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A GENERALIZATION OF BOURGAIN'S CIRCULAR MAXIMAL THEOREM

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1. Introduction and statement of results

For $0 < \delta < \frac{1}{2}$, $1 < r < 2$, and $f \in \mathcal{S}$ define

$$\begin{aligned} C(x,r) &= \{y \in \mathbb{R}^2 \ : \ |x-y| = r\}, \\ C_\delta(x,r) &= \{y \in \mathbb{R}^2 \ : \ r(1-\delta) < |x-y| < r(1+\delta)\}, \\ \mathcal{M}_\delta f(x) &= \sup_{1<r<2} \frac{1}{|C_\delta(x,r)|} \left| \int_{C_\delta(x,r)} f(y)\, dy \right|, \\ \mathcal{M} f &= \sup_{1<r<2} |d\sigma_r * f|, \end{aligned}$$

where $d\sigma_r$ is the normalized surface measure on $r\,S^1$. It is easy to see that $\mathcal{M}$ is not bounded on L^2 (see Example 1.1 below). A well-known result of Bourgain [1] asserts that $\mathcal{M}$ is bounded on L^p for $2 < p \le \infty$. We will consider the question of boundedness of $\mathcal{M}$ and $\mathcal{M}_\delta$ from L^p to L^q. Unless stated to the contrary, we will be dealing only with functions defined on $\mathbb{R}^2$.

Absolute constants will be denoted by C, and the notation $\simeq$ will mean $=$ up to a constant.

Example 1.1. 1. Let $f = \chi_{C_\delta(0,1)}$. Then $\mathcal{M}_\delta f(x) \simeq 1$ on $|x| < \delta$. Hence $||f||_p \simeq \delta^{\frac{1}{p}}$ and $||\mathcal{M}_\delta f||_q \ge C^{-1}\,\delta^{\frac{2}{q}}$.

2. Let $f = \chi_R$, where R is the rectangle centered at 0 with dimensions δ times $\delta^{\frac{1}{2}}$. Then $\mathcal{M}_\delta f(x) \simeq \delta^{\frac{1}{2}}$ provided $|x_1| \simeq 1$ and $|x_2| < \delta^{\frac{1}{2}}$. Hence $||f||_p \simeq \delta^{\frac{3}{2p}}$ and $||\mathcal{M}_\delta f||_q \simeq \delta^{\frac{1}{2}(1+\frac{1}{q})}$.

3. Let $f = \chi_{B(0,\delta)}$. Then $\mathcal{M}_\delta f(x) \simeq \delta$ for $|x| \simeq 1$ and thus $||f||_p \simeq \delta^{\frac{2}{p}}$, $||\mathcal{M}_\delta f||_q \simeq \delta$.

4. Let $f(x) = (|1-|x|| + \delta)^{-\frac{1}{2}} \chi_{B(0,2)\setminus B(0,1)}(x)$. Then

$$\mathcal{M}_\delta f(x) \ge C^{-1} \left| \log \frac{\delta}{|x|} \right| \, |x|^{-\frac{1}{2}} \ \text{ if } \ 2\delta \le |x| < 1. \tag{1.1}$$

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To see this write f as

$$f \simeq \delta^{-\frac{1}{2}} \sum_{1 \le 2^j \le \delta^{-1}} 2^{-j/2} \chi_{C_{2^j\delta}(0,1+(2^j-1)\delta)}. \tag{1.2}$$

Taking the average of f over the annulus $C_\delta(x, 1+|x|)$ and considering the contribution of each dyadic shell in (1.2) separately yields (1.1). Hence $||f||_2 \simeq |\log\delta|^{\frac{1}{2}}$ and $||\mathcal{M}_\delta f||_2 \ge C^{-1}\,|\log\delta|$.

In view of these examples one might make the following conjecture (see Figure 1).

Conjecture 1.2. *For any* $f \in L^1 \cap L^\infty(\mathbb{R}^2)$

$$||\mathcal{M}f||_q \le C\,||f||_p \quad \textit{in region I}, \tag{1.3}$$

$$||\mathcal{M}_\delta f||_q \le C\,\delta^{\frac{2}{q}-\frac{1}{p}}\,||f||_p \quad \textit{in region II}, \tag{1.4}$$

$$||\mathcal{M}_\delta f||_q \le C\,\delta^{\frac{1}{2}(1+\frac{1}{q}-\frac{3}{p})}\,||f||_p \quad \textit{in region III}, \tag{1.5}$$

$$||\mathcal{M}_\delta f||_q \le C\,\delta^{1-\frac{2}{p}}\,||f||_p \quad \textit{in region IV}. \tag{1.6}$$

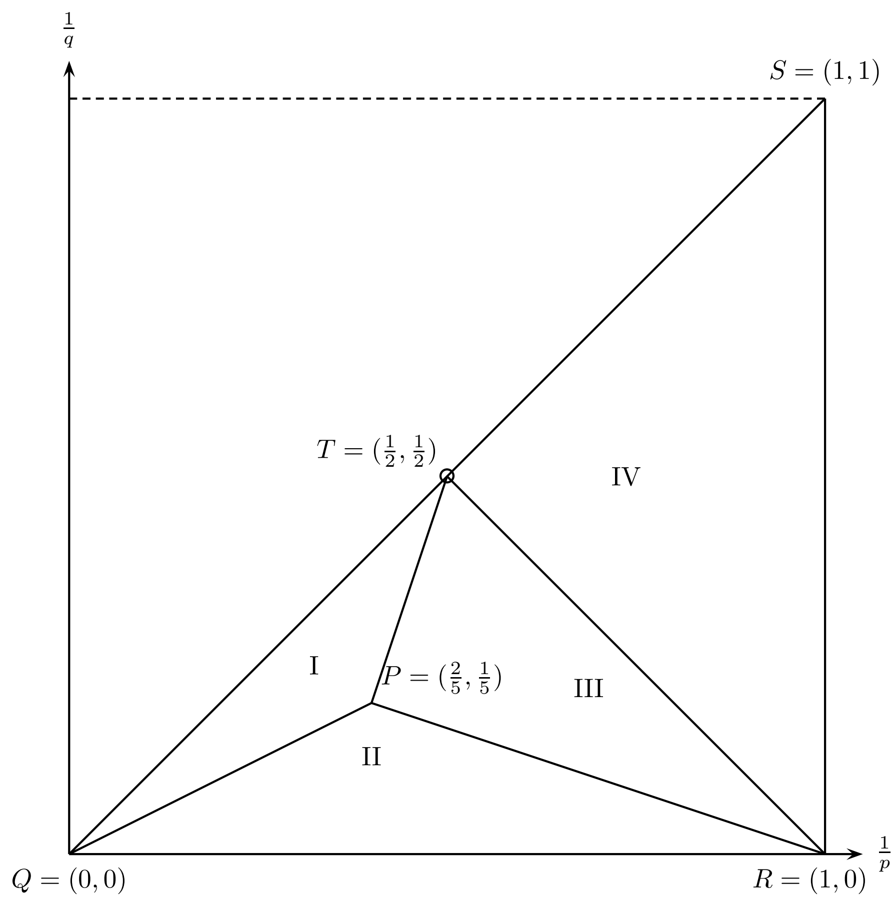

FIGURE 1. Regions of boundedness in Theorem 1.3

Regions I, III, and IV do not contain the point $T = (\frac{1}{2}, \frac{1}{2})$, where we have the well-known, optimal inequality (see Bourgain [1] and [2] and Example 1.1 above)

$$||\mathcal{M}_\delta f||_2 \le C\,|\log\delta|^{\frac{1}{2}}||f||_2. \tag{1.7}$$

Otherwise the boundaries are part of the regions. We will prove the following theorem (by C_ϵ we will always mean a constant depending only on ϵ).

Theorem 1.3. *For any $f \in L^1 \cap L^\infty(\mathbb{R}^2)$ and any $\epsilon > 0$,*

$$||\mathcal{M}f||_q \le C\,||f||_p \quad \text{in region } I \setminus (QP \cup PT), \tag{1.8}$$

$$||\mathcal{M}_\delta f||_q \le C_\epsilon\,\delta^{\frac{2}{q}-\frac{1}{p}-\epsilon}\,||f||_p \quad \text{in region } II, \tag{1.9}$$

$$||\mathcal{M}_\delta f||_q \le C_\epsilon\,\delta^{\frac{1}{2}(1+\frac{1}{q}-\frac{3}{p})-\epsilon}\,||f||_p \quad \text{in region } III, \tag{1.10}$$

$$||\mathcal{M}_\delta f||_q \le C\,\delta^{1-\frac{2}{p}}\,||f||_p \quad \text{in region } IV. \tag{1.11}$$

Remark 1.4. • In certain cases the $\delta^{-\epsilon}$–term can be replaced by a suitable power of $|\log\delta|$, but we do not elaborate on this.

• It can be shown by modifying the proof of Theorem 1.3 that the optimal estimates (i.e., (1.9) with $\epsilon = 0$) hold in the region $II \cap \{\frac{1}{q} < \frac{1}{6}\} \setminus (QP \cup PR)$. In [7] this somewhat technical argument is carried out in detail.

• The most interesting statement in Theorem 1.3 is probably the estimate

$$\|\mathcal{M}_\delta f\|_5 \le C_\epsilon \delta^{-\epsilon}\|f\|_{5/2} \tag{1.12}$$

for all $f \in L^1 \cap L^\infty(\mathbb{R}^2)$ and any $\epsilon > 0$. It is easy to see that (1.12) would follow from Sogge's sharp local smoothing conjecture [8]. Let

$$A_\alpha f(t,x) = \int_{\mathbb{R}^2} e^{2\pi i(x\cdot\xi + t|\xi|)}\,\frac{\hat{f}(\xi)}{(1+|\xi|)^\alpha}\,d\xi.$$

Then Sogge's conjecture says that

$$\|A_\alpha f\|_{L^4([1,2]\times\mathbb{R}^2)} \le C_\alpha\|f\|_{L^4(\mathbb{R}^2)} \quad \text{for} \quad \alpha > 0. \tag{1.13}$$

Interpolating (1.13) with the easy estimate

$$\|A_\alpha f\|_{L^\infty([1,2]\times\mathbb{R}^2)} \le C_\alpha\|f\|_{L^1(\mathbb{R}^2)} \quad \text{for} \quad \alpha \ge 3/2$$

shows that one might expect that

$$\|A_\alpha f\|_{L^q([1,2]\times\mathbb{R}^2)} \le C_{\alpha,q}\|f\|_{L^p(\mathbb{R}^2)} \quad \text{for} \quad q = 3p' \quad \text{and} \quad \alpha > 6(1/4 - 1/q). \tag{1.14}$$

(1.12) would follow from the special case $p = 5/2$, $q = 5$ of (1.14) via the usual Sobolev embedding argument in t, cf. [6]. Note that $q = 3p'$ is the same relation as in the Carleson–Sjölin theorem [3]. Moreover, it is possible to prove (1.14) for $q \ge 5$ by Carleson–Sjölin type arguments. This is shown in a forthcoming paper of C. Sogge and the author.

The proof of Theorem 1.3 is based on a combinatorial argument from Kolasa and Wolff [4] combined with a localized version of the L^2 estimate (1.7). For the δ–free bounds (1.8) we interpolate the $(5/2, 5)$ inequality with an estimate obtained from the local smoothing theorem in Mockenhaupt, Seeger, and Sogge [6].

This paper is organized as follows. In Section 2 we introduce the notion of multiplicity μ of a family of annuli. It is shown that certain estimates for μ are equivalent to $L^p \to L^q$ bounds on $\mathcal{M}_\delta$. Section 3 contains the localized L^2 inequality and a bound on the multiplicity is derived from it. In Section 4 we establish the main result of this paper, i.e., the restricted weak type $(5/2, 5)$ estimate. This is accomplished by combining the combinatorial argument from [4] (which is based on Marstrand's three circle lemma [5]) with the localized inequality from Section 3. Theorem 1.3 then follows by various interpolation arguments in Section 5.

Finally, we would like to mention some consequences of Theorem 1.3. Detailed arguments can be found in [7]. Firstly, it is well-known that one can pass from suitable $L^p \to L^q$ bounds on $\mathcal{M}_\delta$ to corresponding bounds on the global maximal averages $\sup_{0<r<\infty} r^a |d\sigma_r * f|$ by Littlewood-Paley theory, see [1]. The value of a is determined by scaling. Instead of the powers $\delta^{-\alpha}$ in Theorem 1.3 one obtains Sobolev norms on the right-hand side with α derivatives. Secondly, it is easy to see that these estimates on the global circular maximal operator imply estimates for maximal functions associated with more regular averages, e.g., the two–dimensional wave equation. The type of averages considered in [7] are those from Stein [10], and the method is essentially Stein's interpolation theorem.

2. The combinatorial method

Fix $E \subset [0,1]^2$ and $0 < \lambda \le 1$. Let $\{x_j\}_{j=1}^M$ be a maximally δ-separated set in

$$F = \{x \in \mathbb{R}^2 \ : \ (\mathcal{M}_\delta \chi_E)(x) > \lambda\}$$

and let $r_j \in (1,2)$ be chosen so that

$$|C_\delta(x_j, r_j) \cap E| > \lambda |C_\delta(x_j, r_j)|$$

for $j = 1, 2, \ldots, M$. Henceforth we will write C_j^* instead of $C_\delta(x_j, r_j) \cap E$ and C_j instead of $C_\delta(x_j, r_j)$. We introduce the multiplicity function

$$\Phi = \sum_{j=1}^M \chi_{C_j^*}.$$

Following [4] we define μ to be the smallest integer for which there exist at least $M/2$ values of j such that

$$|\{C_j^* \ : \ \Phi \le \mu\}| \ge \frac{\lambda}{2} |C_j|.$$

Clearly, we can then also find at least $M/2$ values of j for which

$$|\{C_j^* \ : \ \Phi \ge \mu\}| \ge \frac{\lambda}{2} |C_j|. \tag{2.1}$$

The combinatorial method attempts to bound μ from above, typically in terms of λ, M, and δ. Since

$$\mu |E| \ge \int_{\{E \, : \, \Phi \le \mu\}} \Phi = \sum_j |\{C_j^* \ : \ \Phi \le \mu\}| \ge C^{-1} \lambda M \delta, \tag{2.2}$$

this will imply a lower bound on $|E|$. The following lemma characterizes the estimates of μ required for $L^p \to L^q$ boundedness of $\mathcal{M}_\delta$.

Lemma 2.1. *Let $0 \le \alpha$ and $\beta < 1$. Then $\mu \le A \lambda^{-\alpha} M^\beta$ implies*

$$||\mathcal{M}_\delta f||_{q,\infty} \le C \ A^{\frac{1}{p}} \delta^{-\gamma} ||f||_{p,1} \quad \textit{for all } f \in L^1 \cap L^\infty,$$

where $p = \alpha + 1$, $q = p(1-\beta)^{-1}$ *and* $\gamma = \frac{1}{p} - \frac{2}{q}$. *We also have the following converse. Suppose that for some fixed* $\rho > 0$, $1 \le q \le \infty$, $1 \le p < \infty$, *and all* $f \in L^1 \cap L^\infty$

$$||\mathcal{M}_\delta f||_q \le C\,\delta^{-\rho}||f||_p. \tag{2.3}$$

Then

$$\mu \le C\,\lambda^{1-p} M^{1-\frac{p}{q}} \delta^{-p\rho+1-2\frac{p}{q}}. \tag{2.4}$$

Proof. For the first statement we need to show

$$|\{\mathcal{M}_\delta \chi_E > \lambda\}|^{\frac{1}{q}} \le C\ \delta^{-\gamma} A^{\frac{1}{p}} \lambda^{-1} |E|^{\frac{1}{p}}. \tag{2.5}$$

Since $\{x_j\}_{j=1}^M$ was chosen to be a maximally δ-separated sequence in $\{\mathcal{M}_\delta \chi_E > \lambda\}$, it follows that

$$|\{\mathcal{M}_\delta \chi_E > \lambda\}| \le C\,M\delta^2.$$

In view of (2.2), i.e., $|E| \ge C^{-1}\,\mu^{-1}\lambda M \delta$, and our assumption on μ we conclude that the right-hand side of (2.5) is

$$\ge C^{-1}\ \delta^{-\gamma} A^{\frac{1}{p}} \lambda^{-1} (A^{-1}\lambda^{1+\alpha} M^{1-\beta}\delta)^{\frac{1}{p}} \simeq (M\delta^2)^{\frac{1}{q}}.$$

To prove the second statement, we distinguish two cases. First assume that

$$|E_1| = |\{E\ :\ \Phi \ge \mu\}| \le \mu^{-1}\lambda M\delta.$$

Applying hypothesis (2.3) to the function $f = \chi_{E_1}$ and using (2.1) we obtain

$$\lambda(\delta^2 M)^{\frac{1}{q}} \le C\,\delta^{-\rho}|E_1|^{\frac{1}{p}} \le C\,\delta^{-\rho}(\mu^{-1}\lambda M\delta)^{\frac{1}{p}},$$

which implies the desired inequality (2.4).

In the other case, i.e., $|E_1| \ge \mu^{-1}\lambda M\delta$, we use duality. Note that the dual statement to (2.3) is

$$\|\sum_j a_j \chi_{C_\delta(y_j,\rho_j)}\|_{p'} \le C\,\delta^{-1-\rho} \tag{2.6}$$

for all δ-separated $\{y_j\}$ in $[0,1]^2$, all $\{a_j\}$ which satisfy $\delta^2 \sum_j |a_j|^{q'} \le 1$, and all $\rho_j \in (1,2)$. Let $y_j = x_j$, $\rho_j = r_j$, and $a_j = (\delta^2 M)^{-\frac{1}{q'}}$ for $j = 1, 2, \ldots, M$. Then by (2.6)

$$\mu(\mu^{-1}\lambda M\delta)^{\frac{1}{p'}} \le \mu|E_1|^{\frac{1}{p'}} \le \|\Phi\|_{p'} \le C\,\delta^{-1-\rho}(\delta^2 M)^{\frac{1}{q'}},$$

which implies (2.4). □

At this point it might be instructive to consider those bounds on μ that correspond to the points P, R, S, T in Figure 1. By Lemma 2.1,

$$P: \quad p = 5/2,\ q = 5, \quad \mu \le C\,\lambda^{-\frac{3}{2}} M^{\frac{1}{2}}, \tag{2.7}$$

$$R: \quad p = 1,\ q = \infty, \quad \mu \le C\,M, \tag{2.8}$$

$$S: \quad p = 1,\ q = 1, \quad \mu \le C\,\delta^{-2}, \tag{2.9}$$

$$T: \quad p = 2,\ q = 2, \quad \mu \le C\,\lambda^{-1}\delta^{-1}. \tag{2.10}$$

Not surprisingly, inequalities (2.8) and (2.9) are trivial, whereas (2.10) follows (up to a $|\log\delta|^{\frac{1}{2}}$ factor) from (1.7). Our main goal will be to show (2.7) (the result below will involve a $|\log\delta|$ factor, though). In order to do this we will need an improved version of the L^2 statement, i.e., inequality (2.10).

3. The L^2 theory

Before formulating the result, we consider an example.

Example 3.1. Let $10\,\delta < 10\rho < r < \frac{1}{2}$ and define

$$E = \{x \in \mathbb{R}^2 \;:\; 1-\rho < |x| < 1\}$$

and $\lambda = \sqrt{\frac{\rho}{r}}$. It is easy to see that $F = \{\mathcal{M}_\delta \chi_E > \lambda\} \simeq B(0,r)$ and $M \simeq \frac{r^2}{\delta^2}$. To determine μ, note that Φ will be approximately constant on

$$E_1 = \{x \;:\; 1-\rho < |x| < 1-\rho/2\}.$$

Hence

$$\mu|E_1| \simeq \int_{E_1} \Phi \simeq \lambda M \delta.$$

Thus

$$\mu \simeq \delta^{-1}\rho^{-\frac{1}{2}} r^{\frac{3}{2}} = \lambda^{-1}\delta^{-1} r.$$

We will prove below that this improved version of (2.10) holds in general (up to a $|\log\delta|$ factor) with r replaced by the typical distance of two intersecting annuli (for a precise version of this see the discussion following Corollary 3.6). To this end we need a refined version of the L^2 inequality (1.7). First we recall a result from [2].

Lemma 3.2. *Let $K \in L^1(\mathbb{R}^d)$ assuming $\widehat{K}$ differentiable. Define for $j \in \mathbb{Z}$*

$$\alpha_j = \sup_{|\xi|\simeq 2^j} |\widehat{K}(\xi)|,$$
$$\beta_j = \sup_{|\xi|\simeq 2^j} |\langle \nabla \widehat{K}(\xi), \xi\rangle|.$$

Then for any fixed j and $f \in \mathcal{S}$ such that $\mathrm{supp}(\hat{f}) \subset \{\mathbb{R}^d \;:\; 2^{j-1} < |\xi| < 2^{j+1}\}$

$$\|\sup_{t\simeq 1} |f * K_t|\|_{L^2(\mathbb{R}^d)} \le C\,\alpha_j^{\frac{1}{2}}(\alpha_j+\beta_j)^{\frac{1}{2}}\|f\|_{L^2(\mathbb{R}^d)}.$$

By well-known decay properties of $\widehat{d\sigma_r}$ (see (3.3) and (3.4) below) Lemma 3.2 implies that

$$\|\mathcal{M}f\|_{L^2(\mathbb{R}^2)} \le C\,\|f\|_{L^2(\mathbb{R}^2)} \tag{3.1}$$

for any $f \in \mathcal{S}$ whose Fourier transform is supported in $\{\mathbb{R}^2 \colon 2^{j-1} < |\xi| < 2^{j+1}\}$ for some $j > 0$. The following proposition shows that this estimate can be improved if one restricts the maximal function to a small ball. We prove this fact by combining Bourgain's original argument with Lemma 3.4 below. Later we will exploit the equivalence of bounds on the multiplicity μ and $p \to q$ estimates for $\mathcal{M}_\delta$, as described in Lemma 2.1, to derive the improved bound on μ alluded to in Example 3.1 above.

Proposition 3.3. *There exists an absolute constant C_0 such that for any $j = 1,2,\ldots,$ all $f \in \mathcal{S}$ with* $\mathrm{supp}(\hat{f}) \subset \{\mathbb{R}^2 \colon 2^{j-1} < |\xi| < 2^{j+1}\}$, *and all $0 < t \le 1$, $x_0 \in \mathbb{R}^2$*

$$\|\mathcal{M}f\|_{L^2(B(x_0,t))} \le C_0\, t^{\frac{1}{2}}\|f\|_{L^2(\mathbb{R}^2)}. \tag{3.2}$$

Proof. We may assume that $x_0 = 0$. Choose cut-off functions $\psi \in C_0^\infty(\mathbb{R}^2)$ with $\psi = 1$ on $B(0,1)$, $\eta \in C_0^\infty(1/2, 4)$ so that $\eta = 1$ on $(1,2)$, and $\phi \in \mathcal{S}$ such that $\mathrm{supp}(\hat{\phi}) \subset \{1/4 < |\xi| < 4\}$ and $\hat{\phi} = 1$ on $\{1/2 < |\xi| < 2\}$. Fix j for the remainder of the proof. Define

$$A_r^j f(x) = \psi(t^{-1}x)\eta(r) \int_{\mathbb{R}^2} e^{2\pi i x\cdot\xi}\, \widehat{d\sigma}(r|\xi|)\hat{\phi}(2^{-j}\xi)\hat{f}(\xi)\, d\xi.$$

Let $\{r_\tau\}_\tau$ be a 2^{-j} net in $[1,2]$. Suppose $r_\tau \le r < r_{\tau+1}$. Then

$$|A_r^j f| \le 2^j \int_{r_\tau}^{r_{\tau+1}} |A_\rho^j f|\, d\rho + \int_{r_\tau}^{r_{\tau+1}} \left| \frac{d}{d\rho} A_\rho^j f \right| d\rho$$

and thus

$$\begin{aligned} \sup_{1<r<2} |A_r^j f|^2 &\le 2^j \int_1^2 |A_\rho^j f|^2\, d\rho + 2^{-j} \int_1^2 \left| \frac{d}{d\rho} A_\rho^j f \right|^2 d\rho \\ &= 2^j A + 2^{-j} B. \end{aligned}$$

It is well known that $\widehat{d\sigma}$ has the representation (see, e.g., [9], Theorem 1.2.1)

$$\widehat{d\sigma}(\xi) = \Re\{e^{2\pi i|\xi|}\omega(|\xi|)\} \tag{3.3}$$

with $\omega \in C^\infty(0,\infty)$ and

$$\left| \frac{d^k\, \omega(s)}{ds^k} \right| \le C\,(1+|s|)^{-\frac{1}{2}-k} \tag{3.4}$$

for all $k = 0, 1, 2, \ldots$. Using (3.3) it is easy to see that the integral of A can be written as

$$\int_{\mathbb{R}^2} A\, dx = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} K(\xi,\tilde{\xi})\hat{\phi}(2^{-j}\xi)\hat{\phi}(2^{-j}\tilde{\xi})\hat{f}(\xi)\overline{\hat{f}(\tilde{\xi})}\, d\xi\, d\tilde{\xi},$$

where

$$K(\xi,\tilde{\xi}) = t^2 \int_{-\infty}^{\infty}\int_{\mathbb{R}^2} e^{2\pi i(tx\cdot(\xi-\tilde{\xi})+r(|\xi|-|\tilde{\xi}|))}\psi^2(x)\eta^2(r)\omega(r|\xi|)\overline{\omega(r|\tilde{\xi}|)}\, dx\, dr.$$

Integrating by parts with respect to x and r in the previous expression and applying the decay estimates (3.4) shows that

$$|K(\xi,\tilde{\xi})| \le C\, 2^{-j} t^2 (1 + t|\xi - \tilde{\xi}|)^{-2}(1 + |\,|\xi| - |\tilde{\xi}|\,|)^{-2},$$

provided $|\xi| \simeq |\tilde{\xi}| \simeq 2^j$. Lemma 3.4 below and Schur's lemma yield

$$\int_{\mathbb{R}^2} A\, dx \le C\, 2^{-j} t \|f\|_2^2.$$

Carrying out the differentiation with respect to ρ in the term B above and applying (3.4) one obtains in a similar fashion

$$\int_{\mathbb{R}^2} B\, dx \le C\, 2^{j} t \|f\|_2^2,$$

and the proposition follows. □

The following lemma was used in the previous proof in order to provide the desired improvement in the L^2 estimate obtained by localizing to a small ball. Roughly speaking, the inequality below is true because the $|\,|\xi| - |\tilde{\xi}|\,|$ factor reduces the two-dimensional scaling in the integral to one dimension. We prove this fact by integrating over shells of different radii with center at $\tilde\xi$ and estimating the various pieces separately.

Lemma 3.4. *Let* $0 < t < 1$. *Then*

$$\sup_{\tilde\xi} \int_{\mathbb{R}^2} \Big(1 + t|\xi - \tilde\xi|\Big)^{-2} \Big(1 + \big|\,|\xi| - |\tilde\xi|\,\big|\Big)^{-2}\, d\xi \le C\, t^{-1}.$$

Proof. Fix a $\tilde\xi \in \mathbb{R}^2$. Then, on the one hand,

$$\begin{aligned}
&\int_{\{\xi\,:\,|\xi-\tilde\xi|\le|\tilde\xi|/2\}} \Big(1 + t|\xi - \tilde\xi|\Big)^{-2} \Big(1 + \big|\,|\xi| - |\tilde\xi|\,\big|\Big)^{-2}\, d\xi \\
&\quad\simeq \sum_{2^j \le |\tilde\xi|/2} (1 + t2^j)^{-2} \int_{\{\xi\,:\,|\xi-\tilde\xi|\simeq 2^j\}} (1 + |\,|\xi| - |\tilde\xi|\,|)^{-2}\, d\xi \\
&\quad\simeq \sum_{2^j \le |\tilde\xi|/2} (1 + t2^j)^{-2} 2^j |\tilde\xi|^{-1} \int_{|\tilde\xi|-2^j}^{|\tilde\xi|+2^j} (1 + |r - |\tilde\xi||)^{-2} r\, dr \\
&\quad\simeq \sum_{2^j \le |\tilde\xi|} (1 + t2^j)^{-2} 2^j \simeq \sum_{2^j t \le 1} 2^j + \sum_{t^{-1} \le 2^j \le |\tilde\xi|} t^{-2} 2^{-j} \\
&\quad\simeq t^{-1} + t^{-2} t \simeq t^{-1}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_{\{\xi\,:\,|\xi-\tilde\xi|\ge|\tilde\xi|/2\}} \Big(1 + t|\xi - \tilde\xi|\Big)^{-2} \Big(1 + \big|\,|\xi| - |\tilde\xi|\,\big|\Big)^{-2}\, d\xi \\
&\quad\simeq \int_{\{\xi\,:\,|\xi-\tilde\xi|\ge|\tilde\xi|/2\,,|\,|\xi|-|\tilde\xi|\,|\le|\tilde\xi|/2\}} \Big(1 + t|\tilde\xi|\Big)^{-2} \Big(1 + \big|\,|\xi| - |\tilde\xi|\,\big|\Big)^{-2}\, d\xi \\
&\qquad + \int_{\{\xi\,:\,|\,|\xi|-|\tilde\xi|\,|\ge|\tilde\xi|/2\}} \Big(1 + t|\xi - \tilde\xi|\Big)^{-2} \Big(1 + \big|\,|\xi| - |\tilde\xi|\,\big|\Big)^{-2}\, d\xi \\
&\quad= A + B.
\end{aligned} \tag{3.5}$$

The first term in (3.5) can be estimated as follows:

$$\begin{aligned}
A &\simeq \Big(1 + t|\tilde\xi|\Big)^{-2} \int_{|\tilde\xi|/2}^{3|\tilde\xi|/2} \Big(1 + |r - |\tilde\xi||\Big)^{-2} r\, dr \\
&\simeq \Big(1 + t|\tilde\xi|\Big)^{-2} |\tilde\xi| \le C\, t^{-1}.
\end{aligned}$$

For the second term compute

$$\begin{aligned}
B &\simeq \int_{\{\xi\,:\,\frac{3}{2}|\tilde\xi|\le|\xi|\}} (1 + t|\xi|)^{-2} (1 + |\xi|)^{-2}\, d\xi \\
&\qquad + \int_{\{\xi\,:\,\frac{1}{2}|\tilde\xi|\ge|\xi|\}} \Big(1 + t|\tilde\xi|\Big)^{-2} \Big(1 + |\tilde\xi|\Big)^{-2}\, d\xi \\
&\simeq \int_{\{\xi\,:\,t^{-1}\le|\xi|\}} (t|\xi|)^{-2} (1 + |\xi|)^{-2}\, d\xi + \int_{\{\xi\,:\,\frac{3}{2}|\tilde\xi|\le|\xi|\le t^{-1}\}} (1 + |\xi|)^{-2}\, d\xi \\
&\qquad + \Big(1 + t|\tilde\xi|\Big)^{-2} \Big(1 + |\tilde\xi|\Big)^{-2} |\tilde\xi|^2 \\
&\le C\,(1 + |\log t| + 1) \le C\, t^{-1},
\end{aligned}$$

and the lemma follows. ∎

Remark 3.5. The proof of Proposition 3.3 above shows that (3.2) is essentially equivalent to the following estimate for the two-dimensional wave equation. Let u solve

$$\Box\, u = 0, \;\; u(0) = f, \; u_t(0) = 0.$$

Then there exists an absolute constant C_0 such that

$$\int_0^1 \int_{B(x_0,r)} |u(x,t)|^2\, dx\, dt \le C_0\, r \|f\|^2_{L^2(\mathbb{R}^2)} \tag{3.6}$$

for all $0 < r \le 1$. It might be interesting to ask whether such an estimate can hold in L^p with $p \neq 2$. Interpolating (3.6) with Sogge's sharp local smoothing conjecture [8], i.e.,

$$\int_0^1 \int_{\mathbb{R}^2} |u(x,t)|^4\, dx\, dt \le C_\epsilon\, \|f\|^4_{L^4_\epsilon} \tag{3.7}$$

with $\epsilon > 0$, yields

$$\left(\int_0^1 \int_{B(x_0,r)} |u(x,t)|^p\, dx\, dt\right)^{\frac{1}{p}} \le C_\epsilon\; r^{\frac{2}{p}-\frac{1}{2}} \|f\|_{L^p_\epsilon} \tag{3.8}$$

for $2 \le p \le 4$, $x_0 \in \mathbb{R}^2$, $0 < r \le 1$, $\epsilon > 0$ and all $f \in \mathcal{S}$. Solving the wave equation above with initial condition f equal to a smooth version of $\chi_{C_\delta(0,r)}$ shows that the exponent $\frac{2}{p} - \frac{1}{2}$ is optimal. Moreover, as in the case of local smoothing, (3.8) cannot hold for $p \notin [2,4]$ or with $\epsilon = 0$ if $p > 2$.

It is standard to pass from f as in the statement of Proposition 3.3 to general $f \in L^2$. This is done in the following corollary.

Corollary 3.6. *There exists an absolute constant C_0 such that for any $f \in L^2(\mathbb{R}^2)$, $x_0 \in \mathbb{R}^2$, and $0 < \delta, t < 1$,*

$$\|\mathcal{M}_\delta f\|_{L^2(B(x_0,t))} \le C_0\, t^{\frac{1}{2}} |\log \delta|^{\frac{1}{2}} \|f\|_2. \tag{3.9}$$

The equivalent dual statement to (3.9) *is:*

$$\|\sum_j a_j \chi_{C_\delta(y_j,\rho_j)}\|_{L^2(\mathbb{R}^2)} \le C_0\, |\log \delta|^{\frac{1}{2}} \delta^{-1} t^{\frac{1}{2}} \tag{3.10}$$

for all δ-separated $\{y_j\}$ in $B(x_0,t)$, all $\{a_j\}$ for which $\delta^2 \sum_j |a_j|^2 \le 1$, and all $\rho_j \in (1,2)$.

Proof. Choose $\phi \in \mathcal{S}$ such that $\mathrm{supp}(\hat{\phi})$ is compact, $\hat{\phi} \ge 0$, $\phi \ge 0$, and $\hat{\phi} \ge 1$ on $B(0,1)$. Given $f \in L^2(\mathbb{R}^2)$, $f \ge 0$ let $f = \sum_{j=0}^\infty f_j$ be a Littlewood-Paley decomposition, i.e., $\mathrm{supp}(\hat{f}_0) \subset \{|\xi| < 2\}$ and $\mathrm{supp}(\hat{f}_j) \subset \{\mathbb{R}^2 \colon 2^{j-1} < |\xi| < 2^{j+1}\}$ for $j = 1, 2, \ldots$. Let $\chi_{\delta,r} = \delta\, d\sigma_r * \phi_\delta$, where we have used the notation $\phi_\delta(x) = \delta^{-2}\phi(\delta^{-1}x)$. Then clearly

$$\chi_{C_\delta(0,r)} \le C\, \chi_{\delta,r}.$$

If M denotes the usual Hardy-Littlewood maximal operator it is easy to see that

$$\begin{aligned}
\|\mathcal{M}_\delta f\|_{L^2(B(x_0,t))} &\le C\,\|Mf_0\|_{L^2(B(x_0,t))} \\
&\quad + \sum_{1<2^j\le C\,\delta^{-1}} \|\sup_{1<r<2} |\chi_{\delta,r} * f_j|\|_{L^2(B(x_0,t))} \\
&\le C\,t\,\|Mf_0\|_\infty + \sum_{1<2^j\le C\,\delta^{-1}} \|\mathcal{M} f_j\|_{L^2(B(x_0,t))} \\
&\le C\,t\,\|f_0\|_\infty + t^{\frac12} \sum_{1<2^j\le C\,\delta^{-1}} \|f_j\|_2 \qquad (3.11)\\
&\le C\,t\,\|f_0\|_2 + t^{\frac12}|\log\delta|^{\frac12}\left(\sum_{1<2^j\le C\,\delta^{-1}} \|f_j\|_2^2\right)^{\frac12} \\
&\le C\,t^{\frac12}|\log\delta|^{\frac12}\|f\|_2.
\end{aligned}$$

In line (3.11) we have used a special case of Bernstein's inequality, namely

$$\|f_0\|_\infty \le C\,\|f_0\|_2.$$

Finally, (3.10) is an immediate consequence of duality. □

In order to obtain information on μ from (3.9) we will determine the typical distance of the centers of two intersecting annuli in any collection of annuli. More precisely, we can specify the distance of the centers and the angle of intersection of those annuli that contribute most to the multiplicity function Φ. Following [4], we will accomplish this by applying the pigeon hole principle to our family of annuli satisfying (2.1). Define $\bar\lambda = |\log\delta|^{-2}\lambda/2$, $\bar\mu = |\log\delta|^{-2}\mu$, $\overline{M} = |\log\delta|^{-2}M/2$. Furthermore, for all $i,j \in \{1,2,\ldots,M\}$ we let (for the meaning of Δ see Lemma 4.2 below)

$$\begin{aligned}
\Delta_{i,j} &= \max(\delta, |\,|x_i-x_j| - |r_i-r_j|\,|), \\
S^j_{t,\epsilon} &= \{i\ :\ C_i\cap C_j \ne \varnothing\ ,\ t/2 \le |x_i-x_j| \le t\ ,\ \epsilon \le \Delta_{i,j} \le 2\epsilon\}, \qquad (3.12)\\
\Phi^j_{t,\epsilon} &= \sum_{i\in S^j_{t,\epsilon}} \chi_{C_i^*}
\end{aligned}$$

(recall that $C_i = C_\delta(x_i,r_i)$ and $C_i^* = E\cap C_i$). The pigeon hole principle asserts that there are numbers $t\in[\delta,1]$, $\epsilon\in[\delta,1]$ such that

$$|\{C_j^*\ :\ \Phi^j_{t,\epsilon} \ge \bar\mu\}| \ge \bar\lambda |C_j| \qquad (3.13)$$

for at least $\overline{M}$ values of j, say $1\le j\le \overline{M}$. Indeed, let j be one of the at least $M/2$ indices satisfying (2.1), i.e.,

$$|\{C_j^*\ :\ \Phi \ge \mu\}| \ge \frac{\lambda}{2}|C_j|.$$

Let $x\in C_j^*$ so that $\Phi(x)\ge\mu$. We conclude that for some choice of t and ϵ depending only on x and j we have

$$\Phi^j_{t,\epsilon}(x) \ge \bar\mu.$$

For if not, then (in the following sum t and ϵ are dyadic)

$$\Phi(x) = \sum_{t,\epsilon\in[\delta,1]} \Phi^j_{t,\epsilon}(x) < |\log\delta|^2\bar\mu = \mu,$$

contradicting the choice of x. Similarly, we see that (3.13) holds for any j as above and for some choice of t and ϵ depending only on j. Finally, applying the pigeon hole principle in j yields that there are t and ϵ such that (3.13) holds for at least $\overline{M}$ values of j. Otherwise, the number of j's satisfying (2.1) would have to be strictly less than $|\log \delta|^2 \overline{M} = M/2$. Henceforth we will fix ϵ and t to be those numbers.

By essentially the same argument as in the second part of Lemma 2.1 we can now establish the refined version of (2.10).

Lemma 3.7. *The multiplicity μ satisfies the following apriori estimate with absolute constants C and b :*

$$\mu \leq C|\log \delta|^b \lambda^{-1}\delta^{-1}t. \tag{3.14}$$

Proof. Let $\{z_i\}$ be a t-net and consider the quantities

$$\begin{aligned} M_1(i) = &\ \operatorname{card}\{1 \leq j \leq \overline{M} \ : \ x_j \in B(z_i, t)\}, \\ M_2(i) = &\ \operatorname{card}\{1 \leq j \leq M \ : \ x_j \in B(z_i, 2t)\}. \end{aligned}$$

Then, clearly,

$$\sum_i M_1(i) \simeq \overline{M} \quad \text{and} \quad \sum_i M_2(i) \simeq M. \tag{3.15}$$

Since $\overline{M} = |\log \delta|^{-2} M/2$, we conclude from (3.15) that there is a point of the net, say z_0, such that $M_1 = M_1(0)$ and $M_2 = M_2(0)$ satisfy $M_1 \geq C^{-1}|\log \delta|^{-2} M_2$. Define

$$\Phi_1 = \sum_{j\,:\,|x_j - z_0| \leq 2t} \chi_{C_j^*}.$$

As in Lemma 2.1 we distinguish two cases. If

$$|E_1| = |\{E \ : \ \Phi_1 \geq \bar{\mu}\}| \leq \bar{\mu}^{-1}\bar{\lambda} M_2 \delta,$$

then by Corollary 3.6 (setting $x_0 = z_0$)

$$\|\mathcal{M}_\delta \chi_{E_1}\|_{L^2(B(z_0,t))} \leq C_0\, |\log \delta|^{\frac{1}{2}} t^{\frac{1}{2}} |E_1|^{\frac{1}{2}}. \tag{3.16}$$

The expression on the left is $\geq C^{-1}\,\bar{\lambda}(\delta^2 M_1)^{\frac{1}{2}}$. Indeed, for any $1 \leq j \leq \overline{M}$ such that $x_j \in B(z_0, t)$ we have

$$\Phi_1 \geq \Phi^j_{t,\epsilon}.$$

Thus (3.13) implies that

$$|\{C_j^* \ : \ \Phi_1 \geq \bar{\mu}\}| \geq \bar{\lambda}|C_j|,$$

or equivalently

$$|C_j \cap E_1| \geq \ \bar{\lambda}|C_j|, \quad \text{or} \quad \mathcal{M}_\delta \chi_{E_1}(x_j) \geq \ \bar{\lambda}.$$

Since the $\{x_j\}$ are δ-separated, our claim follows. On the other hand, the right side of (3.16) is

$$\leq C_0\, |\log \delta|^{\frac{1}{2}} t^{\frac{1}{2}} (\bar{\mu}^{-1}\bar{\lambda} M_2 \delta)^{\frac{1}{2}}$$

by our assumption on $|E_1|$. Recalling the definition of $M_1, M_2, \bar{\lambda}$ etc., we obtain (3.14).

If $|E_1| \geq \bar{\mu}^{-1}\bar{\lambda}M_2\delta$ we use duality, i.e., (3.10). Letting $x_0 = z_0$ in (3.10), replacing t with $2t$, setting $y_j = x_j$, $\rho_j = r_j$, and $a_j = (\delta^2 M_2)^{-\frac{1}{2}}$ for $j = 1, \ldots, M_2$, or $a_j = 0$ otherwise, we obtain

$$\|\Phi_1\|_2 \leq C_0\,|\log\delta|^{\frac{1}{2}}\delta^{-1}t^{\frac{1}{2}}(\delta^2 M_2)^{\frac{1}{2}}.$$

The left-hand side is

$$\geq \bar{\mu}|\{E\ :\ \Phi_1 \geq \bar{\mu}\}|^{\frac{1}{2}} = \bar{\mu}|E_1|^{\frac{1}{2}} \geq \bar{\mu}(\bar{\mu}^{-1}\bar{\lambda}M_2\delta)^{\frac{1}{2}}$$

and the lemma follows. □

In [7] it is shown how to obtain (3.14) without the Fourier transform, using only geometric/combinatorial methods. The main tool turns out to be a two circle lemma.

4. The three circle lemma

In the previous section implicit information about circles was used to prove an L^2 bound on the maximal function and thus a bound on the multiplicity μ. In this section we will attempt to use explicit geometric properties of circles in order to bound μ. The procedure we apply here was discovered by Kolasa and Wolff [4]. Although this section is essentially self-contained, the reader might wish to read Section 3 of [4], in particular Proposition 3.1, which provides the underlying idea for the proof of Proposition 4.3 below. Lemma 4.1 (Marstrand's three circle lemma) is the main geometric tool in the argument below. It is a quantitative version of the following fact (known in incidence geometry as the circles of Apollonius):

Given any three circles which are not internally tangent at a single point, there are at most two circles which are internally tangent to the three given ones (we say that two circles in the plane are internally tangent if they are tangent and the smaller one is contained inside the larger one).

The number ϵ in Lemma 4.1 controls the degree of internal tangency, whereas λ separates the "points of tangency" (see Figure 2). $N_\delta(S)$ denotes the δ entropy of the set S, i.e., the cardinality of a maximally δ-separated set in S. We merely sketch a proof of Lemma 4.1 and refer the reader to Marstrand [5], Lemma 5.2 for further details. For a version applying to families of curves satisfying Sogge's cinematic curvature condition see [4], Lemma 3.1.

Lemma 4.1. *Let $(x_j, r_j)_{j=1}^3 \in \mathbb{R}^2 \times (1,2)$ and fix $0 < \lambda, \epsilon < 1$. Consider the set*

$$S = \Bigg\{ x \in \mathbb{R}^2 \setminus \bigcup_{j=1}^{3} B(x_j,\epsilon)\ :\ \exists\, r \in (1,2) \quad \textit{with}\ \ ||x_i - x| - |r_i - r|| < \epsilon \\ \textit{for } i = 1,2,3 \textit{ and } \quad |e_i(x,r) - e_j(x,r)| > \lambda \textit{ for } i \neq j,\ i,j = 1,2,3 \Bigg\}.$$

Here

$$e_i(x,r) = \frac{x_i - x}{|x_i - x|}\operatorname{sgn}(r - r_i).$$

Then

$$N_\delta(S) \leq C\ \left(\frac{\epsilon}{\delta}\right)^2 \lambda^{-3} \quad \textit{for any } 0 < \delta \leq \epsilon.$$

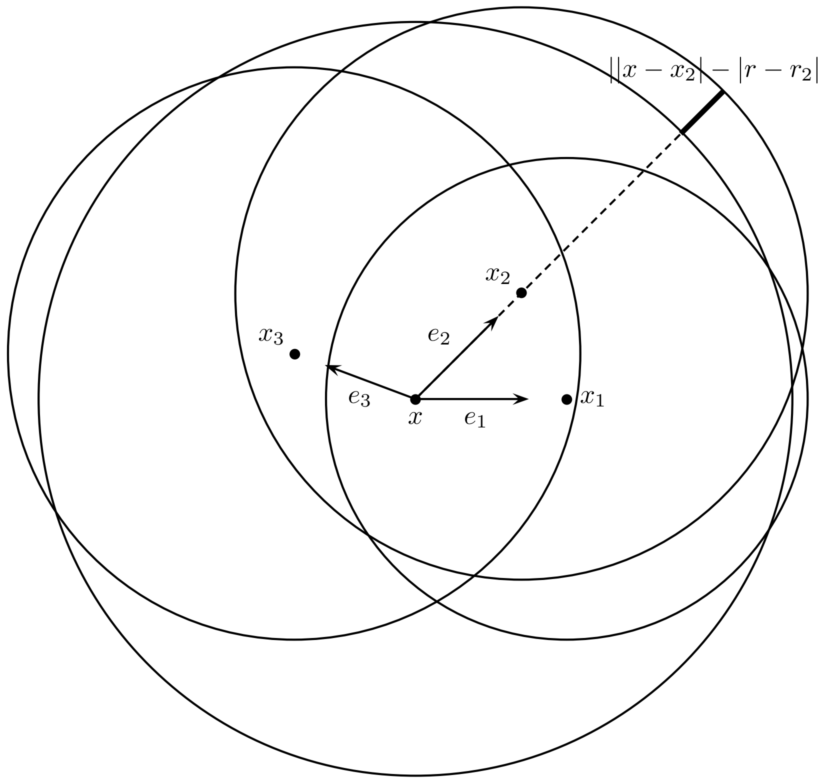

FIGURE 2. Marstrand's three circle lemma

Proof. Let

$$\Omega = \{(x,r) \in \mathbb{R}^2 \times (1,2) \ : |x - x_j| > 3\epsilon, r \neq r_i, \\ |e_i(x,r) - e_j(x,r)| > \lambda \text{ for } i \neq j \ , i,j = 1,2,3\}$$

and $F : \Omega \to \mathbb{R}^3$ be defined by

$$F(x,r) = (|x_i - x| - |r_i - r|)_{i=1}^3.$$

It is easy to see that the Jacobian JF of F satisfies

$$JF \simeq |e_1 - e_2||e_1 - e_3||e_2 - e_3| > \lambda^3.$$

Since $\text{card}(F^{-1}(p)) \leq C_0$ for some absolute constant C_0 and all $p \in \mathbb{R}^3$, we conclude that

$$|F^{-1}(B(0,2\epsilon))| \leq C\,\epsilon^3\lambda^{-3}.$$

According to the definition of S there exists a function $r : S \to (1,2)$ such that for every $x \in S$ we have $|F(x, r(x))| < \epsilon$. Then clearly

$$\{(x,r) : x \in S\,,\ |r - r(x)| < \epsilon\} \subset F^{-1}(B(0,2\epsilon)) \cup \left\{ (x,r) : x \in S \cap \bigcup_{j=1}^{3} B(x_j, 3\epsilon)\,,\ |r - r(x)| < \epsilon \right\}$$

and thus $|S| \le C\,\epsilon^2 \lambda^{-3}$. □

The following lemma contains bounds on the diameter and the area of $C_\delta(x,r) \cap C_\delta(y,s)$. In various forms it appears in several papers on this subject; see, e.g., [1], [4], [5], [8]. Since the exact version we use here does not seem to be contained explicitly in any of these references, we provide a proof for the reader's convenience. We will use the notation

$$\Delta = \max(||x-y| - |r-s||, \delta).$$

Lemma 4.2. *Suppose $x, y \in \mathbb{R}^2$, $x \neq y$, $|x-y| \le \frac{1}{2}$, and $r, s \in (1,2)$, $r \neq s$, $0 < \delta < 1$. There is an absolute constant A such that*

1. *$C_\delta(x,r) \cap C_\delta(y,s)$ is contained in a δ neighborhood of an arc of length $\le A\sqrt{\frac{\Delta}{|x-y|}}$ centered at the point $x - r\,\mathrm{sgn}(r-s)\frac{x-y}{|x-y|}$.*
2. *The area of intersection satisfies*

$$|C_\delta(x,r) \cap C_\delta(y,s)| \le A \frac{\delta^2}{\sqrt{\Delta |x-y|}}. \tag{4.1}$$

Proof. Let $z \in C_\delta(x,r) \cap C_\delta(y,s)$. Then $|z-x| = r_1$ and $|z-y| = s_1$ where $|r - r_1| < \delta$ and $|s - s_1| < \delta$. By simple algebra

$$2(z-x)\cdot(y-x) = r_1^2 - s_1^2 + |y-x|^2. \tag{4.2}$$

Assume $r < s$. Then (4.2) implies

$$2r_1|x-y|(1 - \cos\angle(z-x, x-y)) = (r_1 + |x-y|)^2 - s_1^2$$

and thus

$$\angle(z-x, x-y) \simeq \sqrt{\frac{|x-y| - (s_1 - r_1)}{|x-y|}} \le C\sqrt{\frac{\Delta}{|x-y|}}. \tag{4.3}$$

If $r > s$ one estimates $\angle(z-x, y-x)$ in a similar fashion.

If $\Delta \le 10\delta$ the bound (4.1) follows from the first statement of the lemma. Otherwise consider $\alpha = \angle(z-x, x-y)$ as a function of r_1 and s_1. Taking partial derivatives in (4.2) yields

$$\frac{\partial \alpha}{\partial r_1}\, r_1|x-y|\sin\alpha = r_1 + |x-y|\cos\alpha,$$
$$\frac{\partial \alpha}{\partial s_1}\, r_1|x-y|\sin\alpha = -s_1.$$

Thus

$$\left|\frac{\partial \alpha}{\partial r_1}\right| + \left|\frac{\partial \alpha}{\partial s_1}\right| \le C\,(|\alpha||x-y|)^{-1} \simeq (\Delta|x-y|)^{-\frac{1}{2}}.$$

The last equality is true since $\Delta > 10\delta$ implies that (4.3) holds with $\simeq$ instead of $\leq$. Since r_1 and s_1 vary in a δ interval, α will be contained in an interval of length $\leq C\,\frac{\delta}{\sqrt{\Delta|x-y|}}$ and (4.1) follows. □

Proposition 4.3 below is the main result of this paper.

Proposition 4.3. *$\mathcal{M}_\delta$ is of restricted weak type $(5/2, 5)$, i.e., for any $f \in L^1 \cap L^\infty(\mathbb{R}^2)$*

$$\|\mathcal{M}_\delta f\|_{5,\infty} \leq C\,|\log\delta|^b\,\|f\|_{5/2,1}, \tag{4.4}$$

where b and C are absolute constants.

Proof. In this proof we let B_δ denote a constant of the form $C|\log\delta|^b$, where the values of C and b are allowed to vary depending on the context. By Lemma 2.1 we need to show

$$\mu \leq B_\delta\,\lambda^{-\frac{3}{2}}M^{\frac{1}{2}}. \tag{4.5}$$

C and b are determined implicitly in the calculation below. This will follow from the combinatorial argument in [4], which is based on the three circle lemma, and the refined L^2 bound from above. A is the absolute constant from Lemma 4.2.

$$\underline{\text{Case 1:}}\quad \bar{\lambda} \leq 100A\left(\frac{\epsilon}{t}\right)^{\frac{1}{2}}. \tag{4.6}$$

On the one hand, by (3.13) and Lemma 4.2

$$\bar{\mu}\bar{\lambda}\delta \leq C\int_{C_j}\sum_{i\in S^j_{t,\epsilon}}\chi_{C_i^*} \leq C\,\mathrm{card}(S^j_{t,\epsilon})\frac{\delta^2}{\sqrt{\epsilon t}} \leq C\,M\frac{\delta^2}{\sqrt{\epsilon t}}. \tag{4.7}$$

On the other hand, by Lemma 3.7

$$\bar{\mu} \leq B_\delta\,\bar{\lambda}^{-1}\delta^{-1}t.$$

Thus

$$\bar{\mu}\bar{\lambda}\left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}}\left(\frac{t}{\delta}\right)^{\frac{1}{2}} \leq B_\delta\,\min\left(M, \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}}\left(\frac{t}{\delta}\right)^{\frac{3}{2}}\right).$$

Hence, if

$$M \leq \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}}\left(\frac{t}{\delta}\right)^{\frac{3}{2}}, \tag{4.8}$$

then

$$\begin{aligned}\bar{\mu} &\leq B_\delta\,\bar{\lambda}^{-1}\left(\frac{\delta}{\epsilon}\right)^{\frac{1}{2}}\left(\frac{\delta}{t}\right)^{\frac{1}{2}}M \\ &= B_\delta\,\bar{\lambda}^{-\frac{3}{2}}M^{\frac{1}{2}}\bar{\lambda}^{\frac{1}{2}}M^{\frac{1}{2}}\left(\frac{\delta}{\epsilon}\right)^{\frac{1}{2}}\left(\frac{\delta}{t}\right)^{\frac{1}{2}} \\ &\leq B_\delta\,\bar{\lambda}^{-\frac{3}{2}}M^{\frac{1}{2}}\left(\frac{\epsilon}{t}\right)^{\frac{1}{4}}\left(\frac{\epsilon}{\delta}\right)^{\frac{1}{4}}\left(\frac{t}{\delta}\right)^{\frac{3}{4}}\left(\frac{\delta}{\epsilon}\right)^{\frac{1}{2}}\left(\frac{\delta}{t}\right)^{\frac{1}{2}} \\ &= B_\delta\,\bar{\lambda}^{-\frac{3}{2}}M^{\frac{1}{2}},\end{aligned} \tag{4.9}$$

where we have used (4.6) and (4.8) in line (4.9) to replace $\bar\lambda^{\frac12}$ and $M^{\frac12}$, respectively. If, on the other hand,

$$M \ge \left(\frac{\epsilon}{\delta}\right)^{\frac12}\left(\frac{t}{\delta}\right)^{\frac32}, \tag{4.10}$$

then

$$\begin{aligned}\bar\mu \le\ & B_\delta\,\bar\lambda^{-1}\delta^{-1}t = B_\delta\,\bar\lambda^{-\frac32}\bar\lambda^{\frac12}\delta^{-1}t\\ \le\ & B_\delta\,\bar\lambda^{-\frac32}\left(\frac{\epsilon}{t}\right)^{\frac14}\delta^{-1}t \le B_\delta\,\bar\lambda^{-\frac32}\left(\frac{\delta^2 M}{t^2}\right)^{\frac12}\delta^{-1}t \qquad (4.11)\\ =\ & B_\delta\,\bar\lambda^{-\frac32}M^{\frac12}.\end{aligned}$$

Here we have used (4.6) and then (4.10) in line (4.11).

$$\underline{\text{Case 2:}}\quad \bar\lambda \ge 100A\left(\frac{\epsilon}{t}\right)^{\frac12}. \tag{4.12}$$

Following [4] we let

$$\begin{aligned}Q = \{(j,i_1,i_2,i_3)\ :\ 1\le j\le \overline{M},\ i_1,i_2,i_3\in S^j_{t,\epsilon}\ \text{ and the distance between}\\ \text{any two of the sets }\ C_j\cap C_{i_1}, C_j\cap C_{i_2}, C_j\cap C_{i_3}\ \text{is at least } \bar\lambda/20\}.\end{aligned}$$

Suppose $(j,i_1,i_2,i_3)\in Q$. Then Lemma 4.2 implies that any two of the

$$e_i = x_j - r_j\,\mathrm{sgn}(r_j-r_i)\frac{x_j-x_i}{|x_j-x_i|}$$

for $i=i_1,i_2,i_3$ are separated by a distance $\bar\lambda/20$. Indeed, by that lemma, e_i is the center of $C_i\cap C_j$ and in view of (3.12), for any $i\in S^j_{t,\epsilon}$

$$\mathrm{diam}(C_i\cap C_j)\le 2A\sqrt{\frac{\epsilon}{t}}\le \bar\lambda/50 \tag{4.13}$$

by (4.12). Lemma 4.1 therefore implies that

$$\mathrm{card}(Q)\le C\,\left(\frac{\epsilon}{\delta}\right)^2\bar\lambda^{-3}M^3. \tag{4.14}$$

On the other hand, we claim that

$$\mathrm{card}(Q)\ge C^{-1}\,\overline{M}\left(\bar\mu\frac{\bar\lambda\delta}{\delta^2/\sqrt{\epsilon t}}\right)^3. \tag{4.15}$$

This would clearly follow from

$$\begin{aligned}\min_{1\le j\le\overline{M}}\mathrm{card}(\{(i_1,i_2,i_3)\in(S^j_{t,\epsilon})^3\ :\ \text{the distance between any two}\\ \text{of the sets } C_j\cap C_{i_1}, C_j\cap C_{i_2}, C_j\cap C_{i_3}\ \text{is at least } \bar\lambda/20\})\qquad (4.16)\\ \ge C^{-1}\left(\bar\mu\frac{\bar\lambda\delta}{\delta^2/\sqrt{\epsilon t}}\right)^3.\end{aligned}$$

Denote the set on the left-hand side by $Q^{(j)}$ and fix any j as above. By (4.7) the number of possible choices of i_1 is

$$\mathrm{card}(S^j_{t,\epsilon})\ge C^{-1}\,\bar\mu\frac{\bar\lambda\delta}{\delta^2/\sqrt{\epsilon t}}.$$

Suppose that $(i_1, i_2, i_3) \in Q^{(j)}$. We claim that

$$\operatorname{card}(\{i \in S^j_{t,\epsilon} \ : \ (i_1, i_2, i) \in Q^{(j)}\}) \geq C^{-1}\, \bar{\mu} \frac{\bar{\lambda}\delta}{\delta^2/\sqrt{\epsilon t}}. \tag{4.17}$$

To prove (4.17) let R_1 and R_2 be "rectangles" in C_j of length $\bar{\lambda}/5$ and width δ centered at e_{i_1} and e_{i_2}, respectively. Using (4.13) we conclude that

$$i \in S^j_{t,\epsilon}, \qquad C_i \cap R_\tau = \varnothing \quad \text{for} \quad \tau = 1, 2$$

implies that

$$\operatorname{dist}(C_j \cap C_{i_\tau}, C_j \cap C_i) > \bar{\lambda}/20 \quad \text{for} \quad \tau = 1, 2.$$

Since

$$|\{C^*_j \setminus (R_1 \cup R_2) \ : \ \Phi^j_{t,\epsilon} \geq \bar{\mu}\}| \geq \frac{\bar{\lambda}}{2}|C_j|,$$

(4.17) follows from (4.7) (simply replace (3.13) with the previous inequality). Estimating the number of admissible choices of i_2 given a fixed i_1 in a similar fashion proves (4.16) and thus (4.15). We infer from (4.14) and (4.15) that

$$\bar{\mu}^3 \leq B_\delta\, \bar{\lambda}^{-6} \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}} \left(\frac{\delta M^{\frac{1}{2}}}{t}\right)^{\frac{3}{2}} M^{\frac{5}{4}}. \tag{4.18}$$

Combining (4.18) and (3.14) yields

$$\bar{\mu} \leq B_\delta \, \min\left(\bar{\lambda}^{-1}\delta^{-1}t, \bar{\lambda}^{-2}\left(\frac{\epsilon}{\delta}\right)^{\frac{1}{6}} \left(\frac{\delta M^{\frac{1}{2}}}{t}\right)^{\frac{1}{2}} M^{\frac{5}{12}}\right).$$

Hence, if

$$\bar{\lambda} \leq \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{6}} \left(\frac{\delta M^{\frac{1}{2}}}{t}\right)^{\frac{3}{2}} M^{-\frac{1}{12}}, \tag{4.19}$$

we conclude that

$$\begin{aligned}
\bar{\mu} &\leq B_\delta\, \bar{\lambda}^{-1}\delta^{-1}t \leq B_\delta\, \bar{\lambda}^{-\frac{3}{2}}\bar{\lambda}^{\frac{3}{4}}\bar{\lambda}^{-\frac{1}{4}}\left(\frac{t}{\delta M^{\frac{1}{2}}}\right) M^{\frac{1}{2}} \\
&\leq B_\delta\, \bar{\lambda}^{-\frac{3}{2}} \left(\left(\frac{\epsilon}{\delta}\right)^{\frac{1}{6}} \left(\frac{\delta M^{\frac{1}{2}}}{t}\right)^{\frac{3}{2}} M^{-\frac{1}{12}}\right)^{\frac{3}{4}} \qquad (4.20)\\
&\quad \cdot \left(\left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}} \left(\frac{\delta M^{\frac{1}{2}}}{t}\right)^{\frac{1}{2}} M^{-\frac{1}{4}}\right)^{-\frac{1}{4}} \left(\frac{t}{\delta M^{\frac{1}{2}}}\right) M^{\frac{1}{2}} \\
&= B_\delta\, \bar{\lambda}^{-\frac{3}{2}} M^{\frac{1}{2}}.
\end{aligned}$$

The expressions in (4.20) are obtained by estimating $\bar{\lambda}$ by (4.19) and (4.12), respectively.

If, on the other hand,

$$\bar{\lambda} \geq \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{6}} \left(\frac{\delta M^{\frac{1}{2}}}{t}\right)^{\frac{3}{2}} M^{-\frac{1}{12}}, \tag{4.21}$$

then

$$\bar{\mu} \le B_\delta\, \bar{\lambda}^{-2} \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{6}} \left(\frac{\delta M^{\frac{1}{2}}}{t}\right)^{\frac{1}{2}} M^{\frac{5}{12}}$$

$$\le B_\delta\, \bar{\lambda}^{-2} \left(\bar{\lambda} \left(\frac{\delta M^{\frac{1}{2}}}{t}\right)^{-\frac{3}{2}} M^{\frac{1}{12}}\right)^{\frac{1}{4}} \tag{4.22}$$

$$\cdot \left(\bar{\lambda}^{\frac{1}{3}} \left(\frac{\delta M^{\frac{1}{2}}}{t}\right)^{-\frac{1}{6}} M^{\frac{1}{12}}\right)^{\frac{3}{4}} \left(\frac{\delta M^{\frac{1}{2}}}{t}\right)^{\frac{1}{2}} M^{\frac{5}{12}}$$

$$= B_\delta\, \bar{\lambda}^{-\frac{3}{2}} M^{\frac{1}{2}}.$$

To obtain (4.22), use (4.21) and the inequality

$$\left(\frac{\epsilon}{\delta}\right)^{\frac{1}{6}} \left(\frac{\delta M^{\frac{1}{2}}}{t}\right)^{\frac{1}{6}} \le C\, \bar{\lambda}^{\frac{1}{3}} M^{\frac{1}{12}},$$

which follows from (4.12). Consequently, we have established (4.5) and the proposition follows. □

5. Proof of Theorem 1.3

The following lemma states that instead of averaging over δ annuli we can average over a mollified version of $d\sigma_r$ which is essentially concentrated on a δ annulus.

Lemma 5.1. *Fix a radial function $\phi \in \mathcal{S}(\mathbb{R}^2)$. Suppose that for fixed $1 \le p \le q \le \infty$, $\alpha < 3$*

$$\|\mathcal{M}_\delta f\|_q \le C\, \delta^{-\alpha} \|f\|_p$$

for all $0 < \delta < 1$, $f \in L^1 \cap L^\infty$. Then

$$\|\mathcal{M}(f * \phi_\delta)\|_q \le C\, \delta^{-\alpha} \|f\|_p$$

for all $0 < \delta < 1$, $f \in \mathcal{S}$.

Proof. Write $\phi(|x|) = \phi(x)$. We construct a radial, nonincreasing majorant for ϕ as follows. Let $\rho(r) = r^2 |\phi'(r)|$ and define

$$\psi(|x|) = \int_{|x|}^{\infty} |\phi'(r)|\, dr$$

or equivalently

$$\psi(x) = \int_0^{\infty} (\chi_B)_r\, (x) \rho(r)\, dr,$$

where B is the unit ball in $\mathbb{R}^2$. Note that

$$\int_{\mathbb{R}^2} \psi(x)\, dx = \int_0^{\infty} \rho(r)\, dr = \int_0^{\infty} \psi(r) r\, dr.$$

Let $f \in \mathcal{S}$. Then

$$\sup_{1<t2} |d\sigma_t * (\phi_\delta * f)| \le \left\{ \int_0^{\delta^{-1}} + \int_{\delta^{-1}}^{\infty} \right\} \sup_{1<t<2} |[d\sigma_t * (\chi_B)_{r\delta}] * |f||\, \rho(r)\, dr$$

$$= A + B.$$

On the one hand

$$\begin{aligned}\|A\|_q &\le C \int_0^{\delta^{-1}} \|\mathcal{M}_{10r\delta}|f|\|_q \, \rho(r)\, dr \\ &\le C\,\delta^{-\alpha} \int_0^{\delta^{-1}} r^{-\alpha}\rho(r)\, dr\, \|f\|_p \\ &\le C\,\delta^{-\alpha}\|f\|_p\end{aligned}$$

since $\alpha < 3$. On the other hand, by Young's inequality with $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{s}$,

$$\begin{aligned}\|B\|_q &\le C \int_{\delta^{-1}}^{\infty} \|\,(\chi_B)_{10r\delta} * |f|\|_q \, \rho(r) dr \\ &\le C \int_{\delta^{-1}}^{\infty} \|\,(\chi_B)_{10r\delta}\,\|_s \|f\|_p \, \rho(r) dr \\ &\le C \int_{\delta^{-1}}^{\infty} (\delta r)^{-\frac{2}{s'}} \rho(r) dr \, \|f\|_p \\ &\le C\,\|f\|_p\end{aligned}$$

and the lemma follows. □

Proof of Theorem 1.3. Statements (1.9), (1.10) of Theorem 1.3 follow via Marcinkiewicz's theorem from the estimates at the points Q, R, T, P (see Figure 1). To prove (1.8), suppose we are given any $f \in \mathcal{S}$. Let

$$f = \sum_{j=0}^{\infty} f_j$$

be a Littlewood-Paley decomposition, i.e., $\text{supp}(\hat{f}_0) \subset \{\mathbb{R}^2 \colon |\xi| < 2\}$ and $\text{supp}(\hat{f}_j) \subset \{\mathbb{R}^2 \colon 2^{j-1} < |\xi| < 2^{j+1}\}$ for $j = 1, 2, \ldots$. On the one hand, (1.10), (1.9), and Lemma 5.1 imply

$$||\mathcal{M} f_j||_q \le C_\epsilon\, 2^{j\epsilon} ||f_j||_p \quad \text{if } \left(\frac{1}{p}, \frac{1}{q}\right) \in QP \cup PT \text{ (see Figure 1)} \tag{5.1}$$

for any $\epsilon > 0$ and $j = 1, 2, \ldots$. On the other hand, by the local smoothing theorem in [6] (see also [2] and [8])

$$||\mathcal{M} f_j||_p \le C\; 2^{-j\beta} \|f_j\|_p, \tag{5.2}$$

where $2 < p < \infty$, $\beta = \beta(p) > 0$, and $j = 1, 2, \ldots$. Interpolating (5.2) with (5.1) yields

$$||\mathcal{M} f_j||_q \le C\; 2^{-j\gamma} ||f_j||_p \quad \text{if } \left(\frac{1}{p}, \frac{1}{q}\right) \in \text{region I}\backslash QP \cup PT \tag{5.3}$$

for some $\gamma = \gamma(p, q) > 0$. Furthermore,

$$||\mathcal{M} f_0||_q \le C\; ||f_0||_p \tag{5.4}$$

by the Hardy-Littlewood and Bernstein inequalities. Finally, (1.8) follows from inequalities (5.3) and (5.4) by Littlewood-Paley theory.

Up to a $|\log \delta|$ factor, (1.11) follows by interpolating the estimate at T, i.e., (1.7), with the ones at the endpoints R and S:

$$\|\mathcal{M}_\delta f\|_1 + \|\mathcal{M}_\delta f\|_\infty \le C\,\delta^{-1}\|f\|_1. \tag{5.5}$$

To obtain the sharp estimates, let $f = \sum_0^\infty f_j$ be as above. The analogue of (5.5) is

$$\|\mathcal{M} f_j\|_1 + \|\mathcal{M} f_j\|_\infty \le C\, 2^j \|f_j\|_1, \tag{5.6}$$

which can be shown by a standard application of stationary phase, cf. [7]. Interpolating (5.6) with the L^2 bound (3.1) yields

$$\|\mathcal{M} f_j\|_q \le C\, 2^{j(\frac{2}{p}-1)} \|f_j\|_p \quad \text{ if } \left(\frac{1}{p}, \frac{1}{q}\right) \in \text{region IV.} \tag{5.7}$$

(1.11) now follows from (5.7) by the same type of argument as in the proof of Corollary 3.6 provided $1 < p$. The estimates on the segment SR follow from the ones at the endpoints. We skip the details. □

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