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A COMPACTIFICATION OVER $\overline{M_g}$ OF THE UNIVERSAL MODULI SPACE OF SLOPE-SEMISTABLE VECTOR BUNDLES

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Contents

0. Introduction

0.1. **Compactifications of moduli problems.** Initial statements of moduli problems in algebraic geometry often do not yield compact moduli spaces. For example, the moduli space M_g of nonsingular, genus $g \geq 2$ curves is open. Compact moduli spaces are desired for several reasons. Degeneration arguments in moduli require compact spaces. Also, there are more techniques available to study the global geometry of compact spaces. It is therefore valuable to find natural compactifications of open moduli problems. In the case of M_g, there is a remarkable compactification due to P. Deligne and D. Mumford. A connected, reduced, nodal curve C of arithmetic genus $g \geq 2$ is *Deligne-Mumford stable* if each nonsingular rational component contains at least three nodes of C. $\overline{M_g}$, the moduli space of Deligne-Mumford stable genus g curves, is compact and includes M_g as a dense open set.

There is a natural notion of stability for a vector bundle E on a nonsingular curve C. Let the slope μ be defined as follows: $\mu(E) = degree(E)/rank(E)$. E is *slope-stable (slope-semistable)* if

$$\mu(F) < (\leq)\ \mu(E) \tag{1}$$

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for every proper subbundle F of E. When the degree and rank are not coprime, the moduli space of slope-stable bundles is open. $U_C(e,r)$, the moduli space of slope-semistable bundles of degree e and rank r, is compact. An open set of $U_C(e,r)$ corresponds bijectively to isomorphism classes of stable bundles. In general, points of $U_C(e,r)$ correspond to equivalence classes (see section (1.1)) of semistable bundles.

The moduli problem of pairs (C,E), where E is a slope-semistable vector bundle on a nonsingular curve C, cannot be compact. No allowance is made for curves that degenerate to nodal curves. A natural compactification of this moduli problem of pairs is presented here.

0.2. **Compactification of the moduli problem of pairs.** let $\mathbf{C}$ be a fixed algebraically closed field. As before, let M_g be the moduli space of nonsingular, complete, irreducible, genus $g \geq 2$ curves over the field $\mathbf{C}$. For each $[C] \in M_g$, there is a natural projective variety, $U_C(e,r)$, parametrizing degree e, rank r, slope-semistable vector bundles (up to equivalence) on C. For $g \geq 2$, let $U_g(e,r)$ be the set of equivalence classes of pairs (C,E), where $[C] \in M_g$ and E is a slope-semistable vector bundle on C of the specified degree and rank. A good compactification, K, of the moduli set of pairs $U_g(e,r)$ should satisfy at least the following conditions:

(i) K is a projective variety that functorially parametrizes a class of geometric objects.
(ii) $U_g(e,r)$ functorially corresponds to an open dense subset of K.
(iii) There exists a morphism $\eta : K \to \overline{M_g}$ such that the natural diagram commutes:

$$\begin{array}{ccc} U_g(e,r) & \longrightarrow & K \\ \downarrow & & \downarrow \eta \\ M_g & \longrightarrow & \overline{M_g} \end{array}$$

(iv) For each $[C] \in M_g$, there exists a functorial isomorphism

$$\eta^{-1}([C]) \cong U_C(e,r)/Aut(C).$$

The main result of this paper is the construction of a projective variety $\overline{U_g(e,r)}$ that parametrizes equivalence classes of slope-semistable, torsion free sheaves on Deligne-Mumford stable, genus g curves and satisfies conditions (i-iv) above.

The definition of slope-semistability of torsion free sheaves (due to C. Seshadri) is given in section (1.1).

0.3. **The method of construction.** An often successful approach to moduli constructions in algebraic geometry involves two steps. In the first step, extra data is added to rigidify the moduli problem. With the additional data, the new moduli problem is solved by a Hilbert or Quot scheme. In the second step, the extra data is removed by a group quotient. Geometric Invariant Theory is used to study the quotient problem in the category of algebraic schemes. In good cases, the final quotient is the desired moduli space.

In order to rigidify the moduli problem of pairs, the following data is added to (C,E):

(i) An isomorphism $\mathbf{C}^{N+1} \xrightarrow{\sim} H^0(C, \omega_C^{10})$,
(ii) An isomorphism $\mathbf{C}^n \xrightarrow{\sim} H^0(C,E)$.

Note ω_C is the canonical bundle of C. The numerical invariants of the moduli problem of pairs are the genus g, degree e, and rank r. The rigidified problem should have no more numerical invariants. Hence, N and n must be determined by g, e, and r. Certainly, $N = 10(2g-2) - g$ by Riemann-Roch. It is assumed $H^1(E) = 0$ and E is generated by global sections. In the end, it is checked these assumptions are consequences of the stability condition for sufficiently high degree bundles. We see $n = \chi(E) = e + r(1-g)$.

The isomorphism of (i) canonically embeds C in $\mathbf{P}^N = \mathbf{P}^N_{\mathbf{C}}$. The isomorphism of (ii) exhibits E as a canonical quotient

$$\mathbf{C}^n \otimes \mathcal{O}_C \to E \to 0.$$

The basic parameter spaces in algebraic geometry are Hilbert and Quot schemes. Subschemes of a fixed scheme X are parametrized by Hilbert schemes $Hilb(X)$. Quotients of a fixed sheaf F on X are parametrized by Quot schemes $Quot(X, F)$. The rigidified curve C can be parametrized by a Hilbert scheme H of curves in $\mathbf{P}^N$, and the rigidified bundle E can be parametrized by a Quot scheme $Quot(C, \mathbf{C}^n \otimes \mathcal{O}_C)$ of quotients on C. In fact, Quot schemes can be defined in a relative context. Let U_H be the universal curve over the Hilbert scheme H. The family of Quot schemes, $Quot(C, \mathbf{C}^n \otimes \mathcal{O}_C)$, defined as $C \hookrightarrow \mathbf{P}^N$ varies in H is simply the relative Quot scheme of the universal curve over the Hilbert scheme: $Quot(U_H \to H, \mathbf{C}^n \otimes \mathcal{O}_{U_H})$. This relative Quot scheme is the parameter space of the rigidified pairs (up to scalars). The Quot scheme set up is discussed in detail in section (1.3).

The actions of $GL_{N+1}(\mathbf{C})$ on $\mathbf{C}^{N+1}$ and $GL_n(\mathbf{C})$ on $\mathbf{C}^n$ yield an action of $GL_{N+1} \times GL_n$ on the rigidified data. There is an induced product action on the relative Quot scheme. It is easily seen the scalar elements of the groups act trivially on the Quot scheme. $\overline{U_g(e,r)}$ is constructed via the quotient:

$$Quot(U_H \to H, \mathbf{C}^n \otimes \mathcal{O}_{U_H})/SL_{N+1} \times SL_n. \tag{2}$$

There is a projection of the rigidified problem of pairs $\{(C, E)$ with isomorphisms (i) and (ii)$\}$ to a rigidified moduli problem of curves $\{C$ with isomorphism (i)$\}$. The projection is GL_{N+1}-equivariant with respect to the natural GL_{N+1}-action on the rigidified data of curves. The Hilbert scheme H is a parameter space of the rigidified problem of curves (up to scalars). By results of Gieseker ([Gi]) reviewed in section (1.2), the quotient H/SL_{N+1} is $\overline{M_g}$. A natural morphism $\overline{U_g(e,r)} \to \overline{M_g}$ is therefore obtained. Gieseker's results require the choice in isomorphism (i) of at least the 10-canonical series.

The technical heart of the paper is the study of the Geometric Invariant Theory problem (2). The method is divide and conquer. The action of SL_n alone is first studied. The SL_n-action is called the fiberwise G.I.T. problem. It is solved in sections (2) - (6). The action of SL_{N+1} alone is then considered. There are two pieces in the study of the SL_{N+1} action. First, Gieseker's results in [Gi] are used in an essential way. Second, the abstract G.I.T. problem of SL_{N+1} acting on $\mathbf{P}(Z) \times \mathbf{P}(W)$ where Z, W are representations of SL_{N+1} is studied. If the linearization is taken to be $\mathcal{O}_{\mathbf{P}(Z)}(k) \otimes \mathcal{O}_{\mathbf{P}(W)}(1)$ where $k >> 1$, there are elementary set theoretic relationships between the stable and unstable loci of $\mathbf{P}(Z)$ and $\mathbf{P}(Z) \times \mathbf{P}(W)$. These relationships are determined in section (7). Roughly speaking, the abstract lemmas are used to import the invariants Gieseker has determined in [Gi] to the problem at hand. In section (8), the solution of fiberwise problem is combined with the study of the SL_{N+1}-action to solve the product G.I.T. problem (2).

0.4. **Relationship with past results.** In [G-M], D. Gieseker and I. Morrison propose a different approach to a compactification over $\overline{M_g}$ of the universal moduli space of slope-semistable bundles. The moduli problem of pairs is rigidified by adding only the data of an isomorphism $\mathbf{C}^n \to H^0(C, E)$. By further assumptions on E, an embedding into a Grassmannian is obtained

$$C \hookrightarrow \mathbf{G}(rank(E), H^0(C, E)^*).$$

The rigidified data is thus parametrized by a Hilbert scheme of a Grassmannian. The GL_n-quotient problem is studied to obtain a moduli space of pairs.

Recent progress along this alternate path has been made by D. Abramovich, L. Caporaso, and M. Teixidor ([A], [Ca], [T]). A compactification, $\overline{P_{g,e}}$, of the universal Picard variety is constructed in [Ca]. There is a natural isomorphism

$$\nu : \overline{P_{g,e}} \to \overline{U_g(e,1)}.$$

This isomorphism is established in section (10). In the rank 2 case, the approach of [G-M] yields a compactification not equivalent to $\overline{U_g(e,2)}$ ([A]). The higher rank constructions of [G-M] have not been completed. They are certainly expected to differ from $\overline{U_g(e,r)}$.

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1. The quotient construction

1.1. **Definitions.** Let C be a genus $g \geq 2$, Deligne-Mumford stable curve. A coherent sheaf E on C is *torsion free* if

$$\forall x \in C, \quad depth_{\mathcal{O}_x}(E_x) = 1,$$

or equivalently, if there does not exist a subsheaf

$$0 \to F \to E$$

such that $dim(Supp(F)) = 0$. Let

$$C = \bigcup_1^q C_i,$$

where the curves C_i are the irreducible components of C. Let ω_i be the degree of the restriction of the canonical bundle ω_C to C_i. Let r_i be the the rank of E at the generic point of C_i. The *multirank* of E is the q-tuple $(r_1, \ldots, r_q)$. E is of *uniform rank* r if $r_i = r$ for each C_i. If E is of uniform rank r, define the degree of E by

$$e = \chi(E) - r(1-g).$$

A torsion free sheaf E is defined to be *slope-stable (slope-semistable)* if for each nonzero, proper subsheaf

$$0 \to F \to E$$

with multirank $(s_1, \dots, s_q)$, the following inequality holds:

$$\frac{\chi(F)}{\sum_1^q s_i\omega_i} < \frac{\chi(E)}{\sum_1^q r_i\omega_i}, \tag{3}$$

respectively,

$$\left(\frac{\chi(F)}{\sum_1^q s_i\omega_i} \leq \frac{\chi(E)}{\sum_1^q r_i\omega_i}\right).$$

The above is Seshadri's definition of slope-(semi)stability in the case of canonical polarization. In case E is a vector bundle on a nonsingular curve C, the slope-(semi)stability condition (1) of section (0.1) and condition (3) above coincide. A slope-semistable sheaf has a Jordan-Holder filtration with slope-stable factors. Two slope-semistable sheaves are *equivalent* if they possess the same set of Jordan-Holder factors. Two equivalence classes are said to be *aut-equivalent* if they differ by an automorphism of the underlying curve C.

For $g \geq 2$ and each pair of integers $(e,\ r \geq 1)$, a projective variety $\overline{U_g(e,r)}$ and a morphism

$$\eta : \overline{U_g(e,r)} \to \overline{M_g}$$

satisfying the following properties are constructed in Theorem (8.2.1). There is a functorial, bijective correspondence between the points of $\overline{U_g(e,r)}$ and aut-equivalence classes of slope-semistable, torsion free sheaves of uniform rank r and degree e on Deligne-Mumford stable curves of genus g. The image of an aut-equivalence class under η is the moduli point of the underlying curve.

$\overline{U_g(e,r)}$ and η will be constructed via Geometric Invariant Theory. The G.I.T. problem is described in sections (1.2-1.7). The solution is developed in sections (2-8) of the paper. Basic properties of $\overline{U_g(e,r)}$ are studied in section (9). In particular, the equivalence of $\overline{U_g(e,1)}$ and $\overline{P_{g,e}}$ is established in section (10.3).

1.2. **Gieseker's construction.** We review Gieseker's beautiful construction of $\overline{M_g}$. Fix a genus $g \geq 2$. Define:

$$d = 10(2g-2),$$
$$N = d - g.$$

Consider the Hilbert scheme $H_{g,d,N}$ of genus g, degree d, curves in $\mathbf{P}^N_{\mathbf{C}}$. Let

$$H_g \subset H_{g,d,N}$$

denote the locus of nondegenerate, 10-canonical, Deligne-Mumford stable curves of genus g. H_g is naturally a closed subscheme of the open locus of nondegenerate, reduced, nodal curves. In fact, H_g is a nonsingular, irreducible, quasi-projective variety ([Gi]). The symmetries of $\mathbf{P}^N$ induce a natural $SL_{N+1}(\mathbf{C})$-action on H_g. D. Gieseker has studied the quotient H_g/SL_{N+1} via geometric invariant theory. It is shown in [Gi] that, for suitable linearizations, H_g/SL_{N+1} exists as a G.I.T. quotient and is isomorphic to $\overline{M_g}$.

1.3. **Relative quot schemes.** Let U_H be the universal curve over H_g. We have a closed immersion

$$U_H \hookrightarrow H_g \times \mathbf{P}^N$$

and two projections:

$$\mu : U_H \to H_g,$$
$$\nu : U_H \to \mathbf{P}^N.$$

Let $\mathcal{O}_U$ be the structure sheaf of U_H. The Grothendieck relative Quot scheme is central to our construction. We will be interested in relative Quot schemes of the form

$$Quot(\mu : U_H \to H_g,\ \mathbf{C}^n \otimes \mathcal{O}_U,\ \nu^*(\mathcal{O}_{\mathbf{P}^N}(1)),\ f), \tag{4}$$

where f is a Hilbert polynomial with respect to the μ-relatively very ample line bundle $\nu^*(\mathcal{O}_{\mathbf{P}^N}(1))$. We denote the Quot scheme in (4) by $Q_g(\mu, n, f)$.

We recall the basic properties of the Quot scheme. There is a canonical projective morphism $\pi : Q_g(\mu, n, f) \to H_g$. The fibered product $Q_g(\mu, n, f) \times_{H_g} U_H$ is equipped with two projections:

$$\theta : Q_g(\mu, n, f) \times_{H_g} U_H \to Q_g(\mu, n, f),$$

$$\phi : Q_g(\mu, n, f) \times_{H_g} U_H \to U_H$$

and a universal θ-flat quotient

$$\mathbf{C}^n \otimes \mathcal{O}_{Q\times U} \simeq \phi^*(\mathbf{C}^n \otimes \mathcal{O}_U) \to \mathcal{E} \to 0. \tag{5}$$

Let ξ be a (closed) point of $Q_g(\mu, n, f)$. The point $\pi(\xi) \in H_g$ corresponds to the 10-canonical, Deligne-Mumford stable curve $U_{\pi(\xi)}$. Restriction of the universal quotient sequence (5) to $U_{\pi(\xi)}$ yields a quotient

$$\mathbf{C}^n \otimes \mathcal{O}_{U_{\pi(\xi)}} \to \mathcal{E}_\xi \to 0$$

with Hilbert polynomial

$$f(t) = \chi(\mathcal{E}_\xi \otimes \mathcal{O}_{U_{\pi(\xi)}}(t)).$$

The above is a functorial bijective correspondence between points $\xi \in Q_g(\mu, n, f)$ and quotients of $\mathbf{C}^n \otimes \mathcal{O}_{U_{\pi(\xi)}}$, $\pi(\xi) \in H_g$, with Hilbert polynomial f.

1.4. **Group actions.** Denote the natural actions of SL_{N+1} on H_g and U_H by:

$$\begin{array}{ccc} U_H \times SL_{N+1} & \xrightarrow{a_U} & U_H \\ \downarrow{\scriptstyle \mu\times id} & & \downarrow{\scriptstyle \mu} \\ H_g \times SL_{N+1} & \xrightarrow{a_H} & H_g \end{array}$$

Also define:

$$\overline{\mu} : U_H \times SL_{N+1} \overset{\mu\times inv}{\longrightarrow} H_g \times SL_{N+1} \xrightarrow{a_H} H_g,$$

$$\overline{\pi} : Q_g(\mu, n, f) \times SL_{N+1} \overset{\pi\times id}{\longrightarrow} H_g \times SL_{N+1} \xrightarrow{a_H} H_g.$$

There is a natural isomorphism between the two fibered products

$$\big(Q_g(\mu, n, f) \times SL_{N+1}\big) \times_{H_g} U_H \simeq Q_g(\mu, n, f) \times_{H_g} \big(U_H \times SL_{N+1}\big),$$

where the projection maps to H_g are $(\overline{\pi}, \mu)$ and $(\pi, \overline{\mu})$ in the first and second products respectively. The inversion in the definition of $\overline{\mu}$ is required for the isomorphism of the fibered products. There is a natural commutative diagram

$$\begin{array}{ccc} U_H \times SL_{N+1} & \xrightarrow{\overline{a}_U} & U_H \\ \downarrow{\scriptstyle \overline{\mu}} & & \downarrow{\scriptstyle \mu} \\ H_g & \xrightarrow{id} & H_g \end{array}$$

where

$$\overline{a}_U : U_H \times SL_{N+1} \overset{id\times inv}{\longrightarrow} U_H \times SL_{N+1} \xrightarrow{a_U} U_H.$$

We therefore obtain a natural map of schemes over $\mathbf{C}$:

$$\varrho : (Q_g(\mu, n, f) \times SL_{N+1}) \times_{H_g} U_H \to Q_g(\mu, n, f) \times_{H_g} U_H.$$

By the functorial properties of $Q_g(\mu, n, f)$, the ϱ-pull-back of the universal quotient sequence (5) on $Q_g(\mu, n, f) \times_{H_g} U_H$ yields a natural group action:

$$Q_g(\mu, n, f) \times SL_{N+1} \to Q_g(\mu, n, f).$$

Hence, the natural SL_{N+1}-action on H_g lifts naturally to $Q_g(\mu, n, f)$. There is a natural $SL_n(\mathbf{C})$-action on $Q_g(\mu, n, f)$ induced by the $SL_n(\mathbf{C})$-action on the tensor product $\mathbf{C}^n \otimes \mathcal{O}_U$. In fact, the SL_{N+1}-action and the SL_n-action commute on $Q_g(\mu, n, f)$. The commutation is most easily seen in the explicit linearized projective embedding developed below in section (1.6). Hence, there exists a well-defined $SL_{N+1} \times SL_n$-action. For suitable choices of n, f, and linearization, a component of the quotient $Q_g(\mu, n, f)/(SL_{N+1} \times SL_n)$ will be $\overline{U_g(e, r)}$.

1.5. **Relative embeddings.** Following [Gr], a family of relative projective embeddings of $Q_g(\mu, n, f)$ over H_g is constructed. Since the inclusion

$$Q_g(\mu, n, f) \times_{H_g} U_g \hookrightarrow Q_g(\mu, n, f) \times \mathbf{P}^N$$

is a closed immersion, the universal quotient $\mathcal{E}$ can be extended by zero to $Q_g(\mu, n, f) \times \mathbf{P}^N$. Let

$$\theta_{\mathbf{P}} : Q_g(\mu, n, f) \times \mathbf{P}^N \to Q_g(\mu, n, f)$$

be the projection. The universal quotient sequence (5) induces the following sequence on $Q_g(\mu, n, f) \times \mathbf{P}^N$:

$$0 \to \mathcal{K} \to \mathbf{C}^n \otimes \mathcal{O}_{Q \times \mathbf{P}^N} \to \mathcal{E} \to 0.$$

Since $\mathcal{E}$ and $\mathcal{O}_{Q \times \mathbf{P}^N}$ are $\theta_{\mathbf{P}}$-flat, $\mathcal{K}$ is $\theta_{\mathbf{P}}$-flat. By the semicontinuity theorems for $\theta_{\mathbf{P}}$-flat, coherent sheaves, there exists an integer t_α such that for each $t > t_\alpha$ and each $\xi \in Q_g(\mu, n, f)$:

$$h^1(\mathbf{P}^N, \mathcal{K}_\xi \otimes \mathcal{O}_{\mathbf{P}^N}(t)) = 0, \tag{6}$$

$$h^0(\mathbf{P}^N, \mathcal{E}_\xi \otimes \mathcal{O}_{\mathbf{P}^N}(t)) = f(t), \tag{7}$$

$$h^1(\mathbf{P}^N, \mathcal{E}_\xi \otimes \mathcal{O}_{\mathbf{P}^N}(t)) = 0, \tag{8}$$

$$\mathbf{C}^n \otimes H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(t)) \to H^0(\mathbf{P}^N, \mathcal{E}_\xi \otimes \mathcal{O}_{\mathbf{P}^N}(t)) \to 0. \tag{9}$$

The surjection of (9) follows from (6) and the long exact cohomology sequence. For each $t > t_\alpha$, there is a well defined algebraic morphism (on points)

$$i_t : Q_g(\mu, n, f) \to \mathbf{G}(f(t), (\mathbf{C}^n \otimes H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(t)))^*)$$

defined by sending $\xi \in Q_g(\mu, n, f)$ to the subspace

$$H^0(\mathbf{P}^N, \mathcal{E}_\xi \otimes \mathcal{O}_{\mathbf{P}^N}(t))^* \subset (\mathbf{C}^n \otimes H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(t)))^*.$$

By the theorems of Cohomology and Base Change, it follows from conditions (6-8) there exists a surjection

$$\mathbf{C}^n \otimes H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(t)) \otimes \mathcal{O}_Q \simeq \theta_{\mathbf{P}*}(\mathbf{C}^n \otimes \mathcal{O}_{Q \times \mathbf{P}^N}(t)) \to \theta_{\mathbf{P}*}(\mathcal{E} \otimes \mathcal{O}_{\mathbf{P}^N}(t)) \to 0,$$

where $\theta_{\mathbf{P}*}(\mathcal{E} \otimes \mathcal{O}_{\mathbf{P}^N}(t))$ is a locally free, rank $f(t)$ quotient. The universal property of the Grassmannian defines i_t as a morphism of schemes. It is known that there exists an integer t_β such that for all $t > t_\beta$, the morphism $\pi \times i_t$ is a closed embedding:

$$\pi \times i_t : Q_g(\mu, n, f) \to H_g \times \mathbf{G}(f(t), (\mathbf{C}^n \otimes H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(t)))^*).$$

The morphisms $\pi \times i_t,\ t > t_\beta(g, n, f)$, form a countable family of relative projective embeddings of the Quot scheme $Q_g(\mu, n, f)$ over H_g.

1.6. **Gieseker's linearization.** Since the Hilbert scheme $H_{d,g,N}$ is the Quot scheme

$$Quot(\mathbf{P}^N \to Spec(\mathbf{C}),\ \mathcal{O}_{\mathbf{P}^N}, \mathcal{O}_{\mathbf{P}^N}(1),\ h(s) = ds - g + 1),$$

there are closed embeddings for $s > s_\alpha$:

$$i'_s : H_{d,g,N} \to \mathbf{G}(h(s), H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(s))^*).$$

By results of Gieseker, an integer $\overline{s}(g)$ can be chosen so that the SL_{N+1}-linearized G.I.T. problem determined by $i'_{\overline{s}}$ has two properties:

(i) H_g is contained in the stable locus.
(ii) H_g is closed in the semistable locus.

In order to make use of (i) and (ii) above, we will only consider immersions of the type $i'_{\overline{s}}$.

For each large t, there exists an immersion:

$$i_{\overline{s},t} : Q_g(\mu, n, f) \to \mathbf{G}(h(\overline{s}), H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(\overline{s}))^*) \times \mathbf{G}(f(t), (\mathbf{C}^n \otimes H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(t)))^*).$$

By the Plücker embeddings, we obtain

$$j_{\overline{s},t} : Q_g(\mu, n, f) \to \mathbf{P}(\bigwedge^{h(\overline{s})} H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(\overline{s}))^*) \times \mathbf{P}(\bigwedge^{f(t)} (\mathbf{C}^n \otimes H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(t)))^*).$$

The fact that the SL_{N+1} and SL_n-actions commute on $Q_g(\mu, n, f)$ now follows from the observation that these actions commute on $\mathbf{C}^n \otimes H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(t))^*$.

1.7. **The G.I.T. problem.** Let C be a Deligne-Mumford stable curve of genus $g \geq 2$. For any coherent sheaf F on C, it is not hard to see:

$$\chi(F \otimes \omega_C^t) = \chi(F) + (\sum_1^q s_i \omega_i) \cdot t, \tag{10}$$

where $(s_1, \ldots, s_q)$ is the multirank of F ([Se]). Equation (10) and the slope inequalities of section (1.1) yield a natural correspondence

$$(C, E) \to (C, E \otimes \omega_C^t)$$

between slope-semistable, uniform rank r, torsion free sheaves of degrees e and $e + rt(2g-2)$. Therefore, it suffices to construct $\overline{U_g(e, r)}$ for $e >> 0$.

The strategy for studying the G.I.T. quotient

$$Q_g(\mu, n, f)/(SL_{N+1} \times SL_n)$$

is as follows. First a rank $r \geq 1$ is chosen. Then the degree $e > e(g, r)$ is chosen very large. The Hilbert polynomial is determined by:

$$f_{e,r}(t) = e + r(1-g) + r10(2g-2)t. \tag{11}$$

For $[C] \in H_g$, $f_{e,r}(t)$ is the Hilbert polynomial of degree e, uniform rank r, torsion free sheaves on C with respect to $\mathcal{O}_{\mathbf{P}^N}(1)$. The integer n is fixed by the Euler characteristic, $n = f_{e,r}(0)$. As remarked in section (0.3) of the introduction, n will equal $h^0(C, E)$ for semistable pairs. Let

$$\hat{t}(g, r, e) \;=\; t_\beta(g, n = f_{e,r}(0), f_{e,r})$$

be the constant defined in section (1.5) for $Q_g(\mu, f_{e,r}(0), f_{e,r})$. A very large $t > \hat{t}(g, r, e)$ is then chosen. Selecting e and t are the essential choices that make the G.I.T. problem well-behaved. Finally, to determine a linearization of the $SL_{N+1} \times SL_n$-action on the image of $j_{\overline{s},t}$, weights must be chosen on the two projective spaces. These are chosen so that almost all the weight is on the first,

$$\mathbf{P}(\bigwedge^{h(\overline{s})} H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(\overline{s}))^*). \tag{12}$$

Since SL_n acts only on the second factor of the product, the weighting is irrelevant to the G.I.T. problem for the SL_n-action alone. The SL_n action is studied in sections (2)-(6). Since SL_{N+1} acts on both factors, the weighting is very relevant to the SL_{N+1}- G.I.T. problem. General results of section (7) show that in the case of extreme weighting, information on the stable and unstable loci of the SL_{N+1}-action on first factor can be transferred to the SL_{N+1}-action on the product of the factors. Gieseker's study of the SL_{N+1}-action on the first factor (12) can therefore be used. In section (8), knowledge of the SL_n and SL_{N+1} G.I.T problems is combined to solve the $SL_{N+1} \times SL_n$ G.I.T. problem on $Q_g(\mu, n, f)$.

2. The fiberwise G.I.T. problem

2.1. **The fiberwise result.** The fiber of $\pi : Q_g(\mu, n, f) \to H_g$ over a point $[C] \in H_g$ is the Quot scheme

$$Q_g(C, n, f) = Quot(C \to Spec(\mathbf{C}),\ \mathbf{C}^n \otimes \mathcal{O}_C,\ \omega_C^{10},\ f).$$

For large t, the morphism i_t embeds $Q_g(C, n, f)$ in

$$\mathbf{G}(f(t), (\mathbf{C}^n \otimes Sym^t(H^0(C, \omega_C^{10})))^*).$$

The embedding i_t yields an SL_n-linearized G.I.T. problem on $Q_g(C, n, f)$. Before examining the global G.I.T. problem for the construction of $\overline{U_g(e, r)}$, we will study this fiberwise G.I.T. problem. The main result is:

Theorem 2.1.1. *Let $g \geq 2$, $r > 0$ be integers. There exist bounds $e(g, r) > r(g-1)$ and $t(g, r, e) > \hat{t}(g, r, e)$ such that for each pair $e > e(g, r)$, $t > t(g, r, e)$ and any $[C] \in H_g$, the following holds:*

A point $\xi \in Q_g(C, n = f_{e,r}(0), f_{e,r})$ corresponding to a quotient

$$\mathbf{C}^n \otimes \mathcal{O}_C \to E \to 0$$

is G.I.T. stable (semistable) for the SL_n-linearization determined by i_t if and only if E is a slope-stable (slope-semistable), torsion free sheaf on C and

$$\psi : \mathbf{C}^n \otimes H^0(C, \mathcal{O}_C) \to H^0(C, E)$$

is an isomorphism.

P. Newstead has informed the author that a generalization of this fiberwise G.I.T. problem has been solved recently by C. Simpson in [Si]. A slight twist in our approach is the uniformity of bound needed for each $[C] \in H_g$. The proof will be developed in many steps.

2.2. **The Numerical Criterion.** Stability for a point ξ in a linearized G.I.T. problem can be checked by examining certain limits of ξ along 1-parameter subgroups. This remarkable fact leads to the Numerical Criterion. The most general form of the Numerical Criterion is presented in section (7.2). A more precise version for the fiberwise G.I.T. problem is stated here.

Fix a vector space Z with the trivial SL_n-action. In the applications below,

$$Z \stackrel{\sim}{=} Sym^t(H^0(C,\omega_C^{10})).$$

Consider the linearized SL_n-action on $\mathbf{G}(k,(\mathbf{C}^n \otimes Z)^*)$ obtained from the standard representation of SL_n on $\mathbf{C}^n$. Let $\xi \in \mathbf{G}(k,(\mathbf{C}^n \otimes Z)^*)$. The element ξ corresponds to a k-dimensional quotient

$$\rho_\xi : \mathbf{C}^n \otimes Z \to K_\xi.$$

Let $\overline{v} = (v_1, \ldots, v_n)$ be a basis of $\mathbf{C}^n$ with integer weights $\overline{w} = (w(v_1), \ldots, w(v_n))$. For combinatorial convenience, the additional condition that the weights sum to zero is avoided here. The representation weights of the corresponding 1-parameter subgroup of SL_n are given by rescaling: $e_i = w(v_i) - \sum_i w(v_i)/n$. An element $a \in \mathbf{C}^n \otimes Z$ is said to be $\overline{v}$-*pure* if it lies in a subspace of the form $v_i \otimes Z$. The weight, $w(a)$, of such an element is defined to be $w(v_i)$. The Numerical Criterion yields:

(1) ξ is unstable if and only if there exists a basis $\overline{v}$ of $\mathbf{C}^n$ and weights $\overline{w}$ with the following property. For any k-tuple of $\overline{v}$-pure elements $(a_1, \ldots, a_k)$ such that $(\rho_\xi(a_1), \ldots, \rho_\xi(a_k))$ is a basis of K_ξ, the inequality

$$\sum_{i=1}^{n} \frac{w(v_i)}{n} < \sum_{j=1}^{k} \frac{w(a_j)}{k}$$

is satisfied.

(2) ξ is stable (semistable) if and only if for every basis $\overline{v}$ of $\mathbf{C}^n$ and any nonconstant weights $\overline{w}$ the following holds. There exist $\overline{v}$-pure elements $(a_1, \ldots, a_k)$ such that $(\rho_\xi(a_1), \ldots, \rho_\xi(a_k))$ is a basis of K_ξ and

$$\sum_{i=1}^{n} \frac{w(v_i)}{n} \ > (\geq) \ \sum_{j=1}^{k} \frac{w(a_j)}{k}.$$

See, for example, [N] or [M-F].

2.3. **Step I.** The instability arguments will use the following lemma.

Lemma 2.3.1. *Let $g \geq 2$, $r > 0$, $e > r(g-1)$ be integers, $[C] \in H_g$. Suppose $\xi \in Q_g(C, n = f_{e,r}(0), f_{e,r})$ corresponds to a quotient*

$$\mathbf{C}^n \otimes \mathcal{O}_C \to E \to 0.$$

Let $U \subset \mathbf{C}^n$ be a subspace. Let $\psi(U \otimes H^0(C,\mathcal{O}_C)) = W \subset H^0(C,E)$. Let G be the subsheaf of E generated by W. For any $t > \hat{t}(g,e,r)$ the following holds: if

$$\frac{dim(U)}{n} > \frac{h^0(C, G \otimes \omega_C^{10t})}{f_{e,r}(t)}, \tag{13}$$

then ξ is G.I.T. unstable for the SL_n-linearization determined by i_t.

Proof. Let $u = dim(U)$. Inequality (13) implies $0 < u < n$. Let $\overline{v}$ be a basis of $\mathbf{C}^n$ such that $(v_1, \ldots, v_u)$ is a basis of u. Select weights as follows: $w(v_i) = 0$ for $1 \leq i \leq u$ and $w(v_i) = 1$ for $u+1 \leq i \leq n$. We now use the Numerical Criterion for the SL_n-action on $\mathbf{G}(f_{e,r}(t), (\mathbf{C}^n \otimes Sym^t(H^0(C, \omega_C^{10})))^*)$. The element ξ corresponds to a quotient

$$\psi^t : \mathbf{C}^n \otimes Sym^t(H^0(C, \omega_C^{10})) \to H^0(C, E \otimes \omega_C^{10t}) \to 0.$$

Suppose $(a_1, \ldots, a_{f_{e,r}(t)})$ is a tuple of $\overline{v}$-pure elements mapped by ψ^t to a basis of $H^0(C, E \otimes \omega_C^{10t})$. All a_j's have weight 1 except those contained in $U \otimes Sym^t(H^0(C, \omega_C^{10}))$ which have weight 0. The number of a_j's of weight 0 is hence bounded by $h^0(C, G \otimes \omega_C^{10t})$. Since

$$\sum_{i=1}^{n} \frac{w(v_i)}{n} = 1 - \frac{u}{n} < 1 - \frac{h^0(C, G \otimes \omega_C^{10t})}{f_{e,r}(t)} \leq \sum_{j=1}^{f_{e,r}(t)} \frac{w(a_j)}{f_{e,r}(t)},$$

the Numerical Criterion implies ξ is unstable. □

Proposition 2.3.1. *Let $g \geq 2$, $r > 0$ be integers. For each pair $e > r(g-1)$, $t > \hat{t}(g, r, e)$ and any $[C] \in H_g$, the following holds:*
If $\xi \in Q_g(C, n = f_{e,r}(0) f_{e,r})$ corresponds to a quotient

$$\mathbf{C}^n \otimes \mathcal{O}_C \to E \to 0,$$

where $\psi : \mathbf{C}^n \otimes H^0(C, \mathcal{O}_C) \to H^0(C, E)$ is not injective, then ξ is G.I.T. unstable for the SL_n-linearization determined by i_t.

Proof. Suppose ψ is not injective. Let $U \otimes H^0(C, \mathcal{O}_C)$ be the nontrivial kernel of ψ. The assumptions of Lemma (2.3.1) are easily checked since $W = 0$ and G is the zero sheaf. ξ is G.I.T unstable by Lemma (2.3.1). □

3. Cohomology bounds

3.1. **The bounds.** In order to further investigate the fiberwise SL_n-action, we need to control the first cohomology in various ways.

Lemma 3.1.1. *Let $g \geq 2$, $R > 0$ be integers. There exists an integer $b(g, R)$ with the following property. If E is a coherent sheaf on a Deligne-Mumford stable, genus g curve C such that:*

(i) *E is generated by global sections.*
(ii) *E has generic rank less than R on each irreducible component of C.*

Then $h^1(C, E) < b(g, R)$.

Proof. Since ω_C is ample and of degree $2g-2$, there is a bound $q(g) = 2g-2$ on the number of components of C. Since E is generated by global sections and has bounded rank, there exists an exact sequence:

$$\bigoplus_{1}^{qR} \mathcal{O}_C \to E \to \tau \to 0,$$

where $Supp(\tau)$ has at most dimension zero. Hence

$$h^1(C, E) \leq qR \cdot h^1(C, \mathcal{O}_C).$$

Since $h^1(C, \mathcal{O}_C) = g$, $b(g, R) = (2g-2)Rg + 1$ will have the required property. □

Lemma 3.1.2. *Let $g \geq 2$, $R > 0$ be integers. Let E be a coherent sheaf on a Deligne-Mumford stable, genus g curve C satisfying (i) and (ii) of Lemma (3.1.1). Suppose F is a subsheaf of E generated by global sections. Then*

$$|\chi(F)| < |\chi(E)| + b(g, R).$$

Proof. By Lemma (3.1.1), $h^1(C, F) < b(g, R)$. Therefore, $-b(g, R) < \chi(F)$. By Lemma (3.1.1) applied to E,

$$\chi(F) \leq h^0(C, F) \leq h^0(C, E) < \chi(E) + b(g, R) \leq |\chi(E)| + b(g, R).$$

The result follows. □

Lemma 3.1.3. *Let $g \geq 2$, $R > 0$, χ be integers. There exists an integer $p(g, R, \chi)$ with the following property. Let E be any coherent sheaf on any Deligne-Mumford stable, genus g curve C satisfying (i) and (ii) of Lemma (3.1.1) and satisfying $\chi(E) = \chi$. Let F be any subsheaf of E generated by k global sections:*

$$\mathbf{C}^k \otimes \mathcal{O}_C \to F \to 0. \tag{14}$$

Then for all $t > p(g, R, \chi)$:

(i) $h^1(C, F \otimes \omega_C^{10t}) = 0$.
(ii) $\mathbf{C}^k \otimes Sym^t(H^0(C, \omega_C^{10})) \to H^0(C, F \otimes \omega_C^{10t}) \to 0$.

Proof. Let $C = \bigcup_1^q C_i$. Let ω_i be the degree of ω_C restricted to C_i. Let $(s_1, \ldots, s_q)$ be the multirank of F. By (10) of section (1.7), the Hilbert polynomial of F with respect to ω_C^{10} is:

$$\chi(F \otimes \omega_C^{10t}) = \chi(F) + (\sum_1^q s_i \omega_i) \cdot 10t.$$

By Lemma (3.1.2), $|\chi(F)| < |\chi| + b(g, R)$. Also

$$0 \leq s_i < R, \quad 1 \leq q \leq 2g - 2, \quad 1 \leq \omega_i \leq 2g - 2.$$

Therefore the data g, R, and χ determine a finite collection of Hilbert polynomials

$$\{f_1, \ldots, f_m\}$$

that contains the Hilbert polynomial of every allowed sheaf F.

The morphism (14) yields a natural map $\psi : \mathbf{C}^k \to H^0(C, F)$. Let

$$im(\psi) = V \subset H^0(C, F).$$

We note that (14) can be factored:

$$\mathbf{C}^k \otimes \mathcal{O}_C \to V \otimes \mathcal{O}_C \to F \to 0.$$

Since (ii) is surjective if and only if the analogous map in which $\mathbf{C}^k$ is replaced by V is surjective, we can assume

$$k \leq h^0(C, F) \leq h^0(C, E) < |\chi| + b(g, R).$$

Suppose F is a coherent sheaf on a Deligne-Mumford stable, genus g curve satisfying:

(a) F is generated by $k < |\chi| + b(g, R)$ global sections: $\mathbf{C}^k \otimes \mathcal{O}_C \to F \to 0$.
(b) F has Hilbert polynomial f (with respect to ω_C^{10}).

Then there exists an integer $\bar{t}(g, k, f)$ such that for all $t > \bar{t}(g, k, f)$:

(i) $h^1(C, F \otimes \omega_C^{10t}) = 0$.
(ii) $\mathbf{C}^k \otimes Sym^t(H^0(C, \omega_C^{10})) \to H^0(C, F \otimes \omega_C^{10t}) \to 0$.

The existence of $\bar{t}(g,k,f)$ follows from statements (8) and (9) of section (1.5) applied to the Quot scheme $Q_g(\mu,k,f)$. Now let

$$p(g,R,\chi)=max\{\ \bar{t}(g,k,f_j)\mid 1\le k\le|\chi|+b(g,R),\ 1\le j\le m\}.$$

It follows easily that $p(g,R,\chi)$ has the required property. □

3.2. **Step II.** We apply these cohomology bounds along with the Numerical Criterion in another simple case. First, two definitions:

Define $R(g,r)=r(2g-2)+1$. If E is a coherent sheaf on C with multirank (r_i) and Hilbert polynomial $f_{e,r}$ with respect to ω_C^{10}, then by (10) of section (1.7),

$$\sum r_i\omega_i=r(2g-2).$$

Therefore, $r_i<R(g,r)$ for each i.

If E is a coherent sheaf on C, there is canonical sequence

$$0\to\tau_E\to E\to E'\to 0,$$

where τ_E is the *torsion subsheaf* of E and E' is torsion free.

Proposition 3.2.1. *Let $g\ge 2$, $r>0$, $e>r(g-1)$ be integers. There exists a bound $t_0(g,r,e)>\hat{t}(g,r,e)$ such that for each $t>t_0(g,r,e)$, and $[C]\in H_g$ the following holds:*

If $\xi\in Q_g(C,n=f_{e,r}(0),f_{e,r})$ corresponds to a quotient

$$\mathbf{C}^n\otimes\mathcal{O}_C\to E\to 0, \tag{15}$$

where $\psi\big(\mathbf{C}^n\otimes H^0(C,\mathcal{O}_C)\big)\cap H^0(C,\tau_E)\neq 0$, then ξ is G.I.T. unstable for the SL_n-linearization determined by i_t.

Proof. Let $U\subset\mathbf{C}^n$ be a 1 dimensional subspace such that

$$\psi(U\otimes H^0(C,\mathcal{O}_C))=W\subset H^0(C,\tau_E).$$

Let G be the subsheaf of E generated by W. For all t,

$$h^0(C,G\otimes\omega_C^{10t})\le h^0(C,\tau_E\otimes\omega_C^{10t})=h^0(C,\tau_E).$$

By Lemma (3.1.1),

$$h^0(C,\tau_E)\le h^0(C,E)=\chi(E)+h^1(C,E)<f_{e,r}(0)+b(g,R(g,r)).$$

There certainly exists a $t_0(g,r,e)>\hat{t}(g,r,e)$ satisfying $\forall t>t_0(g,r,e)$,

$$\frac{1}{n}>\frac{f_{e,r}(0)+b(g,R(g,r))}{f_{e,r}(t)}.$$

The Proposition is now a consequence of Lemma (2.3.1). □

4. Slope-unstable, torsion free sheaves

4.1. **Step III.** Propositions (2.3.1) and (3.2.1) conclude G.I.T. instability from certain undesirable properties of points in $Q_g(C,n,f_{e,r})$. In this section, G.I.T. instability is concluded from slope-instability in the case where ψ is an isomorphism and E is torsion free. The case where ψ is not an isomorphism (and E is arbitrary) is analyzed in section (5) where G.I.T. instability is established. The above results (for suitable choices of constants and linearizations) show only points of $Q_g(C,n,f_{e,r})$ where ψ is an isomorphism and E is a slope-semistable torsion free sheaf may be G.I.T. semistable. The G.I.T. (semi)stability results are established in section (6).

Proposition 4.1.1. *Let $g \geq 2$, $r > 0$ be integers. There exist bounds $e_1(g,r) > r(g-1)$ and $t_1(g,r,e) > \hat{t}(g,r,e)$ such that for each pair $e > e_1(g,r)$, $t > t_1(g,r,e)$ and any $[C] \in H_g$, the following holds:*

If $\xi \in Q_g(C, n = f_{e,r}(0), f_{e,r})$ corresponds to a quotient

$$\mathbf{C}^n \otimes \mathcal{O}_C \to E \to 0,$$

where $\psi : \mathbf{C}^n \otimes H^0(C, \mathcal{O}_C) \to H^0(C,E)$ is an isomorphism and E is a slope-unstable, torsion free sheaf, then ξ is G.I.T. unstable for the SL_n-linearization determined by i_t.

4.2. **Lemmas and the proof.** The proof of Proposition (4.1.1) requires two Lemmas which are used to apply Lemma (2.3.1). First a destabilizing subsheaf F of E is selected. F determines a filtration: $W = H^0(C,F) \subset H^0(C,E)$. $H^0(C,E)$ is identified with $\mathbf{C}^n$ by ψ. Let $U = \psi^{-1}(W)$. If F is generated by global sections, the vanishing theorems of section (3) can be applied. Riemann-Roch then shows the conditions of Lemma (2.3.1) for U, W follow from the destabilizing property of F (Lemma (4.2.2)). In fact, the vanishing argument is valid when F is generically generated by global sections. Lemma (4.2.1) guarantees that a destabilizing subsheaf F generically generated by global sections exists if E is of high degree.

Lemma 4.2.1. *Let $g \geq 2$, $r > 0$ be integers. There exists an integer $e_1(g,r) > r(g-1)$ such that for each $e > e_1(g,r)$ and $[C] \in H_g$ the following holds:*

If E is a slope-unstable, torsion free sheaf on C with Hilbert polynomial $f_{e,r}$ (with respect to ω_C^{10}), then there exists a nonzero, proper destabilizing subsheaf F of E and an exact sequence

$$0 \to \overline{F} \to F \to \tau \to 0, \tag{16}$$

where $\overline{F}$ is generated by global sections and $Supp(\tau)$ has at most dimension zero.

Proof. Since E is slope-unstable, there exists a nonzero, proper destabilizing subsheaf,

$$0 \to F \to E.$$

Let C be the union of components $\{C_i\}$ where $1 \leq i \leq q$, and let (s_i), (r_i) be the multiranks of F, E. Since E is torsion free and F is nonzero, the multirank of F is not identically zero. Since the Hilbert polynomial of E is $f_{e,r}$, we see (by section (3.2)) $R(g,r) = r(2g-2)+1$ satisfies $\forall i \;\; r_i < R(g,r)$. F can be chosen to have minimal multirank in the following sense. If F' is a nonzero subsheaf of F with multirank (s'_i) such that $\exists j \;\; s'_j < s_j$, then F' is not destabilizing. Let $\overline{F}$ be the subsheaf of F generated by the global sections $H^0(C,F)$. Since F is destabilizing:

$$h^0(C,F) \geq \chi(F) > \chi(E) \cdot \left(\frac{\sum s_i \omega_i}{r(2g-2)} \right) = (e + r(1-g)) \cdot \left(\frac{\sum s_i \omega_i}{r(2g-2)} \right).$$

Hence if $e > r(g-1)$, $h^0(C,F) > 0$ and $\overline{F}$ is nonzero. We now assume $e > r(g-1)$. Let $(\overline{s}_i)$ be the nontrivial multirank of $\overline{F}$. The sequence (16) has the required properties if and only if $(\overline{s}_i) = (s_i)$. Suppose $\exists j, \;\; \overline{s}_j < s_j$. Then $\overline{F}$ is not destabilizing, so

$$\chi(\overline{F}) \leq (e + r(1-g)) \cdot \left(\frac{\sum \overline{s}_i \omega_i}{r(2g-2)} \right).$$

We obtain

$$\chi(\overline{F}) < h^0(C,F) \cdot \left(\frac{\sum \overline{s}_i \omega_i}{\sum s_i \omega_i} \right) \leq h^0(C,F) \cdot \left(\frac{r(2g-2)-1}{r(2g-2)} \right).$$

The last inequality follows from the fact

$$0 < \sum \overline{s}_i \omega_i < \sum s_i \omega_i \leq r(2g-2).$$

Since $\overline{F}$ is generated by global sections, Lemma (3.1.1) yields

$$h^1(C, \overline{F}) < b(g, R(g,r)) = b.$$

We conclude,

$$h^0(C, \overline{F}) < b + h^0(C, F) \cdot \left(\frac{r(2g-2)-1}{r(2g-2)} \right).$$

Since $h^0(C, \overline{F}) = h^0(C, F)$ and

$$h^0(C, F) > \frac{e + r(1-g)}{r(2g-2)} \ ,$$

we obtain the bound

$$\left(\frac{e + r(1-g)}{r(2g-2)} \right) \cdot \left(\frac{1}{r(2g-2)} \right) < b.$$

Hence

$$e_1(g,r) = b(g, R(g,r)) \cdot (r^2(2g-2)^2) + r(g-1)$$

has the property required by the lemma. □

Lemma 4.2.2. *Let $g \geq 2$, $r > 0$, $e > e_1(g,r)$ be integers. There exists an integer $t_1(g,r,e) > \hat{t}(g,r,e)$ such that for each $t > t_1(g,r,e)$ and $[C] \in H_g$, the following holds:*

If $\xi \in Q_g(C, n = f_{e,r}(0), f_{e,r})$ corresponds to a quotient

$$\mathbf{C}^n \otimes \mathcal{O}_C \to E \to 0,$$

where E is a slope-unstable, torsion free sheaf on C, then there exists a nonzero, proper subsheaf

$$0 \to F \to E$$

such that

$$\frac{h^0(C,F)}{n} > \frac{h^0(C, F \otimes \omega_C^{10t})}{f_{e,r}(t)} \ . \tag{17}$$

Proof. Let $t_1(g,r,e) > p(g, R(g,r), \chi = f_{e,r}(0))$ be determined by Lemma (3.1.3). Take F to be a nonzero, proper, destabilizing subsheaf of E for which there exists a sequence

$$0 \to \overline{F} \to F \to \tau \to 0, \tag{18}$$

where $\overline{F}$ is generated by global sections and $Supp(\tau)$ has dimension zero. Such F exist by Lemma (4.2.1). Since $\overline{F}$ is a subsheaf of E and is generated by global sections, Lemma (3.1.3) yields for any $t > t_1$, $h^1(C, \overline{F} \otimes \omega_C^{10t}) = 0$. By the exact sequence in cohomology of (18), $h^1(C, F \otimes \omega_C^{10t}) = 0$. Let (s_i) be the (nontrivial) multirank of F. We have

$$h^0(C, F \otimes \omega_C^{10t}) = \chi(F) + (\sum s_i \omega_i) 10t,$$

$$f_{e,r}(t) = \chi(E) + r(2g-2)10t.$$

We obtain

$$\chi(F) \cdot f_{e,r}(t) - \chi(E) \cdot h^0(C, F \otimes \omega_C^{10t})$$
$$= \chi(F) \cdot r(2g-2)10t - \chi(E) \cdot (\sum s_i \omega_i)10t > 0.$$

The last inequality follows from the destabilizing property of F. Hence

$$\frac{\chi(F)}{\chi(E)} > \frac{h^0(C, F \otimes \omega_C^{10t})}{f_{e,r}(t)}.$$

Since $h^0(C, F) \geq \chi(F)$ and $\chi(E) = n$, the lemma is proven. □

We can now apply Lemma (2.3.1).

Proof. [Of Proposition (4.1.1)] Let $e_1(g, r)$ be as in Lemma (4.2.1). For $e > e_1(g, r)$, let $t_1(g, r, e)$ be determined by Lemma (4.2.2). Suppose $t > t_1(g, r, e)$. Let F be the subsheaf of E determined by Lemma (4.2.2). Let $U \subset \mathbf{C}^n = \psi^{-1}(H^0(C, F))$. Since ψ is an isomorphism, $dim(U) = h^0(C, F)$. Let G be the subsheaf generated by the global sections $H^0(C, F)$. Certainly

$$h^0(C, F \otimes \omega_C^{10t}) > h^0(C, G \otimes \omega_C^{10t}).$$

Lemmas (4.2.2) and (2.3.1) are now sufficient to conclude the desired G.I.T. instability. □

5. Special, torsion bounded sheaves

5.1. **Lemmas.** As always, let $n = f_{e,r}(0) = \chi(E)$. If

$$\psi : \mathbf{C}^n \otimes H^0(C, \mathcal{O}_C) \to H^0(C, E)$$

is injective but not an isomorphism, then $h^1(C, E) \neq 0$. We now investigate this case and conclude G.I.T. instability for the corresponding points of $Q_g(C, n, f_{e,r})$. The strategy is the following. Since $H^1(C, E)$ is dual to $Hom(E, \omega_C)$, the latter must be nonzero. In Lemma (5.1.1), the kernel of a nonzero element of $Hom(E, \omega_C)$ is analyzed to produce a very destabilizing subsheaf F of E. Lemma (2.3.1) is then applied as in section (4). In order to carry out the above plan, the torsion of E must be treated with care.

For any coherent sheaf E on C, let $0 \to \tau_E \to E$ be the torsion subsheaf. E is said to have *torsion bounded* by k if $\chi(\tau_E) < k$. Let $R(g, r) = r(2g-2) + 1$ as defined in section (3.2).

Lemma 5.1.1. *Let $g \geq 2$, $r > 0$ be integers. There exists an integer $e_2(g, r) > r(g-1)$ such that for each $e > e_2(g, r)$ and $[C] \in H_g$, the following holds:*

If E is a coherent sheaf on C with Hilbert polynomial $f_{e,r}$ (with respect to ω_C^{10}) satisfying

(i) *$h^1(C, E) \neq 0$,*
(ii) *E has torsion bounded by $b(g, R(g, r))$,*

then there exists a nonzero, proper subsheaf F of E with multirank (s_i) not identically zero such that

(i) *F is generated by global sections,*
(ii)

$$\frac{\chi(F) - b(g, R(g, r))}{\sum s_i \omega_i} > \frac{\chi(E)}{r(2g-2)} + 1 \ .$$

Proof. Since by Serre duality $H^1(C,E)^* \cong Hom(E,\omega_C)$, there exists a nonzero morphism of coherent sheaves:

$$\sigma : E \to \omega_C.$$

We have $0 \to \sigma(E) \to \omega_C$, where $\sigma(E) \neq 0$. Since ω_C is torsion free, $\sigma(E)$ has multirank not identically zero. Note

$$\chi(\sigma(E)) \leq h^0(C,\sigma(E)) \leq h^0(C,w_C) = g.$$

Consider the exact sequence:

$$0 \to K \to E \to \sigma(E) \to 0.$$

Since $\chi(K) = \chi(E) - \chi(\sigma(E))$,

$$\chi(K) \geq \chi(E) - g.$$

For $e > r(g-1)+g$, $\chi(K) > 0$ and $K \neq 0$. Let F be the subsheaf generated by the global sections of K. $\chi(K) > 0$ implies $F \neq 0$. Let $b = b(g, R(g,r))$. We have

$$\chi(F) > h^0(C,F) - b = h^0(C,K) - b \geq \chi(K) - b \geq \chi(E) - b - g.$$

For $e > r(g-1)+2b+g$, $\chi(F) > b$. Now assume $e > r(g-1)+2b+g$. By the bound on the torsion of E, F is not contained in τ_E. Let (s_i) be the multirank of F. Since F is not torsion, the multirank is not identically zero. In fact, since $\sigma(E)$ has multirank not identically zero,

$$0 < \sum s_i\omega_i < r(2g-2).$$

We conclude

$$\begin{aligned}\frac{\chi(F)-b}{\sum s_i\omega_i} &> \left(\frac{\chi(E)-2b-g}{r(2g-2)}\right)\cdot\left(\frac{r(2g-2)}{\sum s_i\omega_i}\right)\\ &\geq \left(\frac{\chi(E)-2b-g}{r(2g-2)}\right)\cdot\left(\frac{r(2g-2)}{r(2g-2)-1}\right).\end{aligned}$$

For large e depending only on g and r,

$$\left(\frac{\chi(E)-2b-g}{r(2g-2)}\right)\cdot\left(\frac{r(2g-2)}{r(2g-2)-1}\right) > \frac{\chi(E)}{r(2g-2)} + 1\ .$$

We omit the explicit bound. □

An analogue of Lemma (4.2.2) is now proven.

Lemma 5.1.2. *Let $g \geq 2$, $r > 0$, $e > e_2(g,r)$ be integers. There exists an integer $t_2(g,r,e) > t_0(g,r,e)$ such that for each $t > t_2(g,r,e)$ and $[C] \in H_g$, the following holds:*

If $\xi \in Q_g(C, n = f_{e,r}(0), f_{e,r})$ corresponds to a quotient

$$\mathbf{C}^n \otimes \mathcal{O}_C \to E \to 0,$$

where E is a coherent sheaf on C satisfying

(i) *$\psi : \mathbf{C}^n \otimes H^0(C,\mathcal{O}_C) \to H^0(C,E)$ is injective,*
(ii) *$h^1(C,E) \neq 0$,*
(iii) *E has torsion bounded by $b(g,(R,g,r))$,*

then there exists a nonzero subspace $W \subset \psi\Big(\mathbf{C}^n \otimes H^0(C, \mathcal{O}_C)\Big)$ *generating a nonzero, proper subsheaf* $0 \to G \to E$ *such that*

$$\frac{dim(W)}{n} > \frac{h^0(C, G \otimes \omega_C^{10t})}{f_{e,r}(t)}\ . \tag{19}$$

Proof. Let F be the subsheaf of E determined by Lemma (5.1.1). Let

$$W = im(\psi) \cap H^0(C, F).$$

Since

$$h^0(C, E) < \chi(E) + b(g, R(g, r)),$$

and ψ is injective,

$$dim(W) > h^0(C, F) - b \geq \chi(F) - b.$$

Note by condition (ii) on F in Lemma (5.1.1), $dim(W) > 0$. Let

$$t > p(g, R(g, r), \chi = f_{e,r}(0)).$$

Since F is generated by global sections,

$$h^1(C, F \otimes \omega_C^{10t}) = 0$$

by Lemma (3.1.3). We have

$$h^0(C, F \otimes \omega_C^{10t}) = \chi(F) + (\sum s_i \omega_i) 10t,$$

$$f_{e,r}(t) = \chi(E) + r(2g-2)10t.$$

We compute

$$\begin{aligned} &(\chi(F) - b) \cdot f_{e,r}(t) - \chi(E) \cdot h^0(C, F \otimes \omega_C^{10t}) \\ &\quad = (\chi(F) - b) \cdot r(2g-2)10t - \chi(E) \cdot (\sum s_i \omega_i) 10t - b \cdot \chi(E) \\ &\quad > r(2g-2) \cdot (\sum s_i \omega_i) \cdot 10t - b \cdot \chi(E). \end{aligned}$$

The last inequality follows from condition (ii) on F in Lemma (5.1.1). If also

$$t > b \cdot \chi(E) = b(g, R(g, r)) \cdot (e + r(1-g)),$$

then

$$\frac{\chi(F) - b}{\chi(E)} > \frac{h^0(C, F \otimes \omega_C^{10t})}{f_{e,r}(t)}\ .$$

Let G be the subsheaf of F generated by W. Since $dim(W) > \chi(F) - b$, $n = \chi(E)$, and

$$h^0(C, F \otimes \omega_C^{10t}) \geq h^0(C, G \otimes \omega_C^{10t}),$$

the proof is complete. ∎

5.2. **Step IV.**

Proposition 5.2.1. *Let $g \geq 2$, $r > 0$ be integers. There exist bounds $e_2(g,r) > r(g-1)$ and $t_2(g,r,e) > t_0(g,r,e)$ such that for each pair $e > e_2(g,r)$, $t > t_2(g,r,e)$ and any $[C] \in H_g$, the following holds:*

If $\xi \in Q_g(C, n = f_{e,r}(0), f_{e,r})$ corresponds to a quotient

$$\mathbf{C}^n \otimes \mathcal{O}_C \to E \to 0$$

where $h^1(E,C) \neq 0$, then ξ is G.I.T. unstable for the SL_n-linearization determined by i_t.

Proof. Let $e_2(g,r)$ be given by Lemma (5.1.1). For $e > e_2(g,r)$, let $t_2(g,r,e)$ be given by Lemma (5.1.2). Let

$$\psi : \mathbf{C}^n \otimes H^0(C, \mathcal{O}_C) \to H^0(C,E)$$

be the map on global sections. If ψ has a nontrivial kernel, ξ is unstable by Proposition (2.3.1). We can assume ψ is injective. Note that $im(\psi)$ has codimension less than $b(g, R(g,r))$ in $H^0(C,E)$. If $0 \to \tau \to E$ is a torsion subsheaf such that $h^0(C,\tau) = \chi(\tau) \geq b(g, R(g,r))$, then

$$im(\psi) \cap H^0(\tau, C) \neq 0.$$

In this case, since $t > t_0(g,r,e)$, ξ is unstable by Proposition (3.2.1). We can assume E has torsion bounded by b. We now can apply Lemma (5.1.2). Let $W \subset im(\psi)$ be determined by Lemma (5.1.2). Let $U = \psi^{-1}(W)$. Since ψ is injective, $dim(U) = dim(W)$. Lemmas (5.1.2) and (2.3.1) now imply the desired G.I.T. instability. □

6. Slope-semistable, torsion free sheaves

6.1. **Step V.** Let $g \geq 2$, $r > 0$ be integers. Let

$$e > max(e_1(g,r), e_2(g,r)),$$

$$t > max(t_0(g,r,e), t_1(g,r,e), t_2(g,r,e))$$

be determined by Propositions (2.3.1, 3.2.1, 4.1.1, 5.2.1). We now conclude the only possible semistable points in the SL_n-linearized G.I.T. problem determined by

$$i_t : Q_g(C, n = f_{e,r}(0), f_{e,r}) \to \mathbf{G}(f_{e,r}(t), (\mathbf{C}^n \otimes Sym^t(H^0(C, w_C^{10})))^*)$$

are elements $\xi \in Q_g(C, n, f_{e,r})$ that correspond to quotients

$$\mathbf{C}^n \otimes \mathcal{O}_C \to E \to 0,$$

where

$$\psi : \mathbf{C}^n \otimes H^0(C, \mathcal{O}_C) \to H^0(C,E)$$

is an isomorphism and E is a slope-semistable, torsion free sheaf. In order for ξ to be semistable, ψ must be injective by Proposition (2.3.1). Surjectivity is equivalent to $h^1(C,E) = 0$. By Proposition (5.2.1), ψ must be surjective. Since ψ is an isomorphism, E must be torsion free by Proposition (3.2.1). Finally, by Proposition (4.1.1), E must be slope-semistable. We now establish the converse.

Proposition 6.1.1. *Let $g \geq 2$, $r > 0$ be integers. There exist bounds $e_3(g,r) > r(g-1)$ and $t_3(g,r,e) > \hat{t}(g,r,e)$ such that for each pair $e > e_3(g,r)$, $t > t_3(g,r,e)$ and any $[C] \in H_g$, the following holds:*

If $\xi \in Q_g(C, n = f_{e,r}(0), f_{e,r})$ corresponds to a quotient

$$\mathbf{C}^n \otimes \mathcal{O}_C \to E \to 0,$$

where

$$\psi : \mathbf{C}^n \otimes H^0(C, \mathcal{O}_C) \to H^0(C, E)$$

is an isomorphism and E is a slope-stable (slope-semistable), torsion free sheaf, then ξ is a G.I.T. stable (semistable) point for the SL_n-linearization determined by i_t.

6.2. **Lemmas.** For the proof of G.I.T. (semi)stability, the fundamental step is the inequality of Lemma (6.2.2) for every subsheaf F of E generated by global sections. Note this is the reverse of the inequality required by Lemma (2.3.1). Lemma (6.2.2) follows by vanishing, Riemann-Roch, and the slope-(semi)stability of E when F is nonspecial. In case $h^1(C,F) \neq 0$, an analysis in Lemma (6.2.1) utilizing $Hom(F, \omega) \neq 0$ yields the required additional information. The Numerical Criterion of section (2.2) and Lemma (6.2.2) reduce the stability question to a purely combinatorial result established in Lemma (6.2.3).

Lemma 6.2.1. *Let $g \geq 2$, $r > 0$ be integers. Let q be an integer. There exists an integer $e_3(g,r,q)$ such that for each $e > e_3(g,r,q)$ and $[C] \in H_g$, the following holds:*

If E is a slope-semistable, torsion free sheaf on C with Hilbert polynomial $f_{e,r}$ (with respect to ω_C^{10}) and

$$0 \to F \to E$$

is a nonzero subsheaf with multirank (s_i) satisfying $h^1(C,F) \neq 0$, then:

$$\frac{\chi(F) + q}{\sum s_i \omega_i} < \frac{\chi(E)}{r(2g-2)} - 1. \tag{20}$$

Proof. Since $h^1(C,F) \neq 0$, there exists a nontrivial morphism

$$\sigma : F \to \omega_C.$$

Consider the subsheaf $0 \to \sigma(F) \to \omega_C$ where $\sigma(F) \neq 0$. By the proof of Lemma (5.1.1), $\chi(\sigma(F)) \leq g$. Consider the exact sequence

$$0 \to K \to F \to \sigma(F) \to 0.$$

If $K = 0$, then

$$\frac{\chi(F) + q}{\sum s_i \omega_i} < g + q.$$

Therefore, if

$$e > r(g-1) + r(2g-2)(g+q+1),$$

the case $K = 0$ is settled. Also, the case $\chi(F) \leq g$ is settled. Now suppose $K \neq 0$ and $\chi(F) - g > 0$. We have $\chi(F) - g \leq \chi(K)$. Let (s'_i) be the nontrivial multirank of K. Since $\sigma(F)$ is of nontrivial multirank we have:

$$0 < \sum s'_i \omega_i < \sum s_i \omega_i \leq r(2g-2).$$

We obtain:

$$\left(\frac{\chi(F)-g}{\sum s_i\omega_i}\right)\cdot\left(\frac{r(2g-2)}{r(2g-2)-1}\right) \leq \left(\frac{\chi(F)-g}{\sum s_i\omega_i}\right)\cdot\left(\frac{\sum s_i\omega_i}{\sum s'_i\omega_i}\right) \leq \frac{\chi(K)}{\sum s'_i\omega_i} .$$

Using the slope-semistability of E with respect to K, we conclude:

$$\frac{\chi(F)-g}{\sum s_i\omega_i} \leq \left(\frac{\chi(E)}{r(2g-2)}\right)\cdot\left(\frac{r(2g-2)-1}{r(2g-2)}\right) .$$

It is now clear, for large e depending only on g and r and q, the inequality (20) is satisfied. □

Lemma 6.2.2. *Let $g \geq 2$, $r > 0$. Let $b = b(g, R(g,r))$. Let $e > e_3(g,r,b) > r(g-1)$. There exists an integer $t_3(g,r,e) > \hat{t}(g,r,e)$ such that for each $t > t_3(g,r,e)$ and $[C] \in H_g$, the following holds:*

If $\xi \in Q_g(C, n = f_{e,r}(0), f_{e,r})$ corresponds to a quotient

$$\mathbf{C}^n \otimes \mathcal{O}_C \to E \to 0,$$

where E is a torsion free, slope-semistable sheaf on C and $0 \to F \to E$ is a nonzero, proper subsheaf generated by global sections, then

$$\frac{h^0(C,F)}{n} \leq \frac{h^0(C, F\otimes\omega_C^{10t})}{f_{e,r}(t)} .$$

If E is slope-stable,

$$\frac{h^0(C,F)}{n} < \frac{h^0(C, F\otimes\omega_C^{10t})}{f_{e,r}(t)} .$$

Proof. Suppose $t > p(g, R(g,r), \chi = f_{e,r}(0))$. Let (s_i) be the nontrivial multirank of F. Since F is generated by global sections, the vanishing guaranteed by Lemma (3.1.3) yields

$$h^0(C, F\otimes\omega_C^{10t}) = \chi(F) + (\sum s_i\omega_i)10t.$$

The Hilbert polynomial can be expressed:

$$f_{e,r}(t) = \chi(E) + r(2g-2)10t.$$

First consider the case where $h^1(C,F) = 0$. Then $h^0(C,F) = \chi(F)$. We compute

$$\begin{aligned}&\chi(F)\cdot f_{e,r}(t) - \chi(E)\cdot h^0(C, F\otimes\omega_C^{10t})\\ &\quad = \chi(F)\cdot r(2g-2)10t - \chi(E)\cdot(\sum s_i\omega_i)10t \ < (\leq)\ 0,\end{aligned}$$

where E is slope-stable, (slope-semistable). Hence

$$\frac{h^0(C,F)}{\chi(E)} = \frac{\chi(F)}{\chi(E)} < (\leq)\ \frac{h^0(C, F\otimes\omega_C^{10t})}{f_{e,r}(t)} .$$

Since $n = \chi(E)$, the nonspecial case is thus settled. Now suppose $h^1(C,F) \neq 0$. Lemma (6.2.1) now applies to F. We compute

$$\begin{aligned}&(\chi(F)+b)\cdot f_{e,r}(t) - \chi(E)\cdot h^0(C, F\otimes\omega_C^{10t})\\ &\quad = (\chi(F)+b)\cdot r(2g-2)10t - \chi(E)\cdot(\sum s_i\omega_i)10t + b\cdot\chi(E)\\ &\quad < -(\sum s_i\omega_i)\cdot r(2g-2)\cdot 10t + b\cdot\chi(E).\end{aligned}$$

For $t > b \cdot \chi(E) = b \cdot (e + r(1-g))$,

$$\frac{\chi(F)+b}{\chi(E)} < \frac{h^0(C, F \otimes \omega_C^{10t})}{f_{e,r}(t)} .$$

Since $h^0(C,F) < \chi(F) + b$,

$$\frac{h^0(C,F)}{\chi(E)} < \frac{\chi(F)+b}{\chi(E)}.$$

The proof is complete. □

We require a simple combinatorial lemma.

Lemma 6.2.3. *Let $n \geq 2$ be an integer. Let*

$$W_1 \leq W_2 \leq \ldots \leq W_n, \quad W_1 < W_n$$

be integers. Let $\{\alpha_i\}$, $\{\beta_i\}$ be rational numbers such that

(i) $\sum_1^n \beta_i = \sum_1^n \alpha_i$.
(ii) $\forall\ 1 \leq m \leq n-1, \quad \sum_1^m \beta_i \ < (\leq)\ \sum_1^m \alpha_i$.

Then:

$$\sum_1^n \beta_i \cdot W_i \ > (\geq)\ \sum_1^n \alpha_i \cdot W_i.$$

Proof. Use discrete Abel summation:

$$\sum_{i=1}^n \beta_i \cdot W_i = (\sum_{i=1}^n \beta_i) \cdot W_n - \sum_{m=1}^{n-1} \left((\sum_1^m \beta_i) \cdot (W_{m+1} - W_m) \right)$$
$$> (\geq) (\sum_{i=1}^n \alpha_i) \cdot W_n - \sum_{m=1}^{n-1} \left((\sum_1^m \alpha_i) \cdot (W_{m+1} - W_m) \right) = \sum_{i=1}^n \alpha_i \cdot W_i.$$

The middle inequality follows from (i) and (ii) above. □

6.3. **Proof of Proposition (6.1.1).**

Proof. Let $e_3(g,r) = e_3(g,r,b(g,R(g,r)))$ be determined by Lemma (6.2.1). For $e > e_3(g,r)$, let $t_3(g,r,e) > p(g,R(g,r),\chi = f_{e,r}(0))$ be given by Lemmas (6.2.2) and (3.1.3). We will apply the Numerical Criterion to the linearized SL_n-action on

$$\mathbf{G}(f_{e,r}(t), (\mathbf{C}^n \otimes Sym^t(H^0(C, \omega_C^{10})))^*).$$

The element ξ corresponds to the quotient:

$$\psi^t : \mathbf{C}^n \otimes Sym^t(H^0(C, \omega_C^{10})) \to H^0(C, E \otimes \omega_C^{10t}) \to 0.$$

Let $\overline{v} = (v_1, \ldots, v_n)$ be a basis of $\mathbf{C}^n$. Let $(w(v_1), \ldots, w(v_n))$ be weights satisfying

$$w(v_1) \leq w(v_2) \leq \ldots \leq w(v_n), \quad w(v_1) < w(v_n).$$

To apply the Numerical Criterion for (semi)stability, an $f_{e,r}(t)$-tuple of $\overline{v}$-pure elements of $\mathbf{C}^n \otimes Sym^t(H^0(C, \omega_C^{10}))$ projecting to a basis of $H^0(C, E \otimes \omega_C^{10})$ and satisfying the weight inequality (2) of section (2.2) must be shown to exist.

For $1 \leq i \leq n$, let F_i denote the subsheaf of E generated by $\psi(\bigoplus_{j=1}^i v_j \otimes H^0(C, \mathcal{O}_C))$. By the surjectivity guaranteed by (ii) of Lemma (3.1.3),

$$\psi^t : \bigoplus_{j=1}^i v_j \otimes Sym^t(C, H^0(\omega_C^{10})) \to H^0(C, F_i \otimes \omega_C^{10t}) \to 0. \tag{21}$$

Define for $1 \leq i \leq n$, $A_i = h^0(C, F_i \otimes \omega_C^{10t})$.

The required $f_{e,r}(t)$-tuple $(a_1, a_2, \ldots, a_{f_{e,r}(t)})$ is constructed as follows. Select elements

$$(a_1, \ldots, a_{A_1}) \in \; v_1 \otimes Sym^t(H^0(C, \omega_C^{10}))$$

such that $(\psi^t(a_1), \ldots, \psi^t(a_{A_1}))$ determines a basis of $H^0(C, F_1 \otimes \omega_C^{10t})$. Select

$$(a_{A_1+1}, \ldots, a_{A_2}) \in \; v_2 \otimes Sym^t(H^0(C, \omega_C^{10}))$$

such that $(\psi^t(a_1), \ldots, \psi^t(a_{A_2}))$ determines a basis of $H^0(C, F_2 \otimes \omega_C^{10t})$. Continue selecting

$$(a_{A_i+1}, \ldots, a_{A_{i+1}}) \in \; v_{i+1} \otimes Sym^t(H^0(C, \omega_C^{10}))$$

such that $(\psi^t(a_1), \ldots, \psi^t(a_{A_{i+1}}))$ determines a basis of $H^0(C, F_{i+1} \otimes \omega_C^{10t})$. Note that if $A_i = A_{i+1}$, then $(\psi^t(a_1), \ldots, \psi^t(a_{A_i}))$ already determines a basis of $H^0(C, F_{i+1} \otimes \omega_C^{10t})$ and no elements of $v_{i+1} \otimes Sym^t(H^0(C, \omega_C^{10}))$ are chosen. This selection is possible by the surjectivity of (21).

Let $\alpha_1 = A_1/f_{e,r}(t)$ and $\alpha_i = (A_i - A_{i-1})/f_{e,r}(t)$ for $2 \leq i \leq n$. We have

$$\sum_{i=1}^{n} \alpha_i w(v_i) = \sum_{j=1}^{f_{e,r}(t)} \frac{w(a_j)}{f_{e,r}(t)}.$$

Let $\beta_i = (1/n)$. Note

$$\sum_{1}^{n} \beta_i = \sum_{1}^{n} \alpha_i = 1.$$

Let $1 \leq m \leq n-1$. Since ψ is an isomorphism, $m \leq h^0(C, F_m)$. Suppose $F_m \neq E$. Then Lemma (6.2.2) yields

$$\frac{m}{n} \; < (\leq) \; \frac{A_m}{f_{e,r}(t)} \; . \tag{22}$$

If $F_m = E$, then $A_m = f_{e,r}(t)$ and the inequality (22) holds trivially $(m \leq n-1)$. So for all $1 \leq m \leq n-1$, we have

$$\sum_{1}^{m} \beta_i \; < (\leq) \; \sum_{1}^{m} \alpha_i.$$

Lemma (6.2.3) yields

$$\sum_{i=1}^{n} \frac{w(v_i)}{n} = \sum_{1}^{n} \beta_i w(v_i) \; > (\geq) \; \sum_{1}^{n} \alpha_i w(v_i) = \sum_{j=1}^{f_{e,r}(t)} \frac{w(a_j)}{f_{e,r}(t)}.$$

By the Numerical Criterion, ξ is G.I.T. stable (semistable). $\square$

6.4. **Step VI.** Only one step remains in the proof of Theorem (2.1.1). It must be checked that strict slope-semistability of E implies strict G.I.T. semistability.

Lemma 6.4.1. *Let $g \geq 2$, $r > 0$ be integers. There exists an integer $e_4(g, r)$ such that for each $e > e_4(g, r)$ and $[C] \in H_g$, the following holds:*

If E is any slope-semistable, torsion free sheaf on a C with Hilbert polynomial $f_{e,r}$ (with respect to ω_C^{10t}) and $0 \to F \to E$ is a nonzero subsheaf with multirank (s_i) satisfying

$$\frac{\chi(F)}{\sum s_i \omega_i} = \frac{\chi(E)}{r(2g-2)}, \tag{23}$$

then

(i) $h^1(C,F)=0$.
(ii) F *is generated by global sections.*

Proof. Suppose F is a nonzero subsheaf of E satisfying (23). If $e > e_3(g,r,0)$, by Lemma (6.2.1), $h^1(C,F)=0$. Now let $x \in C$ be a point. We have an exact sequence

$$0 \to m_x F \to F \to \frac{F}{m_x F} \to 0.$$

Since F is torsion free and C is nodal, it is not hard to show that

$$dim\left(\frac{F}{m_x F}\right) < 2 \cdot R(g,r).$$

Since $(F/m_x F)$ is torsion, $m_x F$ has the same multirank as F. Also

$$\chi(m_x F) > \chi(F) - 2R.$$

By (23),

$$\frac{\chi(m_x F)+2R}{\sum s_i \omega_i} > \frac{\chi(E)}{r(2g-2)}.$$

If $e > e_3(g,r,2R)$, $h^1(C, m_x F)=0$ by Lemma (6.2.1). In this case F is generated by global sections. We can therefore choose $e_4(g,r)=e_3(g,r,2R)$. □

Proposition 6.4.1. *Let $g \geq 2$, $r>0$ be integers. There exist bounds $e_4(g,r)$ and $t_4(g,r,e)$ such that for each pair $e > e_4(g,r)$, $t > t_4(g,r,e)$ and any $[C] \in H_g$, the following holds:*

If $\xi \in Q_g(C, n=f_{e,r}(0), f_{e,r})$ corresponds to a quotient

$$\mathbf{C}^n \otimes \mathcal{O}_C \to E \to 0,$$

where

$$\psi : \mathbf{C}^n \otimes H^0(C,\mathcal{O}_C) \to H^0(C,E)$$

is an isomorphism and E is a strictly slope-semistable, torsion free sheaf, then ξ is a G.I.T. strictly semistable point for the SL_n-linearization determined by i_t.

Proof. Since ξ is G.I.T. semistable for $e > e_3(g,r)$ and $t > t_3(g,r,e)$, it suffices to find a semistabilizing 1-parameter subgroup. If $e > e_4(g,r)$, then, for any nonzero, proper semistabilizing subsheaf $0 \to F \to E$, we have $h^0(C,F)=\chi(F)$ and F is generated by global sections. It is now easy to see that the flag $0 \subset H^0(C,F) \subset H^0(C,E)$ with weights $\{0,1\}$ determines semistabilizing data for large $t > t_4(g,r,e)$. □

We have now shown for the bounds:

$$e(g,r) = max\{e_i(g,r) \mid 1 \leq i \leq 4\},$$

$$t(g,r,e) = max\{t_i(g,r,e) \mid 0 \leq i \leq 4\}$$

the claim of Theorem (2.1.1) holds. This completes the proof of Theorem (2.1.1).

By Lemma (6.4.1), each slope-semistable, torsion free sheaf E on C with Hilbert polynomial $f_{e,r}$ appears in $Q_g(C, f_{e,r}(0), f_{e,r})^{SS}_t$ for $e > e(g,r)$, $t > t(g,r,e)$. It is now clear the SL_n-orbits of $Q_g(C, f_{e,r}(0), f_{e,r})^{SS}_t$ correspond exactly to the slope-semistable, torsion free sheaves on C with Hilbert polynomial $f_{e,r}$.

6.5. **Seshadri's construction.** In [Se], C. Seshadri has studied the SL_n-action on $Q_g(C, n = f_{e,r}(0), f_{e,r})$ via a covariant construction. For $e >> 0$, he finds a G.I.T. (semi)stable locus that coincides exactly with the G.I.T. (semi)stable locus of Theorem (2.1.1). These results appear in Theorem 18 of chapter 1 of [Se] for nonsingular curves and Theorem 16 of chapter 6 for singular curves. The collapsing of semistable orbits is determined by the Zariski topology. Seshadri shows that

(i) If E_t is a flat family of slope-semistable, torsion free sheaves on C such that the Jordan-Holder factors of $E_{t\neq 0}$ are $\{F_j\}$, then the Jordan-Holder factors of E_0 are also $\{F_j\}$.
(ii) If E is a slope-semistable, torsion free sheaf on C with Jordan-Holder factors $\{F_j\}$, then there exists a flat family of slope-semistable, torsion free sheaves E_t such that:

$$E_{t\neq 0} \cong E, \quad E_0 \cong \bigoplus_j F_j \quad .$$

Statement (ii) is proven by constructing flat families over extension groups. It follows from these two results that the points of our quotient

$$Q_g(C, n = f_{e,r}(0), f_{e,r})_t^{SS}/SL_n$$

for $e > e(g,r)$, $t > t(g,r,e)$ naturally parametrize slope-semistable, torsion free sheaves with Hilbert polynomial $f_{e,r}$ up to equivalence given by Jordan-Holder factors.

7. Two results in geometric invariant theory

7.1. **Statements.** Let V, Z, and W be finite dimensional $\mathbf{C}$-vector spaces. Consider two rational representations of $SL(V)$:

$$\zeta : SL(V) \to SL(Z),$$

$$\omega : SL(V) \to SL(W).$$

These representations define natural $SL(V)$-linearized actions on $\mathbf{P}(Z)$ and $\mathbf{P}(W)$. There is an induced $SL(V)$-action on the product $\mathbf{P}(Z) \times \mathbf{P}(W)$. Since

$$Pic(\mathbf{P}(Z) \times \mathbf{P}(W)) = \mathbf{Z} \oplus \mathbf{Z},$$

there is a 1-parameter choice of linearization. For $a, b \in \mathbf{N}^+$, let $[a,b]$ denote the linearization given by the line bundle $\mathcal{O}_{\mathbf{P}(Z)}(a) \otimes \mathcal{O}_{\mathbf{P}(W)}(b)$. Subscripts will be used to indicate linearization. Let

$$\rho_Z : \mathbf{P}(Z) \times \mathbf{P}(W) \to \mathbf{P}(Z)$$

be the projection on the first factor.

Proposition 7.1.1. *There exists an integer $k_S(\zeta, \omega)$ such that for all $k > k_S$:*

$$\rho_Z^{-1}(\mathbf{P}(Z)^S) \subset (\mathbf{P}(Z) \times \mathbf{P}(W))^S_{[k,1]}.$$

Proposition 7.1.2. *There exists an integer $k_{SS}(\zeta, \omega)$ such that for all $k > k_{SS}$:*

$$(\mathbf{P}(Z) \times \mathbf{P}(W))^{SS}_{[k,1]} \subset \rho_Z^{-1}(\mathbf{P}(Z)^{SS}).$$

D. Edidin has informed the author that Proposition (7.1.1) is essentially equivalent to Theorem 2.18 of [M-F].

7.2. **q-stability.** Let $\lambda : \mathbf{C}^* \to SL(V)$ be a 1-parameter subgroup. Let $dim(V) = a$. It is well known there exists a basis $\overline{v} = (v_1, \ldots , v_a)$ of V such that λ takes the form

$$\lambda(t)(v_i) = t^{e_i} \cdot v_i, \quad t \in \mathbf{C}^*.$$

Denote the tuple $(e_1, \ldots , e_a)$ by $\overline{e}$. The exponents satisfy the determinant 1 condition, $\sum_{i=1}^{a} e_i = 0$. Let $|\overline{e}| = max\{|e_i|\}$. For the representation $\zeta : SL(V) \to SL(Z)$, there exists a basis $\overline{z} = (z_1, \ldots , z_b)$ such that $\zeta \circ \lambda$ takes the form

$$\zeta \circ \lambda(t)(z_j) = t^{f_j} \cdot z_j, \quad t \in \mathbf{C}^*.$$

Again, $\sum_{j=1}^{b} f_j = 0$. The pairs $\{\overline{v}, \overline{e}\}$ and $\{\overline{z}, \overline{f}\}$ are said to be *diagonalizing data* for λ and $\zeta \circ \lambda$ respectively.

Let $[z] \in \mathbf{P}(Z)$ correspond to the one dimensional subspace of Z spanned by $z \neq 0$. By the Mumford-Hilbert Numerical Criterion, $[z]$ is a stable (semistable) point for the ζ-induced linearization on $\mathbf{P}(Z)$ if and only if for every 1-parameter subgroup $\lambda : \mathbf{C}^* \to SL(V)$, the following condition holds: let $\{\overline{z}, \overline{f}\}$ be diagonalizing data for $\zeta \circ \lambda$ and let $z = \sum_{j=1}^{b} \xi_j \cdot z_j$, then there exists an index j for which $\xi_j \neq 0$ and $f_j < 0$ ($f_j \leq 0$).

Let $q > 0$ be a real number. The point $[z]$ is defined to be *q-stable* for the ζ-induced linearization if and only if for every 1-parameter subgroup $\lambda : \mathbf{C}^* \to SL(V)$ the following condition holds: let $\{\overline{v}, \overline{e}\}$ and $\{\overline{z}, \overline{f}\}$ be diagonalizing data for λ and $\zeta \circ \lambda$ and let $z = \sum_{j=1}^{b} \xi_j \cdot z_j$, then there exists an index j for which $\xi_j \neq 0$ and $f_j < -q \cdot |\overline{e}|$. For $q > 0$, let $\mathbf{P}(Z)^{qS}$ denote the q-stable locus for the ζ-induced linearization.

Proposition (7.1.1) will be established in two steps:

Lemma 7.2.1. *There exists $q(\zeta) > 0$ such that $\mathbf{P}(Z)^{qS} = \mathbf{P}(Z)^S$.*

Lemma 7.2.2. *For any $q > 0$, there exists an integer $k_{qS}(q, \omega)$ such that for all $k > k_{qS}$:*

$$\rho_Z^{-1}(\mathbf{P}(Z)^{qS}) \subset (\mathbf{P}(Z) \times \mathbf{P}(W))^S_{[k,1]}.$$

Lemmas (7.2.1) and (7.2.2) certainly imply Proposition (7.1.1).

7.3. **Proofs of Lemmas (7.2.1) and (7.2.2).** Let U be a finite dimensional $\mathbf{Q}$-vector space. Let $L = \{l_i\}$ be a finite set of elements of U^*. The set L is said to be a *stable configuration* if

$$\forall\ 0 \neq u \in U, \ \ \exists i \ \ l_i(u) < 0.$$

If $\overline{u} = (u_1, u_2, \ldots , u_d)$ is a basis of U, define a norm $| \ |_{\overline{u}} : U \to \mathbf{Q}^{\geq 0}$ by

$$|u|_{\overline{u}} = max\{|\gamma_i|\}, \quad \text{where} \ \ u = \sum_{1}^{k} \gamma_i u_i.$$

Lemma 7.3.1. *Suppose $L = \{l_i\}$ is a stable configuration in U. Let $\overline{u}$ be a basis of U. Then there exists $q > 0$ depending upon L and $\overline{u}$ such that*

$$\forall\ 0 \neq u \in U, \ \ \exists i \ \ l_i(u) < -q \cdot |u|_{\overline{u}}. \tag{24}$$

Proof. Let $U \subset U_{\mathbf{R}} \cong U \otimes_{\mathbf{Q}} \mathbf{R}$. Suppose there exists an element $0 \neq u \in U_{\mathbf{R}}$ and a decomposition $L = L' \cup L''$ satisfying:

(i) $\forall l \in L', \ \ l(u) = 0.$
(ii) $\forall l \in L'', \ l(u) > 0.$

Since the locus $\{z \in U_{\mathbf{R}} \mid \forall l \in L',\ l(z) = 0\}$ is a rational subspace and the locus $\{z \in U_{\mathbf{R}} \mid \forall l \in L'',\ l(z) > 0\}$ is open, there must exist an element $0 \neq \hat{u} \in U$ satisfying (i) and (ii). Since L is a stable configuration in U, such $\hat{u}$ do not exist. It follows

$$\forall\ 0 \neq u \in U_{\mathbf{R}},\quad \exists i\ \ l_i(u) < 0.$$

Let S be the unit box in $U_{\mathbf{R}}$: $S = \{u \in U_{\mathbf{R}} \mid |u|_{\overline{u}} = 1\}$. Define a function $g : S \to \mathbf{R}^-$ by

$$g(s) = min\ \{l(s) \mid l \in L\}.$$

The function g is continuous and strictly negative. Since S is compact, g achieves a maximum value $-m$ on S for some $m > 0$. The bound $q = m/2$ clearly satisfies (24). □

Proof. [Of Lemma (7.2.1)] The proof consists of two simple pieces. First, a basis $\overline{v}$ of V is fixed. By applying Lemma (7.3.1), it is shown there exists a $q > 0$ such that the stability of $[z]$ implies the q-stability inequality for all 1-parameter subgroups of $SL(V)$ diagonal with respect $\overline{v}$. Second, it is checked that this q suffices for any selection of basis.

Let $\overline{v} = (v_1, \ldots , v_a)$ be a basis of V. Let

$$U = \{(e_1, \ldots , e_a) \mid\ \ e_i \in \mathbf{Q}, \sum_1^a e_i = 0\}.$$

There exist linear functions $\{l_1, \ldots , l_b\}$ on U and a basis $\overline{z} = (z_1, \ldots , z_b)$ of Z satisfying the following: if $\lambda : \mathbf{C}^* \to SL(V)$ is any 1-parameter subgroup with diagonalizing data $(\overline{v}, \overline{e})$, then the diagonalizing data of $\zeta\circ\lambda$ is $(\overline{z}, (l_1(\overline{e}), \ldots , l_b(\overline{e})))$. Let $\{L_1, \ldots , L_B\}$ be the set of distinct stable configurations in $\{l_1, \ldots , l_b\}$. That is, for all $1 \leq J \leq B$, $L_J \subset \{l_1, \ldots , l_b\}$ and L_J is a stable configuration in U. Let $\overline{u} = (u_1, \ldots , u_{a-1})$ be a basis of U of the following form:

$$u_1 = (-1, 1, 0, \ldots , 0),\ \ldots\ ,\ u_{a-1} = (-1, 0, \ldots , 0, 1).$$

Note $|\overline{e}| \leq a \cdot |\overline{e}|_{\overline{u}}$ for $\overline{e} \in U$. By Lemma (7.3.1), there exists $q_J > 0$ such that (24) holds for each stable configuration L_J. Let

$$q = \frac{1}{a} \cdot min\{q_J\}.$$

Suppose $[z] \in \mathbf{P}(Z)^S$. Let $z = \sum_1^b \xi_j z_j$. By the Numerical Criterion, the stability of $[z]$ implies the set $\{l_j | \xi_j \neq 0\}$ is a stable configuration in U equal to some L_J. For any 1-parameter subgroup with diagonalizing data $(\overline{v}, \overline{e})$, the diagonalizing data of $\zeta \circ \lambda$ is $(\overline{z}, (l_1(\overline{e}), \ldots , l_b(\overline{e})))$. By (24), we see there exists an $l_i \in L_J$ such that

$$l_i(\overline{e}) < -q_J \cdot |\overline{e}|_{\overline{u}} \leq -q \cdot |\overline{e}|.$$

Suppose $\overline{v}'$ is another basis of V. Then, up to scalars, there exists an element $\gamma \in SL(V)$ satisfying $\gamma(\overline{v}) = \overline{v}'$. It is now clear that

$$(\zeta(\gamma)(\overline{z}), (l_1(\overline{e}), \ldots , l_b(\overline{e})))$$

is diagonalizing data for $\zeta \circ \lambda$ where λ has diagonalizing data $(\overline{v}', \overline{e})$. Since the set $\{l_1, \ldots , l_b\}$ is independent of $\overline{v}$, the above analysis is valid for any 1-parameter subgroup. We have shown that $[z] \in \mathbf{P}(Z)^{qS}$. □

Lemma 7.3.2. *Let $\omega : SL(V) \to SL(W)$ be a rational representation. There exists an $M_\omega > 0$ with the following property. Let $\lambda : \mathbf{C}^* \to SL(V)$ be any 1-parameter subgroup. Let $(\overline{v}, \overline{e})$ and $(\overline{w}, \overline{h})$ be diagonalizing data for λ and $\omega \circ \lambda$. Then $|h| < M_\omega \cdot |e|$.*

Proof. Let $\overline{v}$ be a basis of V. Let U be as in the proof of Lemma (7.2.1). There exist linear functions $\{l_1, \ldots, l_c\}$ on U and a basis $\overline{w} = (w_1, \ldots, w_c)$ of W satisfying the following: if $\lambda : \mathbf{C}^* \to SL(V)$ is any 1-parameter subgroup with diagonalizing data $(\overline{v}, \overline{e})$, then the diagonalizing data of $\omega \circ \lambda$ is $(\overline{w}, (l_1(\overline{e}), \ldots, l_c(\overline{e})))$. Choose M_ω so

$$\forall j, \quad |l_j(\overline{e})| < M_\omega \cdot |\overline{e}|.$$

As in the proof of Lemma (7.2.1), the set of linear functions does not depend on $\overline{v}$. The proof is complete. □

Proof. [Of Lemma (7.2.2)] It is clear that if an element $[z] \in \mathbf{P}(Z)$ is q-stable for the ζ-induced linearization, then $[z^k] \in \mathbf{P}(Sym^k(Z))$ is kq-stable for the $Sym^k(\zeta)$-induced linearization. Let M_ω be determined by Lemma (7.3.2) for the representation ω. Let $k_{qS} = M_\omega / q$. We check for $k > k_{qS}$,

$$\rho_Z^{-1}(\mathbf{P}(Z)^{qS}) \subset (\mathbf{P}(Z) \times \mathbf{P}(W))^S_{[k,1]}.$$

The linearization $[k, 1]$ corresponds to the embedding:

$$\mathbf{P}(Z) \times \mathbf{P}(W) \to \mathbf{P}(Sym^k(Z) \otimes W),$$

$$[z] \times [w] \to [z^k \otimes w].$$

Let $[z] \in \mathbf{P}(Z)^{qS}$ and $[w] \in \mathbf{P}(W)$. Let $\lambda : \mathbf{C}^* \to SL(V)$ be any 1-parameter subgroup. Let $\overline{e}$ be the diagonalized exponents of λ. Let $(\overline{z}^*, \overline{f}^*)$ and $(\overline{w}, \overline{h})$ be the diagonalizing data of $Sym^k(\zeta) \circ \lambda$ and $\omega \circ \lambda$. Since $[z^k]$ is kq-stable for the $Sym^k(\zeta)$-induced linearization, there exists an index μ satisfying:

- (i) The basis element z^*_μ has a nonzero coefficient in the expansion of $[z^k]$.
- (ii) $f^*_\mu < -kq \cdot |\overline{e}| < -M_\omega \cdot |\overline{e}|$.

Let w_ν be any basis element that has a nonzero coefficient in the expansion of w. Note $z^*_\mu \otimes w_\nu$ is an element of the diagonalizing basis $\overline{z}^* \otimes \overline{w}$ of

$$(Sym^k(\zeta) \otimes \omega) \circ \lambda$$

having nonzero coefficient in the expansion of $z^k \otimes w$. The exponent corresponding to $z^*_\mu \otimes w_\nu$ is simply $f^*_\mu + h_\nu$. Since

$$|h_\nu| \leq |\overline{h}| < M_\omega \cdot |e|,$$

condition (ii) above implies the exponent is strictly negative. By the Numerical Criterion, $[z^k \times w]$ is stable. The lemma is proven. □

7.4. **Proof of Proposition (7.1.2).** Let $\zeta : SL(V) \to SL(Z)$ be a rational representation as above. An element $[z] \in \mathbf{P}(Z)$ is $(e_1, \ldots, e_a)$-*unstable* for the ζ-induced linearization if there exists a destabilizing 1-parameter subgroup $\lambda : \mathbf{C}^* \to SL(V)$ with diagonalizing data $(\overline{v}, \overline{e})$: if $(\overline{z}, \overline{f})$ is diagonalizing data for $\zeta \circ \lambda$ and $z = \sum_1^b \xi_j \cdot z_j$, then $\xi_j \neq 0$ implies $f_j > 0$. Let $\mathbf{P}(Z)^{\overline{e}UN} \subset \mathbf{P}(Z)$ denote $\overline{e}$-unstable locus.

From the Numerical Criterion, every unstable point is $\overline{e}$-unstable for some a-tuple $\overline{e} = (e_1, \ldots, e_a)$. We need a simple finiteness result:

Lemma 7.4.1. *Consider the ζ-linearized G.I.T. problem on $\mathbf{P}(Z)$. There exists a finite set of a-tuples, $\mathcal{P}$, such that*

$$\bigcup_{\overline{e}\in\mathcal{P}} \mathbf{P}(Z)^{\overline{e}UN} = \mathbf{P}(Z)^{UN}.$$

Proof. We first show that $\mathbf{P}(Z)^{\overline{e}UN}$ is a constructible subset of $\mathbf{P}(Z)$. Fix a 1-parameter subgroup $\lambda : \mathbf{C}^* \to SL(V)$ with diagonalizing data $(\overline{v}, \overline{e})$. Let $(\overline{z}, \overline{f})$ be diagonalizing data for $\zeta \circ \lambda$. Let H be the projective subspace of $\mathbf{P}(Z)$ spanned by the set $\{z_j | f_j > 0\}$. Certainly $H \subset \mathbf{P}(Z)^{\overline{e}UN}$. Since every 1-parameter subgroup of $SL(V)$ with diagonalized exponents $\overline{e}$ is conjugate to λ, we see the map

$$\kappa : SL(V) \times H \to \mathbf{P}(Z)$$

defined by:

$$\kappa(y, [z]) = [\zeta(y)(z)]$$

is surjective onto $\mathbf{P}(Z)^{\overline{e}UN}$.

The unstable locus, $\mathbf{P}(Z)^{UN}$, is closed. Also, $\mathbf{P}(Z)^{UN}$ is the countable union of the $\mathbf{P}(Z)^{\overline{e}UN}$. Over an uncountable algebraically closed field, any algebraic variety that is countable union of constructible subsets is actually the union of finitely many of them. Therefore a finite set of a-tuples, $\mathcal{P}$, with the demanded property exists in the case $\mathbf{C}$ is uncountable.

There always exists a field extension $\mathbf{C} \to \mathbf{C}'$ where $\mathbf{C}'$ is an uncountable algebraically closed field. By base extension,

$$\zeta_{\mathbf{C}'} : SL_{\mathbf{C}'}(V \otimes_{\mathbf{C}} \mathbf{C}') \to SL_{\mathbf{C}'}(Z \otimes_{\mathbf{C}} \mathbf{C}').$$

Since $\mathbf{C}$ is algebraically closed, it is easy to see that the $\mathbf{C}$-valued closed points of $\mathbf{P}_{\mathbf{C}'}(Z \otimes_{\mathbf{C}} \mathbf{C}')^{\overline{e}UN}$ are simply the points of $\mathbf{P}_{\mathbf{C}}(Z)^{\overline{e}UN}$. Hence, the assertion for $\mathbf{C}'$ implies the assertion for $\mathbf{C}$. This settles the general case. □

Proof. [Of Lemma (7.1.2)] Let $\mathcal{P}$ be determined by Lemma (7.4.1) for the representation ζ. Let M_ω be determined by Lemma (7.3.2) for the representation ω. Let N_ζ satisfy

$$\forall \overline{e} \in \mathcal{P}, \quad N_\zeta > |\overline{e}|.$$

Let $k_{SS} = M_\omega \cdot N_\zeta$. Suppose $k > k_{SS}$. For each element

$$[z] \times [w] \in \mathbf{P}(Z)^{UN} \times \mathbf{P}(W),$$

we must show that $[z^k \otimes w]$ is unstable for the $Sym^k(\zeta) \otimes \omega$-induced linearization on $\mathbf{P}(Sym^k(Z) \otimes W)$. Since $[z] \in \mathbf{P}(Z)^{UN}$, there exists an $\overline{e} \in \mathcal{P}$ such that $[z]$ is $\overline{e}$-unstable for the ζ-induced linearization on $\mathbf{P}(Z)$. Let $\lambda : \mathbf{C}^* \to SL(V)$ be a 1-parameter subgroup with diagonalized exponents $\overline{e}$ that destabilizes $[z]$. Let $(\overline{z}, \overline{f})$ and $(\overline{w}, \overline{h})$ be diagonalizing data for $\zeta \circ \lambda$ and $\omega \circ \lambda$. Let $z = \sum_1^b \xi_s \cdot z_s$ and $w = \sum_1^c \sigma_t \cdot w_t$ be the basis expansions. Since λ destabilizes $[z]$, we see

$$\xi_s \neq 0 \quad \Rightarrow \quad f_s > 0. \tag{25}$$

A diagonalizing basis of $Sym^k(\zeta) \circ \lambda$ can be constructed by taking homogeneous monomials of degree k in $\overline{z}$. Denote this basis with the corresponding exponents by $(\overline{z}^*, \overline{f}^*)$. Then $\overline{z}^* \otimes \overline{w}$ is a diagonalizing basis of

$$(Sym^k(\zeta) \otimes \omega) \circ \lambda.$$

We must show that every nonzero coefficient of the expansion of $z^k \otimes w$ in the basis $\overline{z}^* \otimes \overline{w}$ corresponds to a positive exponent. Suppose the basis element $z^*_{s^*} \otimes w_t$ has a nonzero coefficient. The element $z^*_{s^*}$ must correspond to a homogeneous polynomial of degree k in those z_s for which $\xi_s \neq 0$. Therefore, by (25), the exponent $f^*_{s^*}$ is not less than k. The exponent corresponding to $z^*_{s^*} \otimes w_t$ is $f^*_{s^*} + h_t$. Since

$$|h_t| \leq |\overline{h}| < M_\omega \cdot |\overline{e}_p| < M_\omega \cdot N_\zeta = k,$$

$$f^*_{s^*} + h_t > 0.$$

The proof is complete. □

8. The construction of $\overline{U_g(e,r)}$

8.1. **Uniform rank.** Define

$$Q^r_g(\mu, n, f_{e,r}) \subset Q_g(\mu, n, f_{e,r})$$

to be the subset corresponding to quotients

$$\mathbf{C}^n \otimes \mathcal{O}_C \to E \to 0,$$

where E has uniform rank r on C. Certainly $Q^r_g(\mu, n, f_{e,r})$ is $SL_{N+1} \times SL_n$ -invariant.

Lemma 8.1.1. *$Q^r_g(\mu, n, f_{e,r})$ is open and closed in $Q_g(\mu, n, f_{e,r})$. ($Q^r_g(\mu, n, f_{e,r})$ is a union of connected components.)*

Proof. Let $\kappa : \mathcal{C} \to \mathcal{B}$ be a projective, flat family of Deligne-Mumford stable genus g curves over an irreducible curve. Let $\mathcal{E}$ be a κ-flat coherent sheaf of constant Hilbert polynomial $f_{e,r}$ (with respect to $\omega^{10}_{\mathcal{C}/\mathcal{B}}$). Suppose there exists a $b^* \in \mathcal{B}$ such that $\mathcal{E}_{b^*}$ has uniform rank r on $\mathcal{C}_{b^*} = C$. Let $\{\mathcal{C}_i\}$ be the irreducible components of $\mathcal{C}$. Since $\kappa : \mathcal{C}_i \to \mathcal{B}$ is surjective of relative dimension 1, each $\mathcal{C}_i$ contains a component of C. Since the function $r(z) = dim_{k(z)}(\mathcal{E} \otimes k(z))$ is upper semicontinuous on $\mathcal{C}_i$, there is an open set $U_i \subset \mathcal{C}_i$ where $r(z) \leq r$. It follows there exists an open set $U \subset \mathcal{B}$ such that $\forall b \in U$, the rank of $\mathcal{E}_b$ on each component of $\mathcal{C}_b$ is at most r. If $\exists b'$ such that $\mathcal{E}_{b'}$ is not of uniform rank r, then (by semicontinuity) $\exists i$ so that $r(z)$ is strictly less than r on an open $W \subset \mathcal{C}_i$. For b in the nonempty intersection $U \cap \kappa(W)$, ranks of $\mathcal{E}_b$ are at most r on each component and strictly less than r on at least one component. By equation (10) of section (1.7), $\mathcal{E}_b$ can not have Hilbert polynomial $f_{e,r}$. Thus $\forall b \in \mathcal{B}$, $\mathcal{E}_b$ has uniform rank r. The Lemma is proven. □

8.2. **Determination of the semistable locus.** Select $e > e(g,r)$ and $t > t(g,r,e)$. As usual, let $n = f_{e,r}(0)$. Let

$$Z = \bigwedge^{h(\overline{s})} H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(\overline{s}))^*,$$

$$W = \bigwedge^{f_{e,r}(t)} (\mathbf{C}^n \otimes H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(t)))^*.$$

Consider the immersion

$$j_{\overline{s},t} : Q^r_g(\mu, n, f_{e,r}) \to \mathbf{P}(\bigwedge^{h(\overline{s})} H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(\overline{s}))^*) \times \mathbf{P}(\bigwedge^{f_{e,r}(t)} (\mathbf{C}^n \otimes H^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(t)))^*)$$

defined in section (1.6). Recall $\overline{s}$ is the linearization determined by Gieseker. There are three group actions to examine. In what follows, the superscripts $\{S', SS'\}$ will denote stability and semistability with respect to the SL_{N+1}-action. Similarly, $\{S'', SS''\}$ will correspond to the SL_n-action, and $\{S, SS\}$ will correspond to the $SL_{N+1} \times SL_n$-action.

The strategy for obtaining the desired $SL_{N+1}\times SL_n$-semistable locus is as follows. Consider first the SL_{N+1}-action. For suitable linearization, it will be shown that $Q^r_g(\mu,n,f_{e,r})$ is contained in the SL_{N+1}-stable locus and is closed in SL_{N+1}-semistable locus. This assertion is a consequence of Gieseker's conditions on H_g ((i), (ii) of section (1.6)) and the results of section (7). Next, $\overline{U_g(e,r)}$ is defined as the G.I.T. quotient of $Q^r_g(\mu,n,f_{e,r})^{SS}$ by $SL_{N+1}\times SL_n$. $\overline{U_g(e,r)}$ is a projective variety. Finally, in Proposition (8.2.1), it is shown that the SL_n and $SL_{N+1}\times SL_n$ semistable loci coincide on $Q^r_g(\mu,n,f_{e,r})$. Similarly, the stable loci coincide. The results on the fiberwise G.I.T. problem now yield a geometric identification of the stable and semistable loci for the $SL_{N+1}\times SL_n$- G.I.T. problem.

By Propositions (7.1.1) and (7.1.2), an integer $k>\{k_{S'},k_{SS'}\}$ can be found so that:

(26) $$\rho_Z^{-1}(\mathbf{P}(Z)^{S'})\subset(\mathbf{P}(Z)\times\mathbf{P}(W))^{S'}_{[k,1]},$$

(27) $$(\mathbf{P}(Z)\times\mathbf{P}(W))^{SS'}_{[k,1]}\subset\rho_Z^{-1}(\mathbf{P}(Z)^{SS'}).$$

By (i) of section (1.6), $H_g\subset\mathbf{P}(Z)^{S'}$. Now (26) above yields

(28) $$H_g\times\mathbf{P}(W)\subset(\mathbf{P}(Z)\times\mathbf{P}(W))^{S'}_{[k,1]}.$$

Therefore,

(29) $$Q^r_g(\mu,n,f_{e,r})\subset(\mathbf{P}(Z)\times\mathbf{P}(W))^{S'}_{[k,1]}.$$

By (ii) of section (1.6), H_g is closed in $\mathbf{P}(Z)^{SS'}$. Hence $H_g\times\mathbf{P}(W)$ is closed in $\rho_Z^{-1}(\mathbf{P}(Z)^{SS'})$. By (27) and the projectivity of $Q^r_g(\mu,n,f_{e,r})$ over H_g:

$$Q^r_g(\mu,n,f_{e,r})\text{ is closed in }(\mathbf{P}(Z)\times\mathbf{P}(W))^{SS'}_{[k,1]}.$$

Since

$$(\mathbf{P}(Z)\times\mathbf{P}(W))^{SS}_{[k,1]}\subset(\mathbf{P}(Z)\times\mathbf{P}(W))^{SS'}_{[k,1]},$$

it follows that

(30) $$Q^r_g(\mu,n,f_{e,r})^{SS}_{[k,1]}\text{ is closed in }(\mathbf{P}(Z)\times\mathbf{P}(W))^{SS}_{[k,1]}.$$

We define

$$\overline{U_g(e,r)}=Q^r_g(\mu,n,f_{e,r})^{SS}_{[k,1]}/(SL_{N+1}\times SL_n).$$

By (30), $\overline{U_g(e,r)}$ is a projective variety.

We now identify the locus $Q^r_g(\mu,n,f_{e,r})^{SS}_{[k,1]}$. Certainly

$$Q^r_g(\mu,n,f_{e,r})^{S}_{[k,1]}\subset Q^r_g(\mu,n,f_{e,r})^{S''}_{[k,1]},$$

$$Q^r_g(\mu,n,f_{e,r})^{SS}_{[k,1]}\subset Q^r_g(\mu,n,f_{e,r})^{SS''}_{[k,1]}.$$

In fact:

Proposition 8.2.1. *There are two equalities:*

$$Q^r_g(\mu,n,f_{e,r})^{S}_{[k,1]}=Q^r_g(\mu,n,f_{e,r})^{S''}_{[k,1]},$$

$$Q^r_g(\mu,n,f_{e,r})^{SS}_{[k,1]}=Q^r_g(\mu,n,f_{e,r})^{SS''}_{[k,1]}.$$

Proof. We apply the Numerical Criterion. Let

$$\zeta^k : SL_{N+1} \times SL_n \to SL_{N+1} \to SL(Sym^k(Z)),$$
$$\omega : SL_{N+1} \times SL_n \to SL(W)$$

denote the two representations. Let $\xi \in Q^r_g(\mu, n, f_{e,r})^{S''}_{[k,1]}$. Recalling the morphisms defined in section (1.5),

$$j_{\overline{s},t}(\xi) = (\pi(\xi), i_t(\xi)).$$

Let $\lambda : \mathbf{C}^* \to SL_{N+1} \times SL_n$ be a nontrivial 1-parameter subgroup given by

$$\lambda_1 : \mathbf{C}^* \to SL_{N+1},$$
$$\lambda_2 : \mathbf{C}^* \to SL_n.$$

Let $\{m_i\}$ be a diagonalizing basis for $\zeta^k \circ \lambda$ with weights $\{w(m_i)\}$. Let $\{n_j\}$ be a diagonalizing basis for the $\mathbf{C}^* \times \mathbf{C}^*$ representation $\omega \circ (\lambda_1 \times \lambda_2)$. Let $w_1(n_j)$ and $w_2(n_j)$ denote the weights of the induced $\mathbf{C}^*$ representations $\omega \circ (\lambda_1 \times 1)$ and $\omega \circ (1 \times \lambda_2)$. The weights $\{w(n_j)\}$ of the $\mathbf{C}^*$ representation $\omega \circ \lambda$ are given by $w(n_j) = w_1(n_j) + w_2(n_j)$. Finally let $\{\overline{m}_i\}$ and $\{\overline{n}_j\}$ denote the elements of the diagonalizing bases that appear with nonzero coefficient in the expansions of $\pi(\xi)$ and $i_t(\xi)$. There are three cases.

(1) $\lambda_1 = 1$. Since ξ is a stable point for the SL_n-action, there is a $\overline{n}_j$ with $w_2(\overline{n}_j) < 0$. We see
$$w(\overline{m}_i \otimes \overline{n}_j) = w(\overline{m}_i) + w_1(\overline{n}_j) + w_2(\overline{n}_j) = w_2(\overline{n}_j) < 0$$
for any $\overline{m}_i$.

(2) $\lambda_2 = 1$. By (29), ξ is a stable point for the SL_{N+1}-action. Hence there exists a pair $\overline{m}_i$, $\overline{n}_j$ so that
$$w(\overline{m}_i \otimes \overline{n}_j) = w(\overline{m}_i) + w_1(\overline{n}_j) < 0.$$

(3) $\lambda_1 \neq 1$, $\lambda_2 \neq 1$. Since ξ is a stable point for the SL_n-action, there is a $\overline{n}_j$ with $w_2(\overline{n}_j) < 0$. By (28), $(\pi(\xi) \otimes \overline{n}_j)$ is a stable point for the SL_{N+1}-action. Hence there exists an element $\overline{m}_i$ so that
$$w(\overline{m}_i) + w_1(\overline{n}_j) < 0.$$
Therefore,
$$w(\overline{m}_i \otimes \overline{n}_j) = w(\overline{m}_i) + w_1(\overline{n}_j) + w_2(\overline{n}_j) < 0.$$

By the Numerical Criterion, $\xi \in Q^r_g(\mu, n, f_{e,r})^S_{[k,1]}$. The proof for the semistable case is identical. □

By Theorem (2.1.1), we see the points of $Q^r_g(\mu, n, f_{e,r})^{SS}_{[k,1]}$ correspond exactly to quotients

$$\mathbf{C}^n \otimes \mathcal{O}_C \to E \to 0,$$

where E is slope-semistable, torsion free sheaf of uniform rank r on a 10-canonical, Deligne-Mumford stable, genus g curve $C \subset \mathbf{P}^N$ and

$$\psi : \mathbf{C}^n \otimes H^0(C, \mathcal{O}_C) \to H^0(C, E)$$

is an isomorphism. Similarly for $Q^r_g(\mu, n, f_{e,r})^S_{[k,1]}$.

We now examine orbit closures. Suppose $\overline{\xi} \in Q^r_g(\mu, n, f_{e,r})^{SS}_{[k,1]}$ lies in the $SL_{N+1} \times SL_n$-orbit closure of $\xi \in Q^r_g(\mu, n, f_{e,r})^{SS}_{[k,1]}$. Let

$$\gamma = (\gamma_1, \gamma_2) : \triangle - \{p\} \to SL_{N+1} \times SL_n$$

be a morphism of a nonsingular, pointed curve such that

$$Lim_{z\to p}(\gamma(z)\cdot\xi) = \overline{\xi}.$$

It follows that

$$Lim_{z\to p}(\gamma_1(z)\cdot\pi(\xi)) = \pi(\overline{\xi}).$$

Since $H_g \subset \mathbf{P}(Z)^S$, $\pi(\overline{\xi})$ lies in the SL_{N+1}-orbit of $\pi(\xi)$. After a possible base change, we can assume γ_1 extends over $p \in \triangle$ to

$$\overline{\gamma_1} : \triangle_p \to SL_{N+1}.$$

Since $Q^r_g(U_{\pi(\xi)}, n, f_{e,r})$ is projective,

$$\mu : \triangle - \{p\} \to Q^r_g(U_{\pi(\xi)}, n, f_{e,r})$$

defined by $\mu(z) = \gamma_2(z)\cdot\xi$ extends to $\triangle_p$. Let $\hat{\xi} = Lim_{z\to p}(\gamma_2(z)\cdot\xi) = \mu(p)$. By considering the map

$$\overline{\gamma_1}\cdot\mu : \triangle_p \to Q^r_g(\mu, n, f_{e,r})$$

defined by

$$\overline{\gamma_1}\cdot\mu(z) = \overline{\gamma_1}(z)\cdot\mu(z)$$

we obtain

$$\begin{aligned}\overline{\gamma_1}(p)\cdot\hat{\xi} = \overline{\gamma_1}\cdot\mu(p) &= Lim_{z\to p}(\gamma_1(z)\cdot\gamma_2(z)\cdot\xi)\\ &= Lim_{z\to p}(\gamma(z)\cdot\xi) = \overline{\xi}.\end{aligned}$$

We have shown the SL_{N+1}-orbit of $\overline{\xi}$ meets the SL_n-orbit closure of ξ. If ξ, $\overline{\xi}$ correspond to a slope-semistable, torsion free quotients E, $\overline{E}$ on C, $\overline{C} \subset \mathbf{P}^N$, certainly C, $\overline{C}$ must be projectively equivalent. The elements of the SL_{N+1}-orbit of $\overline{\xi}$ that lie over C are simply the images of $\overline{E}$ under automorphisms of C. Now from section (6.5), we conclude two semistable orbits ξ and $\overline{\xi}$ are identified in the quotient $\overline{U_g(e,r)}$ if and only if

$$\pi(\xi) \equiv \pi(\overline{\xi}) \equiv [C]$$

and the corresponding semistable, torsion free quotient sheaves E, $\overline{E}$ have Jordan-Holder factors that differ by an automorphism of C. We see:

Theorem 8.2.1. $\overline{U_g(e,r)}$ *parametrizes aut-equivalence classes of slope-semistable, torsion free sheaves of uniform rank r and degree e on Deligne-Mumford stable curves of genus g.*

Finally, since

$$Q^r_g(\mu, n, f_{e,r})^{SS}_{[k,1]} \to H_g \to \overline{M_g}$$

is an $SL_{N+1}\times SL_n-$ invariant morphism, there exists a map

$$\eta : \overline{U_g(e,r)} \to \overline{M_g}.$$

9. Basic properties of $\overline{U_g(e,r)}$

9.1. **The functor.** Let $\overline{\mathcal{U}_g(e,r)}$ be the functor that associates to each scheme S the set of equivalence classes of the following data:

(1) A flat family of Deligne-Mumford stable genus g curves, $\mu : \mathcal{C} \to S$.
(2) A μ-flat coherent sheaf $\mathcal{E}$ on $\mathcal{C}$ such that:
  (i) $\mathcal{E}$ is of constant Hilbert polynomial $f_{e,r}$ (with respect to $\omega^{10}_{\mathcal{C}/S}$).
  (ii) $\mathcal{E}$ is a slope-semistable, torsion free sheaf of uniform rank r on each fiber.

Two such data sets are equivalent if there exists an S isomorphism $\phi : \mathcal{C} \to \mathcal{C}'$ and a line bundle L on S so that $\mathcal{E} \cong \phi^*(\mathcal{E}') \otimes \mu^* L$.

Theorem 9.1.1. *There exists a natural transformation*

$$\phi_U : \overline{\mathcal{U}_g(e,r)} \to Hom(*, \overline{U_g(e,r)}).$$

$\overline{U_g(e,r)}$ *is universal in the following sense. If Z is a scheme and $\phi_Z : \overline{\mathcal{U}_g(e,r)} \to Hom(*, Z)$ is a natural transformation, there exists a unique morphism $\gamma : \overline{U_g(e,r)} \to Z$ such that the transformations ϕ_Z and $\gamma \circ \phi_U$ are equal.*

Proof. Let $e > e(g,r)$. Let $\mu : \mathcal{C} \to S$ and $\mathcal{E}$ on C satisfy (1) and (2) above. Note $\mu_*(\omega^{10}_{\mathcal{C}/S})$ is a locally free sheaf of rank $N+1 = 10(2g-2) - g + 1$. Since $\mathcal{E}_s$ is nonspecial for each $s \in S$, $\mu_*(\mathcal{E})$ is locally free of rank $n = f_{e,r}(0)$. Choose an open cover $\{W_i\}$ of S trivializing both $\mu_*(\omega^{10}_{\mathcal{C}/S})$ and $\mu_*(\mathcal{E})$. Let $V_i = \mu^{-1}(W_i)$. For each i, we obtain isomorphisms:

$$\mathbf{C}^{N+1} \otimes \mathcal{O}_{W_i} \cong \mu_*(\omega^{10}_{\mathcal{C}/S})|_{W_i} \tag{31}$$

$$\mathbf{C}^{n} \otimes \mathcal{O}_{W_i} \cong \mu_*(\mathcal{E})|_{W_i}. \tag{32}$$

These isomorphisms yield surjections:

$$\mathbf{C}^{N+1} \otimes \mathcal{O}_{V_i} \cong \mu^*\mu_*(\omega^{10}_{\mathcal{C}/S})|_{V_i} \to \omega^{10}_{\mathcal{C}/S}|_{V_i} \to 0,$$
$$\mathbf{C}^{n} \otimes \mathcal{O}_{V_i} \cong \mu^*\mu_*(\mathcal{E})|_{V_i} \to \mathcal{E}|_{V_i} \to 0.$$

The first surjection embeds V_i in $W_i \times \mathbf{P}^N$. By the universal property of the Quot scheme $Q_g(\mu, n, f_{e,r})$ and the second surjection, there exists a map

$$\phi_i : W_i \to Q_g(\mu, n, f_{e,r}).$$

For $t > t(g,r,e)$, $\phi_i(W_i) \subset Q^r_g(\mu, n, f_{e,r})^{SS}_{[k,1]}$. On the overlaps $W_i \cap W_j$, ϕ_i and ϕ_j differ by a morphism

$$W_i \cap W_j \to PSL_{N+1} \times PSL_n$$

corresponding to the choice of trivialization in (31) and (32). Hence there exists a well defined morphism

$$\phi : S \to \overline{U_g(e,r)}.$$

The functoriality of the universal property of the Quot scheme implies ϕ is functorially associated to the data $\mathcal{E}$ and $\mu : \mathcal{C} \to S$. We have shown there exists a natural transformation

$$\phi_U : \overline{\mathcal{U}_g(e,r)} \to Hom(*, \overline{U_g(e,r)}).$$

Suppose $\phi_Z : \overline{\mathcal{U}_g(e,r)} \to Hom(*, Z)$ is a natural transformation. There exists a canonical element of $\delta \in \overline{\mathcal{U}_g(e,r)}(Q^r_g(\mu, n, f_{e,r})^{SS}_{[k,1]})$ corresponding to the universal family on the Quot scheme. The morphism

$$\phi_Z(\delta) : Q^r_g(\mu, n, f_{e,r})^{SS}_{[k,1]} \to Z$$

is $SL_{N+1} \times SL_n$-invariant. Hence $\phi_Z(\delta)$ descends to $\gamma : \overline{U_g(e,r)} \to Z$. The two transformations ϕ_Z and $\gamma \circ \phi_U$ agree on δ. Naturality now implies $\phi_Z = \gamma \circ \phi_U$. $\square$

By previous considerations, there are natural transformations

$$t_l : \overline{\mathcal{U}_g(e,r)} \to \overline{\mathcal{U}_g(e + rl(2g-2), r)}$$

given by $\mathcal{E} \to \mathcal{E} \otimes \omega^l_{\mathcal{C}/S}$. By Theorem (9.1.1), these induce natural isomorphisms

$$t_l : \overline{U_g(e,r)} \to \overline{U_g(e + rl(2g-2), r)}.$$

The arguments in the above proof imply a useful lemma:

Lemma 9.1.1. *Let $\mu : \mathcal{C} \to S$ be a flat family of Deligne-Mumford stable, genus $g \geq 2$ curves. Let $\mathcal{E}$ be a μ-flat coherent sheaf on $\mathcal{C}$. The condition that $\mathcal{E}_s$ is a slope-semistable torsion free sheaf of uniform rank on $\mathcal{C}_s$ is open on S.*

Proof. Suppose $\mathcal{E}_{s_0}$ is a slope-semistable sheaf of uniform rank r on $\mathcal{C}_{s_0}$ for some $s_0 \in S$. There exists an integer m such that

(i) $h^1(\mathcal{E}_s \otimes \omega^m_{\mathcal{C}_s}, \mathcal{C}_s) = 0$ for all $s \in S$.
(ii) $\mathcal{E}_s \otimes \omega^m_{\mathcal{C}_s}$ is generated by global section for all $s \in S$.
(iii) $degree(\mathcal{E}_{s_0} \otimes \omega^m_{\mathcal{C}_{s_0}}) > e(g,r)$.

It suffices to prove the lemma for $\mathcal{E} \otimes \omega^m_{\mathcal{C}/S}$. Let $f_{e,r}$ be the Hilbert polynomial of $\mathcal{E} \otimes \omega^m_{\mathcal{C}/S}$. By the proof of Theorem (9.1.1), there exists an open set $W \subset S$ containing s_0 and a morphism

$$\phi : W \to Q_g(\mu, n = f_{e,r}(0), f_{e,r})$$

such that $\mathcal{E} \otimes \omega^m_{\mathcal{C}/S}$ is isomorphic to the ϕ-pull back of the universal quotient. Since $\phi(s_0) \in Q^r_g(\mu, n, f_{e,r})^{SS}_{[k,1]}$ and the latter is open, the lemma is proven. $\square$

9.2. **Deformations of torsion free sheaves and the irreducibility of $\overline{U_g(e,r)}$.** We study deformation properties of uniform rank, torsion free sheaves on nodal curves.

Lemma 9.2.1. *Let $\mu : S \to Spec(\mathbf{C}[t])$ be a μ-flat, nonsingular surface. If $\mathcal{E}$ is a μ-flat sheaf on S such that the restriction $\mathcal{E}/t\mathcal{E} = \mathcal{E}_0$ is torsion free, then $\mathcal{E}_0$ is locally free.*

Proof. Let $z \in \mu^{-1}(0)$. Since S is a nonsingular surface, the local ring $\mathcal{O}_{S,z}$ is regular of dimension 2. Consider the $\mathcal{O}_{S,z}$-module $\mathcal{E}_z$. Since $\mathcal{E}$ is μ-flat, t is a $\mathcal{E}_z$-regular element. Since $\mathcal{E}_0 = \mathcal{E}/t\mathcal{E}$ is torsion free, $depth_{\mathcal{O}_{S,z}}(\mathcal{E}_z) \geq 2$. We have the Auslander-Buchsbaum relation ([M]):

$$proj.\ dim_{\mathcal{O}_{S,z}}(\mathcal{E}_z) + depth_{\mathcal{O}_{S,z}}(\mathcal{E}_z) = dim(\mathcal{O}_{S,z}) = 2.$$

We conclude $proj.\ dim_{\mathcal{O}_{S,z}}(\mathcal{E}_z) = 0$. Hence $\mathcal{E}_z$ is free over $\mathcal{O}_{S,z}$. It follows that $\mathcal{E}_0$ is locally free. $\square$

Lemma (9.2.1) shows that it is not possible to deform a torsion free, non-locally free sheaf on a nodal curve to a locally free sheaf on a nonsingular curve if the deformations at the nodes have local equations of the form $(xy - t) \subset Spec(\mathbf{C}[x,y,t])$. However, the next lemma shows such deformations exist locally if the deformations of the nodes have local equations of the form $(xy - t^2)$. In Lemma (9.2.3), it is shown these local deformations can be globalized.

Lemma 9.2.2. *Let $S \subset Spec(\mathbf{C}[x,y,t])$ be the subscheme defined by the ideal $(xy-t^2)$. Let $\mu : S \to Spec(\mathbf{C}[t])$. Let $\zeta = (0,0,0) \in S$. There exists a μ-flat sheaf $\mathcal{E}$ on S such that $\mathcal{E}_{t\neq 0}$ is locally free and $\mathcal{E}_0 \cong m_\zeta$, where m_ζ is the maximal ideal defining ζ on S_0.*

Proof. There exists a section L of μ defined by the ideal $(x-t, y-t)$. Let $\mathcal{E}$ be the coherent sheaf corresponding to this ideal. We have the exact sequence:

$$0 \to \mathcal{E} \to \mathcal{O}_S \to \mathcal{O}_L \to 0. \tag{33}$$

Since $\mathcal{O}_S$ is torsion free over $\mathbf{C}[t]$, so is $\mathcal{E}$. $\mathcal{E}$ is therefore μ-flat. Since $\mathcal{O}_L$ is μ-flat, sequence (33) remains exact after restriction to the special fiber. Hence $\mathcal{E}_0 \cong m_\zeta$. □

Lemma 9.2.3. *Let C be a Deligne-Mumford stable curve of genus $g \geq 2$. Let E be a slope-semistable torsion free sheaf of uniform rank r on C. Then there exists a family $\mu : \mathcal{C} \to \triangle_0$ and a μ-flat coherent sheaf $\mathcal{E}$ on $\mathcal{C}$ such that :*

(1) *$\triangle_0$ is a pointed curve.*
(2) *$\mathcal{C}_0 \cong C$, $\forall t \neq 0$ $\mathcal{C}_t$ is a complete, nonsingular, irreducible genus g curve.*
(3) *$\mathcal{E}_0 \cong E$, $\forall t \neq 0$ $\mathcal{E}_t$ is a slope-semistable torsion free sheaf of rank r.*

Proof. Let $z \in C$ be a node. Since E is torsion free and of uniform rank r, it follows from Propositions (2) and (3) of chapter (8) of [Se]:

$$E_z \cong \bigoplus_{1}^{a_z} \mathcal{O}_{C_z} \oplus \bigoplus_{a_z+1}^{r} m_z, \tag{34}$$

where m_z is the localization of the ideal of the node z. To simplify the deformation arguments, let $\mathbf{C}$ be the field of complex numbers. Let $B(d) \subset \mathbf{C}^2$ be the open ball of radius d with respect to the Euclidean norm; let $B(d_1, d_2) \subset \mathbf{C}^2$ be the open annulus. Disjoint open Euclidean neighborhoods, $z \in U_z \subset C$, can be chosen for each node of C satisfying:

(i) U_z is analytically isomorphic to $B(d_z) \cap (xy = 0) \subset \mathbf{C}^2$.
(ii) $E|_{U_z} \cong \bigoplus_1^{a_z} \mathcal{O}_{U_z} \oplus \bigoplus_{a_z+1}^{r} m_z$.

Let $V_z \subset U_z$ be the closed neighborhood of radius $d_z/2$. Let $W = C \setminus \cup V_z$. A deformation of C can be given by the open cover:

$$\{W \times \triangle_0\} \cup \{Def_z | z \in C_{ns}\},$$

where Def_z (to be defined below) is an open subset of

$$B(d_z) \times \triangle_0 \ \cap \ (xy - t^2 = 0) \qquad \subset \quad \mathbf{C}^2 \times \triangle_0$$

containing $(0,0,0)$. Def_z is a local smoothing at the node z. Define

$$K_z \ = \ B\left(\frac{d_z}{2}, \frac{d_z}{2} + \epsilon_z\right) \times \triangle_0 \ \cap \ (xy = 0) \ \subset \ \mathbf{C}^2 \times \triangle_0.$$

Note $B(r,s)$ is the annulus. Let μ denote the projection to $\triangle_0$. For $\epsilon_z > 0$ (small with respect to δ_z) and $|t| < \delta_z$, it is not hard to find an isomorphism γ_z commuting with μ:

$$\gamma_z : K_z \to D_z \subset B(d_z) \times \triangle_0 \ \cap \ (xy - t^2 = 0)$$

such that

$$B\left(\frac{d_z}{3}\right) \times \triangle_0 \ \cap \ D_z = \emptyset \tag{35}$$

and γ_z extends the identity map on the locus $t = 0$. Such a γ_z can be constructed by considering the holomorphic flow of an algebraic vector field on $(xy - t^2 = 0)$. The space

$$B(d_z) \times \triangle_0 \ \cap \ (xy - t^2 = 0) \quad \setminus \ D_z$$

is disconnected. Let Def_z be the complement of the component not containing $(0,0,0)$. The isomorphism γ_z determines a patching of $W \times \triangle_0$ and Def_z along $K_z \simeq D_z$ in the obvious manner. The constructed family satisfies claims (1) and (2) of the lemma.

E_0 can be extended trivially on $W \times \triangle_0$ to $\mathcal{E}|_W$. $\mathcal{E}|_{K_z}$ is trivial by condition (ii). By Lemma (9.2.2), m_z can be flatly extended to a line bundle L_z on Def_z. By (35) and the construction of Lemma (9.2.2), L_z can be assumed to be trivial on D_z. By patching

$$\bigoplus_{1}^{a_z} \mathcal{O}_{Def_z} \oplus \bigoplus_{a_z+1}^{r} L_z$$

along $K_z \simeq D_z$, $\mathcal{E}$ can be defined such that $\mathcal{E}_0 \cong E$ and $\mathcal{E}_{t\neq 0}$ is locally free. Indeed, such a patching exists for $t = 0$ by condition (ii). The patching can be extended trivially along K_z since

$$K_z = K_{z,t=0} \times \triangle_0.$$

Now condition (3) follows by Lemma (9.1.1). For a general ground field, the étale topology must be used. □

Proposition 9.2.1. *$\overline{U_g(e,r)}$ is an irreducible variety.*

Proof. Consider the morphism $\pi_{SS} : Q^r_g(\mu, n = f_{e,r}(0), f_{e,r})^{SS}_{[k,1]} \to H_g$. By Proposition (24) of chapter (1) of [Se], the scheme

$$\pi^{-1}_{SS}([C]) = Q_g(C, n, f_{e,r})^{SS}$$

is irreducible for each nonsingular curve C, $[C] \in H_g$. Since the locus $H^0_g \subset H_g$ of nonsingular curves is irreducible, $\pi^{-1}_{SS}(H^0_g)$ is irreducible. By Lemma (9.2.3), $\pi^{-1}_{SS}(H^0_g)$ is dense in $Q^r_g(\mu, n, f_{e,r})^{SS}_{[k,1]}$. Finally, since there is a surjective morphism

$$Q^r_g(\mu, n, f_{e,r})^{SS}_{[k,1]} \to \overline{U_g(e,r)},$$

we conclude $\overline{U_g(e,r)}$ is irreducible. □

9.3. **The normality of $\overline{U_g(e,1)}$.** We need an infinitesimal analogue of Lemma (9.2.1). First, we establish some notation. Let

$$R = \mathbf{C}[[x,y]][\epsilon]/(xy - \epsilon, \epsilon^2),$$

Let $A = R/\epsilon R \cong \mathbf{C}[[x,y]]/(xy)$. Let $m = (x,y)$ be the maximal ideal of A. There is a natural, flat inclusion of rings $\mathbf{C}[\epsilon]/(\epsilon^2) \to R$. For a $\mathbf{C}[\epsilon]/(\epsilon^2)$-module M, let $*\epsilon$ denote the M endomorphism given by multiplication by ϵ.

Lemma 9.3.1. *There does not exist a $\mathbf{C}[\epsilon]/(\epsilon^2)$-flat R-module E such that*

$$E/\epsilon E \cong m$$

as A-modules.

Proof. Suppose such an E exists. Let

$$\alpha : E \to E/\epsilon E \xrightarrow{\sim} m.$$

Let e_x, $e_y \in E$ satisfy $\alpha(e_x) = x$ and $\alpha(e_y) = y$. We obtain a morphism

$$\beta : R \oplus R \to E$$

defined by $\beta(1,0) = e_x$ and $\beta(0,1) = e_y$. Since ϵ is nilpotent, β is surjective by Nakayama's Lemma. We have an exact sequence

$$0 \to N \to R \oplus R \to E \to 0.$$

Since E and $R \oplus R$ are $\mathbf{C}[\epsilon]/(\epsilon^2)$-flat, N is $\mathbf{C}[\epsilon]/(\epsilon^2)$-flat. Hence, there is an *exact* sequence

$$0 \to \epsilon N \to N \xrightarrow{*\epsilon} N \tag{36}$$

obtained by tensoring $0 \to (\epsilon) \to \mathbf{C}[\epsilon]/(\epsilon^2) \xrightarrow{*\epsilon} \mathbf{C}[\epsilon]/(\epsilon^2)$ with N. Since $\beta\big((y,0)\big) \in \epsilon E$, there exists an element $n = (y + \epsilon f(x,y), \epsilon g(x,y))$ in N. Consider $xn \in N$:

$$xn = (xy + \epsilon x f(x,y), \epsilon x g(x,y)) = (\epsilon(1 + x f(x,y)), \epsilon x g(x,y)).$$

Note $\epsilon xn = 0$. By the exactness of (36), there exists an $\overline{n} \in N$ satisfying $\epsilon \overline{n} = xn$. Since $R \oplus R$ is flat over $\mathbf{C}[\epsilon]/(\epsilon^2)$, any such $\overline{n}$ must be of the form

$$\overline{n} = (\ 1 + x f(x,y) + \epsilon \hat{f}(x,y),\ \ x g(x,y) + \epsilon \hat{g}(x,y)\ ).$$

Since $\alpha \circ \beta(\overline{n}) = x + x^2 f(x,y) \neq 0$ in m, $\overline{n}$ cannot lie in N. We have a contradiction. No such E can exist. □

To prove $\overline{U_g(e,1)}$ is normal, it suffices to show $Q^1_g(\mu, n = f_{e,1}, f_{e,1})^{SS}_{[k,1]}$ is nonsingular. The nonsingularity is established by computing the dimension of $Q^1_g(\mu, n, f_{e,1})^{SS}_{[k,1]}$ and then bounding the dimension of the Zariski tangent space at each point. The Zariski tangent spaces are controlled by a study of the differential $d\pi_{SS}$, where π_{SS} is the canonical map

$$\pi_{SS} : Q^1_g(\mu, n, f_{e,1})^{SS}_{[k,1]} \to H_g.$$

Lemma 9.3.2. *$Q^1_g(\mu, n = f_{e,1}(0), f_{e,1})^{SS}_{[k,1]}$ is nonsingular.*

Proof. Consider the universal quotient sequence over $Q^1_g(\mu, n, f_{e,1})^{SS}_{[k,1]}$,

$$0 \to \mathcal{F} \to \mathbf{C}^n \otimes \mathcal{O}_{Q^{ss} \times U} \to \mathcal{E} \to 0.$$

Let $\xi \in Q^1_g(\mu, n, f_{e,1})^{SS}_{[k,1]}$ be a closed point and let $C = U_{\pi(\xi)}$. ξ corresponds to a quotient

$$0 \to \mathcal{F}_\xi \to \mathbf{C}^n \otimes \mathcal{O}_C \to \mathcal{E}_\xi \to 0. \tag{37}$$

There is a natural identification of the Zariski tangent space to $Q^1_g(C, n, f_{e,1})$ at ξ:

$$T_\xi(Q^1_g(C, n, f_{e,1})) \cong H^0(C, \underline{Hom}(\mathcal{F}_\xi, \mathcal{E}_\xi))$$

(see [Gr]). If $h^1(C, \underline{Hom}(\mathcal{F}_\xi, \mathcal{E}_\xi)) = 0$ and the deformations of

$$\mathbf{C}^n \otimes \mathcal{O}_C \to \mathcal{E}_\xi \to 0 \tag{38}$$

are locally unobstructed, then ξ is a nonsingular point of $Q^1_g(C, n, f_{e,1})$. If $\mathcal{E}_\xi$ is locally free, the deformations of (38) are locally unobstructed. Sequence (37) yields:

(39)

$$0 \to \underline{Hom}(\mathcal{E}_\xi, \mathcal{E}_\xi) \to \underline{Hom}(\mathbf{C}^n \otimes \mathcal{O}_C, \mathcal{E}_\xi) \to \underline{Hom}(\mathcal{F}_\xi, \mathcal{E}_\xi) \to \underline{Ext}^1(\mathcal{E}_\xi, \mathcal{E}_\xi) \to 0.$$

Since $\underline{Ext}^1(\mathcal{E}_\xi, \mathcal{E}_\xi)$ is torsion and

$$h^1(C, \underline{Hom}(\mathbf{C}^n \otimes \mathcal{O}_C, \mathcal{E}_\xi)) = n \cdot h^1(C, \mathcal{E}_\xi) = 0,$$

we obtain $h^1(C, \underline{Hom}(\mathcal{F}_\xi, \mathcal{E}_\xi)) = 0$.

From (34), at each node $z \in C$, $\mathcal{E}_\xi$ is either locally free or locally isomorphic to m_z. Let s be the number of nodes where $\mathcal{E}_{\xi,z} \cong m_z$. Using (39), we compute $\chi(\underline{Hom}(\mathcal{F}_\xi, \mathcal{E}_\xi))$ in terms of s:

$$\chi(\underline{Hom}(\mathcal{F}_\xi, \mathcal{E}_\xi)) = -\chi(\underline{Hom}(\mathcal{E}_\xi, \mathcal{E}_\xi)) + \chi(\underline{Hom}(\mathbf{C}^n \otimes \mathcal{O}_C, \mathcal{E}_\xi)) + \chi(\underline{Ext}^1(\mathcal{E}_\xi, \mathcal{E}_\xi)).$$

It is clear $\chi(\underline{Hom}(\mathbf{C}^n \otimes \mathcal{O}_C, \mathcal{E}_\xi)) = n^2$. Let $A = \mathbf{C}[[x,y]]/(xy)$ and $m = (x,y) \subset A$ as above. It is not hard to establish:

$$Ext^1_A(m,m) = \mathbf{C}^2, \tag{40}$$

$$0 \to A \xrightarrow{i} Hom_A(m,m) \to \mathbf{C} \to 0 \tag{41}$$

where i is the natural inclusion. Since $\underline{Ext}^1(\mathcal{E}_\xi, \mathcal{E}_\xi)$ is a torsion sheaf supported at each $z \in C$ where $\mathcal{E}_{\xi,z} \cong m_z$ with stalk (40),

$$\chi(\underline{Ext}^1(\mathcal{E}_\xi, \mathcal{E}_\xi)) = 2s.$$

There is a natural inclusion

$$0 \to \mathcal{O}_C \xrightarrow{i} \underline{Hom}(\mathcal{E}_\xi, \mathcal{E}_\xi) \to \delta \to 0.$$

Since $\mathcal{E}_\xi$ is of rank 1, δ is a torsion sheaf supported at the nodes where $\mathcal{E}_\xi$ is not locally free. At these nodes, δ can be determined locally by (41). Hence

$$\chi(\underline{Hom}(\mathcal{E}_\xi, \mathcal{E}_\xi)) = 1 - g + s.$$

Summing the Euler characteristics yields:

$$h^0(C, \underline{Hom}(\mathcal{F}_\xi, \mathcal{E}_\xi)) = g - 1 - s + n^2 + 2s = n^2 - 1 + g + s.$$

If C is a nonsingular curve, $\mathcal{E}_\xi$ is locally free on C. The above results show that ξ is a nonsingular point of $Q^1_g(C, n, f_{e,1})$. Thus $dim(Q^1_g(\mu, n, f_{e,1})^{SS}_{[k,1]}) = n^2 - 1 + g + dim(H_g)$.

Let $\xi \in Q^1_g(\mu, n, f_{e,1})^{SS}_{[k,1]}$ be a closed point with the notation given above. We examine the exact differential sequence:

$$0 \to T_\xi(\pi^{-1}_{SS}[C]) \to T_\xi(Q^1_g(\mu, n, f_{e,1})^{SS}_{[k,1]}) \xrightarrow{d\pi_{SS}} T_{[C]}(H_g).$$

Recall H_g is nonsingular. By Lemma (9.3.1) and the surjection of $T_{[C]}(H_g)$ onto the miniversal deformation space of C,

$$dim(im(d\pi_{SS})) \leq dim(H_g) - s$$

(s is the number of nodes where $\mathcal{E}_\xi$ is not locally free). By previous results:

$$dim(T_\xi(\pi^{-1}_{SS}[C])) = n^2 - 1 + g + s.$$

It follows:

$$\begin{aligned} dim(T_\xi(Q^1_g(\mu, n, f_{e,1})^{SS}_{[k,1]})) &\leq (n^2 - 1 + g + s) + (dim(H_g) - s) \\ &= dim(Q^1_g(\mu, n, f_{e,1})^{SS}_{[k,1]}). \end{aligned} \tag{42}$$

Equality must hold in equation (42). ξ is therefore a nonsingular point of $Q^1_g(\mu, n, f_{e,1})^{SS}_{[k,1]}$. $\square$

As a consequence, we obtain:

Proposition 9.3.1. *$\overline{U_g(e,1)}$ is normal.*

10. The isomorphism between $\overline{U_g(e,1)}$ and $\overline{P_{g,e}}$

10.1. **A review of $\overline{P_{g,e}}$.** In this section, the compactification of the universal Picard variety of degree e line bundles, $\overline{P_{g,e}}$, described in [Ca] is considered. Let e be large enough to guarantee the existence and properties of $\overline{P_{g,e}}$ and $\overline{U_g(e,1)}$. A natural isomorphism $\nu : \overline{P_{g,e}} \to \overline{U_g(e,1)}$ will be constructed.

We follow the notation of [Ca], [Gi]. A Deligne-Mumford *quasi-stable*, genus g curve C is a Deligne-Mumford semistable, genus g curve with destabilizing chains of length at most one. Let $\psi : C \to C_s$ be the canonical contraction to the Deligne-Mumford stable model. For each complete subcurve $D \subset C$, let $D^c = \overline{C \setminus D}$. Define:

$$k_D = \#(D \cap D^c).$$

Let $\omega_{C,D}$ be the degree of the canonical bundle ω_C restricted to D. Let L be a degree e line bundle on C. Denote by L_D the restriction of L to D. Let e_D be the degree of L_D. L has *(semi)stable multidegree* if for each complete, proper subcurve $D \subset C$, the following holds:

$$e_D - e \cdot \left(\frac{\omega_{C,D}}{2g-2}\right) (\leq) < k_D/2. \tag{43}$$

Consider the Hilbert scheme $H_{g,e,M}$ of degree e, genus g curves in $\mathbf{P}^M$, where $M = e - g + 1$. In [Gi], it is shown there exists an open locus $Z_{g,e} \subset H_{g,e,M}$ parametrizing nondegenerate, Deligne-Mumford quasi-stable, genus g curves $C \subset \mathbf{P}^M$ satisfying:

(i) $h^1(C, \mathcal{O}_C(1)) = 0$.
(ii) $\mathcal{O}_C(1)$ is of semistable multidegree on C.

In [Ca], $\overline{P_{g,e}}$ is constructed as the G.I.T. quotient $\overline{P_{g,e}} \stackrel{\sim}{=} Z_{g,e}/SL_{M+1}$ (for a suitable linearization). $\overline{P_{g,e}}$ is a moduli space of line bundles of semistable multidegree on Deligne-Mumford quasi-stable curves (up to equivalence) compactifying the universal Picard variety.

The construction of the isomorphism $\nu : \overline{P_{g,e}} \to \overline{U_g(e,1)}$ proceeds as follows. If L is a very ample line bundle of semistable multidegree on a Deligne-Mumford quasi-stable curve C, then $\psi_*(L)$ is a slope-semistable torsion free sheaf of uniform rank 1 on C_s. This is the result of Lemma (10.2.1). The map ν is constructed by globalizing this correspondence. There is a universal curve

$$U_Z \hookrightarrow Z_{g,e} \times \mathbf{P}^M.$$

A deformation study shows $Z_{g,e}$ and U_Z are nonsingular quasi-projective varieties ([Ca], Lemma 2.2, p.609). There exists a canonical contraction map $\psi : U_Z \to U^s_Z$ over $Z_{g,e}$. The map ψ contracts each fiber of U_Z over $Z_{g,e}$ to its Deligne-Mumford stable model. U^s_Z is a flat, projective family of Deligne-Mumford stable curves over $Z_{g,e}$. Let $L = \mathcal{O}_{U_Z}(1)$ and $\mathcal{E} = \psi_*(L)$. In Lemma (10.2.6), $\mathcal{E}$ is shown to be a flat family of slope-semistable torsion free sheaves of uniform rank 1 and degree e over $Z_{g,e}$. Care is required in establishing flatness. The argument depends upon Zariski's theorem on formal functions and the criterion of Lemma (10.2.5). By the universal property of $\overline{U_g(e,1)}$, $\mathcal{E}$ induces a map $\nu_Z : Z_{g,e} \to \overline{U_g(e,1)}$. Since ν_Z is SL_{M+1}-invariant, a map $\nu : \overline{P_{g,e}} \to \overline{U_g(e,1)}$ is obtained.

It remains to prove ν is an isomorphism. Since $\overline{U_g(e,1)}$ is normal by Proposition (9.3.1), it suffices to show ν is bijective. Surjectivity is clear. Injectivity is established in section (10.3).

10.2. **The construction of ν.** Multidegree (semi)stability corresponds to slope-(semi)stability in the following manner:

Lemma 10.2.1. *Let C be a Deligne-Mumford quasi-stable curve. If L is a very ample, degree e line bundle on C of (semi)stable multidegree, then $E = \psi_*(L)$ is a slope-(semi)stable, torsion free sheaf of uniform rank 1 and degree e on C_s. Also, if L is of strictly semistable multidegree, then E is strictly slope-semistable.*

First we need a simple technical result. For each complete subcurve D of C, define the sheaf F_D on C by the sequence:

$$0 \to F_D \to L \to L_{D^c} \to 0.$$

F_D is the subsheaf of sections of L with support on D. In fact, F_D is exactly the subsheaf of sections of L_D vanishing on $D \cap D^c$. Therefore $degree(F_D) = e_D - k_D$. Note by Riemann-Roch, $\chi(F_D) = degree(F_D) + 1 - g_D$, where g_D is the arithmetic genus of D. We obtain,

$$e_D = \chi(F_D) + g_D - 1 + k_D. \tag{44}$$

Lemma 10.2.2. *Let C be a Deligne-Mumford quasi-stable curve. Let L be a very ample line bundle of semistable multidegree. For every complete subcurve $D \subset C$, $R^1\psi_*(F_D) = 0$.*

Proof. A fiber of ψ is either a point or a destabilizing $\mathbf{P}^1$. By inequality (43) and ampleness, the restriction of L to a destabilizing $\mathbf{P}^1$ is $\mathcal{O}_{\mathbf{P}^1}(1)$. Let P be a destabilizing $\mathbf{P}^1$ of C. There are five cases:

(i) $P \subset D^c$, $P \cap D = \emptyset$. Then $F_D|_P = 0$.
(ii) $P \subset D^c$, $P \cap D \neq \emptyset$. Then $F_D|_P$ is torsion.
(iii) $P \subset D$, $P \cap D^c = \emptyset$. Then $F_D|_P = \mathcal{O}_P(1)$.
(iv) $P \subset D$, $\#|P \cap D^c| = 1$. Then $F_D|_P = \mathcal{O}_P$.
(v) $P \subset D$, $\#|P \cap D^c| = 2$. Then $F_D|_P = \mathcal{O}_P(-1)$.

In each case, $h^1(P, F_D|_P) = 0$. The vanishing of $h^1(P, F_D|_P)$ and the simple character of ψ imply $R^1\psi_*(F_D) = 0$. □

Proof. [Of Lemma (10.2.1)] Let C^1 be the union of the destabilizing $\mathbf{P}^1$'s of C. By Lemma (10.2.2), $R^1\phi_*(F_{C^1}) = 0$. Since each destabilizing P in C^1 is of type (v) in the proof of Lemma (10.2.2), $F_{C^1}|_P = \mathcal{O}_P(-1)$. It follows that $\psi_*(F_{C^1}) = 0$. By the long exact sequence associated to

$$0 \to F_{C^1} \to L \to L_{C^{1c}} \to 0,$$

it follows $E \cong \psi_*(L_{C^{1c}})$. Since $L_{C^{1c}}$ is torsion free on C^{1c} and the morphism $C^{1c} \to C_s$ is finite, E is torsion free. Certainly, E is of uniform rank 1.

Let $0 \to G \to E$ be a proper subsheaf. Let $D_s \subset C_s$ be the support of G. If $D_s = C_s$, the inequality of slope-stability follows trivially. We can assume D_s is a complete, proper subcurve. Let $D = \psi^{-1}(D_s)$. $D \subset C$ is a complete, proper subcurve. It is clear that G is a subsheaf of $\psi_*(F_D)$ with torsion quotient. Therefore, it suffices to check slope-(semi)stability for $\psi_*(F_D)$. Certainly

$$h^0(C_s, \psi_*(F_D)) = h^0(C, F_D).$$

By Lemma (10.2.2), $R^1\psi_*(F_D) = 0$. Thus by a degenerate Leray spectral sequence ([H], Ex. 8.1, p.252),

$$h^1(C_s, \psi_*(F_D)) = h^1(C, F_D).$$

Hence $\chi(F_D) = \chi(\psi_*(F_D))$. Similarly $\chi(L) = \chi(\psi_*(L))$. Inequality (43) and equation (44) yield:

$$(\chi(F_D) + g_D - 1 + k_D) - (\chi(L) + g - 1) \cdot \left(\frac{2g_D - 2 + k_D}{2g - 2} \right) \ (\leq) <\ k_D/2.$$

After some manipulation, we see

$$\frac{\chi(F_D)}{\omega_{D,C}} \ (\leq) <\ \frac{\chi(L)}{2g-2}\ .$$

The above results yield

$$\frac{\chi(\psi_*(F_D))}{\omega_{D_s,C_s}} \ (\leq) <\ \frac{\chi(\psi_*(L))}{2g-2}\ .$$

Hence, $\psi_*(L)$ is slope-(semi)stable. The final claim about strict semistability also follows from the proof. □

Let $\psi : U_Z \to U_Z^s$, $L = \mathcal{O}_{U_Z}(1)$, and $\mathcal{E} = \psi_*(L)$ be as defined in section (10.1). We now establish that $\mathcal{E}$ is flat over $Z_{g,e}$. The vanishing of $R^1\psi_*(L)$ is proved by Zariski's theorem on formal functions in Lemma (10.2.4). The flatness criterion of Lemma (10.2.5) is then applied to obtain the required flatness.

First we need an auxiliary result. Let $[C] \in Z_{e,g}$ and let $P \subset U_Z$ be a destabilizing $\mathbf{P}^1$ of C. The conormal bundle, N_P^*, of P in U_Z is locally free (P, U_Z are nonsingular). Recall a locally free sheaf $\bigoplus \mathcal{O}_{\mathbf{P}^1}(a_i)$ on $\mathbf{P}^1$ is said to be non-negative if each $a_i \geq 0$.

Lemma 10.2.3. *N_P^* is non-negative.*

Proof. Let T_U and T_Z denote the tangent sheaves of U_Z and $Z_{e,g}$. Let $\rho : U_Z \to Z_{e,g}$ be the natural morphism. There is a differential map

$$d\rho : T_U \to \rho^*(T_Z).$$

Restriction to P yields a (non-exact) sequence

$$0 \to T_P \to T_U|_P \to \rho^*(T_Z)|_P.$$

Certainly, $\rho^*(T_Z)|_P \cong \bigoplus \mathcal{O}_P$. We obtain a map

$$\alpha : N_P \to \bigoplus \mathcal{O}_P.$$

Let $\hat{P} \subset P$ be the locus of nonsingular points of C. Since the morphism ρ is smooth on $\hat{P}$, $\alpha|_{\hat{P}}$ is an isomorphism of sheaves. Since N_P is a torsion free sheaf, α must be an injection of sheaves. It follows easily N_P is non-positive. Hence N_P^* is non-negative. □

In fact, an examination of the deformation theory yields $N_P^* \cong \mathcal{O}_P(1) \oplus \mathcal{O}_P(1) \oplus I$ where I is a trivial bundle. We will need only the non-negativity result.

Lemma 10.2.4. $R^1\psi_*(L) = 0$.

Proof. Let $\zeta \in U^s_Z$. It suffices to prove

$$R^1\psi_*(L)_\zeta = 0$$

in case ζ is a node of stable curve destabilized in U_Z. Let m be the ideal of ζ. $\psi^{-1}(m)$ is the ideal of the nonsingular destabilizing $P = \mathbf{P}^1$. Let P_n denote the subscheme of U_Z defined by $\psi^{-1}(m^n) \cong \psi^{-1}(m)^n$. Let L_n be the restriction of L to P_n. By Zariski's Theorem on formal functions:

$$R^1\psi_*(L)_\zeta\hat{} \;\; \cong \;\; \overset{lim}{\leftarrow} H^1(P_n, L_n).$$

Since completion is faithfully flat for noetherian local rings, it suffices to show for each $n \geq 1$, $h^1(P_n, L_n) = 0$. As above, denote the conormal bundle of P in U_Z by N^*_P. Since the varieties in question are nonsingular, there is an isomorphism on P:

$$m^{n-1}/m^n \cong Sym^{n-1}(N^*_P).$$

Since the pair (C, L_C) is of semistable multidegree and P is a destabilizing $\mathbf{P}^1$, $L_1 \cong \mathcal{O}_P(1)$. Hence $h^1(P_1, L_1) = 0$. There is an exact sequence on P_n for each $n \geq 2$:

$$0 \to m^{n-1}/m^n \otimes L_n \to L_n \to L_{n-1} \to 0.$$

There is a natural identification

$$m^{n-1}/m^n \otimes L_n \cong Sym^{n-1}(N^*_P) \otimes_{\mathcal{O}_P} \mathcal{O}_P(1).$$

From the non-negativity of N^*_P (Lemma (10.2.3)), we see

$$h^1(P_n, m^{n-1}/m^n \otimes L_n) = 0.$$

By the induction hypothesis

$$h^1(P_n, L_{n-1}) = h^1(P_{n-1}, L_{n-1}) = 0.$$

Thus $h^1(P_n, L_n) = 0$. The proof is complete. □

Lemma 10.2.5. *Let $\phi : B_1 \to B_2$ be a projective morphism of schemes over A. If F is a sheaf on B_1 flat over A and $\forall\ i \geq 1,\ \ R^i\phi_*(F) = 0$, then $\phi_*(F)$ is flat over A.*

Proof. We can assume A and B_2 are affine and $B_1 \cong \mathbf{P}^k_{B_2}$. Let $\mathcal{U}$ be the standard $k+1$ affine cover of B_2. There is a Cech resolution computing the cohomology of F on B_1:

(45) $$0 \to H^0(B_1, F) \to C^0(\mathcal{U}, F) \to C^1(\mathcal{U}, F) \to \ldots \to C^k(\mathcal{U}, F) \to 0.$$

Since $R^i\phi_*(F) = 0$ for $i \geq 1$, the resolution (45) is exact. Since F is A-flat, the Cech modules $C^j(\mathcal{U}, F)$ are all A-flat. Exactness of (45) implies $H^0(B_1, F) \cong \phi_*(F)$ is A-flat. □

Lemma 10.2.6. *$\mathcal{E}$ is a flat family of slope-semistable, torsion free sheaves of uniform rank 1 over $Z_{e,g}$.*

Proof. By Lemmas (10.2.4) and (10.2.5), $\mathcal{E}$ is flat over $Z_{e,g}$. Let $[C] \in Z_{e,g}$. We have a diagram:

(46)
$$\begin{array}{ccc} C & \xrightarrow{i_C} & U_Z \\ \downarrow{\scriptstyle \psi_C} & & \downarrow{\scriptstyle \psi} \\ C_s & \xrightarrow[i_{C_s}]{} & U^s_Z \end{array}$$

If $i^*_{C_s}(\mathcal{E}) \cong \psi_{C*}(i^*_C(L))$, then the proof is complete by Lemma (10.2.1). There is a natural morphism of sheaves

$$\gamma_C : i^*_{C_s}(\mathcal{E}) \to \psi_{C*}(i^*_C(L)).$$

We first show γ_C is a surjection. Since, by Lemma (10.2.1), $\psi_{C*}(i^*_C(L))$ is a slope-semistable torsion free sheaf of degree e, $\psi_{C*}(i^*_C(L))$ is generated by global sections. There is a natural identification

$$H^0(C_s, \psi_{C*}(i^*_C(L))) \cong H^0(C, \mathcal{O}_C(1)).$$

By the nondegeneracy of C and the nonspeciality of $\mathcal{O}_C(1)$, $H^0(C, \mathcal{O}_C(1))$ is canonically isomorphic to $H^0(\mathbf{P}^M, \mathcal{O}_{\mathbf{P}^M}(1))$. These sections extend over U_Z and thus appear in $i^*_{C_s}(\mathcal{E})$. Therefore, γ_C is a surjection.

Since ψ is an isomorphism except at destabilizing $\mathbf{P}^1$'s of U_Z, the kernel of γ_C is a torsion sheaf on C_s. Flatness of $\mathcal{E}$ over $Z_{e,g}$ implies $\chi(i^*_{C_s}(\mathcal{E}))$ is independent of $[C] \in Z_{e,g}$. By Lemma (10.2.1), $\chi(\psi_{C*}(i^*_C(L)))$ is independent of $[C] \in Z_{e,g}$. Over the open locus of nonsingular curves $[C] \in Z_{e,g}$, ψ is an isomorphism thus:

$$\chi(i^*_{C_s}(\mathcal{E})) = \chi(\psi_{C*}(i^*_C(L))). \tag{47}$$

By the above considerations, (47) holds for every $[C] \in Z_{e,g}$. Hence, the torsion kernel of γ_C must be zero. γ_C is an isomorphism. The proof is complete. □

By combining Lemma (10.2.6) with Theorem (9.1.1), there exists a natural morphism $\nu_Z : Z_{e,g} \to \overline{U_g(e,1)}$. Since ν_Z is SL_{M+1}-invariant, ν_Z descends to $\nu : \overline{P_{g,e}} \to \overline{U_g(e,1)}$. Certainly ν is surjective. Since $\overline{U_g(e,1)}$ is normal, ν is an isomorphism if and only if ν is injective. The injectivity of ν will be established in section (10.3).

10.3. **Injectivity of ν.** Let C be a Deligne-Mumford quasi-stable genus g curve. Let $D \subset C$ be a complete subcurve. A node $z \in D$ is an *external* node of D if $z \in D^c$. $P \subset D$ is a *destabilizing* $\mathbf{P}^1$ *of* D if P is a destabilizing $\mathbf{P}^1$ of C. A destabilizing $\mathbf{P}^1$ of D is *external* if $P \cap D^c \neq \emptyset$. Let $\hat{D}$ denote D minus all the external destabilizing $\mathbf{P}^1$'s of D.

(C, L) is a *semistable pair* if C is Deligne-Mumford quasi-stable and L is a very ample line bundle of semistable multirank. The semistable pairs (C, L) and (C', L') are isomorphic if there exists an isomorphism $\gamma : C \to C'$ such that $\gamma^*(L') \cong L$. A complete subcurve $D \subset C$ is an *extremal subcurve* of the semistable pair (C, L) if equality holds in (43):

$$e_D - e \cdot \left(\frac{\omega_{C,D}}{2g-2} \right) \ = \ k_D/2.$$

The semistable pair (C, L) is said to be *maximal* if the following condition is satisfied: if $z \in C$ is an external node of an extremal subcurve, z is contained in a destabilizing $\mathbf{P}^1$. Let $\psi : C \to C_s$ be the stable contraction. By Lemma (10.2.1), $\psi_*(L)$ is a slope-semistable, torsion free sheaf of uniform rank 1. Let $J(\psi_*(L))$ be the associated set of slope-stable Jordan-Holder factors.

(C, J) is a *Jordan-Holder pair* if C is a Deligne-Mumford stable curve and J is a set of slope-stable, torsion free sheaves. As before, the Jordan-Holder pairs (C, J) and (C', J') are isomorphic if there exists an isomorphism $\gamma : C \to C'$ such that $\gamma^*(J') \cong J$.

Lemma 10.3.1. *Let (C, L), (C', L') be maximal semistable pairs. If $(C_s, J(\psi_*(L)))$ and $(C'_s, J(\psi'_*(L')))$ are isomorphic Jordan-Holder pairs, then (C, L) and (C', L') are isomorphic semistable pairs.*

Proof. Consider a Jordan-Holder filtration of $E = \psi_*(L)$ on C_s:

$$0 = E_0 \subset E_1 \subset E_2 \subset \ldots \subset E_n = E.$$

Let $A_i = Supp(E_i)$. For each $1 \le i \le n$,

$$\frac{\chi(E_i)}{w_{A_i, C_s}} = \frac{\chi(E)}{2g-2}.$$

By the proof of Lemma (10.2.1), we see the $B_i = \psi^{-1}(A_i)$ are extremal subcurves of (C, L) for $1 \le i \le n$. As before, let F_{B_i} be the subsheaf of sections of L with support on B_i. From the proof of Lemma (10.2.1), it follows $E_i \cong \psi_*(F_{B_i})$. For $1 \le i \le n$, let $X_i = \overline{A_i \setminus A_{i-1}}$ and $Y_i = \overline{B_i \setminus B_{i-1}}$. We see $Supp(E_i/E_{i-1}) = X_i$. If $z \in X_i$ is an internal node of X_i, z is destabilized by ψ if and only if (E_i/E_{i-1}) is locally isomorphic to m_z at z. If $z \in X_i$ is an external node, then there are two cases. If $z \in A_{i-1}$, then $\psi^{-1}(z) \cong \mathbf{P}^1 \not\subset Y_i$. If $z \in A_i^c$, then $\psi^{-1}(z) \cong \mathbf{P}^1 \subset Y_i$. These conclusions follow from the maximality of (C, L). It is now clear C_s and the Jordan-Holder factor E_i/E_{i-1} determine $\hat{Y}_i$ completely. Also, the $\hat{Y}_i$ are connected by destabilizing $\mathbf{P}^1$'s. We have shown (C_s, J) determines C up to isomorphism. We will show below in Lemmas (10.3.2-10.3.3) that $L_{\hat{Y}_i}$ is determined up to isomorphism by E_i/E_{i-1}. Since the $\hat{Y}_i$ are connected by destabilizing $\mathbf{P}^1$'s, the isomorphism class of the pair (C, L) is determined by the line bundles $L_{\hat{Y}_i}$. This completes the proof of the lemma. $\square$

Lemma 10.3.2. *There is an isomorphism $\psi_*(L_{\hat{Y}_i}) \cong E_i/E_{i-1}$.*

Proof. We keep the notation of the previous Lemma. Certainly $Supp(F_{B_i}/F_{B_{i-1}}) = Y_i$. Let p_i be the divisor $B_{i-1} \cap Y_i \subset Y_i$. Let q_i be the divisor $B_i^c \cap Y_i \subset Y_i$. Since the points of p_i lie on destabilizing $\mathbf{P}^1$'s joining Y_i to Y_{i-1} and the points of q_i lie on destabilizing $\mathbf{P}^1$'s joining Y_i to Y_{i+1}, we note $p_i \cap q_i = \emptyset$. There is an isomorphism $L_{Y_i} \cong F_{Y_i} \otimes \mathcal{O}_{Y_i}(p_i + q_i)$. Since there is an exact sequence:

$$0 \to F_{Y_i} \to (F_{B_i}/F_{B_{i-1}}) \to p_i \to 0,$$

we see

$$F_{Y_i} = (F_{B_i}/F_{B_{i-1}}) \otimes \mathcal{O}_{Y_i}(-p_i).$$

Thus

$$L_{Y_i} \cong (F_{B_i}/F_{B_{i-1}}) \otimes \mathcal{O}_{Y_i}(q_i).$$

Since B_i is an extremal subcurve of C and the pair (C, L) is maximal, we see q_i lies on external destabilizing $\mathbf{P}^1$'s of Y_i. Hence $L_{\hat{Y}_i}$ is isomorphic to $(F_{B_i}/F_{B_{i-1}})_{\hat{Y}_i}$. We have an exact sequence on Y_i:

$$0 \to I_{\hat{Y}_i} \otimes (F_{B_i}/F_{B_{i-1}}) \to (F_{B_i}/F_{B_{i-1}}) \to (F_{B_i}/F_{B_{i-1}})_{\hat{Y}_i} \to 0.$$

Let P be an external destabilizing $\mathbf{P}^1$ of Y_i. If P meets B_{i-1} then $P \subset B_{i-1}$. Thus each such P meets B_i^c. It is now not hard to see $I_{\hat{Y}_i} \otimes (F_{B_i}/F_{B_{i-1}})$ restricts to $\mathcal{O}_P(-1)$ on each such P. Therefore, by familiar arguments,

$$\psi_*((F_{B_i}/F_{B_{i-1}})_{\hat{Y}_i}) \cong \psi_*(F_{B_i}/F_{B_{i-1}}).$$

By Lemma (10.2.2), $R^1\psi_*(F_{B_{i-1}}) = 0$. Hence

$$\psi_*(F_{B_i}/F_{B_{i-1}}) \cong (\psi_*(F_{B_i})/\psi_*(F_{B_{i-1}})) \cong E_i/E_{i-1}.$$

Following all the isomorphisms yields the lemma. □

Lemma 10.3.3. *Let (C, L) be a semistable pair. Let $D \subset C$ satisfying $\hat{D} = D$. Then L_D is determined up to isomorphism by $\psi_*(L_D)$.*

Proof. Let D^1 be the union of the destabilizing $\mathbf{P}^1$'s of D. Note all these $\mathbf{P}^1$'s are internal. Let $D' = \overline{D \setminus D^1}$ and denote the restriction of ψ to D' by ψ'. Consider the sequence on D:

$$0 \to I_{D'} \otimes L_D \to L_D \to L_{D'} \to 0.$$

Since $I_{D'} \otimes L_D$ restricts to $\mathcal{O}_{\mathbf{P}^1}(-1)$ on each destabilizing $\mathbf{P}^1$ of D, we see

$$\psi_*(I_{D'} \otimes L_D) = R^1\psi_*(I_{D'} \otimes L_D) = 0.$$

Thus $\psi'_*(L_{D'}) \cong \psi_*(L_{D'}) \cong \psi_*(L_D)$. Since ψ' is a finite affine morphism,

$$\beta : \psi'^{\,*}(\psi'_*(L_{D'})) \to L_{D'} \to 0.$$

Let τ be the torsion subsheaf of $\psi'^{\,*}(\psi'_*(L_{D'}))$. Since β is generically an isomorphism and $L_{D'}$ is torsion free on D', we see

$$L_{D'} \cong (\psi'^{\,*}(\psi'_*(L_{D'}))/\tau).$$

We have shown that $L_{D'}$ is determined up to isomorphism by $\psi_*(L_D)$. It is clear that $L_{D'}$ determines L_D up to isomorphism. □

Let $\rho : Z_{e,g} \to \overline{P_{g,e}}$ be the quotient map. Let $\zeta \in \overline{P_{g,e}}$. It follows from the results of [Ca] (Lemma 6.1, p. 640) that there exists a $[C] \in Z_{g,e}$ satisfying:

(i) $\rho([C]) = \zeta$.
(ii) $(C, L = \mathcal{O}_C(1))$ is a maximal semistable pair.

Let $\psi_C : C \to C_s$ be the stable contraction. Let $E = \psi_*(L)$. Let J be the Jordan-Holder factors of E on C_s. From the definition of ν, $\nu(\zeta)$ is the element of $\overline{U_g(e,1)}$ corresponding to the isomorphism class of the data (C_s, J). By Lemmas (10.3.2-10.3.3), the isomorphism class of (C, L) is determined by the isomorphism class of (C_s, J). Therefore ν is injective. By the previous discussion, ν is an isomorphism.

Theorem 10.3.1. *There is a natural isomorphism $\nu : \overline{P_{g,e}} \to \overline{U_g(e,1)}$.*

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