat{y}$, and the *secondary* witness $\hat{y}'$, both in some α-state in $\widehat{\mathcal{C}}_\alpha$ when first appointed. We let p_s denote the value of the parameter p at the end of stage s (e.g. $\hat{y}_s$, $\hat{y}'_s$, $\Psi_s(i,j)$, and $\psi_s(i,j)$) and we let $p_s\downarrow$ ($p_s\uparrow$) denote that the value of parameter p is defined (undefined) at the end of stage s. It is assumed that any parameter p retains its value during stage s unless specified otherwise.

At some stage $s+1$ for certain $j < s$ if $j \geq |W_{i,s}|$ we may define $\Psi_{s+1}(i,j) = 0$ and $\psi(i,j) > \hat{y}_{s+1}$. At some later stage $t+1 > s+1$ if $j < |W_{i,t}|$ and

$$\hat{y}'_t \text{ is defined}, \tag{56}$$

we put $\hat{y}$ into $\widehat{J}_{\alpha,t+1} - \widehat{J}_{\alpha,t}$ (thus starting the journey of $\hat{y}$ toward B as in §7). When $\hat{y} \in B_{v+1} - B_v$ at some stage $v+1 \geq t+1$ we redefine $\Psi_w(i,j) = 1$ and $\psi_w(i,j) = 0$ for all stages $w \geq v+1$. We also define $\hat{y}_{v+1} = \hat{y}'_v$ ($= \hat{y}'_t$ using (56)) and define

$$\psi_{v+1}(i,k) > \hat{y}_{v+1}, \tag{57}$$

for all $k < s$ such that $\hat{y}_v < \psi_v(i,k)$ and $\Psi_{v+1}(i,k) = 0$. Hence, (57) continues to hold for all k such that $\Psi_{v+1}(i,k) = 0$ (*i.e.*, such that $\psi_{v+1}(i,k)\downarrow > 0$). We let $\hat{y}'_{v+1}$ be undefined and we later redefine $\hat{y}'$ under Step $\hat{7}$ and (47). However, such $\hat{x}$ may not appear until much later. While $\hat{y}'$ is undefined then by (56) we do not allow $\hat{y}$ to enter $\widehat{J}_\alpha$ or B. This action ensures that for $\alpha \subset f$, $(a.e.\ s)[\hat{y}_s\downarrow]$, and hence

$$(a.e.\ j)(\forall s)[\psi_s(i,j)\downarrow > 0 \implies \hat{y}_s < \psi_s(i,j)],$$

so that if later $|W_{i,t}| > j$, we can always redefine $\Psi(i,j) = 1$ by putting $\hat{y}$ into B. This guarantees (55).

8.2. **The construction for Theorem 1.3.** We use here the basic coding construction as in the Refined Coding Theorem 7.6 but with the additional Steps $\hat{9}$, $\widehat{10}$ defined below, which will clearly satisfy conditions $(\widehat{\mathrm{P}}1)$–$(\widehat{\mathrm{P}}3)$ and $(\widehat{\mathrm{Q}}1)$–$(\widehat{\mathrm{Q}}5)$. We define $g(\alpha, s) = 1$ for all α and s. Hence, we have $|\widehat{L}_{\alpha,s}| \leq 2$ for all s. Let $\hat{y}_{\alpha,s}$ and $\hat{y}'_{\alpha,s}$ denote $\hat{y}_{\alpha,1,s}$ and $\hat{y}'_{\alpha,1,s}$ respectively so

$$\hat{y}_{\alpha,s} = (\mu\hat{x})[\hat{x} \in \widehat{L}_{\alpha,s}], \text{ and } \hat{y}'_{\alpha,s} = (\mu\hat{x})[\hat{x} > \hat{y}_{\alpha,s} \ \ \& \ \ \hat{x} \in \widehat{L}_{\alpha,s}], \tag{58}$$

if these elements exist.

Step $\hat{9}$. (To define $\psi_{s+1}(i,j)$ if $\hat{y}_{\alpha,s}$ is defined.) Suppose α, i, and j are such that $|\alpha| = 5i$, $\hat{y}_{\alpha,s}\downarrow$ (and hence α is $\mathcal{C}$-consistent), and

(9.1) $\psi_s(i,j)\uparrow$, and

(9.2) $j < |R_{\alpha,s}|$.

Action. Choose the least such triple $\langle \alpha, i, j\rangle$. Define

(9.3) $\Psi_{s+1}(i,j) = \begin{cases} 1 & \text{if } j < |W_{i,s}|, \\ 0 & \text{otherwise.} \end{cases}$

(9.4) If $\Psi_{s+1}(i,j) = 1$, define $\psi_{s+1}(i,j) = 0$.

(9.5) If $\Psi_{s+1}(i,j) = 0$, define

$$\psi_{s+1}(i,j) = (\mu z)(\forall \delta \leq_L \alpha)[\hat{y}_{\delta,s+1}\downarrow \implies \hat{y}_{\delta,s+1} < \hat{z}].$$

Step $\widehat{10}$. (To activate $\hat{y}_{\alpha,s}$.) Suppose $|\alpha| = 5i$, and $\hat{y}_{\alpha,s}\downarrow$, and

(10.1) $\hat{y}_{\alpha,s} \notin \widehat{J}_{\alpha,s}$,

(10.2) $\hat{y}'_{\alpha,s}\downarrow$,

(10.3) $\hat{y}_{\alpha,s} < \psi_s(i,j)$, and

(10.4) $\Psi_s(i,j)\downarrow = 0$ and $j < |W_{i,s}|$.

Action. Put $\hat{y}_{\alpha,s}$ in $\widehat{J}_{\alpha,s+1}$.

8.3. **The verification for Theorem 1.3.** Note that condition (9.2) prevents Step $\hat{9}$ from defining $\psi(i,j)$ for more than finitely many j until a new element is added to Y_λ (namely at the next application of Step 11) so Lemma 5.6 still holds. Now Steps $\hat{9}$ and $\widehat{10}$ clearly satisfy conditions $(\widehat{\mathrm{P}}1)$–$(\widehat{\mathrm{P}}3)$ and $(\widehat{\mathrm{Q}}1)$–$(\widehat{\mathrm{Q}}5)$, so A is automorphic to B by Theorem 7.6(i). It remains to see that B is high.

Lemma 8.1. *Assume $\alpha \subset f$ and $|\alpha| = 5i$. Then*

(a.e. j) $(\forall s)[\psi_s(i,j)\downarrow > 0 \implies \hat{y}_{\alpha,s} < \psi_s(i,j)]$.

Proof. Let v_α be as in Theorem 7.6. By Theorem 7.6(vi) choose $s_1 \geq v_\alpha$ such that for all $s \geq s_1$, $\hat{y}_{\alpha,s}\downarrow$. By (9.2) $\psi_{s_1}(i,j)\uparrow$ for all $j >$ some j_0. For each $j > j_0$ Step $\hat{9}_\gamma$ will define $\psi_{s+1}(i,j)$ for some $s+1 > s_1$ and some γ, $\alpha \leq_L \gamma$, such that $|\gamma| = |\alpha| = 5i$, and by (9.5),

$$\psi_{s+1}(i,j)\downarrow = \hat{z} > 0 \implies \hat{y}_{\alpha,s+1} < \hat{z}, \tag{59}$$

if $\Psi_{s+1}(i,j) = 0$. Now if $\psi_{t+1}(i,j)$ is ever redefined at some stage $t+1 > s+1$, then (9.5) again applies at stage $t+1$ so (59) continues to hold with t in place of s. □

Lemma 8.2. *For all i and j in ω,*

(i) $\psi(i,j) = \lim_s \psi_s(i,j)$ *exists, and*

(ii) $\mathrm{Inf}(i) = \lim_j \Psi(i,j)$.

Proof. Fix i. Choose $\alpha \subset f$, $|\alpha| = 5i$. Choose s_1 as in Lemma 8.1. Hence, for every j, $\psi_s(i,j)\downarrow$ for some $s \geq s_1$ because if $\psi(i,j)$ is undefined, then Step $\hat{9}_\alpha$ eventually will apply and define it. If $j < |W_{i,s}|$ and $\psi(i,j)$ is defined or redefined at stage $s+1$, then for all $t \geq s+1$, $\Psi_t(i,j) = 1$ and $\psi_t(i,j) = 0$ by (9.3)–(9.5).

Case 1. $|W_i| < \infty$. Let $j_0 = |W_i|$, and choose $s_2 \geq s_1$ such that $|W_{i,s_2}| = j_0$. Choose $s_3 \geq s_2$ such that for all $j < j_0$ and all $s \geq s_3$, $\psi_s(i,j) = \psi_{s_3}(i,j)$. Hence, by (10.4), $\hat{y}_{\alpha,s}$ never enters $\widehat{J}_{\alpha,s+1}$ at any stage $s+1 \geq s_3$. Thus, by Theorem 7.6(v),

$$\hat{y}_\alpha = \lim_s \hat{y}_{\alpha,s} = \hat{y}_{\alpha,s_3}, \tag{60}$$

and no $\hat{y}_\beta$, $\beta <_L \alpha$, changes in value after stage s_1. For every $j \geq j_0$ as soon as $\Psi_s(i,j)\downarrow$ we have $\Psi_t(i,j) = 0$ and $\psi_t(i,j) > 0$ for all $t \geq s$ by (9.3)–(9.5). To see that $\lim_t \psi_t(i,j)$ exists, fix $j \geq j_0$ and $s_4 \geq s_3$ such that for all $s \geq s_4$, $\Psi_s(i,j) = 0$ and (9.2) holds for j. If $\psi(i,j)$ is ever redefined at any stage $s_5 + 1 > s_4$, then by (9.5) and (60), and choice of s_1, $\psi_s(i,j) = \psi_{s_5}(i,j)$ for all $s \geq s_4$. If not, then $\psi_s(i,j) = \psi_{s_4}(i,j)$ for all $s \geq s_4$.

Case 2. $|W_i| = \infty$. Clearly, $\lim_s \psi_s(i,j)$ exists for all j by the sentence preceding Case 1. To see that $\Psi(i,j) = 1$ for a.e. j, fix j such that Lemma 8.1 holds for j. Choose $t \geq s_1$ such that $j < |R_{\alpha,t}|$ and $j < |W_{j,t}|$. If $\Psi_t(i,j) = 0$, then at some stage $v+1 \geq t$ Step $\widehat{10}$ will apply to $\langle \alpha, i, j\rangle$ causing $\hat{y}_{\alpha,v}$ to enter $\widehat{J}_{\alpha,v+1}$. But by the Refined Coding Theorem 7.6(iii), $\hat{y}_{\alpha,v}$ will enter B at some stage $w+1 \geq v+1$ so by (9.4) we redefine $\psi_{w+1}(i,j) = 0$ and $\Psi_{w+1}(i,j) = 1$, and they retain these values forever. □

9. Avoiding a downward cone

The Refined Coding Theorem 7.6 yields a very short proof of the following theorem of Harrington which was announced in [28, page 379] but was never written up or published.

Theorem 9.1 (Harrington)**.** *For all r.e. sets A and C such that $\emptyset <_{\mathrm{T}} A$ and $C <_{\mathrm{T}} K$ there is an r.e. set $B \simeq_{\Delta^0_3} A$ such that $B \not\leq_{\mathrm{T}} C$.*

Proof. We shall meet for all e the requirement,

$$\{e\}^C = B \implies K \leq_{\mathrm{T}} B \oplus C.$$

Let $\{C_s\}_{s\in\omega}$ and $\{K_s\}_{s\in\omega}$ be recursive enumerations of C and K. Let B_s be as in Theorem 7.6. Define the usual length of agreement function,

$$\ell(e,s) = \max\{x : (\forall y < x)[\{e\}^{C_s}_s(y)\downarrow = B_s(y)]\}.$$

For every $\alpha \in T$, $|\alpha| = 5e$, define $g(\alpha,s) = \ell(e,s)$. Add to the basic coding construction of Theorem 7.6 the following step, which clearly satisfies conditions $(\widehat{\mathrm{P}}1)$–$(\widehat{\mathrm{P}}3)$ and $(\widehat{\mathrm{Q}}1)$–$(\widehat{\mathrm{Q}}4)$, but not necessarily $(\widehat{\mathrm{Q}}5)$.

Step $\widehat{9}$. If $|\alpha| = 5e$, $i \in K_s$, and $\hat{y}_{\alpha,i,s}\downarrow$, then put $\hat{y}_{\alpha,i,s}$ in $\widehat{J}_{\alpha,s+1}$.

Assume that $\{e\}^C = B$. Choose $\alpha \subset f$, $|\alpha| = 5e$. Then $\lim_s \ell(e,s) = \infty$, so $\lim_s g(\alpha,s) = \infty$. Hence, for all i, $t(\alpha,i) < \infty$ (as defined in Definition 7.4), and clearly $t(\alpha,i)$ is computable in $B \oplus C$. Fix i. To compute whether $i \in K$, find $s > \max\{v_\alpha, t(\alpha,i)\}$ such that $\hat{y}_{\alpha,i,s}\downarrow$. Now $i \in K$ iff $y_{\alpha,i,s} \in B$ by Theorem 7.6(ii), (iii), (iv), and (v). Hence, $K \leq_{\mathrm{T}} B \oplus C \leq_{\mathrm{T}} C$, contrary to the hypothesis $C <_{\mathrm{T}} K$ that C is incomplete. Therefore $\{e\}^C \neq B$. Let $x = (\mu y)[\{e\}^C(y) \neq B(y)]$. Thus, $\liminf_s g(\alpha,s) = x$. Hence B is automorphic to A by Theorem 7.6(i). □

By combining the method of this theorem with that for Theorem 1.3 in §8 we can easily prove the following combined theorem, whose details we leave to the reader.

Theorem 9.2. *For all r.e. sets A and C such that $\emptyset <_{\mathrm{T}} A$ and $C <_{\mathrm{T}} K$ there is a high r.e. set $B \simeq_{\Delta^0_3} A$ such that $B \not\leq_{\mathrm{T}} C$.*

10. Prompt sets and a proof of Theorem 1.2

Definition 10.1. An r.e. set A is *prompt* if A is of promptly simple degree, *i.e.*, $A \equiv_{\mathrm{T}} S$ for a promptly simple set S (as defined in [28, Definition XIII.1.2]) and A is *tardy* otherwise. Similarly, an r.e. degree is *prompt* if it contains a prompt set and is *tardy* otherwise. By the Promptly Simple Degree Theorem [28, Theorem XIII.1.7(iii)] A being prompt is equivalent to the following property which for this paper we may take as the definition. Let $\{A_s\}_{s\in\omega}$ be any recursive enumeration of A. Then there is a recursive function p such that for all s, $p(s) \geq s$, and for all e,

$$W_e \text{ infinite} \implies (\exists^\infty x)\,(\exists s)\,[x \in W_{e,\,\mathrm{at}\ s}\ \ \&\ \ A_s{\restriction}x \neq A_{p(s)}{\restriction}x], \tag{61}$$

namely infinitely often A "promptly permits" on some element $x \in W_e$.

This section is devoted to using the Coding Theorem 7.5 to prove Theorem 1.2 that a prompt set A is automorphic to a complete set B, which for convenience we restate here as Theorem 10.2. It is also possible using the effective automorphism machinery of [28, Ch. XV] to construct such a B which is *effectively* automorphic to A but we do not carry this out here. This generalizes the result of Cholak, Downey, and Stob [4] which asserted the same conclusion under the stronger hypothesis that A is promptly simple, rather than merely prompt. In §11 we show that the proof here works for a strictly larger class of sets beyond the prompt sets which we call *almost prompt* sets.

Theorem 10.2. *If A is any prompt r.e. set, then A is automorphic to a complete set B.*

Proof. Let A be a prompt set. As in §6 and §7 let $U_0 = A$, $\rho = f{\restriction}1$, so $U_\rho = A$ (by our assignment of indices (4)), and let B denote $\widehat{U}_\rho$. We shall arrange the construction so that $K \leq_{\mathrm{T}} B$.

For the rest of this paper we replace the sets $\mathcal{W}^{\#}_\alpha$ of Definition 6.1 and $\mathcal{C}_\alpha$ of Definition 6.2 by the following new versions $\mathcal{W}^{\#}_\alpha$ and $\mathcal{D}_\alpha$, respectively.

Definition 10.3. Let $\mathcal{W}^{\#}_\alpha$ be that set of $\nu_1 = \langle \alpha, \sigma_1, \tau_1\rangle \in \mathcal{M}_\alpha$ such that

$$0 \notin \sigma_1\ \ \&\ \ (\exists \nu_2 = \langle \alpha, \sigma_2, \tau_1\rangle)[\nu_2 \in \mathcal{M}_\alpha\ \ \&\ \ \sigma_1 \subset \sigma_2\ \ \&\ \ 0 \in \sigma_2].$$

Definition 10.4. Define $\mathcal{D}_\alpha$ exactly as $\mathcal{C}_\alpha$ of Definition 6.2 except with the new version of $\mathcal{W}^{\#}_\alpha$ in clause (i). Namely, let $\mathcal{D}_\alpha$ be the set of $\nu_1 = \langle \alpha, \sigma_1, \tau_1\rangle$ such that $\nu_1 \in \mathcal{M}_\alpha$ and

(i) $\nu_1 \in \mathcal{W}^{\#}_\alpha$,
(ii) $\neg(\exists \nu_2 \in \mathcal{M}_\alpha)[\nu_1 <_B \nu_2]$,
(iii) $\nu_1 \notin \mathcal{N}_\alpha =_{\mathrm{dfn}} \mathcal{R}_\alpha \cup \mathcal{B}_\alpha$.

(The condition (i) for $\mathcal{D}_\alpha$ is similar to that for $\mathcal{C}_\alpha$ but simpler. Both conditions assert that $0 \notin \sigma_1$, so that if $\nu(\alpha, x, s) = \nu_1$, then x is not yet in $U_\rho = A$, and that RED has a winning strategy for putting x in $U_\rho = A$. However, in the case of $\mathcal{D}_\alpha$ this strategy involves only *one* move by RED, namely changing x from α-state ν_1 to ν_2, which is an α-legal red move because $\nu_2 \in \mathcal{M}_\alpha$.)

For the rest of this paper we replace all instances of $\mathcal{C}_\alpha$ in §6 and §7 by $\mathcal{D}_\alpha$. For example, in Step 6 and Step $\hat{7}$ we replace $\mathcal{C}_\alpha$ by $\mathcal{D}_\alpha$, and we replace $\mathcal{C}$-consistent everywhere by $\mathcal{D}$-consistent defined as follows.

Definition 10.5. (i) A node $\alpha \in T$ is *$\mathcal{D}$-consistent* if $\alpha = \lambda$ or $\mathcal{D}_\alpha \neq \emptyset$, and *$\mathcal{D}$-inconsistent* otherwise.

(ii) A node $\alpha \in T$ is *consistent* if α is $\mathcal{M}$-consistent (Definition 2.8), $\mathcal{R}$-consistent (Definition 2.9), and also $\mathcal{D}$-consistent.

The rest of the proof is divided into two parts. In the first part we use that A is prompt to get $\mathcal{D}_\alpha \neq \emptyset$ for $\alpha \subset f$ analogous to Lemma 6.4 for A nonrecursive. In the second part we use $\mathcal{D}_\alpha \neq \emptyset$ for $\alpha \subset f$ to code $K \leq_{\mathrm{T}} B$ analogously as we used $\mathcal{C}_\alpha \neq \emptyset$ for $\alpha \subset f$ to code information into B when A is nonrecursive in Theorems 1.3 and 9.1 in §8 and §9.

10.1. **Using A prompt to get $\mathcal{D}_\alpha \neq \emptyset$ for $\alpha \subset f$.** For the automorphism construction of §3 we let $\{U_{n,s}\}_{n>0,s\in\omega}$ be a given recursive enumeration of all r.e. sets as before, but we change the enumeration $\{U_{0,s}\}_{s\in\omega}$ of $U_0 = A$ to achieve $\mathcal{D}_\alpha \neq \emptyset$ for $\alpha \subset f$. Fix $\alpha \neq \lambda$. Let $k \in \omega$ and let F be a finite set of α-states $\nu = \langle \alpha, \sigma, \tau\rangle$ such that $0 \notin \sigma$ (*i.e.*, ν is an α-state of $\overline{A} = \overline{U}_\rho$). We use integers to code the finite sets F and nodes $\alpha \in T$ and we identify $\langle \alpha, F, k\rangle$ with an integer i coding it. For each $\langle \alpha, F, k\rangle$ define a recursive function,

$$\ell(\langle \alpha, F, k\rangle, s) = \max\{x : x \geq k \ \ \& \ \ (\forall y)_{k\leq y<x}\, [y \in U_{\rho,s} \ \vee \ \nu(\alpha, y, s) \in F]\}.$$

We shall define an r.e. sequence of r.e. sets $\{Z_i\}_{i\in\omega}$. By the Recursion Theorem and the Slowdown Lemma [28, Lemma XIII.1.5] we may assume that we have a recursive function $H(i)$ such that for all i, $W_{H(i)} = Z_i$, and $W_{H(i),s} \cap Z_{i,\text{ at } s} = \emptyset$, namely any element enumerated in Z_i appears strictly later in $W_{H(i)}$. Now add to Step 11 in §3 the following Substep 11F to be performed at the end of stage $s+1$ if Step 11 is performed there. (Of course, it is the blue player who is speeding up the enumeration of A.)

Substep 11F. (Speeding up A.) Let v be the maximum stage $w \leq s$ such that Step 11 was performed at stage w, if such exists, and 0 otherwise. For each $i = \langle \alpha, F, k\rangle$, $i < s$, such that

$$\ell(i, s) > \max\{\ell(i, t) : t < v\}, \tag{62}$$

enumerate $\ell(i, s)$ in $Z_{i,s+1}$. Let

$$t_{i,s} = (\mu u)[\ell(i, s) \in W_{H(i),\text{ at } u}],$$

if (62) holds for i, and $t_{i,s} = s + 2$ otherwise. Now $t_{i,s} \geq s + 2$ by the Slowdown Lemma. Let $t = \max\{t_{i,s} : i < s\}$. Compute $p(t)$ for the function p given in (61). For each $x \in A_{p(t)}$ put x in $U_{0,s+1}$.

(Note that adding Substep 11F does not interfere with the construction or proof of the Automorphism Theorem 4.2 or Coding Theorem 7.5 because Substep 11F does not enumerate an element x in any red set U_α or blue set $\widehat{V}_\alpha$ and does not move x among the S^i_α. It only changes the enumeration $\{U_{0,s}\}_{s\in\omega}$ of $U_0 = A$ by making $U_{0,s} \supseteq A_s$, and the construction and proof of Theorem 7.5 must work with *any* recursive enumeration $\{U_{n,s}\}_{n,s\in\omega}$ of the r.e. sets.)

Lemma 10.6. *Suppose the construction of the Coding Theorem* 7.5 *is done but with $\mathcal{D}_\alpha$ everywhere in place of $\mathcal{C}_\alpha$ (e.g. in Step* 6 *and Step* $\hat{7}$*), with the additional*

Substep 11F *as above added to Step* 11*, and with* $A = U_0$ *prompt. Then for all* α*,* $\lambda \neq \alpha \subset f$*,* $\mathcal{D}_\alpha \neq \emptyset$*.*

Proof. Suppose $\alpha \subset f$, $\alpha \neq \lambda$, but $\mathcal{D}_\alpha = \emptyset$. Then α is $\mathcal{D}$-inconsistent. By (42) α is a terminal node on T so $S_\alpha = R_\alpha$. By Lemmas 5.8 and 5.4(v), $S_{\alpha,\infty} =^* \omega$ and no $x \in S_{\alpha,s}$, $s > v_\alpha$, later leaves S_α. Let $F = \{\nu_1, \nu_2, \dots, \nu_n\}$ be the set of α-states $\nu = \langle \alpha, \sigma, \tau \rangle$ which are well-resided on $\overline{A}$ (*i.e.*, the α-states in $\mathcal{M}_\alpha - \mathcal{K}_\alpha$ such that $0 \notin \sigma$). Define

$$k = (\mu m)(\forall x \geq m)[x \in \overline{A} \implies \nu(\alpha, x) \in F].$$

Let $i = \langle \alpha, F, k \rangle$. Now $\lim_s \ell(i,s) = \infty$. Choose $w \geq v_\alpha$ such that $A{\restriction}k = A_w{\restriction}k$. Now there are infinitely many stages $s+1 > w$ such that (62) holds for $\ell(i,s)$ and Substep 11F applies to i at stage s+1. For infinitely many such applications by (61) there will be some element $x < \ell(i,s)$, such that $x \in U_{0,s+1} - U_{\rho,s}$. Hence, there is some *single* $\nu_1 \in F$, $\nu_1 = \langle \alpha, \sigma_1, \tau_1 \rangle$, such that for infinitely many elements x, $\nu(\alpha, x, s) = \nu_1$ and Substep 11F applies to x at some stage $s+1 > w$ and $x \in U_{0,s+1} - U_{\rho,s}$. Since $s > v_\alpha$ and $\alpha \subset f$, Steps 1, 2, and 3 cannot apply to x by the same argument as in Lemma 5.11. Thus, by the ordering of the steps, x will remain in α-state ν_1 until Step 4 applies to x, which must happen at some stage $t+1 > s+1$ because $x \in U_{0,s+1} - U_{\rho,s}$. Hence, $\nu(\alpha, x, t) = \nu(\alpha, x, s) = \nu_1$, and $\nu(\alpha, x, t+1) = \nu_2$ where $\nu_2 = \langle \alpha, \sigma_2, \tau_1 \rangle$, $\sigma_2 \supseteq \sigma_1 \cup \{0\}$.

Since this happens for infinitely many x, $\nu_2 \in \mathcal{M}_\alpha$. Thus, ν_2 witnesses that ν_1 satisfies clause (i) of Definition 10.4. By the definition of F, $\nu_1 \notin \mathcal{K}_\alpha \supseteq \mathcal{N}_\alpha$ so ν_1 satisfies clause (iii) of Definition 10.4, and by Step 6, ν_1 satisfies clause (ii) as in Lemma 6.4. Hence, α is $\mathcal{D}$-consistent. □

10.2. **Using $\widehat{\mathcal{D}}_\alpha \neq \emptyset$ to code K into B.** Assume $\alpha \subset f$. Since $\mathcal{D}_\alpha \neq \emptyset$ by Lemma 10.6, we have by (12), (21), and (22) that $\widehat{\mathcal{D}}_\alpha \neq \emptyset$ where $\widehat{\mathcal{D}}_\alpha = \{\hat{\nu} : \nu \in \mathcal{D}_\alpha\}$. Choose any $\hat{\nu}_1 = \langle \alpha, \hat{\sigma}_1, \hat{\tau}_1 \rangle \in \widehat{\mathcal{D}}_\alpha$. By the dual of Definition 10.4 and (13) we have

$$0 \notin \hat{\sigma}_1 \ \ \& \ \ (\exists \hat{\nu}_2 = \langle \alpha, \hat{\sigma}_2, \hat{\tau}_1 \rangle)[\hat{\nu}_2 \in \widehat{\mathcal{M}}_\alpha \ \ \& \ \ \hat{\sigma}_1 \subset \hat{\sigma}_2 \ \ \& \ \ 0 \in \hat{\sigma}_2], \tag{63}$$

$$\neg(\exists \hat{\nu}_2 \in \widehat{\mathcal{M}}_\alpha)[\hat{\nu}_1 <_R \hat{\nu}_2], \quad \text{namely } \hat{\nu}_1 \text{ is red maximal}, \tag{64}$$

$$\hat{\nu}_1 \notin \widehat{\mathcal{N}}_\alpha =_{\text{dfn}} \widehat{\mathcal{R}}_\alpha \cup \widehat{\mathcal{B}}_\alpha. \tag{65}$$

For Step $\hat{7}$ we define $g(\alpha, s) = 1$ for all α and s as in §8. By the Coding Theorem 7.5 we get α-witnesses $\hat{y}_{\alpha,s} = \hat{x}$ in state $\hat{\nu}_1 \in \mathcal{D}_\alpha$. By (64) $\hat{\nu}_1$ is maximal with respect to α-legal red moves so RED cannot change the α-state of $\hat{x}$. By (65), $\hat{\nu}_1 \notin \widehat{\mathcal{B}}_\alpha$ so BLUE does not have to change the state, and can hold $\hat{x}$ in α-state $\hat{\nu}_1$ and hence in $\overline{B}$ forever if he chooses. However, by (63), BLUE can move $\hat{x}$ from α-state $\hat{\nu}_1$ to $\hat{\nu}_2$ and hence to B whenever he likes. Fix a function $\widehat{F}_\alpha$ such that

$$(\forall \hat{\nu}_1 \in \widehat{\mathcal{D}}_\alpha)[\widehat{F}_\alpha(\hat{\nu}_1) = \hat{\nu}_2 \text{ satisfies (63)}]. \tag{66}$$

The new function $\widehat{F}_\alpha$ will be used in the action of Step $\hat{8}$ which will now be performed during any application of the following Steps $\hat{9}$ and $\widehat{10}$. (We do not need Step $\hat{8}$ itself now because when BLUE wants to enumerate some witness $\hat{z}$ in B he simply does so directly, rather than by putting it in $\widehat{J}_\alpha$.)

Step $\widehat{9}$. Suppose $|\alpha| = 5i$, $\hat{y}_{\alpha,s}{\downarrow} = \hat{x}$, and $\nu(\alpha, \hat{x}, s) = \hat{\nu}_1$. If $i \in K_s$, perform the action of Step $\hat{8}$ of §7.2 on $\hat{x}$ at stage s+1. (Namely, choose the least such pair $\langle \alpha, x \rangle$. Let $\widehat{F}_\alpha(\hat{\nu}_1) = \hat{\nu}_2 = \langle \alpha, \widehat{\sigma}_2, \widehat{\tau}_2 \rangle$. Enumerate $\hat{x}$ in $\widehat{U}_{\delta,s+1}$ for all $\delta \subseteq \alpha$ such that

$\hat{e}_\delta \in \widehat{\sigma}_2$. Since $\hat{x} \in B_{s+1} - B_s$, move $\hat{x}$ from $\widehat{S}^0_\alpha$ to $\widehat{S}^1_\alpha$, and let $\hat{x}$ be cancelled as an α-witness.)

Step $\widehat{10}$. Suppose $\hat{y}_{\alpha,s}\downarrow = \hat{x}$, $\hat{y}'_{\alpha,s}\downarrow = \hat{x}'$, and that at stage s+1 either:

(i) $\hat{x}$ and $\hat{x}'$ will be removed from $\widehat{S}^0_\alpha$ (and from $\widehat{L}_\alpha$) because Step 11C applies to α (*i.e.*, $f_{s+1} <_L \alpha$); or

(ii) one of $\hat{x}$ and $\hat{x}'$ will be pulled from $\widehat{S}^0_\alpha$ to some $\widehat{S}_\beta$, $\beta <_L \alpha$, under Step $\hat{1}_\beta$. (Necessarily $\hat{x}$ not $\hat{x}'$ will be pulled since $\hat{x} < \hat{x}'$, both are in the same α-state, and at most one element is pulled at a time.)

Then perform the action of Step $\hat{8}$ of §7.2 on $\hat{x}'$, as in Step $\hat{9}$ above. (Both $\hat{x}$ and $\hat{x}'$ will be cancelled as α-witnesses at stage s+1 under Step 11C or Step $\hat{1}_\beta$ (1.12).)

(Strictly speaking Step $\widehat{10}$ is not a new step but rather a modification to the earlier Steps 11 and $\hat{1}$ since the action of Step $\widehat{10}$ must be performed at that point in the construction when the latter Steps 11 or $\hat{1}$ apply.)

Lemma 10.7. *Suppose the construction of the Coding Theorem* 7.5 *is done but with $\mathcal{D}_\alpha$ everywhere in place of $\mathcal{C}_\alpha$ (e.g. in Step* 6 *and Step* $\hat{7}$*), with the additional Substep* 11F*, with Steps $\widehat{9}$ and $\widehat{10}$ as above, and with $A = U_0$ of promptly simple degree. Let $B = \widehat{U}_\rho$, where $\rho = f{\restriction}1$. Then*

(i) *A is automorphic to B, and*

(ii) *$K \leq_{\mathrm{T}} B$.*

Proof. (i) Substep 11F does not affect the Coding Theorem construction by the remark immediately following Substep 11F. Step $\hat{9}$ clearly satisfies conditions ($\widehat{\mathrm{P}}1$)–($\widehat{\mathrm{P}}3$). So does Step $\widehat{10}$ but since its action is performed at a different point in the construction, we need to verify that it does not interfere. Suppose $\hat{y}_{\alpha,s}\downarrow = \hat{x}$, $\hat{y}'_{\alpha,s}\downarrow = \hat{x}'$, and that at stage s+1, $\hat{x}$ is pulled to $\widehat{S}_\beta$ by Step $\widehat{1}_\beta$ for some $\beta <_L \alpha$. Then $\hat{x}$ is not enumerated in any sets at stage $s+1$, so the previous argument for $\widehat{S}_\beta$ is not affected. By Step $\widehat{1}$ (1.12), $\hat{x}'$ is moved from $\widehat{S}^0_\alpha$ to $\widehat{S}^1_\alpha$ as before and cancelled as an α-witness, but $\hat{x}'$ is not being pulled to any $\widehat{S}_\beta$. Also by Step $\widehat{10}$ (i), $\hat{x}'$ is enumerated in blue sets to achieve α-state $F_\alpha(\hat{\nu}_1)$, but this enumeration is α-legal, and so satisfies condition ($\widehat{\mathrm{P}}2$). Hence, the previous arguments for both β and α are not affected by Step $\widehat{10}$ (i). If Step $\widehat{10}$ (ii) applies, then similarly both $\hat{x}$ and $\hat{x}'$ have this blue enumeration before being moved from $\widehat{S}_\alpha$, but neither is being pulled to any $\widehat{S}_\beta$, both are being cancelled as α-witnesses, and this enumeration is α-legal, so it cannot affect the previous argument. Finally, $g(\alpha, s) = 1$ for all α and s, so by the proof of Theorem 7.5(i), A is automorphic to B.

(ii) We claim that for all $i \in \omega$,

$$i \in \overline{K} \iff (\exists\alpha)_{|\alpha|=5i}\ (\exists s)\ [\hat{y}_{\alpha,s}\downarrow \in \overline{B}\ \ \&\ \ \hat{y}'_{\alpha,s}\downarrow \in \overline{B}]. \tag{67}$$

First suppose $i \in \overline{K}$. Choose $\alpha \subset f$, $|\alpha| = 5i$. Now Step $\widehat{9}$ never applies to α, and Step $\widehat{10}$ does not apply to α at any stage $s > v_\alpha$. Hence, by the proof of the Coding Theorem 7.5(viii), $\lim_s \hat{y}_{\alpha,s}\downarrow = \hat{x}$ and $\lim_s\ \hat{y}'_{\alpha,s}\downarrow = \hat{x}'$ and $\hat{x}, \hat{x}' \in \overline{B}$.

Now suppose (67) holds, and let $\hat{y}_{\alpha,s} = \hat{x}$, and $\hat{y}'_{\alpha,s} = \hat{x}'$. Neither $\hat{x}$ nor $\hat{x}'$ can be removed from $\widehat{L}_\alpha$ by Step $\hat{1}$ (1.12) or by Step 11C at any stage $t > s$ else by Step $\widehat{10}$ one of $\hat{x}$, $\hat{x}'$ must enter B. Hence, by the same proof as in the Coding Theorem 7.5(iii) and (iv), both $\hat{x}$ and $\hat{x}'$ remain forever in $\widehat{L}_\alpha$ and in $\overline{B}$. But then $i \in \overline{K}$ because if $i \in K_t$ for some $t > s$, then by Step $\hat{9}$ one of $\hat{x}$, $\hat{x}'$ must enter B. This completes the proof of the lemma and of Theorem 10.2. $\square$

Notice that in the above proof we must appeal to the proof rather than merely the statement of the Coding Theorem 7.5 because here we have no set $\widehat{J}_\alpha$ and the work previously performed by Step $\widehat{8}$ is now done during Steps $\widehat{9}$ and $\widehat{10}$ so the conditions ($\widehat{\mathrm{Q}}2$) and ($\widehat{\mathrm{Q}}4$) do not strictly hold. (This is why we use the Coding Theorem 7.5 here rather than the Refined Coding Theorem 7.6 as we did in §10 and §11.) However, if we make the following notational changes, then the obvious modification of the former proof (which we omit) still establishes the following theorem.

Theorem 10.8 (Prompt Coding Theorem)**.** *Let $A = U_0$ be a prompt set. In the statement of the Coding Theorem* 7.5 *replace everywhere $\mathcal{C}_\alpha$ by $\mathcal{D}_\alpha$, and $\widehat{J}_{\alpha,s}$ by B_s; add to conditions* ($\widehat{\mathrm{Q}}2$) *and* ($\widehat{\mathrm{Q}}4$) *the clause "unless simultaneously $\hat{x}$ is enumerated in B". Then all the conclusions of the Coding Theorem* 7.5 *hold except in* (ii) *for the inclusion $\widehat{J}_{\alpha,s} \subseteq \widehat{L}_{\alpha,s} \subseteq \widehat{S}^0_{\alpha,s}$.* (*Note that conclusions* (iv), (v), *and* (vi) *are now tautologies.*) □

(The reason we were able to achieve $K \leq_{\mathrm{T}} B$ in Theorem 10.2 using $\mathcal{D}_\alpha$ but not with $\mathcal{C}_\alpha$ is the following. When both $\hat{y}_{\alpha,s} = \hat{x}$, and $\hat{y}'_{\alpha,s} = \hat{x}'$ for *any* α such that $|\alpha| = 5i$, we define $\Psi_s(i) = K_s(i)$ and define the use function $\psi_s(i) = \hat{x}'$. If later either $\hat{x}$ or $\hat{x}'$ will be removed from $\widehat{L}_\alpha$, or if i enters K, then we must put $\hat{x}$ or $\hat{x}'$ into B according to Step $\widehat{10}$ in order to correct $\psi(i)$. For $\mathcal{D}_\alpha$ this is an α-legal move which can be performed immediately whether or not $\alpha \subset f$. For $\mathcal{C}_\alpha$ this action requires considerable time delay and is only guaranteed to succeed as in Theorem 7.5(vi) if $\alpha \subset f$, which we cannot determine effectively when we first must define $\psi_s(i)$. It is precisely this difficulty which was exploited by Harrington and Soare in [8] to construct a nontrivial $\mathcal{E}$-definable property $Q(A)$ which guarantees that A is incomplete.)

11. Almost prompt sets

11.1. **Almost prompt sets and complete sets.**

Definition 11.1. (i) A set $X \leq_{\mathrm{T}} K$ is *n-r.e.* if $X = \lim_s X_s$ for some recursive sequence $\{ X_s \}_{s\in\omega}$ such that for all x, $X_0(x) = 0$ and

$$\mathrm{card}\{ s : X_s(x) \neq X_{s+1}(x) \} \leq n.$$

For example, the only 0-r.e. set is $\emptyset$, the 1-r.e. sets are the usual r.e. sets, and the 2-r.e. sets are the d.r.e. sets.

(ii) Such a sequence $\{X_s\}_{s\in\omega}$ is called an *n-r.e. presentation* of X.

It is well known and easy to show [28, Exercise III.3.8, p. 38] that for $n > 0$, X is n-r.e. iff

$$X = (W_{e_1} - W_{e_2}) \cup (W_{e_3} - W_{e_4}) \cup \ldots \cup W_{e_{2k+1}}, \text{ or} \tag{68}$$

$$X = (W_{e_1} - W_{e_2}) \cup (W_{e_3} - W_{e_4}) \cup \ldots \cup (W_{e_{2k+1}} - W_{e_{2k+2}}), \tag{69}$$

according as $n = 2k + 1$ is odd or $n = 2k + 2$ is even.

Definition 11.2. For $n = 0$ let $X^0_0 = \emptyset$. For $n > 0$ and $e = \langle e_1, e_2, \ldots, e_n\rangle$ define

$$X^n_e = (W_{e_1} - W_{e_2}) \cup \ldots, \tag{70}$$

as in (68) or (69) according as n is odd or even. We say that $\langle n, e\rangle$ is an *n-r.e. index* for X^n_e. Let

$$X^n_{e,s} = (W_{e_1,s} - W_{e_2,s}) \cup \dots . \tag{71}$$

Definition 11.3. Let A be an r.e. set and let $\{A_s\}_{s\in\omega}$ be a recursive enumeration of A. We say A is *almost prompt*, abbreviated *a.p.*, if there is a nondecreasing recursive function $p(s)$ such that for all n and e,

$$X^n_e = \overline{A} \implies (\exists x)(\exists s)[x \in X^n_{e,s} \ \ \& \ \ x \in A_{p(s)}]. \tag{72}$$

Note that, as in the case of promptly simple, this definition is independent of the enumeration of A; if $p(s)$ works for the enumeration $\{A_s\}_{s\in\omega}$, and if $\{A'_s\}_{s\in\omega}$ is another enumeration of A, define $p'(s) = (\mu t)[A'_t \supseteq A_p(s)]$. We may think of Definition 11.3 as asserting that A will p-promptly hit every approximation $\{X^n_{e,s}\}_{s\in\omega}$ for every n-r.e. set $X^n_e = \overline{A}$ where the recursive approximation $X^n_{e,s}$ is determined by the *standard enumeration* $\{W_{e,s}\}_{e,s\in\omega}$ of the r.e. sets. The next lemma shows that if we specify another collection of n-r.e. sets $\{\widehat{X}^n_e\}_{n,e\in\omega}$, by some recursive approximation $\{\widehat{X}^n_{e,s}\}_{n,e,s\in\omega}$, then there is a recursive function q such that A will q-promptly hit $\{\widehat{X}^n_{e,s}\}_{n,e,s\in\omega}$ if $\widehat{X}^n_e = \overline{A}$.

Lemma 11.4 (Conversion Lemma)**.** *Assume that A is almost prompt via $\{A_s\}_{s\in\omega}$ and p. Suppose that $\{Y_{e,s}\}_{e,s\in\omega}$ is a strong array of finite sets, $Y_e = \bigcup_{s\in\omega} Y_{e,s}$, and there is a recursive function $h(n,e,i)$ such that for every $\langle n,e\rangle$ the n-r.e. set $\widehat{X}^n_e$ and its recursive approximation $\widehat{X}^n_{e,s}$ are defined by*

$$\widehat{X}^n_e = (Y_{h(n,e,1)} - Y_{h(n,e,2)}) \cup (Y_{h(n,e,3)} - Y_{h(n,e,4)}) \cup \dots , \tag{73}$$

$$\widehat{X}^n_{e,s} = (Y_{h(n,e,1),s} - Y_{h(n,e,2),s}) \cup (Y_{h(n,e,3),s} - Y_{h(n,e,4),s}) \cup \dots . \tag{74}$$

Then there is a nondecreasing recursive function $q(s)$ such that

$$\widehat{X}^n_e = \overline{A} \implies (\exists x)(\exists s)[x \in \widehat{X}^n_{e,s} \ \ \& \ \ x \in A_{q(s)}], \tag{75}$$

or equivalently,

$$\widehat{X}^n_e =^* \overline{A} \implies (\exists^\infty x)(\exists s)[x \in \widehat{X}^n_{e,s} \ \ \& \ \ x \in A_{q(s)}]. \tag{76}$$

Proof. We first shall define r.e. sets $Z^{n,e,i} = Y_{h(n,e,i)}$, and by the Recursion Theorem there is a recursive function $H(n,e,i)$ such that for all n, e and i, $W_{H(n,e,i)} = Z^{n,e,i}$, and a recursive function $G(n,e)$ such that

$$X^n_{G(n,e)} = (W_{H(n,e,1)} - W_{H(n,e,2)}) \cup (W_{H(n,e,3)} - W_{H(n,e,4)}) \cup \dots ,$$

so $X^n_{G(n,e)} = \widehat{X}^n_e$ because $W_{H(n,e,i)} = Z^{n,e,i} = Y_{h(n,e,i)}$ for all $n,e,i \in \omega$. We define $t(s)$ at stage s of the following construction. Then we define $q(s) = p(t(s+1))$.

Stage $s = 0$. Define $t(0) = 0$, and $Z^{n,e,i}_0 = \emptyset$ for all n,e,i. (Without loss of generality we may assume that $Y_{h(n,e,i),0} = \emptyset$ for all n,e,i, since otherwise we replace $Y_{h(n,e,i),s}$ by $\widehat{Y}_{h(n,e,i),s}$ where $\widehat{Y}_{h(n,e,i),0} = \emptyset$ and $\widehat{Y}_{h(n,e,i),s+1} = Y_{h(n,e,i),s}$.)

Stage $s+1$. For each $e,n \le s$, do substep i for every $i \le n$, first for each even i in increasing order of i, then for each odd i in increasing order of i.

Substep i. For each $x \le s$ if $x \in Y_{h(n,e,i),s+1} - Z^{n,e,i}_s$, put x in $Z^{n,e,i}_{s+1}$, define $t(n,e,i,x,s) = (\mu t)[x \in W_{H(n,e,i,),t}]$, and note that $t(n,e,i,x,s) > s+1$ by the

Slowdown Lemma [28, Lemma XIII.1.5]. Otherwise, define $t(n,e,i,x,s) = s+2$. Define

$$t(s+1) = (\mu t)[t > t(s) \ \ \& \ \ t \geq \ \max\{t(n,e,i,x,s) : n,e,i,x \leq s\}].$$

Note that $t(s+1) > t(s)$ and $t(s+1) > s+1$.

Claim 1. *For all n, e, s,*
(i) $\widehat{X}^n_{e,s} = X^n_{G(n,e),t(s)}$, *and*
(ii) $(\widehat{X}^n_{e,s} \cup \widehat{X}^n_{e,s+1}) \supseteq \bigcup \{X^n_{G(n,e),v} : t(s) \leq v \leq t(s+1)\}$.

Proof. Now (i) follows for all s because for all $i \leq n$, $W_{H(n,e,i),t(s+1)} = Z^{n,e,i}_{s+1} = Y_{h(n,e,i),s+1}$. For (ii) suppose $x \in X^n_{G(n,e),v}$ for some v, $t(s) \leq v \leq t(s+1)$. If $v = t(s)$, then $x \in \widehat{X}^n_{e,s}$ by (i) for s. If $v > t(s)$, then $x \in W_{H(n,e,2i),v} - W_{H(n,e,2i+1),v}$ for some $2i \leq n$. Hence, $x \in Z^{n,e,2i}_{s+1} - Z^{n,e,2i+1}_{s+1}$. Hence, by the order in which we perform substep j (with all even j being performed first before any odd j) we have $x \in Y_{h(n,e,2i),s+1} - Y_{h(n,e,2i+1),s+1}$. Thus, $x \in \widehat{X}^n_{e,s+1}$ by (74). This proves (ii). □

Claim 2. *If $\widehat{X}^n_e = \overline{A}$, then $(\exists x)(\exists s)[x \in \widehat{X}^n_{e,s} \ \ \& \ \ x \in A_{q(s)}]$.*

Proof. Assume $\widehat{X}^n_e = \overline{A}$. Then $X^n_{G(n,e)} = \overline{A}$, because $X^n_{G(n,e)} = \widehat{X}^n_e$. Hence,

$$(\exists x)(\exists v)[x \in X^n_{G(n,e),v} \ \ \& \ \ x \in A_{p(v)}], \tag{77}$$

by (72). Fix such x and v and find the unique s such that $t(s) \leq v < t(s+1)$. Now $x \in (\widehat{X}^n_{e,s} \cup \widehat{X}^n_{e,s+1})$ by Claim 1. But $v \leq t(s+1)$ implies $p(v) \leq p(t(s+1)) = q(s) \leq q(s+1)$, so $x \in A_{q(s)} \subseteq A_{q(s+1)}$. Thus, x is an instance of (75) for either s or $s+1$. Thus, (75) is satisfied. □

To see that (76) is satisfied we use a proof similar to that in [28, Theorem XIII.1.7(iii)]. For every $n, e, k \in \omega$ define,

$$\widetilde{X}^n_{e,k} = \overline{A} \upharpoonright k \ \cup \ (\widehat{X}^n_e \cap [k,\infty)),$$

and

$$\widetilde{X}^n_{e,k,s} = \overline{A} \upharpoonright k \ \cup \ (\widehat{X}^n_{e,s} \cap [k,\infty)),$$

where $\{Y_{e,s}\}_{e,s\in\omega}$ has been suitably adjusted to achieve (74) for $\widetilde{X}^n_{e,k,s}$. If $\widehat{X}^n_e =^* \overline{A}$, then $(\exists k_0)(\forall k \geq k_0)[\widetilde{X}^n_{e,k} = \overline{A}]$, so (75) applied to $\{\widetilde{X}^n_{e,k,s}\}$ produces one x_k satisfying (75) for $\{\widehat{X}^n_{e,s}\}$, and the set $\{x_k\}_{k\geq k_0}$ verifies (76) for $\{\widehat{X}^n_{e,s}\}$. □

Theorem 11.5. *If A is any almost prompt set, then A is automorphic to a complete set B.*

Proof. It suffices to prove that Lemma 10.6 holds with A *prompt* replaced by A *almost prompt* because then the remainder of the proof is the same as in §10.2. Fix $\alpha \neq \lambda$. Let F be a finite set of α-states $\nu = \langle \alpha, \sigma, \tau \rangle$ such that $0 \notin \sigma$ (*i.e.*, α-states of $\overline{A} = \overline{U}_\rho$). We use integers to code the finite sets F and nodes $\alpha \in T$ and we identify $\langle \alpha, F \rangle$ with an integer i coding it. For $i = \langle \alpha, F \rangle$ define

$$Y^i_s = \{x : \nu(\alpha, x, s) \in F\}. \tag{78}$$

By our assignment of indices e_β, $\hat{e}_\beta$ in (4) and (5), there are at most $|\alpha|$ many indices e_β, or $\hat{e}_\beta$ for $\beta \subseteq \alpha$, so there are at most $n = 2^{|\alpha|}$ many different α-states. Hence, the recursive sequence $\{Y^i_s\}_{s\in\omega}$ witnesses that $Y^i =_{\text{dfn}} \lim_s Y^i_s$ is n-r.e. for $n = 2^{|\alpha|}$. Thus, as in (70) and (71) there are r.e. sets Y^i_j, $1 \le j \le n$, such that

$$Y^i_s = (Y^i_{1,s} - Y^i_{2,s}) \cup \dots . \tag{79}$$

We slightly modify the automorphism construction presented in §3 as follows.
Stage $s+1 = 2t$. Do the next step in the construction exactly as in §3.
Stage $s+1 = 2t+1$. Define Y^i_s as above. Perform no action in the automorphism construction, so $Y^i_s = Y^i_{s+1}$. From Y^i_s and Y^i_{s+1} define $t(s+1)$ and $q(s) = p(t(s+1))$ as in the Conversion Lemma 11.4. For each $x \in A_{q(s)}$, put x in $U_{0,s+1}$.

Hence, by the Conversion Lemma 11.4 applied to the strong array $\{Y^i_{j,s}\}_{i,j,s\in\omega}$ the recursive function $q(s)$ satisfies,

$$Y^i =^* \overline{A} \implies (\exists^\infty x)(\exists s)[x \in Y^i_s \ \&\ x \in A_{q(s)} \subseteq U_{0,s+1}]. \tag{80}$$

Lemma 11.6. *Suppose that $A = U_0$ is an a.p. set. Suppose the construction of the Coding Theorem 7.5 is done but with the modification for odd and even stages as above to define the enumeration $\{U_{0,s}\}_{s\in\omega}$ of U_0, and with $\mathcal{D}_\alpha$ everywhere in place of $\mathcal{C}_\alpha$. Then for all α, $\lambda \ne \alpha \subset f$, $\mathcal{D}_\alpha \ne \emptyset$.*

Proof. Suppose $\alpha \subset f$, $\alpha \ne \lambda$, but $\mathcal{D}_\alpha = \emptyset$. Then α is $\mathcal{D}$-inconsistent. By (42) α is a terminal node on T so $S_\alpha = R_\alpha$. By Lemmas 5.8 and 5.4(v), $S_{\alpha,\infty} =^* \omega$ and no $x \in S_{\alpha,s}$, $s > v_\alpha$, later leaves S_α. Let $F = \{\nu_1, \nu_2, \dots, \nu_n\}$ be the set of α-states $\nu = \langle \alpha, \sigma, \tau\rangle$ which are well resided on $\overline{A}$ (*i.e.*, the α-states in $\mathcal{M}_\alpha - \mathcal{K}_\alpha$ such that $0 \notin \sigma$).

For $i = \langle \alpha, F\rangle$ define Y^i and $Y^i_{j,s}$ as in (78) and (79). Now $Y^i =^* \overline{A}$. Hence, by (80) there is a single $\nu_1 \in F$, $\nu_1 = \langle \alpha, \sigma_1, \tau_1\rangle$ such that for infinitely many elements x, $\nu(\alpha, x, s) = \nu_1 \in F$, so $x \notin U_{\rho,s}$, but $x \in U_{0,s+1}$.

Since $s > v_\alpha$ and $\alpha \subset f$, Steps 1, 2, and 3 cannot apply to x by the same argument as in Lemma 5.11. Thus, by the ordering of the steps, x will remain in α-state ν_1 until Step 4 applies to x, which must happen at some stage $t+1 > s+1$ because $x \in U_{0,s+1} - U_{\rho,s}$, so $x \in U_{\rho,t+1} - U_{\rho,t}$. Hence, $\nu(\alpha, x, t) \ge_R \nu(\alpha, x, s) = \nu_1$, and $\nu(\alpha, x, t+1) = \nu_2$ where $\nu_2 = \langle \alpha, \sigma_2, \tau_1\rangle$, $\sigma_2 \supseteq \sigma_1 \cup \{0\}$.

Since this happens for infinitely many x, $\nu_2 \in \mathcal{M}_\alpha$. Thus, ν_2 witnesses that ν_1 satisfies clause (i) of Definition 10.4. By the definition of F, $\nu_1 \notin \mathcal{K}_\alpha \supseteq \mathcal{N}_\alpha$ so ν_1 satisfies clause (iii), and by Step 6, ν_1 satisfies clause (ii) as in Lemma 6.4. Hence, α is $\mathcal{D}$-consistent. □

This completes the proof of Lemma 11.6. The rest of Theorem 11.5 follows exactly as in §10.2. □

Notice that the construction for Theorem 11.5 just before Lemma 11.6 compared to the Step 11F for prompt illustrates the difference between A being prompt versus being merely almost prompt. Step 11F only had to apply at occasional stages (namely those when Step 11 applies), because when it applied we could challenge the promptness of A to produce infinitely often a p-prompt reply. Here the a.p. hypothesis gives us infinitely many q-prompt replies but we cannot actively produce one so we must do the q-speedup at *every* stage of the automorphism construction or else all the speedups may occur at stages when we are not prepared.

11.2. **Properties of almost prompt sets.**

Theorem 11.7. *If A is any r.e. set of promptly simple degree, then A is almost prompt.*

Proof. Using (61) let A be an r.e. set of promptly simple degree, $\{A_s\}_{s\in\omega}$ a recursive enumeration of A, and $p(s)$ a nondecreasing recursive function such that for all e,

$$W_e \text{ infinite} \implies (\exists^\infty x)\,(\exists s)\,[x \in W_{e,\text{ at } s} \ \&\ A_s{\restriction}x \neq A_{p(s)}{\restriction}x], \tag{81}$$

We shall define a nondecreasing recursive function $q(s)$ such that A, $\{A_s\}_{s\in\omega}$, and $q(s)$ satisfy the Definition 11.3 of A being a.p. Define a recursive function

$$\ell(\langle n,e\rangle, s) = \max\{x : (\forall y)_{y<x}\,[y \in A_s \ \vee\ x \in X^n_{e,s}]\}.$$

For $i = \langle n,e\rangle$ we shall define an r.e. sequence of r.e. sets $\{Z_i\}_{i\in\omega}$. By the Recursion Theorem we may assume that we have a recursive function $H(i)$ such that $W_{H(i)} = Z_i$ for all $i \in \omega$. For each $i < s$ such that

$$\ell(i,s) > \max\{\ell(i,t) : t < s\}, \tag{82}$$

enumerate $\ell(i,s)$ in $Z_{i,s+1}$. Let

$$t_{i,s} = (\mu u)[\ell(i,s) \in W_{H(i),u}],$$

if (82) holds for i, and $t_{i,s} = s+2$ otherwise. Now $t_{i,s} \geq s+2$ by the Slowdown Lemma [28, Lemma XIII.1.5]. Define

$$q(s) = (\mu v)[v > \max\{q(t) : t < s\} \ \&\ v \geq \max\{p(t_{i,s}) : i < s\}].$$

Clearly, $q(s)$ is recursive and nondecreasing. Fix $i = \langle n,e\rangle$ such that $X^n_e = \overline{A}$. Then $\lim_s \ell(i,s) = \infty$. At infinitely many stages s we enumerate $\ell(i,s)$ in $Z_{i,s+1}$, so by (81) and our definition of $q(s)$, we must get

$$(\exists x)(\exists s)[x \in X^n_{e,s} \ \&\ x \in A_{q(s)}].$$

Thus, (72) and Definition 11.3 are satisfied so A is a.p. □

Theorem 11.8. *($\exists$ a tardy r.e. set A)($\forall$r.e. $Z \geq_{\mathrm{T}} A$)[Z is almost prompt].*

Proof. We construct A and B nonrecursive r.e. sets whose degrees form a minimal pair using the usual negative requirements N_e, $e \in \omega$, and negative restraint function $r(e,s)$ as in the usual minimal pair construction in [28, Theorem IX.1.2]. The positive requirement $P_{2e+1} : B \neq \{e\}$ and strategy to meet it are the same as before. Let $\{(\Phi_j, Z_j)\}_{j\in\omega}$ enumerate all pairs (Φ, Z) such that Φ is a partial recursive functional and Z is an r.e. set. Let $\varphi_j(x)$ be the use function for $\Phi_j(x)$. In the minimal pair construction we made A merely nonrecursive by meeting the requirement $P_{2e} : A \neq \{e\}$. Now we shall construct A so that if $\Phi_j^{Z_j} = A$, then Z_j is a.p. via g_j, where

$$g_j(s) = (\mu t > s)(\forall y \leq s)[\Phi_{j,t}^{Z_{j,t}}(y) = A_t(y)].$$

(Since A itself is a.p. it is of course nonrecursive.) Given $i = \langle j,n,e\rangle$ our new positive requirement on A is:

$$P_{2i} : \qquad \Phi_j^{Z_j} = A \ \&\ X^n_e = \overline{Z}_j \implies (\exists y)(\exists s)[y \in X^n_{e,s} \cap Z_{j,g_j(s)}].$$

Define

$$\ell^A(j,s) = \max\{x : (\forall y < x)[\Phi_{j,s}^{Z_{j,s}}(y) = A_s(y)]\}.$$

We say requirement P_{2i}, $i = \langle j,n,e\rangle$, *requires attention at stage* $s+1$ if $A_s \cap \omega^{[i]} = \emptyset$ (*i.e.*, P_{2i} has not yet received attention), and there exists $x \in \omega^{[i]}$ such that:

$$x > r(i,s); \tag{83}$$

$$\ell^A(j,s) > x; \text{ and} \tag{84}$$

$$(\forall y \le \varphi_{i,s}(x))[\, y \in (X^n_{e,s} \cup Z_{j,s})\,]. \tag{85}$$

The construction at stage $s+1$ is as before. We choose the least n such that requirement P_n requires attention, and the least corresponding x, and enumerate x in A_{s+1} if n is even and in B_{s+1} if n is odd.

It follows as in [28, Lemma 1, p. 155] that for all i, $r(i) = \liminf_s r(i,s)$ is finite. It remains to see that for all i requirement P_{2i} is met, since P_{2i+1} is satisfied as before. Fix i and s_0 such that for all $k < 2i$, requirement P_k is satisfied and never receives attention after stage s_0, but P_{2i} is not satisfied.

Assume $\Phi_j^{Z_j} = A$ and $X^n_e = \overline{Z_j}$. Let $i = \langle j,n,e\rangle$. Then there exists $x \in \omega^{[i]} - A_s$ such that (83)–(85) hold of x and i at some stage $s+1 > s_0$. Then P_{2i} receives attention at stage $s+1$ and $x \in A_{s+1} - A_s$. Let $z = \varphi_{i,s}(x)$. Now by (83),

$$\Phi_{j,s}^{Z_{j,s}\setminus z}(x)\downarrow = A_s(x) = 0 \ne 1 = A_{s+1}(x).$$

Since $\Phi_j^{Z_j} = A$, we must have

$$\Phi_{j,t}^{Z_{j,t}}(x)\downarrow = A(x) = 1,$$

for some $t > s$. By (85), we must have

$$(\exists y \le z)[y \in (X^n_{e,s} - Z_{j,s}) \ \ \& \ \ y \in Z_{j,g_j(s)}].$$

Hence, requirement P_{2i} is satisfied. □

This proof illustrates a crucial difference between A being prompt versus A being a.p. We can make A tardy as above because the restraint function $r(i,s)$ can permanently restrain at most finitely many elements. However, if the opponent attempts to make $X^n_e = \overline{A}$, then for each $x \in X^n_e$, x must lie in $X^n_{e,s}$ for almost all s. Hence, for every such x we have *cofinitely many* stages s to achieve $x \in A_{p(s)}$ in order to arrange that A is a.p., and there are infinitely many such x since we build A coinfinite.

Corollary 11.9. *The class of r.e. sets A such that A is tardy* (i.e., *such that A is recursive or* $\deg(A)$ *is half of a minimal pair*) *is not invariant under automorphisms of $\mathcal{E}$, and hence is not $\mathcal{E}$-definable.*

Proof. By [28, Theorem XIII.2.2] if A is r.e., then $\deg(A)$ is half of a minimal pair iff A is tardy and nonrecursive. By Theorem 11.8 there is a nonrecursive tardy r.e. set A such that for all $B \equiv_{\mathrm{T}} A$, B is a.p.; by Theorem 11.5, B is automorphic to a complete set C; and hence $\deg(C)$ is not half of a minimal pair. □

Ambos-Spies and Nies [1] exhibited a property $P(A)$ which holds of an r.e. set A iff $\deg(A)$ is half of a minimal pair. Hence, like the property $Q(A)$ in Theorem 1.1, $P(A)$ guarantees that A is incomplete. However, unlike $Q(A)$ the property $P(A)$ was not defined in the language of $\mathcal{E}$ but required an extra predicate. By Corollary 11.9 there can be no $\mathcal{E}$-*definable* property defining this class of r.e. sets.

Theorem 11.10. *If A is low and simple, then A is almost prompt.*

Proof. Let $H_{\overline{A}} = \{e : W_e \cap \overline{A} \neq \emptyset\}$. If A is low (or even if $\overline{A}$ is semi-low), then $H_{\overline{A}} \leq_{\mathrm{T}} \mathbf{0}'$. Let $g(e,s)$ be a recursive function such that $\lim_s g(e,s)$ is the characteristic function of $H_{\overline{A}}$.

Let $W_a = A$. Fix s. Define $W_{j(e,k)} = W_e \cap [k,\infty)$, and $g(e,k,s) = g(j(e,k),s)$. For every $x, e \leq s$, if $x \in W_{e,s+1} - W_{e,s}$, let

$$t(x,e,s) = (\mu t \geq s)[x \in W_{a,t} \ \vee \ (\forall k \leq x)[g(e,k,t) = 1]],$$

and $t(x,e,s) = s$ otherwise. If $W_e \subseteq^* A$, then the second clause fails for almost all x and s. Let $h(s) = \max\{t(x,e,s) : x, e \leq s\}$. Define $A_s = W_{a,h(s)}$. Putting the enumerations $\{A_s\}_{s\in\omega}$, and $\{W_{e,s}\}_{e,s\in\omega}$ into Definition 2.2, we have

$$(\forall e)[W_e \subseteq^* A \implies W_e \setminus A =^* \emptyset]. \tag{86}$$

We claim that A is a.p. via the identity function $p(s) = s$. Suppose that

$$\overline{A} = X_e^n = (W_{e_1} - W_{e_2}) \cup (W_{e_3} - W_{e_4}) \dots , \tag{87}$$

as in (70). Without loss of generality we may assume that the indices e_j have been adjusted so that

$$(\forall j)_{1\leq j<n}(\forall s)[W_{e_j} \supseteq W_{e_{j+1}} \ \ \& \ \ W_{e_j,s} \supseteq W_{e_{j+1},s}]. \tag{88}$$

Choose the maximum odd $j \leq n$ such that $W_{e_j} - W_{e_{j+1}}$ is infinite (*i.e.*, choose the rightmost parenthetical component of (87) which is infinite). By (87), (88), and the maximality of j, $W_{e_{j+1}} \subseteq^* A$, so by (86) we have $W_{e_{j+1}} \setminus A =^* \emptyset$. But since A is simple the infinitely many elements in $Z =_{\mathrm{dfn}} W_{e_j} \setminus A$ cannot all remain in $\overline{A}$ forever. By the maximality of j and (88) they cannot move to another component of X_e^n. Hence, infinitely many of the $x \in Z \searrow A$ must be in $A \setminus W_{e_{j+1}}$ and must therefore witness $x \in X_{e,s}^n \cap A_s$. □

Lemma 11.11. *If A is almost prompt, and B is r.e., then $C = A \oplus B$ is almost prompt.*

Proof. Let $C = A \oplus B =_{\mathrm{dfn}} \{2x : x \in A\} \cup \{2x+1 : x \in B\}$. Let A be a.p. via $\{A_s\}_{s\in\omega}$ and $p(s)$. Let $\{B_s\}_{s\in\omega}$ be a recursive enumeration of B. Let $C_s = A_s \cup B_s$. Given X_e^n as in (70), define $Y_e^n = \{x : 2x \in X_e^n\}$ and define $Y_{e,s}^n = \{x : 2x \in X_{e,s}^n\}$. Let $q(s)$ be obtained from $p(s)$ and $\{Y_{e,s}^n\}_{n,e,s\in\omega}$ as in the Conversion Lemma 11.4. Assume $X_e^n = \overline{C}$. Then $Y_e^n = \overline{A}$. Hence, $x \in Y_{e,s}^n \cap A_{q(s)}$ for some x and s, because A is a.p., so $2x \in X_{e,s}^n \cap C_{q(s)}$. Thus, C is a.p. via $q(s)$. □

Theorem 11.12. *In every nonzero r.e. degree* $\mathbf{d}$ *there exists an almost prompt set A.*

Proof. By the Robinson Jump Interpolation Theorem [28, Theorem VIII.4.4], choose a low r.e. degree $\mathbf{b} \leq \mathbf{d}$. Choose $B \in \mathbf{b}$ low and simple, and choose $D \in \mathbf{d}$ r.e. Let $A = B \oplus D$. Then $A \in \mathbf{d}$ and A is a.p. by Theorem 11.11. □

11.3. **Very tardy sets.**

Definition 11.13. Let A be an r.e. set and let $\{A_s\}_{s\in\omega}$ be a recursive enumeration of A.

(i) We say A is *very tardy* if A is not almost prompt, namely if for every nondecreasing recursive function $p(s)$,

$$(\exists n)(\exists e)[X^n_e = \overline{A} \ \ \& \ \ (\forall y)(\forall s)[y \in X^n_{e,s} \implies y \notin A_{p(s)}]]. \tag{89}$$

(ii) We say A is *n-tardy* if in (i) the fixed n works uniformly for *all* such functions p, namely for every nondecreasing recursive function $p(s)$, there exists e such that the matrix of (89) holds.

The main idea about a very tardy set A is that if $x \in X^n_{e,s}$, then x can later enter A, but x must first undergo a delay until at least stage $p(s) + 1$ before doing so. Since prompt sets are almost prompt it follows that very tardy sets are tardy, hence the name. Note that A is 0-tardy iff $A = \omega$, and A is 1-tardy iff A is recursive. The 2-tardy and 3-tardy sets play a special role in our work.

In [8] Harrington and Soare introduced a property $Q(A)$ which holds of some nonrecursive sets and guarantees that A is incomplete. In [10] and [11] we shall show that $Q(A)$ implies that A is 2-tardy, and that if A is a small major subset of C and is 2-tardy, then $Q(A)$ holds via C. Thus, the property of 2-tardy is what we want to ensure incompleteness of A, but unfortunately the property of being 2-tardy is not $\mathcal{E}$-definable, so we needed to pass to $Q(A)$ to achieve an equivalent $\mathcal{E}$-definable property. This connection between $Q(A)$ and A being 2-tardy will also be used in [10] where we prove Theorem 1.4.

As in [8] we say (in the style of [5]) that an r.e. set B is *hemi-Q*, written $HQ(B)$, if there is an r.e. set A satisfying $Q(A)$ such that A can be split into the disjoint union of nonrecursive r.e. sets B and C. Note that if $HQ(B)$, then $B \leq_{\mathrm{T}} A$ so B must be incomplete because A is. Since $HQ(B)$ is $\mathcal{E}$-definable, any automorphic image of B must also be incomplete. If $HQ(B)$, then B is 3-tardy.

It is easy to prove that if A and B are very tardy, then so are $A \cap B$ and $A \cup B$, just as the tardy sets are also closed under union and intersection. However, the almost prompt sets are closed under neither union nor intersection, just as the prompt sets are closed under neither union nor intersection. In contrast the *promptly simple* sets form a filter [28, Exer. XIII.1.12].

12. Related results and open questions

Harrington explained in outline form the Δ^0_3-automorphism method to P. Cholak who subsequently developed in [2] and [3] an alternative version and extension of the method and used it to prove some related results. In particular, Cholak proved that for every high r.e. degree $\mathbf{d}$ and every coinfinite r.e. set A there is an r.e. set B in $\mathbf{d}$ such that $\mathcal{L}^*(A) \cong \mathcal{L}^*(B)$. This is quite interesting because it represents the next development in the program initiated by Martin [23] who proved it for the case of A a maximal set, and Lachlan [14] who extended it for the case of A hh-simple (*i.e.*, of $\mathcal{L}^*(A)$ a Boolean algebra). Maass, Shore, and Stob [22] had proved that the conclusion could not be strengthened to assert $A \simeq B$. Cholak also used his version of the method to give an alternative proof of Theorem 1.3. In his announcement Cholak also raised the following three questions:

1. (Item 6.) For all promptly simple high degrees $\mathbf{h}$ and for all promptly simple sets A is there an r.e. set $B \in \mathbf{h}$ such that $A \simeq_{\Delta^0_3} B$?

2. (Item 8.) For all high degrees $\mathbf{h}$ is there a nonrecursive r.e. set A such that for all r.e. sets B with $\deg(B) \geq \mathbf{h}$, $A \not\simeq_{\Delta^0_3} B$?
3. (Item 11.) Let A be a promptly simple set and and $\mathbf{p}$ a promptly simple degree. If $\overline{A}$ is semi-low$_2$ and has the outer splitting property, then is there an r.e. set B with $\deg(B) \leq \mathbf{p}$, such that $A \simeq_{\Delta^0_3} B$?

We negatively answer these questions by Theorem 1.10, Theorem 1.4, and Theorem 1.9 respectively.

For any $A \subseteq \omega$ let $\mathcal{E}^A$ be the lattice of subsets of ω which are r.e. in A. T. Hammond [6] used the effective automorphism machinery with the improvement to semi-low$_{1.5}$ by Maass [19] to prove that $\mathcal{E}^A \cong_{eff} \mathcal{E}^B$ if and only if $A' \equiv_{\mathrm{T}} B'$. Recently Hammond and Harrington have used the Δ^0_3-automorphism method to prove that if $A'' \equiv_{\mathrm{T}} B''$, then $\mathcal{E}^A \cong \mathcal{E}^B$ by an isomorphism which is recursive in A''.

Downey and Stob [5] introduced a property $HHM(A)$, half-hemimaximal of an r.e. set A, and proved that such sets are automorphic to complete sets. The properties $HHM(A)$ and almost prompt overlap (because low simple sets have both properties) but do not coincide, because there is an atomless set which is prompt and hence a.p. but no HHM set is atomless.

Question 1. *Can Theorem 11.10 be strengthened to show that if A is low$_2$ and simple, then A is almost prompt?*

Question 2. *Characterize those r.e. sets A such that A is automorphic to a complete set.*

The key property needed in the proof seems to be something like $\mathcal{D}_\alpha \neq \emptyset$ as in Lemma 10.6 and Lemma 11.6, but it is not clear which external property of A this corresponds to. Notice that both the HHM and a.p. properties guarantee something like this but these apparently do not exhaust the possibilities. Nevertheless, further study of them may yield insight about how to code a complete set into a given orbit.

Toward this end Harrington and Soare have considered the property of A being d-simple defined by Lerman and Soare [17]. A coinfinite set A is *d-simple* if for all X there exists $Y \subseteq X$ such that

(i) $X \cap \overline{A} = Y \cap \overline{A}$, and
(ii) $(\forall Z)[Z - X \text{ infinite} \implies (Z - Y) \cap A \neq \emptyset]$.

Let $\mathbf{D}$ be the class of degrees containing a d-simple set. Lerman and Soare showed [17] that $\mathbf{D}$ includes the high degrees but $\mathbf{D}$ splits the low degrees. Since any d-simple set is clearly simple it follows by Theorem 11.10 that any low d-simple set is automorphic to a complete set.

Question 3. *Is* every *d-simple set automorphic to a complete set?*

This question is not of great intrinsic interest itself, but it appears to be on the cutting edge of the symmetry between the methodologies for generating automorphisms and for producing invariant properties (such as $Q(A)$), and may therefore be useful in gaining insight into the completeness phenomenon and Question 2.

Question 4. *Find an $\mathcal{E}$-definable property which defines those degrees containing an r.e. set which is not automorphic to a complete one.*

Harrington and Soare believe that they will be able to answer this question by constructing an $\mathcal{E}$-definable property which is similar to the property $HQ(A)$.

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Abstract. A set A of nonnegative integers is *computably enumerable (c.e.)*, also called *recursively enumerable (r.e.)*, if there is a computable method to list its elements. Let $\mathcal{E}$ denote the structure of the computably enumerable sets under inclusion, $\mathcal{E} = (\{W_e\}_{e\in\omega}, \subseteq)$. Most previously known automorphisms Φ of the structure $\mathcal{E}$ of sets were effective (computable) in the sense that Φ has an effective presentation. We introduce here a new method for generating noneffective automorphisms whose presentation is Δ^0_3, and we apply the method to answer a number of long open questions about the orbits of c.e. sets under automorphisms of $\mathcal{E}$. For example, we show that the orbit of every noncomputable (*i.e.*, nonrecursive) c.e. set contains a set of high degree, and hence that for all $n > 0$ the well-known degree classes $\mathbf{L}_n$ (the low$_n$ c.e. degrees) and $\overline{\mathbf{H}}_n = \mathbf{R} - \mathbf{H}_n$ (the complement of the high$_n$ c.e. degrees) are noninvariant classes.

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