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THE ANALOGUE OF THE STRONG SZEGÖ LIMIT THEOREM ON THE 2- AND 3-DIMENSIONAL SPHERES

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Introduction

Let S^1 denote the circle $\mathbb{R}/2\pi\mathbb{Z}$ and let $\mathcal{P}_n$ denote the space of functions on S^1 spanned by $\{ e^{im\theta} : 0 \le m \le n \}$. Write P_n for the orthogonal projection $L^2(S^1) \to \mathcal{P}_n$. For $f \in L^1(S^1)$ let $\hat{f}_n$ denote the n^{th} Fourier coefficient of f;

$$\hat{f}_n = \int_{S^1} f(\theta) e^{-in\theta} \frac{d\theta}{2\pi}.$$

For a function f on S^1 let $[f]$ denote the operator *multiplication by* f.

The strong Szegö limit theorem. *If the function $f : S^1 \to \mathbb{C}$ has a logarithm satisfying*

$$\sum_{m\in\mathbb{Z}} |m| \, |\widehat{\log f}_m|^2 < \infty,$$

then

$$\log\det P_n[f]P_n = (n+1)\int_0^{2\pi} \log f(\theta)\,\frac{d\theta}{2\pi} + \frac{1}{2}\sum_{m\in\mathbb{Z}} |m|\,\widehat{\log f}_m \widehat{\log f}_{-m} + \circ(1) \quad \textit{modulo } 2\pi i \quad \textit{as} \quad n\to\infty. \tag{0.1}$$

This theorem was first proved by G. Szegö [**3**] for positive functions f in the class $C^{1+\alpha}$ with $\alpha > 0$. Conditions on f were relaxed by several people (see, for example, [**7**] and [**4**]), until the sharp result above was obtained (see [**2**]).

Let us make a small technical remark. It is certainly true that for some continuous functions f, (0.1) holds precisely, not just mod $2\pi i$. It is not difficult to see that the spectrum of $P_n[f]P_n$ is contained in the closed convex hull of the image of f. If this set does not contain the origin, then there is a branch of the logarithm defined on it which can be used to define $\log f$ and $\log P_n[f]P_n$. If we replace $\log\det P_n[f]P_n$ in (0.1) by $\operatorname{trace}\log P_n[f]P_n$, the resulting formula holds precisely.

We also remark that if we instead define $\mathcal{P}_n$ to be the space spanned by $\{e^{im\theta} : |m| \le n\}$, then formula (0.1) holds if we replace the factor $(n+1)$ in the leading order term by $(2n+1)$. In this paper, we will prove the analogue of this result on the 2- and 3-dimensional spheres.

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For $N = 2, 3, \ldots, S^N$ denotes the unit sphere in $\mathbb{R}^{N+1}$. For N fixed, let σ denote surface measure on S^N, and let $d\mu = d\sigma/(\sigma(S^N))$. Let $\langle\ ,\ \rangle$ denote the standard inner product on $L^2(S^N, d\mu)$. For points x, y in S^N, let $d(x, y)$ denote the distance between x and y on S^N, i.e. the angle between x and y in $\mathbb{R}^{N+1}$.

For $n = 0, 1, 2, \ldots$, let $\mathcal{P}_n$ denote the space of polynomials of degree at most n on $\mathbb{R}^{N+1}$ restricted to S^N, and let P_n denote the orthogonal projection $L^2(S^N, d\mu) \to \mathcal{P}_n$; this is a spectral projection of Δ, the Laplace-Beltrami operator on S^N. Let d_n denote the dimension of $\mathcal{P}_n$; when $N = 2$, $d_n = (n+1)^2$, and when $N = 3$, $d_n = \frac{1}{6}(n+1)(n+2)(2n+3)$.

Frequently in this paper, a polynomial on $\mathbb{R}^{N+1}$ restricted to S^N will simply be called a *polynomial on* S^N.

Let $H^{1/2} = H^{1/2}(S^N)$ denote the Sobolev space of functions f on S^N such that the norm $\|(I - \Delta)^{1/4} f\|_{L^2}$ is finite.

Theorem 0.1. *For $N = 2$ or 3, if $f \in C(S^N) \cap H^{1/2}(S^N)$ is such that the closed convex hull of the image of f does not contain the origin, then*

$$\begin{aligned} \operatorname{trace} \log P_n[f]P_n \ &= d_n \int_{S^N} \log f(x)\, d\mu(x) \\ &\quad + \alpha_N\, n^{N-1} \iint_{S^N \times S^N} \frac{(\log f(x) - \log f(y))^2}{\sin^{N-1} d(x,y) \sin^2 \frac{d(x,y)}{2}}\, d\mu(x) d\mu(y) \\ &\quad + \circ\left(n^{N-1}\right) \qquad \textit{as } n \to \infty, \end{aligned} \tag{0.2}$$

where $\alpha_2 = 1/(4\pi)$, $\alpha_3 = 1/(32)$.

We remark that the assumption that f is continuous is technical, and one might hope to remove it. One might also hope to remove the assumption on the image of f, and expect (0.2) to hold whenever $\log f \in H^{1/2}$ (i.e. whenever the second integral on the right hand side exists), even though this is *not* the case for the analogue of the strong Szegö limit theorem on a multi-dimensional torus, as noted in [**10**].

To show the relationship between formulas (0.1) and (0.2), we remark that the second term on the right hand side of (0.1) can be written as

$$\frac{1}{8} \int_0^{2\pi} \int_0^{2\pi} \frac{(\log f(\theta) - \log f(\phi))^2}{\sin^2 \frac{\theta - \phi}{2}} \frac{d\theta}{2\pi} \frac{d\phi}{2\pi},$$

while the second term on the right hand side of (0.2) has the form given in (1.5). It seems likely that formula (0.2) is also true on S^N with $N > 3$; in (1.4) we write down what the constants α_N ought to be in general.

To put Theorem 0.1 in context we will describe some related results. Write $\mathcal{P}_{(n)}$ for the space of spherical harmonics on S^N of degree precisely n, and $P_{(n)}$ for the orthogonal projection $L^2(S^N, d\mu) \to \mathcal{P}_{(n)}$. Let B be a self-adjoint pseudodifferential operator of order zero on S^N which commutes with Δ. In [**14**], it is shown that for a smooth function F on $\mathbb{R}$, $\operatorname{trace} F(P_{(n)} B P_{(n)})$ has a complete asymptotic expansion in n. A weaker but more general result is proved in [**8**]: let A be a positive self-adjoint elliptic pseudo-differential operator of order 1 on a smooth compact N-dimensional manifold M without boundary. Let $\lambda_1 \leq \lambda_2 \leq \ldots$ denote the eigenvalues of A, let $\mathcal{P}_\lambda$ denote the space spanned by the eigenfunctions of A with eigenvalue at most λ, and let P_λ denote the orthogonal projection $L^2(M) \to \mathcal{P}_\lambda$. Let B be a self adjoint pseudo-differential operator of order zero, for example multiplication by some smooth real function f, and write $K = [-\|B\|, \|B\|] \subset \mathbb{R}$.

Theorem. *There exists an integer r and $C > 0$ such that if $F \in C^r(K)$ then*

$$\Big| \operatorname{trace} F(P_\lambda B P_\lambda) \;-\; \frac{\lambda^N}{(2\pi)^N} \int_{a_0(x,\xi)<1} F(b_0(x,\xi))\, dx d\xi \,\Big| \;\le\; C(\lambda^{N-1}+1)\|f\|_{C^r(K)},$$

where a_0 and b_0 are the principal symbols of A and B.

Now let M be a manifold with density $d\mu$, let $\{\mathcal{P}_n\}$ be a sequence (or continuum) of subspaces of $L^2(M, d\mu)$, write P_n for the orthogonal projection $L^2(M, d\mu) \to \mathcal{P}_n$, and for a function f on M, write $[f]$ for the operator *multiplication by f*. In this situation, the "analogue of the strong Szegö limit theorem" would be a formula giving the second order asymptotics of $\operatorname{trace}\log P_n[f]P_n$ for f in some class of functions for which the operator $\log P_n[f]P_n$ is trace class. In a moment, we will list some cases when such a formula exists. First, however, we remark that as a general rule, if we know the second order asymptotics of

$$\operatorname{trace}\log P_n[f]P_n$$

for all sufficiently regular functions f which are sufficiently close to a constant function, or if we know the second order asymptotics of

$$\operatorname{trace}(P_n[f]P_n)^k, \qquad\qquad \text{for all} \quad k = 1, 2, \ldots$$

for all sufficiently regular functions f, then we can deduce the second order asymptotics of

$$\operatorname{trace} F(P_n[f]P_n)$$

for harmonic functions F defined on a neighbourhood of 0, and all sufficiently small and regular functions f. We will not try to make the above statement precise, but examples of this idea can be seen in Proposition 0.2 and Section 3.

In the following cases, the second order asymptotics of $\operatorname{trace}\log P_n[f]P_n$ are known: in [**16**] and [**17**], $M = \mathbb{R}^N$ with Lebesgue measure and $\mathcal{P}_n$ is the space of functions whose Fourier transform is supported on $n\Omega$, where Ω is any fixed set with sufficient regularity. (In this case, a complete asymptotic formula is known.) In [**1**], [**9**] and [**10**], M is the flat N-dimensional torus with Lebesgue measure and $\mathcal{P}_n$ is the space of functions with Fourier series supported on $n\Omega$, where Ω is any set in $\mathbb{R}^N$ satisfying certain weak conditions. In [**4**], M is the interval $[-1, 1]$ with measure $(1-x^2)^{\nu-1/2}$, and $\mathcal{P}_n$ is the space spanned by the ultraspherical polynomials of index ν and degree at most n. In [**6**], M is a sufficiently regular closed Jordan curve with arclength measure $d\mu$ and $\mathcal{P}_n$ is the space of polynomials in the complex variable z of degree at most n. In [**11**], M is the interval $[-1, 1]$ with Lebesgue measure and $\mathcal{P}_n$ is the space spanned by the first n eigenfunctions of a non-singular Sturm-Liouville operator.

One might ask what the second order terms of

$$\operatorname{trace}\log P_n[f]P_n \quad \text{and} \quad \operatorname{trace}(P_n[f]P_n)^k$$

look like in all these cases. The answer to these questions was useful in proving Theorem 0.1. In all the cases listed above, the second order term of trace $\log P_n[f]P_n$ is quadratic in $\log f$; more precisely, the second order asymptotic formula has the form

$$\begin{aligned}\text{trace } \log P_n[f]P_n \;&=\; (\dim \mathcal{P}_n)\, L(\log f) \;+\; b_n\, B(\log f, \log f)\\ &\quad+\; \text{lower order error},\end{aligned}$$

where L is a linear functional ($L(g) =$ the "average" of g), the constants b_n do not depend on f and $B(\ ,\ )$ is a bilinear functional. Perhaps there is a simple reason why this should be true, but I am not aware of one. The reason that this is a useful observation is that if the second order term of trace $\log P_n[f]P_n$ exists and is quadratic in $\log f$, then there exist *a priori* second order asymptotic formulas for trace$(P_n[f]P_n)^k$. In the case of Theorem 0.1, it is possible to run this backwards; to prove that these formulas for trace$(P_n[f]P_n)^k$ hold and then use them to deduce the asymptotics of trace $\log P_n[f]P_n$.

Now we explain these ideas more fully. First we will derive *a priori* second order asymptotic formulas for trace$(P_n[f]P_n)^k$ assuming that a second order term for trace $\log P_n[f]P_n$ exists and is quadratic in $\log f$. We will also show how to write the asymptotic formulas in terms of the integral kernels $K_n(x, y)$ of the projections P_n.

Notice that in terms of the integral kernels $K_n(x, y)$, we have

$$\begin{aligned}\text{trace } P_n[f]P_n \;&=\; \int_M K_n(x,x) f(x)\, d\mu(x), \qquad (0.3)\\ \text{trace } P_n[f]P_n[g]P_n \;&=\; \iint_{M\times M} |K_n(x,y)|^2 f(x) g(y)\, d\mu(x) d\mu(y),\end{aligned}$$

and, by using the reproducing property of $K_n(x, y)$,

$$\begin{aligned}(0.4)\quad &\text{trace } P_n[f](I - P_n)[g]P_n\\ &\quad=\; \frac{1}{2}\iint_{M\times M} |K_n(x,y)|^2 (f(x) - f(y))(g(x) - g(y))\, d\mu(x) d\mu(y).\end{aligned}$$

The asymptotics of the above quantities can be calculated if we have sufficiently good asymptotics for the kernel $K_n(x, y)$.

Proposition 0.2. *Suppose W is a Banach algebra of functions on M with*

$$W \;\subseteq\; L^\infty(V), \qquad \|\ \|_W \;\geq\; \|\ \|_\infty, \qquad 1 \in W.$$

Suppose there exists a continuous linear functional L on W, a continuous bilinear functional $B(\ ,\ )$ on $W \times W$, sequences of numbers a_n, b_n, c_n and a neighbourhood $\mathcal{U}$ of 1 in W such that for every $f \in \mathcal{U}$,

$$(0.5)\quad \text{trace} \log P_n[f]P_n \;=\; a_n\, L(\log f) \;+\; b_n\, B(\log f, \log f) \;+\; \circ(c_n) \qquad \text{as } n \to \infty.$$

Then for every $g \in W$ we have

$$\begin{aligned}a_n\, L(g) \;&=\; \text{trace}\, P_n[g]P_n \;+\; \circ(c_n),\\ b_n\, B(g, g) \;&=\; \tfrac{1}{2}\,\text{trace } P_n[g](I - P_n)[g]P_n \;+\; \circ(c_n), \qquad (0.6)\end{aligned}$$

so

$$\text{(0.7)}\quad \operatorname{trace} \log P_n[f]P_n = \int_M K_n(x,x)\log f(x)\,d\mu(x) + \frac{1}{4}\iint_{M\times M} |K_n(x,y)|^2(\log f(x)-\log f(y))^2\,d\mu(x)d\mu(y) + \circ(c_n).$$

Furthermore, for $k = 2, 3, \ldots,$

(0.8)

$$\begin{aligned}
&\operatorname{trace}\,(P_n[f]P_n)^k \\
&= \operatorname{trace}\ P_n[f^k]P_n - \sum_{j=1}^{k-1}\frac{1}{j}\operatorname{trace}\ P_n[f^j](I-P_n)[f^{k-j}]P_n + \circ(c_n) \\
&= \int_M K_n(x,x)f^k(x)\,d\mu(x) \\
&\quad + \sum_{j=1}^{k-1}\frac{1}{2j}\iint_{M\times M}|K_n(x,y)|^2(f^j(x)-f^j(y))(f^{k-j}(x)-f^{k-j}(y))\,d\mu(x)d\mu(y) \\
&\qquad + \circ(c_n).
\end{aligned}$$

Proof. To prove the proposition, fix g in W, write (0.5) with f replaced by $1 - tg$, and expand the function log, which occurs in both sides of the equation, as a power series. Formulas (0.6) result from equating coefficients of t and t^2, and (0.3) and (0.4) give (0.7). Equating higher powers of t yields (0.8).

Now we give an outline of the proof of Theorem 0.1. Our strategy is to prove that the key formula (0.8) holds (with $c_n = n^{N-1}$). It is an easier matter to deduce (0.7) from (0.8), and to compute the second order asymptotics of the terms on the right hand side of (0.7), thus proving the theorem. Here are some more details of this plan: the two terms on the right hand side of (0.7) came from (0.3) and (0.4). To compute (0.3) is easy since, for the sphere, $K_n(x,x) = d_n$; so

$$\text{(0.9)}\qquad \int_{S^N} K_n(x,x)f(x)\,d\mu(x) = d_n\int_{S^N} f(x)\,dx.$$

We will begin in Section 1 by using known asymptotics of the kernel $K_n(x,y)$ to obtain the second order asymptotics of (0.4). The result is

$$\begin{aligned}
&\frac{1}{4}\iint_{S^N\times S^N}|K_n(x,y)|^2(f(x)-f(y))(g(x)-g(y)\,d\mu(x)d\mu(y) \\
&\quad = \alpha_N\, n^{N-1}\iint_{S^N\times S^N}\frac{(f(x)-f(y))(g(x)-g(y))}{\sin^{N-1}d(x,y)\sin^2\frac{d(x,y)}{2}}\,d\mu(x)d\mu(y) \\
&\qquad + \circ(n^{N-1}),
\end{aligned}$$

where α_N is defined below (0.2), and in (1.4) for $N > 3$.

In Section 2 we will obtain the second order asymptotics of

$$\operatorname{trace}\,(P_n[f]P_n)^k \qquad \text{as } n\to\infty, \qquad \text{for } k = 3, 4, \ldots,$$

for polynomials f. This is the key step and the most delicate. The asymptotic formula which we will obtain is (0.8), which stated precisely in this context is as follows.

Lemma 0.3. *If f is a polynomial on S^N, where $N = 2$ or 3, then*

$$\operatorname{trace}\,(P_n[f]P_n)^k \;=\; \operatorname{trace}\; P_n[f^k]P_n \;-\; \sum_{j=1}^{k-1}\frac{1}{j}\operatorname{trace}\; P_n[f^j](I-P_n)[f^{k-j}]P_n \;+\; \circ(n^{N-1}) \qquad \text{as } n\to\infty.$$

To prove Lemma 0.3, we will show that if f and f_0 are polynomials on S^N, where $N = 2$ or 3, then

$$\operatorname{trace}\,(P_n[f])^j(I-P_n)[f_0] \;=\; \frac{1}{j}\operatorname{trace}\; P_n[f^j](I-P_n)[f_0] \;+\; \circ(n^{N-1}). \tag{0.10}$$

Combined with the identity

$$(P_n[f]P_n)^k \;=\; P_n[f^k]P_n \;-\; \sum_{j=1}^{k-1}(P_n[f])^j(I-P_n)[f^{k-j}]P_n,$$

this clearly proves the lemma. In fact we will prove a slightly stronger, linearized version of (0.10): if $f_0,\ldots,f_j$ are polynomials on S^N, where $N = 2$ or 3, then

$$\sum_\sigma \operatorname{trace}\; P_n[f_{\sigma_1}]\ldots P_n[f_{\sigma_j}](I-P_n)[f_0]P_n \;=\; \operatorname{trace}\; P_n[f_1\ldots f_j](I-P_n)[f_0]P_n \;+\; \circ(n^{N-1}), \tag{0.11}$$

where the sum is over σ in the set of permutations generated by the cycle $(1,\ldots,j)$. It is clear that (0.11) implies (0.10).

The proof of (0.11) is based on the proof of the strong Szegö limit theorem (on S^1) by M. Kac; see [**7**]. We will take a moment to explain the approach. The way Kac proves the theorem for functions f close to the constant function 1 is to expand both sides of (0.1), first expanding $\log P_n[f]P_n$ and $\log f$ as power series in $P_n[1-f]P_n$ and $(1-f)$ respectively, and then expanding f as a Fourier series. The resulting two expressions are simplified using the fact that $e^{im^1x}e^{imx} = e^{i(m^1+m)x}$, and they are both eventually expressed in the form

$$\sum_{k=0}^{\infty}\;\sum_{-\infty<m_1\le m_2\le\cdots\le m_k<\infty} c_{m_1\ldots m_k}(n)\hat f_{m_1}\ldots\hat f_{m_k}$$

for some coefficients $c_{m_1\ldots m_k}(n)$. Formula (0.1) is established by obtaining certain bounds on these coefficients and by showing that for fixed $m_1,\ldots,m_k$, the coefficients $c_{m_1\ldots m_k}(n)$ coming from each side of (0.1) are equal, for n sufficiently large. This final step amounts to a combinatorial identity of Hunt and Dyson, known today as the *Kac formula*, which states that if $m_1,\ldots,m_j$ are real numbers, then

$$\begin{aligned}\sum_\sigma \Big(\; &\min\{0,\, m_{\sigma_1},\, m_{\sigma_1}+m_{\sigma_2},\,\ldots,\, m_{\sigma_1}+m_{\sigma_2}+\cdots+m_{\sigma_j}\}\\ &-\;\min\{0,\, m_{\sigma_1},\, m_{\sigma_1}+m_{\sigma_2},\,\ldots,\, m_{\sigma_1}+m_{\sigma_2}+\cdots+m_{\sigma_{j-1}}\}\;\Big)\\ =\;&\min\{0,\, m_1+\cdots+m_j\},\end{aligned} \tag{0.12}$$

where the sum is over σ in the set of permutations generated by the cycle $(1,\dots,j)$.

We return now to the proof of Theorem 0.1. To prove (0.11), we expand the polynomial f in spherical harmonics. For this purpose we fix bases for the spherical harmonics on S^N, with basis functions $Y_{\mathbf{m}}$ indexed by lattice points $\mathbf{m}$. On S^2 the lattice is $\mathbb{Z}^2$ and on S^3 it is a sub-lattice of $\mathbb{Z}^3$. Having expanded f in spherical harmonics, we simplify the resulting expression. The difficulty here which is not present on S^1 is that complicated coefficients result when one expresses the product of two spherical harmonics as a linear combination of the basis spherical harmonics. For the bases we use, the coefficients occurring in such linear combinations are explicitly known. The perfect situation, from the point of view of performing reductions analogous to those of Kac, would be if there existed a constant $\Gamma_{\mathbf{m}_1}^{\mathbf{m}^1}$ for each pair of indices $\mathbf{m}_1$ and $\mathbf{m}^1$ such that for all indices $\mathbf{m}$,

$$Y_{\mathbf{m}_1} \cdot Y_{\mathbf{m}} \;=\; \sum_{\mathbf{m}^1} \Gamma_{\mathbf{m}_1}^{\mathbf{m}^1} Y_{\mathbf{m}^1+\mathbf{m}};$$

i.e. if the operator $[Y_{\mathbf{m}_1}]$ acting on basis spherical harmonics corresponded to a fixed linear combination of shifts in the index. Of course this is not the case, but it turns out that in some precise sense, $[Y_{\mathbf{m}_1}]$ can be locally well approximated by a fixed linear combination of shifts. This is made precise in Lemma 2.2.

Finally, in Section 3 we show how to extend the results of Section 2 for polynomials f to functions $f \in C(S^N)\cap H^{1/2}(S^N)$, and then we show how to go from the asymptotics obtained for trace $(P_n[f]P_n)^k$ to the asymptotics for trace $\log P_n[f]P_n$. The methods in this final section are elementary and general.

Theorem 0.1 appeared in my thesis [**10**]. I would like to thank Alice Chang and Tom Wolff for suggesting that I work on this problem.

1. Spherical harmonics and projection kernels

For $N = 2, 3, \dots$ fixed, and $n = 0, 1, 2, \dots$, we let $\mathcal{H}_n = \mathcal{H}_n(N)$ denote the space of spherical harmonics of degree n on S^N, which can be defined in several equivalent ways. It is the space of restrictions to S^N of homogeneous harmonic polynomials on $\mathbb{R}^{N+1}$ of degree n, the space of eigenfunctions of the Laplace-Beltrami operator on S^N with eigenvalue $n(n+N-1)$, and it is also $\mathcal{P}_n \ominus \mathcal{P}_{n-1} = \mathcal{P}_n \cap \mathcal{P}_{n-1}^{\perp}$. We have $\dim \mathcal{H}_n = d_n - d_{n-1}$; the dimensions of $\mathcal{H}_n(2)$ and $\mathcal{H}_n(3)$ are $2n+1$ and $(n+1)^2$ respectively.

For details of the facts that follow, see [**12**]. The integral kernel of the orthogonal projection $L^2(S^N, d\mu) \to \mathcal{H}_n(N)$ will be denoted by $Z_n(x,y) = Z_n^{(N)}(x,y)$. By symmetry, the kernels $Z_n(x,y)$ and $K_n(x,y) = \sum_{m=0}^{n} Z_m(x,y)$ depend only on the distance $d(x,y)$. It will be convenient to define functions on $[0,\pi]$;

$$\begin{aligned} Z_n(\theta) =& Z_n^{(N)}(\theta) = Z_n^{(N)}(x,y), \\ K_n(\theta) =& K_n^{(N)}(\theta) = K_n^{(N)}(x,y), \qquad \text{where } d(x,y) = \theta. \end{aligned}$$

The generating function for the functions $Z_n^{(N)}$ is the Poisson kernel:

$$\sum_{n=0}^{\infty} r^n Z_n^{(N)}(\theta) \;=\; \frac{1-r^2}{(1-2r\cos\theta + r^2)^{(N+1)/2}}.$$

It will also be useful to introduce functions $P_n^{(\lambda)}(\theta)$ defined by

$$\sum_{n=0}^{\infty} r^n P_n^{(\lambda)}(\theta) \;=\; \frac{1}{(1-2r\cos\theta + r^2)^{\lambda}}. \tag{1.1}$$

(The functions $P_n^{(\lambda)}(\cos^{-1} x)$ are the standard *ultraspherical polynomials.*) From the generating functions, one sees that

$$\begin{aligned} Z_n^{(N)}(\theta) &= \frac{2n+N-1}{N-1} P_n^{((N-1)/2)}(\theta), \\ K_n^{(N)}(\theta) &= P_n^{((N+1)/2)}(\theta) + P_{n-1}^{((N+1)/2)}(\theta). \end{aligned} \tag{1.2}$$

Let $N \geq 2$ be fixed.

Lemma 1.1. *If $f, g \in H^{1/2}(S^N)$, then*

$$\begin{aligned} &\frac{1}{4}\iint_{S^N\times S^N} |K_n(x,y)|^2 (f(x)-f(y))(g(x)-g(y))\, d\mu(x)d\mu(y) \\ &\quad = \alpha_N\, n^{N-1} \iint_{S^N\times S^N} \frac{(f(x)-f(y))(g(x)-g(y))}{\sin^{N-1} d(x,y)\, \sin^2 \frac{d(x,y)}{2}}\, d\mu(x)d\mu(y) \;+\; \circ(n^{N-1}), \end{aligned} \tag{1.3}$$

where

$$\alpha_N = \frac{1}{2^{N+2}\left(\Gamma(\frac{N+1}{2})\right)^2} \;=\; \frac{(\sigma(S^N))^2}{8(2\pi)^{N+1}}. \tag{1.4}$$

Writing f_m and g_m for the orthogonal projections of f and g onto the space $\mathcal{H}_m$, another expression for the right hand side of (1.3) is given by

$$\begin{aligned} &\iint_{S^N\times S^N} \frac{(f(x)-f(y))(g(x)-g(y))}{\sin^{N-1} d(x,y)\, \sin^2 \frac{d(x,y)}{2}}\, d\mu(x)d\mu(y) \\ &\qquad\qquad = \sum_{m=1}^{\infty} \beta_m \int_{S^N} f_m(x) g_m(x)\, d\mu(x), \end{aligned} \tag{1.5}$$

where

$$\text{for } N \text{ even,} \qquad \beta_m = \begin{cases} 4\frac{\left(\Gamma(\frac{m+1}{2})\right)^2}{(\frac{m}{2}-1)!(\frac{m}{2})!}\frac{(m+1)(m+3)\dots(m+N-1)}{(m+2)(m+4)\dots(m+N-2)}, & m \text{ even}, \\ 4\frac{\Gamma(\frac{m}{2})\Gamma(\frac{m}{2}+1)}{\left((\frac{m-1}{2})!\right)^2}\frac{m(m+2)\dots(m+N-2)}{(m+1)(m+3)\dots(m+N-3)}, & m \text{ odd}, \end{cases}$$

$$\text{for } N \text{ odd,} \qquad \beta_m = \begin{cases} 4\frac{m(m+2)\dots(m+N-1)}{(m+1)(m+3)\dots(m+N-2)}, & m \text{ even}, \\ 4\frac{(m+1)(m+3)\dots(m+N-2)}{(m+2)(m+4)\dots(m+N-3)}, & m \text{ odd}. \end{cases} \tag{1.6}$$

In particular, for $N = 3$,

$$\beta_m = \begin{cases} 4(m+1-\frac{1}{m+1}), & m \text{ even}, \\ 4(m+1), & m \text{ odd}, \end{cases}$$

and for all N we have $\beta_m \sim 4m$ as $m \to \infty$. Furthermore, for all n and $f \in H^{1/2}$,

$$\iint_{S^N \times S^N} |K_n(x,y)|^2 (f(x) - f(y))^2 \, d\mu(x) d\mu(y) \ \le \ C\, n^{N-1} \|f\|_{H^{1/2}}, \tag{1.7}$$

where C is independent of n and f.

Proof. We will use elementary techniques and well-known asymptotic formulas. We begin by proving (1.5).

For $0 < \theta < \pi$, define the sub-manifold M_θ of $S^N \times S^N$ by $M_\theta = \{(x,y) : d(x,y) = \theta\}$. Let σ_θ denote surface measure on M_θ. To begin with, suppose that $f, g \in C^\infty(S^N)$ and write

$$F(\theta) = \int_{M_\theta} (f(x) - f(y))(g(x) - g(y)) \, d\sigma_\theta(x,y).$$

Suppose that h is any function in $L^\infty[0,\pi]$. The co-area formula gives

$$\begin{aligned}
&\iint_{S^N \times S^N} (f(x) - f(y))(g(x) - g(y)) h(d(x,y)) \, d\sigma(x) d\sigma(y) \\
&\qquad = \int_{\theta=0}^{\pi} \int_{M_\theta} \frac{(f(x) - f(y))(g(x) - g(y))}{|\nabla_{(x,y)} d(x,y)|} \, d\sigma_\theta(x,y) h(\theta) d\theta \\
&\qquad = \frac{1}{\sqrt{2}} \int_0^\pi F(\theta) h(\theta) \, d\theta.
\end{aligned}$$

Now with respect to the measure

$$d\nu(\theta) = \frac{\sin^{N-1}\theta \, d\theta}{\int_0^\pi \sin^{N-1}(\phi) \, d\phi},$$

the functions $h(\theta)$ and $F(\theta)/\sin^{N-1}\theta$ are square integrable. Since the functions $Z_m(\theta)/\sqrt{Z_m(0)}$, $m = 0, 1, 2, \ldots$, form an orthonormal base for $L^2([0,\pi], d\nu(\theta))$, Parseval's formula gives

$$\begin{aligned}
\frac{1}{\sqrt{2}} \int_0^\pi F(\theta) h(\theta) \, d\theta &= \frac{\int_0^\pi \sin^{N-1}\phi \, d\phi}{\sqrt{2}} \int_0^\pi h(\theta) \frac{F(\theta)}{\sin^{N-1}\theta} \, d\nu(\theta) \\
&= \frac{1}{\sqrt{2} \int_0^\pi \sin^{N-1}\phi \, d\phi} \sum_{m=0}^\infty \int_0^\pi \frac{Z_m(\theta)}{Z_m(0)} h(\theta) \sin^{N-1}\theta \, d\theta \int_0^\pi Z_m(\theta) F(\theta) \, d\theta.
\end{aligned}$$

Now

$$\begin{aligned}
&\frac{1}{\sqrt{2}} \int_0^\pi Z_m(\theta) F(\theta) \, d\theta \\
&\qquad = \iint_{S^N \times S^N} Z_m(d(x,y)) (f(x) - f(y))(g(x) - g(y)) \, d\sigma(x) d\sigma(y) \\
&\qquad = \begin{cases} -2(\sigma(S^N))^2 \int_{S^N} f_m(x) g_m(x) \, d\mu(x), & m \neq 0, \\ 2(\sigma(S^N))^2 \big(\int_{S^N} f(x) g(x) \, d\mu(x) - f_0 g_0 \big), & m = 0; \end{cases}
\end{aligned}$$

so

$$\frac{1}{\sqrt{2}}\int_0^\pi \frac{F(\theta)}{\sin^{N-1}\theta} h(\theta)\,d\theta$$
$$= \frac{2((\sigma(S^N))^2}{\int_0^\pi \sin^{N-1}\phi\,d\phi}\sum_{m=0}^\infty \int_0^\pi (1-\frac{Z_m(\theta)}{Z_m(0)})h(\theta)\sin^{N-1}\theta\,d\theta \int_{S^N} f_m(x)g_m(x)\,d\mu(x),$$

and hence

$$\iint_{S^N\times S^N}(f(x)-f(y))(g(x)-g(y))h(d(x,y))\,d\mu(x)d\mu(y)$$
$$= \sum_{m=1}^\infty \beta_m(h)\int_{S^N} f_m(x)g_m(x)\,d\mu(x),$$

where

$$\beta_m(h) = \frac{2}{\int_0^\pi \sin^{N-1}\phi\,d\phi}\int_0^\pi (1-\frac{Z_m(\theta)}{Z_m(0)})h(\theta)\sin^{N-1}\theta\,d\theta.$$

Now $\int_{S^N} f_m(x)g_m(x)\,dx$ is rapidly decreasing as $m\to\infty$. Since $Z_m(\theta)$ attains its maximum value at 0, the function $1-(Z_m(\theta)/Z_m(0))$ is bounded, vanishes at $\theta=0$ and its first derivative also vanishes there. Using the bound

$$\|Z_m^{(N)\prime\prime}\|_{L^\infty} = \mathrm{O}(m^{N+2}), \qquad m\to\infty,$$

we get

$$1-\frac{Z_m(\theta)}{Z_m(0)} \;=\; \mathrm{O}(m^2)\sin^2\frac{\theta}{2},$$

so

$$\int_0^\pi (1-\frac{Z_m(\theta)}{Z_m(0)})h(\theta)\sin^{N-1}\theta\,d\theta \;= \mathrm{O}(m^2)\|h(\theta)\sin^{N-1}\theta\sin^2\tfrac{\theta}{2}\|_\infty.$$

Approximating the function $1/(\sin^{N-1}\theta\sin^2\frac{\theta}{2})$ by bounded functions in an appropriate fashion, we get (1.5) with the coefficients β_m defined by

$$\beta_m \;=\; \frac{2\pi}{\int_0^\pi \sin^{N-1}\theta\,d\theta}\;\int_0^\pi (1-\frac{Z_m(\theta)}{Z_m(0)})\frac{1}{\sin^2\frac{\theta}{2}}\,\frac{d\theta}{\pi}$$
$$= \frac{-4\pi}{\int_0^\pi \sin^{N-1}\theta\,d\theta}\int_0^\pi \frac{Z_m'(\theta)}{Z_m(0)}\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\,\frac{d\theta}{\pi} \;=\; \frac{2^{N+1}\left(\Gamma(\frac{N+1}{2})\right)^2}{(N-1)!}\frac{\widetilde{Z_m'}(0)}{Z_m(0)},$$

where for functions G on $[0,\pi]$ we set

$$\widetilde{G}(0) \;=\; -\int_0^\pi G(\theta)\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\,\frac{d\theta}{\pi}.$$

To derive (1.6), from (1.2) we have

$$\frac{\widetilde{Z_m'}(0)}{Z_m(0)} \;=\; \frac{\widetilde{P_m^{(\lambda)\prime}}(0)}{P_m^{(\lambda)}(0)}, \qquad \lambda = \frac{N-1}{2},$$

and using (1.1) we see that

$$P_m^{(\lambda)}(0) = \binom{m+2\lambda-1}{m} = \binom{m+N-2}{m}$$

and

$$\begin{aligned}\sum_{m=0}^{\infty} r^m \widetilde{P_m^{(\lambda)\prime}}(0) &= \int_0^{\pi} \frac{2\lambda r \sin\theta}{((1-re^{i\theta})(1-re^{-i\theta}))^{\lambda+1}} \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} \frac{d\theta}{\pi} \\ &= 2\lambda \Re \int_0^{\pi} \frac{r(1+z)}{((1-rz)(1-r/z))^{\lambda+1}} \frac{d\theta}{\pi}, \qquad z = e^{i\theta}, \\ &= 2\lambda \sum_{j,k=0}^{\infty} r^{j+k+1}\gamma_j\gamma_k \, \Re \int_0^{\pi} (1+z) z^{j-k} \frac{d\theta}{\pi},\end{aligned}$$

where

$$\sum_{k=0}^{\infty} \gamma_k w^k = \frac{1}{(1-w)^{\lambda+1}},$$

i.e.,

$$\gamma_k = \frac{1}{\Gamma(\lambda+1)} \frac{\Gamma(k+\lambda+1)}{k!} = \frac{1}{\Gamma(\frac{N+1}{2})} \frac{\Gamma(k+\frac{N+1}{2})}{k!}.$$

Equating powers of r, we get

$$\begin{aligned}\widetilde{P_0^{(\lambda)\prime}}(0) &= 0, \\ \widetilde{P_{2k}^{(\lambda)\prime}}(0) &= 2\lambda\gamma_k\gamma_{k-1}, \qquad k > 0, \\ \widetilde{P_{2k+1}^{(\lambda)\prime}}(0) &= 2\lambda\gamma_k^2, \qquad k \geq 0,\end{aligned}$$

and, putting everything together,

$$\beta_m = \begin{cases} 2^{N+1} \frac{m!}{(m+N-2)!} \frac{\Gamma(\frac{m+N-1}{2})\Gamma(\frac{m+N+1}{2})}{(\frac{m}{2}-1)!(\frac{m}{2})!}, & m \text{ even}, \\ 2^{N+1} \frac{m!}{(m+N-2)!} \left(\frac{\Gamma(\frac{m+N}{2})}{(\frac{m-1}{2})!}\right)^2, & m \text{ odd}, \end{cases}$$

from which we get (1.6). We remark that $\beta_m > 0$ for all $m > 0$ and $\beta_m \sim 4m$ as $m \to \infty$, so $(\sum \beta_m \int |f_m|^2)^{1/2}$ is equivalent to the norm $\|(-\Delta)^{1/4} f\|_{L^2}$.

We assumed so far that $f, g \in C^\infty(S^N)$. For $f \in C^\infty(S^N)$, setting $g = \overline{f}$ in (1.5), we have

$$\text{(1.8)} \qquad \iint_{S^N \times S^N} \frac{|f(x)-f(y)|^2}{\sin^{N-1} d(x,y) \, \sin^2 \frac{d(x,y)}{2}} \, d\mu(x) d\mu(y) = \sum_{m=0}^{\infty} \beta_m \int_{S^N} |f_m(x)|^2 \, d\mu(x).$$

By standard arguments, the above identity and (1.5) hold for any $f, g \in H^{1/2}$.

We now prove (1.3), and begin by showing that for N fixed, the projection kernels $K_n(\theta)$ have the following asymptotics:

$$\begin{aligned}(1.9)\quad K_n(\theta) \;=\; &\delta_N \frac{n^{(N-1)/2}}{\sin^{(N-1)/2}\theta\,\sin\frac{1}{2}\theta}\cos((n+N/2)\theta-\pi N/4\,-\,\pi/4)\\ &+\;\left(\theta^{-(N+3)/2}+(\pi-\theta)^{-(N+1)/2}\right)\mathrm{O}(n^{(N-3)/2}),\qquad cn^{-1}\le\theta\le\pi-cn^{-1},\end{aligned}$$

$$K_n(\theta) \;=\; \begin{cases}\mathrm{O}(n^N), & 0\le\theta\le cn^{-1}\\ \mathrm{O}(n^{N-1}), & \pi-cn^{-1}\le\theta\le\pi,\end{cases}$$

where c is a positive number and

$$\delta_N \;=\; \frac{2^{(N+1)/2}\Gamma(\frac{N}{2}+1)}{\pi^{1/2}N!} \;=\; 2\sqrt{2\alpha_N},$$

where α_N is defined in (1.4). In the interval $0<\theta<cn^{-1}$, this is just the trivial bound on the kernel. As for the other intervals, from [**13**] (8.21.17) we have the following:

$$\begin{aligned}P_n^{((N+1)/2)}(\theta) =&\frac{2^{N/2}\Gamma(\frac{N}{2}+1)}{N!}\frac{(n+N)!}{n!(n+\frac{N+1}{2})^{N/2}}\frac{\theta^{1/2}}{\sin^{(N+1)/2}\theta}J_{N/2}((n+\tfrac{N+1}{2})\theta)\\ (1.10)\qquad &+\begin{cases}\theta^{-(N-1)/2}\,\mathrm{O}(n^{(N-3)/2}), & cn^{-1}\le\theta\le\pi-\epsilon,\\ \theta^2\,\mathrm{O}(n^N), & 0\le\theta\le cn^{-1},\end{cases}\end{aligned}$$

where c and ϵ are fixed positive numbers, and $J_\alpha(z)$ is the Bessel function defined by

$$J_\alpha(z) \;=\; \frac{(z/2)^\alpha}{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^{+1}(1-t^2)^{\alpha-\frac{1}{2}}e^{izt}\,dt.$$

Now

$$(1.11)\qquad J_\alpha(z) \;=\; \begin{cases}\mathrm{O}(|z|^\alpha), & z\to 0,\\ \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\cos(z-\alpha\pi/2-\pi/4)\;+\;\mathrm{O}(z^{-\frac{3}{2}}), & z\to\infty.\end{cases}$$

Combining these formulas, we get

$$\begin{aligned}P_n^{((N+1)/2)}(\theta) \;=\; &\frac{2^{(N+1)/2}\Gamma(\frac{N}{2}+1)}{\pi^{1/2}N!}\frac{n^{(N-1)/2}}{\sin^{(N+1)/2}\theta}\cos((n+\tfrac{N+1}{2})\theta-\tfrac{(N+1)\pi}{4})\\ &+\;\theta^{-(N+3)/2}\,\mathrm{O}(n^{(N-3)/2}),\quad cn^{-1}\le\theta\le\pi-\epsilon,\end{aligned}$$

and from this and (1.2) we get (1.9) in the range $0\le\theta\le\pi-\epsilon$. To get the behavior of $K_n(\theta)$ in the range $\pi-\epsilon<\theta\le\pi$, notice that if n is even then $P_n^{(\lambda)}(\pi-\theta)=P_n^{(\lambda)}(\theta)$, and if n is odd then $P_n^{(\lambda)}(\pi-\theta)=-P_n^{(\lambda)}(\theta)$. Hence

$$K_n(\theta) \;=\; (-1)^n\;(\;P_n^{((N+1)/2)}(\pi-\theta)\;-\;P_{n-1}^{((N+1)/2)}(\pi-\theta)\;).$$

Using the identity $J'_\alpha(z) = (\alpha/z)J_\alpha(z) - J_{\alpha+1}(z)$ and (1.11), we get

$$\begin{aligned} &J_\alpha(z+w) - J_\alpha(z-w) \\ &\quad = \begin{cases} \mathrm{O}(|z|^{\alpha-1}|w|), & 2|w| < |z| \to 0, \\ \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \big(\cos(z+w-\alpha\pi/2-\pi/4) \,-\, \cos(z-w-\alpha\pi/2-\pi/4)\big) & \\ \qquad +\ \mathrm{O}(|w|z^{-\frac{3}{2}}), & 2|w| < |z| \to \infty. \end{cases} \end{aligned}$$

From this and (1.10), it is easy to deduce (1.9) in the ranges $\pi - \epsilon < \theta < \pi - cn^{-1}$ and $\pi - cn^{-1} \le \theta \le \pi$.

From (1.9) we get the asymptotics of the square of the kernel:

$$\begin{aligned} K_n^2(\theta) \ &= \ 4\alpha_N \frac{n^{N-1}}{\sin^{N-1}\theta\, \sin^2 \frac{1}{2}\theta} \big(1 + \cos((2n+N)\theta - \pi N/2 - \pi/2) \big) \\ &\qquad +\ (\theta^{-(N+2)} + (\pi-\theta)^{-N})\, \mathrm{O}(n^{N-2}), \\ &\qquad\qquad\qquad\qquad\qquad\qquad cn^{-1} \le \theta \le \pi - cn^{-1}, \\ K_n^2(\theta) \ &= \ \begin{cases} \mathrm{O}(n^{2N}), & 0 \le \theta < cn^{-1}, \\ \mathrm{O}(n^{2N-2}), & \pi - cn^{-1} \le \pi. \end{cases} \end{aligned}$$

To prove (1.3), we saw in (1.8) that $f \in H^{1/2}(S^N)$ implies

$$\iint_{S^N\times S^N} \frac{|f(x)-f(y)|^2}{\sin^{N-1}\theta\, \sin^2\frac{1}{2}\theta}\, d\mu(x)d\mu(y) \ < \ \infty.$$

Let M_θ and σ_θ be defined as above; $M_\theta = \{(x,y) \in S^N \times S^N : d(x,y) = \theta\}$ and σ_θ is surface measure on M_θ. By the co-area formula, for any integrable function G on $S^N \times S^N$ we have

$$\iint_{S^N\times S^N} G(x,y)\, d\sigma(x)d\sigma(y) \ = \ \int_0^\pi \int_{M_\theta} \frac{G(x,y)}{\sqrt{2}}\, d\sigma_\theta\, d\theta.$$

For $f,\, g \in H^{1/2}(S^N)$ write

$$F(\theta) \ = \ \frac{1}{\sin^{N-1}\theta\, \sin^2\frac{\theta}{2}} \int_{M_\theta} (f(x)-f(y))(g(x)-g(y))\, d\sigma_\theta.$$

Then $F \in L^1([0,\pi], d\theta)$, and we have

$$\begin{aligned} &\iint_{S^N\times S^N} |K_n(x,y)|^2 (f(x)-f(y))(g(x)-g(y))\, d\sigma(x)d\sigma(y) \\ &= \ 4\alpha_N\, n^{N-1} \int_0^\pi \frac{F(\theta)}{\sqrt{2}}\, d\theta \ - \ \frac{4\alpha_N n^{N-1}}{\sqrt{2}} \Big(\int_0^{cn^{-1}} F(\theta)\, d\theta \,+\, \int_{\pi-cn^{-1}}^\pi F(\theta)\, d\theta \Big) \\ &\quad +\ \mathrm{O}(n^{N-1}) \int_0^\pi F(\theta) \cos((2n+N)\theta - \pi N/2 - \pi/2)\, d\theta \\ &\quad +\ \mathrm{O}(n^{N-2}) \int_{cn^{-1}}^{\pi-cn^{-1}} (\theta^{-1} + (\pi-\theta)^{-1}) |F(\theta)|\, d\theta \\ &\quad +\ \mathrm{O}(n^{2N}) \int_0^{cn^{-1}} |F(\theta)|\theta^{N+1}\, d\theta \ +\ \mathrm{O}(n^{2N-2}) \int_{\pi-cn^{-1}}^\pi |F(\theta)|(\pi-\theta)^{N-1}\, d\theta. \end{aligned}$$

The first term on the right hand side is equal to

$$4\alpha_N\, n^{N-1} \iint_{S^N\times S^N} \frac{(f(x)-f(y))(g(x)-g(y))}{\sin^{N-1} d(x,y)\, \sin^2\frac{d(x,y)}{2}}\, d\sigma(x) d\sigma(y).$$

It is not hard to check that each of the other terms is $\circ(n^{N-1})$ as $n\to\infty$, and that (1.7) holds.

The rest of this section is devoted to describing the bases we will use for the spherical harmonics on S^2 and S^3, and stating explicit formulas for the coefficients that arise when the product of two spherical harmonics is expanded in spherical harmonics. We will be far more interested in these coefficients than in the formulas for the spherical harmonics themselves, which we only include for the sake of completeness. All the formulas and the theory behind them can be found in [**15**], Chapter 3.

The *Euler angles* (θ,ϕ,ψ) with $0\le\theta<\pi$, $0\le\phi<2\pi$, $0\le\psi<4\pi$ are a coordinate system for S^3 defined by

$$(\,\theta,\phi,\psi\,) \leftrightarrow (\,\cos\tfrac{\theta}{2}\cos\tfrac{\phi+\psi}{2}\,,\ \cos\tfrac{\theta}{2}\sin\tfrac{\phi+\psi}{2}\,,\ \sin\tfrac{\theta}{2}\cos\tfrac{\phi-\psi}{2}\,,\ \sin\tfrac{\theta}{2}\sin\tfrac{\phi-\psi}{2}\,).$$

Normalized surface measure is given by

$$\frac{\sin\theta\, d\theta}{2}\frac{d\phi}{2\pi}\frac{d\psi}{4\pi}.$$

For the familiar *spherical coordinates* (θ,ϕ) on S^2, normalized surface measure is given by

$$\frac{\sin\theta\, d\theta}{2}\frac{d\phi}{2\pi}.$$

In both dimensions $N=2$ and $N=3$, the basis spherical harmonics will be indexed by certain points in a lattice. Let

$$\begin{aligned}\Lambda(2) &= \mathbb{Z}^2,\\ \Lambda(3) &= \{\,(m,p,q)\,:\, m\in\mathbb{Z},\ \ p,q,m-p,m-q\,\text{even}\,\},\\ \Lambda_* &= \{\,(m,p)\,:\, m\in\mathbb{Z},\ \ p,m-p\,\text{even}\,\}.\end{aligned}$$

The letters m,p,q, with or without subscripts or superscripts will be reserved for the coordinates of lattice points. We will always use the notation

$$\mathbf{m}=(m,p),\qquad \mathbf{m}_\ell=(m_\ell,p_\ell)\qquad \mathbf{m}^\ell=(m^\ell,p^\ell)\qquad \mathbf{m}'=(m',p')$$

for points in $\Lambda(2)$ or Λ_*, and

$$\mathbf{m}=(m,p,q),\qquad \mathbf{m}_\ell=(m_\ell,p_\ell,q_\ell)\qquad \mathbf{m}^\ell=(m^\ell,p^\ell,q^\ell)\qquad \mathbf{m}'=(m',p',q')$$

for points in $\Lambda(3)$. $\mathcal{I}(2)$, $\mathcal{I}(3)$ and $\mathcal{I}_*$ will denote the "cones" of lattice points in $\Lambda(2)$, $\Lambda(3)$ and Λ_* given by

$$\begin{aligned}\mathcal{I}(2) &= \{\,(m,p)\in\Lambda(2)\,:\,|p|\le m\,\},\\ \mathcal{I}(3) &= \{\,(m,p,q)\in\Lambda(3)\,:\,|p|,|q|\le m\,\},\\ \mathcal{I}_* &= \{\,(m,p)\in\Lambda_*\,:\,|p|\le m\,\}.\end{aligned}$$

$\mathcal{I}(2)$ will index the basis of spherical harmonics on S^2 and $\mathcal{I}(3)$ will index the basis of spherical harmonics on S^3. Moreover, $\{\,\mathbf{m}=(m,p)\in\mathcal{I}(2) : m=n\,\}$ will index the basis for $\mathcal{H}_n(2)$ (there are $2n+1$ points in this set), while $\{\,\mathbf{m}=(m,p,q)\in\mathcal{I}(3) : m=n\,\}$ will index the base for $\mathcal{H}_n(3)$ (there are $(n+1)^2$ points in this set).

For each index $\mathbf{m}$ in $\mathcal{I}(2)$ or $\mathcal{I}(3)$, we will now give a formula for the basis spherical harmonic $Y_{\mathbf{m}}$ on S^2 or S^3 respectively:

$$\begin{aligned} Y_{(p,m)}(\theta,\phi) &= \sqrt{2m+1}\; t^m_{p0}(\theta,\phi,0), \\ Y_{(p,q,m)}(\theta,\phi,\psi) &= \sqrt{m+1}\; t^{m/2}_{p/2\;q/2}(\theta,\phi,\psi), \end{aligned} \tag{1.12}$$

where for a point (m,p,q) such that $(2m,2p,2q)\in\mathcal{I}(3)$, we have

$$t^m_{pq}(\theta,\phi,\psi) = e^{-i(p\phi+q\psi)}\,P^m_{pq}(\cos\theta), \tag{1.13}$$

where

$$\begin{aligned} P^m_{pq}(z) &= i^{-p-q}\left(\frac{(m-p)!(m-q)!}{(m+p)!(m+q)!}\right)^{\frac{1}{2}}\left(\frac{1+z}{1-z}\right)^{\frac{p+q}{2}} \\ &\quad\times\sum_{\ell=\max\{p,q\}}^{m}\frac{(m+\ell)!\,i^{2\ell}}{(m-\ell)!(\ell-p)!(\ell-q)!}\left(\frac{1-z}{2}\right)^{\ell}, \end{aligned}$$

and where the square root is chosen so that $\big((1+z)/(1-z)\big)^{(p+q)/2}$ is positive when $-1<z<1$. P^m_{pq} is closely related to a Jacobi polynomial. We have $\|Y_{\mathbf{m}}\|_{L^2}=1$.

We now describe the coefficients arising in the linearization of products of spherical harmonics. We have

$$Y_{\mathbf{m}_1}Y_{\mathbf{m}} = \sum_{\mathbf{m}^1}\Gamma^{\mathbf{m}^1}_{\mathbf{m}_1}(\mathbf{m})Y_{\mathbf{m}^1+\mathbf{m}} \tag{1.14}$$

for coefficients $\Gamma^{\mathbf{m}^1}_{\mathbf{m}_1}(\mathbf{m})$ given by

$$\Gamma^{\mathbf{m}^1}_{\mathbf{m}_1}(\mathbf{m}) = \int_{S^3} Y_{\mathbf{m}_1}(x)Y_{\mathbf{m}}(x)\overline{Y_{\mathbf{m}^1+\mathbf{m}}(x)}\,d\mu(x).$$

We extend the definition of $\Gamma^{\mathbf{m}^1}_{\mathbf{m}_1}(\mathbf{m})$ to all $\mathbf{m}$, $\mathbf{m}^1\in\Lambda(N)$ by setting it equal to 0, unless $\mathbf{m}$, $\mathbf{m}^1+\mathbf{m}\in\mathcal{I}(N)$. By Cauchy-Schwarz we see that

$$|\Gamma^{\mathbf{m}^1}_{\mathbf{m}_1}(\mathbf{m})| \le \|Y_{\mathbf{m}_1}\|_{L^\infty}.$$

Now $Y_{\mathbf{m}_1}Y_{\mathbf{m}}$ is a polynomial on S^N of degree at most m_1+m, so $\Gamma^{\mathbf{m}^1}_{\mathbf{m}_1}(\mathbf{m})=0$ unless $m^1+m\le m_1+m$. By also considering the products $Y_{\mathbf{m}_1}(x)\overline{Y_{\mathbf{m}^1+\mathbf{m}}(x)}$ and $Y_{\mathbf{m}}(x)\overline{Y_{\mathbf{m}^1+\mathbf{m}}(x)}$, we find that

$$\Gamma^{\mathbf{m}^1}_{\mathbf{m}_1}(\mathbf{m})=0 \quad \text{unless} \quad |m-m_1|-m\le m^1\le m_1. \tag{1.15}$$

It can be seen from (1.12) and (1.13) that

$$\begin{aligned} &\text{for } N=2, \qquad \Gamma^{\mathbf{m}^1}_{\mathbf{m}_1}(\mathbf{m}) = 0 \quad \text{unless} \quad p^1 = p_1, \\ (1.16)\qquad & \\ &\text{for } N=3, \qquad \Gamma^{\mathbf{m}^1}_{\mathbf{m}_1}(\mathbf{m}) = 0 \quad \text{unless} \quad p^1 = p_1 \text{ and } q^1 = q_1. \end{aligned}$$

We will need more precise information about the coefficients $\Gamma^{\mathbf{m}^1}_{\mathbf{m}_1}(\mathbf{m})$, so here is the explicit formula:

$$\Gamma^{\mathbf{m}^1}_{\mathbf{m}_1}(\mathbf{m}) \;=\; F_0(\mathbf{m})F_1(\mathbf{m})F_2(\mathbf{m})$$

where

$$(1.17)\qquad \begin{array}{ll} \text{for } N=2, & \begin{aligned} F_0(\mathbf{m}) &= \left(\tfrac{(2m_1+1)(2m+1)}{2m^1+2m+1}\right)^{1/2} \\ F_1(\mathbf{m}) &= C(m_1, m, m^1+m; p_1, p, p_1+p) \\ F_2(\mathbf{m}) &= C(m_1, m, m^1+m; 0,0,0), \end{aligned} \\ & \\ \text{for } N=3, & \begin{aligned} F_0(\mathbf{m}) &= \left(\tfrac{(m_1+1)(m+1)}{m^1+m+1}\right)^{1/2} \\ F_1(\mathbf{m}) &= C(\tfrac{m_1}{2}, \tfrac{m}{2}, \tfrac{m^1+m}{2}; \tfrac{p_1}{2}, \tfrac{p}{2}, \tfrac{p_1+p}{2}) \\ F_2(\mathbf{m}) &= C(\tfrac{m_1}{2}, \tfrac{m}{2}, \tfrac{m^1+m}{2}; \tfrac{q_1}{2}, \tfrac{q}{2}, \tfrac{q_1+q}{2}) \end{aligned} \end{array}$$

and the *Clebsch-Gordan coefficient* $C(m_1, m, m^1+m; p_1, p, p_1+p)$ is given by

$$\begin{aligned} C(m_1, m, m^1+m; p_1, p, p_1+p) &= \left(\frac{(m_1+m^1)!(m_1-m^1)!}{(m_1+p_1)!(m_1-p_1)!}\right)^{1/2} \\ &\times\ (2(m+m^1)+1)^{1/2} \\ &\times\ \left(\frac{(m+p)!}{(m+p+m^1+p_1)!}\frac{(m-p+m^1+p_1)!}{(m-p)!}\frac{(2m+m^1+m_1+1)!}{(2m+m^1-m_1)!}\right)^{1/2} \\ &\times\ (-1)^{p_1-m^1} \sum_{s=0}^{\min\{m_1+m^1, m-p+m^1-p_1\}} \frac{(-1)^s}{s!(m_1+m^1-s)!} \\ (1.18)\qquad &\qquad \times\ \frac{(2m+2m^1-s)!}{(2m+m_1+m^1-s+1)!}\frac{(m-p+m^1+m_1-s)!}{(m-p+m^1-p_1-s)!}, \end{aligned}$$

for (m,p), (m_1,p_1), $(m^1+m, p^1+p) \in \mathcal{I}_*$. (This formula can be found in [**15**], (3.8.14″).)

2. Asymptotics for $\operatorname{trace}(P_n[f]P_n)^k$ when f is a polynomial

In this section, we will prove (0.11) and hence Lemma 0.3. We will assume throughout that the dimension N is equal to 2 or 3. To prove (0.11), by linearity we need only consider the case when the functions $f_0, \dots, f_j$ are spherical harmonics, i.e., we just need to prove that for any fixed $\mathbf{m}_0, \dots, \mathbf{m}_j \in \mathcal{I}(N)$,

$$\begin{aligned} &\operatorname{trace}\, P_n[Y_{\mathbf{m}_1}\dots Y_{\mathbf{m}_j}](I-P_n)[Y_{\mathbf{m}_0}]P_n \\ &\quad - \sum_\sigma \operatorname{trace}\; P_n[Y_{\mathbf{m}_{\sigma_1}}]\dots P_n[Y_{\mathbf{m}_{\sigma_j}}](I-P_n)[Y_{\mathbf{m}_0}]P_n \\ &\qquad\qquad\qquad = \circ(n^{N-1}) \quad \text{as } n\to\infty, \end{aligned}$$

where the sum is over σ in the set of permutations generated by the cycle $(1, \ldots, j)$. Now we have

$$\text{trace } P_n\,[Y_{\mathbf{m}_{\sigma_1}}]\cdots P_n\,[Y_{\mathbf{m}_{\sigma_j}}]\ (I-P_n)\ [Y_{\mathbf{m}_0}]\,P_n = \sum_{\mathbf{m}} \langle P_n\,[Y_{\mathbf{m}_{\sigma_1}}]\cdots P_n\,[Y_{\mathbf{m}_{\sigma_j}}]\,(I-P_n)\,[Y_{\mathbf{m}_0}]\,Y_{\mathbf{m}}\ ,\ Y_{\mathbf{m}}\,\rangle, \tag{2.1}$$

where for $N = 2$, the sum is over those $\mathbf{m} = (m, p) \in \mathcal{I}(2)$ with $m \leq n$, and for $N = 3$, it is over those $\mathbf{m} = (m, p, q) \in \mathcal{I}(3)$ with $m \leq n$. Clearly each term in the sum is bounded by $2\|Y_{\mathbf{m}_0}\|_\infty \cdots \|Y_{\mathbf{m}_j}\|_\infty$, so at first glance we see that (2.1) is $\mathrm{O}(n^N)$. We can easily do better; by (1.15), $(I-P_n)[Y_{\mathbf{m}_0}]Y_{\mathbf{m}} = 0$ unless $m_0+m > n$. (Recall the notation: $\mathbf{m}_0 = (m_0, p_0)$ or (m_0, p_0, q_0).) Since the number of indices $\mathbf{m}$ with $n - m_0 < m \leq n$ is $\mathrm{O}(n^{N-1})$, we see that (2.1) is $\mathrm{O}(n^{N-1})$.

Similar remarks apply to $\operatorname{trace} P_n[Y_{\mathbf{m}_1} \ldots Y_{\mathbf{m}_j}](I - P_n)[Y_{\mathbf{m}_0}]$. We have

$$\begin{aligned} &\text{trace } P_n\,[Y_{\mathbf{m}_1}\cdots Y_{\mathbf{m}_j}]\ (I-P_n)\ [Y_{\mathbf{m}_0}] \\ &\qquad - \sum_{\sigma} \operatorname{trace} P_n\,[Y_{\mathbf{m}_{\sigma_1}}]\cdots P_n\,[Y_{\mathbf{m}_{\sigma_j}}]\ (I-P_n)\ [Y_{\mathbf{m}_0}] \\ &= \sum_{\mathbf{m}} \Bigg(\langle P_n\,[Y_{\mathbf{m}_1}\cdots Y_{\mathbf{m}_j}]\,(I-P_n)\,[Y_{\mathbf{m}_0}]\,Y_{\mathbf{m}}\ ,\ Y_{\mathbf{m}}\,\rangle \\ &\qquad - \sum_{\sigma} \langle P_n\,[Y_{\mathbf{m}_{\sigma_1}}]\cdots P_n\,[Y_{\mathbf{m}_{\sigma_j}}]\,(I-P_n)\,[Y_{\mathbf{m}_0}]\,Y_{\mathbf{m}}\ ,\ Y_{\mathbf{m}}\,\rangle \Bigg), \end{aligned} \tag{2.2}$$

where the sum is over those $\mathbf{m} \in \mathcal{I}(N)$ with $n - m_0 < m \leq n$. Our goal is to prove that this expression is $\circ(n^{N-1})$ as $n \to \infty$. So far we know that it is $\mathrm{O}(n^{N-1})$. To see where the cancellation occurs, we first express each of the operators $[Y_{\mathbf{m}_\ell}]$ as a linear combination of "shifts". By (1.14) we have

$$Y_{\mathbf{m}_\ell} Y_{\mathbf{m}} = \sum_{\mathbf{m}^\ell} \Gamma_{\mathbf{m}_\ell}^{\mathbf{m}^\ell}(\mathbf{m})\ S^{\mathbf{m}^\ell}\,Y_{\mathbf{m}}, \tag{2.3}$$

where $S^{\mathbf{m}^\ell}$ denotes the shift operator:

$$S^{\mathbf{m}^\ell} : Y_{\mathbf{m}} \to \begin{cases} Y_{\mathbf{m}^\ell+\mathbf{m}} & \text{if } \mathbf{m}^\ell + \mathbf{m}\ \in \mathcal{I}(N), \\ 0 & \text{otherwise,} \end{cases}$$

and where the sum is over $\mathbf{m}^\ell \in \Lambda(N)$. In fact, by (1.15) and (1.16), $\Gamma_{\mathbf{m}_\ell}^{\mathbf{m}^\ell}(\mathbf{m}) = 0$ unless $|m^\ell| < m_\ell$, $p^\ell = p_\ell$ and, in the case $N = 3$, $q^\ell = q_\ell$. The set of $\mathbf{m}^\ell$ for which these conditions are satisfied is finite—that is to say, for $\mathbf{m}_\ell$ fixed (2.3) holds where the sum is over a finite set of indices $\mathbf{m}^\ell$; a finite set which does not depend on $\mathbf{m}$.

To compute the expression

$$\langle P_n\,[Y_{\mathbf{m}_1}\cdots Y_{\mathbf{m}_j} Y_{\mathbf{m}_0}]\,Y_{\mathbf{m}}\ ,\ Y_{\mathbf{m}}\,\rangle$$

we expand each operator $[Y_{\mathbf{m}_{\sigma_\ell}}]$ in terms of shift operators. We get

$$\begin{aligned}&\langle P_n\,[Y_{\mathbf{m}_1}\cdots Y_{\mathbf{m}_j}Y_{\mathbf{m}_0}]\,Y_{\mathbf{m}}\ ,\ Y_{\mathbf{m}}\,\rangle\\ &\qquad=\sum_{\mathbf{m}^0,\dots,\mathbf{m}^j}\Gamma^{\mathbf{m}^1}_{\mathbf{m}_1}(\mathbf{m}^2+\cdots+\mathbf{m}^j+\mathbf{m}^0+\mathbf{m})\cdots\Gamma^{\mathbf{m}^j}_{\mathbf{m}_j}(\mathbf{m}^0+\mathbf{m})\Gamma^{\mathbf{m}^0}_{\mathbf{m}_0}(\mathbf{m})\\ &\qquad\qquad\qquad\qquad\qquad\times\ \langle\,P_n\,S^{\mathbf{m}^1}\cdots S^{\mathbf{m}^j}S^{\mathbf{m}^0}\,Y_{\mathbf{m}}\ ,\ Y_{\mathbf{m}}\,\rangle\\ &\qquad=\sum_{\mathbf{m}^0,\dots,\mathbf{m}^j}\Delta^{\mathbf{m}^1,\dots,\mathbf{m}^j,\mathbf{m}^0}_{\mathbf{m}_1,\dots,\mathbf{m}_j,\mathbf{m}_0}(\mathbf{m})\ \langle\,P_n\,S^{\mathbf{m}^1}\cdots S^{\mathbf{m}^j}S^{\mathbf{m}^0}\,Y_{\mathbf{m}}\ ,\ Y_{\mathbf{m}}\,\rangle,\end{aligned}$$

where $\Delta^{\mathbf{m}^1,\dots,\mathbf{m}^j,\mathbf{m}^0}_{\mathbf{m}_1,\dots,\mathbf{m}_j,\mathbf{m}_0}(\mathbf{m})$ is defined to be the product of Γ's in the second line and the sums are over those indices $\mathbf{m}^0,\dots,\mathbf{m}^j$ lying in a fixed finite subset of $\Lambda(N)$, which depends on $\mathbf{m}_0,\dots,\mathbf{m}_j$ but not on $\mathbf{m}$. Similarly,

$$\begin{aligned}&\langle P_n\,[Y_{\mathbf{m}_1}\cdots Y_{\mathbf{m}_j}]\,P_n\ \ [Y_{\mathbf{m}_0}]\,Y_{\mathbf{m}}\ ,\ Y_{\mathbf{m}}\,\rangle\\ &\ =\sum_{\mathbf{m}^0,\dots,\mathbf{m}^j}\Delta^{\mathbf{m}^1,\dots,\mathbf{m}^j,\mathbf{m}^0}_{\mathbf{m}_1,\dots,\mathbf{m}_j,\mathbf{m}_0}(\mathbf{m})\ \langle P_n\,S^{\mathbf{m}^1}\cdots S^{\mathbf{m}^j}\,P_n\,S^{\mathbf{m}^0}\,Y_{\mathbf{m}}\ ,\ Y_{\mathbf{m}}\,\rangle,\\ &\langle P_n\,[Y_{\mathbf{m}_{\sigma_1}}]\cdots P_n\,[Y_{\mathbf{m}_{\sigma_j}}]\,[Y_{\mathbf{m}_0}]\,Y_{\mathbf{m}}\ ,\ Y_{\mathbf{m}}\,\rangle\\ &\ =\sum_{\mathbf{m}^0,\dots,\mathbf{m}^j}\Delta^{\mathbf{m}^{\sigma_1},\dots,\mathbf{m}^{\sigma_j},\mathbf{m}^0}_{\mathbf{m}_{\sigma_1},\dots,\mathbf{m}_{\sigma_j},\mathbf{m}_0}(\mathbf{m})\langle P_n\,S^{\mathbf{m}^{\sigma_1}}\cdots P_n\,S^{\mathbf{m}^{\sigma_j}}\ S^{\mathbf{m}^0}\,Y_{\mathbf{m}}\,,\,Y_{\mathbf{m}}\,\rangle,\\ &\langle P_n\,[Y_{\mathbf{m}_{\sigma_1}}]\cdots P_n\,[Y_{\mathbf{m}_{\sigma_j}}]\,P_n\,[Y_{\mathbf{m}_0}]\,Y_{\mathbf{m}}\ ,\ Y_{\mathbf{m}}\,\rangle\\ &\ =\sum_{\mathbf{m}^0,\dots,\mathbf{m}^j}\Delta^{\mathbf{m}^{\sigma_1},\dots,\mathbf{m}^{\sigma_j},\mathbf{m}^0}_{\mathbf{m}_{\sigma_1},\dots,\mathbf{m}_{\sigma_j},\mathbf{m}_0}(\mathbf{m})\langle P_n\,S^{\mathbf{m}^{\sigma_1}}\cdots P_n\,S^{\mathbf{m}^{\sigma_j}}\ P_n\,S^{\mathbf{m}^0}\,Y_{\mathbf{m}}\,,\,Y_{\mathbf{m}}\,\rangle.\end{aligned}$$

These sums are all over a fixed, finite set of indices $\mathbf{m}^0,\dots,\mathbf{m}^j$. Consider the expression

(2.4)

$$\begin{aligned}&\sum_{\mathbf{m}}\Bigg(\ \Delta^{\mathbf{m}^1,\dots,\mathbf{m}^j,\mathbf{m}^0}_{\mathbf{m}_1,\dots,\mathbf{m}_j,\mathbf{m}_0}(\mathbf{m})\,\langle\,P_n\,S^{\mathbf{m}^1}\cdots S^{\mathbf{m}^j}\,(I-P_n)\,S^{\mathbf{m}^0}\,Y_{\mathbf{m}}\ ,\ Y_{\mathbf{m}}\,\rangle\\ &-\ \sum_{\sigma}\Delta^{\mathbf{m}^{\sigma_1},\dots,\mathbf{m}^{\sigma_j},\mathbf{m}^0}_{\mathbf{m}_{\sigma_1},\dots,\mathbf{m}_{\sigma_j},\mathbf{m}_0}(\mathbf{m})\,\langle\,P_n\,[S^{\mathbf{m}^{\sigma_1}}]\cdots P_n\,[S^{\mathbf{m}^{\sigma_j}}]\,(I-P_n)\,[S^{\mathbf{m}^0}]\,Y_{\mathbf{m}}\,,\,Y_{\mathbf{m}}\,\rangle\ \Bigg),\end{aligned}$$

where the first sum is over those $\mathbf{m}$ with $n-m_0<m<n$. If one sums this over $\mathbf{m}^0,\dots,\mathbf{m}^j$ then one gets (2.2); so in order to prove that (2.2) is $\circ(n^{N-1})$, we just need to show that (2.4) is $\circ(n^{N-1})$ for fixed $\mathbf{m}^0,\dots,\mathbf{m}^j$. For the rest of this section we will assume that $\mathbf{m}^0,\dots,\mathbf{m}^j$ are fixed.

Notice that if $S^{\mathbf{m}^1}\cdots S^{\mathbf{m}^j}S^{\mathbf{m}^0}\,Y_{\mathbf{m}}\neq 0$, then none of the shifts $S^{\mathbf{m}^\ell}$ takes the index $\mathbf{m}^{\ell-1}+\cdots+\mathbf{m}^j+\mathbf{m}^0+\mathbf{m}$ outside the indexing set $\mathcal{I}(N)$, and

$$S^{\mathbf{m}^1}\cdots S^{\mathbf{m}^j}S^{\mathbf{m}^0}\,Y_{\mathbf{m}}=S^{\mathbf{m}^1+\cdots+\mathbf{m}^j+\mathbf{m}^0}\,Y_{\mathbf{m}}.$$

On the other hand, if $S^{\mathbf{m}^1}\cdots S^{\mathbf{m}^j}S^{\mathbf{m}^0}\,Y_{\mathbf{m}}=0$, then one of the shifts $S^{\mathbf{m}^\ell}$ must take an index $\mathbf{m}^{\ell-1}+\cdots+\mathbf{m}^j+\mathbf{m}^0+\mathbf{m}$ outside $\mathcal{I}(N)$, and we have $\Delta^{\mathbf{m}^1,\dots,\mathbf{m}^j,\mathbf{m}^0}_{\mathbf{m}_1,\dots,\mathbf{m}_j,\mathbf{m}_0}(\mathbf{m})=0$.

Thus if $\Delta^{\mathbf{m}^1,\ldots,\mathbf{m}^j,\mathbf{m}^0}_{\mathbf{m}_1,\ldots,\mathbf{m}_j,\mathbf{m}_0}(\mathbf{m}) \neq 0$, then $S^{\mathbf{m}^1}\cdots S^{\mathbf{m}^j} S^{\mathbf{m}^0}\, Y_{\mathbf{m}} \neq 0$,

$$\begin{aligned}\langle\, P_n\, S^{\mathbf{m}^1}\cdots S^{\mathbf{m}^j} S^{\mathbf{m}^0}\, Y_{\mathbf{m}}\ ,\ Y_{\mathbf{m}}\,\rangle\ &= \langle\, P_n\, S^{\mathbf{m}^1+\cdots+\mathbf{m}^j+\mathbf{m}^0}\, Y_{\mathbf{m}}\ ,\ Y_{\mathbf{m}}\,\rangle\\ &= \begin{cases} 1 & \text{if } \mathbf{m}^0+\cdots+\mathbf{m}^j = 0 \ \text{ and } m\le n,\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

and

$$\langle\, S^{\mathbf{m}^1}\cdots S^{\mathbf{m}^j}\, P_n\, S^{\mathbf{m}^0}\, Y_{\mathbf{m}}\ ,\ Y_{\mathbf{m}}\,\rangle\ =\ \langle\ S^{\mathbf{m}^1+\cdots+\mathbf{m}^j}\, P_n\, S^{\mathbf{m}^0}\, Y_{\mathbf{m}}\ ,\ Y_{\mathbf{m}}\,\rangle.$$

We can interchange a shift with a projection as follows:

$$S^{\mathbf{m}^\ell}\, P_n\ =\ P_{m^\ell+n}\, S^{\mathbf{m}^\ell};$$

hence

$$\begin{aligned}P_n\, S^{\mathbf{m}^1+\cdots+\mathbf{m}^j}\, P_n\ &=\ P_n P_{m^1+\cdots+m^j+n}\ S^{\mathbf{m}^1+\cdots+\mathbf{m}^j}\\ &=\ P_{n+\min\{0,\, m^1+\cdots+m^j\}}\ S^{\mathbf{m}^1+\cdots+\mathbf{m}^j},\end{aligned}$$

and

$$\begin{aligned}&\langle\ S^{\mathbf{m}^1+\cdots+\mathbf{m}^j}\, P_n\, S^{\mathbf{m}^0}\, Y_{\mathbf{m}}\ ,\ Y_{\mathbf{m}}\,\rangle\\ &\qquad = \begin{cases} 1 & \text{if } \mathbf{m}^0+\cdots+\mathbf{m}^j = 0 \text{ and } m\le n+\min\{0,\, m^1+\cdots+m^j\},\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Similarly, if $\Delta^{\mathbf{m}^{\sigma_1},\ldots,\mathbf{m}^{\sigma_j},\mathbf{m}^0}_{\mathbf{m}_{\sigma_1},\ldots,\mathbf{m}_{\sigma_j},\mathbf{m}_0}(\mathbf{m}) \neq 0$, then $S^{\mathbf{m}^{\sigma_1}}\cdots S^{\mathbf{m}^{\sigma_j}} S^{\mathbf{m}^0}\, Y_{\mathbf{m}} \neq 0$, and by passing all projections to the left of all shifts we get

$$\begin{aligned}&\langle P_n\,[S^{\mathbf{m}^{\sigma_1}}]\cdots P_n\,[S^{\mathbf{m}^{\sigma_j}}]\,[S^{\mathbf{m}^0}]\,Y_{\mathbf{m}}\,,\,Y_{\mathbf{m}}\,\rangle\\ &\qquad = \begin{cases} 1 & \text{if } \mathbf{m}^0+\cdots+\mathbf{m}^j=0\\ & \text{and } m\le n+\min\{0,\, m^{\sigma_1}, m^{\sigma_1}+m^{\sigma_2},\ldots, m^{\sigma_1}+\cdots+m^{\sigma_{j-1}}\},\\ 0 & \text{otherwise,}\end{cases}\\ &\langle P_n\,[S^{\mathbf{m}^{\sigma_1}}]\cdots P_n\,[S^{\mathbf{m}^{\sigma_j}}]\,P_n\,[S^{\mathbf{m}^0}]\,Y_{\mathbf{m}}\,,\,Y_{\mathbf{m}}\,\rangle.\\ &\qquad = \begin{cases} 1 & \text{if } \mathbf{m}^0+\cdots+\mathbf{m}^j=0\\ & \text{and } m\le n+\min\{0,\, m^{\sigma_1}, m^{\sigma_1}+m^{\sigma_2},\ldots, m^{\sigma_1}+\cdots+m^{\sigma_j}\},\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Putting all this together, we see that (2.4) is equal to 0 unless $\mathbf{m}^0+\cdots+\mathbf{m}^j=0$, when it equals

(2.5)

$$\begin{aligned}\sum_{\{\mathbf{m}:n-m_0<m\le n\}} \Bigg\{ &\Delta^{\mathbf{m}^1,\ldots,\mathbf{m}^j,\mathbf{m}^0}_{\mathbf{m}_1,\ldots,\mathbf{m}_j,\mathbf{m}_0}(\mathbf{m})\,\big(\,\mathcal{X}_{m\le n}(m)\ -\ \mathcal{X}_{m\le n+\min\{0,m^1+\cdots+m^j\}}(m)\,\big)\\ &-\ \sum_\sigma \Delta^{\mathbf{m}^{\sigma_1},\ldots,\mathbf{m}^{\sigma_j},\mathbf{m}^0}_{\mathbf{m}_{\sigma_1},\ldots,\mathbf{m}_{\sigma_j},\mathbf{m}_0}(\mathbf{m})\big(\ \mathcal{X}_{m\le n+\min\{0,\,m^{\sigma_1},\ldots,m^{\sigma_1}+\cdots+m^{\sigma_{j-1}}\}}(m)\\ &\qquad -\ \mathcal{X}_{m\le n+\min\{0,\,m^{\sigma_1},\ldots,m^{\sigma_1}+\cdots+m^{\sigma_j}\}}(m)\ \big)\ \Bigg\},\end{aligned}$$

where $\mathcal{X}$ denotes the indicator function. We must show that this is $\circ(n^{N-1})$ as $n \to \infty$.

For $N = 2$ or 3, and $0 \le \ell \le j$, write

$$\begin{aligned}
\Gamma_\ell(\mathbf{m}) &= \Gamma^{\mathbf{m}^\ell}_{\mathbf{m}_\ell}(\mathbf{m}),\\
\Delta(\mathbf{m}) &= \Delta^{\mathbf{m}^1,\ldots,\mathbf{m}^j,\mathbf{m}^0}_{\mathbf{m}_1,\ldots,\mathbf{m}_j,\mathbf{m}_0}(\mathbf{m}),\\
\Delta_\sigma(\mathbf{m}) &= \Delta^{\mathbf{m}^{\sigma_1},\ldots,\mathbf{m}^{\sigma_j},\mathbf{m}^0}_{\mathbf{m}_{\sigma_1},\ldots,\mathbf{m}_{\sigma_j},\mathbf{m}_0}(\mathbf{m}),\\
\Delta^\Sigma(n) &= \sum_{\{\mathbf{m}\in\mathcal{I}(N):m=n\}} \Delta(\mathbf{m}),\\
\Delta^\Sigma_\sigma(n) &= \sum_{\{\mathbf{m}\in\mathcal{I}(N):m=n\}} \Delta_\sigma(\mathbf{m}).
\end{aligned}$$

We have $|\Delta(\mathbf{m})|, |\Delta_\sigma(\mathbf{m})| \le \|\mathbf{m}_0\|_\infty \ldots \|\mathbf{m}_j\|_\infty$, so $\Delta^\Sigma(n)$ and $\Delta^\Sigma_\sigma(n)$ are $\mathrm{O}(n^{N-1})$. The expression (2.5) is equal to

$$\begin{aligned}
\sum_{n-m_0<m\le n} \Bigg\{ &\Delta^\Sigma(m)\mathcal{X}_{n+\min\{0,m^1+\cdots+m^j\}<m\le n}(m) \;-\; \sum_\sigma \Delta^\Sigma_\sigma(m)\\
&\times \mathcal{X}_{n+\min\{0,\,m^{\sigma_1},\ldots,m^{\sigma_1}+\cdots+m^{\sigma_j}\}<m\le n+\min\{0,\,m^{\sigma_1},\ldots,m^{\sigma_1}+\cdots+m^{\sigma_{j-1}}\}}(m) \Bigg\}.
\end{aligned}$$

We write this as

$$\begin{aligned}
&\Delta^\Sigma(n) \sum_{n-m_0<m\le n} \Bigg\{ \mathcal{X}_{n+\min\{0,m^1+\cdots+m^j\}<m\le n}(m)\\
&\quad - \sum_\sigma \mathcal{X}_{n+\min\{0,\,m^{\sigma_1},\ldots,m^{\sigma_1}+\cdots+m^{\sigma_j}\}<m\le n+\min\{0,\,m^{\sigma_1},\ldots,m^{\sigma_1}+\cdots+m^{\sigma_{j-1}}\}}(m) \Bigg\}\\
&\quad + \sum_{n-m_0<m\le n} \Bigg\{ (\Delta^\Sigma(m)-\Delta^\Sigma(n))\mathcal{X}_{n+\min\{0,m^1+\cdots+m^j\}<m\le n}(m)\\
&\qquad - \sum_\sigma (\Delta^\Sigma_\sigma(m)-\Delta^\Sigma(n))\\
&\qquad \times \mathcal{X}_{n+\min\{0,\,m^{\sigma_1},\ldots,m^{\sigma_1}+\cdots+m^{\sigma_j}\}<m\le n+\min\{0,\,m^{\sigma_1},\ldots,m^{\sigma_1}+\cdots+m^{\sigma_{j-1}}\}}(m) \Bigg\}\\
&\quad = (\mathrm{I}) \;+\; (\mathrm{II}).
\end{aligned}$$

We will examine (I). Now, $\mathbf{m}^0+\cdots+\mathbf{m}^j = 0$, so $|m_1+\cdots+m_j| \le m_0$, and

$$\begin{aligned}
&\sum_{n-m_0<m\le n} \Bigg\{ \mathcal{X}_{n+\min\{0,m^1+\cdots+m^j\}<m\le n}(m)\\
&\quad - \sum_\sigma \mathcal{X}_{n+\min\{0,\,m^{\sigma_1},\ldots,m^{\sigma_1}+\cdots+m^{\sigma_j}\}<m\le n+\min\{0,\,m^{\sigma_1},\ldots,m^{\sigma_1}+\cdots+m^{\sigma_{j-1}}\}}(m) \Bigg\}\\
&\quad = \min\{0, m^1+\cdots+m^j\} \;+\; \sum_\sigma \big(\min\{\,0,\, m^{\sigma_1},\ldots,m^{\sigma_1}+\cdots+m^{\sigma_{j-1}}\,\}\\
&\qquad\qquad - \min\{0,\, m^{\sigma_1},\ldots,m^{\sigma_1}+\cdots+m^{\sigma_j}\}\big) \;=\; 0,
\end{aligned}$$

by the Kac formula (0.12), so term (I) is equal to 0. Clearly, to show that (II) is $\circ(n^{N-1})$, it suffices to show the following:

Lemma 2.1. *For any permutation σ of $1, \ldots, j$, we have*

$$\sum_{\{\,m\,:\,n-m_0<m\leq n\,\}} \left| \Delta_\sigma^\Sigma(m) \;-\; \Delta^\Sigma(n) \right| \;=\; \circ(n^{N-1}) \qquad \textit{as } n\to\infty.$$

Proof of Lemma 2.1. We will show that

$$\left| \Delta_\sigma^\Sigma(n) \;-\; \Delta^\Sigma(n) \right| \;=\; \circ(n^{N-1}) \qquad \text{as } n\to\infty \tag{2.6}$$

and

$$\sup_{\{\,n_0\,:n-m_0<m'\leq n\,\}} \left| \Delta^\Sigma(m') \;-\; \Delta^\Sigma(n) \right| \;=\; \circ(n^{N-1}) \qquad \text{as } n\to\infty. \tag{2.7}$$

Write $M = m_0 + \cdots + m_j + 2$. For any function $F : \Lambda \to \mathbb{C}$ define the oscillation of F at $\mathbf{m} \in \Lambda$ to be

$$\omega_F(\mathbf{m}) \;=\; \sup\{|\; F(\mathbf{m}') \;-F(\mathbf{m})| : \mathbf{m}' \in \Lambda,\; |\mathbf{m}-\mathbf{m}'| \leq M\}.$$

It is easy to see that the oscillation of a product satisfies the Leibniz rule:

$$\omega_{FG}(\mathbf{m}) \;\leq\; \omega_F(\mathbf{m})\,\|G\|_\infty \;+\; \|F\|_\infty\,\omega_G(\mathbf{m}).$$

We will see that (2.6) and (2.7) are a consequence of the following:

Lemma 2.2.

$$\sum_{\{\mathbf{m}:m=n\}} \omega_{\Gamma_\ell}(\mathbf{m}) \;=\; \circ(n^{N-1}) \qquad \textit{as } n\to\infty.$$

To show that this indeed implies (2.6) and (2.7), first note that

$$\Delta(\mathbf{m}) \;=\; \Gamma_0(\mathbf{m}_{(0)}+\mathbf{m})\cdots\Gamma_j(\mathbf{m}_{(j)}+\mathbf{m}),$$

where for $0\leq \ell\leq j$, $\mathbf{m}_{(\ell)} \in \Lambda(N)$ is a fixed index. Since Γ_ℓ is bounded, Lemma 2.2 and the Leibniz rule imply that

$$\sum_{\{\mathbf{m}:m=n\}} \omega_{\Delta}(\mathbf{m}) \;=\; \circ(n^{N-1}) \qquad \text{as } n\to\infty. \tag{2.8}$$

Similarly, if σ is fixed then $\Delta_\sigma(\mathbf{m}) \;=\; \Gamma_0(\mathbf{m}^{(0)}+\mathbf{m})\cdots\Gamma_j(\mathbf{m}^{(j)}+\mathbf{m})$, where for $0\leq\ell\leq j$, $\mathbf{m}^{(\ell)}\in\Lambda$ is a fixed index. Now $|\mathbf{m}_{(\ell)}|$, $|\mathbf{m}^{(\ell)}|\leq M$, and the functions Γ_ℓ are bounded. We have

$$\begin{aligned}
&\left| \Delta_\sigma^\Sigma(n) \;-\; \Delta^\Sigma(n) \right| \\
&\leq \sum_{\{\mathbf{m}:m=n\}} \left|\Gamma_0(\mathbf{m}_{(0)}+\mathbf{m})\cdots\Gamma_j(\mathbf{m}_{(j)}+\mathbf{m}) - \Gamma_0(\mathbf{m}^{(0)}+\mathbf{m})\cdots\Gamma_j(\mathbf{m}^{(j)}+\mathbf{m})\,\right| \\
&\leq\; C\sum_{\ell=0}^{j}\sum_{\{\mathbf{m}:m=n\}} \omega_{\Gamma_\ell}(\mathbf{m}) \;=\; \circ(n^{N-1}) \quad \text{as } n\to\infty,
\end{aligned}$$

which proves (2.6).

We will prove (2.7) when $N=3$; the case $N=2$ is similar. Let $n-m_0<m\le n$, so $|n-m|<M-2$. We want to bound $|\Delta^\Sigma(m)-\Delta^\Sigma(n)|$. The cases $n-m$ even and $n-m$ odd are slightly different. We have

$$\begin{aligned}
&\left|\Delta^\Sigma(m)\ -\ \Delta^\Sigma(n)\right| \\
&= \Bigg|\sum_{\{\mathbf{m}':m'=m\}}\Delta(\mathbf{m}')\ -\ \sum_{\{\mathbf{m}':m'=n\}}\Delta(\mathbf{m}')\Bigg| \\
&\le \begin{cases} \sum_{\{p,q:(n,p,q)\in\mathcal{I}(3)\}}|\Delta(m,p,q)-\Delta(n,p,q)| & \text{if } n-m \text{ even},\\ \sum_{\{p,q:(n,p,q)\in\mathcal{I}(3)\}}|\Delta(m,p+1,q+1)-\Delta(n,p,q)| & \text{if } n-m \text{ odd}.\end{cases} \\
&\quad\le \sum_{\{\mathbf{m}':m'=n\}}\omega_\Delta(\mathbf{m}')\ =\ \circ(n^{N-1}),
\end{aligned}$$

by (2.8).

Proof of Lemma 2.2. Without loss of generality $\ell=1$. The number of indices $\mathbf{m}\in\Lambda(N)$ with $m=n$ and $\omega_\Delta(\mathbf{m})\neq 0$ is $\mathrm{O}(n^{N-1})$. Define the "reduced cones" $\mathcal{I}^t$, for $0<t\le 1$, by

$$\begin{aligned}
\mathcal{I}^t(3)\ &=\ \{\,\mathbf{m}=(m,p,q)\in\mathcal{I}(3):|p|,|q|\le tm\,\},\\
\mathcal{I}^t(2)\ &=\ \{\,\mathbf{m}=(m,p)\in\mathcal{I}(2):|p|\le tm\,\},\\
\mathcal{I}^t_*\ &=\ \{\,\mathbf{m}=(m,p)\in\mathcal{I}_*:|p|\le tm\,\}.
\end{aligned}$$

For $N=2$ or 3 and any fixed $t<1$, we have

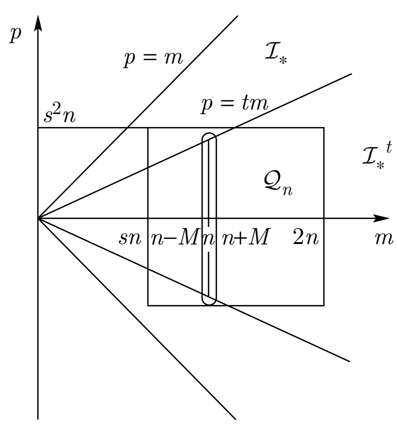
$$\sup_{\{\mathbf{m}\in\mathcal{I}^t(N):m=n\}}\omega_{\Gamma_j}(\mathbf{m})\ =\ \circ(n^{-1})\qquad\text{as}\ \ n\to\infty. \tag{2.9}$$

(This is not true with $t=1$.) This easily implies Lemma 2.2. To prove (2.9) from (1.17), we see that Γ_1 is a product of bounded functions F_0, F_1, and F_2, and clearly

$$\sup_{\{\mathbf{m}\in\mathcal{I}_*:m=n\}}\omega_{F_0}(\mathbf{m})\ =\ \circ(n^{-1})\qquad\text{as}\ \ n\to\infty.$$

Write

$$F(m,p)\ =\ \begin{cases} C(m_1,m,m^1+m;p_1,p,p_1+p) & \text{if } (m,p)\in\mathcal{I}_*,\\ 0 & \text{if } (m,p)\in\Lambda_*\setminus\mathcal{I}_*.\end{cases}$$



Using the Leibniz rule for the oscillation, it is not hard to see that (2.9) will follow if we show that

$$\sup_{\{\mathbf{m}=(m,p)\in\mathcal{I}_*^t:m=n\}} \omega_F(\mathbf{m}) \;=\; \circ(n^{-1}) \qquad \text{as } n\to\infty, \tag{2.10}$$

for any fixed $t<1$. Fix $t<1$ and pick s with $0<s<1$ and $s^2>t$. Define the rectangle of lattice points $\mathcal{Q}_n$ by

$$\mathcal{Q}_n \;=\; \{\,(m,p)\in\mathcal{I}_* \;:\; sn\le m\le 2n,\ |p|\le s^2 n\,\}.$$

To prove (2.10), we will show that for n sufficiently large, there is a function G_n defined on

$$\mathcal{Q}=\{(x,y)\in\mathbb{R}^2 : s\le y\le 2,\ |x|\le s^2\},$$

such that

$$\|\,|\nabla G_n|\,\|_\infty \quad \text{is uniformly bounded over n} \tag{2.11}$$

and

$$F(p,m)=G_n(\tfrac{p}{n},\tfrac{m}{n})+O(n^{-1}) \qquad \text{on } \mathcal{Q}_n.$$

That this does indeed prove (2.10) is easily seen, for we can pick n sufficiently large so that if $\mathbf{m}=(m,p)\in\mathcal{I}_*^t$ and $\mathbf{m}'=(m',p')\in\Lambda_*$ is such that $|\mathbf{m}-\mathbf{m}'|<M$, then $\mathbf{m},\mathbf{m}'\in\mathcal{Q}_n$, and hence

$$\begin{aligned}|F(p,m)-F(p',m')| \;&\le\; |G_n(\tfrac{p}{n},\tfrac{m}{n})-G_n(\tfrac{p'}{n},\tfrac{m'}{n})|+O(n^{-1})\\ &\le\; \tfrac{M}{n}\|\,|\nabla G_n|\,\|_\infty+O(n^{-1}).\end{aligned}$$

To show that such functions G_n exist, we examine the expression for the Clebsch-Gordan coefficient, (1.18):

$$\begin{aligned}F(p,m)=&\left\{(m+c_1)\frac{(m+p)!}{(m+p+c_2)!}\,\frac{(m-p+c_3)!}{(m-p)!}\,\frac{(2m+c_4)!}{(2m+c_5)!}\right\}^{\frac12}\\ &\times\ \sum_{s=0}^{c_{10}}c_{11}(s)\frac{(m-p+c_6-s+1)\cdots(m-p+c_6-s+c_7)}{(2m+c_8-s+1)\cdots(2m+c_8-s+c_9)}\end{aligned}$$

where $c_1,\ldots,c_{10},c_{11}(s)$ are constants with $c_7,c_9\ge 1$. This expression for $F(p,m)$ is a product of two factors. The factor involving the sum can be written as

$$\frac{1}{(2m+c_8-c_{10}+1)\cdots(2m+c_8+c_9)}\,J(p,m)$$

where J is a polynomial. We notice that

$$\frac{1}{(2m+c_8-c_{10}+1)\cdots(2m+c_8+c_9)} \;=\; (2m)^{-c_9-c_{10}}(1+O(m^{-1})).$$

The factor involving the square root is

$$m^{(c_4-c_5+1)/2}(m+p)^{-c_2/2}(m-p)^{c_3/2}(1+\mathrm{O}(m^{-1}))$$

on $\{ (p,m) : |p| < sm \}$. Absorbing constants into J gives

$$F(p,m) \;=\; m^{c_{12}}(m+p)^{c_{13}}(m-p)^{c_{14}}J(p,m)(1+\mathrm{O}(m^{-1}))$$

on $\{ (p,m) : |p| < \nu m \}$, for constants c_{12}, c_{13}, c_{14}, and since $F(p,m)$ is uniformly bounded on $\mathcal{I}_*$, we get

$$F(p,m) \;=\; m^{c_{12}}(m+p)^{c_{13}}(m-p)^{c_{14}}J(p,m) \;+\; \mathrm{O}(m^{-1}).$$

Define G_n on $\mathcal{Q}$ by

$$G_n(x,y) = n^{c_{12}+c_{13}+c_{14}}y^{c_{12}}(y+x)^{c_{13}}(y-x)^{c_{14}}J(nx,ny).$$

Then $F(p,m) = G_n(\frac{p}{n},\frac{m}{n}) + \mathrm{O}(n^{-1})$ on $\mathcal{Q}_n$, and we just need to show (2.11). Clearly

$$y^{c_{12}}(y+x)^{c_{13}}(y-x)^{c_{14}}$$

and

$$\left|\nabla\big(y^{c_{12}}(y+x)^{c_{13}}(y-x)^{c_{14}}\big)\right|$$

are uniformly bounded above and below on $\mathcal{Q}$. Since $J_n(x,y) = J(nx,ny)$ is a polynomial on $\mathcal{Q}$ of degree at most the degree of J, there is a constant c_0 independent of n such that

$$\|\,|\nabla J_n|\,\|_\infty \;\le\; c_0\|J_n\|_\infty$$

on $\mathcal{Q}$. Hence there is a constant c independent of n such that

$$\|\,|\nabla G_n|\,\|_\infty \;\le\; c\|G_n\|_\infty.$$

Now F is uniformly bounded on $\mathcal{I}_*$ so G_n is bounded by a constant c' independent of n on

$$\frac{1}{n}\mathcal{Q}_n \;=\; \{(\tfrac{x}{n},\tfrac{y}{n}) \in \mathcal{Q} : (x,y) \in \mathcal{Q}_n\}.$$

Since any point of $\mathcal{Q}$ is at most $\sqrt{2}/n$ from a point of $\frac{1}{n}\mathcal{Q}_n$, we get

$$\|\,|\nabla G_n|\,\|_\infty \;\le\; c\|G_n\|_\infty \;\le\; c\big(c' + \tfrac{\sqrt{2}}{n}\|\,|\nabla G_n|\,\|_\infty\big)$$

and hence (2.11).

3. Asymptotics for $\operatorname{trace}\log P_n[f]P_n$ when $f \in C(S^N)\cap H^{1/2}(S^N)$

For $N=2$ or 3 fixed, let W be the space $C(S^N)\cap H^{1/2}(S^N)$ with norm $\|\cdot\|_W = \|\cdot\|_{L^\infty} + \|\cdot\|_{H^{1/2}}$.

In this section we will extend the key result of Section 2 to functions $f \in W$.

Lemma 3.1. *If f is a function in W and $0 \le j \le k < \infty$, then*

$$\operatorname{trace}\,\big(P_n[f]\big)^j(I-P_n)[f^{k-j}]P_n \;-\; \frac{1}{j}\;\operatorname{trace}\;P_n[f^j](I-P_n)[f^{k-j}]P_n \;=\; \circ(n^{N-1}) \qquad \text{as } n\to\infty.$$

Furthermore, we will prove the following:

Lemma 3.2. *If $f \in W$ is such that the closed convex hull of the image of f does not contain the origin, then*

$$\begin{aligned}\operatorname{trace}\log P_n[f]P_n \;&=\; \operatorname{trace} P_n[\log f]P_n \\ &\quad+\; \frac{1}{2}\operatorname{trace}\, P_n[\log f](I-P_n)[\log f]P_n \;+\; \circ(n^{N-1}).\end{aligned}$$

Combined with (0.3), (0.4), (0.9) and (0.10), this completes the proof of Theorem 0.1.

In the proofs of these two lemmas, we will make frequent use of the following definitions and facts which can be found in [**5**] on page 187.

Proposition. *Let $(M, d\mu)$ be a finite measure space. For an operator T on $L^2(M, d\mu)$, with integral kernel $K(x,y)$, and an orthonormal base z_k, $k = 1, 2, \ldots$, of $L^2(M, d\mu)$, the Hilbert-Schmidt norm of T is*

$$\|T\|_2 \;=\; \left(\iint_{M\times M} |K(x,y)|^2\, d\mu(x) d\mu(y)\right)^{1/2} \;=\; \left(\sum_{k=1}^{\infty} \|Tz_k\|_2^2\right)^{1/2}.$$

$\|T\|$ will denote the operator norm of T.

If T_1 and T_2 are Hilbert-Schmidt, then T_1T_2 is trace class,

$$|\operatorname{trace} T_1T_2| \;\le\; \|T_1\|_2\|T_2\|_2,$$

and

$$\operatorname{trace} T_1T_2 \;=\; \operatorname{trace} T_2T_1.$$

If T_1 is bounded and T_2 is Hilbert-Schmidt, then T_1T_2 is Hilbert-Schmidt and

$$\|T_1T_2\|_2 \;\le\; \|T_1\|\|T_2\|_2.$$

Proof of Lemma 3.1. Write

$$\begin{aligned} t_n^0(f) \;&=\; t_n^0(j,k;f) \;=\; \frac{1}{n^{N-1}}\ \operatorname{trace}\,(P_n[f])^j(I-P_n)[f^{k-j}]P_n, \\ t_n^1(f) \;&=\; t_n^1(j,k;f) \;=\; \frac{1}{n^{N-1}}\ \operatorname{trace}\ P_n[f^j](I-P_n)[f^{k-j}]P_n. \end{aligned} \tag{3.1}$$

We want to prove that for all $j \le k$ fixed and $f \in W$, we have

$$t_n^0(f) \;-\; \frac{1}{j}\, t_n^1(f) \;=\; \circ(1) \qquad \text{as} \;\; n \to \infty. \tag{3.2}$$

From Section 2, we know that this holds when f is a polynomial on S^N. Using the remarks above (1.8), it is easily seen that W is a Banach algebra, so the Stone-Weierstrass theorem implies that the polynomials on S^N are dense in W. We will show that if $\|f\|_W, \|g\|_W < M$, then for $\alpha = 0, 1$,

$$|t_n^\alpha(f) - t_n^\alpha(g)| \;\le\; C\ M^{k-1}\|f-g\|_W, \tag{3.3}$$

where C is independent of M and n. A simple argument then gives (3.2) for all $f \in W$.

To prove (3.3), we have

$$\begin{aligned}
&\|P_n[f](I-P_n)\|_2 \ \le \ \|P_n[f]-[f]P_n\|_2 \\
&= \ \left(\iint_{M\times M} |K_n(x,y)|^2 |f(x)-f(y)|^2 \, d\mu(x)d\mu(y)\right)^{1/2} \ \le \ Cn^{(N-1)/2}\|f\|_{H^{1/2}},
\end{aligned}$$

so for functions f and g in W, we have

$$\begin{aligned}
\|P_n[fg](I-P_n)\|_2 \ &\le \ \|P_n[f]P_n[g](I-P_n)\|_2 \ + \ \|P_n[f](I-P_n)[g](I-P_n)\|_2 \\
&\le \ \|f\|_{L^\infty}\|P_n[g](I-P_n)\|_2 \ + \ \|g\|_{L^\infty}\|P_n[f](I-P_n)\|_2.
\end{aligned}$$

By induction we see that if $f_\ell \in W$ for $1 \le \ell \le j$, then

$$\begin{aligned}
\|P_n[f_1\cdots f_j](I-P_n)\|_2 \ &\le \ \sum_{\ell=1}^{j} \|P_n[f_\ell](I-P_n)\|_2 \prod_{k\ne\ell} \|f_k\|_{L^\infty} \\
&\le \ C\, n^{(N-1)/2} \sum_{\ell=1}^{j} \|f_\ell\|_{H^{1/2}} \prod_{k\ne\ell} \|f_k\|_{L^\infty}.
\end{aligned} \tag{3.4}$$

From this we see that

$$\|P_n[f_1\cdots f_j](I-P_n)\|_2 \le C\, n^{(N-1)/2}\|f_1\|_W \ldots \|f_j\|_W,$$

and if $\|f\|_W, \|g\|_W \le M$ then

$$\begin{aligned}
&n^{N-1}|t_n^1(f)-t_n^1(g)| \\
&= \ \left|\operatorname{trace} P_n[f^j-g^j](I-P_n)[f^{k-j}]P_n + \operatorname{trace} P_n[g^j](I-P_n)[f^{k-j}-g^{k-j}]P_n\right| \\
&\le \ \|P_n[f-g][f^{j-1}+\cdots+g^{j-1}](I-P_n)\|_2\|(I-P_n)[f^{k-j}]P_n\|_2 \\
&\qquad + \ \|P_n[g^j](I-P_n)\|_2\|(I-P_n)[f-g][f^{k-j-1}+\cdots+g^{k-j-1}]P_n\|_2 \\
&\le \ Cn^{N-1}M^{k-1}\|f-g\|_W.
\end{aligned}$$

The case $\alpha = 0$ can be proved in a similar way.

Proof of Lemma 3.2. By a simple scaling argument, Lemma 3.2 is is equivalent to the following:

Lemma 3.2′. *If $f \in W$ and $\|f\|_\infty < 1$, then*

$$\begin{aligned}
\operatorname{trace}\log P_n[1-f]P_n \ &= \ \operatorname{trace} P_n[\log(1-f)]P_n \\
&\quad + \ \frac{1}{2}\operatorname{trace} P_n[\log(1-f)](I-P_n)[\log(1-f)]P_n \ + \ \circ(n^{N-1}).
\end{aligned}$$

Proof of Lemma 3.2′. Let t^0 and t^1 be defined as in (3.1). We have

$$\begin{aligned}
|\, t_n^0(j,k;f)\,| \ &\le \ C(k-j)\,\|f\|_{L^\infty}^{k-2}\|f\|_{H^{1/2}}^2, \\
|\, t_n^1(j,k;f)\,| \ &\le \ Cj(k-j)\,\|f\|_{L^\infty}^{k-2}\|f\|_{H^{1/2}}^2,
\end{aligned} \tag{3.5}$$

where C does not depend on j, k or f. To see this, by (3.4) we have

$$\|(I-P_n)[f^{k-j}]P_n\|_2 \;\le\; Cn^{(N-1)/2}\|f\|_\infty^{k-j-1}\|f\|_{H^{1/2}},$$

so

$$\begin{aligned} n^{N-1}|\,t_n^0(j,k;f)\,| \;&=\; |\,\mathrm{trace}(P_n[f])^j(I-P_n)[f^{k-j}]P_n\,| \\ &\le\; \|\,(P_n[f])^{j-1}\|\|\,P_n[f](I-P_n)\,\|_2\,\|\,(I-P_n)[f^{k-j}]P_n\,\|_2 \\ &\le\; Cn^{N-1}(k-j)\|f\|_{L^\infty}^{k-2}\|f\|_{H^{1/2}}^2. \end{aligned}$$

The proof of the second inequality is similar.

Now the operator norm of $P_n[f]P_n$ is bounded by $\|f\|_\infty$, so since $\|f\|_\infty < 1$, we have

$$\begin{aligned} &\mathrm{trace}\log P_n[1-f]P_n \;-\; \mathrm{trace}\,P_n[\log(1-f)]P_n \\ &\quad=\; \sum_{k=1}^{\infty}\frac{1}{k}\big(\,-\mathrm{trace}(P_n[f]P_n)^k \;+\; \mathrm{trace}\,P_n[f^k]P_n\,\big) \\ &\quad=\; \sum_{k=1}^{\infty}\frac{1}{k}\sum_{j=1}^{k-1}\mathrm{trace}(P_n[f])^j(I-P_n)[f^{k-j}]P_n \\ &\quad=\; n^{N-1}\sum_{k=1}^{\infty}\frac{1}{k}\sum_{j=1}^{k-1}t_n^0(j,k;f) \end{aligned}$$

and

$$\begin{aligned} &\mathrm{trace}\;P_n[\log(1-f)](I-P_n)[\log(1-f)]P_n \\ &\quad=\; \sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{j}\frac{1}{\ell}\,\mathrm{trace}\;\;P_n[f^j](I-P_n)[f^\ell]P_n \\ &\quad=\; \sum_{k=2}^{\infty}\frac{1}{k}\sum_{j=1}^{k-1}\Big(\frac{1}{j}+\frac{1}{k-j}\Big)\,\mathrm{trace}\;\;P_n[f^j](I-P_n)[f^{k-j}]P_n \\ &\quad=\; 2\sum_{k=2}^{\infty}\frac{1}{k}\sum_{j=1}^{k-1}\frac{1}{j}\,\mathrm{trace}\;\;P_n[f^j](I-P_n)[f^{k-j}]P_n \\ &\quad=\; 2n^{N-1}\sum_{k=2}^{\infty}\frac{1}{k}\sum_{j=1}^{k-1}\frac{1}{j}\;t_n^1(j,k;f), \end{aligned}$$

so

$$\begin{aligned} &|\;\mathrm{trace}\;\log P_n[1-f]P_n \;-\; \mathrm{trace}\;\;P_n[\log(1-f)]P_n \\ &\qquad\qquad -\;\frac{1}{2}\,\mathrm{trace}\;\;P_n[\log(1-f)](I-P_n)[\log(1-f)]P_n\;| \\ &\le\; n^{N-1}\sum_{k=2}^{\infty}\frac{1}{k}\sum_{j=1}^{k-1}\Big|\;t_n^0(j,k;f)\;-\;\frac{1}{j}\;t_n^1(j,k;f)\;\Big|. \end{aligned}$$

By (3.2), (3.5) and the dominated convergence theorem we see that this final expression is $\circ(n^{N-1})$ as $n\to\infty$.

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