

THETA DICHOTOMY FOR UNITARY GROUPS

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INTRODUCTION

Some recent work of Gross and Prasad [14] suggests that the root numbers attached to certain symplectic representations of the Weil-Deligne group of a local field F control certain branching rules for representations of orthogonal groups over F . On a global level, this local phenomenon should have implications for the structure of the value at the center of symmetry for certain L-functions of arithmetic interest [12]. This conjectural picture is based, to some extent, on the now classic work of Tunnell [47] and Waldspurger [49] as well as on the case of the triple product L-function [13, 34, 17]. In all of these examples, the local root number detects the existence of a certain type of invariant linear functional. It is possible to set up analogous conjectures for the isometry groups of Hermitian or quaternion-hermitian forms.

It turns out that root numbers also play a role in the local and global theta correspondence. Roughly speaking, certain local root numbers should control the occurrence of representations in the local theta correspondence between groups of the ‘same size’, e.g., for dual pairs of the form $(Sp(n), O(2n+1))$ and $(U(n), U(n))$. As will be seen, the theta correspondence for such pairs is connected with a certain induced representation $I_n(s, \chi)$ at the point $s = 0$ on the unitary axis. By contrast, the correspondence for the pairs $(Sp(n), O(2n))$ and $(Sp(n), O(2n+2))$, discussed by Prasad [35], is connected to the behavior of a similar induced representation at the points $\pm\frac{1}{2}$, in which case no epsilon factor arises.

In this paper we consider the local theta correspondence for unitary groups in the non-archimedean case. Let F be a non-archimedean local field of characteristic not equal to 2, and let E be a quadratic extension of F . For further notation, see the notation section at the end of this introduction. Let V and W be E vector spaces of dimensions m and n equipped, respectively, with a Hermitian form $(\ , \) : V \times V \rightarrow E$ and a skew-Hermitian form $\langle \ , \ \rangle : W \times W \rightarrow E$. Then the isometry groups $G(V)$ and $G(W)$ of the spaces V and W form a dual reductive pair in the symplectic group $Sp(\mathbb{W})$, where $\mathbb{W} = V \otimes_E W$, viewed as an F vector space of dimension $2mn$ and equipped with the symplectic form

$$(0.1) \quad \langle \langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle \rangle = \frac{1}{2} \text{tr}_{E/F}((v_1, v_2) \overline{\langle w_1, w_2 \rangle}).$$

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For a fixed non-trivial additive character ψ of F , let ω_ψ be the smooth Weil representation of the metaplectic cover $Mp(\mathbb{W})$ (we view this as an extension of $Sp(\mathbb{W})$ by \mathbb{C}^1) realized on a space S . For any choice of a pair of characters $\chi = (\chi_1, \chi_2)$ of E^\times whose restrictions to F^\times satisfy $\chi_1|_{F^\times} = \epsilon_{E/F}^n$, and $\chi_2|_{F^\times} = \epsilon_{E/F}^m$, there exists a homomorphism

$$(0.2) \quad \tilde{\iota}_\chi : G(V) \times G(W) \longrightarrow Mp(\mathbb{W})$$

lifting the natural map

$$(0.3) \quad \iota : G(V) \times G(W) \longrightarrow Sp(\mathbb{W}),$$

and so, we obtain a representation ω_χ of $G(V) \times G(W)$ on S . For an irreducible admissible representation π of $G(W)$, we say that π occurs in the local theta correspondence for the dual pair $G(V), G(W)$ (and write $\Theta_\chi(\pi, V) \neq 0$) if π occurs as a quotient of ω_χ . The local Howe duality principle then asserts that there is a uniquely determined irreducible admissible representation $\theta_\chi(\pi, V)$ of $G(V)$ such that

$$(0.4) \quad Hom_{G(V) \times G(W)}(\omega_\chi, \theta_\chi(\pi, V) \otimes \pi) \neq 0,$$

and that the representations $\theta_\chi(\pi_1, V)$ and $\theta_\chi(\pi_2, V)$ corresponding to distinct π 's are distinct. This principle is known to hold whenever the residue characteristic of F is not 2 [50].

For a fixed dimension m there are precisely two isomorphism classes of (non-degenerate) Hermitian spaces over E , and these are distinguished by the coset $(-1)^{\frac{m(m-1)}{2}} \det V \cdot NE^\times$ in F^\times/NE^\times , where $\det V$ denotes the determinant of the matrix $((v_i, v_j))$ of inner products for any E basis $\{v_1, \dots, v_m\}$. We denote these spaces by V_m^\pm , so that

$$(0.5) \quad \epsilon_{E/F} \left((-1)^{\frac{m(m-1)}{2}} \det V_m^\pm \right) = \pm 1,$$

and we consider the collection of representations $\theta_\chi(\pi, V_m^\pm)$ attached to a fixed representation π of $G(W)$ as m and \pm vary.

The main dichotomy phenomenon referred to in the title is the following:

Theta dichotomy (Corollary 4.4). *Suppose that π is a supercuspidal representation of $G(W)$, where $\dim_E W = n$. Then, for a fixed character χ of E^\times with $\chi|_{F^\times} = \epsilon_{E/F}^n$, precisely one of the representations $\theta_\chi(\pi, V_n^\pm)$ is non-zero.*

Here we are taking $\chi_1 = \chi_2 = \chi$, in a slight abuse of the notation above.

Thus, the supercuspidal part of the sum $S_\chi(V_n^+) \oplus S_\chi(V_n^-)$ of the Weil representations of $U(n)$ attached to the two Hermitian spaces of dimension n provides a ‘model’ for the supercuspidal representations for $U(n)$, in the sense that one has

$$(0.6) \quad \left(S_\chi(V_n^+) \oplus S_\chi(V_n^-) \right)_{s.c.} \simeq \bigoplus_{\pi \text{ s.c.}} \pi \otimes \theta_\chi(\pi, V_n^\pm).$$

Here we take the non-zero space $\theta_\chi(\pi, V_n^\pm)$.

In fact, theta dichotomy should hold for all irreducible admissible representations π , and we prove this for many non-supercuspidals (Corollary 4.4). For example, we also show that for any π , and for $m \geq n$, at least one of the representations $\theta_\chi(\pi, V_m^+)$ and $\theta_\chi(\pi, V_m^-)$ is non-zero (Corollary 4.5).

The non-vanishing of $\theta_\chi(\pi, V_n^\pm)$ is controlled by a local root number which is defined as follows. For an irreducible admissible representation π of $G(W)$ and a character χ of E^\times , Piatetski-Shapiro and Rallis [31, 32, 33] defined a family of zeta integrals $\{ Z(s, \phi, \Phi) \}$ and obtained from it an L-factor, $L(s, \pi, \chi) = L_{PSR}(s, \pi, \chi)$, and ϵ -factor, $\epsilon(s, \pi, \chi, \psi) = \epsilon_{PSR}(s, \pi, \chi, \psi; W)$. If the space W is unramified (see the notation section below), if χ is unramified, and if π is an unramified principal series representation, then

$$(0.7) \quad L_{PSR}(s, \pi, \chi) = L(s, BC(\pi) \otimes \chi),$$

and

$$(0.8) \quad \epsilon_{PSR}(s, \pi, \chi, \psi; W) = \epsilon(s, BC(\pi) \otimes \chi, \psi_E),$$

where the factors on the right are the standard L and ϵ -factors [19, 11] for the base change, [5], $BC(\pi)$ of π to $GL_n(E)$, twisted by χ . Here we note that the definition of ϵ given in [32] depends on an auxiliary choice, and the dependence on that choice is not made explicit there. In section 6, we determine this dependence and give an unambiguous definition of $\epsilon_{PSR}(s, \pi, \chi, \psi; W)$, which is, of course, crucial for us. This factor actually depends on the isomorphism class of the space W ((0.19) and Proposition 6.11).

Epsilon Dichotomy (Theorem 6.1). *For a supercuspidal representation π of $G(W)$, and for a fixed character χ of E^\times with $\chi|_{F^\times} = \epsilon_{E/F}^n$, $n = \dim_E W$,*

$$(0.9) \quad \theta_\chi(\pi, V) \neq 0 \iff \epsilon_{PSR}(\frac{1}{2}, \pi, \chi, \psi; W) = \epsilon_{E/F}(-2)^n \cdot \epsilon_{E/F}(\det V).$$

Here V is a Hermitian space with $\dim_E V = n$.

Again, we expect that this result holds for all irreducible admissible representations of $G(W)$.

This dichotomy phenomenon provides information about the structure of the local theta correspondence for unitary groups. Some of the nicest properties of that local correspondence involve its compatibility with *Witt towers*. Let $V_{r,r}$ denote the Hermitian space of dimension $2r$ over E with maximal isotropic subspaces of dimension r , so that, with the notation introduced above, $V_{r,r} \simeq V_{2r}^+$. The collections of spaces $\{ V_{2r}^+ \mid r \geq 0 \}$ and $\{ V_{2r}^- \mid r \geq 1 \}$ are then the two Witt towers of even dimensional Hermitian spaces. Similarly, $V_1^\pm \simeq E$ is the one dimensional Hermitian space over E with Hermitian form $(x, y) = \alpha \bar{x}y$, where $\alpha \in F^\times$ with $\epsilon_{E/F}(\alpha) = \pm 1$, and $V_{2r+1}^\pm \simeq V_1^\pm + V_{r,r}$. The collections $\{ V_{2r+1}^+ \mid r \geq 0 \}$ and $\{ V_{2r+1}^- \mid r \geq 0 \}$ are then the two Witt towers of odd dimensional Hermitian spaces. If we fix an element $\delta \in E^\times$ with $\bar{\delta} = -\delta$, then to every skew-Hermitian space W , $\langle \ , \ \rangle$, over E we may associate a Hermitian space $W' = W$ with $(x, y)' = \delta^{-1} \langle y, x \rangle$. We then let W_n^\pm be the skew-Hermitian space such that $(W_n^\pm)' \simeq V_n^\pm$. If n is even, this space is independent of the choice of δ , while, if n is odd, it depends on that choice.

Now suppose again that π is a supercuspidal representation of $G(W)$, fix a character χ of E^\times with $\chi|_{F^\times} = \epsilon_{E/F}^{m_{00}}$, $m_{00} = 0, 1$, and consider the collection of theta lifts $\theta_\chi(\pi, V_m^\pm)$ as V_m^\pm runs over a fixed Witt tower $\{V_m^\pm\}$, with m of the same parity as m_{00} . Let $m_0^\pm(\chi)$ be the smallest m such that $\theta_\chi(\pi, V_m^\pm) \neq 0$. Then it is known [29] that $\theta_\chi(\pi, V_{m_0^\pm(\chi)}^\pm)$ is a supercuspidal representation of $G(V_{m_0^\pm(\chi)}^\pm)$ and that, for $m = m_0^\pm(\chi) + 2t$, $t > 0$, the representation $\theta_\chi(\pi, V_m^\pm)$ is non-zero and not supercuspidal. Thus, if m_{00} has the same parity as $n = \dim_E W$, theta dichotomy implies that one element of the set $\{m_0^+(\chi), m_0^-(\chi)\}$ is $\leq n$ (the ‘early’ tower for π and χ) while the other is $> n$ (the ‘late’ tower for π and χ). If, say, $m_0^+(\chi) \leq n$ and $n_1 := m_0^-(\chi) > n$, then the representation $\pi_1 = \theta_\chi(\pi, V_{n_1}^-)$ is a supercuspidal representation of $G(V_{n_1}^-)$, and we can repeat the construction, considering the theta lifts of π_1 to the two skew-Hermitian Witt towers. In the following statement we identify Hermitian and skew-Hermitian forms W_m^\pm and V_n^\pm , as above, and we again take $\chi = \chi_1 = \chi_2$. By the procedure just described, we obtain:

Chains of supercuspidals (Section 5). *For any supercuspidal representation π of $G(W)$, and choice of character χ of E^\times with $\chi|_{F^\times} = \epsilon_{E/F}^n$, $n = \dim_E W$, there exists a sequence of dimensions $n = n_0 < n_1 < n_2 < \dots < n_i < \dots$ and supercuspidal representations $\pi = \pi_0, \pi_1, \dots, \pi_i, \dots$ of $G(V_{n_i}^\pm)$ such that*

$$(0.10) \quad \theta_\chi(\pi_i, V_{n_{i+1}}^\pm) = \pi_{i+1}.$$

Here, for example, if $W \simeq W_n^+$, the sequence of signs is

$$+, \epsilon, -, -\epsilon, +, \dots$$

For example, in the case $n = 1$, the first step in such a chain can be found in Rogawski [41]. This is explained in detail in section 7. In general, Speculation 7.5 suggests that

$$(0.11) \quad m_0^+(\chi) + m_0^-(\chi) = 2n + 2.$$

If we assume that $\theta_\chi(\pi, V_n^\pm)$ is also supercuspidal, i.e., that π does not come from a smaller unitary group via the theta correspondence, then Speculation 7.5 implies that the sequence of dimensions will be

$$(0.12) \quad n, n + 2, n + 6, n + 12, \dots$$

Relation (0.11) and (0.12) have now been proved for π supercuspidal by Rallis and the second author [24]. We note that there is an analogous construction when the Hermitian and skew-Hermitian spaces have dimensions of opposite parity. This is described at the end of section 7. Moreover, a similar phenomenon occurs for reductive dual pairs of type (Sp, O) .

In section 7, we combine dichotomy with several other basic principles and conjectures to give some speculations about the functoriality properties of the theta correspondence for unitary groups. It is known, for example, that the theta correspondence does not always preserve L-packets, whereas, according to conjectures of Adams [1], which are based on fairly extensive evidence in the archimedean

case, the grouping of representations into A-packets should be preserved. It seems that theta dichotomy is at the root of the failure of functoriality on the level of L-packets. For example, for n odd, the L-group will not distinguish the groups $G(V_n^+)$ and $G(V_n^-)$ —they can be canonically identified as groups—whereas the theta correspondence behaves very differently in the two towers. The speculations of section 7 suggest that it should be possible, nonetheless, to predict the behavior of L-packets, in the extended sense [48], under the correspondence. In particular, we give a conjecture, Speculation 7.7, about the L-parameters of the supercuspidals in a chain. Specifically, let W_F be the Weil group of F , let $W'_F = SL_2(\mathbb{C}) \times W_F$, and let $\varphi_0 : W'_F \rightarrow {}^L G(W_n)$ be the L-parameter for $\pi = \pi_0$. Assume, as above, that π does not come from a smaller unitary group via the theta correspondence. Then the L-parameter for π_r should be

$$(0.13) \quad \varphi_r = \varphi_0 \oplus \chi(\rho_2 \oplus \rho_4 \oplus \cdots \oplus \rho_{2r}),$$

where ρ_b is the $(b-1)$ -st symmetric power of the standard representation of $SL_2(\mathbb{C})$.

For a given supercuspidal π of $G(W_n)$, the dichotomy principle says that, if $\epsilon = \pm 1$ is the sign for the early tower, i.e., if $\theta_\chi(\pi, V_n^\epsilon) \neq 0$, then $m_0^\epsilon = n - 2r$, for some $r \geq 0$. It turns out that how ‘early’ π occurs is reflected in the L-factor. For any character χ of E^\times , let $\tilde{\chi}(x) = \chi(\bar{x})^{-1}$. The following result is the analogue for unitary groups of an unpublished result of Rallis and the second author for dual pairs of type (Sp, O) .

Early lifts and poles (Theorem 6.2). *For a supercuspidal representation π of $G(W)$, for any character χ of E^\times , and for $s \in \mathbb{C}$, assume that the pair (s, χ) is normalized as at the beginning of section 6.*

- (i) *If $\chi \neq \tilde{\chi}$, then $L_{PSR}(s, \pi, \chi) = 1$.*
- (ii) *If $\chi = \tilde{\chi}$, and $\chi|_{F^\times} = \epsilon_{E/F}^n$ let*

$$(0.14) \quad m_0 = m_0(\chi) = \min\{m_0^+(\chi), m_0^-(\chi)\} = n - 2r.$$

Then, if $r > 0$,

$$(0.15) \quad L_{PSR}(s, \pi^\vee, \chi) = (1 - q_E^{-(s+\frac{1}{2})})^{-1} (1 - q_E^{-(s+\frac{3}{2})})^{-1} \cdots (1 - q_E^{-(s+r-\frac{1}{2})})^{-1}.$$

If $r = 0$, $L_{PSR}(s, \pi^\vee, \chi) = 1$.

- (iii) *If $\chi = \tilde{\chi}$, and $\chi|_{F^\times} = \epsilon_{E/F}^{n-1}$ let*

$$(0.16) \quad m_0 = m_0(\chi) = \min\{m_0^+(\chi), m_0^-(\chi)\} = n - 2r + 1.$$

Then, if $r > 0$,

$$(0.17) \quad L_{PSR}(s, \pi^\vee, \chi) = (1 - q_E^{-s})^{-1} (1 - q_E^{-(s+1)})^{-1} \cdots (1 - q_E^{-(s+r-1)})^{-1}.$$

If $r = 0$, $L_{PSR}(s, \pi^\vee, \chi) = 1$.

Note that in case (iii), any representation of $G(W)$ has $\theta_\chi(\pi, V_{n+1}^\pm) \neq 0$ for at least one choice of \pm , and this should usually be the first non-zero theta lift, i.e., one should usually have $r = 0$. Similarly, in case (ii), one expects to have $r = 0$.

The L and ϵ -factors occurring in our results are those defined by Piatetski-Shapiro and Rallis. One would like to prove the following:

Comparison Conjecture. For all irreducible admissible representations π of $G(W)$, and characters χ of E^\times , let $BC(\pi)$ be the base change of π to $GL_n(E)$. Then

$$(0.18) \quad L_{PSR}(s, \pi, \chi) = L(s, BC(\pi) \otimes \chi).$$

If n is odd, then

$$(0.19) \text{ (odd)} \quad \epsilon_{PSR}(s, \pi, \chi, \psi; W) = \chi(-\det W) |\det W|_E^{s-\frac{1}{2}} \omega_\pi(-1) \epsilon(s, BC(\pi) \otimes \chi, \psi_E).$$

If n is even, let $\epsilon(W) = +1$ if $G(W)$ is quasi-split, $\epsilon(W) = -1$ if not. Then

$$(0.19) \text{ (even)} \quad \epsilon_{PSR}(s, \pi, \chi, \psi; W) = \epsilon(W) |\det W|_E^{s-\frac{1}{2}} \omega_\pi(-1) \epsilon(s, BC(\pi) \otimes \chi, \psi_E).$$

Here, if the skew-Hermitian form on W has matrix $\delta \mathbf{a}$, with \mathbf{a} Hermitian, then $\det W = \delta^n \det \mathbf{a}$.

The base change involved here is well known for $n = 1$, and is due to Rogawski [41] for $n = 2$ and 3. In general, it can be defined implicitly using the global method of [5], although it is difficult to verify that the resulting local base change is independent of its imbedding into a global situation.

In the case $n = 1$ we verify this conjecture in section 8, Proposition 8.4, by an amusing calculation. With this additional information, our general results on theta and epsilon dichotomy specialize to a new proof of the local results of Rogawski [42] and Moen [30] for the dual pair $(U(1), U(1))$. Our proof has the advantage of being purely local, and it does not involve any calculations with Gauss sums.

In the case $n = 2$, our results (almost) specialize to those of [15], except for the fact that things in [15] are expressed in terms of the similitude group $GU(2)$. Again, the methods of the present paper yield new proofs, although we do not prove the comparison conjecture for $n = 2$.

Finally, it should be noted that the global results of Rogawski [42], Gelbart-Rogawski [6] for $n = 1$ and of Harris [15, 16] for $n = 2$, as well as the global characterization of ‘early’ theta lifts via poles in the case of (Sp, O) dual pairs [23], should have generalizations to unitary groups for all n , by methods which are by now standard, based on the local results of this paper and a suitable generalization of the Siegel-Weil formula [23] to the unitary case.

We now give a section by section summary. One of the main technical difficulties involved in the study of the theta correspondence for unitary groups is the non-uniqueness of the splitting homomorphism (0.2). In particular, one has to keep careful track of the behavior of the splitting when restricting to subgroups, e.g., in a seesaw construction. In section 1 and in the appendix, we describe, in (1.16) and the discussion preceding it, how to associate a splitting to a pair of characters, as above, and we make a fairly thorough analysis of the compatibility of the splittings, so defined, with restrictions to subgroups, direct sums, etc. These results, which are based on [22], are essential to what follows and will be fundamental in future applications as well. In section 2, we consider the Weil representation defined using the splitting homomorphism defined by χ . We determine its behavior under pull-back via doubling and under the Mœglin-Vignéras-Waldsurger (MVW) involution (Proposition 2.2). If $W, \langle \cdot, \cdot \rangle$ is a skew-Hermitian space, as above, let $W_- = W$

with skew-Hermitian form $-\langle \cdot, \cdot \rangle$, and let

$$(0.20) \quad i : G(W) \times G(W) = G(W) \times G(W_-) \longrightarrow G(W + W_-)$$

be the natural homomorphism. For a Hermitian space V , and choice of χ_1 , we have Weil representations $\omega_{V,W,\chi}$ of $G(V) \times G(W)$ and $\omega_{V,W+W_-,\chi}$ of $G(V) \times G(W + W_-)$, and the latter can be realized on the Schwartz space $S(V^n)$ with $G(V)$ acting by the twist by χ of its natural linear action. Let $R_n(V, \chi)$ be the maximal quotient of $S(V^n)$ on which $G(V)$ acts by a multiple of χ . This representation of $G(W + W_-) \simeq U(n, n)$ is a submodule of a certain degenerate principal series representation $I_n(s, \chi)$, the structure of which was completely determined in [25]. For example, if $m = n$,

$$(0.21) \quad I_n(0, \chi) = R_n(V_n^+, \chi) \oplus R_n(V_n^-, \chi),$$

where V_n^\pm is as above. The relevant results of [25] are summarized in Proposition 4.1. In section 3, we prove that for an irreducible admissible representation π of $G(W)$,

$$(0.22) \quad \Theta_\chi(\pi, V) \neq 0 \iff \text{Hom}_{G(W) \times G(W)}(R_n(V, \chi), \pi \otimes (\chi\pi^\vee)) \neq 0,$$

(Proposition 3.1). The existence of this sort of relation is due to Rallis [36]. Next, in section 4, we prove:

Key Theorem (Theorem 4.3). (i) *If π is any irreducible admissible representation of $G(W)$, then, for all s ,*

$$(0.23) \quad \dim \text{Hom}_{G(W) \times G(W)}(I_n(s, \chi), \pi \otimes (\chi\pi^\vee)) \geq 1.$$

(ii) *If π is supercuspidal, then, for all s ,*

$$(0.24) \quad \dim \text{Hom}_{G(W) \times G(W)}(I_n(s, \chi), \pi \otimes (\chi\pi^\vee)) = 1.$$

Note that it is proved in [32] that, for any π , the dimension of this space of homomorphisms is at most 1 for almost all s . Theta dichotomy for supercuspidals follows immediately from the combination of (0.21), (0.22), and (0.24) at $s = 0$. It has long been held by Rallis and the second author that (0.24) should hold for all π , and this would imply theta dichotomy in general. In fact, we prove (0.24) at $s = 0$ for ‘most’ representations (Theorem 4.3), and hence obtain theta dichotomy for these (Corollary 4.4).

In section 5, we discuss the chains of supercuspidals in detail.

In section 6, we first recall the construction of L and ϵ -factors by the Piatetski-Shapiro, Rallis doubling method of [31], [32], [33]. The main new ingredient here is an explicit calculation (Proposition 6.3) of the normalizing factor for the intertwining operator $M(s, \chi)$, (6.1), which occurs in the functional equation, (iv) of Proposition 6.6 and (6.18), of the family of zeta integrals. This calculation, which is based on the technique of [44], allows us to give a precise definition of the normalized Γ factor $\Gamma^*(s, \pi, \chi, \psi)$ (Proposition 6.6), and hence of the ϵ -factor, (6.19). It should be noted that, when $n = \dim_E W$ is odd, the factor $\epsilon(s, \pi, \chi, \psi) = \epsilon_{PSR}(s, \pi, \chi, \psi; W)$ depends on W (Proposition 6.11). Note that the normalized intertwining operator satisfies the functional equation

$$(0.25) \quad M^*(-s, \check{\chi}) \circ M^*(s, \chi) = 1.$$

The key idea in the proof of epsilon dichotomy is then that, for a supercuspidal representation π of $G(W)$, the unique intertwining map of (0.24), at $s = 0$, is supported on the summand in (0.21) on which the sign of the normalized intertwining operator $M^*(0, \chi)$ agrees with the root number $\epsilon_{PSR}(\frac{1}{2}, \pi, \chi, \psi; W)$. Note that, by (0.25), $M^*(0, \chi)^2 = 1$. The eigenvalue of $M^*(0, \chi)$ on the summand $R_n(V, \chi)$ is given in Proposition 6.8. Finally, for supercuspidal representations, we establish the relation between poles of the L-factor $L_{PSR}(s, \pi, \chi)$ and early local theta lifts (Theorem 6.2) described above.

In section 7, we attempt to give a picture of the behavior of L-packets under the local theta correspondence. Our results on dichotomy and poles of L-factors provide some important clues, and we make a series of speculations (Speculations 7.3–7.7) about how the local theta correspondence for unitary groups should look. Among other things, the validity of these speculations would show that theta dichotomy *forces* L-packets to be broken up in the correspondence (failure of functoriality), but that this phenomenon takes place in a precisely controlled way which is compatible with Adams' conjectures about the preservation of A-packets [1]. Some of the speculations may be accessible at the moment, while others assume the local Langlands conjecture for general unitary groups!

In section 8, we give a rather self contained treatment of the case $n = 1$, a case which is not so trivial, as several authors have observed. There we prove $n = 1$ versions of the key comparison results (0.18) and (0.19) for L and ϵ -factors. There is some chance that our method can be extended to arbitrary n , but some new ideas will be needed.

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Notation. Let F be a non-archimedean local field of characteristic 0 with ring of integers \mathcal{O}_F and maximal ideal \mathcal{P} . Let E/F be a quadratic extension with associated quadratic character $\epsilon_{E/F}$. Let $\tau : x \mapsto \bar{x}$ denote the non-trivial Galois automorphism of E over F . We will sometimes write $\bar{x} = x^\tau$. Let $N_{E/F}$ and $tr_{E/F}$ be the norm and trace maps from E to F . We choose an element $\delta \in E^\times$ such that $\bar{\delta} = -\delta$. Then $\delta^2 = \Delta \in F^\times$ and $\epsilon_{E/F}(x) = (x, \Delta)_F$ is the quadratic character of F^\times associated to E/F by local class field theory. Here $(x, y)_F$ is the Hilbert symbol

for F . We let E^1 be the kernel of the norm map and recall that the sequence

$$(n.0) \quad 1 \longrightarrow F^\times \longrightarrow E^\times \longrightarrow E^1 \longrightarrow 1$$

$$x \mapsto x/\bar{x}$$

is exact. If μ is a character of E^1 we let

$$(n.1) \quad \tilde{\mu}(x) = \mu(x/\bar{x}).$$

Similarly, if χ is a character of E^\times , we will sometimes write χ^o for its restriction to F^\times .

Fix, once and for all, a non-trivial additive character ψ of F and extend it to E via the trace. We will let $n(\psi)$ be the largest integer n such that ψ is trivial on \mathcal{P}^{-n} . We say that ψ is unramified if $n(\psi) = 0$. Also, for $a \in F^\times$, let $\psi_a(x) = \psi(ax)$.

We will usually write V (resp. W) for an E vector space with a Hermitian (resp. skew-Hermitian) form $(\ , \)$ (resp. $\langle \ , \ \rangle$). All such forms are assumed to be non-degenerate. We take V to be a right E vector space, with $\dim_E V = m$, and hence have

$$(n.2) \quad (v_1a, v_2b) = \bar{a}(v_1, v_2)b,$$

for $a, b \in E$. Similarly, W will be a left E vector space, with $\dim_E W = n$, and

$$(n.3) \quad \langle aw_1, bw_2 \rangle = a\langle w_1, w_2 \rangle \bar{b}.$$

These conventions come into play in the definitions of tensor products. Also, if W is a given skew-Hermitian space and if δ is chosen as above, then we can define a Hermitian space W' by $W' = W$ (viewed as a right vector space via $wa = aw$) and

$$(n.4) \quad (w_1, w_2) = \delta^{-1}\langle w_2, w_1 \rangle.$$

Thus there will be a symmetry to many of our constructions, but there may be a dependence on the choice of δ .

For a Hermitian space V or skew-Hermitian space W , $G(V)$ and $G(W)$ will denote the isometry groups. Note that, if $a \cdot V$, $a \in F^\times$ is the space V with Hermitian form $a(\ , \)$, then $G(a \cdot V) = G(V)$.

For a fixed dimension m , there are two isomorphism classes of non-degenerate Hermitian forms over E , and these are distinguished by the quantity $\det V \in F^\times/NE^\times$. If $m = \dim_E V = 1$, the two classes of Hermitian forms are given by $V_1^\pm = E$ with $(x, y)_+ = \bar{x}y$ and $(x, y)_- = a\bar{x}y$, where $a \in F^\times - NE^\times$, i.e., $\epsilon_{E/F}(a) = -1$. In general, for $m = 2r + 1$ odd, the two classes of Hermitian forms are given by

$$(n.5) \quad V_m^\pm = V_1^\pm + V_{r,r},$$

where $V_{r,r}$ is the Hermitian space of dimension $2r$ with Hermitian form given by

$$(n.6) \quad \begin{pmatrix} 0 & 1_r \\ 1_r & 0 \end{pmatrix},$$

for a suitable choice of basis. Note that

$$(n.7) \quad \epsilon_{E/F}((-1)^{\frac{m(m-1)}{2}} \det V_m^\pm) = \pm 1.$$

Also, if $a \in F^\times$ with $\epsilon_{E/F}(a) = -1$, then

$$(n.8) \quad a \cdot V_{2r+1}^\pm \simeq V_{2r+1}^\mp.$$

If m is even, we again denote representatives of the two classes of Hermitian forms by V_m^+ and V_m^- , where the sign \pm is determined by (n.7). Note that $V_{2r}^+ = V_{r,r}$; we refer to this space as the split space. If E/F is unramified, we refer to the space V_m^+ as the unramified Hermitian space of dimension m .

For a fixed choice of δ , we can transfer this classification to skew-Hermitian spaces by setting $W = W_n^\pm$ if $W' = V_n^\pm$. If $n = \dim_E W$ is even, the resulting notation is independent of δ and W_{2r}^+ , the split space, has matrix

$$(n.9) \quad \begin{pmatrix} 0 & 1_r \\ -1_r & 0 \end{pmatrix},$$

for a suitable choice of basis. On the other hand, if n is odd, then the space $W = W_n^+$ depends on the choice of δ , via (n.8)! This fact must be reflected in our formulas.

The groups $G(V_m^+)$ and $G(W_n^+)$ are quasisplit for all m and n , while the groups $G(V_{2r}^-)$ and $G(W_{2r}^-)$ are not.

For an admissible representation σ of a p-adic group G , we let σ^\vee denote the contragredient. Our induced representations are always normalized, i.e. are (pre-)unitary for (pre-)unitary induction data.

We write μ_n for the group of n -th roots of unity.

For any non-trivial additive character η of F and $a \in F^\times$, we will let

$$(n.10) \quad \gamma_F(a, \eta) = \frac{\gamma_F(\eta_a)}{\gamma_F(\eta)} \in \mu_4,$$

where $\gamma_F(\eta) \in \mu_8$ is the Weil index of the character of second degree $x \mapsto \eta(x^2)$. Here $(\eta_a)(x) = \eta(ax)$. Recall that

$$(n.11) \quad \gamma_F(ab, \eta) = (a, b)_F \gamma_F(a, \eta) \gamma_F(b, \eta).$$

§1. COCYCLES AND SPLITTINGS

In this section we record some essential facts concerning the restriction of the metaplectic covering to a dual pair of unitary groups. Of particular importance (and delicacy) is the behavior of these splittings for the seesaw dual pairs of the doubling construction.

Let $W, \langle \cdot, \cdot \rangle$ (resp. $V, (\cdot, \cdot)$) be a non-degenerate skew-Hermitian (resp. Hermitian) space over E with $\dim_E W = n$ (resp. $\dim_E V = m$). Let

$$(1.1) \quad \mathbb{W} = V \otimes_E W$$

with the symplectic form

$$(1.2) \quad \langle\langle \cdot, \cdot \rangle\rangle = \frac{1}{2} \text{tr}_{E/F}(\langle \cdot, \cdot \rangle \otimes \langle \cdot, \cdot \rangle^\tau),$$

as usual. Let

$$(1.3) \quad 1 \longrightarrow \mathbb{C}^1 \longrightarrow Mp(\mathbb{W}) \longrightarrow Sp(\mathbb{W}) \longrightarrow 1$$

denote the metaplectic cover of $Sp(\mathbb{W})$, which we view as an extension of $Sp(\mathbb{W})$ by \mathbb{C}^1 , in order to have more room to modify our cocycles (cf. [22]). It should be noted that, in contrast to the usual twofold covering, this group now has non-trivial characters.

We want to construct the Weil representation for the dual reductive pair $U(V), U(W)$ in $Sp(\mathbb{W})$, and, in particular, to be careful about the dependence of this representation on a choice of a splitting of the restriction of the metaplectic cover $Mp(\mathbb{W}) \rightarrow Sp(\mathbb{W})$ to $U(V)$ and to $U(W)$. For convenience, we write $G(V) = U(V)$ and $G(W) = U(W)$, etc. for the unitary groups involved.

We will begin with the group $G(W)$. For a fixed Hermitian space V , let

$$(1.4) \quad \iota_V : G(W) \longrightarrow Sp(\mathbb{W})$$

be the natural homomorphism. If $\dim_E V = m$, we choose a character $\chi = \chi_V$ of E^\times such that

$$(1.5) \quad \chi_V|_{F^\times} = \epsilon_{E/F}^m,$$

where $\epsilon_{E/F}$ is the quadratic character associated to the extension E/F . This choice determines a lift

$$(1.6) \quad \begin{array}{ccc} & Mp(\mathbb{W}) & \\ & \nearrow \tilde{\iota}_{V, \chi_V} & \downarrow \\ G(W) & \xrightarrow{\iota_V} & Sp(\mathbb{W}) \end{array}$$

by a procedure which we will review in a moment. Note that two choices χ_1 and χ_2 of χ_V differ by a character μ of E^\times which is trivial on F^\times , i.e., $\chi_2 = \mu\chi_1$. Then μ defines a character μ' on E^1 by

$$(1.7) \quad \mu' \left(\frac{x}{\bar{x}} \right) = \mu(x),$$

and we have

$$(1.8) \quad \tilde{\iota}_{V, \chi_2} = (\mu' \circ \det) \cdot \tilde{\iota}_{V, \chi_1},$$

where $\mu' \circ \det$ takes values in the central \mathbb{C}^1 in $Mp(\mathbb{W})$.

We now describe the construction of $\tilde{\iota}_{V, \chi}$, following [22]. Let W_- denote the space W with the skew-Hermitian form $-\langle \cdot, \cdot \rangle$. Note that the identity map on W gives an anti-isometry $W = W_-$. Then let

$$(1.9) \quad \mathbb{W}_- = V \otimes_E W_-$$

so that $\mathbb{W} = \mathbb{W}_-$ (an anti-isometry). Let ι_V be the natural embedding of $G(W)$ into $Sp(\mathbb{W})$. We then obtain a commutative diagram

$$(1.10) \quad \begin{array}{ccc} G(W + W_-) & \xrightarrow{\iota_V} & Sp(\mathbb{W} + \mathbb{W}_-) \\ \uparrow i & & \uparrow i \\ G(W) \times G(W) & \xrightarrow{\iota_V \times \iota_V} & Sp(\mathbb{W}) \times Sp(\mathbb{W}) \\ \uparrow \Delta & & \uparrow \Delta \\ G(W) & \longrightarrow & Sp(\mathbb{W}) \end{array}$$

Here we identify $G(W)$ and $G(W_-)$ (resp. $Sp(\mathbb{W})$ and $Sp(\mathbb{W}_-)$) via the identity (anti-isometry) of W and W_- (resp. \mathbb{W} and \mathbb{W}_-). The metaplectic covering $Mp(\mathbb{W} + \mathbb{W}_-)$ splits over each of these subgroups [22], and we want to determine the compatibilities among the splittings.

Let

$$(1.11) \quad W + W_- = X + Y$$

be the polarization of the split skew-Hermitian space $W + W_-$ where X is the graph of minus the identity map from W to W , and Y is the graph of the identity map. This choice of polarization induces a polarization $\mathbb{X} + \mathbb{Y}$ of $\mathbb{W} + \mathbb{W}_-$, with $\mathbb{X} = V \otimes_E X$ and $\mathbb{Y} = V \otimes_E Y$.

We fix a non-trivial additive character ψ of F . The choice of maximal isotropic subspace \mathbb{Y} in $\mathbb{W} + \mathbb{W}_-$ then determines an isomorphism

$$(1.12) \quad Mp(\mathbb{W} + \mathbb{W}_-) \simeq Sp(\mathbb{W} + \mathbb{W}_-) \times \mathbb{C}^1$$

as sets [40]. Here the multiplication on the right hand side is described by the cocycle $c_{\mathbb{Y}}(g_1, g_2)$ given by the Leray invariant [22], [40]

$$(1.13) \quad c_{\mathbb{Y}}(g_1, g_2) = \gamma_F(\eta \circ L(\mathbb{Y}, \mathbb{Y}g_2^{-1}, \mathbb{Y}g_1)).$$

Here $\eta = \frac{1}{2}\psi$, as in [22]. Moreover, having chosen the character χ with $\chi|_{F^\times} = \epsilon_{E/F}^m$, $m = \dim_E V$, as above, we also obtain an explicit homomorphism [22]

$$(1.14) \quad \begin{aligned} \tilde{\iota}_{V,\chi} : G(W + W_-) &\longrightarrow Mp(\mathbb{W} + \mathbb{W}_-) \simeq Sp(\mathbb{W} + \mathbb{W}_-) \times \mathbb{C}^1 \\ g &\mapsto \tilde{\iota}_{V,\chi}(g) \simeq (\iota_V(g), \beta_{V,\chi}(g)), \end{aligned}$$

lifting ι_V . Here

$$(1.15) \quad \beta_{V,\chi}(g) = \chi_V(x(g)) \gamma_F(\eta \circ RV)^{-j(g)},$$

where the notation is as in [22]. In particular,

$$(1.16) \quad \gamma_F(\eta \circ RV) = (\Delta, \det V)_F \gamma_F(-\Delta, \eta)^m \gamma_F(-1, \eta)^{-m}.$$

Now the inverse image in $Mp(\mathbb{W} + \mathbb{W}_-)$ of the image of $Sp(\mathbb{W}) \times 1$ (resp. $1 \times Sp(\mathbb{W})$) in $Sp(\mathbb{W} + \mathbb{W}_-)$ is isomorphic to $Mp(\mathbb{W})$. This isomorphism is canonical if we specify its restriction to the central \mathbb{C}^1 . We choose a lift \tilde{i} of i so that

$$(1.17) \quad \tilde{i} : Mp(\mathbb{W}) \times Mp(\mathbb{W}) \longrightarrow Mp(\mathbb{W} + \mathbb{W}_-)$$

restricts to the map

$$(1.18) \quad \begin{aligned} \mathbb{C}^1 \times \mathbb{C}^1 &\longrightarrow \mathbb{C}^1, \\ (\epsilon_1, \epsilon_2) &\mapsto \epsilon_1 \bar{\epsilon}_2 \end{aligned}$$

on the central \mathbb{C}^1 's.

Then the restriction of $\tilde{\iota}_{V,\chi}$ to $G(W) \times 1$ yields a splitting homomorphism

$$(1.19) \quad \tilde{\iota}_{V,\chi} : G(W) \longrightarrow Mp(\mathbb{W}),$$

lifting ι_V . Similarly, the restriction of $\tilde{\iota}_{V,\chi}$ to $1 \times G(W)$ yields a splitting homomorphism

$$(1.20) \quad \tilde{\iota}_{V,\chi,-} : G(W) \longrightarrow Mp(\mathbb{W}).$$

Thus we have a diagram

$$(1.21) \quad \begin{array}{ccc} G(W + W_-) & \xrightarrow{\tilde{\iota}_{V,\chi}} & Mp(\mathbb{W} + \mathbb{W}_-) \\ \uparrow i & & \uparrow \tilde{i} \\ G(W) \times G(W) & \xrightarrow{\tilde{\iota}_{V,\chi} \times \iota_{V,\chi,-}} & Mp(\mathbb{W}) \times Mp(\mathbb{W}) \end{array}$$

lifting the upper part of (1.10).

We first determine the relation between $\tilde{\iota}_{V,\chi}$ and $\tilde{\iota}_{V,\chi,-}$.

Lemma 1.1. $\tilde{\iota}_{V,\chi,-} = \chi^{-1} \cdot \tilde{\iota}_{V,\chi}$.

Proof. Let

$$(1.22) \quad c : \mathbb{W} + \mathbb{W}_- \longrightarrow \mathbb{W} + \mathbb{W}_-$$

be the anti-isometry induced by switching W and W_- , i.e., sending the pair $(w_1, w_2) \in W \oplus W_-$ to $(w_2, w_1) \in W \oplus W_-$. This map acts by $+1$ on \mathbb{Y} and by -1 on \mathbb{X} . Thus $-c$ has matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on $\mathbb{X} + \mathbb{Y}$. Conjugation by $-c$ induces an automorphism sw of $Sp(\mathbb{W} + \mathbb{W}_-)$ such that the diagram

$$(1.23) \quad \begin{array}{ccc} Sp(\mathbb{W} + \mathbb{W}_-) & \xrightarrow{sw} & Sp(\mathbb{W} + \mathbb{W}_-) \\ \uparrow i & & \uparrow i \\ Sp(\mathbb{W}) \times Sp(\mathbb{W}) & \xrightarrow{\text{switch}} & Sp(\mathbb{W}) \times Sp(\mathbb{W}) \\ \uparrow \iota_V \times \iota_V & & \uparrow \iota_V \times \iota_V \\ G(W) \times G(W) & \xrightarrow{\text{switch}} & G(W) \times G(W) \end{array}$$

commutes. □

The following result is easily checked.

Lemma 1.2. *There is a unique lift \tilde{sw} , which restricts to complex conjugation on the central \mathbb{C}^1 , of the automorphism sw of $Sp(\mathbb{W} + \mathbb{W}_-)$ given by conjugation by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. For the isomorphism*

$$Mp(\mathbb{W} + \mathbb{W}_-) \simeq Sp(\mathbb{W} + \mathbb{W}_-) \times \mathbb{C}^1$$

determined by \mathbb{Y} , \tilde{sw} is given by:

$$(g, \epsilon) \mapsto (Ad(c)g, \bar{\epsilon}).$$

Moreover, the diagram

$$\begin{array}{ccc}
 Mp(\mathbb{W} + \mathbb{W}_-) & \xrightarrow{s\tilde{w}} & Mp(\mathbb{W} + \mathbb{W}_-) \\
 \uparrow \tilde{i} & & \uparrow \tilde{i} \\
 Mp(\mathbb{W}) \times Mp(\mathbb{W}) & \xrightarrow{switch} & Mp(\mathbb{W}) \times Mp(\mathbb{W}) \\
 \uparrow \tilde{\iota}_{V,\chi} \times \tilde{\iota}_{V,\chi,-} & & \uparrow \tilde{\iota}_{V,\chi} \times \tilde{\iota}_{V,\chi,-} \\
 G(W) \times G(W) & \xrightarrow{switch} & G(W) \times G(W)
 \end{array}$$

(1.24)

commutes.

For $g \in G(W)$, we now compute:

$$\begin{aligned}
 s\tilde{w} \circ \tilde{i}(1, \tilde{\iota}_{V,\chi,-}(g)) &= s\tilde{w} \circ \tilde{\iota}_{V,\chi}(i(1, g)) \\
 &= s\tilde{w} \left(\iota_V(i(1, g)), \beta_{V,\chi}(i(1, g)) \right) \\
 &= (\iota_V(i(g, 1)), \overline{\beta_{V,\chi}(i(1, g))}).
 \end{aligned}$$

(1.25)

We then use the following relations satisfied by the basic functions j and x of [22]:

$$j(g) := j(i(g, 1)) = j(i(1, g)),$$

(1.26)

$$x(i(1, g)) = (-1)^{j(g)} x(i(g, 1)),$$

(1.27)

and

$$\beta_{V,\chi}(i(1, g)) = \chi_V(-1)^{j(g)} \beta_{V,\chi}(i(g, 1)).$$

(1.28)

On the other hand,

$$\overline{\beta_{V,\chi}(g)} = \chi_V(-1)^{j(g)} \beta_{V,\chi^{-1}}(g) = \chi_V(-1)^{j(g)} \chi_V(\det g)^{-1} \beta_{V,\chi}(g).$$

(1.29)

Continuing our calculation, we have:

$$(\iota_V(i(g, 1)), \overline{\beta_{V,\chi}(i(1, g))}) = (\iota_V(i(g, 1)), \chi(\det g)^{-1} \beta_{V,\chi}(i(g, 1))),$$

(1.30)

as claimed. □

The following result will also play an important role below.

Lemma 1.3. *Let $P_{\mathbb{Y}}$ be the maximal parabolic subgroup of $Sp(\mathbb{W} + \mathbb{W}_-)$ which preserves \mathbb{Y} , and let $M_{\mathbb{Y}}$ be the Levi factor of $P_{\mathbb{Y}}$ which preserves the complete polarization*

$$\mathbb{W} + \mathbb{W}_- = \mathbb{X} + \mathbb{Y}.$$

Then, for $g \in G(W)$, $\iota_V \circ i \circ \Delta(g) \in M_{\mathbb{Y}}$, and

$$\tilde{\iota}_{V,\chi}(i(\Delta(g))) = \left(\left(\begin{array}{cc} 1_V \otimes {}^t \bar{g}^{-1} & 0 \\ 0 & 1_V \otimes g \end{array} \right), \chi_V(\det g) \right) \in Sp(\mathbb{W} + \mathbb{W}_-) \times \mathbb{C}^1.$$

Proof. It is clear that $\iota_V(i(\Delta(g)))$ preserves the complete polarization, and that it acts on $\mathbb{Y} = V \otimes \Delta(W)$ by $1_V \otimes g$. The element $i(\Delta(g)) \in G(W + W_-)$ similarly preserves Y and acts on Y by g . Thus $j(i(\Delta(g))) = 0$ and $x(g) = \det(g)$. Thus

$$(1.31) \quad \beta_{V,\chi}(i(\Delta(g))) = \chi_V(\det(g)),$$

as claimed. □

Finally, we need to understand the effect of the involutions considered by Moeglin-Vigneras-Waldspurger [29] (the MVW involutions). Let $g_0 \in GL_F(W)$ and $h_0 \in GL_F(V)$ be τ -linear automorphisms such that

$$(1.32) \quad \langle w_1 g_0, w_2 g_0 \rangle = \langle w_2, w_1 \rangle, \quad \text{for all } w_1 \text{ and } w_2 \in W$$

and

$$(1.33) \quad (h_0 v_1, h_0 v_2) = (v_2, v_1), \quad \text{for all } v_1 \text{ and } v_2 \in V.$$

Note that, since g_0 and h_0 are both τ -linear, $h_0 \otimes g_0 \in GL_F(\mathbb{W})$ is well defined, and that

$$(1.34) \quad \langle (h_0 \otimes g_0)z_1, (h_0 \otimes g_0)z_2 \rangle = \langle z_2, z_1 \rangle = -\langle z_1, z_2 \rangle$$

for all $z_1, z_2 \in \mathbb{W}$, so that $h_0 \otimes g_0$ is an element of $GS p(\mathbb{W})$ with scale -1 . Let $A : Sp(\mathbb{W}) \rightarrow Sp(\mathbb{W})$ be the outer automorphism $A = Ad(h_0 \otimes g_0)$. Note that there is a unique lift \tilde{A} of A to $Mp(\mathbb{W})$ which is anti-linear on \mathbb{C}^1 .

Lemma 1.4. *The following diagram commutes:*

$$\begin{array}{ccc} G(W) & \xrightarrow{\tilde{\iota}_{V,\chi}} & Mp(\mathbb{W}) \\ Ad(g_0) \downarrow & & \downarrow \tilde{A} \\ G(W) & \xrightarrow{\tilde{\iota}_{V,\chi}} & Mp(\mathbb{W}) \end{array}$$

Proof. We begin by doubling the diagram of the Lemma, writing $U = W + W_-$ and $\mathbb{U} = \mathbb{W} + \mathbb{W}_-$,

$$(1.35) \quad \begin{array}{ccccc} G(W) \times G(W) & \xrightarrow{i} & G(U) & \longrightarrow & Mp(\mathbb{U}) \\ A_0 \times A_0 \downarrow & & B_0 \downarrow & & \tilde{B} \downarrow \\ G(W) \times G(W) & \xrightarrow{i} & G(U) & \longrightarrow & Mp(\mathbb{U}). \end{array}$$

Here $B_0 = Ad(g_0 \times g_0)$ and $B = Ad(h_0 \otimes (g_0 \times g_0))$. It suffices to prove that the right hand square in this diagram commutes. Moreover, since the anti-linear maps $\gamma : U \rightarrow U$ such that $\langle u_1 \gamma, u_2 \gamma \rangle = \langle u_2, u_1 \rangle$ form a single $G(U)$ coset, it suffices to check the commutativity for a single such γ . If $e_1, \dots, e_n, e'_1, \dots, e'_n$ is a standard ‘symplectic’ basis for U , let γ be the E anti-linear isomorphism such that $e_i \gamma = e_i$ and $e'_i \gamma = -e'_i$ for all i . Let Y be the E span of the e'_i ’s, $\mathbb{Y} = V \otimes Y$, and let

$$(1.36) \quad Mp(\mathbb{U}) \simeq Sp(\mathbb{U}) \times \mathbb{C}^1$$

be the associated isomorphism. Let $C_0 = Ad(\gamma)$ on $G(U)$, and let $C = Ad(h_0 \otimes \gamma)$ on $Sp(\mathbb{U})$. Then, since $h_0 \otimes \gamma$ preserves \mathbb{Y} , the lift \tilde{C} of C to $Mp(\mathbb{U})$ is given by

$$(1.37) \quad (g, \epsilon) \mapsto (C \cdot g, \bar{\epsilon}),$$

while the map $\tilde{\iota}_{V,\chi}$ is given by $g \mapsto (\iota_V(g), \beta_{V,\chi}(g))$. Now, if we let $c : U \rightarrow U$ be the E -linear similitude whose action on the standard basis is the same as that of γ , then, for $g \in G(U)$,

$$(1.38) \quad C_0 \cdot g = Ad(c) \cdot \bar{g}.$$

Now

$$(1.39) \quad \begin{aligned} x(\bar{g}) &= \overline{x(g)}, \\ x(Ad(c) \cdot g) &= (-1)^{j(g)} x(g), \end{aligned}$$

and $j(\bar{g}) = j(Ad(c) \cdot g) = j(g)$. Thus

$$(1.40) \quad \begin{aligned} \beta_{V,\chi}(C_0 \cdot g) &= \chi(x(Ad(c) \cdot \bar{g})) \gamma_F(\eta \circ RV)^{-j(g)} \\ &= \chi(-1)^{j(g)} \chi(\overline{x(g)}) \gamma_F(\eta \circ RV)^{-j(g)} \\ &= \chi(-1)^{j(g)} \chi(x(g))^{-1} \gamma_F(\eta \circ RV)^{-j(g)}. \end{aligned}$$

Since

$$(1.41) \quad \gamma_F(\eta \circ RV) = (\Delta, \det V)_F \gamma_F(-\Delta, \eta)^m \gamma_F(-1, \eta)^{-m},$$

we find that

$$(1.42) \quad \overline{\gamma_F(\eta \circ RV)} = (-1, \Delta)_F^m \gamma_F(\eta \circ RV).$$

Thus, since $\chi(-1)^{j(g)} = (-1, \Delta)_F^m \chi^{j(g)}$, we have

$$(1.43) \quad \beta_{V,\chi}(C_0 \cdot g) = \overline{\beta_{V,\chi}(g)}.$$

This proves the required commutativity. □

So far we have only discussed the splitting $\tilde{\iota}_{V,\chi}$ over $G(W)$, where W is the skew-Hermitian space in the pair (V, W) . To obtain a splitting over $G(V)$, we use the following construction. Recall that we have fixed δ with $\bar{\delta} = -\delta$. Let $W' = W$ with *Hermitian* form

$$(1.44) \quad (w'_1, w'_2)' = \delta^{-1} \langle w_2, w_1 \rangle,$$

and let $V' = V$ with skew-Hermitian form

$$(1.45) \quad \langle v'_1, v'_2 \rangle = \delta \langle v_2, v_1 \rangle.$$

Here, for $w \in W$ (resp. $v \in V$), we let w' (resp. v') denote the corresponding element of W' (resp. V'). Applying the previous constructions to the pair (W', V') , we obtain a splitting

$$(1.46) \quad \tilde{\iota}_{W',\chi'} : G(V) = G(V') \rightarrow Mp(\mathbb{V}'),$$

where $\mathbb{V}' = W' \otimes_E V'$, and where χ' is a character of E^\times whose restriction to F^\times is $\epsilon_{E/F}^n$, for $n = \dim_E W$. Note that the map $v \otimes w \mapsto w' \otimes v'$ defines an isometry $\mathbb{W} \rightarrow \mathbb{V}'$, since

$$(1.47) \quad \begin{aligned} (w'_1, w'_2)' \overline{\langle v'_1, v'_2 \rangle}' &= \delta^{-1} \langle w_2, w_1 \rangle \overline{\delta \langle v_2, v_1 \rangle} \\ &= (v_1, v_2) \overline{\langle w_1, w_2 \rangle}. \end{aligned}$$

Thus we have an isomorphism $\alpha : Mp(\mathbb{V}') \rightarrow Mp(\mathbb{W})$ (restricting to the identity map on \mathbb{C}^1), and we define

$$(1.48) \quad \tilde{t}_{W, \chi'} = \alpha \circ \tilde{t}_{W', \chi'} : G(V) \rightarrow Mp(\mathbb{W}).$$

The various results of this section carry over to $\tilde{t}_{W, \chi'}$.

§2. WEIL REPRESENTATIONS

We may now define the Weil representations of the dual pairs $(G(V), G(W))$, $(G(W + W_-), G(V))$, etc., and we determine their relations via the doubling seesaw.

For our fixed additive character ψ of F , let (ω_ψ, S) be the Weil representation of $Mp(\mathbb{W})$ or of $Mp(\mathbb{W} + \mathbb{W}_-)$. In the latter case, we consider the standard Schrödinger model associated to the complete polarization $\mathbb{X} + \mathbb{Y}$, and so we take $S = S(\mathbb{X}) = S(V^n)$. Here the identification of $\mathbb{X} = V \otimes X$ is obtained from a standard E -basis for $W + W_-$ (cf. [22], sections 4 and 5). Note that, in both cases, the central \mathbb{C}^1 acts by its holomorphic identity character, i.e.,

$$(2.1) \quad \omega_\psi(z) = \text{multiplication by } z.$$

We fix a pair of characters $\chi = (\chi_n, \chi_m)$ of E^\times with

$$(2.2) \quad \chi_m|_{F^\times} = \epsilon_{E/F}^m \quad \text{and} \quad \chi_n|_{F^\times} = \epsilon_{E/F}^n,$$

where $m = \dim_E V$ and $n = \dim_E W$. These characters define splitting homomorphisms

$$(2.3) \quad \begin{aligned} \tilde{t}_{V, \chi_m} : G(W) &\rightarrow Mp(\mathbb{W}), \\ \tilde{t}_{V, \chi_m} : G(W + W_-) &\rightarrow Mp(\mathbb{W} + \mathbb{W}_-), \end{aligned}$$

and

$$(2.4) \quad \begin{aligned} \tilde{t}_{W, \chi_n} : G(V) &\rightarrow Mp(\mathbb{W}), \\ \tilde{t}_{W, \chi_n} : G(V + V_-) &\rightarrow Mp(\mathbb{W} + \mathbb{W}_-). \end{aligned}$$

We will often write χ in place of χ_m or χ_n , since the choice of the relevant element of the pair should be clear from context. Using these, we define representations

$$(2.5) \quad \omega_{V, \chi} = \omega_\psi \circ \tilde{t}_{V, \chi}$$

of $G(W)$ or of $G(W + W_-)$ and

$$(2.6) \quad \omega_{W, \chi} = \omega_\psi \circ \tilde{t}_{W, \chi}$$

of $G(V)$ or of $G(V + V_-)$. Hopefully the relevant group will also be clear from the context.

The following facts are well known:

Lemma 2.1. (i) ([40]) *Under the homomorphism*

$$\tilde{i} : Mp(\mathbb{W}) \times Mp(\mathbb{W}) \longrightarrow Mp(\mathbb{W} + \mathbb{W}_-),$$

of (1.17),

$$\omega_\psi \circ \tilde{i} = \omega_\psi \otimes \omega_\psi^\vee,$$

where ω_ψ^\vee is the contragredient of ω_ψ .

(ii) ([29]) *If π is an irreducible admissible representation of $G(W)$, then*

$$\pi \circ Ad(g_0) \simeq \pi^\vee,$$

where $Ad(g_0)$ is as in (1.32). Similarly, if σ is an irreducible admissible representation of $G(V)$, then

$$\sigma \circ Ad(h_0) \simeq \sigma^\vee,$$

where h_0 is as in (1.33).

Remark. Here (ii) is a deep result of Mœglin-Vignéras-Waldspurger, in spite of the label here. Also, in (i), the anti-linearity (1.18) of \tilde{i} on the second central \mathbb{C}^1 is essential.

The results of section 1 now yield the following information on pullbacks:

Proposition 2.2. (i) *As representations of $G(W) \times G(W)$,*

$$\omega_{V,\chi} \circ i \simeq \omega_{V,\chi} \otimes (\chi_m \cdot (\omega_{V,\chi})^\vee),$$

and, as representations of $G(V) \times G(V)$,

$$\omega_{W,\chi} \circ i \simeq \omega_{W,\chi} \otimes (\chi_n \cdot (\omega_{W,\chi})^\vee).$$

(ii) *The representation $\omega_{V,\chi} \circ i \circ \Delta$ of $G(W)$ is isomorphic to the twist by χ_m of the linear action of $G(W)$ on $S(V^n)$, i.e., for $\varphi \in S(V^n)$ and $x \in V^n$,*

$$\omega_{V,\chi}(i(g, g))\varphi(x) = \chi_m(\det g) \cdot \varphi(x^t \bar{g}^{-1}).$$

Similarly, the representation $\omega_{W,\chi} \circ i \circ \Delta$ of $G(V)$ is isomorphic to the twist by χ_n of the linear action of $G(V)$ on $S(V^n)$, i.e., for $\varphi \in S(V^n)$ and $x \in V^n$,

$$\omega_{W,\chi}(i(h, h))\varphi(x) = \chi_n(\det h) \cdot \varphi(h^{-1}x).$$

(iii) *For the MVW involutions,*

$$\omega_{V,\chi} \circ Ad(g_0) \simeq (\omega_{V,\chi})^\vee \quad \text{and} \quad \omega_{W,\chi} \circ Ad(h_0) \simeq (\omega_{W,\chi})^\vee.$$

Proof. We have, for example,

$$\begin{aligned} \omega_{V,\chi} \circ i &= \omega_\psi \circ \tilde{l}_{V,\chi} \circ i \\ (2.7) \quad &= \omega_\psi \circ \tilde{i} \circ (\tilde{l}_{V,\chi} \times \tilde{l}_{V,\chi,-}) \\ &= \omega_\psi \circ \tilde{i} \circ (\tilde{l}_{V,\chi} \times \chi_m^{-1} \cdot \tilde{l}_{V,\chi}) \\ &= \omega_{V,\chi} \otimes (\chi_m \cdot (\omega_{V,\chi})^\vee), \end{aligned}$$

where we note that the anti-linear action of the central \mathbb{C}^1 for ω_ψ^\vee inverts the χ_m^{-1} in the last step.

To prove (iii), we consider the diagram of Lemma 1.4 and its analogue for $\tilde{\iota}_{W,\chi}$. Then

$$\begin{aligned}
 (2.8) \quad \omega_{V,\chi} \circ Ad(g_0) &= \omega_\psi \circ \tilde{\iota}_{V,\chi} \circ Ad(g_0) \\
 &= \omega_\psi \circ \tilde{A} \circ \tilde{\iota}_{V,\chi} \\
 &= (\omega_\psi)^\vee \circ \tilde{\iota}_{V,\chi} \\
 &= (\omega_{V,\chi})^\vee.
 \end{aligned}$$

□

§3. THE THETA CORRESPONDENCE

For convenience, we now write $(\omega_{V,W,\chi}, S_{V,W,\chi})$ for the Weil representation $\omega_{V,\chi} \otimes \omega_{W,\chi}$ on the space $S_{V,W,\chi}$. Note that the Weil representations $\omega_{V,W+W_-, \chi}$ of $(G(V), G(W + W_-))$ and $\omega_{V+V_-, W, \chi}$ of $(G(V + V_-), G(W))$ are both realized on the space $S(V^n)$.

If π is an irreducible admissible representation of $G(W)$, let $S_{V,W,\chi}(\pi)$ be the maximal quotient of $S_{V,W,\chi}$ on which $G(W)$ acts as a multiple of π . Write

$$(3.1) \quad S_{V,W,\chi}(\pi) \simeq \pi \otimes \Theta_\chi(\pi, V),$$

where $\Theta_\chi(\pi, V)$ is a representation of $G(V)$. We set $\Theta_\chi(\pi, V) = 0$ if π does not occur as a quotient of $S_{V,W,\chi}$. The Howe duality principle asserts that

- (i) $\Theta_\chi(\pi, V)$ is a finitely generated admissible representation of $G(V)$.
- (ii) $\Theta_\chi(\pi, V)$ has a unique proper maximal $G(V)$ -invariant subspace and a unique irreducible quotient $\theta_\chi(\pi, V)$.
- (iii) The correspondence $\pi \mapsto \theta_\chi(\pi, V)$ defines a bijection between the subsets of $Irr(G(W))$ and $Irr(G(V))$ consisting of irreducible admissible representations which occur as quotients of $S_{V,W,\chi}$.

This is known to hold if the residue characteristic of F is not 2 [50]. Here we only assume a consequence of (i):

- (i)' As a representation of $G(V)$, $\Theta_\chi(\pi, V)$ admits an irreducible quotient σ .

In fact, when π is supercuspidal, we will prove (Proposition 5.4, below) that (i)' is always satisfied.

Now consider the dual pair $(G(W + W_-), G(V))$, and note that, by Proposition 2.2, for the Weil representation $\omega_{V,W+W_-, \chi}$ described there, the group $G(V)$ acts via the twist by χ_n of its linear action in $S(V^n)$. Let

$$(3.2) \quad R_n(V, \chi_m) = S(V^n)(\chi_n) = \Theta_\chi(\chi_n, W + W_-)$$

be the maximal quotient on which $G(V)$ acts by the character χ_n . Alternatively, $R_n(V, \chi_m)$ is the maximal quotient of $S(V^n)$, with the linear action of $G(V)$, on which $G(V)$ acts trivially. Note that $R_n(V, \chi_m)$ is known [25] to be an admissible representation of $G(W + W_-)$ of finite length and with a unique irreducible quotient in all cases, i.e., for any residue characteristic. Observe that it is the character χ_m , with $\chi_m|_{F^\times} = \epsilon_{E/F}^m$, $m = \dim_E V$, which determines the action of $G(W + W_-)$ in this space. We will usually abuse notation and write $R_n(V, \chi)$ for this representation.

A key result is then the following:

Proposition 3.1. *Assume (i)'. Then, for any irreducible admissible representation π of $G(W)$,*

$$\Theta_\chi(\pi, V) \neq 0 \iff \text{Hom}_{G(W) \times G(W)}(R_n(V, \chi), \pi \otimes (\chi \cdot \pi^\vee)) \neq 0.$$

Proof. By definition, we have

$$(3.3) \quad \Theta_\chi(\pi, V) \neq 0 \iff \text{Hom}_{G(W)}(\omega_{V,W,\chi}, \pi) \neq 0,$$

and so, since we assume (i)',

$$(3.4) \quad \Theta_\chi(\pi, V) \neq 0 \iff \text{Hom}_{G(W) \times G(V)}(\omega_{V,W,\chi}, \pi \otimes \sigma) \neq 0,$$

for some irreducible admissible representation σ of $G(V)$. If $\lambda : \omega_{V,W,\chi} \rightarrow \pi \otimes \sigma$ is a non-zero intertwining map, then, applying the MVW involution to both factors, we have a non-zero $\lambda^0 : \omega_{V,W,\chi}^\vee \rightarrow \pi^\vee \otimes \sigma^\vee$. Then, writing $G = G(W)$ and $H = G(V)$ and assuming (i)' in the first step, we have

$$\begin{aligned} & \text{Hom}_G(\omega_{V,\chi}, \pi) \neq 0 \\ \iff & \text{Hom}_{G \times H}(\omega_{V,W,\chi}, \pi \otimes \sigma) \neq 0 \\ (3.5) \quad \implies & \text{Hom}_{G \times G \times H \times H}(\omega_{V,W,\chi} \otimes \omega_{V,W,\chi}^\vee, \pi \otimes \pi^\vee \otimes \sigma \otimes \sigma^\vee) \neq 0 \\ \implies & \text{Hom}_{G \times G \times H^\Delta}(\omega_{V,W,\chi} \otimes \omega_{V,W,\chi}^\vee, \pi \otimes \pi^\vee \otimes \mathbf{1}) \neq 0 \\ \iff & \text{Hom}_{G \times G \times H^\Delta}(i^* \omega_{V,W+W_-, \chi}, \pi \otimes (\chi \cdot \pi^\vee) \otimes \chi) \neq 0 \\ \iff & \text{Hom}_{G \times G}(i^* R_n(V, \chi), \pi \otimes (\chi \cdot \pi^\vee)) \neq 0. \end{aligned}$$

To complete the proof, note that if any of the last three *Hom*'s are non-zero, then

$$(3.6) \quad \text{Hom}_{G \times G}(\omega_{V,\chi} \otimes \omega_{V,\chi}^\vee, \pi \otimes \pi^\vee) \neq 0,$$

by forgetting the action of $G(V)$. But the non-vanishing of this last space easily implies that

$$(3.7) \quad \text{Hom}_{G(W)}(\omega_{V,\chi}, \pi) \neq 0,$$

and we are done. □

Remark. The argument for this proof has its origin in [36].

§4. THETA DICHOTOMY

Let P_Y be the maximal parabolic subgroup of $G(W + W_-)$ which stabilizes Y . The Levi factor M_Y of P_Y which stabilizes the complete polarization $W + W_- = X + Y$ is isomorphic to $GL(n, E)$ via restriction to $X \simeq E^n$. We write $m(a) \in M_Y$ for the element corresponding to $a \in GL(n, E)$. For $s \in \mathbb{C}$ and for any character χ of E^\times , let $I_n(s, \chi)$ be the degenerate principal series consisting of smooth functions on $G(W + W_-)$ which satisfy

$$(4.1) \quad \Phi(nm(a)g, s) = \chi(\det a) |\det a|_E^{s+\frac{n}{2}} \Phi(g, s),$$

where $n \in N_Y$, the unipotent radical of P_Y .

For V of dimension m over E , as before, set $s_0 = \frac{m-n}{2}$, and assume that

$$(4.2) \quad \chi|_{F^\times} = \epsilon_{E/F}^m.$$

These conventions will be fixed throughout this section. Then the map

$$(4.3) \quad \begin{aligned} S(V^n) &\longrightarrow I_n(s_0, \chi), \\ \varphi &\longmapsto \omega_{V, \chi}(g)\varphi(0), \end{aligned}$$

factors through $R_n(V, \chi)$, and, by [29], induces an injection

$$(4.4) \quad R_n(V, \chi) \hookrightarrow I_n(s_0, \chi).$$

Recall that the isometry class of the Hermitian space V of dimension m over E is determined by the sign

$$(4.5) \quad \epsilon(V) = \epsilon_{E/F}((-1)^{\frac{m(m-1)}{2}} \det V),$$

where $\det V = \det((x_i, x_j)) \in F^\times / NE^\times$, for any E -basis x_1, \dots, x_m for V . We will write V_m^+ and V_m^- for the corresponding pair of Hermitian spaces. The following result is proved in [25].

Proposition 4.1. (i) *If $m \leq n$, then $R_n(V, \chi)$ is irreducible, and*

$$R_n(V_m^+, \chi) \oplus R_n(V_m^-, \chi)$$

is the maximal completely reducible submodule of $I_n(s_0, \chi)$.

(ii) *If $m = n$, then*

$$I_n(0, \chi) = R_n(V_n^+, \chi) \oplus R_n(V_n^-, \chi).$$

(iii) *If $n < m < 2n$, then*

$$I_n(s_0, \chi) = R_n(V_m^+, \chi) + R_n(V_m^-, \chi),$$

and $R_n(V_m^+, \chi) \cap R_n(V_m^-, \chi)$ is the unique irreducible submodule of $I_n(s_0, \chi)$. Moreover, the normalized intertwining operator

$$M^{**}(s_0, \chi) : I_n(s_0, \chi) \longrightarrow I_n(-s_0, \chi)$$

(see [25] and Lemma 6.7 (i) below) has kernel $R_n(V_m^+, \chi) \cap R_n(V_m^-, \chi)$. Also

$$M^{**}(s_0, \chi) \left(R_n(V_m^\pm, \chi) \right) = R_n(V_{n-m}^\pm, \chi).$$

(iv) *If $m = 2n$, then V_{2n}^+ is the split Hermitian space of dimension m , and*

$$I_n(s_0, \chi) = R_n(V_{2n}^+, \chi).$$

Moreover, $R_n(V_{2n}^-, \chi)$ is the unique irreducible submodule of $I_n(s_0, \chi)$, and the quotient $I_n(s_0, \chi) / R_n(V_{2n}^-, \chi)$ is one dimensional, with $G(W + W_-)$ acting via the character $\chi \circ \det$.

(v) *If $m > 2n$, then $I_n(s_0, \chi) = R_n(V_m^\pm, \chi)$ is irreducible.*

Corollary 4.2. *Let π be an irreducible admissible representation of $G = G(W)$. For $1 \leq m \leq n$, let $s_0 = \frac{m-n}{2}$. Then*

$$\begin{aligned} \dim \operatorname{Hom}_{G \times G}(I_n(-s_0, \chi), \pi \otimes (\chi \cdot \pi^\vee)) \\ \geq \dim \operatorname{Hom}_{G \times G}(R_n(V_m^+, \chi), \pi \otimes (\chi \cdot \pi^\vee)) \\ + \dim \operatorname{Hom}_{G \times G}(R_n(V_m^-, \chi), \pi \otimes (\chi \cdot \pi^\vee)). \end{aligned}$$

Moreover, equality holds if $m = n$.

Now we have the following variant of a result of Piatetski-Shapiro and Rallis [32].

Theorem 4.3. (i) *If π is any irreducible admissible representation of $G(W)$, then, for all s ,*

$$\dim \text{Hom}_{G \times G}(I_n(s, \chi), \pi \otimes (\chi \cdot \pi^\vee)) \geq 1.$$

(ii) *If π is an irreducible supercuspidal representation of $G(W)$, or, more generally, if π does not occur in the boundary at $s = s_0$, (see Definition 4.6 below), then*

$$\dim \text{Hom}_{G \times G}(I_n(s_0, \chi), \pi \otimes (\chi \cdot \pi^\vee)) = 1.$$

Combining Theorem 4.3, Corollary 4.2 and Proposition 3.1, we obtain:

Corollary 4.4 (Theta dichotomy). *Suppose that $m \leq n$, and fix a character χ of E^\times such that $\chi|_{F^\times} = \epsilon_{E/F}^m$. Let π be an irreducible admissible supercuspidal representation of $G(W)$, or, more generally, suppose that π is any irreducible admissible representation which does not occur in the boundary at $s = s_0 = \frac{n-m}{2}$, (see Definition 4.6 below). Then at most one of the two theta lifts $\Theta_\chi(\pi, V_m^\pm)$ is non-zero. Moreover, if $m = n$ then precisely one is non-zero.*

The case $n = 3$ and $m = 1$ or 2 of this result was proved by Gelbart, Rogawski and Soudry by a global method [10].

Corollary 4.5. *If π is any irreducible admissible representation of $G = G(W)$ and if $m \geq n$, then there exists a V with $\dim_E V = m$ such that $\Theta_\chi(\pi, V) \neq 0$.*

Proof. Since, in this case $I_n(s_0, \chi)$ is the sum (not necessarily direct) of the $R_n(V, \chi)$'s, a non-trivial homomorphism from $I_n(s_0, \chi)$ to $\pi \otimes (\chi \cdot \pi^\vee)$ must have non-zero restriction to at least one such space, and Proposition 3.1 again yields the claimed result. □

Note that, by ‘reflection’, the same results hold for supercuspidal representations of $G(V)$ and theta lifts to $G(W)$'s with $\dim_E W = \dim_E V$.

Proof of Theorem 4.3. We postpone the proof of part (i) of the Theorem until section 6, below.

To prove (ii), we write $G = G(W)$. The restriction of the induced representation $I_n(s, \chi)$ to $G \times G$ has a filtration arising from the orbit structure of $P_Y \backslash G(W + W_-)$, the space of isotropic n -planes in $W + W_-$, for the action of $G \times G$ [36],[32], [33]. The orbit of an n -plane Z is determined by the invariant

$$(4.6) \quad d = \dim(Z \cap W) = \dim(Z \cap W_-).$$

Note that $0 \leq d \leq r_0$, where r_0 is the Witt index of W . We write Ω_d for the corresponding orbit, and choose a double coset representative $\delta_d \in G(W + W_-)$ for $\Omega_d = P\delta_d i(G \times G)$. The unique open orbit Ω_0 is the orbit of Y , and the closure of Ω_r is

$$(4.7) \quad \bar{\Omega}_r = \coprod_{r' \geq r} \Omega_{r'}.$$

Let

$$(4.8) \quad I_n(s, \chi) = I_n^{(r_0)}(s, \chi) \supset I_n^{(r_0-1)}(s, \chi) \supset \cdots \supset I_n^{(0)}(s, \chi),$$

be the filtration by support, and let

$$(4.9) \quad Q_n^{(r)}(s, \chi) = I_n^{(r)}(s, \chi)/I_n^{(r-1)}(s, \chi) \simeq \text{Ind}_{St_r}^{G \times G}(\xi_{r,s}).$$

Here St_r is the stabilizer in $G \times G$ of the coset $P_Y \cdot \delta_r$ in $\Omega_r \simeq P_Y \backslash P_Y \delta_r (G \times G)$, and $\xi_{r,s}$ is the character of St_r defined by

$$(4.10) \quad \xi_{r,s}(g) = \chi(\det a) |\det a|_E^{s+\rho_n}$$

where $\delta_r g \delta_r^{-1} = nm(a) \in P_Y$.

Definition 4.6. We say that an irreducible admissible representation π of G **does not occur in the boundary at the point** $s = s_0$ if $\pi \otimes \chi \pi^\vee$ does not occur as a quotient of any $Q_n^{(r)}(s_0, \chi)$ for $r > 0$.

Lemma 4.7. *If π is a supercuspidal representation of G , then π does not occur in the boundary at any point s_0 .*

Proof. For the case of π supercuspidal, we only need the following facts. First, for $r > 0$, St_r contains the full unipotent radical U of a parabolic subgroup of $G \times G$ and $\xi_{r,s}|_U = 1$. Thus

$$(4.11) \quad \text{Hom}_{G \times G}(Q_n^r(s, \chi), \pi \otimes (\chi \cdot \pi^\vee)) = 0.$$

□

Now suppose that π does not occur in the boundary at $s = s_0$, and consider a non-zero homomorphism $\lambda : I_n(s_0, \chi) \rightarrow \pi \otimes (\chi \cdot \pi^\vee)$. If the restriction of λ to $I_n^{(0)}(s_0, \chi)$, the last step in the filtration, were zero, then there would be a smallest r , with $r > 0$, such that the restriction of λ to $I_n^{(r)}(s_0, \chi)$ is non-zero. This restriction would then factor through the quotient $Q_n^{(r)}(s_0, \chi)$, contradicting our assumption on π . Thus, under our assumption on π , the restriction map

$$\text{Hom}_{G \times G}(I_n(s_0, \chi), \pi \otimes (\chi \cdot \pi^\vee)) \rightarrow \text{Hom}_{G \times G}(I_n^{(0)}(s_0, \chi), \pi \otimes (\chi \cdot \pi^\vee))$$

is injective.

For $r = 0$, we take $\delta_0 = 1$ and have $St_0 = \Delta(G(W))$. As in Lemma 1.3,

$$(4.12) \quad \xi_{0,s}(g) = \chi(\det \bar{g}^{-1}) = \chi(\det g).$$

For all s , the map $\Phi(s) \mapsto f_s$ given by

$$(4.13) \quad f_s(g) = \Phi(i(g, 1), s),$$

yields an intertwining isomorphism

$$(4.14) \quad I_n^{(0)}(s, \chi) \xrightarrow{\sim} S(G)(1 \otimes \chi),$$

where $G \times G$ acts on $S(G)(1 \otimes \chi)$ by

$$(4.15) \quad r(g_1, g_2)f(x) = \chi(g_2)f(g_2^{-1}xg_1).$$

But now, for any irreducible admissible representation π , there is a unique non-zero $G \times G$ intertwining map

$$S(G) \longrightarrow \pi \otimes \pi^\vee$$

given by sending $f \in S(G)$ to the functional on $\pi^\vee \otimes \pi$:

$$(4.16) \quad \xi^\vee \otimes \xi \mapsto \int_G f(g) \langle \pi^\vee(g)\xi^\vee, \xi \rangle dg.$$

Thus

$$\dim \text{Hom}_{G \times G}(S(G)(1 \otimes \chi), \pi \otimes (\chi \cdot \pi^\vee)) = 1,$$

and we obtain

$$\dim \text{Hom}_{G \times G}(I_n(s_0, \chi), \pi \otimes (\chi \cdot \pi^\vee)) \leq 1,$$

as required. □

§5. A CURIOUS EXAMPLE

In this section we will show that the dichotomy of Corollary 4.4 gives rise to certain infinite families of related supercuspidal representations!

Let $V_{r,r}$ be the split Hermitian space of dimension $2r$ over E . We will refer to the family of Hermitian spaces $V + V_{r,r}$, $r = 0, 1, \dots$, obtained by adding hyperbolic planes to a fixed space V as the Witt tower, or simply tower, on V . Note that, since $\det(V + V_{r,r}) = (-1)^r \det(V)$, the sign $\epsilon(V_m)$ is constant as V_m varies in a tower. Conversely, the spaces V_m^\pm for a fixed sign ± 1 vary in a tower.

Proposition 5.1. *Fix a Witt tower $\{V_m^\epsilon\}$, $\epsilon = \pm 1$, and a character χ of E^\times such that $\chi|_{F^\times} = \epsilon_{E/F}^m$. Suppose that π is an irreducible admissible supercuspidal representation of $G(W)$.*

- (i) (stable range [18]) *If the Witt index r of V_m satisfies $r \geq n$, then $\Theta_\chi(\pi, V_m^\epsilon) \neq 0$.*
- (ii) (first occurrence [21], [29]) *If $m_0 = m_0^\epsilon(\chi)$ is the smallest value of m for which $\Theta_\chi(\pi, V_m^\epsilon) \neq 0$, then $\Theta_\chi(\pi, V_{m_0}^\epsilon)$ is an irreducible supercuspidal representation of $G(V_{m_0}^\epsilon)$.*
- (iii) (persistence) *If $m \geq m_0$, then $\Theta_\chi(\pi, V_m^\epsilon) \neq 0$.*

Remark. One should be careful to note that (ii) does not apply when π is not supercuspidal, i.e., the first occurrence need not be supercuspidal in general.

Proof of (iii). First observe the following useful fact.

Lemma 5.2. *Suppose that V_1 and V_2 are Hermitian spaces of dimensions m_1 and m_2 , and let $V = V_1 + V_2$, $\mathbb{W}_i = V_i \otimes W$, and $\mathbb{W} = \mathbb{W}_1 + \mathbb{W}_2$. Fix characters χ_1 and χ_2 of E^\times with restrictions $\epsilon_{E/F}^{m_i}$ to F^\times and let $\chi = \chi_1 \cdot \chi_2$. Then the diagram*

$$\begin{array}{ccc} G(W) & \xrightarrow{\tilde{i}_{V,\chi}} & Mp(\mathbb{W}) \\ \Delta \downarrow & & \uparrow i \\ G(W) \times G(W) & \xrightarrow{\tilde{i}_{V_1,\chi_1} \times \tilde{i}_{V_2,\chi_2}} & Mp(\mathbb{W}_1) \times Mp(\mathbb{W}_2) \end{array}$$

commutes, where the map i is holomorphic on both central \mathbb{C}^1 's.

Proof. If we write $U = W + W_-$, $\mathbb{U}_i = V_i \otimes U$, etc., then it suffices to prove the result for the analogous diagram for $G(U)$. But it is easy to check that

$$(5.1) \quad \beta_{V,\chi}(g) = \beta_{V_1,\chi_1}(g) \cdot \beta_{V_2,\chi_2}(g),$$

and the Lemma follows. □

Applying Lemma 5.2 to the space $V_m^\epsilon = V_{m_0}^\epsilon + V_{r,r}$, with $\chi_1 = \chi$ and $\chi_2 = 1$, we have

$$(5.2) \quad \omega_{V_m^\epsilon, W, \chi} \simeq \omega_{V_{m_0}^\epsilon, W, \chi} \otimes \omega_{V_{r,r}, W, 1}.$$

Now, by (ii) of Proposition 2.2, the representation $\omega_{V_{r,r}, W, 1}$ can be realized in the space $S(W^r)$ with the group $G(W)$ acting linearly. Then, if

$$(5.3) \quad \lambda : S_{V_{m_0}^\epsilon, W, \chi} \longrightarrow \pi$$

is a non-zero $G(W)$ -intertwining map, then

$$(5.4) \quad \lambda \otimes \delta_0 : S_{V_{m_0}, W, \chi} \otimes S(W^r) \longrightarrow \pi$$

is a non-zero $G(W)$ -intertwining map, where δ_0 is the delta distribution at the origin on $S(W^r)$. □

The combination of these three principles with the dichotomy of Corollary 4.4 gives rise to a curious phenomenon. Recall that we have fixed $\delta \in E^\times$ with $\delta^2 = \Delta$. We let W_n^\pm be the skew-Hermitian space which becomes isomorphic to V_n^\pm (in the notation introduced in section 4) when we scale the skew-Hermitian form by δ^{-1} . We then obtain four towers $\{V_n^+\}$, $\{V_n^-\}$, $\{W_n^+\}$ and $\{W_n^-\}$ as n runs over positive integers of a fixed parity. Fix χ of the same ‘parity’ as n , i.e., which satisfies (4.4).

Let π_0 be an irreducible supercuspidal representation of $G(W_0)$, where $W_0 = W_{n_0}^{\epsilon_0}$. By dichotomy, there is a unique sign ϵ_1 such that

$$(5.5) \quad \Theta_\chi(\pi_0, V_{n_0}^{\epsilon_1}) = 0,$$

so that the first occurrence of π_0 in the $V_n^{\epsilon_1}$ tower occurs for some integer $n_1 > n_0$, i.e., for $V_1 = V_{n_1}^{\epsilon_1}$,

$$(5.6) \quad \pi_1 = \Theta_\chi(\pi_0, V_{n_1}^{\epsilon_1})$$

is an irreducible supercuspidal representation of $G(V_1)$. By applying the same argument to π_1 , we obtain a supercuspidal π_2 for some $G(W_2)$, where $W_2 = W_{n_2}^{\epsilon_2}$ with $n_2 > n_1$. Note, however, that since π_1 is a theta lift of π_0 , we must have $\epsilon_2 = -\epsilon_0$. Proceeding again, we obtain a supercuspidal π_3 of $G(V_3)$, where $V_3 = V_{n_3}^{\epsilon_3}$, and $n_3 > n_2$, $\epsilon_3 = -\epsilon_1$. This game can be continued indefinitely, and we obtain:

Proposition 5.3. *Suppose that π is an irreducible supercuspidal representation of $G(W)$, where $W = W_{n_0}^{\epsilon_0}$. Fix a character χ of E^\times with $\chi|_{F^\times} = \epsilon_E^{n_0/F}$. Then there exist a sequence of integers $n_0 < n_1 < \dots < n_i < \dots$ and signs $\epsilon_0, \epsilon_1, \dots$ with $\epsilon_i = -\epsilon_{i+2}$ and a sequence of irreducible supercuspidal representations π_i of unitary groups G_i , where*

$$G_i = \begin{cases} G(W_{n_i}^{\epsilon_i}) & \text{if } i \text{ is even,} \\ G(V_{n_i}^{\epsilon_i}) & \text{if } i \text{ is odd,} \end{cases}$$

such that

$$\Theta_\chi(\pi_i, G_{i+1}) = \pi_{i+1},$$

with the obvious slight abuse of notation.

A conjectural explanation of this result from the point of view of Langlands functoriality will be given in section 7 below. In particular, Speculation 7.7 gives a conjectural description of the sequence of dimensions $n_0 < n_1 < \dots$ and L-parameters which arise.

We conclude this section with a proof that the assumption (i)' at the beginning of section 3 always holds for supercuspidal representations.

Proposition 5.4. *Suppose that π is a supercuspidal representation of $G(W)$. Then, as a representation of $G(V)$, the space $\Theta_\chi(\pi, V)$ is either zero or admits a non-zero irreducible quotient.*

Proof. We consider the tower containing the space V , and write $V_r = V_0 + V_{r,r}$ with V_0 anisotropic and $V_{r,r}$ split. For convenience we write S_r for $S_{V_r, W, \chi}$, $\Theta_r(\pi)$ for $\Theta_\chi(\pi, V_r)$, and H_r for $G(V_r)$.

Let r_0 be the smallest value of r for which $\Theta_r(\pi)$ is non-zero. Then, by the calculations of [21], as extended to the unitary case in [29], all of the Jacquet functors of $\Theta_{r_0}(\pi)$ relative to maximal parabolics, and hence also relative to all proper parabolics, are zero. Thus $\Theta_{r_0}(\pi)$ is a (not necessarily irreducible) supercuspidal representation of H_{r_0} . Any such representation has an irreducible quotient, since the quotient of H_{r_0} by its derived group is compact. Thus we have:

$$(5.7) \quad \lambda_0 : S_{r_0} \longrightarrow S_{r_0}(\pi) \simeq \pi \otimes \Theta_{r_0}(\pi) \longrightarrow \pi \otimes \sigma,$$

for some irreducible supercuspidal representation σ of H_{r_0} . Denote the composite map by λ_0 .

For any $\ell > 0$, let $r = r_0 + \ell$. Then, writing

$$(5.8) \quad V_r = Z + V_{r_0} + Z',$$

where $V_{\ell, \ell} = Z + Z'$, with Z and Z' isotropic, and using the mixed polarization of $V_r \otimes_E W$ derived from Z , we have a mixed model $S_{r_0} \otimes S(W^\ell)$ for S_r . We view elements of this model as functions on W^ℓ valued in S_{r_0} . Let δ_0 be the delta distribution at $0 \in W^\ell$. Then consider the map

$$(5.9) \quad \begin{aligned} S_r &\simeq S_{r_0} \otimes S(W^\ell) \longrightarrow (\pi \otimes \sigma), \\ \lambda_1 : f &\mapsto \lambda_0(\delta_0(f)). \end{aligned}$$

Let $L_r \simeq H_{r_0} \times GL_\ell(E)$ be the Levi factor in H_r which preserves the decomposition (5.8). Then λ_1 is intertwining for L_r , where the $GL_\ell(E)$ factor acts on the right by a character ξ . Since λ_1 factors through $S_r(\pi)$, we obtain a commutative diagram:

$$(5.10) \quad \begin{array}{ccc} S_{r_0} \otimes S(W^\ell) & \longrightarrow & (\pi \otimes \sigma) \\ \downarrow & \nearrow & \uparrow (*) \\ \pi \otimes \Theta_r(\pi) \simeq S_r(\pi) & \longrightarrow & \pi \otimes \text{Ind}_{Q_\ell}^{H_r}(\sigma \otimes \xi). \end{array}$$

Here the bottom map is induced by the map

$$(5.11) \quad \begin{aligned} S_r &\longrightarrow \text{Ind}_{Q_\ell}^{H_r}(\sigma \otimes \xi), \\ f &\longmapsto \left(h \mapsto \lambda_1(\omega_r(h)f) \right), \end{aligned}$$

and the map $(*)$ is given by evaluation at $e \in H_r$. Since the composition of these maps, from $\pi \otimes \Theta_r(\pi)$ to $\pi \otimes \sigma$ is surjective, we obtain a nonzero map

$$(5.12) \quad \Theta_r(\pi) \longrightarrow \text{Ind}_{Q_\ell}^{H_r}(\sigma \otimes \xi),$$

and hence $\Theta_r(\pi)$ must admit a *non-zero* quotient. □

§6. L AND EPSILON FACTORS

In this section we will show that the dichotomy of Corollary 4.4 (in the case $m = n$) is controlled by a certain local root number associated to the representation π of $G(W)$, and that the ‘early’ theta lifts of π are controlled by the poles of a certain local L-factor.

More precisely, the doubling method of Piatetski-Shapiro and Rallis yields a family of zeta integrals which produce, by more or less standard procedures, certain L and ϵ -factors attached to π . If $G(W)$ is the split group and π is an unramified principal series, and if χ is unramified, then the base change $BC(\pi)$ to an unramified principal series representation of $GL(n, E)$ is well defined, and the local zeta integrals yield $L(s, BC(\pi) \otimes \chi)$ and $\epsilon(s, BC(\pi) \otimes \chi, \psi_E)$, the standard L and ϵ -factors for $BC(\pi) \otimes \chi$. The case of the L-factor was discussed by Li [28]. For general representations π , we will write $L(s, \pi, \chi)$ and $\epsilon(s, \pi, \chi, \psi)$ for the factors which arise by the doubling method. The expected relation between these factors and those defined by other means is described in the comparison conjecture (0.18) and (0.19) of the introduction.

Our first main result is then the following local root number characterization of dichotomy.

Theorem 6.1 (epsilon dichotomy). *Let π be an irreducible admissible supercuspidal representation of $G(W)$, or, more generally, suppose that π is any irreducible admissible representation which does not occur in the boundary at $s = 0$ (see Definition 4.6). For $n = \dim_E W$, fix a character χ of E^\times such that $\chi|_{F^\times} = \epsilon_{E/F}^n$.*

(i) *Then*

$$\epsilon\left(\frac{1}{2}, \pi, \chi, \psi\right) = \pm 1.$$

(ii) *Let V be a Hermitian space with $\dim_E V = \dim_E W = n$. Then*

$$\Theta_\chi(\pi^\vee, V) \neq 0 \iff \epsilon\left(\frac{1}{2}, \pi, \chi, \psi\right) = \epsilon_{E/F}(-2)^n \cdot \epsilon_{E/F}(\det V).$$

Our second result links the poles of the doubling L-factor for a supercuspidal representation π to the existence of early theta lifts. First recall that, if we fix a choice of a uniformizing parameter ϖ_E of E , and let $q_E = |\varpi_E|_E^{-1}$, then the pair (s, χ) is called normalized if

$$(6.1) \quad \chi(\varpi_E \cdot \overline{\varpi_E}) = 1$$

and

$$(6.2) \quad \text{Im}(s) \in \left[0, \frac{\pi}{\log(q_E)}\right).$$

Also recall that $\check{\chi}(x) = \chi(\bar{x})^{-1}$, and that $\chi = \check{\chi}$ if and only if $\chi|_{F^\times} = \epsilon_{E/F}$ or 1.

Theorem 6.2. *For a supercuspidal representation π of $G(W)$, and a unitary character χ of E^\times , let $L(s, \pi, \chi)$ be the L -factor defined by doubling. Assume that the pair (s, χ) is normalized.*

- (i) *If $\chi \neq \check{\chi}$, then $L(s, \pi, \chi) = 1$.*
- (ii) *If $\chi = \check{\chi}$, then the poles of $L(s, \pi, \chi)$ are at most simple and lie in the set*

$$X_n(\chi) = \left\{ \frac{m - n + 1}{2} \mid 0 \leq m \leq n - 1, \chi^o = \epsilon_{E/F}^m \right\},$$

where χ^o denotes the restriction of χ to F^\times . Note that this set is precisely the set of poles of $b_n(s, \chi)$. Moreover, a pole occurs at $s_0 = \frac{m-n+1}{2} \in X_n(\chi)$ if and only if there is a space V with $\dim_E V = m$ such that

$$\Theta_\chi(\pi^\vee, V) \neq 0.$$

Note that, if we consider theta lifts for a Witt tower, as in section 5, then by Corollary 4.5, any π will always have a non-trivial theta lift to one of the two possible towers provided $\dim_E V = m \geq n$. If π is supercuspidal, then π has a theta lift to some *smaller* $G(V)$, i.e., to one with $\dim_E V < n$ if and only if $L(s, \pi, \chi) \neq 1$ for some χ . Moreover, by ‘persistence’ ((iii) of Proposition 5.1), the degree of this Euler factor will be $-\lfloor \frac{m-n}{2} \rfloor$, where m is the dimension of the smallest V for which such an ‘early’ theta lift occurs. The compatibility of this picture with functoriality will be discussed in section 7 below.

Before proving these results, we must review the doubling method definition of L and ϵ -factors [32], [33]. We do this in some detail for two reasons. First, Piatetski-Shapiro and Rallis introduced in [33] a certain refinement (the use of “good sections”) of the method of [32]. Second, the normalization of the intertwining operator in [32] depends on an auxiliary choice, and the ϵ -factor depends on this choice in a way which is not determined. As a result, the actual root number remains ambiguous! In our discussion we will give a precise definition of the ϵ -factor, and it is this definition which is used in Theorem 6.1.

In the case $n = 1$, Theorem 6.1 was already obtained by Rogawski [41]. In fact, Rogawski states the result in terms of the base change ϵ -factor, and relates this factor to the earlier result of Moen [30] on the $(U(1), U(1))$ dual pair. We will give a precise description of the relation between the base change ϵ and the doubling ϵ in section 8 below, in the case $n = 1$. Thus, Theorem 6.1 together with the computations of section 8 will yield a purely local proof of Rogawski’s result. In the case $n = 2$, Theorem 6.1 (for similitude groups) is due to Harris [15]. Of course, the original result of this sort is that of Tunnell [47] and Waldspurger [49], and, indeed their results are used by Harris in the case of $GU(2)$.

For the moment, we allow χ to be an arbitrary character of E^\times .

We begin with the normalization of the intertwining operator; for more details the reader should consult section 3 of [25]. For a section $\Phi(s) \in I_n(s, \chi)$ and for $\text{Re}(s) > \rho_n = \frac{n}{2}$, the intertwining operator

$$(6.3) \quad M(s, \chi) : I_n(s, \chi) \longrightarrow I_n(-s, \check{\chi}),$$

for $\check{\chi}(x) = \chi(\bar{x})^{-1}$, is defined by the integral

$$(6.4) \quad M(s, \chi)\Phi(g) = \int_{\text{Herm}_n(E)} \Phi(w_n(x)g, s) dx.$$

Here the measure dx is normalized to be self dual with respect to the pairing $[x, y] \mapsto \psi(\text{tr}(xy))$ on $\text{Herm}_n(E)$. In particular, the additive character ψ should really be included in the notation. For example, if we replace ψ by ψ_a for $a \in F^\times$, then dx , and hence $M(s, \chi)$, changes by a factor of $|a|_F^{n^2}$. Also, for $T \in \text{Herm}_n(E)$, the generalized Whittaker integral is given by

$$(6.5) \quad W_T(s)\Phi(g) = \int_{\text{Herm}_n(E)} \Phi(\text{wn}(x)g, s) \psi(\text{tr}(Tx)) dx,$$

with the same choice of measure. Karel [20] proved that, if $\det T \neq 0$, then $W_T(s)$ has an entire analytic continuation and a functional equation:

$$(6.6) \quad W_T(-s) \circ M(s, \chi) = \kappa_T(s, \chi, \psi) \cdot W_T(s).$$

Note that $W_T(s)$ depends on the choice of ψ . The following result is proved in [25], and is analogous to a result of Sweet [44] for the symplectic group.

Proposition 6.3. *Let $\chi^o = \chi|_{F^\times}$. Then*

$$\begin{aligned} \kappa_T(s, \chi, \psi) &= \chi(\det T)^{-1} |\det T|_F^{-2s} \gamma_F(\Delta, \psi)^{\frac{n(n-1)}{2}} \\ &\quad \times \epsilon_{E/F}(\det T)^{n-1} \prod_{r=0}^{n-1} \rho_F(2s + r - n + 1, \chi^o \cdot \epsilon_{E/F}^r, \psi). \end{aligned}$$

Here, for a character ξ of F^\times , $\rho_F(s, \xi, \psi)$ is Tate’s gamma factor, [45] and (8.19) below, satisfying

$$\rho_F(s, \xi, \psi) = \epsilon(s, \xi, \psi)^{-1} \frac{L(s, \xi)}{L(1 - s, \xi^{-1})}.$$

(We have called it ρ_F to avoid confusion with the Weil invariant γ_F .) Recall that it depends on the additive character ψ . Moreover, for $a \in F^\times$,

$$\kappa_T(s, \chi, \psi_a) = |a|_F^{n^2} \kappa_{aT}(s, \chi, \psi) = \chi(a)^{-n} |a|_F^{-2ns+n^2} \kappa_T(s, \chi, \psi).$$

We set

$$(6.7) \quad \kappa(s, \chi, \psi) = \gamma_F(\Delta, \psi)^{\frac{n(n-1)}{2}} \prod_{r=0}^{n-1} \rho_F(2s + r - n + 1, \chi^o \cdot \epsilon_{E/F}^r, \psi),$$

and define the normalized intertwining operator

$$(6.8) \quad M^*(s, \chi) = M^*(s, \chi, \psi) = \frac{1}{\kappa(s, \chi, \psi)} \cdot M(s, \chi).$$

Note that

$$(6.9) \quad M^*(s, \chi, \psi_a) = \chi(a)^n |a|_F^{2ns} M^*(s, \chi, \psi).$$

We will usually suppress the dependence on ψ from the notation. The functional equation (6.3) then becomes:

$$(6.10) \quad W_T(-s) \circ M^*(s, \chi) = \chi(\det T)^{-1} |\det T|_F^{-s} \cdot \epsilon_{E/F}(\det T)^{n-1} \cdot W_T(s),$$

and so, we obtain

$$(6.11) \quad M^*(-s, \check{\chi}) \circ M^*(s, \chi) = 1.$$

Note that all that has been done so far depends only on the group $U(n, n)$ and does not involve the embedding $i : G \times G \rightarrow U(n, n)$.

Next recall that the local zeta integral is defined as follows. Let π be an irreducible admissible representation of $G = G(W)$ and, for $\xi \in \pi$ and $\xi^\vee \in \pi^\vee$ let $\phi(g) = \langle \pi(g)\xi, \xi^\vee \rangle$ be the corresponding matrix coefficient. For a section $\Phi(s) \in I_n(s, \chi)$, define

$$(6.12) \quad Z(s, \xi, \xi^\vee, \Phi) = \int_G \phi(g) \Phi(i(g, 1), s) dg.$$

This integral converges for large $\text{Re}(s)$ and defines a non-zero element

$$(6.13) \quad Z(s) \in \text{Hom}_{G \times G}(I_n(s, \chi), \pi^\vee \otimes (\chi \cdot \pi)).$$

To define the L and ϵ -factors, Piatetski-Shapiro and Rallis introduce the family of good sections [33]. For a suitable choice of basis, the group $G(W + W_-)$ is isomorphic to the subgroup of $GL(2n, E)$ which preserves the skew-Hermitian form with matrix $\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$. Let K be the maximal compact subgroup of $G(W + W_-)$ obtained by intersecting this group with $GL(2n, \mathcal{O}_E)$. A section $\Phi(s)$ is *standard* if its restriction to K is independent of s . For example, if χ is unramified, let $\Phi^0(s)$ be the standard section whose restriction to K is equal to 1. Recall that if χ and ψ are unramified, then [28]

$$(6.14) \quad M(s, \chi)\Phi^0(s) = \frac{a_n(s, \chi)}{b_n(s, \chi)} \cdot \Phi^0(-s),$$

with

$$(6.15) \quad a_n(s, \chi) = \prod_{r=0}^{n-1} L(2s + r - n + 1, \chi^o \epsilon_{E/F}^r),$$

and

$$(6.16) \quad b_n(s, \chi) = \prod_{r=0}^{n-1} L(2s + n - r, \chi^o \epsilon_{E/F}^r),$$

where χ^o is the restriction of χ to F^\times . Also, if χ and ψ are unramified and $\text{ord det } T = 0$, then [43]

$$(6.17) \quad W_T(s)\Phi^0(e) = \frac{1}{b_n(s, \chi)},$$

where $W_T(s)$ is defined by (6.5).

Definition 6.4. The family of **good sections** consists of

- (a) $\mathbb{C}[q^s, q^{-s}] \cdot$ standard sections,
- (b) $M^*(-s, \check{\chi}) \left(\mathbb{C}[q^s, q^{-s}] \cdot \text{standard sections} \right),$

and, when χ is unramified,

(c) the sections of the form

$$b_n(s, \chi) \cdot \Phi^0(s) * \mu(s),$$

where $\mu(s) \in \mathcal{H}(G(W + W_-), K)[q^s, q^{-s}]$, with $\mathcal{H}(G(W + W_-), K)$ the Hecke algebra with respect to K , which acts on the space of K -invariant sections by convolution on the right.

Lemma 6.5. (i) ([33])

$$M^*(s, \chi) \left(\text{Good sections} \right) = \text{Good sections}.$$

(ii) Moreover,

$$\kappa(s, \chi, \psi) = \gamma_F(\Delta, \psi)^{\frac{n(n-1)}{2}} \frac{a_n(s, \chi)}{b_n(-s, \tilde{\chi})} \cdot \prod_{r=0}^{n-1} \epsilon(2s + r - n + 1, \chi^o \epsilon_{E/F}^r, \psi)^{-1}.$$

Proof. Since $M^*(s, \chi)((a)) = (b)$ and $M^*(s, \chi)((b)) = (a)$ by the functional equation (6.7), it suffices to consider

$$\begin{aligned} M^*(s, \chi) \left(b_n(s, \chi) \cdot \Phi^0(s) * \mu(s) \right) &= b_n(s, \chi) \frac{1}{\kappa(s, \chi, \psi)} \cdot \frac{a_n(s, \chi)}{b_n(s, \chi)} \Phi^0(-s) * \mu(s) \\ (6.18) \qquad \qquad \qquad &= \frac{a_n(s, \chi)}{\kappa(s, \chi, \psi)} \Phi^0(-s) * \check{\mu}(-s). \end{aligned}$$

But now,

$$\begin{aligned} &\frac{a_n(s, \chi)}{\kappa(s, \chi, \psi)} \\ &= \gamma_F(\Delta, \psi)^{-\frac{n(n-1)}{2}} \cdot \prod_{r=0}^{n-1} L(2s + r - n + 1, \chi^o \epsilon_{E/F}^r) \\ &\quad \times \frac{L(-2s - r + n, (\chi^o)^{-1} \epsilon_{E/F}^r)}{L(2s + r - n + 1, \chi^o \epsilon_{E/F}^r)} \cdot \epsilon(2s + r - n + 1, \chi^o \epsilon_{E/F}^r, \psi) \\ (6.19) \qquad \qquad \qquad &= b_n(-s, \tilde{\chi}) \cdot \gamma_F(\Delta, \psi)^{-\frac{n(n-1)}{2}} \cdot \prod_{r=0}^{n-1} \epsilon(2s + r - n + 1, \chi^o \epsilon_{E/F}^r, \psi). \end{aligned}$$

The product of epsilon factors, etc., can be absorbed into the action of $\mathcal{H}[q^s, q^{-s}]$, so we find that $M^*(s, \chi)((c)) = (c)$, as desired. \square

Now, following [32], [33], (but notice that our normalization of the intertwining operator is different!) we have:

Proposition 6.6. (i) For any good section $\Phi(s)$, $Z(s, \xi, \xi^\vee, \Phi) \in \mathbb{C}(q^{-s})$, i.e., the zeta integral is a rational function of q^{-s} .

(ii) For any fixed ξ and ξ^\vee , there exists a good section $\Phi(s)$ such that

$$Z(s, \xi, \xi^\vee, \Phi(s)) = 1.$$

(iii) *The set*

$$\mathcal{I} = \{ Z(s, \xi, \xi^\vee, \Phi) \mid \xi \in \pi, \xi^\vee \in \pi^\vee, \Phi(s) = \text{good} \}$$

is an ideal for $\mathbb{C}[q^s, q^{-s}]$, with a generator $P(q^{-s})^{-1}$, for a unique polynomial $P(x) \in \mathbb{C}[x]$ with $P(0) = 1$. Moreover, there exist ξ, ξ^\vee and a good section $\Phi(s)$ such that

$$Z(s, \xi, \xi^\vee, \Phi) = \frac{1}{P(q^{-s})}.$$

(iv) *There is a functional equation*

$$Z(-s, \xi, \xi^\vee, M^*(s, \chi)\Phi) = \Gamma^*(s, \pi, \chi, \psi) \cdot Z(s, \xi, \xi^\vee, \Phi).$$

Recall that $M^*(s, \chi) = M^*(s, \chi, \psi)$ depends on ψ .

Now define

$$(6.20) \quad L(s + \frac{1}{2}, \pi, \chi) = \frac{1}{P(q^{-s})}.$$

By (iv) and the functional equation (6.7) of $M^*(s, \chi)$,

$$(6.21) \quad \Gamma^*(-s, \pi, \check{\chi}, \psi) \cdot \Gamma^*(s, \pi, \chi, \psi) = 1.$$

Moreover, we have

$$(6.22) \quad \frac{Z(-s, \xi, \xi^\vee, M^*(s, \chi)\Phi)}{L(\frac{1}{2} - s, \pi, \check{\chi})} = \epsilon(s + \frac{1}{2}, \pi, \chi, \psi) \cdot \frac{Z(s, \xi, \xi^\vee, \Phi)}{L(s + \frac{1}{2}, \pi, \chi)},$$

where

$$(6.23) \quad \epsilon(s, \pi, \chi, \psi) = \frac{L(s, \pi, \chi)}{L(1 - s, \pi, \check{\chi})} \cdot \Gamma^*(s - \frac{1}{2}, \pi, \chi, \psi).$$

Note that the functional equation

$$(6.24) \quad \epsilon(1 - s, \pi, \check{\chi}, \psi) \cdot \epsilon(s, \pi, \chi, \psi) = 1$$

follows from (6.21). By Lemma 6.5, the ratios on both sides of (6.22) are entire. Taking a section as in (iii) of Proposition 6.6, we see that $\epsilon(s, \pi, \chi, \psi)$ is a rational function of q^{-s} which is entire and never zero. Thus it is a monomial in q^{-s} . Finally, the functional equation (6.24) implies that

$$(6.25) \quad \epsilon(s, \pi, \chi, \psi) = A \cdot q^{B(s - \frac{1}{2})},$$

for constants A and B . Note that, when $\chi = \check{\chi}$, the combination of (6.24) and (6.25) yields $A = \pm 1$, i.e., statement (i) of Theorem 6.1.

Before continuing, we observe that our definition of $\epsilon(s, \pi, \chi, \psi)$ has one desirable property. Specifically, for $a \in F^\times$, we have

$$(6.26) \quad \epsilon(s, \pi, \chi, \psi_a) = \chi(a)^n |a|_F^{2n(s - \frac{1}{2})} \epsilon(s, \pi, \chi, \psi).$$

This follows from the analogous relation for $M^*(s, \chi, \psi)$. It is compatible with the comparison conjecture (0.19) and relation (1.3.9) in Jacquet [19] for the standard ϵ -factor for $GL_n(E)$, provided we recall that $|a|_E = |a|_F^2$ and that the central character of $BC(\pi) \otimes \chi$ restricts to χ^n on F^\times .

Proof of (ii) of Theorem 6.1. Our goal is to relate the dichotomy of Corollary 4.4 to the root number $\epsilon(\frac{1}{2}, \pi, \chi, \psi)$ defined by doubling. We first note the following facts.

Lemma 6.7. *Fix the character χ .*

(i) [25] *The operator*

$$M^{**}(s, \chi) = \frac{1}{a_n(s, \chi)} M(s, \chi)$$

*is entire; $M^{**}(s_0, \chi)$ is not identically zero for any value of s_0 . Also, by (ii) of Lemma 6.4,*

$$M^*(s, \chi) = (\text{entire}) \cdot b_n(-s, \check{\chi}) \cdot M^{**}(s, \chi).$$

In particular, for a standard section $\Phi(s)$, the poles of $M^(-s, \check{\chi})\Phi$ lie among the poles of $b_n(s, \chi)$.*

(ii) *The poles of the good sections lie among the poles of $b_n(s, \chi)$.*

(iii) *$b_n(s, \chi)$ is holomorphic at $s = 0$ for any (unitary) χ , while, if $\chi|_{F^\times} = \epsilon_{E/F}^n$, then $a_n(s, \chi)$ is holomorphic at $s = 0$. In particular, when $\chi|_{F^\times} = \epsilon_{E/F}^n$ then $M(s, \chi)$ is holomorphic at $s = 0$.*

(iv) *For any good section $\Phi(s)$, there is a neighborhood of $s = 0$ on which $\Phi(s)$ is a linear combination of standard sections with coefficients holomorphic functions of s .*

Proof of (i) of Theorem 4.3. For any irreducible admissible representation π of $G(W)$, and for any fixed $s_0 \in \mathbb{C}$, the zeta integral defines an intertwining map:

$$(6.27) \quad \begin{aligned} Z^*(s_0) : I_n(s_0, \chi) &\longrightarrow \pi^\vee \otimes (\chi \cdot \pi), \\ \Phi &\mapsto \left(\xi \otimes \xi^\vee \mapsto \frac{Z(s, \xi, \xi^\vee, \Phi)}{L(s + \frac{1}{2}, \pi, \chi)} \Big|_{s=s_0} \right), \end{aligned}$$

where an element $\Phi \in I_n(s_0, \chi)$ is first extended to a standard section $\Phi(s) \in I_n(s, \chi)$. If the relation analogous to that of (iv) of Lemma 6.7 holds for good sections and standard sections at $s = s_0$, then the order of pole at $s = s_0$ of the family $Z(s, \xi, \xi^\vee, \Phi)$ for good sections $\Phi(s)$ coincides with that for the family $Z(s, \xi, \xi^\vee, \Phi)$ for standard sections $\Phi(s)$. Thus, by the definition of the L-factor, it follows that $Z^*(s_0)$ is a non-zero element of $Hom_{G \times G}(I_n(s_0, \chi), \pi^\vee \otimes (\chi \cdot \pi))$. On the other hand, suppose that the family $Z(s, \xi, \xi^\vee, \Phi)$ for good sections has a higher order pole at $s = s_0$ than the family $Z(s, \xi, \xi^\vee, \Phi)$ for standard sections $\Phi(s)$, so that $Z^*(s_0)$ will be identically zero. In this case, we can replace $Z^*(s_0)$ by the leading term of Laurent expansion of the family

$$(6.28) \quad \frac{Z(s, \xi, \xi^\vee, \Phi)}{L(s + \frac{1}{2}, \pi, \chi)}$$

for $\Phi(s)$ a standard section. This leading term again defines a non-zero intertwining map, and (i) of Theorem 4.3 is proved. \square

We now specialize (6.27) to the case $s_0 = 0$. If we assume that $\chi|_{F^\times} = \epsilon_{E/F}^n$, then, by (iv) of Lemma 6.7, the order of pole at $s = 0$ of the family $Z(s, \xi, \xi^\vee, \Phi)$ for good sections $\Phi(s)$ coincides with that for the family $Z(s, \xi, \xi^\vee, \Phi)$ for standard sections $\Phi(s)$. Thus, by the definition of the L-factor, it follows that $Z^*(0)$ is a non-zero element of $Hom_{G \times G}(I_n(0, \chi), \pi^\vee \otimes (\chi \cdot \pi))$. We want to consider the restriction of $Z^*(0)$ to the subspaces $R_n(V_n^\pm, \chi)$. For this we note that the functional equation (6.22) implies that

$$(6.29) \quad Z^*(0) \circ M^*(0, \chi) = \epsilon(\frac{1}{2}, \pi, \chi, \psi) \cdot Z^*(0).$$

On the other hand, by (6.11),

$$(6.30) \quad M^*(0, \chi)^2 = 1.$$

Since the $R_n(V_n^\pm, \chi)$'s are irreducible and non-isomorphic submodules of $I_n(0, \chi)$, $M^*(0, \chi)$ must act by a scalar on each of them.

Proposition 6.8.

$$M^*(0, \chi)|_{R_n(V, \chi)} = \epsilon_{E/F}(-2)^n \cdot \epsilon_{E/F}(\det V).$$

Proof. By (6.10), we have

$$(6.31) \quad W_T(0) \circ M^*(0, \chi) = \epsilon_{E/F}(\det T) \cdot W_T(0).$$

On the other hand, $W_T(0)$ is non-zero on $R_n(V, \chi)$ for $V = V_{-2T}$, the Hermitian space of dimension n defined by the Hermitian matrix T , while it is identically zero on the other Hermitian space of dimension n , [25]. Thus we find that

$$(6.32) \quad M^*(0, \chi)|_{R_n(V_{-2T}, \chi)} = \epsilon_{E/F}(\det T),$$

as claimed. □

Comparing this result with the functional equation (6.29), we obtain:

Proposition 6.9.

$$Z^*(0)|_{R_n(V, \chi)} \neq 0 \iff \epsilon(\frac{1}{2}, \pi, \chi, \psi) = \epsilon_{E/F}(-2)^n \cdot \epsilon_{E/F}(\det V).$$

Combining this result with Proposition 3.1, we obtain the second statement of Theorem 6.1. □

Proof of Theorem 6.2. Since the matrix coefficients of a supercuspidal representation π are compactly supported, the only poles of $Z(s, \xi, \xi^\vee, \Phi)$ must arise from the poles of the good section $\Phi(s)$. By (ii) of Lemma 6.7, these must lie among the poles of $b_n(s, \chi)$.

Lemma 6.10. *Assume that the pair (s, χ) is normalized, and write $\chi^o = \chi|_{F^\times}$.*

- (i) *If $\chi \neq \tilde{\chi}$, then $b_n(s, \chi)$ is entire.*
- (ii) *If $\chi|_{F^\times} = \tilde{\chi}$, then the poles of $b_n(s, \chi)$ are simple and occur at the points*

$$X_n(\chi) = \left\{ \frac{m-n}{2} \mid 0 \leq m \leq n-1, \chi^o = \epsilon_{E/F}^m \right\}.$$

Now suppose that $\chi = \tilde{\chi}$ is fixed and that $s_0 = \frac{m-n}{2} \in X_n(\chi)$. Then, in a neighborhood of s_0 , any good section can be written in the form

$$(6.33) \quad \Phi(s) = b_n(s, \chi) \sum_j \alpha_j(s) \Psi_j(s),$$

where the $\Psi_j(s)$ are standard sections and the $\alpha_j(s)$ are holomorphic functions. Moreover, if $\alpha_j(s_0) \neq 0$, then

$$(6.34) \quad \Psi_j(s_0) \in R_n(V_m^+, \chi) \oplus R_n(V_m^-, \chi) \subset I_n(s_0, \chi).$$

Here we note that, by (iii) of Proposition 4.1, the submodule $R_n(V_m^+, \chi) \oplus R_n(V_m^-, \chi)$ is the image of the intertwining operator $M^{**}(-s_0, \chi)$. Moreover, if χ is unramified, this submodule contains the spherical vector $\Phi^0(s_0)$. Thus, the family of zeta integrals $Z(s, \xi, \xi^\vee, \Phi)$ has a pole at $s = s_0$ if and only if there is a standard section $\Psi(s)$ with $\Psi(s_0) \in R_n(V_m^+, \chi) \oplus R_n(V_m^-, \chi)$ such that

$$(6.35) \quad Z(s_0, \xi, \xi^\vee, \Psi) \neq 0.$$

But the restriction of $Z(s_0, \cdot, \cdot, \cdot)$ to

$$(6.36) \quad \pi \otimes \pi^\vee \otimes \left(R_n(V_m^+, \chi) \oplus R_n(V_m^-, \chi) \right)$$

defines an element of

$$(6.37) \quad \text{Hom}_{G \times G}(R_n(V_m^+, \chi) \oplus R_n(V_m^-, \chi), \pi^\vee \otimes (\chi \cdot \pi)).$$

This space is at most one dimensional, by Corollary 4.2 and Theorem 4.3, and it is easy to see that the element defined by $Z(s_0, \cdot, \cdot, \cdot)$ is, in fact, a basis. Applying Proposition 3.1, we obtain the assertion of Theorem 6.2. \square

We conclude this section with some remarks on the way in which the epsilon factor defined by doubling depends on the Hermitian space W . First recall that the discussion of (6.3) – (6.11) takes place on the group $U(n, n)$, and hence is independent of the embedding $i : G(W) \times G(W) \rightarrow U(n, n)$.

For $\alpha \in F^\times$, let W_α denote the space W with skew-Hermitian form $\alpha \langle \cdot, \cdot \rangle$. If $\epsilon_{E/F}(\alpha) = -1$, i.e., if $\alpha \notin N_{E/F}E^\times$, and if n is odd, then the skew-Hermitian spaces W_α and W are not isomorphic, although their isometry groups $G = G(W) = G(W_\alpha)$ are canonically identified. We obtain the diagram:

$$(6.38) \quad \begin{array}{ccccccc} i : & G(W) \times G(W) & \longrightarrow & G(W + \bar{W}) & \longrightarrow & U(n, n) & \\ & \parallel & & \parallel & & \downarrow \text{Ad}(c_\alpha) & \\ i_\alpha : & G(W_\alpha) \times G(W_\alpha) & \longrightarrow & G(W_\alpha + \bar{W}_\alpha) & \longrightarrow & U(n, n), & \end{array}$$

where, for a suitable choice of bases,

$$(6.39) \quad c_\alpha = \begin{pmatrix} 1_n & \\ & \alpha \cdot 1_n \end{pmatrix}.$$

Note that $\text{Ad}(c_\alpha)$ preserves the Siegel parabolic P and carries $I_n(s, \chi)$ to itself, i.e., if $\Phi(s) \in I_n(s, \chi)$, then the function

$$(6.40) \quad \Phi^\alpha(g, s) = \Phi(c_\alpha g c_\alpha^{-1}, s)$$

also lies in $I_n(s, \chi)$. For a matrix coefficient ϕ of π and for a section $\Phi(s) \in I_n(s, \chi)$, let

$$(6.41) \quad Z_\alpha(s, \phi, \Phi) = \int_G \phi(g) \Phi(i_\alpha(g, 1), s) dg = Z(s, \phi, \Phi^\alpha).$$

From the family of zeta integrals $Z_\alpha(s, \phi, \Phi)$ obtained by letting $\Phi(s)$ run over the space of good sections of Definition 6.4, we obtain factors $L_\alpha(s, \pi, \chi)$ and $\epsilon_\alpha(s, \pi, \chi, \psi)$.

Proposition 6.11. (i) For any χ ,

$$\Gamma_\alpha^*(s, \pi, \chi, \psi) = \chi(\alpha)^n |\alpha|_F^{2ns} \Gamma^*(s, \pi, \chi, \psi).$$

In particular, if $L(s, \pi, \chi)$ and $L_\alpha(s, \pi, \chi)$ have poles at $s = \frac{1}{2}$ whose orders have the same parity, e.g., if π is supercuspidal, then

$$\epsilon_\alpha(1/2, \pi, \chi, \psi) = \chi(\alpha)^n \epsilon(1/2, \pi, \chi, \psi).$$

(ii) Suppose that χ is ramified. Then

$$L_\alpha(s, \pi, \chi) = L(s, \pi, \chi),$$

and

$$\epsilon_\alpha(s, \pi, \chi, \psi) = \chi(\alpha)^n |\alpha|_F^{2n(s-\frac{1}{2})} \epsilon(s, \pi, \chi, \psi).$$

Thus, if n is odd and if $\chi|_{F^\times} = \epsilon_{E/F}$, the root numbers $\epsilon(1/2, \pi, \chi, \psi)$ of π , defined by the doubling method, vary with α as π is viewed as a representation of $G(W_\alpha)$!

In fact, we expect that the condition that χ be ramified can be dropped. Note that this result is compatible with the comparison conjecture (0.18) and (0.19).

Proof. We begin with (ii). If χ is ramified, then the good sections are just those defined by (a) and (b) of Definition 6.4.

Lemma 6.12. (i) If

$$\Phi(s) \in \mathbb{C}[q^s, q^{-s}] \cdot (\text{standard sections}),$$

then

$$\Phi^\alpha(s) \in \mathbb{C}[q^s, q^{-s}] \cdot (\text{standard sections}).$$

(ii) If

$$\Phi(s) \in M^*(s, \chi) \left(\mathbb{C}[q^s, q^{-s}] \cdot (\text{standard sections}) \right),$$

then

$$\Phi^\alpha(s) \in M^*(s, \chi) \left(\mathbb{C}[q^s, q^{-s}] \cdot (\text{standard sections}) \right).$$

(iii)

$$M(s, \chi) \Phi^\alpha = \chi(\alpha)^{-1} |\alpha|_F^{-2ns} (M(s, \chi) \Phi)^\alpha.$$

Proof. Let

$$(6.42) \quad K' = K \cap c_\alpha^{-1} K c_\alpha,$$

and write

$$(6.43) \quad K = \prod_i (P \cap K) k_i K',$$

with $k_i \in K$. Also write

$$(6.44) \quad c_\alpha k_i c_\alpha^{-1} = n_i m(a_i) \tilde{k}_i.$$

For a standard section $\Phi(s) \in I_n(s, \chi)$, and for each index i , define a standard section $\Phi_i^\alpha(s)$ with

$$(6.45) \quad \text{supp}(\Phi_i^\alpha(s)) = P k_i K'$$

and

$$(6.46) \quad \Phi_i^\alpha(k_i k', s) = \Phi(\tilde{k}_i c_\alpha k' c_\alpha^{-1}, s).$$

Note that the second expression here is independent of s , since $\Phi(s)$ is standard. Then

$$(6.47) \quad \Phi^\alpha(s) = \sum_i \chi(\det a_i) |\det a_i|^{s+\rho_n} \cdot \Phi_i^\alpha(s),$$

and so, in particular, $\Phi^\alpha(s)$ lies in $\mathbb{C}[q^s, q^{-s}] \cdot (\text{standard sections})$, as claimed in (i).

Next we prove (iii), from which (ii) follows immediately. We have:

$$\begin{aligned} (6.48) \quad M(s, \chi) \Phi^\alpha(g) &= \int_N \Phi^\alpha(w n(b) g, s) db \\ &= \int_N \Phi(c_\alpha w n(b) g c_\alpha^{-1}, s) db \\ &= \int_N \Phi(m(\alpha^{-1}) w n(\alpha^{-1} b) c_\alpha g c_\alpha^{-1}, s) db \\ &= \chi(\alpha)^{-n} |\alpha|_F^{-2n(s+\frac{n}{2})} |\alpha|_F^{n^2} \int_N \Phi(w n(b) c_\alpha g c_\alpha^{-1}, s) db \\ &= \chi(\alpha)^{-n} |\alpha|_F^{-2ns} (M(s, \chi) \Phi)^\alpha(g), \end{aligned}$$

as was to be shown. □

Thus, if χ is ramified, we have shown that

$$(6.49) \quad (\text{good sections})^\alpha = (\text{good sections}),$$

and so, by (6.41), we find the claimed relation between the L-factors.

Next we make no assumptions about χ and consider the functional equation:

$$(6.50) \quad Z_\alpha(-s, \phi, M^*(s, \chi)\Phi) = \Gamma_\alpha^*(s, \pi, \chi, \psi) Z_\alpha(s, \phi, \Phi) = \Gamma_\alpha^*(s, \pi, \chi, \psi) Z(s, \phi, \Phi^\alpha).$$

The left side equals:

$$(6.51) \quad \begin{aligned} Z(-s, \phi, (M^*(s, \chi)\Phi)^\alpha) &= \chi(\alpha)^n |\alpha|_F^{2ns} Z(-s, \phi, M^*(s, \chi)\Phi^\alpha) \\ &= \chi(\alpha)^n |\alpha|_F^{2ns} \Gamma^*(s, \pi, \chi, \psi) Z(s, \phi, \Phi^\alpha). \end{aligned}$$

Here we assume that $\Phi(s)$ is a standard section, so that we may apply (6.48). Comparing these expressions, we find that

$$(6.52) \quad \Gamma_\alpha^*(s, \pi, \chi, \psi) = \chi(\alpha)^n |\alpha|_F^{2ns} \Gamma^*(s, \pi, \chi, \psi),$$

as claimed in (i). If χ is ramified, we have already shown that the L-factors do not change with the scaling by α , and thus, via (6.23), we obtain the behavior of the ϵ factor under such scaling claimed in (ii).

If χ is arbitrary, and if we assume that the two L-factors have orders at $s = \frac{1}{2}$ of the same parity, then the result on the root number claimed in (i) again follows from (6.52) and (6.23). □

§7. FUNCTORIALITY

In this section we review the, mostly conjectural, functoriality properties of the local theta correspondence. In the unitary group case, conjectures about the behavior of A-packets under the correspondence were made by J. Adams [1], and also by Gelbart and Rogawski [6]. After reviewing these conjectures, we hazard a slightly more refined guess about the L-packets involved. We then illustrate this guess in certain very special cases and finally explain what it would say about the L-parameters of the supercuspidals occurring in the ‘curious example’ of section 5.

Throughout this section, we let

$$W'_F = SL_2(\mathbb{C}) \times W_F \quad \text{and} \quad W''_F = W'_F \times SL_2(\mathbb{C}),$$

where W_F is the Weil group of F . Also, for a character μ of E^\times , we write μ° for the restriction $\mu|_{F^\times}$.

First, recall that if $H = G(V)$ is a unitary group, with $\dim_E V = m$, then

$$(7.1) \quad {}^L H = GL_m(\mathbb{C}) \rtimes W_F,$$

where the subgroup W_E of W_F acts trivially, and an element $w_\sigma \in W_F - W_E$ acts on ${}^L H = GL_m(\mathbb{C})$ by

$$(7.2) \quad g \mapsto \Phi_m {}^t g^{-1} \Phi_m^{-1},$$

where

$$(7.3) \quad \Phi_m = \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & \ddots & & \\ & & & & \\ (-1)^{m-1} & & & & \end{pmatrix}.$$

We fix an element $w_\sigma \in W_F - W_E$ once and for all. For a Langlands parameter

$$(7.4) \quad \varphi : W'_F \longrightarrow {}^L H,$$

we will write Π_φ for the (conjecturally) corresponding L-packet. These are, of course, known for $U(1)$, $U(2)$ and $U(3)$ [41]. In the case $m = 1$, for a character η of $U(1) \simeq E^1$, the norm 1 subgroup of E^\times , we let $\tilde{\eta}$ be the character of E^\times obtained by pullback:

$$(7.5) \quad \tilde{\eta}(x) = \eta(x/\bar{x}).$$

Then, for $\varphi : W_F \longrightarrow \mathbb{C}^\times \rtimes W_F$ defined by

$$(7.6) \quad \varphi : w \mapsto \tilde{\eta}(w) \times w \quad \text{and} \quad w_\sigma \mapsto 1 \times w_\sigma,$$

$\Pi_\varphi = \{\eta\}$. The cases of $U(2)$ (due essentially to [26]) and $U(3)$ (due to [41], Chapter 13) are discussed in Chapters 11 and 12–13 of [41].

Finally, if

$$(7.7) \quad \psi : W''_F \longrightarrow {}^L H$$

is an Arthur parameter [2], we let Π_ψ be the (conjecturally) corresponding Arthur packet. Again, these are known for $U(1)$ (trivial), $U(2)$ and $U(3)$, and are discussed in detail in [27]. Recall that for a given Arthur parameter ψ , there is a corresponding Langlands parameter

$$(7.8) \quad \varphi_\psi : W'_F \longrightarrow {}^L H$$

defined by composing ψ with the map $W'_F \longrightarrow W''_F$ given by

$$(7.9) \quad g_1 \times w \mapsto g_1 \times w \times \begin{pmatrix} |w|^{\frac{1}{2}} & \\ & |w|^{-\frac{1}{2}} \end{pmatrix}.$$

Here recall that, according to our convention, the second $SL_2(\mathbb{C})$ is added last.

We recall from [41, Chapter 4, 4.6] that for a unitary group $H = G(V)$, where $\dim_E V = m$, the elliptic endoscopic groups are the quasi-split unitary groups $I = U(a) \times U(b)$, where $a + b = m$. The embedding ${}^L I \longrightarrow {}^L H$ depends on the choice of characters μ_1 and μ_2 of E^\times satisfying:

$$(7.10) \quad \mu_1^o = \epsilon_{E/F}^b \quad \text{and} \quad \mu_2^o = \epsilon_{E/F}^a.$$

Then

$$(7.11) \quad \begin{aligned} (g_1, g_2) \times 1 &\mapsto \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix} \times 1, \\ 1 \times w &\mapsto \begin{pmatrix} \mu_1(w) & \\ & \mu_2(w) \end{pmatrix} \times w, \end{aligned}$$

where $w \in W_E$, and

$$(7.12) \quad 1 \times w_\sigma \mapsto \Phi_{a,b} \cdot \Phi_m^{-1} \times w_\sigma.$$

Note that the conditions on μ_1^o and μ_2^o are ‘crossed’. Here

$$(7.13) \quad \Phi_{a,b} = \begin{pmatrix} \Phi_a & \\ & \Phi_b \end{pmatrix}.$$

Now suppose that $G = G(W)$ and $H = G(V)$ are given, with W skew-Hermitian, $\dim_E W = n$, and V Hermitian with $\dim_E V = m$. Assume that $m \geq n$, and let $I = U(a) \times U(b)$ be the elliptic endoscopic group of H with $a = n$ and $b = m - n$. We define an L-homomorphism ${}^L G \rightarrow {}^L I$ by

$$(7.14) \quad \begin{aligned} g \times 1 &\mapsto (g, 1_b) \times 1, \\ 1 \times w &\mapsto 1 \times w \end{aligned}$$

and $1 \times w_\sigma \mapsto 1 \times w_\sigma$. Composing this with the map ${}^L I \rightarrow {}^L H$ defined above, we obtain a map $\alpha_{m,n}$,

$$(7.15) \quad \begin{aligned} \alpha_{m,n} : {}^L G &\longrightarrow {}^L H, \\ g \times 1 &\mapsto \begin{pmatrix} g & \\ & 1_b \end{pmatrix} \times 1, \\ 1 \times w &\mapsto \begin{pmatrix} \mu_1(w) & \\ & \mu_2(w) \end{pmatrix} \times w \end{aligned}$$

and

$$(7.16) \quad 1 \times w_\sigma \mapsto \Phi_{a,b} \cdot \Phi_m^{-1} \times w_\sigma.$$

This map plays a key role in Adams’ conjecture.

One additional ingredient is needed. For any $b \in \mathbb{Z}_{\geq 1}$, let ρ_b be the symmetric tensor representation of $SL_2(\mathbb{C})$ on $Sym^{b-1}(\mathbb{C}^2)$. Note that $\dim_{\mathbb{C}} \rho_b = b$. We take the monomial basis

$$\frac{(b-1)!}{(b-i-1)!} x^{b-i-1} y^i,$$

$0 \leq i \leq b-1$, for $Sym^{b-1}(\mathbb{C}^2)$, and hence have $\rho_b : SL_2(\mathbb{C}) \rightarrow GL_b(\mathbb{C})$.

Lemma 7.1. *For all $g \in SL_2(\mathbb{C})$,*

$$\Phi_b^{-1}({}^t \rho_b(g)^{-1}) \Phi_b = \rho_b(g).$$

In particular, the representation ρ_b is symplectic for b even and orthogonal for b odd.

Now suppose that χ_1 and χ_2 are characters of E^\times with $\chi_1^o = \epsilon_{E/F}^m$ and $\chi_2^o = \epsilon_{E/F}^n$, so that we obtain a splitting homomorphism

$$(7.17) \quad \tilde{t}_{V,\chi_1} \times \tilde{t}_{W,\chi_2} : G(W) \times G(V) \longrightarrow Mp(\mathbb{W}),$$

as described in section 1 and in the appendix. For simplicity, we will often write χ for the pair of characters (χ_1, χ_2) .

Conjecture 7.2 (Adams’ conjecture). *Suppose that $\psi : W_F'' \longrightarrow {}^L G$ is an Arthur parameter for G and define an Arthur parameter*

$$\theta_\chi(\psi) : W_F'' \longrightarrow {}^L H$$

for H by

$$\theta_\chi(\psi)|_{W_F'} = \alpha_\chi \circ \psi|_{W_F'},$$

and

$$\theta_\chi(\psi)|_{1 \times 1 \times SL_2(\mathbb{C})} = \left(\alpha_\chi \circ \psi|_{1 \times 1 \times SL_2(\mathbb{C})} \right) \oplus \rho_b.$$

Here α_χ is the L -homomorphism defined above with

$$\mu_1 = \chi_1^{-1} \chi_2,$$

and

$$\mu_2 = \chi_2.$$

This is well defined, i.e., the image of w_σ commutes with the image of the second $SL_2(\mathbb{C})$ by Lemma 7.1. Suppose that π is an irreducible admissible representation of $G(W)$ with $\pi \in \Pi_\psi$. Then

$$\theta_\chi(\pi, V) \in \Pi_{\theta_\chi(\psi)}.$$

Note that, if $m - n \geq 2$, the parameter $\theta_\chi(\psi)$ will be non-trivial on Arthur’s $SL_2(\mathbb{C})$, even if the original parameter is trivial on this factor. This corresponds to the fact that the theta lift of a tempered representation may be non-tempered.

In fact, Adams [1] states his conjecture in terms of E-groups, rather than L-groups, and works directly with the representations of the full inverse images of $G(W)$ and $G(V)$ in $Mp(\mathbb{W})$, or rather its twofold analogue. Thus he avoids the choice of the pair of characters χ needed to define the splitting homomorphism and the choice of the pair of characters (μ_1, μ_2) needed to define the L-homomorphism. He also gives a more intrinsic description of the ‘tail’ ρ_b , as the representation of $SL_2(\mathbb{C})$ associated to the principal unipotent conjugacy class in the centralizer of the image under α of \tilde{G} . Our description is more convenient for the calculations we have at hand, and it is an amusing exercise to check that the two versions of the conjecture agree.

A similar surmise about the relation of the theta correspondence to endoscopy was made by Gelbart and Rogawski [6, p. 448], based on their results in the cases $(U(1), U(1))$, $(U(1), U(3))$ and $(U(2), U(3))$.

First we observe that the relationship we have given between the pairs $\chi = (\chi_1, \chi_2)$ and $\mu = (\mu_1, \mu_2)$ is more or less determined. Specifically, suppose that we change χ_i to $\chi_i \nu_i$ for characters ν_1 and ν_2 of E^\times which are trivial on F^\times . Let ν'_i , $i = 1, 2$, denote the corresponding characters of E^1 . Then the splitting homomorphism associated to χ' differs from that associated to χ by the character $\nu' = \nu'_1 \otimes \nu'_2$ of $G(W) \times G(V)$, and hence the Weil representations satisfy

$$(7.18) \quad S_{\chi'} = \nu' \otimes S_\chi.$$

Thus

$$\begin{aligned}
 \sigma_2 = \theta_\chi(\sigma_1, V) &\iff \text{Hom}_{G(W) \times G(V)}(S_\chi, \sigma_1 \otimes \sigma_2) \neq 0 \\
 &\iff \text{Hom}_{G(W) \times G(V)}(S_{\chi'}, (\nu'_1 \sigma_1) \otimes (\nu'_2 \sigma_2)) \neq 0 \\
 (7.19) \qquad \qquad \qquad &\iff \nu'_2 \sigma_2 = \theta_{\chi'}(\nu'_1 \sigma_1, V).
 \end{aligned}$$

This determines how the groups of characters of $G(W) \times G(V)$ must act on our parameters.

Remark. It only remains to check that we have chosen the correct base point. This can be done by considering the correspondence on irreducible principal series and using the compatibility with parabolic induction. In particular, using the standard methods of [37], [21], and the analysis of splittings given in [22] and the appendix to this article, it is easy to verify that the conjecture of Adams is true when π is an irreducible principal series representation and the theta correspondence is defined using an unramified additive character. In particular, the characters μ_1 and μ_2 are correct in this case. We omit the details.

Next we present some speculations about the actual L-parameters involved. We phrase these as speculations rather than as conjectures, since we have not tested them in sufficiently general situations. Throughout this discussion we will assume the Howe duality conjecture, and write $\theta_\chi(\pi, V)$ for the unique irreducible quotient of $\Theta_\chi(\pi, V)$ (or for 0 if the irreducible representation π of $G(W)$ does not occur in the χ -theta correspondence with V). We will also grant the existence of L-packets in the extended sense described by Vogan [48]. Thus, for a given Langlands parameter $\varphi : W'_F \rightarrow {}^L G$, the packet Π_φ will¹ be a finite set of irreducible admissible representations, some of which may be representations of $G^+ = G(W^+)$ and some of $G^- = G(W^-)$, where W^\pm are the two possible skew-Hermitian spaces of dimension n . Moreover, the representations in the set Π_φ will be parameterized by the irreducible characters, taken up to a certain equivalence, of the finite group A_φ , where $A_\varphi = C_\varphi/C_\varphi^0$, for C_φ the centralizer in ${}^\vee G$ of the image of φ . The precise description is given in Vogan [48], p. 344, Definition 4.14 and Conjecture 4.15. In our present case, ${}^\vee G = GL_n(\mathbb{C})$, so that $Z({}^\vee G) \simeq \mathbb{C}^\times$,

$$Z({}^\vee G)^\Gamma \simeq \langle \pm 1 \rangle,$$

and

$$H^1(\Gamma, G) \xrightarrow{\sim} (Z({}^\vee G)^\Gamma)^\wedge \simeq (\langle \pm 1 \rangle)^\wedge.$$

There is a natural map

$$Z({}^\vee G)^\Gamma \rightarrow A_\varphi,$$

and, after suitable choice of base point (Vogan [48] pp. 345–346), the representation π attached to the irreducible representation ξ of A_φ lives on the unitary group $G(W^\epsilon)$, where $\epsilon = \xi(-1)$ is the value of ξ on the image in A_φ of the non-trivial element in $Z({}^\vee G)^\Gamma$.

We will use the same convention for the groups $G(V^\pm)$.

We will make essential use of the involution $\psi \leftrightarrow \psi'$ on the set of Arthur parameters, mentioned in [27]. The parameter π' is obtained by composing ψ with the map $sw : W''_F \rightarrow W''_F$,

$$(7.20) \qquad \qquad \qquad sw : g_1 \times w \times g_2 \mapsto g_2 \times w \times g_1.$$

¹We are now granting the conjectures...

Fix the pair of skew-Hermitian spaces W_n^\pm of dimension n , and consider the two Witt towers $\{V_m^+\}$ and $\{V_m^-\}$ with $\dim_E V_m^\pm = m$. Here we use an obvious variant of the notation introduced at the beginning of section 5. Let $G_n^\pm = G(W_n^\pm)$ and let $H_m^\pm = G(V_m^\pm)$. Fix the pair of characters χ used to define the splitting homomorphism and the theta correspondence for all of the dual pairs (G_n^+, H_m^\pm) and (G_n^-, H_m^\pm) .

Speculation 7.3. *If $m = n$, then the theta correspondence is functorial, i.e., if π lies in an (extended) L-packet $\Pi_\varphi(G_n)$ and if $\chi_1 = \chi_2$, then $\theta_\chi(\pi, V_n^\pm)$ lies in the (extended) L-packet $\Pi_\varphi(H_n)$.*

Speculation 7.4. *If $m = n$, then for any irreducible admissible π , precisely one of the spaces $\theta_\chi(\pi, V_n^+)$ and $\theta_\chi(\pi, V_n^-)$ is non-zero. (If π is supercuspidal, this is Corollary 4.4.) The sign $\epsilon(\pi) = \pm 1$ for which $\theta_\chi(\pi, V_n^{\epsilon(\pi)}) \neq 0$ may depend on χ , and upon the inner form G_n^\pm of which π is a representation, but is otherwise independent of the choice of π within the extended packet. Thus, if $\chi_1 = \chi_2 = \chi$, then the map*

$$\pi \mapsto \theta_\chi(\pi, V_n^{\epsilon(\pi)})$$

permutes the elements of the (extended) L-packet. In particular, this permutation either carries

$$\Pi_\varphi(G_n^+) \leftrightarrow \Pi_\varphi(G_n^+) \quad \text{and} \quad \Pi_\varphi(G_n^-) \leftrightarrow \Pi_\varphi(G_n^-),$$

or

$$\Pi_\varphi(G_n^+) \leftrightarrow \Pi_\varphi(G_n^-) \quad \text{and} \quad \Pi_\varphi(G_n^-) \leftrightarrow \Pi_\varphi(G_n^+).$$

Note that the second case can occur only when

$$|\Pi_\varphi(G_n^+)| = |\Pi_\varphi(G_n^-)|.$$

Suppose that φ is a fixed Langlands parameter for G . For π in the L-packet Π_φ , suppose that $\theta_\chi(\pi, V_n^+) \neq 0$, and hence that $\theta_\chi(\pi, V_n^-) = 0$. In this case, we say that π occurs **late** in the $(-)$ -tower. How late it occurs should be determined by how **early** it occurs in the $(+)$ -tower:

Speculation 7.5. *Let m_0^+ (resp. m_0^-) be the smallest value of m , of the same parity as n , such that $\theta_\chi(\pi, V_m^+) \neq 0$ (resp. $\theta_\chi(\pi, V_m^-) \neq 0$). Then*

$$m_0^+ + m_0^- = 2n + 2.$$

Moreover, the pair $\{m_0^+, m_0^-\}$ depends only on the L-packet Π_φ .

Note that the idea that $\{m_0^+, m_0^-\}$ depends on φ alone is motivated by (ii) of Theorem 6.2.

Remark. In the case of m and n odd, so that the groups $G^+ = G(W_n^+)$ and $G^- = G(W_n^-)$ can be canonically identified, a given representation π of this group should occur twice in the (extended) packet, once as π^+ , a representation of G_n^+ , and once as π^- , a representation of G_n^- . For fixed χ , these two should have non-zero theta lifts to opposite towers on the V side, and should account for the same ‘multiplicity 2’ phenomenon for $G(V_n^+) = G(V_n^-)$. For a given supercuspidal π , this disjunction follows from Theorem 6.1 and Proposition 6.11.

For example, one might expect that for most representations,

$$(7.21) \quad \{m_0^+, m_0^-\} = \{n, n + 2\},$$

i.e., most representations do not show up in the theta correspondence until they are forced to, by Corollary 4.5, and this delays their appearance only one step in the ‘late’ tower.

The following simple example is instructive. Consider groups

$$(7.22) \quad \begin{array}{cccc} G_3^+ & G_3^- & H_3^+ & H_3^- \\ G_1^+ & G_1^- & H_1^+ & H_1^- \end{array}$$

For any character η of E^1 , the extended L-packet consists of the pair $\{\eta^+, \eta^-\}$, where η^\pm is the character η of G_1^\pm . We assume that the characters $\chi_1 = \chi_2 = \chi$, with $\chi^o = \epsilon_{E/F}$, are fixed, and we let

$$(7.23) \quad \epsilon = \epsilon(\frac{1}{2}, \chi\tilde{\eta}^{-1}, \psi_E)\eta(-1)\chi(-\delta)$$

be the sign given by Corollary 8.5. Then, if $\epsilon = 1$, we have the following picture:

$$(7.24) \quad \begin{array}{cccc} \theta_\chi(\eta^+, W_3^+) & \theta_\chi(\eta^-, W_3^-) & \theta_\chi(\eta^+, V_3^+) & \theta_\chi(\eta^-, V_3^-) \\ \theta_\chi(\eta^-, W_3^+) & \theta_\chi(\eta^+, W_3^+) & \theta_\chi(\eta^-, V_3^+) & \theta_\chi(\eta^+, V_3^-) \\ \eta^+ & \eta^- & \eta^+ = \theta_\chi(\eta^+, V_1^+) & \eta^- = \theta_\chi(\eta^-, V_1^-) \end{array}$$

The representations in the top two rows may be identified as:

$$(7.25) \quad \begin{array}{cccc} \pi^{n,+} & \pi^{n,-} & \pi^{n,+} & \pi^{n,-} \\ \pi^{s,+} & \pi^{s,-} & \pi^{s,+} & \pi^{s,-} \end{array},$$

where $\pi^{n,\pm}$ is the non-tempered constituent of the principal series associated to the character

$$(7.26) \quad \begin{pmatrix} a & & & \\ & b & & \\ & & \bar{a}^{-1} & \\ & & & \end{pmatrix} \mapsto \chi(a)|a|_{E}^{\frac{1}{2}}\eta(b),$$

and viewed as a representation of the group G_3^\pm or H_3^\pm . In Rogawski [41, pp.198–199], this representation is denoted by $\pi^n(\xi)$, for a suitable ξ . Similarly, $\pi^{s,\pm}$ is the supercuspidal representation, denoted by $\pi^s(\xi)$ in Rogawski [41], Proposition 13.1.3 (d), p.199, where ξ is determined by the condition that $\{\pi^n(\xi), \pi^s(\xi)\}$ is the A-packet discussed on p. 199, and, again, viewed as a representation of the group G_3^\pm or H_3^\pm .

Finally, let $\pi^{2,\pm}$ ($\pi^2(\xi)$ in [41]) be the square integrable constituent of the principal series containing $\pi^{n,\pm}$ ($\pi^n(\xi)$ in [41]). Then the collection of representations

$$(7.27) \quad \theta_\chi(\pi^{s,-}, W_3^+) \quad \theta_\chi(\pi^{s,+}, W_3^-) \quad \theta_\chi(\pi^{s,-}, V_3^+) \quad \theta_\chi(\pi^{s,+}, V_3^-),$$

coincides with

$$(7.28) \quad \pi^{2,+} \quad \pi^{2,-} \quad \pi^{2,+} \quad \pi^{2,-}.$$

Observe that $\{\pi^{n,+}, \pi^{n,-}\}$ is the extended L-packet associated to the non-tempered parameter

$$(7.29) \quad \begin{aligned} \varphi^n : W'_F &\longrightarrow GL_3(\mathbb{C}) \rtimes W_F, \\ w &\mapsto \begin{pmatrix} \tilde{\eta}(w) & & \\ & \chi(w)|w|^{\frac{1}{2}} & \\ & & \chi(w)|w|^{-\frac{1}{2}} \end{pmatrix} \times w, \end{aligned}$$

$$(7.30) \quad w_\sigma \mapsto \Phi_{1,2}\Phi_3^{-1} \times w_\sigma$$

and trivial on $SL_2(\mathbb{C})$. Note that this parameter is obtained by applying Arthur’s recipe, (7.8) and (7.9), to the A-parameter given by Adams’ conjecture.

Similarly,

$$(7.31) \quad \{\pi^{s,+}, \pi^{2,+}, \pi^{s,-}, \pi^{2,-}\}$$

is the extended discrete series L-packet associated to the parameter

$$(7.32) \quad \begin{aligned} \varphi^2 : W'_F &\longrightarrow GL_3(\mathbb{C}) \rtimes W_F, \\ w &\mapsto \begin{pmatrix} \tilde{\eta}(w) & & \\ & \chi(w) & \\ & & \chi(w) \end{pmatrix} \times w, \end{aligned}$$

$$w_\sigma \mapsto \Phi_{1,2}\Phi_3^{-1} \times w_\sigma,$$

and

$$(7.33) \quad g \mapsto \begin{pmatrix} 1 & \\ & g \end{pmatrix} \times 1,$$

where $g \in SL_2(\mathbb{C})$. Note that this parameter is obtained by applying the involution (7.20) on Arthur parameters to the parameter of Adams' conjecture. The resulting Arthur parameter is trivial on the second $SL_2(\mathbb{C})$, and hence is just an L-parameter. It turns out to be precisely the L-parameter we want.

The situation when the sign ϵ of (7.23) is -1 is similar, with the obvious switch in signs.

Next we consider the case in which m and n have opposite parity. Here we recall that Corollary 4.5 asserts that, for any irreducible admissible representation π of G_n^\pm , at least one of the spaces $\theta_\chi(\pi, V_{n+1}^+)$ and $\theta_\chi(\pi, V_{n+1}^-)$ is non-zero.

Speculation 7.6. *Let m_0^+ (resp. m_0^-) be the smallest value of m , of parity opposite to that of n , such that $\theta_\chi(\pi, V_m^+) \neq 0$ (resp. $\theta_\chi(\pi, V_m^-) \neq 0$). Then*

$$m_0^+ + m_0^- = 2n + 2.$$

Moreover, the pair $\{m_0^+, m_0^-\}$ depends only on the L-packet Π_φ .

Thus, for most L-packets, we would have $m_0^+ = m_0^- = n + 1$. At the other extreme, for n odd and for the trivial representation $\mathbb{1}$, we have $m_0^+ = 0$, since $\mathbb{1}$ comes from the $U(0)$ which lies, formally, at the bottom of the tower of 'split' unitary groups. Thus, Speculation 7.6 predicts that $m_0^- = 2n + 2$, so that $\theta_\chi(\mathbb{1}, V_{2r}^-) = 0$ for $r = 1, 2, \dots, n$. Here we are assuming that $\chi_1 = 1$. In fact, the vanishing of $\theta_\chi(\mathbb{1}, V_{2r}^-)$ in this range has now been proved, [24].

These speculations are not difficult to verify for the dual pairs $(U(1), U(m))$, m even, and $(U(2), U(m))$, m odd.

Next, we describe a guess about the parameters involved in the 'curious example' of section 5. We assume that $\chi_1 = \chi_2 = \chi$ is fixed with $\chi^o = \epsilon_{E/F}^n$. For this, we may suppose that $\pi = \pi_0$ is a supercuspidal representation of $G = G(W^+)$, $\dim_E W = n$ for which (7.21) holds. Thus $\theta_\chi(\pi, V_n^+)$ (say) is supercuspidal, and $\theta_\chi(\pi, V_n^-) = 0$. According to Speculation 7.3, the L-parameter of $\theta_\chi(\pi, V_n^+)$ is the same as that of π_0 . On the other hand, according to Speculation 7.5 and (7.21), $\pi_1 = \theta_\chi(\pi_0, V_{n+2}^-) \neq 0$ and is a supercuspidal representation of $G(V_{n+2}^-)$. Applying Speculation 7.5 again, we obtain $\pi_2 = \theta_\chi(\pi_1, V_{n+6}^-)$, etc. Let $\varphi_0 : W_F' \rightarrow {}^L G_n$ be the L-parameter for π_0 . Then

Speculation 7.7. *Suppose that π is a supercuspidal of G_n for which (7.21) holds. The following list gives the L-parameters of the sequence of supercuspidals constructed from π :*

n	π_0	φ_0
$n + 2$	π_1	$\varphi_0 \oplus \chi \cdot \rho_2$
$n + 6$	π_2	$\varphi_0 \oplus \chi(\rho_2 \oplus \rho_4)$
$n + 12$	π_3	$\varphi_0 \oplus \chi(\rho_2 \oplus \rho_4 \oplus \rho_6)$
\dots	\dots	\dots
$n + r(r + 1)$	π_r	$\varphi_0 \oplus \chi(\rho_2 \oplus \rho_4 \oplus \dots \oplus \rho_{2r})$
\dots	\dots	\dots

Here the parameter φ_r for π_r is given, more precisely, by:

$$\begin{aligned} \varphi_r : W'_F &\longrightarrow {}^L G_{n+r(r+1)}, \\ w &\mapsto \begin{pmatrix} \varphi_{00}(w) & & & \\ & \chi(w) \cdot 1_{r(r+1)} & & \\ & & & \\ & & & \end{pmatrix} \times w, \\ w_\sigma &\mapsto \begin{pmatrix} \varphi_{00}(w_\sigma) & & & \\ & 1_{r(r+1)} & & \\ & & & \\ & & & \end{pmatrix} \cdot \Phi_{n,2,4,\dots,2r} \cdot \Phi_{n+r(r+1)}^{-1} \times w_\sigma, \end{aligned}$$

and

$$g \mapsto \begin{pmatrix} \varphi_{00}(g) & & & & & \\ & \rho_2(g) & & & & \\ & & \rho_4(g) & & & \\ & & & \ddots & & \\ & & & & \rho_{2r}(g) & \end{pmatrix} \times 1.$$

Here $\varphi_0(w) = \varphi_{00}(w) \times w$.

Speculation 7.7 is compatible with Theorem 6.2. Observe that for the supercuspidal π_r of $G_{n+r(r+1)}$, the analogue of (7.21) is

$$(7.34) \quad \{ m_0^+(\pi_r), m_0^-(\pi_r) \} = \{ n + r(r - 1), n + r(r + 3) + 2 \}.$$

By Theorem 6.2, the fact that the earliest theta lift of π_r is to $G_{n+r(r-1)}$ means that

$$(7.35) \quad L(s, \pi_r^\vee, \chi) = (1 - q_E^{-(s+\frac{1}{2})})^{-1} (1 - q_E^{-(s+\frac{3}{2})})^{-1} \dots (1 - q_E^{-(s+\frac{2r-1}{2})})^{-1}.$$

Note that the condition (7.21) on $\pi = \pi_0$ implies that

$$(7.36) \quad L(s, \pi_0^\vee, \chi) = 1,$$

since the earliest theta lift is to G_n^\pm . On the other hand, the L-parameter for π_r^\vee is

$$(7.37) \quad \varphi_0^\vee \oplus \chi^{-1} \left(\rho_2 \oplus \rho_4 \oplus \dots \oplus \rho_{2r} \right).$$

But, by [46], for example,

$$(7.38) \quad L(s, \chi^{-1} \rho_b, \chi) = (1 - q_E^{-(s+\frac{b-1}{2})})^{-1}.$$

Thus the L-factor attached to the restriction of our L-parameter to W'_E and to the standard representation of the L-group $GL_{n+r(r+1)}(\mathbb{C}) \times W_E$ agrees with the doubling L-factor of π_r , as determined by Theorem 6.2.

The phenomena involving L and A-packets in the example discussed in (7.22)–(7.33) above should have an analogue in the general case.

Remark. The L-parameter ϕ_r of the representation π_r introduced in Speculation 7.7 contains each of the representations ρ_i , $i = 2, 4, \dots, 2r$, with multiplicity one. Let ϕ be an L-parameter in which one of the ρ_i occurs with multiplicity greater than one. Then, as one easily verifies, the image of ϕ is contained in a proper Levi subgroup of the corresponding L-group. The full Langlands functoriality conjecture implies that such ϕ is not associated to a discrete series L-packet and, *a fortiori*, does not define a supercuspidal representation. In other words, the sequence of dimensions $n, n+2, n+6, \dots, n+r(r+1)$ proposed in Speculation 7.7 is the “densest” compatible with the Langlands conjectures.

The following is the analogue of Speculation 7.7 in the case of opposite parity. Note that, in this case, chains of supercuspidals once again exist, due to the dichotomy of Corollary 4.4. The ‘bottom’ of such a chain occurs when π is a supercuspidal representation of G_n such that

$$(7.39) \quad m_0^+ = m_0^- = n + 1.$$

The steps in the chain are determined by Speculation 7.6, so that we obtain a chain of supercuspidals π_r , $r = 0, 1, \dots$ on G_m ’s with m given by

$$(7.40) \quad n, n + 1, n + 4, n + 9, \dots, n + r^2, \dots$$

Recall that, in this case, we must fix a pair of characters χ_1 and χ_2 of E^\times such that $\chi_1^o = \epsilon_{E/F}^{n+1}$ and $\chi_2^o = \epsilon_{E/F}^n$. We use $\chi = (\chi_1, \chi_2)$ to define the splittings and theta lifts for the dual pairs $(G_n, G_{n+1}), (G_{n+4}, G_{n+9}), \dots$ and we use $\chi = (\chi_2, \chi_1)$ to define the splittings and theta lifts for the dual pairs $(G_{n+1}, G_{n+4}), (G_{n+9}, G_{n+16}), \dots$

Speculation 7.8. *Suppose that π is a supercuspidal of G_n for which (7.39) holds. The following list gives the L-parameters of the sequence of supercuspidals constructed from π :*

n	π_0		φ_0
$n + 1$	π_1		$(\chi_1^{-1}\chi_2 \cdot \varphi_0) \oplus \chi_2 \cdot \rho_1$
$n + 4$	π_2		$\varphi_0 \oplus \chi_1(\rho_1 \oplus \rho_3)$
$n + 9$	π_3		$(\chi_1^{-1}\chi_2 \cdot \varphi_0) \oplus \chi_2(\rho_1 \oplus \rho_3 \oplus \rho_5)$
\dots	\dots		\dots
$n + r^2$	π_r	$\begin{Bmatrix} \chi_2 \\ \chi_1 \end{Bmatrix}$	$\cdot \left((\chi_1^{-1} \cdot \varphi_0) \oplus \rho_1 \oplus \rho_3 \oplus \dots \oplus \rho_{2r-1} \right)$
\dots	\dots		\dots

Here the parameter φ_r for π_r is given, more precisely, by:

$$\begin{aligned} \varphi_r : W'_F &\longrightarrow {}^L G_{n+r^2}, \\ w &\mapsto \chi_j(w) \cdot \begin{pmatrix} \chi_1(w)^{-1}\varphi_{00}(w) & & & \\ & 1_{r,2} & & \\ & & & \\ & & & \end{pmatrix} \times w, \\ w_\sigma &\mapsto \begin{pmatrix} \varphi_{00}(w_\sigma) & & & \\ & 1_{r,2} & & \\ & & & \\ & & & \end{pmatrix} \cdot \Phi_{n,1,3,\dots,2r-1} \cdot \Phi_{n+r^2}^{-1} \times w_\sigma, \end{aligned}$$

and

$$g \mapsto \begin{pmatrix} \varphi_{00}(g) & & & & \\ & \rho_1(g) & & & \\ & & \rho_3(g) & & \\ & & & \ddots & \\ & & & & \rho_{2r-1}(g) \end{pmatrix} \times 1.$$

Here $\varphi_0(w) = \varphi_{00}(w) \times w$, and $\chi_j = \chi_1$ for r even and χ_2 for r odd.

The analogues of (7.34)–(7.36) are:

$$(7.41) \quad \{ m_0^+(\pi_r), m_0^-(\pi_r) \} = \{ n + (r - 1)^2, n + (r + 1)^2 \},$$

$$(7.42) \quad L(s, \pi_r^\vee, \chi_j) = (1 - q_E^{-s})^{-1} (1 - q_E^{-(s+1)})^{-1} \dots (1 - q_E^{-(s+r-1)})^{-1},$$

and

$$(7.43) \quad L(s, \pi_0^\vee, \chi_1) = 1,$$

where, in (7.42), $\chi_j = \chi_1$ for r even and χ_2 for r odd. Thus, we again have compatibility between Theorem 6.2 and our conjectural Langlands parameter.

§8. COMPARISON OF ϵ -FACTORS, THE CASE $n = 1$

In this section we restrict to the case of $U(1)$ and we determine the relation between the ϵ -factor defined by the doubling method and the usual Tate ϵ -factor of the base change. In fact we give a complete treatment of the (local) doubling construction in this case.

To be more precise, let $W = E$ with the skew-Hermitian form $\langle x, y \rangle = \delta x \bar{y}$, and let $G = G(W) = E^1$ be the associated unitary group. We identify the split skew-Hermitian space $W + W_-$ with the standard split skew-Hermitian space E^2 with skew-Hermitian inner product defined by $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, and obtain the embedding

$$(8.1) \quad i : G \times G \longrightarrow U(1, 1) = H.$$

Specializing the discussion of section 6, we find that the normalizing factor for the intertwining operator reduces to

$$(8.2) \quad \kappa(s, \chi, \psi) = \rho(2s, \chi_0, \psi),$$

and that

$$(8.3) \quad a_1(s, \chi) = L(2s, \chi_0),$$

and

$$(8.4) \quad b_1(s, \chi) = L(2s + 1, \chi_0).$$

Let $P \subset H$ be the Siegel parabolic. Then $P \backslash H$ consists of a single $i(G \times G)$ orbit (!), so that we may write

$$(8.5) \quad H = P \cdot i(G \times 1).$$

Given any character η of $G = E^1$, there is a unique section $\Phi_\eta(s)$ defined by

$$(8.6) \quad \Phi_\eta(nm(a)i(g, 1), s) = \chi(a) |a|_E^{s+\frac{1}{2}} \eta(g).$$

The restriction of the degenerate principal series representation $I_1(s, \chi)$ to $i(G \times 1)$ then decomposes as:

$$(8.7) \quad I(s) = \bigoplus_{\eta} \mathbb{C} \cdot \Phi_\eta(s),$$

and the function $\Phi_\eta(s)$ transforms according to the character $\eta \otimes (\eta^\vee \cdot \chi)$ of $G \times G$. Moreover,

Lemma 8.1. $\Phi_\eta(s)$ is an eigenfunction for the intertwining operator, with

$$M(s, \chi) \Phi_\eta(s) = \mu(s, \chi, \eta) \cdot \Phi_\eta(-s),$$

where

$$\mu(s, \chi, \eta) = \chi(-1) \int_F \chi^{-1} \tilde{\eta} \left(b + \frac{1}{2\delta} \right) \left| b + \frac{1}{2\delta} \right|_E^{-s-\frac{1}{2}} db,$$

and $\tilde{\eta}(x) = \eta(\frac{x}{\delta})$.

Proof. For large $\text{Re}(s)$, we have:

$$(8.8) \quad \mu(s, \chi, \eta) = M(s, \chi) \Phi_\eta(e) = \int_F \Phi_\eta(wn(b), s) db.$$

We calculate the decomposition

$$(8.9) \quad \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} \alpha & \\ & \bar{\alpha}^{-1} \end{pmatrix} \cdot i(g, 1).$$

Recall that

$$(8.10) \quad i(g, 1) = \frac{1}{2} \begin{pmatrix} g+1 & (2\delta)^{-1}(g-1) \\ (2\delta)(g-1) & g+1 \end{pmatrix},$$

so that, comparing bottom rows, we have

$$(8.11) \quad (-1, -b) = \bar{\alpha}^{-1}(\delta(g-1), \frac{1}{2}(g+1)).$$

This gives

$$(8.12) \quad \alpha = -\frac{1}{b-1/2\delta}$$

and

$$(8.13) \quad g = \frac{b+1/2\delta}{b-1/2\delta},$$

and thus

$$(8.14) \quad \begin{aligned} \Phi_\eta(wn(b), s) &= \chi(-1) \cdot \chi^{-1} \left(b + \frac{1}{2\delta} \right) \left| b + \frac{1}{2\delta} \right|_E^{-s-\frac{1}{2}} \eta \left(\frac{b+1/2\delta}{b-1/2\delta} \right) \\ &= \chi(-1) \cdot \chi^{-1} \tilde{\eta} \left(b + \frac{1}{2\delta} \right) \left| b + \frac{1}{2\delta} \right|_E^{-s-\frac{1}{2}} \end{aligned}$$

with $\tilde{\eta}(x) = \eta(\frac{x}{\delta})$. Therefore

$$(8.15) \quad \mu(s, \chi, \eta) = \chi(-1) \int_F \chi^{-1} \tilde{\eta} \left(b + \frac{1}{2\delta} \right) \left| b + \frac{1}{2\delta} \right|_E^{-s-\frac{1}{2}} db,$$

as claimed. □

Proposition 8.2. *Let μ be a character of E^\times and write $\mu_0 = \mu|_{F^\times}$. Then*

$$\int_F \mu(a+2\delta) |a+2\delta|_E^{-s-\frac{1}{2}} da = \mu(4\delta) |4\delta|_E^{-s} \frac{\rho(2s, \mu_0^{-1}, \psi)}{\rho(s+\frac{1}{2}, \mu^{-1}, \psi_E)},$$

where $\rho(s, \lambda, \psi)$ is the Tate ‘gamma’ factor.

Recall that, for any character λ of F^\times , $\rho(s, \lambda, \psi)$ is defined by the relation

$$(8.16) \quad \rho(s, \lambda, \psi) Z_T(1-s, \lambda^{-1}, \hat{f}) = Z_T(s, \lambda, f),$$

where

$$(8.17) \quad Z_T(s, \lambda, f) = \int_F f(x) \lambda(x) |x|_F^{s-1} dx,$$

for a self dual measure dx with respect to ψ , and \hat{f} the Fourier transform with respect to ψ and this measure. The Tate factor for E is defined analogously. Normalizing by the Tate L-factor, we have:

$$(8.18) \quad \frac{Z_T(1-s, \lambda^{-1}, \hat{f})}{L(1-s, \lambda^{-1})} = \frac{1}{\rho(s, \lambda, \psi)} \frac{L(s, \lambda)}{L(1-s, \lambda^{-1})} \cdot \frac{Z_T(s, \lambda, f)}{L(s, \lambda)},$$

and so

$$(8.19) \quad \frac{1}{\rho(s, \lambda, \psi)} = \epsilon(s, \lambda, \psi) \cdot \frac{L(1-s, \lambda^{-1})}{L(s, \lambda)},$$

where $\epsilon(s, \lambda, \psi)$ is the Tate epsilon factor.

Proof. Write

$$(8.20) \quad B(s, \mu) = \int_F \mu(a + 2\delta) |a + 2\delta|_E^{-s-\frac{1}{2}} da,$$

and note that this integral is absolutely convergent for $\text{Re}(s) > \frac{1}{2}$. Choose (non-zero) Schwartz functions $f \in S(F)$ and $\phi \in S(E)$ and let \hat{f} (resp. $\hat{\phi}$) denote the Fourier transform with respect to ψ (resp. ψ_E). We take Haar measures dx on F and dy on E to be self dual with respect to these fixed additive characters. We further assume that $\hat{f}(0) = 0$ and $\hat{\phi}(0) = 0$.

Consider the product

$$(8.21) \quad Z_T(-2s + 1, \mu_0, \hat{f}) \cdot B(s, \mu) \cdot Z_T(s + \frac{1}{2}, \mu^{-1}, \phi),$$

where Z_T denotes the Tate zeta integral. If we assume that s lies in the half plane $\text{Re}(s) > \frac{1}{2}$, then we may write the product of the three factors as:

$$(8.22) \quad \int_F \hat{f}(b) \mu_0(b) |b|_F^{-2s} db \cdot \int_F \mu(a + 2\delta) |a + 2\delta|_E^{-s-\frac{1}{2}} da \cdot \int_E \phi(y) \mu^{-1}(y) |y|^{s-\frac{1}{2}} dy.$$

Note that the condition $\hat{f}(0) = 0$ insures that the first of these integrals is absolutely convergent for all s . Since

$$(8.23) \quad \begin{aligned} & \int_F \mu(a + 2\delta b) |a + 2\delta b|_E^{-s-\frac{1}{2}} da \\ &= \int_F \mu(ab + 2\delta b) |ab + 2\delta b|_E^{-s-\frac{1}{2}} d(ab) \\ &= \mu(b) |b|_F^{-2s-1} |b|_F \int_F \mu(a + 2\delta) |a + 2\delta|_E^{-s-\frac{1}{2}} da \\ &= \mu_0(b) |b|_F^{-2s} \int_F \mu(a + 2\delta) |a + 2\delta|_E^{-s-\frac{1}{2}} da, \end{aligned}$$

we find that the product of the first two factors can be written as

$$(8.24) \quad \int_F \int_F \hat{f}(b) \mu(a + 2\delta b) |a + 2\delta b|_E^{-s-\frac{1}{2}} da db.$$

Setting $z = a + 2\delta b$, we can write this as

$$(8.25) \quad \begin{aligned} C \cdot \int_E \hat{f}(\frac{1}{4} \text{tr}_{E/F}(\delta^{-1}z)) \mu(z) |z|_E^{-s-\frac{1}{2}} dz \\ = C \cdot \mu(2\delta) |2\delta|_E^{-s+\frac{1}{2}} \int_E \hat{f}(\frac{1}{2} \text{tr}_{E/F}(z)) \mu(z) |z|_E^{-s-\frac{1}{2}} dz, \end{aligned}$$

for a positive constant C , defined by $C \cdot dz = da db$, which we must still determine.

Dropping the factor $C \cdot \mu(2\delta)|2\delta|_E^{-s+\frac{1}{2}}$ for the moment, we observe that,

$$\begin{aligned}
 & \int_E \hat{f}(\tfrac{1}{2}tr_{E/F}(yz)) \mu(z) |z|_E^{-s-\frac{1}{2}} dz \\
 (8.26) \quad &= \int_E \hat{f}(\tfrac{1}{2}tr_{E/F}(z)) \mu(y^{-1}z) |y^{-1}z|_E^{-s-\frac{1}{2}} d(y^{-1}z) \\
 &= \mu^{-1}(y)|y|_E^{s-\frac{1}{2}} \int_E \hat{f}(\tfrac{1}{2}tr_{E/F}(z)) \mu(z) |z|_E^{-s-\frac{1}{2}} dz,
 \end{aligned}$$

so that we may write the full product as:

$$\begin{aligned}
 & \int_E \int_E \phi(y) \hat{f}(\tfrac{1}{2}tr_{E/F}(yz)) \mu(z) |z|_E^{-s-\frac{1}{2}} dz dy \\
 (8.27) \quad &= \int_E \int_E \phi(y) \left(\int_F f(c) \psi(-c\tfrac{1}{2}tr_{E/F}(yz)) dc \right) \mu(z) |z|_E^{-s-\frac{1}{2}} dz dy \\
 &= \int_F \int_E f(c) \left(\int_E \phi(y) \psi(-c\tfrac{1}{2}tr_{E/F}(yz)) dy \right) \mu(z) |z|_E^{-s-\frac{1}{2}} dz dc \\
 &= \int_F \int_E f(c) \hat{\phi}(\tfrac{1}{2}cz) \mu(z) |z|_E^{-s-\frac{1}{2}} dz dc \\
 &= \mu(2)|2|_E^{-s+\frac{1}{2}} \int_F f(c)\mu(c)^{-1}|c|_F^{2s-1} dc \int_E \hat{\phi}(z) \mu(z) |z|_E^{-s-\frac{1}{2}} dz \\
 &= \mu(2)|2|_E^{-s+\frac{1}{2}} Z_T(2s, \mu_0^{-1}, f) Z_T(-s + \tfrac{1}{2}, \mu, \hat{\phi}).
 \end{aligned}$$

Note that we have taken advantage of the fact that $\hat{\phi}(0) = 0$ in this manipulation. We find that

$$\begin{aligned}
 (8.28) \quad & Z_T(-2s + 1, \mu_0, \hat{f}) \cdot B(s, \mu) \cdot Z_T(s + \tfrac{1}{2}, \mu^{-1}, \phi) \\
 &= C \mu(4\delta)|4\delta|_E^{-s+\frac{1}{2}} \cdot Z_T(2s, \mu_0^{-1}, f) Z_T(-s + \tfrac{1}{2}, \mu, \hat{\phi}),
 \end{aligned}$$

for the constant C above. Thus,

$$\begin{aligned}
 (8.29) \quad & B(s, \mu) = C \mu(4\delta)|4\delta|_E^{-s+\frac{1}{2}} \cdot \frac{Z_T(2s, \mu_0^{-1}, f)}{Z_T(-2s + 1, \mu_0, \hat{f})} \cdot \frac{Z_T(-s + \tfrac{1}{2}, \mu, \hat{\phi})}{Z_T(s + \tfrac{1}{2}, \mu^{-1}, \phi)} \\
 &= C \mu(4\delta)|4\delta|_E^{-s+\frac{1}{2}} \cdot \frac{\rho(2s, \mu_0^{-1}, \psi)}{\rho(s + \tfrac{1}{2}, \mu^{-1}, \psi_E)}.
 \end{aligned}$$

Finally, we determine the positive constant C , defined by $C \cdot dz = da db$, where dz is the self dual measure on E with respect to ψ_E , da and db are self dual measures on F with respect to ψ , and $z = a + 2b\delta$. Take $\phi(a + 2b\delta) = f_0(a)f_1(b)$ for $f_0, f_1 \in S(F)$. Then, writing $w = x + 2y\delta$, so that

$$(8.30) \quad tr_{E/F}(zw) = 2ax + 8\Delta by,$$

we have

$$\begin{aligned}
 (8.31) \quad & C \cdot \hat{\phi}(w) = \int_E \phi(z) \psi_E(-zw) C \cdot dz \\
 &= \int_F \int_F f_0(a)f_1(b) \psi(-(2ax + 8\Delta by)) da db \\
 &= \hat{f}_0(2x)\hat{f}_1(8\Delta y).
 \end{aligned}$$

Coming back, we have

$$\begin{aligned}
 C^2 \cdot \phi(z) &= \int_E C \cdot \hat{\phi}(w) \psi_E(zw) C \cdot dw \\
 (8.32) \qquad &= \int_F \int_F \hat{f}_0(2x) \hat{f}_1(8\Delta y) \psi(2ax + 8\Delta by) dx dy \\
 &= |2|_F^{-4} |\Delta|_F^{-1} f_0(a) f_1(b) \\
 &= |2|_F^{-4} |\Delta|_F^{-1} \phi(z).
 \end{aligned}$$

Thus

$$(8.33) \qquad C = |2|_E^{-1} |\Delta|_F^{-\frac{1}{2}} = |4\delta|_E^{-\frac{1}{2}}.$$

□

Summarizing, we have:

Corollary 8.3. (i)

$$M(s, \chi) \Phi_\eta = \mu(s, \chi, \eta) \cdot \Phi_\eta(-s),$$

where

$$\mu(s, \chi, \eta) = \eta(-1) \chi(-\delta) |\delta|_E^s \frac{\rho(2s, \epsilon_{E/F}, \psi)}{\rho(s + \frac{1}{2}, \chi \tilde{\eta}^{-1}, \psi_E)}.$$

(ii)

$$M^*(s, \chi) \Phi_\eta = \mu^*(s, \chi, \eta) \cdot \Phi_\eta(-s),$$

where

$$\mu^*(s, \chi, \eta) = \eta(-1) \chi(-\delta) |\delta|_E^s \frac{1}{\rho(s + \frac{1}{2}, \chi \tilde{\eta}^{-1}, \psi_E)}.$$

Note that

$$\mu^*(-s, \check{\chi}, \eta) \mu^*(s, \chi, \eta) = 1.$$

Proof. We have

$$\begin{aligned}
 M^*(s, \chi) \Phi_\eta(e) &= \frac{1}{\kappa(s, \chi, \psi)} \mu(s, \chi, \eta) \\
 (8.34) \qquad &= \frac{1}{\rho(2s, \chi_0, \psi)} \eta(-1) \chi(-\delta) |\delta|_E^s \frac{\rho(2s, \chi_0, \psi)}{\rho(s + \frac{1}{2}, \chi \tilde{\eta}^{-1}, \psi_E)} \\
 &= \eta(-1) \chi(-\delta) |\delta|_E^s \frac{1}{\rho(s + \frac{1}{2}, \chi \tilde{\eta}^{-1}, \psi_E)}.
 \end{aligned}$$

□

We now return to the calculation of the L and ϵ factors defined by doubling.

For a fixed character η of $G = E^1$, only the section $\Phi_{\tilde{\eta}}$ makes a non-zero contribution to the zeta integral. Thus we consider

Proposition 8.4. (i) $\Phi_{\tilde{\eta}}(s)$ is a good section, and

$$Z(s, \eta, \Phi_{\tilde{\eta}}) = 1.$$

(ii) $M^*(-s, \check{\chi}) \Phi_{\tilde{\eta}}$ is a good section, and

$$Z(s, \eta, M^*(-s, \check{\chi}) \Phi_{\tilde{\eta}}) = \mu^*(-s, \check{\chi}, \tilde{\eta}).$$

(iii) *The doubling L-factor is*

$$L_{PSR}(s, \eta, \chi) = L(s, \chi\tilde{\eta}).$$

(iv) *The doubling epsilon factor is:*

$$\epsilon_{PSR}(s, \eta, \chi, \psi) = \eta(-1)\chi(-\delta)|\delta|_E^{s-\frac{1}{2}}\epsilon(s, \chi\tilde{\eta}, \psi_E).$$

Proof. We compute

$$\begin{aligned} Z(s, \eta, \Phi_{\tilde{\eta}}) &= \int_G \eta(g) \Phi_{\tilde{\eta}}(i(g, 1), s) dg \\ (8.35) \qquad &= \int_G \eta(g) \tilde{\eta}(g) dg = 1. \end{aligned}$$

Then

$$\begin{aligned} Z(s, \eta, M^*(-s, \tilde{\chi})\Phi_{\tilde{\eta}}) &= \mu^*(-s, \tilde{\chi}, \tilde{\eta}) Z(s, \eta, \Phi_{\tilde{\eta}}) \\ (8.36) \qquad &= \mu^*(-s, \tilde{\chi}, \tilde{\eta}). \end{aligned}$$

Since

$$\begin{aligned} \mu^*(-s, \tilde{\chi}, \tilde{\eta}) &= \eta(-1)\tilde{\chi}(-\delta)|\delta|_E^{-s} \frac{1}{\rho(-s + \frac{1}{2}, \tilde{\chi}\tilde{\eta}, \psi_E)} \\ (8.37) \qquad &= L(s + \frac{1}{2}, \tilde{\chi}^{-1}\tilde{\eta}^{-1}) \cdot \left(\text{entire} \right), \end{aligned}$$

we see that the doubling L-factor is $L(s, \tilde{\chi}^{-1}\tilde{\eta}^{-1})$. Note that, if λ is any character of E^\times , then, setting $\lambda'(x) = \lambda(\bar{x})$, we have $L(s, \lambda') = L(s, \lambda)$. Also note that $\tilde{\chi}^{-1} = \chi'$ and $\tilde{\eta}^{-1} = \tilde{\eta}'$. Thus the doubling L-factor is as claimed.

Finally, we consider the functional equation:

$$(8.38) \qquad Z(-s, \eta, M^*(s, \chi)\Phi_{\tilde{\eta}}) = \mu^*(s, \chi, \tilde{\eta}),$$

so that

$$(8.39) \qquad \Gamma^*(s, \chi, \eta, \psi) = \mu^*(s, \chi, \tilde{\eta}).$$

Hence

$$\begin{aligned} \epsilon_{PSR}(s + \frac{1}{2}, \eta, \chi, \psi) &= \mu^*(s, \chi, \tilde{\eta}) \cdot \frac{L(s + \frac{1}{2}, \chi\tilde{\eta})}{L(-s + \frac{1}{2}, \tilde{\chi}\tilde{\eta})} \\ (8.40) \qquad &= \eta(-1)\chi(-\delta)|\delta|_E^s \frac{1}{\rho(s + \frac{1}{2}, \chi\tilde{\eta}, \psi_E)} \cdot \frac{L(s + \frac{1}{2}, \chi\tilde{\eta})}{L(-s + \frac{1}{2}, \tilde{\chi}\tilde{\eta})} \\ &= \eta(-1)\chi(-\delta)|\delta|_E^s \epsilon(s + \frac{1}{2}, \chi\tilde{\eta}, \psi_E). \end{aligned}$$

□

Combining (iv) of Proposition 8.4 with Theorem 6.1, we obtain

Corollary 8.5. *Let W (resp. V) be a one dimensional skew-Hermitian (resp. Hermitian) space. Suppose that the skew-Hermitian form on $W = E$ is given by $\langle x, y \rangle = a\delta x\bar{y}$, and that the Hermitian form on $V = E$ is given by $(x, y) = b\bar{x}y$ for a and $b \in F^\times$. Let*

$$\epsilon(V) = \epsilon_{E/F}(b), \quad \text{and} \quad \epsilon(W) = \epsilon_{E/F}(a).$$

Fix a character χ of E^\times whose restriction to F^\times is $\epsilon_{E/F}$. Let $\omega_{V,\chi,\psi}$ be the Weil representation $\omega_{V,\chi,\psi} = \omega_\psi \circ \tilde{\iota}_{V,\chi}$ of $G(W) = E^1$ obtained via the homomorphism

$$\tilde{\iota}_{V,\chi,\psi} : E^1 = G(W) \longrightarrow Mp(\mathbb{W})$$

of (1.17). Then

$$\omega_{V,\chi,\psi} = \bigoplus_{\eta} \mathbb{C}_{\eta},$$

where η runs over all characters of E^1 such that

$$\epsilon(\frac{1}{2}, \chi\tilde{\eta}^{-1}, \psi_E) \cdot \eta(-1) \cdot \chi(-\delta) \cdot \epsilon_{E/F}(-2) = \epsilon(V)\epsilon(W).$$

Here $\tilde{\eta}(x) = \eta(\frac{x}{\bar{x}})$ is the (standard) base change of η to $GL_1(E)$.

Remark. This result is due to Rogawski [42], whose proof was quite different and, in particular, relied on some global arguments and some calculations involving Gauss sums. It has also been verified by direct computation by Tonghai Yang [52].

APPENDIX

In this section we record several useful properties of the splitting.

Suppose that W is a split skew-Hermitian space, and let $\Omega(W)$ be the space of maximal isotropic subspaces of W . For a Hermitian space V , and $\mathbb{W} = V \otimes_E W$, with symplectic form

$$(a.1) \quad \langle\langle \cdot, \cdot \rangle\rangle = \frac{1}{2} tr_{E/F}((\cdot, \cdot) \otimes \overline{\langle \cdot, \cdot \rangle}),$$

let

$$(a.2) \quad \iota_V : G(W) \longrightarrow Sp(\mathbb{W}),$$

be the associated symplectic embedding. If $\dim_E V = m$, let χ be a character of E^\times such that $\chi|_{F^\times} = \epsilon_{E/F}^m$. Then, associated to χ and to a choice of $Y \in \Omega(W)$, we have the splitting homomorphism

$$(a.3) \quad \begin{array}{ccc} & Mp(\mathbb{W}) & \\ & \nearrow \tilde{\iota}_{V,\chi} & \downarrow \\ G(W) & \xrightarrow{\iota_V} & Sp(\mathbb{W}). \end{array}$$

Explicitly, the maximal isotropic subspace $\mathbb{Y} = V \otimes Y$ of \mathbb{W} , associated to Y , determines an isomorphism of sets

$$(a.4) \quad Mp(\mathbb{W}) \simeq Sp(\mathbb{W}) \times \mathbb{C}^1.$$

The splitting homomorphism is given by

$$(a.5) \quad \tilde{\iota}_{V,\chi}(g) = (\iota_V(g), \beta_{V,\chi}(g)),$$

where

$$(a.6) \quad \beta_{V,\chi}(g) = \chi(x(g)) \gamma_F(\eta \circ RV)^{-j(g)},$$

with $x(g)$, $j(g)$, and RV given in [22].

Our first result is the following.

Proposition A.1. *The splitting homomorphism $\tilde{\iota}_{V,\chi}$ is independent of the choice of $Y \in \Omega(W)$, and thus depends only on χ .*

Proof. To begin, for any pair of maximal isotropic subspaces \mathbb{Y}_1 and \mathbb{Y}_2 of \mathbb{W} , we consider the diagram:

$$(a.7) \quad \begin{array}{ccc} Mp(\mathbb{W}) & \xrightarrow{\sim} & Sp(\mathbb{W}) \times \mathbb{C}^1 & (g, z) \\ \parallel & & \downarrow & \downarrow \\ Mp(\mathbb{W}) & \xrightarrow{\sim} & Sp(\mathbb{W}) \times \mathbb{C}^1 & (g, z \lambda(g)), \end{array}$$

where the isomorphism in the first row is determined by \mathbb{Y}_1 , and the isomorphism in the second row is determined by \mathbb{Y}_2 . According to Lemma 4.2 of [22], the factor $\lambda(g)$ is given by

$$(a.8) \quad \begin{aligned} \lambda(g) &= \gamma_F(\eta \circ L(\mathbb{Y}_1, \mathbb{Y}_2 g^{-1}, \mathbb{Y}_2)) \gamma_F(\eta \circ L(\mathbb{Y}_1, \mathbb{Y}_2, \mathbb{Y}_1 g)) \\ &= c_{\mathbb{Y}_1}(\alpha, g\alpha^{-1}) c_{\mathbb{Y}_1}(g, \alpha^{-1}), \end{aligned}$$

where $\mathbb{Y}_2 = \mathbb{Y}_1 \alpha$, for $\alpha \in Sp(\mathbb{W})$. Now assume that $\mathbb{Y}_1 = V \otimes Y_1$ and $\mathbb{Y}_2 = V \otimes Y_2$, for Y_1 and $Y_2 \in \Omega(W)$. We may assume that $\alpha \in G(W)$ (which we temporarily identify with its image under ι_V) with $Y_2 = Y_1 \alpha$. Then,

$$c_{\mathbb{Y}_1}(\alpha, g\alpha^{-1}) = \beta_{V,1}(\alpha g\alpha^{-1}) \beta_{V,1}(\alpha)^{-1} \beta_{V,1}(g\alpha^{-1})^{-1},$$

and

$$(a.9) \quad c_{\mathbb{Y}_1}(g, \alpha) = \beta_{V,1}(g\alpha^{-1}) \beta_{V,1}(g)^{-1} \beta_{V,1}(\alpha^{-1})^{-1},$$

where $\beta_{V,1}(g)$ is the function on $G(W)$ which defines the splitting homomorphism associated to Y_1 . Thus

$$(a.10) \quad \lambda(g) = \beta_{V,1}(\alpha g\alpha^{-1}) [\beta_{V,1}(\alpha) \beta_{V,1}(\alpha^{-1}) \beta_{V,1}(g)]^{-1}.$$

Now we observe that, if $g \in G(W)$, then for any $Y \in \Omega(W)$ with associated functions j and x ,

$$(a.11) \quad j(g^{-1}) = j(g), \quad \text{and} \quad x(g^{-1}) = (-1)^{j(g)} x(g)^{-1}.$$

Thus, we let $j_1(g)$, $x_1(g)$ and $\beta_{V,1}(g)$ be the quantities associated to Y_1 ,

$$(a.12) \quad \begin{aligned} \beta_{V,1}(g^{-1}) &= \chi(x_1(g^{-1})) \gamma_F(\eta \circ RV)^{-j_1(g^{-1})} \\ &= \chi(-1)^{j_1(g)} \chi(x_1(g))^{-1} \gamma_F(\eta \circ RV)^{-j_1(g)}. \end{aligned}$$

Hence, since

$$(a.13) \quad \gamma_F(\eta \circ RV) = (\Delta, \det(V))_F \gamma_F(-\Delta, \eta)^m \gamma_F(-1, \eta)^{-m},$$

we have

$$(a.14) \quad \begin{aligned} \beta_{V,1}(\alpha) \beta_{V,1}(\alpha^{-1}) &= \chi(x_1(\alpha)) \chi(x_1(\alpha^{-1})) \gamma_F(\eta \circ RV)^{-j_1(\alpha)} \gamma_F(\eta \circ RV)^{-j_1(\alpha^{-1})} \\ &= \chi(-1)^{j_1(\alpha)} \gamma_F(\eta \circ RV)^{-2j_1(\alpha)} \\ &= \epsilon_{E/F}(-1)^{mj_1(\alpha)} (-1, -\Delta)_F^{mj_1(\alpha)} (-1, -1)_F^{mj_1(\alpha)} \\ &= 1. \end{aligned}$$

Thus

$$(a.15) \quad \lambda(g) = \beta_{V,1}(\alpha g\alpha^{-1}) \beta_{V,1}(g)^{-1}.$$

On the other hand, we let $j_2(g)$, $x_2(g)$, and $\beta_{V,2}(g)$ be the quantities associated to Y_2 . Let P_1 (resp. P_2) be the stabilizer of Y_1 (resp. Y_2). Then $P_2 = \alpha^{-1} P_1 \alpha$, and

$\tau_{S,2} = \alpha^{-1}\tau_{S,1}\alpha$, where the $\tau_{S,i}$, $i = 1, 2$ are the Weyl elements for P_i , as in [40] and [22]. Thus,

$$(a.16) \quad g = p_2\tau_{S,2}p'_2 \implies \alpha g\alpha^{-1} = \alpha p_2\alpha^{-1}\tau_{S,1}\alpha p'_2\alpha^{-1},$$

so that

$$(a.17) \quad x_1(\alpha g\alpha^{-1}) = x_2(g), \quad \text{and} \quad j_1(\alpha g\alpha^{-1}) = j_2(g),$$

and so

$$(a.18) \quad \beta_{V,1}(\alpha g\alpha^{-1}) = \beta_{V,2}(g).$$

Therefore,

$$(a.19) \quad \lambda(g) = \beta_{V,2}(g)\beta_{V,1}(g)^{-1},$$

as required. □

Corollary A.2. *Suppose that V and χ are fixed, as above. If W is a split space and $W = W_1 + W_2$, with W_1 and W_2 also split, then the following diagram commutes:*

$$\begin{array}{ccc} G(W) & \xrightarrow{\tilde{\iota}_{V,\chi}} & Mp(\mathbb{W}) \\ \uparrow & & \uparrow \\ G(W_1) \times G(W_2) & \xrightarrow{\tilde{\iota}_{V,\chi} \times \tilde{\iota}_{V,\chi}} & Mp(\mathbb{W}_1) \times Mp(\mathbb{W}_2). \end{array}$$

In particular, the restriction of the splitting $\tilde{\iota}_{V,\chi}$ of $G(W)$ to any $G(W_1)$ for a split subspace W_1 in W is the corresponding splitting, determined by χ , for $G(W_1)$.

Proof. By Proposition 1, we can work with a maximal isotropic subspace of W of the form $Y = Y_1 + Y_2$, for $Y_i \in \Omega(W_i)$. If x, j, x_1, j_1, x_2 , and j_2 are the associated functions on $G(W)$, $G(W_1)$ and $G(W_2)$ respectively, then, for $g = (g_1, g_2)$, we clearly have

$$(a.20) \quad x(g) = x(g_1)x(g_2) \quad \text{and} \quad j(g) = j(g_1) + j(g_2).$$

This implies that

$$(a.21) \quad \beta_V(g) = \beta_V(g_1)\beta_V(g_2),$$

as claimed. □

Next, for W an arbitrary skew-Hermitian space, we define $\tilde{\iota}_{V,\chi}$ by the doubling construction of section 4 of [22]. Thus, we let W_- be the space W with the negative of the given form, let

$$(a.22) \quad \widetilde{W} = W + W_-,$$

and let

$$(a.23) \quad i : G(W) \longrightarrow G(\widetilde{W})$$

be the embedding defined by letting $G(W)$ act in the natural way on W and trivially on W_- . We define the splitting $\tilde{\iota}_{V,\chi}$ so that the diagram

$$(a.24) \quad \begin{array}{ccc} G(\widetilde{W}) & \xrightarrow{\tilde{\iota}_{V,\chi}} & Mp(\widetilde{\mathbb{W}}) \\ i \uparrow & & \uparrow \\ G(W) & \xrightarrow{\tilde{\iota}_{V,\chi}} & Mp(\mathbb{W}) \end{array}$$

commutes. Note that, if W is itself split, we have a potential conflicting notation. But, by Corollary A.2, applied to $\widetilde{W} \supset W$, we find that the two definitions coincide!

Corollary A.3. *Suppose that W is any skew-Hermitian space, and that $W = W_1 + W_2$, for skew-Hermitian spaces W_1 and W_2 . For V and χ as above, let $\tilde{v}_{V,\chi}$ be the splitting homomorphism for each of $G(W)$, $G(W_1)$ and $G(W_2)$, determined by doubling. Then the following diagram commutes:*

$$\begin{array}{ccc} G(W) & \xrightarrow{\tilde{v}_{V,\chi}} & Mp(\mathbb{W}) \\ \uparrow & & \uparrow \\ G(W_1) \times G(W_2) & \xrightarrow{\tilde{v}_{V,\chi} \times \tilde{v}_{V,\chi}} & Mp(\mathbb{W}_1) \times Mp(\mathbb{W}_2). \end{array}$$

In particular, the splitting $\tilde{v}_{V,\chi}$ determined by χ is compatible with restriction to $G(W_1) \subset G(W)$, where $W_1 \subset W$ is any non-degenerate subspace.

Proof. Consider the diagram

$$(a.25) \quad \begin{array}{ccc} Mp(\tilde{\mathbb{W}}_1) \times Mp(\tilde{\mathbb{W}}_2) & \longrightarrow & Mp(\tilde{\mathbb{W}}) \\ \tilde{v}_{V,\chi} \uparrow & & \tilde{v}_{V,\chi} \uparrow \\ G(\tilde{W}_1) \times G(\tilde{W}_2) & \longrightarrow & G(\tilde{W}) \\ i \times i \uparrow & & i \uparrow \\ G(W_1) \times G(W_2) & \longrightarrow & G(W), \end{array}$$

obtained by doubling the spaces W , W_1 and W_2 . The bottom square clearly commutes while the top square commutes by Corollary A.2. □

This result should be very useful in applications!

We record here several other technical results. In particular, we determine the dependence of our splittings on scaling of the form on W .

First we change notation for a moment and let W, \langle , \rangle be a symplectic vector space over F of dimension $2n$. For $\alpha \in F^\times$, let W_α be the space W with symplectic form $\alpha \cdot \langle , \rangle$. Then any maximal isotropic subspace Y in W is also an isotropic subspace Y_α in W_α , and we then obtain a diagram:

$$(a.26) \quad \begin{array}{ccc} Mp(W) & \longrightarrow & Sp(W) \times \mathbb{C}^1 \\ \parallel & & \downarrow \\ Mp(W_\alpha) & \longrightarrow & Sp(W_\alpha) \times \mathbb{C}^1, \end{array}$$

of ‘Rao’ isomorphisms.

Proposition A.4. *The right hand arrow in the previous diagram is given by*

$$(g, z) \mapsto (g, z \cdot (\alpha, x(g))_F \gamma_F(\alpha, \eta)^{-j(g)}).$$

Proof. We need to compare the two Rao cocycles:

$$(a.27) \quad c_Y(g_1, g_2) = \gamma_F(\eta \circ L(Y, Y g_2^{-1}, Y g_1))$$

and

$$(a.28) \quad c_{Y_\alpha}(g_1, g_2) = \gamma_F(\eta \circ L(Y_\alpha, Y_\alpha g_2^{-1}, Y_\alpha g_1)).$$

Since

$$(a.29) \quad L(Y_\alpha, Y_\alpha g_2^{-1}, Y_\alpha g_1) = \alpha \cdot L(Y, Y g_2^{-1}, Y g_1) = \alpha \cdot L,$$

we compute:

$$\begin{aligned}
 \gamma_F(\eta \circ \alpha \cdot L) &= \gamma_F(\det(\alpha L), \eta) \gamma_F(\eta)^\ell h_F(\alpha L) \\
 &= (\alpha, \det L)_F^\ell \gamma_F(\eta)^\ell (\alpha, -1)_{F^{\frac{\ell(\ell-1)}{2}}} (\alpha, \det L)_F^{\ell-1} h_F(\ell) \\
 \text{(a.30)} \quad &= (\alpha, \det L)_F \gamma_F(\alpha, \eta)^\ell \gamma_F(\eta \circ L) \\
 &= (\alpha, -1)^t (\alpha, x(g_1 g_2))_F (\alpha, x(g_1))_F (\alpha, x(g_2))_F \\
 &\quad \times \gamma_F(\alpha, \eta)^{j_1 + j_2 - j} (-1, \alpha)^t \gamma_F(\eta \circ L),
 \end{aligned}$$

where we have used the identities,

$$\text{(a.31)} \quad h_F(\alpha L) = (\alpha, -1)_{F^{\frac{\ell(\ell-1)}{2}}} (\alpha, \det L)_F^{\ell-1} h_F(\ell),$$

for the Hasse invariant, and

$$\text{(a.32)} \quad \det(L) = (-1)^t x(g_1 g_2) x(g_1) x(g_2),$$

and

$$\text{(a.33)} \quad \ell = \dim_F L = j_1 + j_2 - j - 2t,$$

from [22]. This quantity is clearly the coboundary of the function

$$\text{(a.34)} \quad g \mapsto (\alpha, x(g))_F \gamma_F(\alpha, \eta)^{-j(g)},$$

as required. □

We now return to our previous notation, and, for $\alpha \in F^\times$, we consider W and the scaled space W_α . We then obtain the diagram

$$\text{(a.35)} \quad \begin{array}{ccccc}
 G(W) & \xrightarrow{\tilde{\iota}_{V,\chi}} & Mp(\mathbb{W}) & \xrightarrow{\sim} & Sp(\mathbb{W}) \times \mathbb{C}^1 \\
 \parallel & & \parallel & & \downarrow \\
 G(W_\alpha) & \xrightarrow{\tilde{\iota}_{V,\chi}^\alpha} & Mp(\mathbb{W}_\alpha) & \xrightarrow{\sim} & Sp(\mathbb{W}_\alpha) \times \mathbb{C}^1.
 \end{array}$$

We want to determine the factor by which it fails to commute. Going around the top of the diagram, we find that $g \in G(W)$ goes to

$$\text{(a.36)} \quad (\iota_V(g), \beta_{V,\chi}(g) \cdot (\alpha, x(\iota_V(g)))_F \gamma_F(\alpha, \eta)^{-j(\iota_V(g))}).$$

Lemma A.5. (i) $x(\iota_V(g)) = N_{E/F}(x(g))^m \cdot (-\Delta)^{mj(g)}$.

(ii) $j(\iota_V(g)) = 2mj(g)$.

Thus, the \mathbb{C}^1 term above becomes

$$\begin{aligned}
 &\beta_{V,\chi}(g) (\alpha, N_{E/F}(x(g)))^m (\alpha, -\Delta)_F^{mj(g)} (-1, \alpha)_F^{mj(g)} \\
 \text{(a.37)} \quad &= \beta_{V,\chi}(g) (\alpha, N_{E/F}(x(g)))^m \chi(\alpha)^{j(g)}.
 \end{aligned}$$

On the other hand, when the form on W is scaled by α , the standard symplectic basis e_1, \dots, e'_n , which we write as e, e' for short, must be changed to, say, $e, \alpha^{-1}e'$, and thus Rao's Weyl element τ , which is defined by $\tau e = -e'$ and $\tau e' = e$, must be modified:

$$\text{(a.38)} \quad \tau_\alpha = \tau \begin{pmatrix} \alpha & \\ & \alpha^{-1} \end{pmatrix}.$$

Thus,

$$\text{(a.39)} \quad g = p_1 \tau_{S,\alpha} p_2 = p_1 \tau_S \begin{pmatrix} \alpha_S & \\ & \alpha_S^{-1} \end{pmatrix} p_2,$$

so that

$$(a.40) \quad x_\alpha(g) = \alpha^{j(g)} x(g)$$

and $j_\alpha(g) = j(g)$, where x_α and j_α are the Rao functions for W_α . Using these quantities in $\beta_{V,\alpha,\chi}(g)$, we find that the image of g around the bottom part of the diagram is:

$$(a.41) \quad (\iota_V(g), \beta_{V,\alpha,\chi}(g)),$$

with

$$(a.42) \quad \beta_{V,\alpha,\chi}(g) = \chi(x_\alpha(g)) \gamma_F(\eta \circ RV)^{-j_\alpha(g)} = \chi(\alpha)^{j(g)} \beta_{V,\chi}(g).$$

Comparing these two results, we conclude the following:

Proposition A.6. *Suppose that V and χ are fixed, and that, for $\alpha \in F^\times$, W_α is the skew-Hermitian space obtained by scaling the form on W by α . Then the splittings*

$$\tilde{\iota}_{V,\chi}, \tilde{\iota}_{V,\alpha,\chi} : G(W) = G(W_\alpha) \longrightarrow Mp(\mathbb{W}) = Mp(\mathbb{W}_\alpha)$$

differ by a quadratic character:

$$\tilde{\iota}_{V,\alpha,\chi}(g) = \eta_\alpha(g)^m \cdot \tilde{\iota}_{V,\chi}(g),$$

where

$$\eta_\alpha(g) = (\alpha, N_{E/F}(x(g)))_F.$$

Note that the twisting character η_α^m is trivial if $m = \dim_E(V)$ is even.

Since the identity

$$(a.43) \quad \det(g) = \left(\frac{x(g)}{x(g)} \right)$$

determines $x(g) \in E^\times$ up to multiplication by an element α of F^\times , the map

$$(a.44) \quad \eta_\alpha(g) = (\alpha, N_{E/F}(x(g)))_F$$

indeed defines a quadratic character η_α of $G(W)$.

Now $U = V \otimes W$ is a skew-Hermitian space over E with inner product:

$$(a.45) \quad \langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = (v_2, v_1) \langle w_1, w_2 \rangle.$$

Here U is a left vector space over E , where $\alpha \cdot v \otimes w = v \otimes \alpha w$. Let $V_1 = E$ with Hermitian form $(x, y)_1 = \bar{x}y$, and note that

$$(a.46) \quad \mathbb{W} = R_{E/F}(V \otimes_E W) \simeq R_{E/F}(V_1 \otimes_E U).$$

If χ_1 is a character of E^\times with $\chi_1|_{F^\times} = \epsilon_{E/F}$, there is a splitting

$$(a.47) \quad G(U) \xrightarrow{\tilde{\iota}_{V_1,\chi_1}} Mp(\mathbb{W}),$$

as usual. Let

$$(a.48) \quad G(V) \times G(W) \xrightarrow{i} G(U)$$

be the natural embedding.

Lemma A.7. *For $m = \dim_E V$, set $\chi_m = \chi_1^m$. Then on $G(W)$,*

$$\tilde{\iota}_{V_1,\chi_1} \circ i = \tilde{\iota}_{V,\chi_m}.$$

Proof. First we assume that W and hence U is split, and choose a polarization $W = X + Y$ and hence $U = V \otimes_E X + V \otimes_E Y$. Then

$$(a.49) \quad \mathbb{Y} = R_{E/F}(V \otimes_E Y) \simeq R_{E/F}(V_1 \otimes_E (V \otimes_E Y)),$$

and we use the isomorphism $Mp(\mathbb{W}) \simeq Sp(\mathbb{W}) \times \mathbb{C}^1$. If we let x_U and j_U be the Rao functions for $G(U)$, then for $g \in G(W)$,

$$(a.50) \quad j_U(i(g)) = mj(g), \quad \text{and} \quad x_U(i(g)) = x(g)^m \cdot (\det V)^{j(g)}.$$

Thus

$$(a.51) \quad \begin{aligned} \beta_{V_1, \chi_1}(i(g)) &= \chi_1(x_U(i(g))) \cdot \gamma_F(\eta \circ RV_1)^{-j_U(i(g))} \\ &= \chi_1(x(g))^m \chi_1(\det V)^{j(g)} \gamma_F(-\Delta, \eta)^{-mj(g)} \gamma_F(-1, \eta)^{mj(g)} \\ &= \chi_m(x(g)) \left((\Delta, \det V)_F \gamma_F(-\Delta, \eta)^m \gamma_F(-1, \eta)^{-m} \right)^{-j(g)} \\ &= \chi_m(x(g)) \gamma_F(\eta \circ RV)^{-j(g)} \\ &= \beta_{V, \chi_m}(g), \end{aligned}$$

as claimed.

Next, if W is arbitrary, we double both W and U and obtain the claimed relation from that just proved for the split case. \square

Now we recall the ‘reflection principle’ used at the end of section 1 to construct a splitting over $G(V)$. As in section 1, let $W' = W$ with *Hermitian* form

$$(a.52) \quad (w'_1, w'_2)' = \delta^{-1} \langle w_2, w_1 \rangle,$$

and let $V' = V$ with skew-Hermitian form

$$(a.53) \quad \langle v'_1, v'_2 \rangle' = \delta(v_2, v_1).$$

Here, for $w \in W$ (resp. $v \in V$), we let w' (resp. v') denote the corresponding element of W' (resp. V'). Let $U' = W' \otimes_E V'$ with

$$(a.54) \quad \begin{aligned} \langle w'_1 \otimes v'_1, w'_2 \otimes v'_2 \rangle' &= (w'_2, w'_1)' \langle v'_1, v'_2 \rangle' \\ &= \langle w_1, w_2 \rangle \langle v_2, v_1 \rangle \\ &= \langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle. \end{aligned}$$

Thus, the map $\alpha : U' \rightarrow U$ given by $w' \otimes v' \mapsto v \otimes w$ is an isometry of skew-Hermitian spaces. We then have the commutative diagram:

$$(a.55) \quad \begin{array}{ccccc} G(V) \times G(W) & \longrightarrow & G(U) & \xrightarrow{\tilde{t}_{V_1, \chi_1}} & Mp(\mathbb{W}) \\ sw \downarrow & & Ad(\alpha) \downarrow & & \downarrow \widetilde{Ad(\alpha)} \\ G(W') \times G(V') & \longrightarrow & G(U') & \xrightarrow{\tilde{t}_{V_1, \chi_1}} & Mp(\mathbb{W}'), \end{array}$$

where the left most arrow just switches factors. Now, by definition,

$$(a.56) \quad \tilde{t}_{W, \chi'} : G(V) \rightarrow Mp(\mathbb{W})$$

is given by

$$(a.57) \quad \tilde{t}_{W, \chi'} = \widetilde{Ad(\alpha)}^{-1} \circ \tilde{t}_{W', \chi'},$$

and, by the previous result,

$$(a.58) \quad \tilde{t}_{W, \chi_n}(h) = \tilde{t}_{V_1, \chi_1}(i(h)).$$

Corollary A.8. *If χ_1 is a character of E^\times with $\chi_1|_{F^\times} = \epsilon_{E/F}$, and if $\chi_m = \chi_1^m$ and $\chi_n = \chi_1^n$, then for $z \in E^1$,*

$$\tilde{\iota}_{V, \chi_m}(z \cdot 1_W) = \tilde{\iota}_{W, \chi_n}(z \cdot 1_V),$$

i.e., if the pair of characters (χ_m, χ_n) is used to define the splitting over $G(V) \times G(W)$, then the splittings agree on the centers of $G(V)$ and $G(W)$. In particular, if irreducible representations π of $G(W)$ and τ of $G(V)$ correspond, i.e., if $\tau = \theta_\chi(\pi, V)$, then they have the same central character.

Corollary A.9. *If $m = n = 1$ and if the same character χ is used to define the splitting for $G(V)$ and $G(W)$, then the theta correspondence is the identity map, $\eta \leftrightarrow \eta$, on those representations which occur.*

Finally, suppose that $W' \simeq V$, and hence $V' \simeq W$. Then we may identify the spaces U and U' , and view $\alpha : U \rightarrow U' = U$ as an element of $G(U)$. We then have the commutative diagram:

$$(a.59) \quad \begin{array}{ccccc} G(V) \times G(W) & \longrightarrow & G(U) & \xrightarrow{\tilde{\iota}_{V_1, \chi_1}} & Mp(\mathbb{W}) \\ sw \downarrow & & Ad(\alpha) \downarrow & & \downarrow Ad(\tilde{\iota}_{V_1, \chi_1}(\alpha)) \\ G(W') \times G(V') & \longrightarrow & G(U) & \xrightarrow{\tilde{\iota}_{V_1, \chi_1}} & Mp(\mathbb{W}). \end{array}$$

Corollary A.10. *If $W' \simeq V$ and if the same character χ is used to define the splittings over $G(V)$ and $G(W)$, then the Weil representation $\omega_{V, W, \chi}$ of $G(V) \times G(W)$ is isomorphic to the representation $\omega_{W', V', \chi}$ of $G(W') \times G(V') \simeq G(V) \times G(W)$.*

Note that both the definition of the splitting over $G(V)$ (resp. $G(W')$) and the definition of W' (resp. V') depend on the choice of δ .

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