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GLOBAL C^∞ IRREGULARITY OF THE $\bar\partial$–NEUMANN PROBLEM FOR WORM DOMAINS

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0. Introduction

Let $\Omega \subset \mathbf{C}^n$ be a bounded, pseudoconvex domain with C^∞ boundary, equipped with the standard Hermitian metric inherited from $\mathbf{C}^n$. The $\bar\partial$–Neumann problem for $(0,1)$ forms in Ω is the boundary value problem

$$\begin{cases} \Box u = f & \text{in } \Omega, \\ u \lrcorner \bar\partial\rho = 0 & \text{on } \partial\Omega, \\ \bar\partial u \lrcorner \bar\partial\rho = 0 & \text{on } \partial\Omega, \end{cases}$$

where ρ is a defining function for Ω, $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$, u, f are $(0,1)$ forms, and $\lrcorner$ denotes the interior product of forms. Under the stated hypotheses on Ω, this problem is uniquely solvable for every $f \in L^2(\Omega)$. The Neumann operator N, mapping f to the solution u, is continuous on $L^2(\Omega)$. The Bergman projection B is the orthogonal projection of $L^2(\Omega)$ onto the closed subspace of L^2 holomorphic functions on Ω, and is related to N by $B = I - \bar\partial^* N \bar\partial$.

N and B are C^∞ pseudolocal if Ω is strictly pseudoconvex, or more generally, is of finite type [Ca1]. Both preserve $C^\infty(\overline{\Omega})$ under certain weaker hypotheses [BS2], [Ca2]. For any pseudoconvex, smoothly bounded Ω and any finite exponent s, there exists a strictly positive weight $w \in C^\infty(\overline{\Omega})$ such that the Neumann operator and Bergman projection with respect to the Hilbert space $L^2(\Omega, w(x)dx)$ map the Sobolev space $H^t(\Omega)$ boundedly to $H^t(\Omega)$, for all $0 \le t \le s$ [K1]. It has remained an open question whether N and B, defined with respect to the standard metric, preserve $C^\infty(\overline{\Omega})$ without further hypotheses on Ω. An affirmative answer would have significant consequences [BL].

Theorem. *There exist pseudoconvex, smoothly bounded domains $\Omega \Subset \mathbf{C}^2$ for which the Neumann operator on $(0,1)$ forms and Bergman projection fail to preserve $C^\infty(\overline{\Omega})$.*

Examples are the worm domains, originally introduced by Diederich and Fornæss [DF] for another purpose[1] but long considered likely candidates for N and B to fail to be globally regular in C^∞.

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[1]Some but not all worm domains have nontrivial Nebenhülle [FS, p. 111], [DF, Theorem 5.1], whereas all worm domains are counterexamples to global regularity.

The proof depends on the observation of Barrett [B] that for each worm domain $\mathcal{W}$, for all sufficiently large s, N and B fail to map $H^s(\mathcal{W})$ boundedly to $H^s(\mathcal{W})$. We establish for each worm domain an *a priori* inequality of the form $\|Nf\|_{H^s} \le C_s\|f\|_{H^s}$, valid for all $f \in C^\infty(\overline{\mathcal{W}})$ such that $Nf \in C^\infty(\overline{\mathcal{W}})$, for a sequence of exponents s tending to ∞. If N were to preserve $C^\infty(\overline{\mathcal{W}})$, then since it is a bounded linear operator on $L^2(\mathcal{W})$ and since $C^\infty(\overline{\mathcal{W}})$ is dense in $H^s(\mathcal{W})$, it would follow that N maps $H^s(\mathcal{W})$ boundedly to itself, for a sequence of values of s tending to ∞, a contradiction. More accurately, our inequality is valid only for certain subspaces of $L^2(\mathcal{W})$ preserved by $\Box$, but this still suffices to contradict the theorem of Barrett.

An analogous counterexample in the real analytic context is already known [Ch1]: there exists a bounded, pseudoconvex domain $\Omega \subset \mathbf{C}^2$ having real analytic boundary, such that the Szegö projection fails to preserve $C^\omega(\partial\Omega)$. That result and its proof are however not closely related to the C^∞ case.

§§1 through 3 review material on worm domains and the $\bar\partial$–Neumann problem, and present some routine but tedious reductions. §4 formalizes a class of two-dimensional problems subsuming those to which the reductions lead. The analysis of those problems is contained in §§5 and 6.

1. Reduction to the boundary

The $\bar\partial$–Neumann problem is a boundary value problem for an elliptic partial differential equation, and as such is amenable to treatment by the method of reduction to a pseudodifferential equation on the boundary. This reduction has been carried out in detail for domains in $\mathbf{C}^2$ by Chang, Nagel and Stein [CNS]. We review here certain of their computations and direct consequences thereof.

Assume $\Omega \subset \mathbf{C}^2$ to be a smoothly bounded domain. The equation $\Box u = f$ on Ω for $(0,1)$ forms in $C^\infty(\overline{\Omega})$ is equivalent to an equation $\Box^+ v = g$ on $\partial\Omega$, where v, g are sections of a certain complex line bundle[2] $\mathcal{B}^{0,1}$. Let ρ be a smooth defining function for Ω and define $\bar\omega_2 = \bar\partial\rho$, and $\bar\omega_1 = (\partial\rho/\partial z_2)d\bar z_1 - (\partial\rho/\partial z_1)d\bar z_2$.

v is related to u by $u = Pv + Gf$, where P, G are respectively Poisson and Green operators for the elliptic system $\Box u = f$ with Dirichlet boundary conditions. In particular, if $f \in C^\infty(\overline{\Omega})$, then $u \in C^\infty(\overline{\Omega})$ if and only if the same holds for v. More precisely, G maps $H^s(\Omega)$ to $H^{s+2}(\Omega)$, while P maps $H^s(\partial\Omega)$ to $H^{s+1/2}(\Omega)$, for each $s \ge 0$. Thus if $f \in H^s$, in order to conclude that $u \in H^s$ it suffices to know that $v \in H^{s-1/2}$.

On the other hand, $g = (\bar\partial Gf \lrcorner \bar\partial\rho)$, restricted to $\partial\Omega$. If $f \in H^s$, then $\bar\partial Gf \in H^{s+1}(\Omega)$, so its restriction to the boundary belongs to $H^{s+1/2}(\partial\Omega)$. Thus in order to show that N preserves $H^s(\Omega)$ it suffices to show that if $\Box^+ v \in H^{s+1/2}(\partial\Omega)$, then $v \in H^{s-1/2}(\partial\Omega)$, assuming always that $s > 1/2$.

On $\partial\Omega$ a Cauchy-Riemann operator is the complex vector field $\bar L = (\partial_{\bar z_1}\rho)\partial_{\bar z_2} - (\partial_{\bar z_2}\rho)\partial_{\bar z_1}$. Define L to be the complex conjugate of $\bar L$. The characteristic variety [3] of $\bar L$ is a real line bundle Γ. Assuming Ω to be pseudoconvex and the set of strictly pseudoconvex points to be dense in $\partial\Omega$, Γ splits smoothly and uniquely as $\Gamma^+ \cup \Gamma^-$, where each fiber of $\Gamma^\pm$ is a single ray, and where Γ^+ is distinguished from Γ^- by

[2] $\mathcal{B}^{0,1}$ is defined to be the quotient of the restriction to $\partial\Omega$ of $T^{0,1}\mathbf{C}^2$, modulo the span of $\bar\partial\rho$. Sections of $\mathcal{B}^{0,1}$ may be identified with scalar-valued functions times $\bar\omega_1$, hence with scalar-valued functions.

[3] By the characteristic variety of a pseudodifferential operator we mean the conic subset of the cotangent bundle on which its principal symbol vanishes.

the requirement that the principal symbol of $[\bar{L}, L]$ is nonpositive on Γ^+, modulo terms spanned by the symbols of the real and imaginary parts of $\bar{L}$ and a term of order 0. Equivalently, the principal symbol of $[\bar{L}, \bar{L}^*]$ is nonnegative on Γ^+, modulo the same kinds of error terms.

$\Box^+$ is a classical pseudodifferential operator of order +1. Its principal symbol vanishes everywhere on Γ^+ but nowhere else. Microlocally in a conic neighborhood of Γ^+, $\Box^+$ takes the form

$$\Box^+ = Q\bar{L}L + F_1\bar{L} + F_2 L + F_3,$$

where Q is an elliptic pseudodifferential operator of order -1, and each F_j is a pseudodifferential operator of order less than or equal to -1. Since $\Box^+$ is elliptic except on Γ^+, for any pseudodifferential operator G of order zero whose symbol vanishes identically in some neighborhood of Γ^+, one has for all $u \in C^\infty$ and all $N < \infty$

$$\|Gu\|_{H^{t+1}(\partial\Omega)} \le C\|\Box^+ u\|_{H^t(\partial\Omega)} + C_N\|u\|_{H^{-N}(\partial\Omega)}. \tag{1.1}$$

Let A be an elliptic pseudodifferential operator of order +1 such that $A \circ Q$ equals the identity on $L^2(\partial\Omega)$, modulo an operator smoothing of infinite order. Composing on the left with A, the equation $\Box^+ v = g$ may be rewritten as $\mathfrak{L}v = \tilde{g}$ microlocally in a conic neighborhood of Γ^+, where

$$\mathfrak{L} = \bar{L}L + B_1\bar{L} + B_2 L + B_3, \tag{1.2}$$

$\|\tilde{g}\|_{H^t} \le C\|g\|_{H^{t+1}} + C_N\|v\|_{H^{-N}}$ for any finite N, and each B_j is an operator of order less than or equal to zero. Therefore, in order to show that the Neumann operator satisfies an *a priori* inequality of the form $\|Nf\|_{H^s(\Omega)} \le C\|f\|_{H^s(\Omega)}$ for all $f \in C^\infty(\overline{\Omega})$ such that $Nf \in C^\infty(\overline{\Omega})$, it suffices to establish an *a priori inequality* for all $v \in C^\infty(\partial\Omega)$ of the form

$$\|v\|_{H^t} \le C\|\mathfrak{L}v\|_{H^t} + C\|v\|_{H^{t'}} + C\|\tilde{Q}v\|_{H^{t+2}}, \tag{1.3}$$

where $t = s - 1/2$, for some $t' < t$ and some pseudodifferential operator $\tilde{Q}$ of order zero whose symbol vanishes identically in some neighborhood of Γ^+.

2. Worm domains

A worm domain in $\mathbf{C}^2$ is an open set of the form

$$\mathcal{W} = \{z : |z_1 + e^{i\log|z_2|^2}| < 1 - \phi(\log|z_2|^2)\},$$

where the function ϕ vanishes identically on some interval $[-r, r]$ of positive length, and is constructed [DF] so as to guarantee that $\mathcal{W}$ will be pseudoconvex with C^∞ boundary, and will be strictly pseudoconvex at every boundary point except those on the exceptional annulus $\mathcal{A} \subset \partial\mathcal{W}$ defined as

$$\mathcal{A} = \{z : z_1 = 0 \text{ and } |\log|z_2|^2| \le r\}.$$

The circle group acts as a group of automorphisms of $\mathcal{W}$ by $z \mapsto R_\theta z = (z_1, e^{i\theta}z_2)$. It acts on functions by $R_\theta f(z) = f(R_\theta z)$, and on $(0,1)$ forms by

$$R_\theta(f_1 d\bar{z}_1 + f_2 d\bar{z}_2) = (R_\theta f_1)d\bar{z}_1 + (R_\theta f_2)e^{-i\theta}d\bar{z}_2.$$

The Hilbert space $L^2_{(0,k)}(\mathcal{W})$ of square integrable $(0,k)$ forms decomposes as the orthogonal direct sum $\bigoplus_{j\in\mathbf{Z}}\mathcal{H}^k_j$, where $\mathcal{H}^k_j$ is the set of all $(0,k)$ forms f satisfying $R_\theta f\equiv e^{ij\theta}f$. Here $\bar\partial$ is an unbounded linear operator from $\mathcal{H}^k_j$ to $\mathcal{H}^{k+1}_j$, B maps $\mathcal{H}^0_j$ to itself, and the Neumann operator N maps $\mathcal{H}^1_j$ to $\mathcal{H}^1_j$ boundedly, for each j.

For each k and each $s\ge 0$, the Sobolev space $H^s(\mathcal{W})$ likewise decomposes as an orthogonal direct sum of subspaces H^s_j. It is known that for any smoothly bounded, pseudoconvex domain $\Omega\subset\mathbf{C}^2$, for any exponent $s\ge 0$, if N maps $H^s(\Omega)$ boundedly to itself, then B also maps $H^s(\Omega)$ boundedly to itself [BS1], where H^s denotes in the first instance a space of one forms, and in the second, a space of functions. Because N,B preserve the summands $\mathcal{H}_j$, the same proof shows[4] that for any fixed j, if N maps the space H^s_j of $(0,1)$ forms boundedly to itself, then B maps the space H^s_j of functions boundedly to itself. Barrett [B] showed that for each worm domain, for all sufficiently large s, for all j, B fails to map H^s_j boundedly to itself. Therefore, in order to prove that N, acting on $(0,1)$ forms, fails to preserve $C^\infty(\overline{\mathcal{W}})$, it suffices to establish the following result for a single index j.

Proposition 1. *For each worm domain there exists a discrete subset $S\subset\mathbf{R}^+$ such that for each $s\notin S$ and each $j\in\mathbf{Z}$ there exists $C_{s,j}<\infty$ such that for every $(0,1)$ form $u\in\mathcal{H}^1_j\cap C^\infty(\overline{\mathcal{W}})$ such that $Nu\in C^\infty(\overline{\mathcal{W}})$,*

$$\|Nu\|_{H^s(\mathcal{W})}\le C_{s,j}\|u\|_{H^s(\mathcal{W})}. \tag{2.1}$$

The defining function $\rho = 1-\phi(\log|z_2|^2)-|z_1+e^{i\log|z_2|^2}|^2$ for $\mathcal{W}$ is invariant under R_θ, as is the $(0,1)$ form $\bar\omega_2$ defined above. $\bar\omega_1$ satisfies $R_\alpha\bar\omega_1=\exp(-i\alpha)\bar\omega_1$ for all α, but it may also be made invariant by multiplying it by the function $(z_1,re^{i\theta})\mapsto e^{i\theta}$, which is smooth in a neighborhood of $\overline{\mathcal{W}}$. We work henceforth with this modified $\bar\omega_1$.

$\Box^+$ commutes with R_θ for all θ. Indeed, $\Box^+v=\bar\partial Pv\,\lrcorner\,\bar\partial\rho$ [CNS]. $\Box$ commutes with R_θ, hence so must P. $\bar\partial$ commutes with R_θ, and the Hermitian metric on $\mathbf{C}^2$ and $\bar\partial\rho$ are likewise R_θ–invariant. Thus all ingredients in the above expression for $\Box^+$ are invariant, hence so is $\Box^+$ itself.

Identify square integrable sections of $\mathcal{B}^{0,1}$ with scalar-valued L^2 functions as above, and decompose $L^2(\partial\mathcal{W})=\bigoplus\mathcal{H}_j(\partial\mathcal{W})$, where $\mathcal{H}_j$ is the subspace of those functions satisfying $R_\theta f\equiv e^{ij\theta}f$. Then $\Box^+$ maps $\mathcal{H}_j(\partial\mathcal{W})\cap C^\infty$ to $\mathcal{H}_j(\partial\mathcal{W})$. We have seen in §1 that Proposition 1 would be a consequence of the validity of (1.3) for all $v\in C^\infty(\partial\mathcal{W})\cap\mathcal{H}_j$, for $t=s-1/2$.

Fix j and assume henceforth that u belongs to $\mathcal{H}^1_j(\mathcal{W})$ and to C^∞. Then the associated boundary function v belongs to $\mathcal{H}_j\cap C^\infty(\partial\mathcal{W})$. Henceforth we work exclusively on the boundary, and simplify notation by writing simply $\mathcal{H}_j$ rather than $\mathcal{H}_j(\partial\mathcal{W})$.

Note that $\bar L, L$ take $\mathcal{H}_j\cap C^\infty$ to $\mathcal{H}_{j+1}$ and to $\mathcal{H}_{j-1}$, respectively. The operator A introduced after (1.1) may be constructed to be R_θ-invariant, for both $\Box^+$ and $\bar L\circ L$ are invariant while $\bar L, L$ are automorphic of certain degrees, so that averaging the equation $\Box^+ = Q\bar LL+F_1\bar L+F_2L+F_3$ with respect to $R_\theta d\theta$ produces an invariant Q and F_3, and operators F_1,F_2 automorphic of the appropriate degrees. Thus $(\bar LL+B_1\bar L+B_2L+B_3)v\in H^t$ microlocally near Γ^+, where B_1,B_2,B_3 map $\mathcal{H}_j$ to $\mathcal{H}_i$ for $i=j-1,j+1,j$ respectively.

[4] This follows from the argument of Boas and Straube [BS1] because all elements of their proof may be chosen to be invariant under the automorphisms R_θ.

Since $\mathcal{W}$ is strictly pseudoconvex at all points not in $\mathcal{A}$, it follows as in [K2] that on the complement of any neighborhood of $\mathcal{A}$, the H^{s+1} norm of v is majorized by $C\|\Box^+ v\|_{H^{t+1}(\partial\mathcal{W})} + C\|v\|_{H^{-N}}$, hence by $C\|\mathfrak{L}v\|_{H^s} + C\|v\|_{H^{-N}} + C\|\tilde{Q}v\|_{H^{s+2}}$ with $\tilde{Q}$ as in (1.3). This estimate is one derivative stronger than that which we seek. In particular, it now suffices to control the H^s norm of v in an arbitrarily small neighborhood U of $\mathcal{A}$, and to do so microlocally near Γ^+.

Fix a C^∞ cutoff function φ supported in a small neighborhood of $\mathcal{A}$ but identically equal to 1 in a smaller neighborhood, and fix an open set V disjoint from a neighborhood of $\mathcal{A}$ such that $\nabla\varphi$ is supported in V. By Leibniz's rule and the pseudolocality of pseudodifferential operators, the H^s norm of $\mathfrak{L}(\varphi v - v)$ is majorized by $C\|v\|_{H^{s+1}(V)} + C\|v\|_{H^{-N}}$ for any $N < \infty$. Thus by replacing v with φv we may reduce matters to the case where v is supported in an arbitrarily small neighborhood W of $\mathcal{A}$. Therefore it suffices to prove the existence of some W and an exponent $s' < s$ such that (1.3) holds (with t replaced by s) for all $v \in C_0^\infty$ supported in W.

In a neighborhood of $\mathcal{A}$ in $\partial\mathcal{W}$ introduce coordinates (x, θ, t), where

$$z_2 = e^{x+i\theta}, \qquad z_1 = e^{i2x}(e^{it}(1 - \phi(2x)) - 1)$$

with $2|x| < r + \delta$ and $|t| < \delta$ for some small $\delta > 0$. In these coordinates $\mathcal{A} = \{t = 0,\ |x| \le r/2\}$. Setting

$$\gamma(x,t) = 2\big[e^{-it} - 1 + \phi(2x) - i\phi'(2x)\big]/\big[1 - \phi(2x)\big],$$

the vector field $\bar{L} = \partial_x + i\partial_\theta + \gamma\partial_t$ annihilates both z_1 and z_2. Hence it differs from what was previously denoted as $\bar{L}$ by multiplication on the left by a nonvanishing factor, which may be verified to have the form $b(x,t)\exp(i\theta)$. For $|x| \le r/2$, $\phi(x) \equiv 0$ and consequently $\gamma(x,t) \equiv 2(e^{-it} - 1)$. Therefore[5]

$$\bar{L} = \partial_x + i\partial_\theta + it\alpha(t)\partial_t \qquad \text{where } |x| \le r/2, \text{ with } \operatorname{Re}\alpha(0) \ne 0. \tag{2.2}$$

The representation $\mathfrak{L} = \bar{L}L + B_1\bar{L} + B_2L + B_3$ in terms of the new $\bar{L}$ remains valid with modified coefficients B_i that now preserve each $\mathcal{H}_j$, once the operator formerly denoted by $\mathfrak{L}$ is multiplied by $|b|^{-2}$.

There exists a unique C^∞ real-valued function μ, independent of θ, such that $[\bar{L}, L] = i\mu(x,t)\partial_t$ modulo the span of the real and imaginary parts of $\bar{L}$. More precisely, since the coefficients of ∂_θ in $[\bar{L}, L]$ and in $\operatorname{Re}\bar{L}$ vanish while the coefficient in $\operatorname{Im}\bar{L}$ is nowhere zero,

$$[\bar{L}, L] = i\mu(x,t)\partial_t + i\nu(x,t)\operatorname{Re}\bar{L} \tag{2.3}$$

for unique real-valued, C^∞ coefficients μ, ν. The pseudoconvexity of $\partial\mathcal{W}$ means that μ does not change sign. Replacing t by $-t$ if necessary, we may assume that $\mu \ge 0$.

Fix an integer k. We identify functions of $(x,t) \in \mathbf{R}^2$ with elements of $\mathcal{H}_k$ via the correspondence $u(x,t) \mapsto u(x,t)e^{ik\theta}$; ∂_θ then becomes multiplication by ik. For the remainder of the paper we work in $\mathbf{R}^2$. Define

$$\mathcal{D} = \{(x,t) : |x| \le r/2 \text{ and } t = 0\}.$$

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[5] Boas and Straube [BS2] have shown the $\bar\partial$–Neumann problem to be globally C^∞ hypoelliptic whenever there exists a real vector field on the boundary that is transverse to the complex tangent space and has a certain favorable commutation property. If $\operatorname{Re}\alpha(0)$ were to vanish, then ∂_t would be such a vector field. Thus nonvanishing of $\operatorname{Re}\alpha(0)$ is for our purpose an essential feature of worm domains.

By incorporating ik into the B_j we may further rewrite $\mathfrak{L}$, when restricted to $\mathcal{H}_k$, as

$$\mathcal{L}_0 = \bar{\ell}\ell + B_1\bar{\ell} + B_2\ell + B_3$$

where $\bar{\ell}$ is a complex vector field in $\mathbf{R}^2$ which, for $|x| \le r/2$, takes the form

$$\bar{\ell} = \partial_x + it\alpha(t)\partial_t.$$

Here ℓ denotes the conjugate of $\bar{\ell}$, each B_j is a pseudodifferential operator of order ≤ 0 in $\mathbf{R}^2$, and $\operatorname{Re}\alpha(0) \ne 0$. Note that the commutator of the real and imaginary parts of $\bar{\ell}$ is not forced to vanish identically, because α is not real-valued. We have $[\bar{\ell}, \ell] = i\mu(x,t)\partial_t + i\nu(x,t)\operatorname{Re}\bar{\ell}$ with the same coefficients as in (2.3).

Letting (ξ, τ) be Fourier variables dual to (x, t), define $\tilde{\Gamma} = \{(x,t,\xi,\tau) : (x,t) \in \mathcal{D} \text{ and } \xi = 0\}$. $\mathcal{L}_0$ is not elliptic at points of $\mathcal{D}$, but is elliptic at most points in its complement; $\tilde{\Gamma}$ is the intersection of the characteristic variety of $\mathcal{L}_0$ with $\{(x,t,\xi,\tau) : (x,t) \in \mathcal{D}\}$. Decompose $\tilde{\Gamma} = \tilde{\Gamma}^+ \cup \tilde{\Gamma}^-$ where $\tilde{\Gamma}^+ = \tilde{\Gamma} \cap \{\tau > 0\}$. Thus the principal symbol of $i\mu\partial_t$, namely $-\mu\tau$, is nonpositive in a conic neighborhood of $\tilde{\Gamma}^+$.

3. Two pseudodifferential manipulations

By an operator we will always mean a classical pseudodifferential operator, that is, one whose symbol admits a full asymptotic expansion in homogeneous terms of integral degrees. $\sigma_j(T)$ denotes the j-th order symbol of T (in the Kohn-Nirenberg calculus), always with respect to the fixed coordinate system (x,t,ξ,τ). Henceforth we work under the convention that A, B, E denote operators whose orders are less than or equal to $0, 0, -1$ respectively, whose meanings are permitted to change freely from one occurrence to the next, even within the same line. A denotes always an operator having the additional property that $\sigma_0(A)(x,t,\xi,\tau) \equiv 0$ for all $(x,t) \in \mathcal{D}$. Any operator of type E may be regarded as one of type A. Two operators are said to agree microlocally in some conic open set if the full symbol of their difference vanishes identically there.

Any operator B may be written in the form $B = \beta(x) + E \circ \ell + A$ microlocally in a conic neighborhood of $\tilde{\Gamma}^+$, where β denotes both a C^∞ function and the operator defined by multiplication by that function. Indeed, $\sigma_0(B)(x,0,0,\tau)$ depends only on $(x, \operatorname{sgn}(\tau))$ and we define $\beta(x)$ to be this quantity for $\tau > 0$. Then where $\tau > 0$ and $|x| \le r/2$, $\sigma_0(B)(x,0,\xi,\tau)$ is divisible by $\xi = -i\sigma_1(\ell)(x,0,\xi,\tau)$. $\sigma_{-1}(E)(x,0,\xi,\tau)$ is then uniquely determined for such x, τ by the equation $\sigma_0(B) = \beta(x) + \sigma_1(\ell) \cdot \sigma_{-1}(E)$. Define E to be any operator of order -1 whose principal symbol satisfies this equation when restricted to $(x,t) \in \mathcal{D}$ and to a conic neighborhood of $\tilde{\Gamma}^+$. Then simply define $A = B - \beta - E \circ \ell$.

Writing $B_1 = \beta_1(x) + E_1 \circ \ell + A_1$ and similarly $B_2 = \beta_2(x) + E_2 \circ \bar{\ell} + A_2$, and expressing $E\ell\bar{\ell} = E\bar{\ell}\ell$ plus an operator of order ≤ 0, we obtain $\mathcal{L}_0 = (I+E)\bar{\ell}\ell + (\beta_1 + A)\bar{\ell} + (\beta_2 + A)\ell + B$. Composing both sides with a parametrix for $I + E$ and modifying the definition of $\mathcal{L}_0$ to include this factor, we have $\mathcal{L}_0 = \bar{\ell}\ell + (\beta_1 + A)\bar{\ell} + (\beta_2 + A)\ell + B$. Writing finally $B = \beta_3(x) + E \circ \ell + A$ results in

$$\mathcal{L}_0 = \bar{\ell}\ell + (\beta_1 + A)\bar{\ell} + (\beta_2 + A)\ell + (\beta_3 + A), \tag{3.1}$$

where the β_j are C^∞ functions depending only on x. $\square$

We next reduce the question of *a priori* H^s inequalities to L^2, simultaneously for all s. Fix an operator Q of order 0 that is elliptic in some conic neighborhood of $\tilde\Gamma^-$, whose symbol vanishes identically in some conic neighborhood of $\tilde\Gamma^+$. Fix an exponent $s > 0$, for which we seek an *a priori* inequality for all $v \in C^\infty$ of the form

$$\|v\|_{H^s} \le C\|\mathcal{L}_0 v\|_{H^s} + C\|v\|_{H^{s'}} + C\|Qv\|_{H^{s+2}} \tag{3.2}$$

for some $s' < s$. Having such an inequality for a sequence of exponents s tending to $+\infty$ would imply Proposition 1 by the preceding discussion. In particular, the H^{s+2} norm of Qv is already under control by virtue of (1.1), while the H^0 norm of v is harmless because the Neumann operator is bounded on L^2, and $\|v\|_{H^{s'}} \le \varepsilon\|v\|_{H^s} + C_{\varepsilon,N}\|v\|_{H^{-N}}$ for any $\varepsilon > 0$ and $N < \infty$.

Fix a C^∞, strictly positive function $m = m(\xi,\tau)$, homogeneous of degree 1 for large $|(\xi,\tau)|$ and identically equal to $(1+\tau^2)^{1/2}$ in a conic neighborhood of $\{\xi = 0\}$. Define Λ^s to be the Fourier multiplier operator on $\mathbf{R}^2$ with symbol $m(\xi,\tau)^s$.

Substituting $v = \Lambda^{-s}u$ and $g = \Lambda^{-s}f$, estimation of the H^s norm of v, modulo a lower order norm, in terms of that of g is equivalent to estimation of the H^0 norm of u in terms of that of f, modulo a negative order norm of u. The equation $\mathcal{L}_0 v = g$ becomes $\Lambda^{-s}\mathcal{L}_0\Lambda^s u = f$. Write $\Lambda^{-s}\bar\ell\ell\Lambda^s = \Lambda^{-s}\bar\ell\Lambda^s \circ \Lambda^{-s}\ell\Lambda^s$ and similarly for other terms, and note that $\Lambda^{-s}(\beta_j + A)\Lambda^s = \beta_j + A$ with the same function β_j.

$\Lambda^{\pm s}$ commutes with ∂_x and with ∂_t, so $\Lambda^{-s}(\partial_x + i\alpha t\partial_t)\Lambda^s = \partial_x + \Lambda^{-s}t\Lambda^s \circ \Lambda^{-s}\alpha\Lambda^s\partial_t$. Applying the Fourier transform gives immediately $\Lambda^{-s}[t,\Lambda^s]\partial_t = -s + E$, microlocally in some conic neighborhood of $\tilde\Gamma^+$, modulo operators smoothing there of infinite order. Since $\Lambda^{-s}[\alpha,\Lambda^s]\partial_t$ is of order 0,

$$\begin{aligned}\Lambda^{-s}t\Lambda^s \circ \Lambda^{-s}\alpha\Lambda^s\partial_t &= \alpha t\partial_t + \Lambda^{-s}[t,\Lambda^s]\alpha\partial_t + t \circ \Lambda^{-s}[\alpha,\Lambda^s] \circ \partial_t + E\\ &= \alpha \circ (t\partial_t - s) + tB + E,\end{aligned}$$

microlocally near $\tilde\Gamma^+$. Thus microlocally near $\tilde\Gamma^+$, $\Lambda^{-s}\mathcal{L}_0\Lambda^s = \mathcal{L}_s$ becomes

$$\mathcal{L}_s = \bar\ell_s\ell_s + (\beta_1 + A)\bar\ell_s + (\beta_2 + A)\ell_s + (\beta_3 + A),$$

where $\bar\ell_s, \ell_s$ are first-order differential operators differing from $\bar\ell, \ell$ respectively by terms of order zero, and taking the forms $\bar\ell_s = \partial_x + i\alpha(t\partial_t - s)$, $\ell_s = \partial_x - i\bar\alpha(t\partial_t - s)$ for $|x| \le r/2$, where α depends only on t and the β_i only on x.

To see that $\bar\ell_s$ does take the form claimed for $|x| \le r/2$, express $\bar\ell = \partial_x + i\alpha t\partial_t$ modulo terms $\gamma(x,t)\partial_x$ and $\gamma(x,t)\partial_t$ where $\gamma \equiv 0$ for $|x| \le r/2$. Then $\Lambda^{-s}[\gamma(x,t)\partial_x, \Lambda^s]$ and $\Lambda^{-s}[\gamma(x,t)\partial_t, \Lambda^s]$ are operators of the type A, since they have nonpositive orders and their symbols of order zero vanish identically for $|x| < r/2$. □

4. A two-dimensional problem and preliminary inequalities

The remainder of the paper consists of a self-contained analysis of a special class of pseudodifferential equations in a real two-dimensional region. We begin by describing the equations in question and fixing notation, which in some respects differs from that of preceding sections.

Fix an interval $I = [-r, r] \subset \mathbf{R}$. Denote by $(x,t) \in \mathbf{R}^2$ coordinates in a neighborhood U of $\mathcal{D} = I \times \{0\}$. The interval $\mathcal{D}$ corresponds to the degenerate annulus embedded in the boundary of the worm domain, and will be the focus of attention. The convention concerning the symbols A, B, E introduced at the outset of §3 remains in force.

Consider a one parameter family of pseudodifferential operators of the form

$$\mathcal{L}_s = \bar{L}L + (\beta_1(x) + A)\bar{L} + (\beta_2(x) + A)L + (\beta_3(x) + A), \tag{4.1}$$

where the β_j are C^∞ functions. Suppose that $\bar{L}$, L are first-order differential operators depending on the real parameter s, and that $-L$ is the formal adjoint of $\bar{L}$, modulo an operator of order zero. Suppose that where $|x| \le r$, they take the special forms[6]

$$\bar{L} = \partial_x + ia(x)(t\partial_t + s) + O(t^2)\partial_t,$$
$$L = \partial_x - i\bar{a}(x)(t\partial_t + s) + O(t^2)\partial_t.$$

Here $O(t^2)$ denotes multiplication by a smooth function divisible by t^2 on the region U. Here a and the coefficients β_j are assumed independent of s, but A and the terms $O(t^2)\partial_t$ are permitted to depend on s.

Assume that

$$\operatorname{Re} a(x) \ne 0 \qquad \text{for all } x \in I, \tag{4.2}$$

and that there exist smooth real-valued coefficients μ, ν such that $[\bar{L}, L] = i\mu(x,t)\partial_t + i\nu(x,t)\operatorname{Re}\bar{L}$, satisfying

$$\mu \ge 0 \text{ at every point of } U. \tag{4.3}$$

Because $\bar{L}^* = -L$ modulo a term of order zero, L has the same real part as $\bar{L}$. A change of variables of the form $(x,t) \mapsto (x, h(x,t))$, with $h(x,0) \equiv 0$ where $|x| \le r$, therefore reduces matters to the case where the real parts of both $\bar{L}$ and L are everywhere parallel to ∂_x, and $\bar{L} = \partial_x + i\tilde{a}(x,t)(t\partial_t + s) + O(t^2)\partial_t$ on $I \times \mathbf{R}$, with $\tilde{a}$ real-valued and nonvanishing. Rewrite $\tilde{a}(x,t) = a(x) + O(t)$, and incorporate the contribution of $O(t)$ into the various terms $O(t^2)\partial_t$ and A. (4.3) is invariant under diffeomorphism and hence μ cannot change sign, so the coefficient of t in the Taylor expansion of $\mu(x,t)$ about $t = 0$ must vanish identically, for $|x| \le r$. This forces $\partial_x a(x) \equiv 0$ there. Thus

$$\bar{L} = \partial_x + ia(t\partial_t + s) + O(t^2)\partial_t \tag{4.4}$$

for $|x| \le r$, where a is a nonzero real constant. Moreover, ∂_x may be expressed in U as a nonvanishing scalar multiple of $\bar{L} + L$, modulo an operator of order zero. From now on we work in these new coordinates.

Define $\Gamma = \{(x,t,\xi,\tau) : (x,t) \in \mathcal{D} \text{ and } \xi = 0\}$. Decompose $\Gamma = \Gamma^+ \cup \Gamma^-$ where $\Gamma^+ = \{\tau > 0\} \cap \Gamma$. Then by (4.3), in some conic neighborhood of Γ^+ the principal symbol of $[\bar{L}, L]$ equals a nonpositive symbol, modulo terms in the span of the symbols of $\bar{L}, L$ and a term of order zero.

The symbol $\|\cdot\|$, with no subscript, denotes the norm in $L^2(U)$, while $\|\cdot\|_t$ denotes any fixed norm for the Sobolev space H^t of functions having t derivatives in L^2 and supported in U. The goal of the remainder of the paper is the following *a priori* estimate.

Proposition 2. *Let $\{\mathcal{L}_s\}$ be a family of operators of the form* (4.1) *satisfying all of the hypotheses introduced above. Then there exists a discrete exceptional set $S \subset [0,\infty)$ such that for any $s \notin S$ and any pseudodifferential operator Q of order* 0 *whose principal symbol is nonzero in some conic neighborhood of Γ^-, there exist*

[6] No assumption is now made on the vanishing or nonvanishing of the coefficient of ∂_t in $\bar{L}$ where $|x| > r$, but the strict pseudoconvexity of $\mathcal{W}$ outside the exceptional annulus was used to reduce Proposition 1 to Proposition 2 below.

$C < \infty$ and a neighborhood W of $\mathcal{D}$ such that for every C^∞ function u supported in W,

$$\|u\| + \|\bar{L}u\| + \|Lu\| \le C \cdot (\|\mathcal{L}_s u\| + \|u\|_{-1} + \|Qu\|_1)\,.$$

The key conclusions are that there is no loss of derivatives in estimating u in terms of $\mathcal{L}_s u$, and that this holds for a sequence of values of s tending to $+\infty$. The assumption that $u \in C^\infty$ is essential. All hypotheses of §4 are satisfied by the family of operators $\mathcal{L}_s$ derived in §2 and §3. Proposition 2 thus implies the validity of (3.2), and hence of Proposition 1, which in turn implies our Theorem.

Our first preliminary estimate is a standard one valid for all $s \in \mathbf{R}$.

Lemma 1. *For each exponent s and each Q there exists $C < \infty$ such that*

$$\|\partial_x u\| \le C\|u\| + C\|\mathcal{L}_s u\| + C\|Qu\|_1$$

for every $u \in C_0^\infty(U)$.

Proof. For $(x,t) \in \mathcal{D}$, $\sigma_2(\mathcal{L}_s)(x,t,\xi,\tau) = 0$ if and only if $(x,t,\xi,\tau) \in \Gamma$. Therefore, the characteristic variety of $\mathcal{L}_s$ in T^*W is contained in an arbitrarily small conic neighborhood of Γ as $\delta \to 0$. Consequently, there exists an operator $\tilde{Q}$ of order zero such that firstly, T^*W is contained in the union of the two regions where $\tilde{Q}$ is elliptic and $\tau > 0$, and secondly, the symbol of $\tilde{Q}$ is supported in the union of the two regions where $\mathcal{L}_s$ is elliptic, and where Q is elliptic.

Since $\mathcal{L}_s$ is elliptic outside a small conic neighborhood of Γ, the H^2 norm of u is majorized away from Γ by $\|\mathcal{L}_s u\| + \|u\|_{-1}$, while in a conic neighborhood of Γ^- the H^1 norm of u is majorized by $\|Qu\|_1 + \|u\|_{-1}$.

Write $\langle f,\, g\rangle = \int_U f\bar{g}\,dx\,dt$. By Gårding's inequality and the fact that $i\mu \cdot i\tau \le 0$ in the support of the symbol of $\tilde{Q}$,

$$\begin{aligned}-\operatorname{Re}\left(\langle \bar{L}Lu,\, u\rangle\right) \ge c\|Lu\|^2 + c\|\bar{L}u\|^2 - C\|\bar{L}u\| \cdot \|u\| \\ - C\|Lu\| \cdot \|u\| - C\|u\|^2 - C\|\tilde{Q}u\|_1^2.\end{aligned}$$

The second condition imposed on $\tilde{Q}$ ensures that

$$\|\tilde{Q}u\|_1 \le C\|\mathcal{L}_s u\| + C\|Qu\|_1 + C\|u\|_{-1}.$$

Estimating $\langle(\mathcal{L}_s - \bar{L}L)u,\, u\rangle$ by Cauchy-Schwarz thus leads to

$$\|\bar{L}u\| + \|Lu\| \le C\|\mathcal{L}_s u\| + C\|u\| + C\|Qu\|_1.$$

But ∂_x may be expressed as a linear combination of $\bar{L}$ and of L modulo an operator of order 0. □

Lemma 2. *There exists $C < \infty$ such that for every $f \in C^1(\mathbf{R})$ and every $\varepsilon > 0$,*

$$\|f\|_{L^2[\varepsilon,2\varepsilon]} \le C\|f\|_{L^2[-2\varepsilon,-\varepsilon]} + C\varepsilon\|\partial_x f\|_{L^2(\mathbf{R})}.$$

Likewise

$$|f(0) - f(-\varepsilon)| \le C\varepsilon^{1/2}\|\partial_x f\|_{L^2}.$$

The conclusions are invariant under translation, and the lemma will be invoked in that more general form.

Proof. For each $x \in (\varepsilon, 2\varepsilon)$, $|f(x) - f(x-3\varepsilon)| \le \int_{-2\varepsilon}^{2\varepsilon} |\partial_x f(y)|\,dy$ and both conclusions follow from the triangle and Cauchy-Schwarz inequalities. □

To simplify notation define

$$\mathfrak{B} = \|\mathcal{L}_s u\| + \|u\|_{-1} + \|Qu\|_1.$$

Let $\delta > 0$ be a small constant to be chosen in §6, and assume u to be supported in

$$W \subset \{(x,t) : |t| < \delta,\ |x| < r + \delta\}.$$

Applying Lemma 2 to the function $x \mapsto u(x,t)$ for each t and applying Lemma 1 gives the following estimate, under the hypotheses of Lemma 1.

Lemma 3.

$$\|\partial_x u\| + \|u\| \le C\|u\|_{L^2(I\times(-\delta,\delta))} + C\mathfrak{B}.$$

5. Limiting operators and Mellin transform

Let a be a nonvanishing C^∞, real-valued function. For $\zeta \in \mathbf{C}$ define the ordinary differential operator

$$\begin{aligned} H_\zeta = {} & (\partial_x + i\zeta a(x))(\partial_x - i\zeta a(x)) \\ & + \beta_1(x)(\partial_x + i\zeta a(x)) + \beta_2(x)(\partial_x - i\zeta a(x)) + \beta_3(x), \end{aligned}$$

acting on functions of $x \in I$. Only the case of constant a will be needed in this paper, but the general case arises in another problem and hence merits discussion.

Definition 1. $\mathfrak{S}$ is defined to be the set of all $\zeta \in \mathbf{C}$ such that there exists a solution g of $H_\zeta g \equiv 0$ on I, satisfying $g(-r) = g(r) = 0$.

For any complex number w we write $\langle w\rangle = (1 + |w|^2)^{1/2}$.

Lemma 4. *$\mathfrak{S}$ is a discrete subset of $\mathbf{C}$, and for any compact subset K of $[0,\infty)$, the set of all $\zeta \in \mathfrak{S}$ having real part in K is finite. For each s such that $\mathfrak{S} \cap (s + i\mathbf{R}) = \emptyset$ there exists $C < \infty$ such that for all $\zeta \in s + i\mathbf{R}$, for all $f, \varphi, \psi \in C^\infty(I)$ satisfying $H_\zeta f = \varphi + \partial_x\psi$, one has*

$$\begin{aligned} & \|f\|_{L^2(I)} + \langle\zeta\rangle^{-1}\|\partial_x f\|_{L^2(I)} \\ & \le C\langle\zeta\rangle^{-1/2}\big(|f(-r)| + |f(r)|\big) + C\langle\zeta\rangle^{-2}\|\varphi\|_{L^2(I)} + C\langle\zeta\rangle^{-1}\|\psi\|_{L^2(I)}. \end{aligned} \tag{5.1}$$

Proof. Throughout this proof, all norms without subscripts denote L^2 norms. The selfadjoint part of H_ζ, applied to f, equals $(\partial_x - \gamma a(x))(\partial_x + \gamma a(x))f$, modulo $O(\langle\gamma\rangle\|f\| + \|\partial_x f\|)$, in the $L^2(I)$ norm. Therefore for s in any fixed compact subset of $\mathbf{R}$ and any $\zeta = s + i\gamma \in s + i\mathbf{R}$, for any f vanishing at both endpoints of I,

$$-\operatorname{Re}\langle H_\zeta f,\, f\rangle \ge \|\partial_x f\|^2 + \gamma^2\int_I |f|^2|a|^2 - O(\langle\gamma\rangle\|f\|^2 + \|f\|\cdot\|\partial_x f\|).$$

The coefficient a vanishes nowhere, while

$$|\langle H_\zeta f,\, f\rangle| = |\langle\varphi + \partial_x\psi,\, f\rangle| \le \|f\|\cdot\|\varphi\| + \|\partial_x f\|\cdot\|\psi\|.$$

Combining the last two inequalities and invoking the Cauchy-Schwarz inequality and small constant – large constant trick, one obtains

$$\gamma^2\|f\| + |\gamma|\cdot\|\partial_x f\| \le C\|\varphi\| + C|\gamma|\cdot\|\psi\| \tag{5.2}$$

for all sufficiently large $|\gamma|$, under the additional hypothesis that f vanishes at both endpoints of I.

There exists a unique solution ϕ_ζ of $H_\zeta\phi_\zeta = 0$ on I, satisfying $\phi_\zeta(-r) = 0$, $\partial_x\phi_\zeta(-r) = 1$. Then $\phi_\zeta(r)$ is an entire holomorphic function of ζ, and $\zeta \in \mathfrak{S} \Leftrightarrow$

$\phi_\zeta(r) = 0$. We have seen that $\zeta \notin \mathfrak{S}$ provided that the imaginary part of ζ is sufficiently large, when the real part stays in a bounded set. Thus $\phi_\zeta(r)$ is nonconstant, so has discrete zeros.[7]

To prove (5.1) let $\zeta = s + i\gamma$ and $f \in C^2(I)$ be given, and decompose $f = g + h$ where $H_\zeta g \equiv 0$ and h vanishes at the endpoints of I. The hypothesis $\mathfrak{S} \cap (s+i\mathbf{R}) = \emptyset$ means that the Dirichlet nullspace of H_ζ is $\{0\}$, so by elementary reasoning we conclude that for each γ there exists $C < \infty$ such that $\|h\| + \|\partial_x h\| \le C\|\varphi\| + \|\psi\|$, since $H_\zeta h = H_\zeta f = \varphi + \partial_x \psi$. Moreover, since H_ζ depends continuously on ζ, C may be taken to be independent of ζ in any compact subset of $\mathbf{C}\backslash\mathfrak{S}$. When $|\gamma|$ is sufficiently large, on the other hand, (5.2) implies $\|h\| + \langle\zeta\rangle^{-1}\|\partial_x h\| \le C\langle\zeta\rangle^{-2}\|\varphi\| + C\langle\zeta\rangle^{-1}\|\psi\|$. Thus the component h of f satisfies (5.1).

We have $H_\zeta g = 0$, so that clearly $\|g\|$ and $\|\partial_x g\|$ are majorized by $C|f(r)| + C|f(-r)|$, uniformly for ζ in any compact set disjoint from $\mathfrak{S}$. Assuming henceforth that $|\gamma|$ is large, the equation gives the inequality

$$\|\partial_x^2 g\| \le C\gamma^2\|g\| + C|\gamma| \cdot \|\partial_x g\|.$$

Integrating by parts as in the proof of (5.2) yields

$$\|\partial_x g\|^2 + \gamma^2\|g\|^2 \le C|g(-r)\partial_x g(-r)| + C|g(r)\partial_x g(r)|. \tag{5.3}$$

To control the right-hand side we use the bound

$$|\partial_x g(-r)| + |\partial_x g(r)| \le C|\gamma|^{1/2}\|\partial_x g\| + C|\gamma|^{-1/2}\|\partial_x^2 g\|.$$

Indeed, setting $v = \partial_x g$, for any $r' \in [r - |\gamma|^{-1}, r]$

$$|v(r) - v(r')| \le C\int_{r'}^{r} |\partial_x v| \le C|\gamma|^{-1/2}\|\partial_x v\|_{L^2}.$$

Then

$$|v(r)| \le |\gamma| \int_{r-|\gamma|^{-1}}^{r} |v(r) - v(r')|\, dr' + |\gamma| \int_{r-|\gamma|^{-1}}^{r} |v(r')|\, dr'$$

and the desired bound follows by Cauchy-Schwarz.

Putting this into (5.3), introducing a parameter $\lambda \in \mathbf{R}^+$ and applying Cauchy-Schwarz yields

$$\begin{aligned}
\gamma^2&\|g\|^2 + \|\partial_x g\|^2 \\
&\le C\lambda|g(-r)|^2 + C\lambda|g(r)|^2 + C\lambda^{-1}|\gamma| \cdot \|\partial_x g\|^2 + C\lambda^{-1}|\gamma|^{-1}\|\partial_x^2 g\|^2 \\
&\le C\lambda|g(-r)|^2 + C\lambda|g(r)|^2 + C\lambda^{-1}|\gamma| \cdot \|\partial_x g\|^2 \\
&\quad + C\lambda^{-1}|\gamma|^{-1}\big(\gamma^4\|g\|^2 + \gamma^2\|\partial_x g\|^2\big) \\
&\le C\lambda|g(-r)|^2 + C\lambda|g(r)|^2 + C\lambda^{-1}|\gamma| \cdot \|\partial_x g\|^2 + C\lambda^{-1}|\gamma|^3\|g\|^2.
\end{aligned}$$

Choose λ to be a large constant times $|\gamma|$. Then the last two terms on the right-hand side may be absorbed into the left, leaving

$$\gamma^2\|g\|^2 + \|\partial_x g\|^2 \le C|\gamma| \cdot |g(-r)|^2 + C|\gamma| \cdot |g(r)|^2.$$

Since $g = f$ at the endpoints of I, this is the desired inequality for g. Adding it to that for h concludes the proof. □

[7] An alternative method of proof would be to combine (5.2) with general results from the perturbation theory of linear operators [Ka], utilizing again the holomorphic dependence of H_ζ on ζ.

Definition 2.

$$S = \{s \in [0,\infty) : \text{ there exists } \gamma \in \mathbf{R} \text{ such that } s - \tfrac{1}{2} + i\gamma \in \mathfrak{S}\}.$$

Lemma 4 guarantees that S is discrete.

Specialize now to the case where $a(x) \equiv a$, the real constant in (4.4). Define

$$\begin{aligned}\mathbb{L}_s =&(\partial_x + ia(t\partial_t + s)) \circ (\partial_x - ia(t\partial_t + s))\\ &+ \beta_1(x)(\partial_x + ia(t\partial_t + s)) + \beta_2(x)(\partial_x - ia(t\partial_t + s)) + \beta_3(x).\end{aligned}$$

Expanding the last term in the expression $\mathbb{L}_s = \mathcal{L}_s + (\mathbb{L}_s - \mathcal{L}_s)$ gives

$$\mathbb{L}_s u = \Phi + \partial_x \Psi,$$

where

$$\begin{aligned}\Phi &= \mathcal{L}_s u + (t\partial_t)^2 Au + t\partial_t Au + Au,\\ \Psi &= t\partial_t Au + Au.\end{aligned} \tag{5.4}$$

To reach (5.4) we may for instance express $t\partial_t \circ O(t^2)\partial_t$ as $(t\partial_t)^2 A + t\partial_t A + A$, since multiplication by t is an operator of the type A. Likewise $[A, t\partial_t] = t[A, \partial_t] + [A, t]\partial_t$ is an operator of type A, because $\sigma_{-1}([A, t]) = c\partial_\tau \sigma_0(A)$ vanishes identically for $(x, t) \in \mathcal{D}$ since $\sigma_0(A)$ itself vanishes there.

The partial Mellin transform of f with respect to the t variable is defined to be

$$\hat{f}(x,\gamma) = \int_0^\infty f(x,t) t^{-i\gamma}\, t^{-1} dt,$$

provided that the integral converges. If $f(x,\cdot) \in C^\infty[0,\infty)$ has bounded support for each x, then the integral defining $\hat{f}(x,\gamma)$ converges absolutely whenever γ has strictly positive imaginary part, and $\hat{f}(x,\gamma)$ extends to a meromorphic function of $\gamma \in \mathbf{C}$, whose only possible poles are at $\gamma = 0, -i, -2i, \ldots$. Clearly

$$(t\partial_t f)\hat{\ }(x,\gamma) = i\gamma \hat{f}(x,\gamma)$$

for all such f. Consequently

$$(\mathbb{L}_s u)\hat{\ }(x,\gamma) = H_{s+i\gamma}\hat{u}(x,\gamma) \qquad \text{for all } \gamma \in \mathbf{C}\backslash\{0, -i, -2i, \ldots\}.$$

The Mellin inversion and Plancherel formulas read

$$f(x,t) = c\int_{\mathbf{R}} \hat{f}(x,\gamma)\, t^{i\gamma}\, d\gamma, \qquad \int_0^\infty |f(x,t)|^2\, t^{-1} dt = c' \int_{\mathbf{R}} |\hat{f}(x,\gamma)|^2\, d\gamma.$$

It follows directly from the definitions that $(t^{1/2} f)\hat{\ }(x,\gamma) = \hat{f}(x, \gamma + \frac{i}{2})$ for all $\gamma \in \mathbf{R}$. Thus the Plancherel identity may be rewritten as

$$\int_0^\infty |f(x,t)|^2\, dt = \int_0^\infty |t^{1/2} f(x,t)|^2\, t^{-1} dt = c' \int_{\mathbf{R}} |\hat{f}(x, \gamma + \tfrac{i}{2})|^2\, d\gamma.$$

6. Proof of the main estimate

We may now estimate u in terms of $\mathcal{L}_s u$. To begin,

$$\iint_{I\times[0,\delta)} |u(x,t)|^2\, dx\, dt = c' \iint_{I\times\mathbf{R}} |\hat{u}(x, \gamma + \tfrac{i}{2})|^2\, d\gamma\, dx.$$

Assume that $s \notin S$, and write $\zeta = s - \frac{1}{2} + i\gamma$. With Φ, Ψ defined as in (5.4), $H_\zeta \hat{u}(x, \gamma + \frac{i}{2}) = \hat{\Phi}(x, \gamma + \frac{i}{2}) + \partial_x \hat{\Psi}(x, \gamma + \frac{i}{2})$. Applying Lemma 4 on I yields for each $\gamma \in \mathbf{R}$

(6.1)
$$\int_I \hat{u}(x, \gamma + \tfrac{i}{2})\, dx \le C \int_I |\hat{\Phi}(x, \gamma + \tfrac{i}{2})|^2 \langle\gamma\rangle^{-4}\, dx + C \int_I |\hat{\Psi}(x, \gamma + \tfrac{i}{2})|^2 \langle\gamma\rangle^{-2}\, dx$$
$$+ C|\hat{u}(-r, \gamma + \tfrac{i}{2})|^2 + C|\hat{u}(r, \gamma + \tfrac{i}{2})|^2.$$

Lemma 5. *Assume $W \subset \{(x,t) : |t| < \delta,\ |x| < r + \delta\}$. Then there exists $C < \infty$ such that*

$$\iint_{I\times\mathbf{R}} |\hat{\Phi}(x, \gamma + \tfrac{i}{2})|^2 \langle\gamma\rangle^{-4}\, d\gamma\, dx \le C\delta^2 \|u\|^2 + C\mathfrak{B}^2$$

and

$$\iint_{I\times\mathbf{R}} |\hat{\Psi}(x, \gamma + \tfrac{i}{2})|^2 \langle\gamma\rangle^{-2}\, d\gamma\, dx \le C\delta^2 \|\partial_x u\|^2 + C\mathfrak{B}^2.$$

Granting the lemma, we conclude from (6.1) that

$$\|u\|^2_{L^2(I\times[0,\delta))} \le C\delta^2\|u\|^2 + C\delta^2\|\partial_x u\|^2 + C\mathfrak{B}^2$$
$$+ C\int_{\mathbf{R}} |u(r,t)|^2\, dt + C\int_{\mathbf{R}} |u(-r,t)|^2\, dt.$$

But by Lemma 2 and the assumption that $u(x,t) \equiv 0$ for $|x| > r + \delta$, this last term is dominated by $C\delta\|\partial_x u\|^2$. Thus

$$\|u\|^2_{L^2(I\times[0,\delta))} \le C\delta^2\|u\|^2 + C\delta\|\partial_x u\|^2 + C\mathfrak{B}^2.$$

All the same reasoning applies on the region $I \times (-\delta, 0]$, after the change of variables $t \mapsto -t$. Thus

$$\|u\|_{L^2(I\times(-\delta,\delta))} \le C\delta^{1/2}(\|u\| + \|\partial_x u\|) + C\mathfrak{B}.$$

Combining this with Lemma 3 gives

$$\|u\| + \|\partial_x u\| \le C\delta^{1/2}(\|u\| + \|\partial_x u\|) + C\mathfrak{B},$$

so choosing δ to be sufficiently small gives $\|u\| \le C\mathfrak{B}$, concluding the proof. □

Proof of Lemma 5. The principal term in the double integral of the lemma for Φ is of course the contribution of $\mathcal{L}_s u$:

$$\iint_{I\times\mathbf{R}} \left|(\mathcal{L}_s u)\widehat{\ }(x, \gamma + \tfrac{i}{2})\right|^2 \langle\gamma\rangle^{-4}\, d\gamma\, dx$$
$$\le \iint_{I\times\mathbf{R}} \left|(\mathcal{L}_s u)\widehat{\ }(x, \gamma + \tfrac{i}{2})\right|^2\, d\gamma\, dx \le C\|\mathcal{L}_s u\|^2,$$

as desired.

Any operator A of order ≤ 0 satisfying $\sigma_0(A)(x,t,\xi,\tau)\equiv 0$ for $(x,t)\in\mathcal{D}$ satisfies

$$\|Au\|^2 \le C\delta^2\|u\|^2 + C\|u\|_{-1}^2$$

for all u supported in W, as $\delta\to 0$. A typical term of Φ resulting from $\mathbb{L}_s u - \mathcal{L}_s u$ is $(t\partial_t)^2 Au$. Its contribution to the first double integral in Lemma 5 is

$$\begin{aligned}\iint_{I\times\mathbf{R}} \left|((t\partial_t)^2Au)\widehat{\ }(x,\gamma+\tfrac{i}{2})\right|^2 \langle\gamma\rangle^{-4}\,d\gamma\,dx &\le C\iint_{I\times\mathbf{R}} \left|(Au)\widehat{\ }(x,\gamma+\tfrac{i}{2})\right|^2\,d\gamma\,dx\\ &= C\iint_{I\times[0,\delta)} |Au(x,t)|^2\,dx\,dt\\ &\le C\delta^2\|u\|^2 + C\|u\|_{-1}^2.\end{aligned}$$

A typical constituent of Ψ is the term $t\partial_t Au$. Its contribution is dominated by

$$C\iint_{I\times\mathbf{R}} \left|(t\partial_t Au)\widehat{\ }(x,\gamma+\tfrac{i}{2})\right|^2 \langle\gamma\rangle^{-2}\,d\gamma\,dx \le C\iint_{I\times\mathbf{R}} \left|(Au)\widehat{\ }(x,\gamma+\tfrac{i}{2})\right|^2\,d\gamma\,dx$$

and the remainder of the calculation is as above. □

Comments. The author can advance no reason why the method of reduction to the boundary should be essential to this analysis. Working directly on $\overline{\mathcal{W}}$ might well result in a shorter proof. On the other hand, the analysis in §§4–6 applies with minor modification to a broader class of equations unconnected with the $\bar\partial$–Neumann problem.

Some refinements of this analysis and related observations will appear in [Ch2].

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