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A POLYNOMIALLY BOUNDED OPERATOR ON HILBERT SPACE WHICH IS NOT SIMILAR TO A CONTRACTION

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§0. Introduction

Let H be a Hilbert space. Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and $\mathbf{T} = \partial D = \{z \mid |z| = 1\}$.

Any operator $T : H \to H$ with $\|T\| \le 1$ is called a contraction. By a celebrated inequality due to von Neumann [vN], we have then, for any polynomial P,

$$\|P(T)\| \le \sup_{z\in D} |P(z)|. \tag{0.1}$$

We say that T is similar to a contraction if there is an invertible operator $S :\ H \to H$ such that $S^{-1}TS$ is a contraction.

An operator $T : H \to H$ is called power bounded if $\sup_{n\ge 1} \|T^n\| < \infty$.

It is called polynomially bounded if there is a constant C such that, for any polynomial P,

$$\|P(T)\| \le C \sup\{|P(z)| \mid z \in \mathbb{C},\ |z| = 1\}. \tag{0.1$'$}$$

Clearly, if T is similar to a contraction, then it is power bounded, and actually, by von Neumann's inequality, it is polynomially bounded.

Let A be the disc algebra, *i.e.* the closure of the space of all (analytic) polynomials in the space $C(\mathbf{T})$ of all continuous functions on $\mathbf{T}$, equipped with the norm

$$\|P\|_\infty = \sup_{z\in\partial D} |P(z)|.$$

Let $L^p(\mathbf{T})$ be the L^p-space relative to normalized Lebesgue measure on $\mathbf{T}$, which we denote by m.

Let $H^p = \{f \in L^p(\mathbf{T}) |\ \hat{f}(n) = 0\ \forall n < 0\}$. If X is a Banach space, we denote by $H^p(X)$, for $p < \infty$, the analogous space of X-valued functions. In the particular case $X = B(H)$ with $H = \ell_2(I)$ (I being an arbitrary set), we denote by $L^\infty(B(H))$ the space of all (classes of) bounded $B(H)$-valued functions of which all matrix coefficients are measurable. This space can be identified isometrically with the dual of the space $L^1(B(H)_*)$ of all Bochner integrable functions with values in the predual of $B(H)$, *i.e.* the space of all trace class operators.

In 1959, Béla Sz.-Nagy [SN] asked whether every power bounded operator T is similar to a contraction. He proved that the answer is positive if T is invertible and if both T and its inverse are power bounded (then T is actually similar to a unitary operator; see [SN]). In 1964, S. Foguel [Fo] (see also [Ha2]) gave a counterexample

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to Sz.-Nagy's question using some properties of Hadamard-lacunary Fourier series. Foguel's example is not polynomially bounded (see [Le]), whence the next question:

Is every polynomially bounded operator similar to a contraction?

This revised version of Sz.-Nagy's original question was popularized by P. Halmos in [Ha1], and since then, many authors refer to it as "the Halmos problem".

If T is polynomially bounded, the map $T \to P(T)$ extends to a unital homomorphism $\rho_T : A \to B(H)$ and $(0.1')$ implies $\|\rho_T\| \le C$. In [Pa2], Paulsen gave a useful criterion for an operator T to be similar to a contraction. He proved that this holds iff the homomorphism ρ_T is "completely bounded" (see below for more background on this notion). In that case, the operator T is called "completely polynomially bounded". In these terms, the above problem becomes: Is every polynomially bounded operator completely polynomially bounded? Or equivalently, is every bounded unital homorphism $\rho : A \to B(H)$ automatically completely bounded?

In [Pe1], [Pe2] (see also [FW]), V. Peller proposes a candidate for a counterexample: let $\Gamma : H^2 \to (H^2)^*$ be a Hankel operator , *i.e.* such that the associated bilinear map (denoted again by Γ) on $H^2 \times H^2$ satisfies $\forall F \in A,\ \forall g, h \in H^2\ \Gamma(gF, h) = \Gamma(g, Fh)$. In other words, if we denote by $M_F : H^2 \to H^2$, for $F \in A$, the operator of multiplication by F, and by ${}^tM_F : (H^2)^* \to (H^2)^*$ its adjoint, we have

$$\Gamma M_F = {}^tM_F\Gamma \qquad \forall F \in A. \tag{0.2}$$

Let $H = (H^2)^* \oplus H^2$. For any polynomial F consider the operator $R(F) : H \to H$ defined by the following block matrix:

$$R(F) = \begin{pmatrix} {}^tM_F & \Gamma M_{F'} \\ 0 & M_F \end{pmatrix}. \tag{0.3}$$

Since the coefficient $F \to \Gamma M_{F'}$ behaves like a derivation, we easily verify that $F \to R(F)$ is a unital homomorphism on polynomials, *i.e.* we have $R(1) = 1$, and, by (0.2), for all polynomials F, G, we have

$$R(FG) = R(F)R(G). \tag{0.4}$$

Hence, under the condition

$$\exists C\ \forall F \text{ polynomial } \|\Gamma M_{F'}\| \le C\|F\|_\infty, \tag{0.5}$$

the mapping $F \to R(F)$ defines a bounded unital homomorphism on A, or equivalently the operator $T_\Gamma = R(z)$ (*i.e.* $R(\varphi_0)$ for φ_0 defined by $\varphi_0(z) = z$ for all z) is polynomially bounded.

By a well-known theorem of Nehari (see e.g. [Ni]), each Hankel operator Γ is associated to a symbol $\varphi \in L^\infty$ so that $\Gamma = \Gamma_\varphi$. (This corresponds to the case $\dim(H) = 1$ in (0.6) below.) We denote simply by T_φ the operator T_Γ with $\Gamma = \Gamma_\varphi$. In [Pe2], Peller shows that if $\varphi' \in \mathrm{BMO}$, then we have (0.5), and consequently T_φ is polynomially bounded. The question whether this implies "similar to a contraction" was then posed by Peller [Pe2] , but Bourgain [Bo2] showed that $\varphi' \in \mathrm{BMO}$ implies that T_φ is similar to a contraction, and very recently (summer of 1995), Aleksandrov and Peller [AP] showed that actually T_φ is polynomially bounded only if $\varphi' \in \mathrm{BMO}$. In conclusion, there is no counterexample in this class.

However, as is well known (cf. e.g. [Ni]) there is a *vectorial* version of Hankel operators: given a function $\varphi \in L^\infty(B(H))$, we can associate to it an operator (usually called a vectorial Hankel operator) $\Gamma_\varphi : H^2(H) \to H^2(H)^*$ still satisfying the identity (0.2), with respect to the multiplication operator M_F considered as acting on $H^2(H)$. More precisely, we can define for any g, h in $H^2(H)$

$$\Gamma_\varphi(g,h) = \int [\varphi(\xi)g(\xi), h(\xi)]\, dm(\xi), \tag{0.6}$$

where $[\cdot,\cdot]$ is a bilinear map on H associated to a fixed isometry between H and H^*. Thus, defining $R(F)$ as in (0.3) above, we still have (0.4) and the associated operator $T_\varphi : H^2(H)^* \oplus H^2(H) \to H^2(H)^* \oplus H^2(H)$ is polynomially bounded provided $\Gamma = \Gamma_\varphi$ satisfies (0.5).

The main result of this paper is that, when H is infinite dimensional, there are counterexamples to the Halmos problem of the form $T = T_\varphi$, with $\varphi \in L^\infty(B(H))$.

Remark. There is some recent related work (on operators of the form T_Γ) by Stafney [St] and also by S. Petrovic, V. Paulsen and Sarah Ferguson whose work I have heard of, through the seminar talks they gave in Texas at various occasions in 1995.

Notation and background. In general, any unexplained notation is standard. Let H, K be two Hilbert spaces. We denote by $B(H,K)$ (resp. $B(H)$) the space of all bounded linear operators from H to K (resp. from H to H) equipped with its usual norm. We denote by H^* the dual of H which, of course, is a Hilbert space canonically identifiable with the complex conjugate $\bar{H}$ of H.

Recall that we denote simply by A the disc algebra. Note that A is a closed subalgebra of H^∞. Often, we implicitly consider a function in H^∞ as extended analytically inside the unit disc, in such a way that we recover the original function on the circle by taking radial (or nontangential) limits almost everywhere. When the original function on the circle is actually in A, its analytic extension is continuous on $\bar{D}$.

Consider a function f in (say) $L^p(\mathbf{T})$ $(1 \le p \le \infty)$. When we sometimes abusively say that f is "analytic", what we really mean is that $f \in H^p$, *i.e.* that f extends analytically inside D. In that case, throughout this paper, the derivative f' of f always means the derivative of the Taylor series of f.

We denote by ℓ_2^n the n-dimensional Hilbert space, and by $\ell_2^n(H)$ the (Hilbertian) direct sum of n copies of H. Moreover, we denote $M_n = B(\ell_2^n)$. Let H, K be Hilbert spaces. Let $S \subset B(H)$ be a subspace. In the theory of operator algebras, the notion of complete boundedness for a linear map $u : S \to B(K)$ has been extensively studied recently. Its origin lies in the work of Stinespring (1955) and Arveson (1969) on completely positive maps (see [Pa1], [Pi4] for more details and references). Let us equip $M_n(S)$ and $M_n(B(K))$ (the spaces of matrices with entries respectively in S and $B(K)$) with the norm induced respectively by $B(\ell_2^n(H))$ and $B(\ell_2^n(K))$.

A map $u\colon S \to B(K)$ is called completely bounded (in short c.b.) if there is a constant C such that the maps $I_{M_n} \otimes u$ are uniformly bounded by C, *i.e.* if we have

$$\sup_n \|I_{M_n} \otimes u\|_{M_n(S) \to M_n(B(K))} \le C,$$

and the c.b. norm $\|u\|_{cb}$ is defined as the smallest constant C for which this holds.

When $\|u\|_{cb} \le 1$, we say that u is completely contractive (or a complete contraction).

Let $\mathcal{A} \subset B(H)$ be a unital subalgebra, and let $\pi : \mathcal{A} \to B(H)$ be a unital homomorphism. Paulsen ([Pa2]) proved that π is completely bounded iff there is an invertible operator $S : H \to H$ such that $S^{-1}\pi(\cdot)S$ is completely contractive.

We now return to polynomially bounded operators. An operator $T\colon\ H \to H$ will be called *completely* polynomially bounded if there is a constant C such that for all n and all $n \times n$ matrices (P_{ij}) with polynomial entries we have

$$\|(P_{ij}(T))\|_{B(\ell_2^n(H))} \le C \sup_{z \in \mathbf{T}} \|(P_{ij}(z))\|_{M_n}, \tag{0.7}$$

where $(P_{ij}(T))$ is identified with an operator on $\ell_2^n(H)$ in the natural way. Recall that M_n is identified with $B(\ell_2^n)$. Note that T is completely polynomially bounded iff the homomorphism $P \to P(T)$ defines a completely bounded homomorphism ρ_T from the disc algebra A into $B(H)$, and $\|\rho_T\|_{cb}$ is equal to the smallest constant C such that (0.7) holds for all n. Here of course we consider A as a subalgebra of the C^*-algebra $C(\mathbf{T})$, which itself can be embedded e.g. in $B(L_2(\mathbf{T}))$ by identifying a function f in $C(\mathbf{T})$ or $L_\infty(\mathbf{T})$ with the operator of multiplication by f on $L_2(\mathbf{T})$.

We can now state Paulsen's criterion:

Theorem 0.1 ([Pa2]). *An operator T in $B(H)$ is similar to a contraction iff it is completely polynomially bounded. Moreover, T is completely polynomially bounded with constant C (as in (0.7) above) iff there is an isomorphism $S\colon\ H \to H$ such that $\|S\|\,\|S^{-1}\| \le C$ and $\|S^{-1}TS\| \le 1$.*

Actually, we only use the easy direction of this criterion, which can be derived as follows from Sz.-Nagy's well-known unitary dilation theorem. Let $T : H \to H$ be a contraction. Sz.-Nagy proved (see [SNF]) that we can find a larger Hilbert space $\hat{H}$ containing H as a subspace, and a unitary operator U on $\hat{H}$, such that for all polynomials P

$$P(T) = P_H P(U)_{|H}.$$

(Here P_H denotes the orthogonal projection from $\hat{H}$ to H.) In particular, this implies von Neumann's inequality (0.1). More generally, for any matrix $[P_{ij}]$ with polynomial entries we have $[P_{ij}(T)] = j^*[P_{ij}(U)]j$, where j^* is the diagonal matrix with diagonal entries all equal to P_H. Therefore,

$$\|[P_{ij}(T)]\| \le \|[P_{ij}(U)]\| \le \sup_{z \in \mathbf{T}} \|(P_{ij}(z))\|_{M_n}.$$

The last inequality is a direct consequence of the fact that U generates a commutative C^*-subalgebra of $B(\hat{H})$, and its spectrum lies in $\mathbf{T}$. Clearly, this implies

$$\|[P_{ij}(S^{-1}TS)]\| \le \|S\|\|S^{-1}\| \sup_{z \in \mathbf{T}} \|(P_{ij}(z))\|_{M_n},$$

which proves that similarity to a contraction implies complete polynomial boundedness.

For any linear map $u : A \to B(H)$ it is easy to check that, for any n and for any finitely supported sequence (a_k) in M_n, we can write:

$$\left\|\sum a_k \otimes u(z^k)\right\|_{B(\ell_2^n(H))} \le \|u\|_{cb} \sup_{z \in \mathbf{T}} \left\|\sum a_k z^k\right\|_{M_n}. \tag{0.8}$$

This formulation will be used below.

Concerning the vectorial "Nehari" theorem, the classical references are [Sa] and [Pag] (cf. [Ni]). See [Pe3], and consult [Tr] for more recent refinements on vectorial

Hankel operators and [Pi2] for Banach-space-valued versions of Nehari's theorem. We refer the reader to the books [SNF], [Pa1] for more background on dilation theory, and to the Lecture Notes volume [Pi4], which describes the "state of the art" on similarity problems until 1995.

§1. Main results

Let H be a Hilbert space. Consider an operator

$$\Gamma\colon\ H^2(H) \to H^2(H)^*,$$

or equivalently a bounded bilinear form $\Gamma\colon\ H^2(H) \times H^2(H) \to \mathbb{C}$. Then Γ is called Hankelian (or a Hankel operator) if for any multiplication operator $M_\varphi\colon\ H^2(H) \to H^2(H)$ by a polynomial or a function φ in A (or in H^∞), we have

$$\forall g, h \in H^2(H) \qquad \Gamma(g\varphi, h) = \Gamma(g, \varphi h). \tag{1.1}$$

Equivalently we have $\Gamma M_\varphi = {}^t M_\varphi \Gamma$. Let $F \mapsto \mathcal{D}(F)$ be a linear mapping from A into $B(H^2(H), H^2(H)^*)$ which is a "derivation" in the following sense:

$$\forall F, G \in A \qquad \mathcal{D}(FG) = {}^t M_F \mathcal{D}(G) + \mathcal{D}(F) M_G. \tag{1.2}$$

Let $\mathcal{H} = H^2(H)^* \oplus H^2(H)$. Then the mapping $R : A \to B(\mathcal{H})$ defined by

$$R(F) = \begin{pmatrix} {}^t M_F & \mathcal{D}(F) \\ 0 & M_F \end{pmatrix}$$

clearly is a unital homomorphism.

Now assume in addition that $\mathcal{D}(F)$ is Hankelian for any F in A, *i.e.* that $\mathcal{D}(F) M_\varphi = {}^t M_\varphi \mathcal{D}(F)$ for all φ in A. Then a simple computation shows by induction that

$$\mathcal{D}(F^2) = 2\mathcal{D}(F) M_F \quad \text{ and } \quad \mathcal{D}(F^n) = n\mathcal{D}(F) M_{F^{n-1}}$$

for all $n \geq 1$. Therefore, for any polynomial P we must have

$$\mathcal{D}(P(F)) = \mathcal{D}(F) M_{P'(F)},$$

where P' is the derived polynomial. Applying this with the function $F = \varphi_0$ defined by $\varphi_0(z) = z$ for all z, we find

$$\mathcal{D}(P) = \mathcal{D}(\varphi_0) M_{P'}.$$

So if we let $\Gamma = \mathcal{D}(\varphi_0)$, we have for all polynomials P

$$\mathcal{D}(P) = \Gamma M_{P'}. \tag{1.3}$$

Conversely, given any Hankel operator Γ, if we define $\mathcal{D}_\Gamma$ by setting $\mathcal{D}_\Gamma(P) = \Gamma M_{P'}$, then $\mathcal{D}_\Gamma(P)$ is Hankelian, satisfies the derivation identity (1.2) for all polynomials F, G, and $\mathcal{D}_\Gamma(\varphi_0) = \Gamma$. (Thus (1.3) defines a one-to-one correspondence between Γ and $\mathcal{D}_\Gamma$.)

Given Γ (and the associated $\mathcal{D}_\Gamma$), let $T_\Gamma\colon\ H^2(H)^* \oplus H^2(H) \to H^2(H)^* \oplus H^2(H)$ be the operator defined by the operator matrix

$$T_\Gamma = \begin{pmatrix} {}^t S & \Gamma \\ 0 & S \end{pmatrix},$$

where $S\colon\ H^2(H) \to H^2(H)$ is the shift operator, *i.e.* $S = M_{\varphi_0}$, and ${}^t S$ denotes its adjoint on the dual space $H^2(H)^*$. We will show in this paper that, in the *vectorial* case (with $\dim(H) = \infty$), there are polynomially bounded operators of this type which are not similar to a contraction. The preceding remarks show that, for any polynomial P,

$$P(T_\Gamma) = \begin{pmatrix} {}^tP(S) & \Gamma M_{P'} \\ 0 & P(S) \end{pmatrix}.$$

Therefore if there is a constant C such that

$$\forall P \qquad \|\Gamma M_{P'}\| \le C\|P\|_\infty, \tag{1.4}$$

then T_Γ is polynomially bounded. Our main result is the following:

Theorem 1.1. *Let (C_n) be any sequence in $B(H)$ such that*

$$\forall(\alpha_n) \in \ell_2 \qquad \left\|\sum \alpha_n C_n\right\|_{B(H)} \le \left(\sum |\alpha_n|^2\right)^{1/2}, \tag{1.5}$$

and let (K_n) be an increasing sequence of positive integers such that for all $n > 1$

$$2^{n-1} < K_n \le 2^n. \tag{1.6}$$

Consider the operator $u\colon\ A \to B(H)$ defined by $u(F) = \sum_{n\ge 1} \hat{F}(K_n)C_n$. Then there is a function $\varphi \in L^\infty(B(H))$ such that the associated vectorial Hankel operator $\Gamma = \Gamma_\varphi\colon\ H^2(H) \to H^2(H)^*$ satisfies (1.4) *for some constant C and moreover is such that for any polynomial P* (*with derived polynomial denoted by P'*)*:*

$$\forall x, y \in H \quad \langle u(P)x, \overline{y}\rangle = \Gamma[P'(1 \otimes x)](1 \otimes y), \tag{1.7}$$

where we denote by $1\otimes x$ the element of $H^2(H)$ corresponding to the function taking constantly the value x, and where $y \mapsto \overline{y}$ is an antilinear isometry on H.

Corollary 1.2. *There is a polynomially bounded operator* (*of the form T_Γ*) *which is not similar to a contraction.*

Proof. Let (C_n) be a sequence of operators satisfying the CAR ("canonical anti-commutation relations"), *i.e.* such that

$$\forall i, j \qquad C_iC_j + C_jC_i = 0, \quad C_i^*C_j + C_jC_i^* = \delta_{ij}I.$$

It is well known that this implies (1.5) (with equality even); see e.g. [BR, p. 11]. Moreover, if $(C_1', \dots, C_n')$ is another n-tuple satisying the CAR, there is a C^*-representation taking $(C_1, \dots, C_n)$ to $(C_1', \dots, C_n')$ (see [BR, Th. 5.2.5]), so that for any $a_1, \dots, a_n$ in M_N (with $N \ge 1$) the value of $\|\sum_1^n a_i \otimes C_i\|_{B(\ell_2^N(H))}$, does not depend on the particular realization of $(C_1, \dots, C_n)$. We denote by $\bar{x}$ the complex conjugate of a matrix x. Equivalently, if $x \in B(H)$, we can identify $\bar{x}$ with ${}^tx^* : H^* \to H^*$.

We claim that, with this choice of (C_n), the mapping u is *not* completely bounded. To see this last fact (also well known; see e.g. [H, Example 3.3]), we first fix n and recall that there is (using the Pauli or Clifford matrices; see also [BR, p. 15]) a realization of $(C_1, \dots, C_n)$ satisfying the CAR in the space of (say) $2^n \times 2^n$ matrices. Then observe that by (0.8) we have

$$\left\|\sum_1^n C_k \otimes \overline{C}_k\right\| = \left\|\sum u(z^{K_k}) \otimes \overline{C}_k\right\| \le \|u\|_{cb} \sup_{|z|=1} \left\|\sum_1^n z^{K_k} \otimes \overline{C}_k\right\| = \|u\|_{cb}\sqrt{n}.$$

However, since we can assume $(C_1, \dots, C_n)$ to be realized in the space of $2^n \times 2^n$ matrices, whose unit matrix is denoted by I, we have

$$\left\| \sum\nolimits_1^n C_k \otimes \overline{C}_k \right\| \geq \frac{\operatorname{tr}\left(\sum_1^n C_k C_k^*\right)}{\operatorname{tr} I} = \frac{n}{2}.$$

(To check this, note that

$$\left\| \sum\nolimits_1^n C_k \otimes \overline{C}_k \right\| = \sup \left| \sum\nolimits_1^n \operatorname{tr}\, (C_k X C_k^* Y) \right|$$

where the supremum runs over all $2^n \times 2^n$ matrices X, Y with Hilbert-Schmidt norm 1, and take $X = Y = I(\operatorname{tr} I)^{-1/2}$.) Thus we conclude that $\|u\|_{cb} \geq \sqrt{n}/2$ for all n, hence u is *not* completely bounded. Therefore, if Γ satisfies (1.7), the mapping $F \mapsto \Gamma M_{F'}$ cannot be c.b., hence a fortiori $F \mapsto F(T_\Gamma)$ is not c.b. on A, which ensures by Paulsen's criterion (easy direction) [Pa2] that T_Γ is not similar to a contraction. □

Remark 1.3. Let $T_\Gamma = \begin{pmatrix} {}^tS & \Gamma \\ 0 & S \end{pmatrix}$ be an operator as in the last corollary. Fix $\varepsilon > 0$, and consider

$$T_{\varepsilon\Gamma} = \begin{pmatrix} {}^tS & \varepsilon\Gamma \\ 0 & S \end{pmatrix}.$$

Then, by (1.4), we have for all polynomials P

$$\|P(T_{\varepsilon\Gamma})\| \leq (1 + \varepsilon C)\|P\|_\infty,$$

hence we can get an operator with polynomially bounded constant arbitrarily close to 1, but still not similar to a contraction.

Remark 1.4. In the preceding argument for Corollary 1.2, we implicitly use the ideas behind a joint result of V. Paulsen and the author (included in [Pa3, Th. 4.1]). The latter result shows in particular the following: let $m = (m(n))_{n\geq 0}$ be a scalar sequence and let $u_m\colon\ A \to B(H)$ be the operator defined by $u_m(F) = \sum_{n\geq 0} \hat{F}(n) C_n m(n)$, with (C_n) satisfying the CAR. Then u_m is c.b. iff $\sum |m(n)|^2 < \infty$.

Let $(m(n))_{n\geq 0}$ be a scalar sequence such that

$$\sup_{n\geq 0} \sum_{2^{n-1} < k \leq 2^n} |m(k)|^2 < \infty. \tag{1.8}$$

It is well known that this condition characterizes the Fourier multipliers from H^1 to H^2 on the circle (see e.g. [D, p. 103]).

With essentially the same arguments as for Theorem 1.1, we can prove the more general

Theorem 1.5. *Let (C_n) be as in Theorem 1.1. Assume that $(m(n))_{n\geq 0}$ satisfies (1.8) as above. Let $u : A \to B(H)$ be the mapping defined by*

$$u(F) = \sum_{n\geq 0} \hat{F}(n) m(n) C_n. \tag{1.9}$$

Then the conclusion of Theorem 1.1 still holds.

See Remark 2.4 below for an indication of proof.

Remark 1.6. Let $\varphi \in L^\infty(B(H))$ be the function associated to (1.9) by Theorem 1.5. By (0.6) and (1.7) the Fourier transform of φ restricted to nonpositive integers is determined, and we have for all $n \geq 1$

$$C_n m(n) = \int \varphi(e^{it}) n e^{i(n-1)t} dm(e^{it}).$$

Hence as a "symbol" in the sense of Nehari's theorem as in (0.6), we can take just as well

$$\varphi(e^{it}) = \sum_{n\geq 1} n^{-1} C_n m(n) e^{-i(n-1)t}.$$

Let $f = \sum_{k\geq 0} \hat{f}(k) z^k \in A$. Let $P_n(f) = \sum_{0\leq k\leq n} \hat{f}(k) z^k$. Let us denote by (e_n) the canonical basis in ℓ_2. Then, assuming (1.8), the key inequality (1.4) is equivalent to the following one, perhaps of independent interest: there is a constant C such that for any analytic function f in BMO

$$\left\| \sum_{n\geq 1} e_n n^{-1} m(n) e^{-i(n-1)t} P_{n-1}(f') \right\|_{\mathrm{BMO}(\ell_2)} \leq C \|f\|_{\mathrm{BMO}}, \tag{1.10}$$

where we have denoted by BMO (resp. $\mathrm{BMO}(\ell_2)$) the classical space of BMO functions on the circle with values in $\mathbb{C}$ (resp. ℓ_2).

Remark 1.7. Let us examine the converse to Theorem 1.5. Fix a sequence (C_n) for which there is a constant $\delta > 0$ such that

$$\forall \alpha = (\alpha_n) \in \ell_2 \qquad \delta(\sum |\alpha_n|^2)^{1/2} \leq \|\sum \alpha_n C_n\|. \tag{1.11}$$

Let $(m(n))_{n\geq 0}$ be any scalar sequence. Then if the conclusion of Theorem 1.1 holds for the mapping u defined in (1.9), we have necessarily (1.8). This follows from [AP]. Indeed, let $\varphi \in L^\infty(B(H))$ be a function associated to such a u as in Theorem 1.1, with associated Hankel operator satisfying (0.5). For any sequence (β_n) in the unit ball of ℓ_2, we have a linear form ξ in $B(H)^*$ with $\|\xi\|_{B(H)^*} \leq 1/\delta$ such that $\xi(C_n) = \beta_n$, for all n. Clearly, the function $\xi(\varphi) \in L^\infty$ defines a Hankel operator $\Gamma_{\xi(\varphi)}$ still satisfying (0.5). By [AP], this implies that $\xi(\varphi')$ is in BMO, or equivalently that $\sum_{n\geq 0} \beta_n m(n) z^n$ is in BMO. Since this holds for all (β_n) in ℓ_2, this means, after transposition, that $(m(n))$ is a (bounded) multiplier from H^1 to $\ell_2 \approx H^2$. As already mentioned, this is equivalent to (1.8) (see e.g. [D, p. 103]). In particular, a sequence (K_n) satisfies the conclusion of Theorem 1.1 (for any (C_n) satisfying (1.5) and (1.11)) if and only if it is the union of finitely many sequences which are subsequences of a sequence verifying (1.6). Equivalently, iff it is a finite union of Hadamard lacunary sequences.

Of course, our results can also be described in terms of Hankel and Toeplitz matrices.

Let G be the Hankel matrix (with operator entries) defined by setting for all $i, j \geq 0$

$$G_{ij} = m(i+j+1)(i+j+1)^{-1} C_{i+j+1}. \tag{1.12}$$

For $f \in A$, let us denote by $T(f)$ the Toeplitz matrix (with scalar entries) defined by setting for all $i, j \geq 0$

$$T(f)_{ij} = \hat{f}(i-j).$$

Then Theorem 1.5 can be reformulated as follows: assuming (1.8), there is a constant C such that for all polynomials F in A we have

$$\|GT(F')\| \le C\|F\|_A. \tag{1.13}$$

Note that, actually, the BMO norm of F can be substituted to $\|F\|_A$ in (1.13); see Remark 2.3 below. To verify (1.12), we simply apply (1.7) to $F(z) = z^n$, to show that the matrix coefficients of Γ, with respect to the decomposition $H^2(H) = \bigoplus_{i\ge 0} z^i H$ are given by (1.12). Then (1.13) reduces to (1.4).

§2. Proof of Theorem 1.1

Let $(B_t)_{t>0}$ be the standard complex Brownian motion starting from the origin, and let

$$T_n = \inf\left\{t > 0 \mid |B_t| = 1 - \frac{1}{2^n}\right\}$$

be the usual stopping time. Let

$$r_n = 1 - \frac{1}{2^n}.$$

We denote by $T = T_\infty$ the exit time from the unit disc and by $(\mathcal{A}_t)_{t>0}$ the Brownian filtration. Let $\mathcal{F}_n = \mathcal{A}_{T_n}$ for all $n \ge 0$. We denote simply by $\mathbf{E}_n$ the conditional expectation with respect to $\mathcal{F}_n$. Note that $B_{T_{n-1}}$ is uniformly distributed on $\{z \mid |z| = r_{n-1}\}$. Moreover for any fixed z with $|z| = r_{n-1}$ the distribution of B_{T_n} conditional to $B_{T_{n-1}} = z$, is given by a homothetic of the Poisson kernel, which we now recall how to compute.

Consider first the disc at the origin. For any analytic function F in A (considered as extended analytically inside D), we can write

$$F(0) = \int_{\partial D} F(\xi)\, dm(\xi) \quad \text{and} \quad F'(0) = \int_{\partial D} \bar{\xi}[F(\xi) - F(0)]\, dm(\xi),$$

where $dm(\xi)$ is normalized Lebesgue measure on ∂D. The Möbius transformations are usually denoted by $\zeta \mapsto \varphi_z(\zeta)$, for any z in D. For simplicity, we set

$$\forall z \in D,\ \forall \zeta \in \mathbf{T} \qquad \Phi(z,\zeta) = \varphi_z(\zeta) = \frac{\zeta + z}{1 + \bar{z}\zeta}.$$

Now consider $0 \le r < s < 1$ and z with $|z| = r$. After a suitable change of variables, we can rewrite the preceding formula as follows:

$$F'(z)s\left(1 - \frac{|z|^2}{s^2}\right) = \int \bar{\xi}[F(s\Phi(z/s,\xi)) - F(z)]\, dm(\xi). \tag{2.1}$$

Let Z_n be the unimodular complex-valued random variable defined by

$$B_{T_n} = r_n \Phi(r_n^{-1} B_{T_{n-1}}, Z_n).$$

Let F be a polynomial. Let us set for all $n \ge 1$

$$dF_n = F(B_{T_n}) - F(B_{T_{n-1}}).$$

Clearly, (F_n) is a martingale, and it is easy to verify that (2.1) implies

$$F'(B_{T_{n-1}})\frac{r_n^2 - r_{n-1}^2}{r_n} = \mathbf{E}[\bar{Z}_n(F(B_{T_n}) - F(B_{T_{n-1}}))|B_{T_{n-1}}]. \tag{2.2}$$

Observe that, given F, G in A, the function $w \mapsto (F(w) - F(z))(G(w) - G(z))$ has a derivative which vanishes at z. Hence (2.2) (applied conditionally to $B_{T_{n-1}} = z$) implies that (almost surely)

$$0 = \mathbf{E}[\bar{Z}_n(F(B_{T_n}) - F(B_{T_{n-1}}))(G(B_{T_n}) - G(B_{T_{n-1}})) \mid B_{T_{n-1}}]. \tag{2.3}$$

A similar vanishing formula holds for triple (or k fold) products of the form

$$w \to (F_1(w) - F_1(B_{T_{n-1}}))(F_2(w) - F_2(B_{T_{n-1}}))(F_3(w) - F_3(B_{T_{n-1}})).$$

Step 1. We will show that there exists a sequence of $\mathcal{F}_n$-adapted bounded complex (random) variables η_n $(n \geq 0)$ such that for any polynomial $F = \sum_{n \geq 0} \widehat{F}(n) z^n$ in A, we have for all $n \geq 1$:

(i) $$\widehat{F}(K_n) = \mathbf{E}(\bar{Z}_n \eta_{n-1}(F(B_{T_n}) - F(B_{T_{n-1}}))),$$

(ii) $\|\eta_{n-1}\|_\infty \leq C'$, where C' is some numerical constant (independent of n).

The variable η_{n-1} will actually depend only on $B_{T_{n-1}}$.

We clearly have for all $r < 1$ and for all $m \geq 0$

$$r^{m-1} m \widehat{F}(m) = \int e^{-i(m-1)t} F'(re^{it})\, dm(e^{it}),$$

where $F = \sum_{n \geq 0} \widehat{F}(n) z^n$ is an arbitrary function in A. Let $\xi_{n-1} = B_{T_{n-1}}/r_{n-1}$ so that $|\xi_{n-1}| = 1$. Taking $m = K_n$ and $r = r_{n-1}$ we obtain

$$(r_{n-1})^{K_n - 1} K_n \hat{F}(K_n) = \int \bar{\xi}^{K_n - 1} F'(r_{n-1}\xi) dm(\xi) = \mathbf{E}(\bar{\xi}_{n-1}^{K_n - 1} F'(B_{T_{n-1}})).$$

By (2.2) this implies for all $n > 1$

$$\hat{F}(K_n) = \mathbf{E}\left[\eta_{n-1} \bar{Z}_n (F(B_{T_n}) - F(B_{T_{n-1}}))\right] \tag{2.4}$$

with

$$\eta_{n-1} = \bar{\xi}_{n-1}^{K_n - 1} \cdot r_n (r_n^2 - r_{n-1}^2)^{-1} \cdot (K_n (r_{n-1})^{K_n - 1})^{-1}.$$

The first terms being irrelevant, we may as well assume that $K_1 = 1$, so that taking $\eta_0 = 2$ identically, we still have (2.4) for $n = 1$. Whence (i). Since we assume $2^{n-1} < K_n \leq 2^n$ for $n > 1$, we clearly have (ii) for some numerical constant C'. This completes Step 1.

Step 2. Let F again be a polynomial. We now consider the Hankelian bilinear forms $\Gamma\colon\ H^2(H) \times H^2(H) \to \mathbb{C}$ and $\mathcal{D}(F)\colon\ H^2(H) \times H^2(H) \to \mathbb{C}$ given, for all g, h in $H^\infty(H)$, by

$$\Gamma(g, h) = \mathbf{E}\left(\sum_{n \geq 1} \bar{Z}_n \eta_{n-1} (B_{T_n} - B_{T_{n-1}})[C_n g(B_T), h(B_T)]\right), \tag{2.5}$$

$$\mathcal{D}(F)(g, h) = \mathbf{E}\left(\sum_{n \geq 1} \bar{Z}_n\, \eta_{n-1}\, dF_n [C_n g, h]\right), \tag{2.6}$$

where again on the right we use a *bilinear* pairing $(x, y) \mapsto [x, y]$ on H associated to a (fixed) *linear* isometry $J\colon\ H \to H^*$.

Concerning the convergence of the series in (2.5) and (2.6), note that $\|F\|_2^2 = \sum_{n \geq 0} \mathbf{E}|dF_n|^2$, and we take the precaution to assume that g, h belong to $H^\infty(H)$ (which is dense in $H^2(H)$). Therefore, by (1.5) the series $\sum_{n \geq 1} \bar{Z}_n\, \eta_{n-1}\, dF_n [C_n g, h]$

is convergent in $L^2(\Omega, P)$, so that (2.5) and (2.6) are well defined. But actually, we will show (see below) that (2.5) and (2.6) define bounded operators from $H^2(H)$ to its dual, so they will eventually make sense for all g, h in $H^2(H)$.

Note that Γ and $\mathcal{D}(F)$ are clearly Hankelian. The main goal of this step is to show that for any polynomial F and all g, h in $H^\infty(H)$ we have

$$\Gamma(F'g, h) = \mathcal{D}(F)(g, h), \tag{2.7}$$

or equivalently that Γ coincides with $\mathcal{D}(F)$ when F is taken to be the identity map (*i.e.* $F(z) = z$ for all z). We denote $F_n = F(B_{T_n}), dF_n = F_n - F_{n-1}$, and similarly $F'_n = F'(B_{T_n})$, $g_n = g(B_{T_n}), \dots$. We also set

$$dz_n = B_{T_n} - B_{T_{n-1}}.$$

It will be convenient to start by a verification of the following

Claim. For all F in A and all g, h in $H^2(H)$ we have

$$\mathbf{E}(\bar{Z}_n\, \eta_{n-1}\; dF_n[C_n g(B_T), h(B_T)]) = \mathbf{E}(\bar{Z}_n\, \eta_{n-1}\; dF_n[C_n g_{n-1}, h_{n-1}]). \tag{2.8}$$

Indeed, since $\bar{Z}_n\, \eta_{n-1}\; dF_n$ is $\mathcal{F}_n$-measurable and since $k \mapsto [C_n g_k, h_k]$ is a martingale, we have

$$\mathbf{E}(\bar{Z}_n\, \eta_{n-1}\; dF_n[C_n g(B_T), h(B_T)]) = \mathbf{E}(\bar{Z}_n\, \eta_{n-1}\; dF_n[C_n g_n, h_n]). \tag{2.9}$$

Then using the identity

$$[C_n g_n, h_n] = [C_n g_{n-1}, h_{n-1}] + [C_n dg_n, h_{n-1}] + [C_n g_{n-1}, dh_n] + [C_n dg_n, dh_n],$$

and developing the right side of (2.9), we can easily verify (2.8) by repeatedly invoking (2.3) (and the observation after (2.3)).

Let us denote $\hat{\eta}_{n-1} = \eta_{n-1}\frac{r_n^2 - r_{n-1}^2}{r_n}$. Then, using (2.2) we deduce from (2.8) that

$$\mathbf{E}(\bar{Z}_n\, \eta_{n-1}\; dF_n[C_n g(B_T), h(B_T)]) = \mathbf{E}(\hat{\eta}_{n-1} F'_{n-1}[C_n g_{n-1}, h_{n-1}]). \tag{2.10}$$

We now return to the proof of the announced identity (2.7). By (2.8) applied first with dz_n in place of dF_n and $F'g$ in place of g we have (denoting $F'_\infty = F'(B_T), g_\infty = g(B_T), \dots$)

$$\mathbf{E}(\bar{Z}_n\, \eta_{n-1}\; dz_n[C_n F'_\infty g_\infty, h_\infty]) = \mathbf{E}(\bar{Z}_n\, \eta_{n-1}\; dz_n[C_n F'_{n-1} g_{n-1}, h_{n-1}]).$$

Now, by (2.10) (applied with dz_n instead of dF_n and $F'g$ instead of g) the last line is

$$= \mathbf{E}(\hat{\eta}_{n-1}[C_n F'_{n-1} g_{n-1}, h_{n-1}]) = \mathbf{E}(\hat{\eta}_{n-1} F'_{n-1}[C_n g_{n-1}, h_{n-1}])$$

which, by (2.10), this time applied properly to dF_n, is equal to

$$= \mathbf{E}(\bar{Z}_n\, \eta_{n-1}\; dF_n[C_n g_\infty, h_\infty]).$$

Summing over n, we obtain (2.7). This completes Step 2.

Step 3. We now turn to the boundedness of $F \mapsto \Gamma M_{F'}$ on A, *i.e.* to (1.4).

Recall that the predual of $B(H)$ can be naturally identified with the projective tensor product $H\widehat{\otimes}H$. Therefore, if g, h are in the unit ball of $H^2(H)$, then $g \otimes h$ can be viewed as an element of the unit ball of $H^1(H\widehat{\otimes}H)$. Thus, we can use the easy direction of the "vectorial Nehari Theorem" (see e.g. [Ni]) to argue that

$$\|\mathcal{D}(F)\| \le \left\| \sum \bar{Z}_n\, \eta_{n-1}\, dF_n\, C_n \right\|_{H^1(H\widehat{\otimes}H)^*}. \tag{2.11}$$

Now let $v\colon\ \ell_2 \to B(H)$ be the mapping defined by $v(e_n) = C_n$, which by (1.5) satisfies $\|v\| \le 1$. Then (2.11) can be estimated through ℓ_2:

$$\|\mathcal{D}(F)\| \le \left\| \sum \bar{Z}_n \, \eta_{n-1} \, dF_n \, e_n \right\|_{H^1(\ell_2)^*}. \tag{2.12}$$

On the other hand, let $H^1_{\mathrm{M}}(\ell_2)$ be the space comprised of all $(\mathcal{F}_n)$-adapted ℓ_2-valued martingales $f = (f_n)_{n\ge 0}$ such that $f^* = \sup \|f_n\|$ is integrable. We equip $H^1_{\mathrm{M}}(\ell_2)$ with the norm $f \mapsto \mathbf{E}(f^*)$. Recall that the natural inclusion $H^1(\ell_2) \to H^1_{\mathrm{M}}(\ell_2)$ taking f to the associated martingale (f_n) is bounded. Indeed, this is entirely classical: let $f \in H^1(\ell_2)$; by Jensen's inequality the sequence $(\|f_n\|^p)$ is a submartingale for any $p > 0$; then choosing $p = 1/2$, the desired result follows from Doob's maximal inequality in L_2 (cf. e.g. [Du, p. 307]). Thus, $f \mapsto \mathbf{E}(f^*)$ is an equivalent norm on $H^1(\ell_2)$. By Burgess Davis's well-known inequality ([Bu], [Du], [Ga]), the latter norm is equivalent to the norm $f \mapsto \mathbf{E}\left(\sum_{n\ge 0} \|df_n\|^2\right)^{1/2}$. Moreover, by Fefferman's theorem (see [FS] and Remark 2.2 below), the dual of $H^1_{\mathrm{M}}(\ell_2)$ can be identified with the space $\mathrm{BMO}_{\mathrm{M}}(\ell_2)$ defined as the space of all ℓ_2-valued martingales $y = (y_n)_{n\ge 0}$ such that for all $n \ge 1$

$$\sup_{n\ge 1} \left\| \mathbf{E}_n(\|y - y_{n-1}\|^2) \right\|_\infty < \infty,$$

equipped with the "norm" (modulo constants)

$$|||y||| = \left(\sup_{n\ge 1} \|\mathbf{E}_n(\|y - y_{n-1}\|^2)\|_\infty \right)^{1/2}.$$

It follows from these remarks that there are numerical constants K' and K such that if we denote by (e_n) the canonical basis of ℓ_2, and if we set $y = \sum \bar{Z}_n \, \eta_{n-1} \, dF_n \, e_n$, then we have

$$\|y\|_{H^1(\ell_2)^*} \le K' \|y\|_{H^1_{\mathrm{M}}(\ell_2)^*} \le K[|||y||| + (\mathbf{E}\|y\|^2)^{1/2}]. \tag{2.13}$$

But by a simple computation we find (recalling (ii))

$$\begin{aligned}
\mathbf{E}_n \|y - y_{n-1}\|^2 &= \mathbf{E}_n \left\| \sum_{k\ge n} e_k(\bar{Z}_k \, \eta_{k-1} \, dF_k - \mathbf{E}_{n-1}(\bar{Z}_k \, \eta_{k-1} \, dF_k)) \right\|^2 \\
&\le 2{C'}^2 \mathbf{E}_n \left(\sum_{k\ge n} |dF_k|^2 + \mathbf{E}_{n-1} \sum_{k\ge n} |dF_k|^2 \right) \\
&\le 2{C'}^2 \left(\mathbf{E}_n |F - F_{n-1}|^2 + \mathbf{E}_{n-1} |F - F_{n-1}|^2 \right) \\
&\le 4{C'}^2 |||F|||^2 \le 16{C'}^2 \|F\|^2_\infty.
\end{aligned}$$

Thus we obtain

$$|||y||| \le 2C' |||F||| \le 4C' \|F\|_\infty; \tag{2.14}$$

similarly we have $(\mathbf{E}\|y\|^2)^{1/2} \le C'\|F\|_2$ and by (2.12) and (2.13) we conclude that $\|\mathcal{D}(F)\| \le 5C'K\|F\|_\infty$. By (2.7), $\|\mathcal{D}(F)\| = \|\Gamma M_{F'}\|$, hence we obtain (1.5). On the other hand, by (i) and (2.7), we have (1.7). This ends the proof of Theorem 1.1. □

Recapitulating, we have actually proved the following statement.

Theorem 2.1. *Let* (η_n) *be any adapted sequence satisfying* (ii). *Let*

$$\mathcal{D}^\eta : \ H^\infty \to B(H^2(H), H^2(H)^*)$$

be the mapping defined by (2.6). *Then* $\mathcal{D}^\eta$ *is bounded, satisfies* (1.2) *and its range is formed of Hankel operators.*

Remark 2.2. The fact that the duality H^1, BMO for martingales (see the classical reference [FS]) extends to the Hilbert-space-valued case is a well known fact. The proof given on the first pages of Garsia's book [Ga] extends almost verbatim. See also [Du], [Pet].

Remark 2.3. Let F be an analytic function on D with boundary values in the classical space of (scalar-valued) BMO over the circle. Then it is well known (see [Pet], [Du]) that the associated martingale (F_n) is in the (scalar-valued) space $\mathrm{BMO}_{\mathrm{M}}$ considered above and (with a suitable choice of norms) we have $\|(F_n)\|_{\mathrm{BMO}_{\mathrm{M}}} \le \|F\|_{\mathrm{BMO}}$. Therefore, we have actually proved Theorem 1.1 with (1.4) replaced by the following stronger inequality:

$$\exists C \ \forall F \text{ polynomial } \|\Gamma M_{F'}\| \le C\|F\|_{\mathrm{BMO}}, \tag{1.4$'$}$$

where (say)

$$\|F\|_{\mathrm{BMO}} = \sup_{z\in D} \left(\int |F(\Phi(z,\xi)) - F(z)|^2 \, dm(\xi) \right)^{1/2}.$$

Remark 2.4. To prove Theorem 1.5, let us define, for all k with $2^{n-1} < k \le 2^n$,

$$\eta_{n-1,k} = \bar{\xi}^k_{n-1} \, r_n \, (r_n^2 - r_{n-1}^2)^{-1} \, (k(r_{n-1})^{k-1})^{-1}.$$

Note that there is a constant C' such that $\|\eta_{n-1,k}\|_\infty \le C'$ for all n and all k with $2^{n-1} < k \le 2^n$. Moreover, if $2^{n-1} < k \le 2^n$, we have, for all F in A,

$$\hat{F}(k) = \mathbf{E}\left[\eta_{n-1,k} \bar{Z}_n dF_n\right].$$

We can now introduce the following modified version of $\mathcal{D}$: for all g, h in $H^2(H)$

$$\widetilde{\mathcal{D}}(F)(g,h) = \mathbf{E}\left(\sum_{n\ge 1} \bar{Z}_n \, dF_n \sum_{2^{n-1}<k\le 2^n} m(k)\eta_{n-1,k} \, [C_k g, h] \right).$$

Then, the same argument as above yields Theorem 1.5.

Remark 2.5. There is a well-known "dictionary" between the classical theory of H^1 and BMO for the disc and the corresponding theory for Brownian martingales. Although I did not see it, it seemed very likely that there should be a way to prove that the operator Γ appearing in Theorem 1.1 satisfies (1.4) using only the classical theory, and effectively this has recently been done by S. Kislyakov (personal communication).

§3. Operator space interpretations and finite-dimensional estimates

In this section, we give various consequences of the previous results for operator spaces or (nonselfadjoint) operator algebras. In the theory of operator spaces, the relevant morphisms are *completely bounded maps* and the corresponding isomorphisms are called *complete isomorphisms*. See e.g. [BP], [ER], [Pa1], [Pi4].

We start with an operator space theoretic reformulation of Theorem 1.1. Fix a number $c > 1$. Let $[A]_c$ be the disc algebra equipped with the operator space (actually operator algebra) structure induced by the embedding

$$A \subset \bigoplus_{\pi \in I(c)} B(H_\pi)$$

taking a to $\bigoplus_{\pi \in I(c)} \pi(a)$, where $I(c)$ is the class of all unital homomorphisms $\pi\colon\ A \to B(H_\pi)$ with $\|\pi\| \leq c$. Note that, as Banach algebras, $[A]_c$ and A are the same with equivalent norms, namely we have $\|a\|_A \leq \|a\|_{[A]_c} \leq c\|a\|_A$ for all a in A. However, as operator spaces, they are quite different. As we will see this is a consequence of Theorem 1.1 (and, by Theorem 0.1, it is equivalent to Corollary 1.2).

Indeed, consider the operator space $\max(\ell_2)$ in the sense of [BP]. The latter can be defined as follows. Let $\mathcal{I}$ be the class of all linear maps $v : \ell_2 \to B(H_v)$ with $\|v\| \leq 1$. Let $J : \ell_2 \to \bigoplus_{v \in \mathcal{I}} B(H_v)$ be the isometric embedding defined by $J(x) = \bigoplus_{v \in \mathcal{I}} v(x)$. Then the operator space $\max(\ell_2)$ can be defined as the range of J. Clearly, by Theorem 1.1, the mapping

$$P\colon\ [A]_c \to \max(\ell_2)$$

defined by $PF = (\hat{F}(K_n))_{n \geq 1}$ is completely bounded (see the proof of Theorem 3.1 below for details). Moreover, P is surjective (cf. e.g. [Pi1, p. 69]).

More generally, as an immediate consequence of Theorem 1.5, we have

Theorem 3.1. *Let $c > 1$. Let $(m(n))_{n \geq 0}$ be any scalar sequence. If $(m(n))_{n \geq 0}$ satisfies* (1.8)*, then the mapping $\mathcal{M} : [A]_c \to \max(\ell_2)$ defined* (*for $F \in A$*) *by $\mathcal{M}(F) = (m(n)\hat{F}(n))$ is completely bounded. However, if we view $\mathcal{M}$ as acting from A to $\max(\ell_2)$, then it is completely bounded iff $\sum |m(n)|^2 < \infty$.*

Proof. It suffices to show that for some constant C, we have for any $v \in \mathcal{I}$, $\|v\mathcal{M}\|_{cb([A]_c, B(H_v))} \leq C$. Let $C_n = v(e_n)$. Note that (1.5) holds, so that Theorem 1.1 and Remark 1.3 imply that there is a homomorphism $\pi \in I(c)$ and operators V_1, V_2 so that we can write, for all F in A,

$$v\mathcal{M}(F) = V_1 \pi(F) V_2.$$

Moreover, we can achieve this with $\|V_1\|\|V_2\| \leq K_c$ for some constant K_c depending only on c. This implies $\|v\mathcal{M}\|_{cb([A]_c, B(H_v))} \leq \|V_1\|\|V_2\| \leq K_c$, hence $\|\mathcal{M}\|_{cb([A]_c, \max(\ell_2))} \leq K_c$. This proves the first part. The second follows from a joint result of V. Paulsen and the author (cf. [Pa3, Th. 4.1]), due independently to M. Junge. $\square$

Corollary 3.2. *For any $c > 1$ we have a complete isomorphism*

$$\frac{[A]_c}{\ker P} \simeq \max(\ell_2).$$

Proof. Indeed, $[A]_c / \ker(P) \approx \ell_2$ as Banach spaces, so that the complete boundedness of the mapping $\max(\ell_2) \to [A]_c / \ker(P)$ is obvious by maximality. For the inverse mapping, the complete boundedness follows immediately from the preceding statement. $\square$

Remark. The preceding two statements remain valid with H^∞ in place of A.

Corollary 3.3. *When $c > 1$, the operator space $[A]_c$ is not exact* (*in the sense of e.g.* [JP]).

Proof. Indeed, by [BP] we know that $\max(\ell_2)^*$ coincides with an operator subspace of a commutative C^*-algebra (namely with $\min(\ell_2)$). Hence, by [JP, Corollary 1.7], if $[A]_c$ was exact, the mapping ${}^tP : \max(\ell_2)^* \to ([A]_c)^*$ would be 2-summing, which is absurd since it is an isomorphism on ℓ_2. $\square$

Remark. Let P_c^d be the subspace of $[A]_c$ spanned by the polynomials of degree at most d. With the notation of [JP], the preceding argument shows that for some absolute constant $\delta > 0$ we have, for all $c > 1$ and all d,

$$\delta(c-1)\sqrt{\ln(d)} \leq d_{SK}(P_c^d).$$

In [Bo2], Bourgain proves an upper estimate for polynomially bounded $n \times n$ matrices, which we will now be able to bound from below. But first we consider another parameter (related to Theorem 0.1) which we will estimate rather precisely. For any n, we will denote by $f(c,n)$ the norm of the identity mapping from $M_n(A)$ to $M_n([A]_c)$. Equivalently, we have

$$f(c,n) = \sup \|(P_{ij}(T))\|_{M_n(B(H))},$$

where the supremum runs over all polynomially bounded operators T with $\|\rho_T\| \leq c$ and over all $n \times n$ matrices (P_{ij}) with polynomial entries such that

$$\sup_{z \in \mathbf{T}} \|(P_{ij}(z))\|_{M_n} \leq 1.$$

Theorem 3.4. *There is an absolute constant $K > 0$ such that for all $c > 1$ and all integers n we have*

$$K^{-1}(c-1)\sqrt{n} \leq f(c,n) \leq Kc\sqrt{n}. \tag{3.1}$$

Proof. Let (a_{ij}) be an $n \times n$ matrix with entries in $B(H)$. It is easy to check that

$$\|(a_{ij})\|_{B(\ell_2^n(H))} \leq \sqrt{n} \sup_i \left\| \left(\sum_j a_{ij} a_{ij}^* \right)^{1/2} \right\|_{B(H)}. \tag{3.2}$$

Let (P_{ij}) be an $n \times n$ matrix of polynomials as above such that $\sup_{z \in \mathbf{T}} \|(P_{ij}(z))\|_{M_n} \leq 1$. Then, for each i and any z in $\mathbf{T}$, we have $\sum_j |P_{ij}(z)|^2 \leq 1$. By a result due to Bourgain (see (21) in [Bo2], this result uses [Bo1, Th. 2.2]) there is a numerical constant K_1 such that for each i

$$\left\| \left(\sum_j P_{ij}(T) P_{ij}(T)^* \right)^{1/2} \right\| \leq K_1 c.$$

Hence by (3.2) we conclude that

$$f(c,n) \leq K_1 c\sqrt{n}.$$

To prove the first inequality in (3.1), we will use the following known fact on random matrices (the idea to use this in this context comes from M. Junge's unpublished independent proof of the already mentioned result from [Pa3]): there is a numerical constant K_2 such that for each n there is an n-tuple of unitary matrices $U_1, \ldots, U_n$ in M_n such that

$$\text{(3.3)} \qquad \forall\ (\alpha_i) \in \ell_2^n \qquad \left\| \sum \alpha_i U_i \right\|_{M_n} \le K_2 \left(\sum_1^n |\alpha_i|^2 \right)^{1/2}.$$

For a proof see e.g. [TJ, p. 323]. Let $C_i = (K_2)^{-1} U_i$ for $i = 1, 2, \ldots, n$ and $C_i = 0$ for $i > n$. Then (1.5) is satisfied, but on the other hand, since the U_i's are (finite-dimensional) unitaries, we have $\|\sum_1^n U_i \otimes \overline{U}_i\| = n$, hence $\|\sum_1^n C_i \otimes \overline{C}_i\| = (K_2)^{-2} n$.

Let $u : A \to M_n$ be the mapping associated to (C_i) as in Theorem 1.1. Then by Theorem 1.1 and Remark 1.3, there is, for some numerical constant K_3, a polynomially bounded operator T with $\|\rho_T\| \le c$ and operators V_1, V_2 with $\|V_1\|\|V_2\| \le K_3(c-1)^{-1}$ such that, for any polynomial P, we have

$$u(P) = V_1 P(T) V_2.$$

On the other hand, arguing as in the proof of Corollary 1.2 we find that there is a matrix (P_{ij}) with $\sup_{z \in \mathbf{T}} \|(P_{ij}(z))\|_{M_n} \le 1$, such that

$$\|(I \otimes u)(P_{ij})\|_{M_n(M_n)} \ge K_2^{-2} \sqrt{n}.$$

Hence this implies

$$\|(P_{ij}(T))\|_{B(\ell_2^n(H))} \ge K_3^{-1} K_2^{-2} \sqrt{n}(c-1),$$

which yields the left side of (3.1) for a suitable constant K. □

In [Bo2], Bourgain proves that if $T \in M_N$ satisfies $\|\rho_T\| \le c$, there is an invertible $S \in M_N$ for which $\|S^{-1} T S\| \le 1$ and

$$\|S^{-1}\|\|S\| \le K c^4 \ln(N+1)$$

for some absolute constant K (independent of N or c). By Theorem 0.1, this is equivalent to

$$\|\rho_T\|_{cb} \le K c^4 \ln(N+1).$$

It is unclear how sharp this estimate is asymptotically. However, as a simple direct consequence of Theorem 1.1, we have

Theorem 3.5. *There is a constant $\delta > 0$ with the following property: for any N and $c > 1$, there is $T \in M_N$ polynomially bounded with $\|\rho_T\| \le c$ such that any S invertible in M_N with $\|S^{-1} T S\| \le 1$ must satisfy*

$$\delta(c-1)\sqrt{\ln(N+1)} \le \|S^{-1}\|\|S\|.$$

Proof. Fix an integer n and let C_i be as in the preceding proof with $\dim(H) = n$ and $C_i = 0$ for all $i > n$. Let $\Gamma = \Gamma_\varphi$ be as in Theorem 1.1. For any polynomial F, we again let $\mathcal{D}(F) = \Gamma_\varphi M_{F'}$. Note that by Remark 1.6, we have $\hat{\varphi}(k) = 0$ for all $k \le -2^n$.

Observe that $\mathcal{D}(F) = \Gamma_\varphi M_{F'} = \Gamma_{\varphi F'}$, so that $\mathcal{D}(F) = 0$ if $F \in z^{2^n+1} A$.

Let us introduce the notation $K = H^2(H)/z^{2^n+1} H^2(H)$ (or equivalently $K = H^2(H) \ominus z^{2^n+1} H^2(H)$) and for any F in A let $Q_F : K \to K$ be the compression of M_F to K. The observation immediately preceding implies further that, for any polynomial F, $\mathcal{D}(F)$ vanishes on both $H^2(H) \times z^{2^n+1} H^2(H)$ and $z^{2^n+1} H^2(H) \times H^2(H)$. Therefore, $\mathcal{D}(F)$ defines unambiguously a linear map $\Delta(F) : K \to K^*$

satisfying $\|\Delta(F)\| \le \|\mathcal{D}(F)\|$, hence (by Theorem 1.1) we have for any polynomial F

$$\|\Delta(F)\| \le C\|F\|_\infty.$$

Finally, let $\mathcal{H} = K^* \oplus K$, let $\varepsilon = C^{-1}(c-1)$, and let $r(F) : \mathcal{H} \to \mathcal{H}$ be defined by

$$r(F) = \begin{pmatrix} {}^tQ_F & \varepsilon\Delta(F) \\ 0 & Q_F \end{pmatrix}.$$

Then, it is easy to check that $F \to r(F)$ is a bounded homomorphism with $\|r\| \le c$ for which there are contractions V_1 and V_2 such that $V_1 r(z^{2^k})V_2 = \varepsilon C_k$ for all $k = 1, 2, \ldots, n$. By the same argument as in the preceding proof, we know that this implies

$$\|r\|_{cb} \ge \varepsilon(K_2)^{-2}\sqrt{n}. \tag{3.4}$$

But on the other hand we have $\dim(\mathcal{H}) = 2\dim(K) = 2(2^n+1)\dim(H) = 2(2^n+1)n$ so that $n \approx \ln(\dim(\mathcal{H}))$, which shows that (3.4) yields the announced result, modulo Paulsen's criterion (cf. Theorem 0.1). □

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Note added on October 3, 1996

After seeing a preliminary version of this paper, K. Davidson and V. Paulsen have found a different and simpler proof of the polynomial boundedness of our examples (*i.e.* of the inequality (1.4) for the vectorial Hankel operators appearing in Theorem 1.1 or Theorem 1.5). The latter proof appears in their preprint entitled "Polynomially bounded operators". Their key idea is the observation that, by the vectorial Nehari theorem ([Pag]), it actually suffices to prove (1.4) in the special case when $C_n = e_{n1}$ for all n. (Here e_{ij} is the infinite matrix with 1 as its (i, j)-coefficient and zero elsewhere.) Then, with this special choice of (C_n), it turns out to be quite easy to show that the associated operators T_Γ are similar to contractions, and a fortiori are polynomially bounded. But, as we already mentioned, by the vectorial Nehari theorem, this implies the polynomial boundedness of T_Γ for *any* sequence (C_n) satisfying (1.5), whence Theorem 1.1.

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Abstract. Let $\varepsilon > 0$. We prove that there exists an operator $T_\varepsilon : \ell_2 \to \ell_2$ such that for any polynomial P we have $\|P(T_\varepsilon)\| \le (1+\varepsilon)\|P\|_\infty$, but T_ε is not similar to a contraction, *i.e.* there does not exist an invertible operator $S:\ \ell_2 \to \ell_2$ such that $\|S^{-1}T_\varepsilon S\| \le 1$. This answers negatively a question attributed to Halmos after his well-known 1970 paper ("Ten problems in Hilbert space"). We also give some related finite-dimensional estimates.

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