

INFINITESIMAL PRESENTATIONS OF THE TORELLI GROUPS

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1. INTRODUCTION

By a theorem of Malcev [33], every torsion free nilpotent group can be imbedded canonically as a discrete, cocompact subgroup of a real nilpotent Lie group. One can therefore associate to a finitely generated group π a tower of nilpotent Lie groups

$$(1) \quad \cdots \rightarrow G_3 \rightarrow G_2 \rightarrow G_1 = H_1(\pi, \mathbb{R})$$

by taking G_k to be the nilpotent Lie group associated to the maximal torsion free quotient of π of length k . Since each nilpotent Lie group is simply connected, the

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tower (1) is determined by the corresponding tower

$$\cdots \rightarrow \mathfrak{g}_3 \rightarrow \mathfrak{g}_2 \rightarrow \mathfrak{g}_1 = H_1(\pi, \mathbb{R})$$

of nilpotent Lie algebras. The inverse limit \mathfrak{g} of this tower is a pronilpotent Lie algebra, called the *Malcev Lie algebra associated to π* . This Lie algebra has the property that the graded Lie algebra $\text{Gr } \mathfrak{g}$ associated with its lower central series is isomorphic to $(\text{Gr } \pi) \otimes \mathbb{R}$, where $\text{Gr } \pi$ is the graded \mathbb{Z} -Lie algebra associated to the filtration of π by its lower central series. The Lie algebra \mathfrak{g} has a canonical \mathbb{Q} -form.

In this paper we give a presentation of the Malcev Lie algebra $\mathfrak{t}_{g,r}^n$ associated to the Torelli group $T_{g,r}^n$ for all $g \geq 6$. Since each Torelli group injects into its unipotent completion (at least when $n + r > 0$), the corresponding Malcev Lie algebra should contain significant information about the group.

Recall that the mapping class group $\Gamma_{g,r}^n$ is defined as follows. Fix a compact orientable surface S of genus g , together with $n + r$ distinct points

$$(2) \quad x_1, \dots, x_n; y_1, \dots, y_r$$

of S and r non-zero tangent vectors v_1, \dots, v_r , where v_j is tangent to S at y_j . The group $\Gamma_{g,r}^n$ is the group of isotopy classes of orientation preserving diffeomorphisms of S that fix each of the points (2) and each of the tangent vectors v_j .¹ The Torelli group $T_{g,r}^n$ is defined to be the kernel of the natural homomorphism

$$(3) \quad \Gamma_{g,r}^n \rightarrow \text{Aut } H_1(S, \mathbb{Z}).$$

Observe that the classical pure braid group P_n is $T_{0,1}^n$.

Our presentation of $\mathfrak{t}_{g,r}^n$ generalizes the well-known presentation of \mathfrak{p}_n , the Malcev Lie algebra of the pure braid group P_n , which is of importance in the theory of Vassiliev invariants (cf. [31, 11, 3]) and was first written down by Kohno [28]. Denote the free Lie algebra generated by indeterminates X_1, \dots, X_m by $\mathbb{L}(X_1, \dots, X_m)$, and that generated by a vector space V by $\mathbb{L}(V)$. Then \mathfrak{p}_n is the completion of the graded Lie algebra

$$\mathbb{L}(X_{ij} : ij \text{ is a two element subset of } \{1, \dots, n\})/R,$$

where R is the ideal generated by the quadratic relations

$$\begin{aligned} & [X_{ij}, X_{kl}] \text{ when } i, j, k \text{ and } l \text{ are distinct;} \\ & [X_{ij}, X_{ik} + X_{jk}] \text{ when } i, j \text{ and } k \text{ are distinct.} \end{aligned}$$

The property that \mathfrak{p}_n is the completion of the associated graded Lie algebra $\text{Gr } \mathfrak{p}_n$ does not hold for the generic group, but does hold for all Torelli groups as we shall see.

It is easiest to first state the result for the absolute Torelli group, $T_g := T_{g,0}^0$. It follows from Dennis Johnson's computation of the first homology of T_g [22] that each graded quotient of the lower central series of \mathfrak{t}_g is a representation of the algebraic group Sp_g . We will give a presentation of $\text{Gr } \mathfrak{t}_g$ in the category of representations of Sp_g . Choose a set $\lambda_1, \dots, \lambda_g$ of fundamental weights of Sp_g . Denote the representation of Sp_g with highest weight $\lambda = \sum n_j \lambda_j$ by $V(\lambda)$. Johnson's fundamental computation is that there is a natural $Sp_g(\mathbb{Z})$ equivariant isomorphism between $H_1(T_g, \mathbb{Q})$ and $V(\lambda_3)$.

¹One can replace each tangent vector in the definition by a boundary component — with this change, the diffeomorphisms are required to be the identity on each boundary component.

For all $g \geq 3$, the representation $\Lambda^2 V(\lambda_3)$ contains a unique copy of $V(2\lambda_2) + V(0)$. Denote the Sp_g invariant complement of this by R_g . Since the quadratic part of the free Lie algebra $\mathbb{L}(V)$ is $\Lambda^2 V$, we can view R_g as being a subspace of the quadratic elements of $\mathbb{L}(V(\lambda_3))$.

Theorem 1.1. *For all $g \neq 2$, \mathfrak{t}_g is the completion of its associated graded $\text{Gr } \mathfrak{t}_g$. When $g \geq 6$, this has presentation*

$$\text{Gr } \mathfrak{t}_g = \mathbb{L}(V(\lambda_3))/(R_g),$$

where R_g is the set of quadratic relations defined above. When $3 \leq g < 6$, the relations in $\text{Gr } \mathfrak{t}_g$ are generated by the quadratic relations R_g , and possibly some cubic relations.

In fact, we will show that in genus 3 there are no quadratic relations and the cubic relations contain a copy of $V(\lambda_3)$.

Dennis Johnson has proved that T_g is finitely generated for all $g \geq 3$, but it is not known for any $g \geq 3$ whether or not T_g is finitely presented. Geoff Mess [34] proved that T_2 is a countably generated free group. (Note that when $g = 0, 1$, T_g is trivial.)

Corollary 1.2. *For all $g \neq 2$, and for all $r, n \geq 0$, $\mathfrak{t}_{g,r}^n$ is finitely presented.* □

In the decorated case, we have the extension

$$(4) \quad 1 \rightarrow \pi_{g,r}^n \rightarrow T_{g,r}^n \rightarrow T_g \rightarrow 1,$$

where $\pi_{g,r}^n$ denotes the fundamental group of the configuration space of n points and r tangent vectors in S . After applying the Malcev Lie algebra functor, we obtain an extension

$$0 \rightarrow \mathfrak{p}_{g,r}^n \rightarrow \mathfrak{t}_{g,r}^n \rightarrow \mathfrak{t}_g \rightarrow 0$$

of pronilpotent Lie algebras, where $\mathfrak{p}_{g,r}^n$ denotes the Malcev Lie algebra of $\pi_{g,r}^n$.

Theorem 1.3. *For all $g \geq 0$ and all $r, n \geq 0$, the Lie algebra $\mathfrak{p}_{g,r}^n$ is the completion of its associated graded $\text{Gr } \mathfrak{p}_{g,r}^n$. The associated graded has a presentation with only quadratic relations.²*

The explicit presentation is given in Section 12. In order to give the presentation for $\mathfrak{t}_{g,r}^n$, we prove that (4) remains exact after taking graded quotients. Thus, in order to give a presentation of $\mathfrak{t}_{g,r}^n$, it suffices to determine the map

$$[\ , \] : (\text{Gr}^1 \mathfrak{t}_g \otimes \text{Gr}^1 \mathfrak{p}_{g,r}^n) \oplus (\text{Gr}^1 \mathfrak{t}_g \otimes \text{Gr}^1 \mathfrak{t}_g) \rightarrow \text{Gr}^2 \mathfrak{t}_{g,r}^n$$

determined by the bracket. We do this in Section 13 to obtain the presentation of $\mathfrak{t}_{g,r}^n$ in general.

Our results complement, and sometimes overlap with, the beautiful work [37, 38, 40, 39] of Shigeyuki Morita who began the study of the “higher Johnson homomorphisms” studied in this paper. Our main theorem allows us to answer several questions about Torelli groups, and to prove a conjecture of Morita. These and other applications are discussed in Section 14.

²After writing the paper, I discovered that Nakamura, Takao and Ueno [41, (2,8.2)] had previously given an essentially equivalent presentation of \mathfrak{p}_g^n . I have retained my original proof of this presentation as it is quite different from and more direct than theirs.

Another feature of the classical case is the existence of a canonical universal integrable connection. Denote the classifying space

$$\mathbb{C}^n - \{(z_1, \dots, z_n) : \text{the } z_i \text{ are not distinct}\}$$

of P_n by X_n . Denote the complex of global meromorphic k -forms on a complex manifold Y by $\Omega^k(Y)$. The universal integrable connection on X_n is given by the \mathfrak{p}_g valued 1-form

$$\sum_{ij} d \log(z_i - z_j) X_{ij} \in \Omega^1(X_n) \otimes \mathfrak{p}_n.$$

It plays a central role in the theory of Vassiliev invariants (cf. [29, 11, 27].) We are able to prove that there is a canonical universal connection form with “scalar curvature” for each $T_{g,r}^n$, provided $g \neq 2$, although, to date, we have not been able to give an explicit formula for it. The universal connection is discussed in Section 15.

The basic approach in this paper is to use Hodge theory. The main technical theorem of the paper is:

Theorem 1.4. *Suppose that $g \neq 2$ and that $r, n \geq 0$. For each choice of a complex structure on the decorated reference surface*

$$(S; x_1, \dots, x_n; y_1, \dots, y_r; v_1, \dots, v_r)$$

there is a \mathbb{Q} -mixed Hodge structure on $\mathfrak{t}_{g,r}^n$ for which the bracket is a morphism of mixed Hodge structures.

This mixed Hodge structure is canonical once one fixes an isomorphism of $\Gamma_{g,r}^n$ with the (orbifold) fundamental group of the moduli space of smooth projective curves of genus g with n marked points, and r non-zero tangent vectors. The theorem is proved using the mixed Hodge structure on the completion of the mapping class group $\Gamma_{g,r}^n$ relative to the homomorphism $\Gamma_{g,r}^n \rightarrow Sp_g(\mathbb{Q})$ induced by (3), the existence of which follows from [19]. The mixed Hodge structure on the Lie algebra $\mathfrak{u}_{g,r}^n$ of the pronipotent radical of the relative completion is lifted to $\mathfrak{t}_{g,r}^n$ using two results from [17]. The first states that we have a central extension

$$0 \rightarrow \mathbb{G}_a \rightarrow \mathfrak{t}_{g,r}^n \rightarrow \mathfrak{u}_{g,r}^n \rightarrow 0$$

when $g \geq 3$. The second gives an explicit relationship between this extension and the algebraic 1-cycle $C - C^-$ in the jacobian of an algebraic curve C . The theory of relative completion is reviewed in Section 3.

When $g \geq 3$, the weight filtration of $\mathfrak{t}_{g,r}^n$ is its lower central series. The fact that the weight graded functors are exact on the category of mixed Hodge structures then allows the reduction to the associated graded with impunity when studying $\mathfrak{t}_{g,r}^n$, $\mathfrak{u}_{g,r}^n$, $\mathfrak{p}_{g,r}^n$ and maps between them.

In order to bound the degrees of relations in \mathfrak{t}_g by N , we need to know that if \mathbb{V} is a variation of Hodge structure of weight n over \mathcal{M}_g (the moduli space of curves) that comes from a rational representation of Sp_g , then the weights on

$$H^2(\mathcal{M}_g, \mathbb{V})$$

are bounded between $2+n$ and $n+N$ — see Section 7. There is no *a priori* uniform bound on the weights of $H^k(X, \mathbb{V})$, where X is a smooth variety and \mathbb{V} is a variation of Hodge structure over X of weight l , as there is in the case of \mathbb{Q} coefficients where the weights are bounded between k and $2k$. For example, if Γ is a finite index

subgroup of $SL_2(\mathbb{Z})$ and X the quotient of the upper half plane by Γ , then the non-trivial weights on $H^1(X, S^n \mathbb{V})$ are $n + 1$ and $2n + 2$ for infinitely many n , as can be seen from results in [51]. Here \mathbb{V} denotes the fundamental representation of SL_2 viewed as a variation of Hodge structure over X of weight 1 and $S^n \mathbb{V}$ its n th symmetric power. Thus, one of the main technical ingredients in the paper is the result of Kabanov [24] (see also [25]) which states that one can take N to be 2 when $g \geq 6$, and 3 when $3 \leq g < 6$.

The existence of the mixed Hodge structure on the Malcev Lie algebra associated to the Torelli group was obtained several years ago. The quadratic relations (proved in Section 10) were subsequently derived in [16]. Morita then proved (unpublished) that when the genus is sufficiently large there are no cubic or quartic relations in \mathfrak{t}_g . Kabanov’s purity theorem allows us to avoid Morita’s involved computations and to show there are no higher order relations.

2. BRAID GROUPS IN POSITIVE GENUS

Throughout this section g will be positive. Suppose that S is a compact oriented surface of genus g , and that r and n are integers ≥ 0 . The configuration space of $m \geq 1$ points on S is

$$F^m(S) = S^m - \Delta,$$

where Δ is the union of the various diagonals $x_i = x_j$. Denote the tangent bundle of S by TS , and the bundle of non-zero tangent vectors by V . The pullback of V to $F^m(S)$ along the j th projection $p_j : F^m(S) \rightarrow S$ will be denoted by V_j . For a subset A of $\{1, \dots, m\}$ denote the fibered product of the V_j , where $j \in A$, by V_A .

The configuration space $F_{g,r}^n$ of n points and r non-zero tangent vectors of S is defined to be the total space of the bundle

$$V_A \rightarrow F^{r+n}(S),$$

where $A = \{n + 1, \dots, n + r\}$. Fix a base point f_o of $F_{g,r}^n$. Define

$$\pi_{g,r}^n = \pi_1(F_{g,r}^n, f_o).$$

When $r = 0$ this is just the group of pure braids with n strings on the surface S . In general, this group can be thought of as the group of pure braids on S with $r + n$ strings where r of the strings are framed. It is a standard fact that the space $F_{g,r}^n$ is an Eilenberg-Mac Lane space of type $K(\pi, 1)$ [5, §1.2].

In contrast with the genus 0 case, we have:

Proposition 2.1. *For each $g \geq 0$, there is a short exact sequence*

$$0 \rightarrow (\mathbb{Z}/(2g - 2)\mathbb{Z})^r \rightarrow H_1(\pi_{g,r}^n, \mathbb{Z}) \xrightarrow{p} H_1(S^{n+r}, \mathbb{Z}) \rightarrow 0,$$

where p is induced by the natural map $F_{g,r}^n \rightarrow S^{n+r}$.

Proof. We first consider the case when $r = 0$. In this case $F_{g,r}^n$ is $S^n - \Delta$. The divisor Δ is the union of the diagonals Δ_{ij} where the i th and j th points of S^n are equal. We therefore have a Gysin sequence

$$\dots \rightarrow H_2(S^n) \xrightarrow{\gamma} \bigoplus_{i < j} \mathbb{Z} \xrightarrow{t} H_1(S^n - \Delta) \rightarrow H_1(S^n) \rightarrow 0.$$

The map γ takes a cycle z to the element of $\bigoplus_{i < j} \mathbb{Z}$ whose ij th term is the intersection number $z \cdot \Delta_{ij}$. The map t takes the generator of the ij th factor to the homology class of a small circle which winds about Δ_{ij} in the positive direction.

Fix a base point x_o of S . For $u \in H_k(S)$ and $i \in \{1, \dots, n\}$, denote by u^i the element

$$x_o \times \cdots \times x_o \times \overset{i}{u} \times x_o \times \cdots \times x_o$$

of $H_k(S^n, \mathbb{Z})$, where u is placed in the i th factor. For elements u and v of $H_1(S)$ and $i, j \in \{1, \dots, n\}$, denote the element

$$x_o \times \cdots \times x_o \times \overset{i}{u} \times x_o \times \cdots \times x_o \times \overset{j}{v} \times x_o \times \cdots \times x_o$$

of $H_2(S, \mathbb{Z})$ by $u^i \times v^j$, where u is placed in the i th factor and v in the j th.

By choosing representatives of u and v which do not pass through the base point, one sees immediately that

$$\gamma : u^i \times v^j \mapsto (u \cdot v) \Delta_{ij}$$

from which it follows that γ is surjective and that t is trivial. This proves the result when $r = 0$.

Observe that

$$\gamma : S^i \mapsto \sum_{j \neq i} \Delta_{ij}.$$

If a and b are elements of $H_1(S, \mathbb{Z})$ with intersection number 1, then

$$S^i - \sum_{j \neq i} a^i \times b^j$$

is in the kernel of γ , and therefore lifts to an element σ_i of $H_2(S^n - \Delta, \mathbb{Z})$.

We prove the general case by induction on r . Our inductive hypothesis is that the result has been proven for $F_{g,s}^m$ when $s < r$, and that there are classes

$$\sigma_1, \dots, \sigma_m \in H_2(F_{g,s}^m, \mathbb{Z})$$

whose images under the maps

$$p_{j*} : H_2(F_{g,s}^m, \mathbb{Z}) \rightarrow H_2(S, \mathbb{Z})$$

induced by the various projections $p_j : F_{g,s}^m \rightarrow S$ satisfy $p_j(\sigma_i) = \delta_{ij}[S]$. We have proved this when $r = 0$.

Suppose that $r > 0$. We have the projection

$$q : F_{g,r}^n \rightarrow F_{g,r-1}^{n+1},$$

which replaces the first tangent vector by its anchor point. This is a principal \mathbb{C}^* bundle. It fits into a commutative square:

$$\begin{array}{ccc} F_{g,r}^n & \longrightarrow & V \\ q \downarrow & & \downarrow \\ F_{g,r-1}^{n+1} & \longrightarrow & S \end{array}$$

One therefore has a map

$$\begin{array}{ccccccccc} H_2(F_{g,r-1}^{n+1}) & \xrightarrow{\gamma^F} & H_0(F_{g,r}^n) & \longrightarrow & H_1(F_{g,r}^n) & \longrightarrow & H_1(F_{g,r-1}^{n+1}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ H_2(S) & \xrightarrow{\gamma^S} & H_0(S) & \longrightarrow & H_1(V) & \longrightarrow & H_1(S) & \longrightarrow & 0 \end{array}$$

of Gysin sequences. The map γ_S is simply multiplication by the Euler characteristic. Since $\sigma_{n+1} \mapsto [S]$, we see that γ_F takes σ_{n+1} to $2 - 2g$. The result follows by induction. \square

Remark 2.2. This result (with $r = 0$) can also be proved by considering the natural fibrations $F_g^{n+1} \rightarrow F_g^n$ obtained by forgetting the last point. The fiber is an n punctured copy of S , and its homology therefore fits into an exact sequence

$$0 \rightarrow \mathbb{Z}^n/\text{diagonal} \rightarrow H_1(S - n \text{ points}) \rightarrow H_1(S) \rightarrow 0.$$

One has to be a little careful as the monodromy action is not trivial. A simple geometric argument shows that the monodromy acts trivially on the kernel and quotient in the sequence above, and therefore is given by a homomorphism

$$\pi_g^n \rightarrow \text{Hom}(H_1(S), \mathbb{Z}^n/\text{diagonal}) \cong H^1(S^n)/\text{diagonal}.$$

By induction on n , $H_1(\pi_g^n)$ is isomorphic to $H_1(S^n)$. Since the monodromy is abelian, it factors through the quotient map

$$\pi_g^n \rightarrow H_1(S)^{\oplus n}.$$

A straightforward geometric argument shows that the action of the latter is given by the map

$$H_1(S)^{\oplus n} \xrightarrow{PD^{\oplus n}} H^1(S)^{\oplus n}/\text{diagonal},$$

where PD denotes Poincaré duality. The coinvariants are therefore given by

$$H_0(F_g^n, H_1(\text{fiber})) = H_1(S).$$

An elementary spectral sequence argument completes the inductive step.

Kohno and Oda [30, p. 208] use this method, but their result contradicts ours as they mistakenly assume that the monodromy representation is trivial.

3. RELATIVE COMPLETION OF MAPPING CLASS GROUPS

In this section we recall the main theorem of [17] which makes precise the relationship between the Malcev completion of $T_{g,r}^n$ and the unipotent radical of the relative Malcev completion of $\Gamma_{g,r}^n$. We first recall the definition of relative Malcev completion, which is due to Deligne. A reference for this material is [17, §§2–4].

Suppose that Γ is a discrete group, S a reductive linear algebraic group over a field F of characteristic zero, and that $\rho : \Gamma \rightarrow S(F)$ is a representation whose image is Zariski dense. The *Malcev completion of Γ over F relative to ρ* is a homomorphism $\tilde{\rho} : \Gamma \rightarrow \mathcal{G}$ of Γ into a proalgebraic group \mathcal{G} , defined over F , which is an extension

$$1 \rightarrow \mathcal{U} \rightarrow \mathcal{G} \xrightarrow{p} S \rightarrow 1$$

of S by a pronipotent group \mathcal{U} such that the diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{\rho} & S \\ \tilde{\rho} \downarrow & & \parallel \\ \mathcal{G} & \xrightarrow[p]{} & S \end{array}$$

commutes. It is characterized by a universal mapping property: If G is a linear (pro)algebraic group over F which is an extension

$$1 \rightarrow U \rightarrow G \rightarrow S \rightarrow 1$$

of S by a (pro)unipotent group, and if $\tau : \Gamma \rightarrow G$ is a homomorphism whose composition with $G \rightarrow S$ is ρ , then there is a unique homomorphism $\mathcal{G} \rightarrow G$ such that the diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{\bar{\rho}} & \mathcal{G} \\ \tau \downarrow & \searrow & \downarrow p \\ G & \longrightarrow & S \end{array}$$

commutes.

When S is the trivial group, the relative completion of Γ coincides with the classical Malcev (or unipotent) completion of Γ .

Suppose that K/F is an extension of fields of characteristic zero. When S is defined over F and $\rho : \Gamma \rightarrow S(F)$, one can ask if the K -form of the completion of Γ relative to ρ is obtained from the F -form by extension of scalars. If this is the case for all such field extensions, we will say that the relative completion of Γ relative to ρ can be defined over F .

The action of the mapping class group on S preserves the intersection pairing $q : H_1(S, \mathbb{Z})^{\otimes 2} \rightarrow \mathbb{Z}$. We therefore have a homomorphism

$$(5) \quad \rho : \Gamma_{g,r}^n \rightarrow \text{Aut}(H_1(S, \mathbb{Z}), q) \cong Sp_g(\mathbb{Z}).$$

For a positive integer l , we define the *level l subgroup* $\Gamma_{g,r}^n[l]$ to be the kernel of the induced map

$$\Gamma_{g,r}^n \rightarrow \text{Aut}(H_1(S, \mathbb{Z}/l\mathbb{Z}), q) \cong Sp_g(\mathbb{Z}/l\mathbb{Z}).$$

Here we interpret $Sp_g(\mathbb{Z}/l\mathbb{Z})$ as the trivial group when $l = 1$.

Theorem 3.1. *For all $g \geq 3$ and all $l \geq 1$, the completion of the mapping class group $\Gamma_{g,r}^n[l]$ relative to the homomorphism $\rho : \Gamma_{g,r}^n[l] \rightarrow Sp_g(\mathbb{Q})$ induced by (5) is defined over \mathbb{Q} . \square*

This result was proved in [17, (4.14)] under the assumption that $g \geq 8$ and that $l = 1$. That the stronger result is true follows from the strengthening [9] of Borel's stability theorem [7, 8] for the symplectic group, stated below, which ensures that the hypothesis [17, (4.10)] is satisfied when $l \geq 1$ and $g \geq 3$.

Theorem 3.2. *Suppose that V is an irreducible rational representation of the algebraic group Sp_g and that Γ is a finite index subgroup of $Sp_g(\mathbb{Z})$. If $k < g$, then $H^k(\Gamma, V)$ vanishes when V is non-trivial, and agrees with the stable cohomology of $Sp_g(\mathbb{Z})$ when V is the trivial representation. \square*

Denote the completion of $\Gamma_{g,r}^n$ relative to ρ by $\tilde{\rho} : \Gamma_{g,r}^n \rightarrow \mathcal{G}_{g,r}^n$. Denote the pronipotent radical of $\mathcal{G}_{g,r}^n$ by $\mathcal{U}_{g,r}^n$, and its Lie algebra by $\mathfrak{u}_{g,r}^n$.

Proposition 3.3. *If $g \geq 3$, then for all $l \geq 1$, the composite*

$$\Gamma_{g,r}^n[l] \hookrightarrow \Gamma_{g,r}^n \rightarrow \mathcal{G}_{g,r}^n$$

is the completion of $\Gamma_{g,r}^n[l]$ relative to the restriction of ρ to $\Gamma_{g,r}^n[l]$.

Proof. This follows directly from results in [17, §4] as we shall explain. Denote the relative completion of $\Gamma_{g,r}^n[l]$ by $\mathcal{G}_{g,r}^n[l]$ and its pronipotent radical by $\mathcal{U}_{g,r}^n[l]$. There is a natural map $\mathcal{U}_{g,r}^n[l] \rightarrow \mathcal{U}_{g,r}^n$, the surjectivity of which follows from (3.2) and [17, (4.6)]. Injectivity follows directly from (3.2) and [17, (4.13)]. \square

We have an extension

$$1 \rightarrow \mathcal{U}_{g,r}^n \rightarrow \mathcal{G}_{g,r}^n \rightarrow Sp_g \rightarrow 1$$

of proalgebraic groups over \mathbb{Q} . The homomorphism $\tilde{\rho}$ induces a map $T_{g,r}^n \rightarrow \mathcal{U}_{g,r}^n$. Denote the classical Malcev completion of $T_{g,r}^n$ by $\mathcal{T}_{g,r}^n$, and its Lie algebra by $\mathfrak{t}_{g,r}^n$. Since $\mathcal{U}_{g,r}^n$ is pronipotent, $\tilde{\rho}$ induces a homomorphism

$$\theta : \mathcal{T}_{g,r}^n \rightarrow \mathcal{U}_{g,r}^n$$

of pronipotent groups.

The following theorem is the main result of [17].³ There it is proved for all $g \geq 8$, but in view of (3.2), it holds for all $g \geq 3$. (Cf. the third footnote on page 76 of [17].)

Theorem 3.4. *For all $g \geq 3$, the homomorphism θ is surjective and has a one-dimensional kernel isomorphic to \mathbb{G}_a which is central in $\mathcal{T}_{g,r}^n$ and is trivial as an $Sp_g(\mathbb{Z})$ module. Moreover, the extensions are all pulled back from that of \mathcal{T}_g ; that is, the diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{G}_a & \longrightarrow & \mathcal{T}_{g,r}^n & \longrightarrow & \mathcal{U}_{g,r}^n \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{G}_a & \longrightarrow & \mathcal{T}_g & \longrightarrow & \mathcal{U}_g \longrightarrow 1 \end{array}$$

commutes. \square

It is a standard fact that the sequence

$$1 \rightarrow \pi_{g,r}^n \rightarrow \Gamma_{g,r}^n \rightarrow \Gamma_g \rightarrow 1$$

is exact, where $\pi_{g,r}^n$ is the braid group defined in Section 2. Restricting to \mathcal{T}_g , we obtain an extension

$$(6) \quad 1 \rightarrow \pi_{g,r}^n \rightarrow T_{g,r}^n \rightarrow T_g \rightarrow 1.$$

Proposition 3.5. *If $g \geq 3$, then the extension (6) induces an exact sequence*

$$0 \rightarrow H_1(\pi_{g,r}^n, \mathbb{Q}) \rightarrow H_1(T_{g,r}^n, \mathbb{Q}) \rightarrow H_1(T_g, \mathbb{Q}) \rightarrow 0.$$

Proof. It follows from (2.1) that the natural map $\pi_{g,r}^n \rightarrow \pi_g^{n+r}$ induces an isomorphism on H_1 with rational coefficients. The corresponding surjection $T_{g,r}^n \rightarrow T_g^{n+r}$ induces a map

$$\begin{array}{ccccccc} H_1(\pi_{g,r}^n, \mathbb{Q}) & \longrightarrow & H_1(T_{g,r}^n, \mathbb{Q}) & \longrightarrow & H_1(T_g, \mathbb{Q}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \parallel & & \\ H_1(\pi_g^{n+r}, \mathbb{Q}) & \longrightarrow & H_1(T_g^{n+r}, \mathbb{Q}) & \longrightarrow & H_1(T_g, \mathbb{Q}) & \longrightarrow & 0 \end{array}$$

³There is a minor error in proof of the case “ $A_{g,r}^n$ implies $A_{h,r}^n$ ” of the proof of [17, (7.4)]. It is easily fixed.

of exact sequences. Since the middle vertical map is a surjection, and the two outside maps are isomorphisms, it follows that the middle map is an isomorphism. It therefore suffices to prove the result when $r = 0$.

To prove this, we need to prove that the map

$$(7) \quad H_1(\pi_g^n, \mathbb{Q}) \rightarrow H_1(T_g^n, \mathbb{Q})$$

is injective. We first remark that this is easily proved when $n = 1$ using the Johnson homomorphism

$$\tau_g^1 : H_1(T_g^1) \rightarrow H_3(\text{Jac } C).$$

(Cf. [20] and [18, §3].) The composition of τ_g^1 with the map

$$H_1(\pi_g^1, \mathbb{Q}) \rightarrow H_3(\text{Jac } C, \mathbb{Q})$$

is easily seen to be the map $H_1(C) \rightarrow H_3(\text{Jac } C)$ which takes a class in H_1 to its Pontrjagin product with the class of C in $H_2(\text{Jac } C)$. Since this map is injective, it follows that (7) is injective when $n = 1$.

The general case follows from this by considering the maps $p_j : H_1(T_g^n) \rightarrow H_1(T_g^1)$ induced by the n forgetful maps $T_g^n \rightarrow T_g^1$. \square

Define $\mathcal{P}_{g,r}^n$ to be the Malcev completion of $\pi_{g,r}^n$ and $\mathfrak{p}_{g,r}^n$ to be the corresponding Malcev Lie algebra. Applying Malcev completion to (6) we obtain a sequence

$$\mathfrak{p}_{g,r}^n \rightarrow \mathfrak{t}_{g,r}^n \rightarrow \mathfrak{t}_g$$

of Malcev Lie algebras.

Proposition 3.6. *If $g \neq 2$, then the sequence*

$$0 \rightarrow \mathfrak{p}_{g,r}^n \rightarrow \mathfrak{t}_{g,r}^n \rightarrow \mathfrak{t}_g \rightarrow 0$$

associated to (6) is exact.

Proof. By [15, (5.6)] it suffices to verify two conditions. First, that T_g acts unipotently on $H^1(\pi_{g,r}^n, \mathbb{Q})$; this follows from (2.1). The second condition there is satisfied if, for example, the extension

$$0 \rightarrow H_1(\pi_{g,r}^n, \mathbb{Q}) \rightarrow G \rightarrow T_g \rightarrow 1$$

obtained by pushing (6) out along $\pi_{g,r}^n \rightarrow H_1(\pi_{g,r}^n, \mathbb{Q})$ is split. In our case this follows from (3.5) as the extension above can be pulled back from the extension

$$0 \rightarrow H_1(\pi_{g,r}^n, \mathbb{Q}) \rightarrow H_1(T_{g,r}^n, \mathbb{Q}) \rightarrow H_1(T_g, \mathbb{Q}) \rightarrow 0$$

which is split for trivial reasons. \square

The standard homomorphism $\Gamma_{g,r}^n \rightarrow \Gamma_g$ induces a homomorphism $\mathcal{G}_{g,r}^n \rightarrow \mathcal{G}_g$ of relative completions. The inclusion $\mathcal{P}_{g,r}^n \rightarrow \mathcal{T}_{g,r}^n$ induces a homomorphism $\mathcal{P}_{g,r}^n \rightarrow \mathcal{U}_{g,r}^n$. We therefore have a sequence $\mathcal{P}_{g,r}^n \rightarrow \mathcal{G}_{g,r}^n \rightarrow \mathcal{G}_g \rightarrow 1$ of proalgebraic groups.

Lemma 3.7. *If $g \geq 3$, then the sequence*

$$1 \rightarrow \mathcal{P}_{g,r}^n \rightarrow \mathcal{G}_{g,r}^n \rightarrow \mathcal{G}_g \rightarrow 1$$

is exact. In particular, the sequence

$$0 \rightarrow \mathfrak{p}_{g,r}^n \rightarrow \mathfrak{u}_{g,r}^n \rightarrow \mathfrak{u}_g \rightarrow 0$$

is exact.

Proof. To prove the result, it suffices to prove that the sequence

$$1 \rightarrow \mathcal{P}_{g,r}^n \rightarrow \mathcal{U}_{g,r}^n \rightarrow \mathcal{U}_g \rightarrow 1$$

is exact. But this follows immediately from (3.4) and (3.6). □

Suppose that \mathfrak{g} is a finitely generated pronilpotent Lie algebra. Denote the group of automorphisms of \mathfrak{g} by $\text{Aut } \mathfrak{g}$. Denote the subgroup of $\text{Aut } \mathfrak{g}$ consisting of the elements that act trivially on $H_1(\mathfrak{g})$ by $L^1 \text{Aut } \mathfrak{g}$. Since the action of an automorphism on the graded quotients of the lower central series is determined by the action on the first graded quotient, $\text{Aut } \mathfrak{g}$ is a proalgebraic group which is an extension

$$1 \rightarrow L^1 \text{Aut } \mathfrak{g} \rightarrow \text{Aut } \mathfrak{g} \rightarrow S \rightarrow 1$$

of a closed subgroup S of $\text{Aut } H_1(\mathfrak{g})$ by the pronilpotent group consisting of those automorphisms of \mathfrak{g} that act trivially on the graded quotients of the lower central series. Its Lie algebra is the Lie algebra $\text{Der } \mathfrak{g}$ of derivations of \mathfrak{g} . This is an extension

$$0 \rightarrow L^1 \text{Der } \mathfrak{g} \rightarrow \text{Der } \mathfrak{g} \rightarrow \mathfrak{s} \rightarrow 0$$

of the Lie algebra \mathfrak{s} of S by the pronilpotent Lie algebra of derivations of \mathfrak{g} that act trivially on the graded quotients of the lower central series of \mathfrak{g} .

If \mathcal{G} is the pronilpotent group corresponding to \mathfrak{g} , then $\text{Aut } \mathcal{G}$ and $\text{Aut } \mathfrak{g}$ are isomorphic, as can be seen using the Baker-Campbell-Hausdorff formula.

Lemma 3.8. *For all $g \geq 0$ the natural action of $\Gamma_{g,r}^n$ on $\pi_{g,r}^n$ induces a representation*

$$\mathcal{G}_{g,r}^n \rightarrow \text{Aut } \mathfrak{p}_{g,r}^n.$$

Proof. Suppose that $g \geq 0$. The mapping class group $\Gamma_{g,r}^n$ acts on $\mathfrak{p}_{g,r}^n$. We therefore have a homomorphism

$$(8) \quad \Gamma_{g,r}^n \rightarrow \text{Aut } \mathfrak{p}_{g,r}^n.$$

By (2.1) we know that

$$\text{Aut } H_1(\mathfrak{p}_{g,r}^n) = \text{Aut } H_1(S)^{\oplus(n+r)}.$$

There is a diagonal copy of Sp_g contained in this group, and it is easy to see that this is the Zariski closure of the image of $\Gamma_{g,r}^n$ in $\text{Aut } H_1(\mathfrak{p}_{g,r}^n)$. It follows that the Zariski closure of the image of (8) is an extension of this diagonal Sp_g by a pronilpotent group. Since the homomorphism from $\Gamma_{g,r}^n$ to this copy of Sp_g is the standard representation, the universal mapping property of the relative completion implies that (8) induces a homomorphism $\mathcal{G}_{g,r}^n \rightarrow \text{Aut } \mathfrak{p}_{g,r}^n$. □

Remark 3.9. When $g = 1$, the results (3.3) and (3.4) are false. That (3.3) and (3.4) fail can be deduced from [19, (10.3)], a special case of which states that there is a natural isomorphism

$$H^1(\mathcal{M}_1[l], S^n \mathbb{V}) \cong (H^1(\mathcal{U}_1[l]) \otimes S^n \mathbb{V})^{SL_2}.$$

Here $\mathcal{M}_1[l]$ denotes the moduli space of elliptic curves with a level l structure, \mathbb{V} denotes the variation of Hodge structure over $\mathcal{M}_1[l]$ of weight 1 corresponding to H^1 of the universal elliptic curve, and $S^n \mathbb{V}$ denotes its n th symmetric power. Since

the level l congruence subgroup of $SL_2(\mathbb{Z})$ is free when $l \geq 4$, it follows by an Euler characteristic argument that $H^1(\mathcal{M}_1[l], S^k\mathbb{V})$ is non-zero whenever $l \geq 4$. Since \mathcal{T}_1 is trivial, it cannot surject onto $\mathcal{U}_1[l]$. So (3.4) fails. Since the rank r_l of the level l subgroup of $SL_2(\mathbb{Z})$ depends on l , and since

$$\dim H^1(\mathcal{M}_1[l], S^n\mathbb{V}) = (r_l - 1) \dim S^k\mathbb{V},$$

it follows that the rank of the $S^n\mathbb{V}$ isotypical part of $H_1(\mathcal{U}_1[l])$ depends on l . So (3.3) does not hold. In Remark 7.2, we will show that each $\mathfrak{u}_1[l]$ is a free Lie algebra.

4. MIXED HODGE STRUCTURES ON TORELLI GROUPS

Denote by $\mathcal{M}_{g,r}^n[l]$ the moduli space of ordered $(n + r + 1)$ -tuples

$$(C; x_1, \dots, x_n; v_1, \dots, v_r),$$

where C is a smooth complex projective curve with a level l structure, the x_j are distinct points of C , and the v_j are non-zero holomorphic tangent vectors of C which are anchored at r distinct points of C which are also distinct from the x_j . We shall omit the l when it is 1, and r and n when they are zero. So, for example, \mathcal{M}_g denotes the moduli space of smooth projective curves of genus g .

For each point x of $\mathcal{M}_{g,r}^n[l]$, there is a natural isomorphism of $\Gamma_{g,r}^n[l]$ with the (orbifold) fundamental group $\pi_1(\mathcal{M}_{g,r}^n[l], x)$ of $\mathcal{M}_{g,r}^n[l]$. We will denote the latter by $\Gamma_{g,r}^n[l](x)$. We shall denote the subgroup of $\Gamma_{g,r}^n(x)$ corresponding to $T_{g,r}^n$ by $T_{g,r}^n(x)$. Denote the relative Malcev completion of $\Gamma_{g,r}^n(x)$ by $\mathcal{G}_{g,r}^n(x)$, its pronilpotent radical by $\mathcal{U}_{g,r}^n(x)$, etc. The Lie algebras corresponding to $T_{g,r}^n(x)$ and $\mathcal{U}_{g,r}^n(x)$ will be denoted by $\mathfrak{t}_{g,r}^n(x)$ and $\mathfrak{u}_{g,r}^n(x)$, respectively.

In this section we prove that for each choice of a point x in $\mathcal{M}_{g,r}^n$, there is a canonical \mathbb{Q} mixed Hodge structure (MHS) on $\mathfrak{t}_{g,r}^n(x)$. The first ingredient in the construction of this MHS is the following theorem, which is proved in [19, (13.1)].

Theorem 4.1. *Suppose that X is a smooth quasi-projective algebraic variety and $(\mathbb{V}, \langle \ , \ \rangle)$ is a polarized variation of Hodge structure over X of geometric origin whose monodromy representation*

$$\rho : \pi_1(X, x_o) \rightarrow \text{Aut}_{\mathbb{R}}(\mathbb{V}_o, \langle \ , \ \rangle)$$

has Zariski dense image. Then the coordinate ring of the completion of $\pi_1(X, x_o)$ relative to ρ and its unipotent radical both have natural real MHSs such that the product, coproduct, and antipode of each are morphisms of MHSs. \square

We will say that a homomorphism $\mathcal{G} \rightarrow \mathcal{H}$ between proalgebraic groups, each of whose coordinate rings is a Hopf algebra in the category of mixed Hodge structures, is a morphism of MHSs if the corresponding map on coordinate rings is. Since $\Gamma_{g,r}^n(x)$ is the orbifold fundamental group of $(\mathcal{M}_{g,r}^n, x)$, the following result is not unexpected.

Theorem 4.2. *For all $g, r, n \geq 0$, and for each choice of a point*

$$x = [C; x_1, \dots, x_n; v_1, \dots, v_r]$$

of $\mathcal{M}_{g,r}^n$, there is a canonical real MHS on the coordinate ring of $\mathcal{G}_{g,r}^n(x)$ for which the product, coproduct and antipode are morphisms of MHS. Moreover, the homomorphisms $\mathcal{G}_{g,r}^n(x) \rightarrow \mathcal{G}_{g,r}^{n-1}(x')$ and $\mathcal{G}_{g,r}^n(x) \rightarrow \mathcal{G}_{g,r-1}^{n+1}(x'')$, induced by forgetting a point or by replacing a tangent vector by its anchor point, are morphisms of mixed Hodge structure.

Proof. Since the mapping class group is not, in general, the fundamental group of $\mathcal{M}_{g,r}^n$, we need to pass to a level. Choose an integer l such that $\Gamma_{g,r}^n[l]$ is torsion free. In this case, the moduli space $\mathcal{M}_{g,r}^n[l]$ is smooth and has fundamental group isomorphic to $\Gamma_{g,r}^n[l]$. Since $\Gamma_{g,r}^n[l]$ is torsion free, there is a universal curve

$$\pi : \mathcal{C} \rightarrow \mathcal{M}_{g,r}^n[l].$$

Take \mathbb{V} to be the dual of the local system $R^1\pi_*\mathbb{Z}$. This is a polarized variation of Hodge structure of weight -1 and is clearly of geometric origin. Its monodromy representation is

$$\rho : \Gamma_{g,r}^n[l] \rightarrow Sp_g(\mathbb{Z}).$$

So by (3.3) and (4.1), there is a canonical real MHS on the coordinate ring of the relative completion $\mathcal{G}_{g,r}^n(x)$ of $\Gamma_{g,r}^n(x)$ for each choice of a point of $\mathcal{M}_{g,r}^n[l]$ that lies over x .

Denote the projection $\mathcal{M}_{g,r}^n[l] \rightarrow \mathcal{M}_{g,r}^n$ by p . The set of lifts of a point x of $\mathcal{M}_{g,r}^n$ to $\mathcal{M}_{g,r}^n[l]$ is permuted transitively by the Galois group $Sp_g(\mathbb{Z}/l\mathbb{Z})$. It follows from the naturality of the MHS on the relative completion that the MHSs on $\mathcal{G}_{g,r}^n(x)$ with respect to any two points of $p^{-1}(x)$ are canonically isomorphic. The MHS on $\Gamma_{g,r}^n(x)$ is therefore independent of the choice of a point of $p^{-1}(x)$, and is therefore canonical.

To show that the MHS on $\Gamma_{g,r}^n(x)$ constructed above is independent of the choice of the level l , suppose that l_1 and l_2 are two levels for which the mapping class group is torsion free. One can then compare the corresponding MHSs by passing to the level corresponding to the least common multiple of l_1 and l_2 . The naturality statement follows directly from [19, (13.12)]. □

Corollary 4.3. *For all $g \geq 0$, and for each choice of a point*

$$x = [C; x_1, \dots, x_n; v_1, \dots, v_r]$$

of $\mathcal{M}_{g,r}^n$, the pronilpotent Lie algebra $\mathfrak{u}_{g,r}^n(x)$ of the pronilpotent radical of $\mathcal{G}_{g,r}^n(x)$ has a canonical real MHS for which the bracket is a morphism of MHS. Moreover, the morphisms $\mathfrak{u}_{g,r}^n(x) \rightarrow \mathfrak{u}_{g,r}^{n-1}(x')$ and $\mathfrak{u}_{g,r}^n(x) \rightarrow \mathfrak{u}_{g,r-1}^{n+1}(x'')$, obtained by forgetting a point or replacing a tangent vector by its anchor point, are morphisms of mixed Hodge structure. □

We shall denote by $\pi_{g,r}^n(x)$ the fundamental group with respect to the base point x of the (orbifold) fiber of the projection $\mathcal{M}_{g,r}^n \rightarrow \mathcal{M}_g$ that contains x . It is isomorphic to the group $\pi_{g,r}^n$ defined in Section 2. Given a point x of $\mathcal{M}_{g,r}^n$, there are two *a priori* different MHSs on $\mathfrak{p}_{g,r}^n(x)$, the Malcev Lie algebra of $\pi_{g,r}^n(x)$. The first is the one obtained from the construction given in [14]. The second arises as $\mathfrak{p}_{g,r}^n(x)$ is the kernel of the natural surjection $\mathfrak{u}_{g,r}^n(x) \rightarrow \mathfrak{u}_g(x)$ — see (3.7). The following assertion follows directly from the naturality properties [19, (13.12)] of the mixed Hodge structure of relative completions.

Proposition 4.4. *These two MHSs are identical.* □

Fix a point x of $\mathcal{M}_{g,r}^n$. Then both $\mathcal{G}_{g,r}^n(x)$ and $\mathfrak{p}_{g,r}^n(x)$ have canonical MHSs. It is natural to expect that the natural action

$$(9) \quad \mathcal{G}_{g,r}^n(x) \rightarrow \text{Aut } \mathfrak{p}_{g,r}^n(x)$$

constructed in (3.8) is compatible with these.

Lemma 4.5. *The action (9) is a morphism of MHS. Consequently, the morphism*

$$\mathfrak{u}_{g,r}^n(x) \rightarrow \text{Der } \mathfrak{p}_{g,r}^n(x)$$

is also a morphism of MHS with respect to the canonical MHSs determined by $x \in \mathcal{M}_{g,r}^n$.

Proof. It follows immediately from (3.7) that $\mathcal{P}_{g,r}^n(x)$ is a normal subgroup of $\mathcal{G}_{g,r}^n(x)$. Since the coordinate ring of $\mathcal{G}_{g,r}^n(x)$ has a natural mixed Hodge structure compatible with its operations, the action of $\mathcal{G}_{g,r}^n(x)$ on $\mathcal{P}_{g,r}^n(x)$ via conjugation is a morphism of MHS. But this action is easily seen to coincide with the canonical action of $\mathcal{G}_{g,r}^n(x)$ on $\mathcal{P}_{g,r}^n(x)$. \square

For a curve C of genus $g \geq 3$, denote by $PH_3(\text{Jac } C, \mathbb{Q})$ the *primitive three-dimensional homology* of its jacobian $\text{Jac } C$ — that is, the subspace of $H_3(\text{Jac } C, \mathbb{Q})$ corresponding to $PH^{2g-3}(\text{Jac } C, \mathbb{Q})$ under Poincaré duality. It has a natural Hodge structure of weight -3 .

Proposition 4.6. *If $g \geq 3$, then for each $x = [C; x_1, \dots, x_n; v_1, \dots, v_r] \in \mathcal{M}_{g,r}^n$ the induced mixed Hodge structure on $H_1(\mathfrak{u}_{g,r}^n(x))$ is of weight -1 and is canonically isomorphic to*

$$PH_3(\text{Jac } C, \mathbb{R}(-1)) \oplus H_1(C, \mathbb{R})^{\oplus(r+n)}.$$

This isomorphism respects the \mathbb{Q} -structures on these groups so that the MHS on $H_1(\mathfrak{u}_{g,r}^n(x))$ can be lifted canonically to a \mathbb{Q} -MHS.

We will see shortly that this \mathbb{Q} -Hodge structure on $H_1(\mathfrak{u}_{g,r}^n(x))$ lifts to a \mathbb{Q} -MHS on $\mathfrak{u}_{g,r}^n(x)$.

Proof. As in the proof of (3.5), we reduce the proof to showing that it is true for \mathfrak{u}_g^1 . Then, by (4.5), the composite

$$(10) \quad H_1(\mathfrak{u}_g^1(x)) \rightarrow W_{-1}H_1(\text{Der } \mathfrak{p}_g^1(x)) \rightarrow \text{Gr}_{-1}^W \text{Der } \mathfrak{p}_g^1(x)$$

is a morphism of MHS. Observe that

$$\text{Gr}_{-1}^W \text{Der } \mathfrak{p}_g^1(x) \subset \text{Hom}(\text{Gr}_{-1}^W \mathfrak{p}_g^1(x), \text{Gr}_{-2}^W \mathfrak{p}_g^1(x)).$$

From the work of Johnson [20] (see also [18, §4]), it follows that (10) is injective, from which the result follows for \mathfrak{u}_g^1 . \square

The fact that $H_1(\mathfrak{u}_{g,r}^n)$ is pure of weight -1 allows us to conclude that the weight filtration of $\mathfrak{u}_{g,r}^n$ is essentially its lower central series. This follows from the following general fact.

Lemma 4.7. *Suppose that \mathfrak{g} is a pronilpotent Lie algebra in the category of mixed Hodge structures with finite dimensional H_1 . If the induced MHS on $H_1(\mathfrak{g})$ is pure of weight -1 , then $W_{-l}\mathfrak{g}$ is the l th term of the lower central series of \mathfrak{g} .*

Proof. Denote the l th term of the lower central series of \mathfrak{g} by $\mathfrak{g}^{(l)}$. Since $H_1(\mathfrak{g})$ is pure of weight -1 , since \mathfrak{g} is pronilpotent, and since the bracket is a morphism of MHS, it follows that $\mathfrak{g} = W_{-1}\mathfrak{g}$. An elementary argument using the Jacobi identity shows that the bracket

$$(11) \quad \mathfrak{g} \otimes \mathfrak{g}^{(l)} \rightarrow \mathfrak{g}^{(l+1)}$$

is surjective. Since the bracket is a morphism of MHS, it follows that $\mathfrak{g}^{(l)} \subseteq W_{-l}\mathfrak{g}$. The fact that $H_1(\mathfrak{g})$ is pure of weight -1 forces $\mathfrak{g}^{(2)} = W_{-2}\mathfrak{g}$. The result now

follows by an induction argument (induct on l) using the fact that (11), being a morphism of MHS, is strict with respect to the weight filtration. \square

Corollary 4.8. *The l th term of the lower central series of $\mathfrak{u}_{g,r}^n$ is $W_{-l}\mathfrak{u}_{g,r}^n$.* \square

This result implies that the weight filtration on $\mathcal{G}_{g,r}^n$ is defined over \mathbb{Q} which implies that this MHS is really defined over \mathbb{Q} .

Corollary 4.9. *The weight filtration of the canonical MHSs on $\mathcal{G}_{g,r}^n(x)$ and $\mathfrak{u}_{g,r}^n(x)$ associated to a point of $\mathcal{M}_{g,r}^n$ are topologically determined and therefore defined over \mathbb{Q} . Consequently, the MHSs on $\mathcal{G}_{g,r}^n(x)$ and $\mathfrak{u}_{g,r}^n(x)$ each have a canonical lift to \mathbb{Q} -MHSs.* \square

We are now ready to lift the MHS from $\mathfrak{u}_{g,r}^n(x)$ to $\mathfrak{t}_{g,r}^n(x)$:

Theorem 4.10. *Suppose that $g \neq 2$ and that $r, n \geq 0$. For each choice of a base point*

$$x = [C; x_1, \dots, x_n; v_1, \dots, v_r]$$

of $\mathcal{M}_{g,r}^n$ there is a canonical \mathbb{Q} -MHS on $\mathfrak{t}_{g,r}^n(x)$ for which the bracket and the quotient map $\mathfrak{t}_{g,r}^n(x) \rightarrow \mathfrak{u}_{g,r}^n(x)$ are morphisms of MHS. Moreover, $W_{-l}\mathfrak{t}_{g,r}^n(x)$ is the l th term of the lower central series of $\mathfrak{t}_{g,r}^n(x)$ and the central \mathbb{G}_a is isomorphic to $\mathbb{Q}(1)$.

Proof. For all $g \geq 0$, we have the exact sequence

$$1 \rightarrow \pi_{g,r}^n \rightarrow T_{g,r}^n \rightarrow T_g \rightarrow 1.$$

When $g = 0, 1$, T_g is the trivial group, so that $T_{g,r}^n$ is isomorphic to $\pi_{g,r}^n$. It follows that in these cases $\mathfrak{t}_{g,r}^n$ is isomorphic to the Malcev Lie algebra $\mathfrak{p}_{g,r}^n$ associated to $\pi_{g,r}^n$. The choice of the base point of $\mathcal{M}_{g,r}^n$ gives the configuration space $(F_{g,r}^n, f_o)$ the structure of a pointed smooth complex algebraic variety. Since $\pi_{g,r}^n$ is the fundamental group of $(F_{g,r}^n, f_o)$, the existence of the MHS on $\mathfrak{t}_{g,r}^n(x)$ when $g = 0, 1$ follows from [14, (6.3.1)].

Now suppose that $g \geq 3$. To construct a MHS on $\mathfrak{t}_{g,r}^n(x)$, it suffices to show that $\mathfrak{t}_g(x)$ has a MHS such that $\mathfrak{t}_g(x) \rightarrow \mathfrak{u}_g(x)$ is a morphism as it follows from (3.4), (3.6) and (3.7) that the diagram

$$\begin{array}{ccc} \mathfrak{t}_{g,r}^n(x) & \longrightarrow & \mathfrak{t}_g(x) \\ \downarrow & & \downarrow \\ \mathfrak{u}_{g,r}^n(x) & \longrightarrow & \mathfrak{u}_g(x) \end{array}$$

is a pullback square in the category of pronilpotent Lie algebras.

It is useful to begin by explaining the philosophy behind the proof. The essential point is that $\Gamma_g(x)$ acts on $\mathfrak{t}_g(x)$ and on $\mathfrak{u}_g(x)$ — the action is induced by the action of $\Gamma_g(x)$ on $T_g(x)$ by conjugation. The central extension

$$(12) \quad 0 \rightarrow \mathbb{G}_a \rightarrow \mathfrak{t}_g(x) \rightarrow \mathfrak{u}_g(x) \rightarrow 0$$

given by (3.4) can be viewed as an extension of local systems over \mathcal{M}_g (in the orbifold sense, of course) where \mathbb{G}_a is a trivial local system. Although we have not proved it yet, $\mathfrak{u}_g(x)$ should be a variation of MHS over \mathcal{M}_g . So we should try to construct the MHS on $\mathfrak{t}_g(x)$ so that (12) is both an extension of local systems and an extension of mixed Hodge structures. This, and the fact that the bracket has to be a morphism of MHS, gives us no choice. We now carry out this program. Note that we will *not* appeal to the assertion that the set of $\mathfrak{u}_g(x)$ form a variation

of MHS over \mathcal{M}_g , but we will use the fact that the $H_1(\mathbf{u}_g(x))$ form a variation of Hodge structure over \mathcal{M}_g .

The first point is that we know that, since $H_1(\mathfrak{t}_g)$ does not contain any copies of the trivial representation, the central \mathbb{G}_a is contained in $[\mathfrak{t}_g, \mathfrak{t}_g]$. The second is that by the computations in §8 of [17] we know that the central \mathbb{G}_a lies in the image of the map

$$\Lambda^2 H_1(\mathfrak{t}_g) \rightarrow \mathrm{Gr}_{-2}^{\mathrm{lcs}} \mathfrak{t}_g$$

induced by the bracket. So in order that the bracket be a morphism of MHS, the central \mathbb{G}_a must be of weight -2 . Since it is one dimensional, we are forced to give it the Hodge structure $\mathbb{Q}(1)$.

Now fix a base point x of \mathcal{M}_g . There is a corresponding \mathbb{Q} -MHS on $\mathbf{u}_g(x)$. To lift this MHS to $\mathfrak{t}_g(x)$, we have to give an element of

$$\mathrm{Ext}_{\mathcal{H}}^1(\mathbf{u}_g(x), \mathbb{Q}(1)),$$

where $\mathrm{Ext}_{\mathcal{H}}$ denotes the Ext group in the category \mathcal{H} of \mathbb{Q} mixed Hodge structures. Applying the functor $\mathrm{Ext}_{\mathcal{H}}^\bullet$ to the sequence

$$0 \rightarrow W_{-2}\mathbf{u}_g(x) \rightarrow \mathbf{u}_g(x) \rightarrow H_1(\mathbf{u}_g(x)) \rightarrow 0,$$

we see that the natural map

$$\mathrm{Ext}_{\mathcal{H}}^1(H_1(\mathbf{u}_g(x)), \mathbb{Q}(1)) \rightarrow \mathrm{Ext}_{\mathcal{H}}^1(\mathbf{u}_g(x), \mathbb{Q}(1))$$

is an isomorphism.

Now let the base point vary. By (4.6), $H_1(\mathbf{u}_g)$ is a variation of \mathbb{Q} -MHS over \mathcal{M}_g (in the orbifold sense) of weight -1 — cf. [18, (9.1)]. Denote the category of admissible variations of \mathbb{Q} -MHS over a smooth variety X by $\mathcal{H}(X)$ and the category of \mathbb{Q} -local systems over X by $\mathcal{L}(X)$. Then, by [18, (8.1)], the forgetful map

$$(13) \quad \mathrm{Ext}_{\mathcal{H}(\mathcal{M}_g)}^1(H_1(\mathbf{u}_g), \mathbb{Q}(1)) \rightarrow \mathrm{Ext}_{\mathcal{L}(\mathcal{M}_g)}^1(H_1(\mathbf{u}_g), \mathbb{Q}(1)) \cong H^1(\Gamma_g, H^1(\mathbf{u}_g))$$

is an isomorphism.⁴ We will lift the MHS on $\mathbf{u}_g(x)$ to $\mathfrak{t}_g(x)$ using an element of the right hand group.⁵ We do this by producing an element of the right hand group which corresponds to the central extension (12).

Denote the k th term of the lower central series of $\mathfrak{t}_g(x)$ by $\mathfrak{t}_g(x)^{(k)}$. We can form the extension

$$(14) \quad 0 \rightarrow \mathfrak{t}_g(x)^{(2)}/\mathfrak{t}_g(x)^{(3)} \rightarrow \mathfrak{t}_g(x)/\mathfrak{t}_g(x)^{(3)} \rightarrow H_1(\mathfrak{t}_g(x)) \rightarrow 0.$$

As has been pointed out above, the image of $\mathfrak{t}_g(x)^{(2)}/\mathfrak{t}_g(x)^{(3)}$ in the central \mathbb{G}_a in $\mathcal{T}_{g,r}^n(x)$ is non-trivial. The kernel of the extension (14) is a rational representation of Sp_g . Since $H_1(\mathfrak{t}_g(x))$ is irreducible, its second exterior power contains exactly one copy of the trivial representation. There is therefore a unique non-zero Sp_g -invariant projection

$$\mathfrak{t}_g(x)^{(2)}/\mathfrak{t}_g(x)^{(3)} \rightarrow \mathbb{G}_a.$$

If we push the extension (14) out along this map, we obtain a canonical element of

$$\mathrm{Ext}_{\mathcal{L}(\mathcal{M}_g)}^1(H_1(\mathbf{u}_g), \mathbb{Q}(1)).$$

⁴This group is one dimensional and generated by the Johnson homomorphism, although we do not really need to know this here — see [18, (5.2)].

⁵The class we seek, not surprisingly, is the one corresponding to the Johnson homomorphism, and is half the class associated to the algebraic cycle $C - C^-$ in $\mathrm{Jac} C$ — see [18, §8].

By the isomorphism (13), we obtain an element of

$$\text{Ext}_{\mathcal{H}(\mathcal{M}_g)}^1(H_1(\mathfrak{u}_g), \mathbb{Q}(1))$$

which allows us, for each point x of \mathcal{M}_g , to lift the MHS on $\mathfrak{u}_g(x)$ associated to x to a MHS on $\mathfrak{t}_g(x)$.

Our last task is to show that the bracket is a morphism of MHS. We only need show that the bracket preserves the Hodge and weight filtrations. First observe that since \mathbb{G}_a is central and contained in W_{-2} , the bracket preserves the weight filtration, and its restriction to $W_{-2}\mathfrak{t}_g(x)$ is a morphism of MHS. It remains to show that the bracket preserves the Hodge filtration. In view of these facts, it suffices to prove that

$$[F^p\mathfrak{t}_g(x), F^q\mathfrak{t}_g(x)] = 0$$

when $p + q > -1$. This is easily deduced from the fact that the map

$$[\ , \] : \Lambda^2 H_1(\mathfrak{t}_g) \rightarrow \text{Gr}_{-2}^W \mathfrak{t}_g(x) \xrightarrow{\text{proj}} \mathbb{G}_a \cong \mathbb{Q}(1)$$

induced by the bracket is a polarization of $H_1(\mathfrak{t}_g(x))$ as it is Sp_g equivariant and non-zero by results in [17, §7]. \square

Remark 4.11. It follows immediately from (4.5) and (4.10) that for each choice of base point in $\mathcal{M}_{g,r}^n$, the canonical morphism $\mathfrak{t}_{g,r}^n(x) \rightarrow \text{Der } \mathfrak{p}_{g,r}^n(x)$ is a morphism of MHS.

5. REVIEW OF CONTINUOUS COHOMOLOGY

In this section, we briefly review the theory of continuous cohomology of discrete groups, which is mainly developed in [15]. It will be our principal tool in proving that Torelli has a presentation with only quadratic relations. As a warm up, we show how it can be used to give a new and simpler proof of Morgan’s theorem that the complex form of the Lie algebra associated to the fundamental group of a smooth variety has a weighted homogeneous presentation with generators of weights equal to those occurring in $H_1(X)$, and relations of weight contained in those of $H_2(X)$.

Define the continuous cohomology of a discrete group π to be the direct limit of the rational cohomology of its finitely generated nilpotent quotients:

$$H_{\text{cts}}^\bullet(\pi, \mathbb{Q}) := \varinjlim H^\bullet(N, \mathbb{Q}),$$

where N ranges over the finitely generated nilpotent quotients of π . There is an obvious natural homomorphism

$$(15) \quad H_{\text{cts}}^\bullet(\pi, \mathbb{Q}) \rightarrow H^\bullet(\pi, \mathbb{Q}).$$

If X is a topological space with fundamental group π , then we also have a natural homomorphism

$$H_{\text{cts}}^\bullet(\pi, \mathbb{Q}) \rightarrow H^\bullet(X, \mathbb{Q})$$

as there is a canonical map $H^\bullet(\pi) \rightarrow H^\bullet(X)$.

Proposition 5.1. *If $H_1(\pi, \mathbb{Q})$ is finite dimensional, then the natural homomorphism (15) is an isomorphism in degree 1 and injective in degree 2. \square*

This is really a restatement of the result of Dennis Sullivan which asserts that the Lie algebra of the 1-minimal model of a space is the Malcev Lie algebra of the fundamental group. A more direct proof can be found, for example, in [15, (5.1)]. We present a new proof because it is elementary.

Proof. The group $H^2(G, \mathbb{Q})$ parameterizes central extensions of G by \mathbb{Q} . Suppose that $\alpha \in H_{\text{cts}}^2(\pi, \mathbb{Q})$ whose image in $H^2(\pi, \mathbb{Q})$ is trivial. Then there is a nilpotent quotient N of π and an element $\tilde{\alpha}$ of $H^2(N, \mathbb{Q})$ that is a lift of α . There is a central extension

$$(16) \quad 1 \rightarrow \mathbb{Q} \rightarrow E \rightarrow N \rightarrow 1$$

corresponding to $\tilde{\alpha}$. The key point to note is that E is nilpotent. To say that the image of α is trivial in $H^2(\pi, \mathbb{Q})$ is to say that the pullback of the extension (16) to π is split. Composing a splitting of this projection with the projection of $\pi \rightarrow N$ gives a homomorphism $\pi \rightarrow E$ which lifts $\pi \rightarrow N$. Denote the image of π in E by \tilde{N} . It is easy to see that the pullback of the extension (16) to \tilde{N} splits. Since \tilde{N} is a nilpotent quotient of π , the class α vanishes. \square

Similarly, we can define the continuous cohomology of a pronilpotent Lie algebra \mathfrak{g} to be the direct limit of the cohomology of its finite-dimensional nilpotent quotients:

$$H_{\text{cts}}^\bullet(\mathfrak{g}) := \varinjlim H^\bullet(\mathfrak{n}),$$

where \mathfrak{n} ranges over the finite-dimensional nilpotent quotients of \mathfrak{g} .

A mild generalization of a theorem of Nomizu [42] states that for each finitely generated nilpotent group N there is a natural isomorphism

$$H^\bullet(\mathfrak{n}) \cong H^\bullet(N, \mathbb{Q}),$$

where \mathfrak{n} is the Lie algebra of the Malcev completion of N . It follows immediately from the definitions that if π is a finitely generated group and \mathfrak{p} the associated Malcev Lie algebra, then there is a natural isomorphism

$$H_{\text{cts}}^\bullet(\mathfrak{p}) \cong H_{\text{cts}}^\bullet(\pi, \mathbb{Q}).$$

The continuous cohomology of a pronilpotent Lie algebra \mathfrak{g} can be computed using the standard complex $\mathcal{C}^\bullet(\mathfrak{g})$ of continuous cochains of \mathfrak{g} . This is defined to be the direct limit of the Chevalley-Eilenberg cochains of the finite-dimensional nilpotent quotients of \mathfrak{g} . Denote the continuous dual of \mathfrak{g} by \mathfrak{g}^* . Then we have a d.g.a. isomorphism

$$\mathcal{C}^\bullet(\mathfrak{g}) = \Lambda^\bullet(\mathfrak{g}^*[-1]);$$

the differential is derivation of degree 1 whose restriction to \mathfrak{g}^* is minus the dual of the bracket.

The definition of continuous cohomology can be extended to the case where the coefficients are \mathbb{Q} modules on which \mathfrak{g} acts via a representation of one of its nilpotent quotients — cf. [15].

Suppose now that $H_1(\mathfrak{g})$ is finite dimensional. If \mathfrak{g} has a MHS, then, by linear algebra, so do $\mathcal{C}^\bullet(\mathfrak{g})$ and $H^\bullet(\mathfrak{g})$. We will call such a pronilpotent Lie algebra a *Hodge Lie algebra*. It follows that if X is an algebraic variety, $x \in X$, then $H_{\text{cts}}^\bullet(\pi_1(X, x), \mathbb{Q})$ has a canonical MHS. One can show that this MHS does not depend on the base point x of X — [15].

Since the weight filtration of a MHS splits canonically over \mathbb{C} , each finite-dimensional Hodge Lie algebra \mathfrak{g} is canonically isomorphic to the graded Lie algebra $\text{Gr}_\bullet^W \mathfrak{g}$ after tensoring with \mathbb{C} . The following result therefore follows by taking inverse limits.

Proposition 5.2. *If \mathfrak{g} is a Hodge Lie algebra, all of whose weights are negative, then there is a canonical Lie algebra isomorphism*

$$\mathfrak{g}_{\mathbb{C}} \cong \prod_{l \geq 1} \text{Gr}_{-l}^W \mathfrak{g}_{\mathbb{C}}. \quad \square$$

Since each choice of a base point of $\mathcal{M}_{g,r}^n$ determines a canonical MHS on $\mathfrak{t}_{g,r}^n$, we have:

Corollary 5.3. *For each choice of a base point of $\mathcal{M}_{g,r}^n$, there is a canonical isomorphism*

$$\mathfrak{t}_{g,r}^n \otimes \mathbb{C} \cong \prod_{l \geq 1} (\text{Gr}_{-l}^W \mathfrak{t}_{g,r}^n \otimes \mathbb{C}). \quad \square$$

The following result is proved, for example, in [10, (11.7)]. In Section 7 we will prove a generalization needed for studying the relations in \mathfrak{t}_g .

Theorem 5.4. *If X is a smooth algebraic variety, then the natural homomorphism*

$$H_{\text{cts}}^\bullet(\pi_1(X), \mathbb{Q}) \rightarrow H^\bullet(X, \mathbb{Q})$$

is a morphism of mixed Hodge structures.

The final two results in this section together will allow us to use continuous cohomology as an effective tool for studying relations in Hodge Lie algebras in general, and \mathfrak{t}_g in particular.

The cochains, and therefore the cohomology, of a graded Lie algebra both have an extra grading, and are therefore bigraded algebras. If \mathfrak{g} is a Hodge Lie algebra, then $\text{Gr}_\bullet^W \mathfrak{g}$ has an extra grading by weight. Since the functor Gr^W is exact on the category of MHS, we have:

Proposition 5.5. *If \mathfrak{g} is a Hodge Lie algebra, then there is a canonical bigraded algebra isomorphism*

$$\text{Gr}_\bullet^W H_{\text{cts}}^\bullet(\mathfrak{g}) \cong H^\bullet(\text{Gr}_\bullet^W \mathfrak{g}). \quad \square$$

If \mathfrak{g} is a graded Lie algebra with negative weights, then we can write \mathfrak{g} as a quotient of the free graded Lie algebra \mathfrak{f} generated by $H_1(\mathfrak{g})$ modulo a homogeneous ideal \mathfrak{r} . Note that we are not assuming that $H_1(\mathfrak{g})$ is pure — in general it will be graded. The group

$$H_0(\mathfrak{f}/\mathfrak{r}) = \mathfrak{r}/[\mathfrak{f}, \mathfrak{r}]$$

is graded. One can obtain a minimal set of relations of \mathfrak{g} by taking the image of any splitting of the projection

$$\mathfrak{r} \rightarrow H_0(\mathfrak{f}/\mathfrak{r}).$$

The following result is an analogue of Hopf’s description of the second homology of a group in terms of a presentation.

Proposition 5.6. *If \mathfrak{g} is a graded Lie algebra with negative weights, then there is a canonical isomorphism of graded vector spaces*

$$H_0(\mathfrak{f}/\mathfrak{r}) \cong H_2(\mathfrak{g}).$$

Proof. There are several ways to see this. One is to look closely at the Chevalley-Eilenberg cochains of \mathfrak{g} . The second is to use the fact that a sub-Lie algebra of a free Lie algebra is free [46, (2.5)] to deduce that, as a Lie algebra, \mathfrak{r} is free. Then apply the Lie algebra analogue of the Hochschild-Serre spectral sequence to the extension

$$0 \rightarrow \mathfrak{r} \rightarrow \mathfrak{f} \rightarrow \mathfrak{g} \rightarrow 0.$$

The details are standard and are omitted. □

Corollary 5.7. *If \mathfrak{g} is a graded Lie algebra with negative weights, then there is an injective linear map*

$$\delta : H_2(\mathfrak{g}) \hookrightarrow \mathbb{L}(H_1(\mathfrak{g}))$$

of graded vector spaces such that \mathfrak{g} has presentation

$$\mathbb{L}(H_1(\mathfrak{g})) / (\text{im } \delta)$$

in the category of graded Lie algebras.

Combining (5.2), (5.7) and the existence of a canonical MHS on the Malcev Lie algebra $\mathfrak{g}(X, x)$ associated to a pointed variety, we obtain one of Morgan’s theorems [35, (10.3)].

Theorem 5.8. *If X is a smooth complex algebraic variety and $x \in X$, then the complex Malcev Lie algebra $\mathfrak{g}(X, x)_{\mathbb{C}}$ associated to $\pi_1(X, x)$ has the property that*

$$\mathfrak{g}(X, x)_{\mathbb{C}} \cong \prod_{l \geq 1} \text{Gr}_{-l}^W \mathfrak{g}_{\mathbb{C}}$$

and there is a homomorphism of graded vector spaces

$$\delta : H_2(X, \mathbb{C}) \rightarrow \mathbb{L}(\text{Gr}_{\bullet}^W H_1(X))$$

such that

$$\text{Gr}_{\bullet}^W \mathfrak{g}_{\mathbb{C}} \cong \mathbb{L}(\text{Gr}_{\bullet}^W H_1(X, \mathbb{C})) / (\delta(\text{Gr}_{\bullet}^W H_2(X, \mathbb{C})))$$

in the category of graded Lie algebras. □

6. REMARKS ON THE REPRESENTATIONS OF \mathfrak{sp}_g

In this section we review some basic facts from the representation theory that we shall need in subsequent sections. A good reference is [13].

Denote the Lie algebra of Sp_g by \mathfrak{sp}_g . The representation theory of the group and the Lie algebra are the same. Denote their common representation ring by $R(\mathfrak{sp}_g)$. Choose a symplectic basis $a_1, \dots, a_g, b_1, \dots, b_g$ of the fundamental representation of \mathfrak{sp}_g . Denote by \mathfrak{h} the torus in \mathfrak{sp}_g consisting of matrices that are diagonal with respect to this basis. Choose coordinates $t = (t_1, \dots, t_g)$ on \mathfrak{h} so that

$$t \cdot a_i = t_i a_i \text{ and } t \cdot b_i = -t_i b_i.$$

The subalgebra of positive nilpotents \mathfrak{n} has as basis the elements $S_{i,j}$ ($i < j$), T_i , and $F_{i,j}$ ($i \neq j$) of \mathfrak{sp}_g , where

$$S_{i,j}(a_j) = a_i, \quad S_{i,j}(b_i) = -b_j, \quad S_{i,j}(\text{other basis vectors}) = 0,$$

$$T_i(b_i) = a_i, \quad T_i(\text{other basis vectors}) = 0,$$

$$F_{i,j}(b_i) = a_j, \quad F_{i,j}(b_j) = a_i, \quad F_{i,j}(\text{other basis vectors}) = 0.$$

A fundamental set of weights of \mathfrak{sp}_g is $\lambda_j : \mathfrak{h} \rightarrow \mathbb{R}, 1 \leq j \leq g$, where λ_j is defined by

$$\lambda_j(t) = t_1 + t_2 + \cdots + t_j.$$

The irreducible representations of \mathfrak{sp}_g correspond to positive integral linear combinations λ of the λ_j . Denote the irreducible representation of \mathfrak{sp}_g with highest weight λ by $V(\lambda)$.

The irreducible representations of \mathfrak{sp}_g can also be indexed by partitions α of an integer n into $\leq g$ parts:

$$n = \alpha_1 + \alpha_2 + \cdots + \alpha_g$$

where

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_g \geq 0.$$

The irreducible representation corresponding to α has highest weight

$$t \mapsto \sum_j \alpha_j t_j.$$

We shall denote the integer

$$\sum_{k=1}^g \alpha_k = \sum_{k=1}^g k n_k$$

by $|\alpha|$ or by $|\lambda|$ according to whether we are using partitions or highest weights. This can be considered as a measure of the size of the corresponding irreducible representation; it is the smallest positive integer d such that $V(\lambda)$ occurs in the d th tensor power of the fundamental representation.

There is a notion of stability of the decomposition of tensor products and Schur functors of representations of symplectic groups. In order to state the result, we need to first define the *depth*, $\delta(V)$, of a representation V of \mathfrak{sp}_g . If the module is irreducible with highest weight $\sum n_k \lambda_k$, define $\delta(V)$ to be the largest d such that $n_d \neq 0$ — or, equivalently, it is the number of rows in the corresponding Young diagram. Define the depth of an arbitrary representation to be the maximum of the depths of its irreducible components.

In order to discuss stability, we will need a stabilization map. When $h \geq g$, define a group homomorphism

$$R(\mathfrak{sp}_g) \hookrightarrow R(\mathfrak{sp}_h)$$

by taking the irreducible representation of \mathfrak{sp}_g corresponding to the partition α to the representation of \mathfrak{sp}_h corresponding to the same partition. Equivalently, take the representation of \mathfrak{sp}_g with highest weight $\sum n_k \lambda_k$ to the representation of \mathfrak{sp}_h with the same highest weight decomposition.

Recall that to each partition β of a positive integer n , one has a Schur functor \mathbb{S}_β defined on the category of representations of each group. For example, if $\beta = [n]$, then \mathbb{S}_β is the n th symmetric power, if $\beta = [1^n]$, then \mathbb{S}_β is the n th exterior power. We shall denote the integer n by $|\beta|$.

The second assertion of the following stability result appears to be folklore — the only proof I know of is in Kabanov’s thesis.

Theorem 6.1. (1) ([13, p. 424]) *If V and W are representations of \mathfrak{sp}_g and $\delta(V) + \delta(W) \leq g$, then the irreducible representations and their multiplicities occurring in the decomposition of $V \otimes W$ are independent of g .*

(2) ([24] — see also [26]) *If V is a representation of \mathfrak{sp}_g and β is a partition with $|\beta|\delta(V) \leq g$, then the decomposition of $\mathbb{S}_\beta V$ into irreducible components is independent of g .* \square

Remark 6.2. Some of the computations of highest weight decompositions in this paper have been made using the computer program LiE from the University of Amsterdam. The computations were performed for a particular g in the stable range. The stability theorem was then used to deduce the decomposition for all g in the stable range. Note that all such computations were checked using several values of g in the stable range. In addition, many of the unstable computations were done using LiE.

By composition with the canonical homomorphism

$$\Gamma_{g,r}^n \rightarrow Sp_g(\mathbb{Q})$$

we see that each representation V of \mathfrak{sp}_g gives rise to a local system over $\mathcal{M}_{g,r}^n$, at least in the orbifold sense. It is a standard fact that each such local system arising from an irreducible representation of \mathfrak{sp}_g is an admissible variation of Hodge structure over $\mathcal{M}_{g,r}^n$ in a unique way up to Tate twist — cf. [18, (9.1)]. It can always be realized as a variation of weight $|\lambda|$, and we shall take this as the default weight.

We would like to discuss the cohomology of $\mathcal{M}_{g,r}^n$ with coefficients in such a local system. To do this, first choose a level l such that $\Gamma_{g,r}^n$ is torsion free. Then $\mathcal{M}_{g,r}^n[l]$ is smooth and the variation of Hodge structure \mathbb{V} corresponding to an irreducible representation V of Sp_g is defined over $\mathcal{M}_{g,r}^n[l]$ and has a natural $Sp_g(\mathbb{Z}/l\mathbb{Z})$ action. From the work of M. Saito [47], we know that $H^k(\mathcal{M}_{g,r}^n[l], \mathbb{V})$ has a canonical mixed Hodge structure with weights $\geq k + \text{weight}(V)$. The action of $Sp_g(\mathbb{Z}/l\mathbb{Z})$ preserves this MHS. So we can define

$$H^\bullet(\mathcal{M}_{g,r}^n, \mathbb{V}) = H^0(Sp_g(\mathbb{Z}/l\mathbb{Z}), H^\bullet(\mathcal{M}_{g,r}^n[l], \mathbb{V}))$$

as a MHS. Note that the underlying group is canonically isomorphic to $H^\bullet(\Gamma_{g,r}^n, V)$.

7. CONTINUOUS COHOMOLOGY OF TORELLI GROUPS

The next step in finding a presentation of $\mathfrak{t}_{g,r}^n$ is to determine the relations in $\text{Gr}_\bullet^W \mathfrak{t}_{g,r}^n$. Since this is a graded Lie algebra generated in degree -1 , the generators of the ideal of relations is homogeneous. In this section we will use a result of Kabanov [24] (see also [25]) about the second cohomology of $\Gamma_{g,r}^n$ to show that the ideal of relations in $\text{Gr}_\bullet^W \mathfrak{t}_{g,r}^n$ is generated by quadratic and cubic generators when $g \geq 3$, and quadratic relations alone when $g \geq 6$. Our principal tool will be the continuous cohomology defined in Section 5.

First some notation. Take X and \mathbb{V} as in the statement of (4.1). Denote the Lie algebra associated to the pronipotent radical of the completion of $\pi_1(X, x)$ relative to ρ by $\mathfrak{u}(x)$. This is a Hodge Lie algebra. The next result is a generalization of (5.4).

Proposition 7.1. *Suppose that \mathbb{W} is an admissible variation of Hodge structure over X which is a subquotient of a tensor power of \mathbb{V} . Then for all $k \geq 0$ and each*

$x \in X$, there is a natural homomorphism

$$H^0(X, H^k_{\text{cts}}(\mathfrak{u}(x)) \otimes W_x) \rightarrow H^k(X, \mathbb{W})$$

which is a morphism of MHS. It is an isomorphism when $k = 1$ and injective when $k = 2$.

Proof. The case $k = 1$ is proved in [19, (10.3),(13.8)]. We will prove the result when $k > 1$ by induction. The most important case for us is when $k = 2$, so we will give that argument in more detail and briefly sketch the remaining cases. We will assume throughout that the reader is familiar with [19]. A convenient auxiliary reference for rational homotopy theory is [14, §2].

We begin by recalling some well-known facts from rational homotopy theory. The base point $x \in X$ determines an augmentation

$$\epsilon_x : E_{\text{fin}}^\bullet(X, \mathcal{O}(P)) \rightarrow \mathbb{R},$$

where P is the principal bundle defined in [19, §4]. We shall write \mathcal{O} instead of $\mathcal{O}(P)$. We can form the bar construction

$$B(E_{\text{fin}}^\bullet(X, \mathcal{O})) := B(\mathbb{R}, E_{\text{fin}}^\bullet(X, \mathcal{O}), \mathbb{R}),$$

where both copies of \mathbb{R} are regarded as $E_{\text{fin}}^\bullet(X, \mathcal{O})$ modules via ϵ_x . The Lie algebra $\mathfrak{u}(x)$ is determined by $B(E_{\text{fin}}^\bullet(X, \mathcal{O}))$ as follows: the dual of

$$H^0(B(E_{\text{fin}}^\bullet(X, \mathcal{O})))$$

is a complete Hopf algebra, $\mathfrak{u}(x)$ is its set of primitive elements. (See, for example, [14, (2.4.5) and §2.6].) There is an augmentation preserving d.g.a. homomorphism

$$(17) \quad \mathcal{C}^\bullet(\mathfrak{u}(x)) \rightarrow E_{\text{fin}}^\bullet(X, \mathcal{O}),$$

unique up to homotopy, which induces the map

$$\theta : H_{\text{cts}}^\bullet(\mathfrak{u}(x)) \rightarrow H^\bullet(E_{\text{fin}}^\bullet(X, \mathcal{O}))$$

on homology. The map θ is an isomorphism in degree 1 and injective in degree 2.⁶ There is a canonical isomorphism

$$H^k(X, \mathbb{W}) \cong H^0(X, H^k(E_{\text{fin}}^\bullet(X, \mathcal{O})) \otimes \mathbb{W})$$

of MHSs for each VHS \mathbb{W} over X whose monodromy representation is the pullback of a rational representation of $\text{Aut}(V_o, q)$ via the representation ρ . Since $\mathfrak{u}(x)$ has a canonical MHS, and since $E_{\text{fin}}^\bullet(X, \mathcal{O})$ is a mixed Hodge complex, each of the domain and target of θ have a canonical MHS. To prove the result, we need only prove that θ is a morphism of MHS.

First we give an intuitive proof. The image of the map

$$\theta^2 : H_{\text{cts}}^2(\mathfrak{u}) \rightarrow H^2(E_{\text{fin}}^\bullet(X, \mathcal{O}))$$

is the subspace of the right hand side generated by the cup product $H^1 \otimes H^1 \rightarrow H^2$, all Massey triple products of 1-forms, all Massey quadruple products of 1-forms, etc. Since the cup product and all Massey k -fold products have domain which is a sub-MHS of $\otimes^k H^1$ and are themselves morphisms, it follows that the image of θ^2 is a MHS. That θ^2 is a morphism follows as θ^1 is an isomorphism of MHS. One can continue in an analogous fashion to prove that each θ^k is a morphism.

We now make this argument precise. The spectral sequence associated to the standard filtration of the bar construction is called the Eilenberg-Moore spectral

⁶In the language of Sullivan [50], the map (17) is the 1-minimal model of $E_{\text{fin}}^\bullet(X, \mathcal{O})$.

sequence (EMss): for an augmented d.g.a. A^\bullet with connected homology, it takes the form

$$E_1^{-s,t} = [\otimes^s H^+(A)]^t \implies H^{t-s}(B(A)).$$

Denote the EMss associated to $\mathcal{C}(\mathbf{u})^\bullet$ by $\{E_r(\mathbf{u})\}$ and the EMss associated to $E_{\text{fin}}^\bullet(X, \mathcal{O})$ by $\{E_r(X)\}$.

The map (17) induces a morphism of Eilenberg-Moore spectral sequences. Each of these is a spectral sequence of MHSs as both the domain and target of (17) are mixed Hodge complexes, but we have to prove that the map between them is a morphism of MHSs. This is the case in total degree 0 as $E_1^{-s,s}$ is $\otimes^s H^1$ and θ^1 is an isomorphism of MHS.

It is a standard fact that

$$H^k(B(\mathcal{C}^\bullet(\mathbf{u}))) = 0$$

when $k > 0$; cf. [14, (2.6.2)] and [6]. Therefore, the $E_\infty^{-1,2}$ term of the associated EMss vanishes. (This is a precise way to say that $H_{\text{cts}}^2(\mathbf{u})$ is generated by Massey products.) The edge homomorphisms

$$H_{\text{cts}}^k(\mathbf{u}) = E_1^{-1,k}(\mathbf{u}) \rightarrow E_r^{-1,k}(\mathbf{u})$$

are all surjective. Let M_r^k be the inverse image in $H_{\text{cts}}^k(\mathbf{u})$ of the image of

$$d_{r-1} : E_{r-1}^{-r,k+r-2}(\mathbf{u}) \rightarrow E_{r-1}^{-1,k}(\mathbf{u}).$$

Then the fact that the higher cohomology of $B(\mathcal{C}^\bullet(\mathbf{u}))$ vanishes implies that whenever $k \geq 2$

$$H_{\text{cts}}^k(\mathbf{u}) = \bigcup_r M_r^k.$$

Since the spectral sequence is a spectral sequence of MHS, each M_r^k is a sub-MHS of $H_{\text{cts}}^k(\mathbf{u})$.

Since both spectral sequences are spectral sequences of MHSs, it follows that the image of

$$H_{\text{cts}}^2(\mathbf{u}) = E_1^{-1,2}(\mathbf{u}) \rightarrow E_1^{-1,2}(X) = H^2(E_{\text{fin}}^\bullet(X, \mathcal{O}))$$

is a sub-MHS and that θ^2 is a morphism of MHS.

If $k > 2$, one can assume by induction that θ^m is a morphism whenever $m < k$. It follows easily that the natural map

$$E_1^{-s,t}(\mathbf{u}) \rightarrow E_1^{-s,t}(X)$$

is a morphism of MHS whenever $-s+t < k-1$, and therefore that its image is a sub-MHS of $E_1^{-s+t}(X)$. But since these spectral sequences are spectral sequences in the category of MHSs, and since $E_\infty^{-1,k}(\mathbf{u})$ vanishes, it follows that θ^k is a morphism. \square

Remark 7.2. This is a continuation of Remark 3.9. The previous result implies that $\mathcal{U}_1[l]$ is a free pronilpotent group as $SL_2(\mathbb{Z})[l]$ has a free subgroup of finite index, which implies that $H^2(\mathcal{M}_1[l], S^n V)$ vanishes for all n . It follows that $H^2(\mathbf{u}_1[l])$ vanishes and from (5.7) that $\mathbf{u}_1[l]$ is free.

We thus have the following version of (7.1) for moduli spaces of curves.

Proposition 7.3. *If $g \geq 3$ and \mathbb{V} is a variation of Hodge structure over $\mathcal{M}_{g,r}^n$, whose monodromy representation comes from a rational representation of Sp_g , then for all k , there is a natural map*

$$H^0(Sp_g, H_{\text{cts}}^k(\mathbf{u}_{g,r}^n(x)) \otimes V_x) \rightarrow H^k(\mathcal{M}_{g,r}^n, \mathbb{V})$$

which is a morphism of MHS. It is an isomorphism when $k = 1$ and injective when $k = 2$. Here V_x denotes the fiber of \mathbb{V} over x . \square

Denote the λ isotypical part of an Sp_g module V by V_λ .

Corollary 7.4. *If $g \geq 3$ and λ is a dominant integral weight of Sp_g , then*

$$\dim \text{Gr}_l^W H_{\text{cts}}^2(\mathbf{u}_{g,r}^n)_\lambda \leq \dim \text{Gr}_{l+|\lambda|}^W H^2(\mathcal{M}_{g,r}^n, \mathbb{V}(\lambda)). \quad \square$$

Proposition 7.5. *For all $g \geq 3$, there is an Sp_g equivariant exact sequence*

$$0 \rightarrow \mathbb{Q}(1) \rightarrow H_{\text{cts}}^2(\mathbf{u}_{g,r}^n(x)) \rightarrow H_{\text{cts}}^2(\mathbf{t}_{g,r}^n(x)) \rightarrow 0$$

of mixed Hodge structures. In particular,

$$\dim \text{Gr}_l^W H_{\text{cts}}^2(\mathbf{u}_{g,r}^n)_\lambda = \dim \text{Gr}_l^W H_{\text{cts}}^2(\mathbf{t}_{g,r}^n)_\lambda + \delta_2^l \delta_0^\lambda.$$

Here we are using Kronecker's delta notation.

Proof. This follows directly from (3.4) and [15, (5.5)]. \square

So, in order to bound the degrees of the relations in $\mathbf{t}_{g,r}^n$ (and $\mathbf{u}_{g,r}^n$), it suffices to give a bound the weights on $H^2(\mathcal{M}_{g,r}^n, \mathbb{V}(\lambda))$. In the absolute case we have the following result of Kabanov [24, 25] which is proved using intersection homology.

Theorem 7.6 (Kabanov). *For each irreducible representation $V(\lambda)$ of Sp_g , we have*

$$\text{Gr}_{k+|\lambda|}^W H^2(\mathcal{M}_g, \mathbb{V}(\lambda)) = 0 \text{ when } \begin{cases} k \neq 2 & \text{and } g \geq 6; \\ k \neq 2, 3 & \text{and } 3 \leq g < 6. \end{cases}$$

Combining Kabanov's result with the previous results, we obtain:

Corollary 7.7. *If $g \geq 3$, then $\text{Gr}_\bullet^W \mathbf{t}_g$ has a presentation with only quadratic and cubic relations, and only quadratic relations when $g \geq 6$. \square*

It is now an easy matter to insert the decorations:

Corollary 7.8. *If $g \geq 3$, then $\text{Gr}_\bullet^W \mathbf{t}_{g,r}^n$ has a presentation with only quadratic and cubic relations, and only quadratic relations when $g \geq 6$. \square*

The corresponding result with $\mathbf{t}_{g,r}^n$ replaced by $\mathbf{u}_{g,r}^n$ also holds.

8. THE LOWER CENTRAL SERIES QUOTIENTS OF A SURFACE GROUP

In this section we gather some information about $\text{Gr } \mathfrak{p}_g^1$ that will be useful when computing relations in $\text{Gr } \mathbf{t}_g$ and $\text{Gr } \mathbf{t}_g^1$. Our basic tool, once again, is continuous cohomology.

A group is called *pseudo-nilpotent* if θ is an isomorphism. A proof of the following result is sketched by Kohno and Oda in [30, (4.1)].

Theorem 8.1. *If $g \geq 1$, then π_g^1 is pseudo-nilpotent. \square*

Even though we will not be needing it, we record the following result which is stated by Kohno and Oda [30, (4.1)]. Their proof is incorrect — cf. (2.2). Nonetheless, the result follows directly from (8.1) and [15, (5.7)].

Corollary 8.2. *If $g = 0$ and $r \geq 1$, or if $g \geq 1$, then, for all $n \geq 0$, each of the decorated pure braid groups $F_{g,r}^n$ is pseudo-nilpotent.* \square

Since $H_1(\mathfrak{p}_g^1)$ is the fundamental representation of \mathfrak{sp}_g , $\text{Gr}_\bullet^W \mathfrak{p}_g^1$ is a graded Lie algebra in $R(\mathfrak{sp}_g)$, and its complex of chains $\Lambda^\bullet \text{Gr}_\bullet^W \mathfrak{p}_g^1$ is a complex in $R(\mathfrak{sp}_g)$.

We shall write \mathfrak{p}_g instead of \mathfrak{p}_g^1 , and π_g instead of π_g^1 . We shall denote the l th weight graded quotient of a Hodge Lie algebra \mathfrak{g} by $\mathfrak{g}(l)$. In particular, we shall denote $\text{Gr}_{-l}^W \mathfrak{p}_g^1$ by $\mathfrak{p}_g(l)$.

Corollary 8.3. *If $g \geq 1$, then, for each $l \geq 3$, the complex*

$$\text{Gr}_{-l}^W \Lambda^\bullet \text{Gr}_\bullet^W \mathfrak{p}_g$$

is an acyclic complex of \mathfrak{sp}_g modules. When $l = 2$, we have an exact sequence

$$0 \rightarrow \mathbb{Q}(1) \rightarrow \Lambda^2 \mathfrak{p}_g(1) \rightarrow \mathfrak{p}_g(2) \rightarrow 0$$

of \mathfrak{sp}_g modules. \square

This result allows us to compute the $\mathfrak{p}_g(l)$ inductively as elements of $R(\mathfrak{sp}_g)$. As before, we fix a set $\lambda_1, \dots, \lambda_g$ of fundamental weights of \mathfrak{sp}_g .

Proposition 8.4. *For all $g \geq 3$, the highest weight decomposition of $\mathfrak{p}_g(l)$ when $1 \leq l \leq 4$ is given by*

$$\mathfrak{p}_g(l) = \begin{cases} V(\lambda_1) & \text{when } l = 1; \\ V(\lambda_2) & \text{when } l = 2; \\ V(\lambda_1 + \lambda_2) & \text{when } l = 3; \\ V(2\lambda_1) + V(2\lambda_1 + \lambda_2) + V(\lambda_1 + \lambda_3) & \text{when } l = 4. \end{cases}$$

Proof. This is a straightforward consequence of (8.3). To show how this works, we prove the case where $l = 3$. In this case, we have the exact sequence

$$0 \rightarrow \Lambda^3 \mathfrak{p}_g(1) \rightarrow \mathfrak{p}_g(1) \otimes \mathfrak{p}_g(2) \rightarrow \mathfrak{p}_g(3) \rightarrow 0$$

in $R(\mathfrak{sp}_g)$. Taking Euler characteristics and applying the result for $l = 2$ and $g \geq 3$, we see that

$$\mathfrak{p}_g(3) = V(\lambda_1) \otimes V(\lambda_2) - \Lambda^3 V(\lambda_1) = V(\lambda_1 + \lambda_2).$$

\square

Since the k th exterior power is the Schur functor corresponding to the Young diagram with k rows and one box in each row, and since $\mathfrak{p}_g(1) = H_1(\pi_g)$ is the fundamental representation of \mathfrak{sp}_g , we obtain the following stability result for the graded quotients of the lower central series of π_g .

Corollary 8.5. *The highest weight decomposition of $\mathfrak{p}_g(l)$ is independent of g when $g \geq l$.* \square

9. THE ACTION OF \mathfrak{t}_g^1 ON \mathfrak{p}_g

In this section we obtain a lower bound for the size of $\text{Gr}_l^W \mathfrak{t}_g$ when $l = 2, 3$ and $g \geq 3$ by studying the action of \mathfrak{t}_g^1 on \mathfrak{p}_g . This will provide an upper bound on the size of the quadratic and cubic relations of $\text{Gr}_\bullet^W \mathfrak{t}_g$. Related results have been obtained by Asada-Kaneko [1], Morita [40] and Asada-Nakamura [2]. We also use a result [2] of Asada and Nakamura to prove that \mathfrak{t}_g is infinite dimensional. (This fact also follows from a recent result of Oda [43].)

First, some notation. Denote the prnilpotent Lie algebra $W_{-1} \text{Der } \mathfrak{p}_g$ by \mathfrak{d}_g , and the quotient of this by inner automorphisms by \mathfrak{o}_g . Once a base point x of \mathcal{M}_g^1 has been chosen, each of these acquires the structure of a Hodge Lie algebra.

We have natural representations

$$\mathfrak{u}_g^1 \rightarrow \mathfrak{d}_g \text{ and } \mathfrak{u}_g \rightarrow \mathfrak{o}_g.$$

These induce homomorphisms of their associated graded Lie algebras.

It is clear that there is an injective homomorphism

$$\mathfrak{d}_g(l) \hookrightarrow \text{Hom}(\mathfrak{p}_g(1), \mathfrak{p}_g(l + 1)).$$

Each element $\delta : \mathfrak{p}_g(1) \rightarrow \mathfrak{p}_g(l + 1)$ determines a derivation $\tilde{\delta}$ of the free Lie algebra $\mathbb{L}(\mathfrak{p}_g(1))$, the second graded quotient of which is isomorphic to $\mathfrak{p}_g(2) \oplus \mathbb{Q}\omega$, where $\mathbb{Q}\omega$ is the unique copy of the trivial representation in $\Lambda^2 \mathfrak{p}_g(1)$. By taking the image of $\tilde{\delta}(\omega)$ under the projection

$$\mathbb{L}(\mathfrak{p}_g(1)) \rightarrow \text{Gr}_\bullet^W \mathfrak{p}_g,$$

we obtain an element $\sigma_g(\delta)$ of $\mathfrak{p}_g(l + 2)$. Observe that δ induces a derivation of $\text{Gr } \mathfrak{p}_g$ if and only if $\sigma_g(\delta)$ vanishes. We therefore have an isomorphism

$$\mathfrak{d}_g(l) \rightarrow \ker \left\{ \text{Hom}(\mathfrak{p}_g(1), \mathfrak{p}_g(l + 1)) \xrightarrow{\sigma_g} \mathfrak{p}_g(l + 2) \right\}.$$

Proposition 9.1. *The map σ_g is surjective. Consequently,*

$$\mathfrak{d}_g(l) = \mathfrak{p}_g(1) \otimes \mathfrak{p}_g(l + 1) - \mathfrak{p}_g(l + 2)$$

in $R(\mathfrak{sp}_g)$.

Proof. Consider the diagram

$$\begin{array}{ccc} \text{Hom}(\mathfrak{p}_g(1), \mathfrak{p}_g(l + 1)) & \xrightarrow{\sigma_g} & \mathfrak{p}_g(l + 2) \\ \downarrow & & \parallel \\ \mathfrak{p}_g(1) \otimes \mathfrak{p}_g(l + 1) & \xrightarrow{[\cdot, \cdot]} & \mathfrak{p}_g(l + 2) \end{array}$$

where the left hand vertical map is induced by the quadratic form $\omega = \sum a_i \wedge b_i$. This diagram commutes as the left hand map satisfies

$$\delta \mapsto \sum_{i=1}^g a_i \otimes \delta(b_i) - b_i \otimes \delta(a_i),$$

which goes to

$$\sigma_g(q) = \sum_{i=1}^g ([\delta(a_i), b_i] + [a_i, \delta(b_i)])$$

under the bracket. Since the bottom map is surjective and all maps are \mathfrak{sp}_g equivariant, the result follows. \square

Combining this with the computation of the first few graded quotients of \mathfrak{p}_g given in (8.4), we obtain the following result.

Corollary 9.2. *For all $g \geq 3$, we have*

$$\mathfrak{d}_g(l) = \begin{cases} V(\lambda_3) + V(\lambda_1) & \text{when } l = 1; \\ V(2\lambda_2) + V(\lambda_2) & \text{when } l = 2; \\ V(2\lambda_1 + \lambda_3) + V(\lambda_1 + \lambda_2) + V(3\lambda_1) & \text{when } l = 3. \end{cases}$$

□

The computation of $\mathfrak{d}_g(1)$ is simply another formulation of the Johnson homomorphism.

It is proven in [1, A', p. 149] that the center of $\text{Gr } \mathfrak{p}_g$ is trivial, so that the inclusion $\mathfrak{p}_g \rightarrow \mathfrak{d}_g$ of the inner automorphisms is injective.

Proposition 9.3. *For all $g \geq 3$ and all $l \geq 1$, $\mathfrak{o}_g(l) = \mathfrak{d}_g(l) - \mathfrak{p}_g(l)$.* □

Combining (8.4) and (9.2), we obtain the following computation.

Corollary 9.4. *For all $g \geq 3$, we have*

$$\mathfrak{o}_g(l) = \begin{cases} V(\lambda_3) & \text{when } l = 1; \\ V(2\lambda_2) & \text{when } l = 2; \\ V(2\lambda_1 + \lambda_3) + V(3\lambda_1) & \text{when } l = 3. \end{cases}$$

□

It does not seem obvious *a priori*, that \mathfrak{t}_g is infinite dimensional.

Proposition 9.5. *For all $g \geq 3$, the image of \mathfrak{t}_g in \mathfrak{o}_g is infinite dimensional.*⁷

Proof. Since $\mathfrak{t}_g \rightarrow \mathfrak{o}_g$ is a morphism of MHS, the image \mathfrak{g} has a MHS. Since Gr_\bullet^W is an exact functor,

$$\text{Gr}_\bullet^W \mathfrak{g} = \text{image of } \{ \text{Gr}_\bullet^W \mathfrak{t}_g \rightarrow \text{Gr}_\bullet^W \mathfrak{o}_g \}.$$

So it suffices to show that each graded quotient of \mathfrak{g} is non-trivial. It follows from the result of Asada and Nakamura [2, Theorem B] that the image of

$$\mathfrak{t}_g^1(l) \rightarrow \mathfrak{d}_g(l)$$

contains the representation $V(2m\lambda_1 + \lambda_3)$ when $l = 2m + 1$, and $V(2m\lambda_1 + 2\lambda_2)$ when $l = 2m + 2$. These representations both have the maximal possible depth, $l + 2$. But the inner automorphisms $\mathfrak{p}_g(l)$ in $\mathfrak{d}_g(l)$ have depth at most l . The result follows. □

We can now bound above the low degree relations in \mathfrak{t}_g .

Proposition 9.6. *For all $g \geq 3$, the image of $\mathfrak{u}_g(l)$ in $\mathfrak{o}_g(l)$ is*

$$\begin{cases} V(\lambda_3) & \text{when } l = 1; \\ V(2\lambda_2) & \text{when } l = 2; \\ V(2\lambda_1 + \lambda_3) & \text{when } l = 3. \end{cases}$$

⁷This result also follows quite directly from a result of Oda [43].

Proof. It follows from (9.5) that when $g \geq 3$, the image of $\mathfrak{u}_g(l) \rightarrow \mathfrak{o}_g(l)$ is non-trivial for all l . Since this map is \mathfrak{sp}_g equivariant, the image of $\mathfrak{u}_g(2)$ must be all of $\mathfrak{o}_g(2)$. Since $\text{Gr}_\bullet^W \mathfrak{u}_g$ is generated by $\mathfrak{u}_g(1)$, and since $V(3\lambda_1)$ does not appear in $\mathfrak{u}_g(1) \otimes \mathfrak{u}_g(2)$, the assertion for $l = 3$ follows. \square

Note that the copy of $V(3\lambda_1)$ is detected by Morita’s trace [40].

10. QUADRATIC RELATIONS

In this section, we find some obvious quadratic relations in \mathfrak{t}_g for each $g \geq 4$. These give a lower bound for the relations in \mathfrak{t}_g . Serendipitously, this coincides with the upper bound (9.6) derived in the previous section, thus yielding all the quadratic relations.

Theorem 10.1. *For all $g \geq 3$, we have*

$$\text{Gr}_{-2}^W \mathfrak{t}_g = \text{Gr}_{-2}^W \mathfrak{u}_g + V(0) = V(2\lambda_2) + V(0).$$

The proof occupies the rest of this section. We prove the result by finding a pair of commuting elements ϕ and ψ of T_g whose class

$$\tau(\phi) \wedge \tau(\psi) \in \Lambda^2 V(\lambda_3)$$

generates the Sp_g complement of $V(2\lambda_2) + V(0)$ for all $g \geq 3$. Since we know, by (9.6), that the quadratic relations are contained in the complement of $V(2\lambda_2) + V(0)$, we have found all quadratic relations.

Lemma 10.2. *If $g \geq 3$, then*

$$\Lambda^2 V(\lambda_3) = \begin{cases} V(\lambda_6) + V(\lambda_4) + V(\lambda_2) + V(\lambda_2 + \lambda_4) + V(2\lambda_2) + V(0) & \text{when } g \geq 6; \\ V(\lambda_4) + V(\lambda_2) + V(\lambda_2 + \lambda_4) + V(2\lambda_2) + V(0) & \text{when } g = 5; \\ V(\lambda_2) + V(\lambda_2 + \lambda_4) + V(2\lambda_2) + V(0) & \text{when } g = 4; \\ V(2\lambda_2) + V(0) & \text{when } g = 3. \end{cases}$$

\square

From (4.10), we know that $V(0)$ occurs in $\mathfrak{t}_g(2)$. By (9.6) and the previous proposition, there is nothing to prove when $g = 3$. So we suppose that $g \geq 4$.

We use the notation introduced in Section 6. Set

$$\omega = a_1 \wedge b_1 + \cdots + a_g \wedge b_g.$$

Proposition 10.3. *When $g \geq 3$, there are elements $\phi_{i,j}$, $1 \leq i < j \leq g$, of the Torelli group whose image under the Johnson homomorphism*

$$\tau_g : H_1(T_g) \rightarrow V(\lambda_3)$$

is given by

$$(g - 1)\tau_g(\phi_{i,j}) = (g - 1)a_i \wedge a_j \wedge b_j - a_i \wedge \omega.$$

Here we are viewing $V(\lambda_3)$ as a submodule of $\Lambda^3 V(\lambda_1)$. Moreover, we can choose them such that $\phi_{1,2}$ and $\phi_{3,4}$ commute when $g \geq 4$.

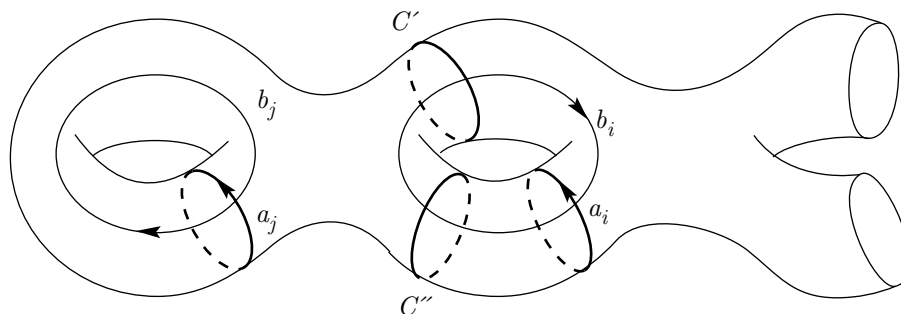


FIGURE 1. ϕ_{ij} is the product of opposite Dehn twists on C' and C''

Proof. For $1 \leq i < j \leq g$ it is easy to construct elements $\phi_{i,j}$ of the pointed Torelli group T_g^1 with

$$\tau_g^1(\phi_{i,j}) = a_i \wedge a_j \wedge b_j \in \Lambda^3 V(\lambda_1).$$

See Figure 1. (To compute $\tau_g^1 : H_1(T_g^1) \rightarrow \Lambda^3 V(\lambda_1)$, use Johnson's original definition in terms of the action of T_g^1 on π_g .) It is also easy to arrange for $\phi_{1,2}$ and $\phi_{3,4}$ to have disjoint supports, and therefore commute. To compute $\tau_g(\phi_{i,j}) \in V(\lambda_3)$, we just use the fact that the maps

$$_ \wedge \omega : V(\lambda_1) \rightarrow \Lambda^3 V(\lambda_1)$$

and

$$p : \Lambda^3 V(\lambda_1) \rightarrow V(\lambda_1)$$

defined by $p(x \wedge y \wedge z) = q(x, y)z + q(y, z)x + q(z, x)y$ are \mathfrak{sp}_g equivariant and satisfy

$$p \circ (_ \wedge \omega) = (g-1) \text{id}.$$

It follows that $V(\lambda_3)$ is the kernel of p and that

$$(g-1)\tau_g(\phi_{i,j}) = (g-1)a_i \wedge a_j \wedge b_j - a_i \wedge \omega.$$

□

Take $\phi = \phi_{1,2}$ and $\psi = \phi_{3,4}$. Since these commute,

$$v := \tau_g(\phi) \wedge \tau_g(\psi) \in \Lambda^2 V(\lambda_3)$$

will lie inside the \mathfrak{sp}_g module of quadratic relations. Denote the \mathfrak{sp}_g submodule of $\Lambda^2 V(\lambda_3)$ generated by v by V . By (10.3),

$$v = [(g-1)a_1 \wedge a_2 \wedge b_2 - a_1 \wedge \omega] \wedge [(g-1)a_3 \wedge a_4 \wedge b_4 - a_3 \wedge \omega].$$

Recall that elements of \mathfrak{sp}_g act on exterior powers as derivations. Note also that for all $X \in \mathfrak{sp}_g$, $X\omega = 0$. We now compute the highest weight decomposition of V .

$\lambda_2 + \lambda_4$. Apply $F_{2,3}$, then $F_{1,4}$, then $T_{2,3}$ to v to get

$$(g-1)^2 [a_1 \wedge a_2 \wedge a_3] \wedge [a_1 \wedge a_2 \wedge a_4] \in \Lambda^2 \Lambda^3 V(\lambda_1)$$

which is a highest weight vector on which \mathfrak{h} acts via the character

$$\lambda_2 + \lambda_4 = (t_1 + t_2) + (t_1 + t_2 + t_3 + t_4).$$

To decompose the rest of V , consider the \mathfrak{sp}_g equivariant map

$$\Lambda^2 V(\lambda_3) \hookrightarrow \Lambda^2 \Lambda^3 V(\lambda_1) \rightarrow \Lambda^6 V(\lambda_1).$$

Denote the image of V under this map by W . It is spanned by the image of v in $\Lambda^6 V(\lambda_1)$. For the time being, we suppose that $g \geq 6$.

λ_6 . In this case, the image of v in W is

$$w := ((g - 1) a_1 \wedge a_2 \wedge b_2 - a_1 \wedge \omega) \wedge ((g - 1) a_3 \wedge a_4 \wedge b_4 - a_3 \wedge \omega).$$

To find a highest weight vector for the representation it generates, first apply $F_{2,5}$, then $F_{4,6}$ to this vector to get the highest weight vector

$$(g - 1)^2 a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge a_5 \wedge a_6$$

of W on which \mathfrak{h} acts via the character $\lambda_6 = t_1 + t_2 + t_3 + t_4 + t_5 + t_6$.

To show that the weights λ_2 and λ_4 occur in V , it is useful to recall that for all $k \geq 2$, there is an \mathfrak{sp}_g equivariant projection

$$(18) \quad \theta_k : \Lambda^k V(\lambda_1) \rightarrow \Lambda^{k-2} V(\lambda_1)$$

which is defined by

$$x_1 \wedge \dots \wedge x_k \mapsto \sum_{1 \leq i < j \leq k} (-1)^{i+j+1} q(x_i, x_j) x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge \hat{x}_j \wedge \dots \wedge x_k.$$

λ_4 . The image of V in $\Lambda^4 V(\lambda_1)$ is generated by $\theta_6(w)$ which is

$$\begin{aligned} & (g - 1)^2 a_1 \wedge a_3 \wedge (a_2 \wedge b_2 + a_4 \wedge b_4) - (g - 1)(g - 3) a_1 \wedge a_3 \wedge (a_2 \wedge b_2 + a_4 \wedge b_4) \\ & - 2(g - 1) a_1 \wedge a_3 \wedge \omega - 2(g - 2) a_1 \wedge a_3 \wedge \omega \\ & = 2(g - 1) a_1 \wedge a_2 \wedge (a_3 \wedge b_3 + a_4 \wedge b_4) - 2 a_1 \wedge a_2 \wedge \omega. \end{aligned}$$

Applying $F_{3,6}$, then $S_{4,6}$, one gets the highest weight vector

$$2(g - 1) a_1 \wedge a_2 \wedge a_3 \wedge a_4$$

on which \mathfrak{h} acts via the character $\lambda_4 = t_1 + t_2 + t_3 + t_4$.

λ_2 . The image of V in $\Lambda^2 V(\lambda_1)$ is generated by the image under θ_4 of $\theta_6(w)$. This is

$$4(g - 1) a_1 \wedge a_3 - 2(g - 2) a_1 \wedge a_3 = 2g a_1 \wedge a_3.$$

Apply $S_{2,3}$ to this to get $2g a_1 \wedge a_2$ upon which \mathfrak{h} acts with highest weight $\lambda_2 = t_1 + t_2$.

We sketch the remaining cases $g = 4, 5$. When $g = 5$, W is generated by the vector

$$(g - 1) (a_1 \wedge a_2 \wedge a_3 \wedge b_2 - a_1 \wedge a_3 \wedge a_4 \wedge b_4) \wedge \omega.$$

By contracting with q as above, it is easy to see that this vector generates a submodule of

$$\omega \wedge \Lambda^4 V(\lambda_1) \cong \Lambda^6 V(\lambda_1)$$

isomorphic to $V(\lambda_4) + V(\lambda_2)$.

When $g = 4$, W is generated by $a_1 \wedge a_3 \wedge \omega^2$. Again, by contracting with q , it is easy to see that this vector generates a submodule of

$$w^2 \wedge \Lambda^2 V(\lambda_1) \subset \Lambda^6 V(\lambda_1)$$

isomorphic to $V(\lambda_2)$.

Remark 10.4. Note that we have determined the quadratic relations for all $g \geq 3$. One should be able to determine the cubic relations when $3 \leq g \leq 5$ by applying similar methods and the fact that the Dehn twist about the separating curve C in Figure 2 commutes with the bounding pair map associated to the curves C' and C'' . The Dehn twist about C is in the kernel of the Johnson homomorphism, but has non-trivial image in the second graded quotient of \mathfrak{t}_g . Note that there have to be cubic relations in genus 3 as there are no quadratic relations, and there has to be one copy of $V(\lambda_3)$ in the cubic relations to ensure the existence of the central \mathbb{G}_a .

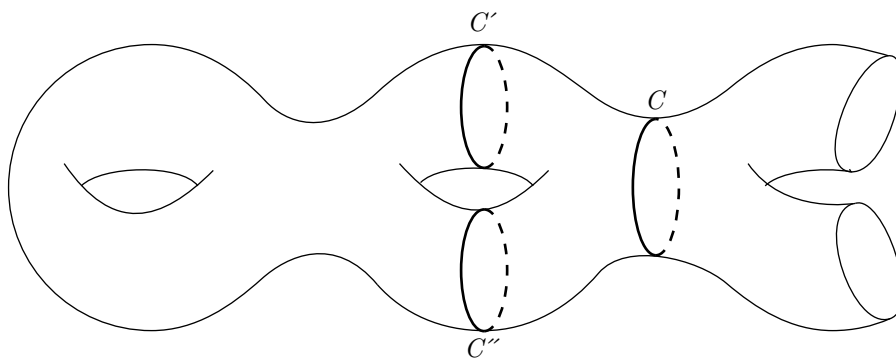


FIGURE 2

11. PRESENTATIONS OF \mathfrak{t}_g , $\mathfrak{t}_{g,1}$ AND \mathfrak{t}_g^1

Recall that $L(V)$ denotes the free Lie algebra generated by the vector space V . In this section we shall give presentations of \mathfrak{t}_g , \mathfrak{t}_g^1 and $\mathfrak{t}_{g,1}$ when $g \geq 6$. This also gives presentations for u_g , u_g^1 and $u_{g,1}$ as $u_{g,r}^n$ by Theorems 3.4 and 4.10. First \mathfrak{t}_g — combining (9.6), (10.1) and (10.3), we have:

Theorem 11.1. *For all $g \geq 6$, $\text{Gr}_{\bullet}^W \mathfrak{t}_g$ is isomorphic to*

$$\mathbb{L}(V(\lambda_3))/R_g$$

as a graded Lie algebra in $R(\mathfrak{sp}_g)$, where R_g is the ideal generated by the quadratic relations

$$V(\lambda_6) + V(\lambda_4) + V(\lambda_2) + V(\lambda_2 + \lambda_4) \subseteq \Lambda^2 V(\lambda_3). \quad \square$$

Since $\mathfrak{t}_g \otimes \mathbb{C} \cong \prod_l \text{Gr}_l^W \mathfrak{t}_g \otimes \mathbb{C}$, this gives the desired presentation of \mathfrak{t}_g for $g \geq 6$.

We next consider \mathfrak{t}_g^1 . Fix a point $[C; x]$ of \mathcal{M}_g^1 so that \mathfrak{t}_g^1 and \mathfrak{p}_g have canonical MHSs. Then the sequence

$$0 \rightarrow \mathfrak{p}_g^1 \rightarrow \mathfrak{t}_g^1 \rightarrow \mathfrak{t}_g \rightarrow 0$$

is an exact sequence of MHSs. Since Gr_{\bullet}^W is an exact functor, the sequence

$$0 \rightarrow \text{Gr}_{\bullet}^W \mathfrak{p}_g^1 \rightarrow \text{Gr}_{\bullet}^W \mathfrak{t}_g^1 \rightarrow \text{Gr}_{\bullet}^W \mathfrak{t}_g \rightarrow 0$$

is exact in $R(\mathfrak{sp}_g)$. Since the sequence of H_1 's is canonically split in $R(\mathfrak{sp}_g)$, there is a canonical lift

$$\mathbb{L}(H_1(\mathfrak{t}_g)) \rightarrow \text{Gr}_{\bullet}^W \mathfrak{t}_g^1$$

of the natural homomorphism $\mathbb{L}(H_1(\mathfrak{t}_g)) \rightarrow \text{Gr}_{\bullet}^W \mathfrak{t}_g$. Since $\mathfrak{p}_g(2)$ is isomorphic to $V(\lambda_2)$, and since this representation does not occur in $\mathfrak{t}_g(2)$, we can take the λ_2 component of the bracket

$$\Lambda^2 H_1(\mathfrak{t}_g) \hookrightarrow \Lambda^2 H_1(\mathfrak{t}_g^1) \rightarrow \mathfrak{t}_g^1(2)$$

to obtain an \mathfrak{sp}_g module map

$$(19) \quad c : \Lambda^2 H_1(\mathfrak{t}_g) \rightarrow \mathfrak{p}_g(2) \cong V(\lambda_2).$$

This and the map

$$(20) \quad H_1(\mathfrak{t}_g) \otimes H_1(\mathfrak{p}_g) \rightarrow \mathfrak{p}_g(2)$$

induced by the bracket completely determine $\text{Gr}_{\bullet}^W \mathfrak{t}_g^1$ given $\text{Gr}_{\bullet}^W \mathfrak{t}_g$ and $\text{Gr}_{\bullet}^W \mathfrak{p}_g$. The map (20) is simply the adjoint of the Johnson homomorphism

$$\tau_g^1 : H_1(\mathfrak{t}_g) \rightarrow \Lambda^3 V \subset \text{Hom}(H_1(\mathfrak{p}_g), \mathfrak{p}_g(2)).$$

So, to give a presentation of \mathfrak{t}_g^1 , we have to determine the map (19). We do this by studying the action of $\mathbb{L}(V(\lambda_3))$ on $\mathbb{L}(V(\lambda_1))$.

Set $V = V(\lambda_1)$. We identify V with $H_1(C)$ and $H_1(T_g^1)$ with $\Lambda^3 V$ via the Johnson homomorphism. Recall that $V(\lambda_3)$ can be realized as the kernel of the map $p : \Lambda^3 V \rightarrow V$ defined by

$$(21) \quad p : v_1 \wedge v_2 \wedge v_3 \mapsto (v_1 \cdot v_2)v_3 + (v_2 \cdot v_3)v_1 + (v_3 \cdot v_1)v_2.$$

We identify $H_1(T_g) \cong \Lambda^3 V/V$ with $V(\lambda_3)$ via the map

$$V(\lambda_3) = \ker p \hookrightarrow \Lambda^3 V \rightarrow \Lambda^3 V/V.$$

The natural action of $\Lambda^3 V$ on $\mathbb{L}(V)$ is defined by

$$e_1 \wedge e_2 \wedge e_3 \mapsto -\{v \mapsto (e_1 \cdot v)[e_2, e_3] + (e_2 \cdot v)[e_3, e_1] + (e_3 \cdot v)[e_1, e_2]\} \\ \in \text{Hom}(V, \Lambda^2 V) \subseteq \text{Der } \mathbb{L}(V).$$

(With this choice of sign, $\sum x \wedge a_j \wedge b_j \mapsto \text{ad}(x)$.) It follows from the definition of the Johnson homomorphism that the composite

$$H_1(T_{g,1}) \xrightarrow{\tau_{g,1}} \Lambda^3 V \hookrightarrow \text{Hom}(V, \Lambda^2 V)$$

is the map induced by the action of $T_{g,1}$ on $\pi_{g,1}$. The action descends to the action of $\text{Gr}_{\bullet}^W \mathfrak{t}_g^1$ on $\text{Gr}_{\bullet}^W \mathfrak{p}_g$.

Define a projection $r : \Lambda^2 V(\lambda_3) \rightarrow V(\lambda_2)$ to be the composite

$$\Lambda^2 V(\lambda_3) \hookrightarrow \Lambda^2 \Lambda^3 V \xrightarrow{\text{mult}} \Lambda^6 V \xrightarrow{\theta_4 \theta_6} \Lambda^2 V \rightarrow V(\lambda_2)$$

where θ_k is the contraction (18) defined in the previous section, and the last map is the standard projection

$$u \wedge v \mapsto u \wedge v - (u \cdot v) \omega / (g - 1).$$

Since there is only one copy of $V(\lambda_2)$ in $\Lambda^2 V(\lambda_3)$, this projection is unique up to a scalar.

Proposition 11.2. *The map (19) is given by*

$$c[u, v] = -\frac{1}{2g + 2} \text{ad}(r(u \wedge v)) \in \text{Hom}(H_1(p_g), \mathfrak{p}_g(3)).$$

In particular, this map is non-zero, and the extensions

$$0 \rightarrow \mathfrak{p}_g \rightarrow \mathfrak{t}_g^1 \rightarrow \mathfrak{t}_g \rightarrow 0 \text{ and } 0 \rightarrow \mathfrak{p}_g \rightarrow \mathfrak{u}_g^1 \rightarrow \mathfrak{u}_g \rightarrow 0$$

are not split.

Sketch of the proof. Denote the degree k part of the free Lie algebra $\mathbb{L}(V)$ by $\mathbb{L}(V)(k)$. Recall that there is a standard exact sequence

$$0 \rightarrow \Lambda^3 V \xrightarrow{j} V \otimes \Lambda^2 V \xrightarrow{b} \mathbb{L}(V)(3) \rightarrow 0.$$

The first map is the “Jacobi identity” map

$$j : v_1 \wedge v_2 \wedge v_3 \mapsto v_1 \otimes v_2 \wedge v_3 + v_2 \otimes v_3 \wedge v_1 + v_3 \otimes v_1 \wedge v_2,$$

and the second map is the bracket. (We are identifying $\mathbb{L}(V)(2)$ with $\Lambda^2 V$ in the standard way.)

Lemma 11.3. *The bracket $[e_1 \wedge e_2 \wedge e_3, f_1 \wedge f_2 \wedge f_3]$ of two elements of $\Lambda^3 V$ as derivations of $\mathbb{L}(V)$ is obtained by summing the expression*

$$\begin{aligned} (e_1 \cdot f_1)(e_2 \otimes [e_3, [f_2, f_3]] - e_3 \otimes [e_2, [f_2, f_3]] + f_2 \otimes [f_3, [e_2, e_3]] - f_3 \otimes [f_2, [e_2, e_3]]) \\ \in V \otimes \mathbb{L}(V)(3) \cong \text{Hom}(V, \mathbb{L}(V)(3)) \end{aligned}$$

cyclically in (e_1, e_2, e_3) and in (f_1, f_2, f_3) . □

We shall view this expression as an element of $(V \otimes V \otimes \Lambda^2 V) / (V \otimes \Lambda^3 V)$. The next step is to write down the projections of this group onto $V(\lambda_2)$.

There are four copies of $V(\lambda_2)$ in $V \otimes V \otimes \Lambda^2 V$. These are detected by the following four projections onto $\Lambda^2 V$:

$$\begin{aligned} p_1 : u_1 \otimes u_2 \otimes u_3 \wedge u_4 &\mapsto (u_1 \cdot u_2)u_3 \wedge u_4, \\ p_2 : u_1 \otimes u_2 \otimes u_3 \wedge u_4 &\mapsto (u_3 \cdot u_4)u_1 \wedge u_2, \\ p_3 : u_1 \otimes u_2 \otimes u_3 \wedge u_4 &\mapsto ((u_1 \cdot u_4)u_2 \wedge u_3 - (u_1 \cdot u_3)u_2 \wedge u_4)/2, \\ p_4 : u_1 \otimes u_2 \otimes u_3 \wedge u_4 &\mapsto ((u_2 \cdot u_3)u_1 \wedge u_4 - (u_2 \cdot u_4)u_1 \wedge u_3)/2. \end{aligned}$$

One can easily check that there are two copies of $V(\lambda_2)$ in $V \otimes \Lambda^3 V$ and that the projections $p_1 - p_3$ and $p_2 - p_4$ vanish on these. This leaves two copies of $V(\lambda_2)$ in $(V \otimes V \otimes \Lambda^2 V) / (V \otimes \Lambda^3 V)$. One of these vanishes in $\text{Hom}(V, \mathfrak{p}_g(3))$ as $V \otimes V \otimes \omega$ projects to zero there. We are now ready to compute.

Since

$$u_j = a_j \wedge a_3 \wedge b_3 - a_j \wedge a_4 \wedge b_4$$

lies in the kernel of the projection p above when $j = 1, 2$, $u_1 \wedge u_2$ is an element of $\Lambda^2 V(\lambda_3)$. The projection r takes $u_1 \wedge u_2$ to $-4 a_1 \wedge a_2$. On the other hand, by straightforward computations using (11.3), we have

$$p_1([u_1, u_2]) = p_2([u_1, u_2]) = 0 \text{ and } p_3([u_1, u_2]) = p_4([u_1, u_2]) = -4 a_1 \wedge a_2.$$

Consequently,

$$(p_1 - p_3)([u_1, u_2]) = (p_2 - p_4)([u_1, u_2]) = 4 a_1 \wedge a_2.$$

Next observe that $\text{ad}[a_1, a_2]$ corresponds to the element

$$-\sum_{j=1}^g (a_j \otimes b_j \otimes a_1 \wedge a_2 - b_j \otimes a_j \otimes a_1 \wedge a_2)$$

of $(V \otimes V \otimes \Lambda^2 V) / (V \otimes \Lambda^3 V)$. Since $\sum [a_j, b_j] = 0$, $\text{ad}[a_1, a_2]$ is also represented by

$$z := -\sum_{j=1}^g (a_j \otimes b_j \otimes a_1 \wedge a_2 - b_j \otimes a_j \otimes a_1 \wedge a_2) - 2 \sum_{j=1}^g a_1 \otimes a_2 \otimes a_j \wedge b_j.$$

By direct computation, we have

$$(p_1 - p_3)(z) = (p_2 - p_4)(z) = -(2g + 2) a_1 \wedge a_2.$$

This concludes the proof of Proposition 11.2. □

Next we consider the case of $\mathfrak{t}_{g,1}$. Fix a point $(C; x, v)$ of $\mathcal{M}_{g,1}$ so that $\mathfrak{t}_{g,1}, \mathfrak{p}_{g,1}$, etc. all have compatible MHSs. By strictness, the sequence

$$0 \rightarrow \text{Gr}_{\bullet}^W \mathfrak{p}_{g,1} \rightarrow \text{Gr}_{\bullet}^W \mathfrak{t}_{g,1} \rightarrow \text{Gr}_{\bullet}^W \mathfrak{t}_g \rightarrow 0$$

is exact in $R(\mathfrak{sp}_g)$. Since the sequence

$$0 \rightarrow \mathbb{Q}(1) \rightarrow \mathfrak{p}_{g,1} \rightarrow \mathfrak{p}_g \rightarrow 0$$

is exact, it follows that $\mathfrak{p}_{g,1}(2)$ is isomorphic to $\Lambda^2 V$ via the bracket.

As in the case of \mathfrak{t}_g^1 , there is a canonical lifting $\mathbb{L}(H_1(\mathfrak{t}_g)) \rightarrow \text{Gr}_{\bullet}^W \mathfrak{t}_{g,1}$ of the natural surjection $\mathbb{L}(H_1(\mathfrak{t}_g)) \rightarrow \text{Gr}_{\bullet}^W \mathfrak{t}_g$. It follows that to give a presentation of $\text{Gr}_{\bullet}^W \mathfrak{t}_{g,1}$ given presentations of \mathfrak{t}_g and $\mathfrak{p}_{g,1}$, it suffices to give the map

$$(22) \quad H_1(\mathfrak{t}_g) \otimes H_1(\mathfrak{p}_{g,1}) \longrightarrow \mathfrak{p}_{g,1}(2)$$

induced by the bracket, together with the λ_2 component

$$(23) \quad \Lambda^2 H_1(\mathfrak{t}_g) \longrightarrow V(\lambda_2) \subset \mathfrak{p}_{g,2}$$

and the invariant part

$$(24) \quad c_0 : \Lambda^2 H_1(\mathfrak{t}_g) \longrightarrow \mathfrak{t}_{g,1}(2)^{Sp_g} \cong \mathbb{Q}(1)^2$$

of the bracket. As in the case of \mathfrak{t}_g^1 , the first map (22) is the adjoint of the Johnson homomorphism and the second (23), by naturality with respect to the projection $\mathfrak{t}_{g,1} \rightarrow \mathfrak{t}_g^1$, is the map c determined in (11.2). It remains to determine the map (24). Observe that the sequence

$$0 \rightarrow \mathfrak{p}_{g,1}(2)^{Sp_g} \rightarrow \mathfrak{t}_{g,1}(2)^{Sp_g} \rightarrow \mathfrak{t}_g(2)^{Sp_g} \rightarrow 0$$

splits canonically as the canonical central \mathbb{G}_a in $\mathfrak{t}_{g,1}$ projects to the canonical central \mathbb{G}_a in \mathfrak{t}_g by (3.4), and because $\mathbb{G}_a = \mathfrak{t}_g(2)^{Sp_g}$. As a generator of $\mathfrak{p}_{g,1}(2)^{Sp_g}$ we take $\sum [a_j, b_j]$.

Fix an invariant bilinear form $\langle \cdot, \cdot \rangle$ on $V(\lambda_3)$ by insisting that

$$\langle a_1 \wedge a_2 \wedge a_3, b_1 \wedge b_2 \wedge b_3 \rangle = 1.$$

We can therefore choose a generator γ of \mathbb{G}_a such that if $u, v \in H_1(\mathfrak{t}_g)$, then the invariant component of $[u, v]$ in $\mathfrak{t}_g(2)$ is $\langle u, v \rangle \gamma$.

Proposition 11.4. *If $u, v \in H_1(\mathfrak{t}_g)$, then*

$$c_0[u, v] = \langle u, v \rangle \gamma - \frac{6\langle u, v \rangle}{g(2g + 1)} \sum_{j=1}^g [a_j, b_j].$$

As in the previous case, we determine the coefficient by studying the action of $\mathbb{L}(V(\lambda_3))$ on $\mathbb{L}(V)$. Note that $\Gamma_{g,1}$ acts on the free group $\pi_1(C - \{x\}, v)$.⁸ We therefore have a representation $\mathfrak{t}_{g,1} \rightarrow \mathfrak{p}(C - \{x\}, v) \cong \mathbb{L}(V)$.⁹ We continue with the notation in the proof of (11.2).

There are two copies of the trivial representation in $V \otimes V \otimes \Lambda^2 V$. The corresponding projections to \mathbb{Q} are:

$$\begin{aligned} q_1 &: u_1 \otimes u_2 \otimes u_3 \wedge u_4 \mapsto (u_1 \cdot u_2)(u_3 \cdot u_4), \\ q_2 &: u_1 \otimes u_2 \otimes u_3 \wedge u_4 \mapsto ((u_1 \cdot u_4)(u_2 \cdot u_3) - (u_1 \cdot u_3)(u_2 \cdot u_4))/2. \end{aligned}$$

There is one copy of the trivial representation in $V \otimes \Lambda^3 V$ and $q_1 - q_2$ vanishes on it. The vectors

$$u_1 = a_1 \wedge a_2 \wedge a_3 \text{ and } u_2 = b_1 \wedge b_2 \wedge b_3$$

both lie in $V(\lambda_3)$ and $\langle u_1, u_2 \rangle = 1$. It follows from the formula (19) that $[u_1, u_2]$ is obtained by summing the expression

$$a_2 \otimes [a_3, [b_2, b_3]] - a_3 \otimes [a_2, [b_2, b_3]] + b_2 \otimes [b_3, [a_2, a_3]] - b_3 \otimes [b_2, [a_2, a_3]]$$

over the cyclic group generated by $(1, 2, 3)$. We have $(q_1 - q_2)([u_1, u_2]) = 6$. On the other hand, $\text{ad} \sum [a_j, b_j]$ is represented by

$$\sum_{j=1}^g \sum_{k=1}^g (b_j \otimes a_k - a_j \otimes b_k) \otimes a_k \wedge b_k.$$

The projection $q_1 - q_2$ takes the value $-g(2g + 1)$ on this. The result follows.

Remark 11.5. The formulas in (11.2) and (11.4) are closely related to those in Theorem 3.1 of Morita’s paper [38].

12. A PRESENTATION OF $\mathfrak{p}_{g,r}^n$

In this section we give an explicit quadratic presentation of the pure braid Lie algebras $\mathfrak{p}_{g,r}^n$ for all $g > 0$. We continue with the notation of Section 2. We fix a complex structure on and a base point of $F_{g,r}^n$ by choosing a point

$$[C; x_1, \dots, x_n; v_1, \dots, v_r]$$

of $\mathcal{M}_{g,r}^n$.

In Section 2 we showed that $H^1(F_{g,r}^n(C))$ is pure of weight 1. We will show that $H^2(F_{g,r}^n(C))$ is pure of weight 2, from which the existence of a quadratic presentation of $\mathfrak{p}_{g,r}^n$ will follow via Morgan’s Theorem (5.8).

First, some notation. Denote the projection of C^n onto its i th factor by p_i . Denote the image of the inclusion

$$p_i^* : H^\bullet(C) \hookrightarrow H^\bullet(C^n)$$

⁸This notation denotes Deligne’s fundamental group of $C - \{x\}$ with base point the tangent vector $v \in T_x C$.

⁹If we put the limit MHS on $\pi_1(C - \{x\}, v)$ associated with the tangent vector v , then this action is a morphism of MHS.

by $H^\bullet(C_i)$. For $x \in H^\bullet(C)$, denote p_i^*x by $x^{(i)}$. Denote the component of Δ where the i th and j th coordinates are equal by Δ_{ij} . Fix a symplectic basis $a_1, \dots, a_g, b_1, \dots, b_g$ of $H_1(C)$, and let $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ be the dual basis of $H^1(C)$. Denote the positive integral generator of $H^2(C)$ by ζ , and the intersection form

$$\sum_{r=1}^g \alpha_r \wedge \beta_r$$

by q . When $i \neq j$, set

$$q_{ij} = \sum_{r=1}^g \alpha_r^{(i)} \wedge \beta_r^{(j)} + \alpha_r^{(j)} \wedge \beta_r^{(i)}.$$

Lemma 12.1. *The Poincaré dual $PD(\Delta_{ij})$ of Δ_{ij} is $\zeta^{(i)} + \zeta^{(j)} - q_{ij}$. □*

This is elementary. Another elementary fact we shall need is the following statement. It is easily proved using a Mayer-Vietoris argument.

Lemma 12.2. *The natural map*

$$\bigoplus_{i < j} H_{2n-3}(\Delta_{ij}) \rightarrow H_{2n-3}(\Delta)$$

is an isomorphism. □

We can therefore write the Gysin map $\gamma : H_{2n-3}(\Delta) \rightarrow H^3(C^n)$ as the sum of the Gysin maps $\gamma_{ij} : H^1(\Delta_{ij}) \rightarrow H^3(C^n)$; the map γ_{ij} being given by cup product with $PD(\Delta_{ij})$.

Lemma 12.3. *The composite $H^1(C) \xrightarrow{p_k^*} H^1(\Delta_{ij}) \xrightarrow{\gamma_{ij}} H^3(C^n)$ is given by*

$$x \mapsto \begin{cases} \zeta^{(i)} \wedge x^{(j)} + \zeta^{(j)} \wedge x^{(i)} & \text{if } k \in \{i, j\}; \\ \zeta^{(i)} \wedge x^{(k)} + \zeta^{(j)} \wedge x^{(k)} - q_{ij} \wedge x^{(k)} & \text{if } k \notin \{i, j\}. \quad \square \end{cases}$$

It follows from (2.1) that the part

$$0 \rightarrow \bigoplus_{i < j} \mathbb{Z} \rightarrow H^2(C^n) \rightarrow H^2(C^n - \Delta) \rightarrow H_{2n-3}(\Delta) \rightarrow H^3(C^n)$$

of the Gysin sequence is exact. Purity of $H^2(F_g^n(C))$ therefore follows from the following proposition.

Proposition 12.4. *The Gysin map $\gamma : H_{2n-3}(\Delta) \rightarrow H^3(C^n)$ is injective.*

Proof. The Gysin sequence can be viewed as the fiber over $[C] \in \mathcal{M}_g$ of an exact sequence of (orbifold) local systems over \mathcal{M}_g . It follows from (12.2) that the monodromy actions of the last two terms of the Gysin sequence above factor through the symplectic group. It is convenient, though not necessary, to decompose these groups under the action of Sp_g .

First note that $H_{2n-3}(\Delta_{ij})$ is isomorphic to $H^1(\Delta_{ij})$, which is isomorphic to $n - 1$ copies of the fundamental representation V . Next,

$$H^3(C^n) = \bigoplus_{i \neq j} (H^2(C_i) \otimes H^1(C_j)) \oplus \bigoplus_{i < j < k} H^1(C_i) \otimes H^1(C_j) \otimes H^1(C_k).$$

Each of the terms $H^2(C_i) \otimes H^1(C_j)$ is a copy of the fundamental representation which we shall denote by V_j^i . The term $H^1(C_i) \otimes H^1(C_j) \otimes H^1(C_k)$ is isomorphic to $V^{\otimes 3}$. It contains 3 copies of V . If, for i, j, k distinct, we set

$$V_{ij}^k = \text{the image of } \left\{ H^1(C_k) \xrightarrow{\wedge^{q_{ij}}} H^3(C^n) \right\},$$

then

$$[H^1(C_i) \otimes H^1(C_j) \otimes H^1(C_k)]_{\lambda_1} = V_{ij}^k \oplus V_{jk}^i \oplus V_{ki}^j.$$

It is now easy to see that γ is injective. Indeed, by (12.3), we see that the images of the maps

$$H^1(C) \xrightarrow{p_i^*} H^1(\Delta_{ij}) \xrightarrow{\gamma_{ij}} H^3(C^n)$$

are independent copies of V , and also, when $k \notin \{i, j\}$, that the image of

$$H^1(C) \xrightarrow{p_k^*} H^1(\Delta_{ij}) \xrightarrow{\gamma_{ij}} H^3(C^n)$$

is congruent to V_{ij}^k modulo the sum of the V_b^a . □

Similarly, one can show that the r Chern classes of the central extensions

$$0 \rightarrow \mathbb{Z}^r \rightarrow \pi_{g,r}^n \rightarrow \pi_g^{r+n} \rightarrow 1$$

are linearly independent in $H^2(\pi_g^{r+n})$ as they correspond to independent copies of the trivial representation in $H^2(F_{g,r}^n)$. It follows that $H^2(F_{g,r}^n)$ is also pure of weight 2.

Assembling all this, we obtain:

Proposition 12.5. *For each choice of a base point $[C]$ of \mathcal{M}_g and for all $g \geq 1$ and $n, r \geq 0$, the natural MHS on $H^1(F_{g,r}^n)$ is pure of weight 1 and that on $H^2(F_{g,r}^n)$ is pure of weight 2. In addition, the cup product*

$$\Lambda^2 H^1(F_{g,r}^n, \mathbb{Q}) \rightarrow H^2(F_{g,r}^n)$$

is surjective. □

From Morgan’s Theorem we deduce that $\mathfrak{p}_{g,r}^n$ has a quadratic presentation for all non-negative g, r and n .¹⁰

Our final task is to determine the relations explicitly. First some notation. The Lie algebra $\mathfrak{p}_{g,r}^n$ is a quotient of the free Lie algebra generated by

$$H_1(\mathfrak{p}_{g,r}^n) \cong H_1(C^{n+r}) \cong \bigoplus_{i=1}^{n+r} H_1(C_i).$$

We shall think of elements of $H_1(C^{n+r})$ as indeterminates, and write them as upper case letters. If $X \in H_1(C)$, we shall denote the corresponding element of $H_1(C_i)$ by $X^{(i)}$. Fix a symplectic basis $A_1, \dots, A_g, B_1, \dots, B_g$ of $H_1(C)$. Denote the intersection number of X and $Y \in H_1(C)$ by $(X \cdot Y)$.

¹⁰In the genus zero case, it is well known that H^1 has weight 2 and H^2 weight 4 as the corresponding classifying spaces are complements of hyperplanes in affine space.

Theorem 12.6. For all $g \geq 1$ and all $r, n \geq 0$,

$$\mathrm{Gr}_{\bullet}^W \mathfrak{p}_{g,r}^n \cong \mathbb{L}(H_1(C)^{\oplus(n+r)})/R,$$

where R is the ideal generated by the relations

$$\begin{aligned} [X^{(i)}, Y^{(j)}] &= [X^{(j)}, Y^{(i)}] && \text{all } i \text{ and } j; \\ [X^{(i)}, Y^{(j)}] &= \frac{(X \cdot Y)}{g} \sum_{k=1}^g [A_k^{(i)}, B_k^{(j)}] && \text{when } i \neq j; \\ \sum_{k=1}^g [A_k^{(i)}, B_k^{(i)}] + \frac{1}{g} \sum_{j \neq i} \sum_{k=1}^g [A_k^{(i)}, B_k^{(j)}] &= 0 && \text{for } 1 \leq i \leq n; \end{aligned}$$

where X and Y are arbitrary elements of $H_1(C)$.

Note that the last relation holds only for those factors corresponding to a marked point, and not those corresponding to a marked tangent vector.

Proof. If \mathfrak{g} is a graded Lie algebra generated in weight -1 and $H_2(\mathfrak{g})$ of weight 2 , then we have an exact sequence

$$0 \rightarrow H_2(\mathfrak{g}) \xrightarrow{\mathrm{cup}^*} \Lambda^2 H_1(\mathfrak{g}) \xrightarrow{\mathrm{bracket}} \mathrm{Gr}_{-2}^W \mathfrak{g} \rightarrow 0,$$

where the first map is the dual of the cup product.¹¹ In our case, the natural injection

$$H^2(\mathfrak{p}_{g,r}^n) \rightarrow H^2(F_{g,r}^n, \mathbb{Q})$$

is an isomorphism because the cup product is surjective. The coproduct is the obvious inclusion of $H_2(F_{g,r}^n, \mathbb{Q})$ into $\Lambda^2 H_1(C^n, \mathbb{Q})$, and the sequence is a sequence of Sp_g modules:

$$0 \rightarrow H_2(F_{g,r}^n, \mathbb{Q}) \rightarrow \Lambda^2 H_1(C^n, \mathbb{Q}) \rightarrow \mathrm{Gr}_{-2}^W \mathfrak{p}_{g,r}^n \rightarrow 0.$$

We will consider one weight at a time. Note that the three weights occurring in $\Lambda^2 H_1(C^n)$ are $0, \lambda_2$ and $2\lambda_1$ — the last being the symmetric square of $H_1(C)$ and the second being the primitive part of $H_2(\mathrm{Jac} C)$. We also have the exact sequence

$$0 \rightarrow H_2(F_{g,r}^n) \rightarrow H_2(C^n) \xrightarrow{\gamma^*} \bigoplus_{i < j} \mathbb{Q} \rightarrow 0$$

of Sp_g modules. The last map is the dual of the Gysin map. It follows that

$$H_2(F_{g,r}^n, \mathbb{Q})_{\lambda} = H_2(C^n, \mathbb{Q})_{\lambda}$$

when λ is $2\lambda_1$ or λ_2 .

The $2\lambda_1$ isotypical component is spanned by elements of the form

$$X^{(i)} \times Y^{(j)} + Y^{(i)} \times X^{(j)}.$$

This gives the first relation:

$$(25) \quad [X^{(i)}, Y^{(j)}] = [X^{(j)}, Y^{(i)}].$$

¹¹There are many ways to see this — the easiest being from the standard complex of Lie algebra cochains. However, the statement holds in greater generality — cf. [49].

Since $V(\lambda_2)$ is the kernel of the symplectic form $\Lambda^2 V(\lambda_1) \rightarrow \mathbb{Q}$, the λ_2 isotypical component of $H_2(C^m)$ is spanned by vectors of the form

$$X^{(i)} \times Y^{(j)} - Y^{(i)} \times X^{(j)} - \frac{(X \cdot Y)}{g} \sum_{k=1}^g \left(A_k^{(i)} \times B_k^{(j)} - B_k^{(i)} \times A_k^{(j)} \right),$$

where $i \neq j$. This gives relations of the form

$$[X^{(i)}, Y^{(j)}] + [X^{(j)}, Y^{(i)}] = \frac{(X \cdot Y)}{g} \sum_{k=1}^g \left([A_k^{(i)}, B_k^{(j)}] + [A_k^{(j)}, B_k^{(i)}] \right)$$

which simplifies to the second relation after applying (25).

For the time being, we assume that $r = 0$. The trivial isotypical component lies in an exact sequence

$$0 \rightarrow H_2(F_{g,r}^n)^{Sp_g} \rightarrow H_2(C^m)^{Sp_g} \xrightarrow{\gamma^*} H^{2g-2}(\Delta) \rightarrow 0.$$

The map γ^* takes $W \in H_2(C^m)$ to the functional

$$\{\Delta_{ij} \mapsto W \cdot \Delta_{ij}\}.$$

Note that

$$H_2(C^m)^{Sp_g} = \bigoplus_{i=1}^n H_2(C_i) \oplus \bigoplus_{i < j} [H_1(C_i) \otimes H_1(C_j)]^{Sp_g}.$$

The first terms has basis the $Z^{(i)}$, where Z denotes the integral generator of $H_2(C)$. The second term has basis consisting of the

$$Q_{ij} := \sum_{k=1}^n \left(A_k^{(i)} \times B_k^{(j)} - B_k^{(i)} \times A_k^{(j)} \right).$$

We next determine a basis of $\ker \gamma^*$.

Observe that

$$Q_{ij} \cdot \Delta_{kl} = \begin{cases} -2g & \text{when } ij = kl; \\ 0 & \text{otherwise,} \end{cases}$$

and

$$Z^{(i)} \cdot \Delta_{kl} = \begin{cases} 1 & \text{when } i \in \{k, l\}; \\ 0 & \text{when } i \notin \{k, l\}. \end{cases}$$

It follows immediately that a basis of

$$H_2(F_g^n)^{Sp_g} = \ker\{H_2(C^m)^{Sp_g} \rightarrow H^2(\Delta)\}$$

consists of the

$$Z^{(i)} + \frac{1}{2g} \sum_{j \neq i} Q_{ij}.$$

These give the relations

$$\sum_{k=1}^g [A_k^{(i)}, B_k^{(i)}] + \frac{1}{2g} \sum_{j \neq i} \sum_{k=1}^g \left([A_k^{(i)}, B_k^{(j)}] + [A_k^{(j)}, B_k^{(i)}] \right) = 0$$

which becomes the third relation after an application of (25).

Finally, the third relation is dual to the first Chern class of the pullback of the tangent bundle of C along $p_i : C^m \rightarrow C$. It follows that in the general case, we do

not get any relations coming from the trivial representation associated to an index corresponding to a tangent vector. \square

We conclude this section with a computation of the generating function of the lower central series of $\pi_{g,r}^n$. This corrects the formula in [30]. (The Galois analogue of this corrected formula is also stated in [41, (2.14)].)

Theorem 12.7. *For all $g \geq 1$ and all $n \geq 1$, we have*

$$\prod_{k=1}^n (1 - 2gt - (k - 2)t^2) = \prod_{l=1}^{\infty} (1 - t^l)^{r_l},$$

where r_l is the rank of the l th graded quotient of the lower central series of π_g^n .

Proof. Since F_g^n is a smooth variety and a rational $K(\pi, 1)$ by (8.2), we can apply (4.7) and the formula [15, (9.7)] to deduce that

$$W_{F_g^n}(t) = \prod_{l=1}^{\infty} (1 - t^l)^{r_l},$$

where, for a graded (variation of) MHS H ,

$$W_H(t) = \sum_{k \geq 0} \chi(\text{Gr}_k^W H) t^k$$

and, for an algebraic variety X , $W_X(t) = W_{H^\bullet(X)}(t)$. (Here χ denotes the Euler characteristic.) The result now follows by induction on n using the fact that

$$W_{C - \{x_1, \dots, x_n\}}(t) = 1 - 2gt + (n - 1)t^2$$

and the following lemma which is proved by induction on the length of the weight filtration of V . \square

Lemma 12.8. *If \mathbb{V} is an admissible unipotent variation of MHS over a smooth variety X , then*

$$W_{H^\bullet(X, \mathbb{V})}(t) = W_X(t)W_{\mathbb{V}}(t). \quad \square$$

13. THE GENERAL CASE

In this section we assemble results from Sections 11 and 12 to obtain a presentation of $\mathfrak{t}_{g,r}^n$ for all $g \geq 6$ and all r and $n \geq 0$. Fix a base point

$$[C; x_1, \dots, x_n; v_1, \dots, v_n]$$

of $\mathcal{M}_{g,r}^n$ so that $\mathfrak{t}_{g,r}^n$ and $\mathfrak{p}_{g,r}^n$, etc. all have mixed Hodge structures. The sequence of Lie algebras

$$0 \rightarrow \mathfrak{p}_{g,r}^n \rightarrow \mathfrak{t}_{g,r}^n \rightarrow \mathfrak{t}_g \rightarrow 0$$

is exact in the category of MHSs, and therefore remains exact after applying Gr_\bullet^W :

$$0 \rightarrow \text{Gr}_\bullet^W \mathfrak{p}_{g,r}^n \rightarrow \text{Gr}_\bullet^W \mathfrak{t}_{g,r}^n \rightarrow \text{Gr}_\bullet^W \mathfrak{t}_g \rightarrow 0.$$

By (5.2), $\mathfrak{t}_{g,r}^n \otimes \mathbb{C}$ is isomorphic to the completion of $\text{Gr}_\bullet^W \mathfrak{t}_{g,r}^n \otimes \mathbb{C}$. So, to find a presentation of $\mathfrak{t}_{g,r}^n$, it suffices to find a presentation of its associated graded. As in §11, there is a lift of the canonical homomorphism $\mathbb{L}(H_1(\mathfrak{t}_g)) \rightarrow \text{Gr}_\bullet^W \mathfrak{t}_g$

to a homomorphism $\mathbb{L}(H_1(\mathfrak{t}_g)) \rightarrow \text{Gr}_\bullet^W \mathfrak{t}_{g,r}^n$. Given presentations of $\text{Gr}_\bullet^W \mathfrak{t}_g$ and $\text{Gr}_\bullet^W \mathfrak{p}_{g,r}^n$, a presentation of $\text{Gr}_\bullet^W \mathfrak{t}_{g,r}^n$ is determined by maps

$$\begin{aligned} a &: H_1(\mathfrak{t}_g) \otimes H_1(\mathfrak{p}_{g,r}^n) \rightarrow \mathfrak{p}_{g,r}^n(2), \\ c &: \Lambda^2 H_1(\mathfrak{t}_g) \rightarrow \mathfrak{p}_{g,r}^n(2)_{\lambda_2}, \\ c_0 &: \Lambda^2 H_1(\mathfrak{t}_g) \rightarrow \mathfrak{t}_{g,r}^n(2)^{Sp_g} \end{aligned}$$

induced by the bracket.

Observe that the homomorphism

$$\mathfrak{p}_{g,r}^n \rightarrow \mathfrak{p}_g^{\oplus(n+r)}$$

induced by the inclusion $F_{g,r}^n(C) \hookrightarrow C^{n+r}$ induces isomorphisms

$$H_1(\mathfrak{p}_{g,r}^n) \cong H_1(\mathfrak{p}_g)^{\oplus(n+r)} \text{ and } \mathfrak{p}_{g,r}^n(2)_{\lambda_2} \rightarrow \mathfrak{p}_g(2)^{\oplus(n+r)}.$$

By a naturality argument, the map a is easily seen to be the adjoint of the map

$$H_1(\mathfrak{t}_g) \rightarrow \text{Hom} \left(\bigoplus_{j=1}^{r+n} H_1(\mathfrak{p}_g), \bigoplus_{j=1}^{r+n} \mathfrak{p}_g(2)_{\lambda_2} \right)$$

which is the direct sum of $n + r$ copies of the Johnson homomorphism.

The map c is simply the sum over all the marked points and tangent vectors

$$\Lambda^2 H_2(\mathfrak{t}_g) \rightarrow \bigoplus_{j=1}^{r+n} \mathfrak{p}_g(2) \cong \mathfrak{p}_{g,r}^n(2)_{\lambda_2}$$

of the maps (19) which is determined in (11.2).

It remains to determine c_0 . As in the case of $\mathfrak{t}_{g,1}$ considered in Section 11, there is a canonical splitting of the sequence

$$0 \rightarrow \mathfrak{p}_{g,r}^n(2)^{Sp_g} \rightarrow \mathfrak{t}_{g,r}^n(2)^{Sp_g} \rightarrow \mathfrak{t}_g^{Sp_g}(2) \rightarrow 0$$

from which we obtain a canonical decomposition

$$\mathfrak{t}_{g,r}^n(2)^{Sp_g} = \mathfrak{p}_{g,r}^n(2)^{Sp_g} \oplus \mathbb{G}_a.$$

As in §11, we identify $H_1(\mathfrak{t}_g)$ with the subspace of $\Lambda^3 V$ which is the kernel of the projection (21), denote by $\langle \cdot, \cdot \rangle$ the unique Sp_g invariant bilinear form on $H_1(\mathfrak{t}_g)$ such that

$$\langle a_1 \wedge a_2 \wedge a_3, b_1 \wedge b_2 \wedge b_3 \rangle = 1,$$

and choose a generator γ of \mathbb{G}_a such that if $u, v \in H_1(\mathfrak{t}_g)$, then the invariant part of $[u, v]$ in $\mathfrak{t}_g(2)$ is $\langle u, v \rangle \gamma$.

Observe that there is an exact sequence

$$0 \rightarrow \bigoplus_{1 \leq i < j \leq r+n} \mathbb{Q}(1) \oplus \bigoplus_{i=1}^r \mathbb{Q}(1) \rightarrow \mathfrak{p}_{g,r}^n(2) \rightarrow \bigoplus_{j=1}^{r+n} \mathfrak{p}_g(2) \rightarrow 0$$

of Sp_g modules. It follows that

$$\mathfrak{p}_{g,r}^n(2)^{Sp_g} \cong \bigoplus_{1 \leq i < j \leq r+n} \mathbb{Q}(1) \oplus \bigoplus_{i=1}^r \mathbb{Q}(1).$$

The terms indexed by $1 \leq i \leq r$ correspond to the r marked tangent vectors; those indexed by $1 \leq i < j \leq r + n$ to the diagonals Δ_{ij} . It is easy to see that the composition

$$\Lambda^2 H_1(\mathfrak{t}_g) \rightarrow \bigoplus_{i=1}^r \mathbb{Q}(1)$$

of c_0 with the projection

$$\mathfrak{t}_{g,r}^n(2)^{Sp_g} \rightarrow \bigoplus_{i=1}^r \mathbb{Q}(1)$$

is the sum of the maps c_0 associated to $\mathfrak{t}_{g,1}$ computed in (11.4). So it remains to determine the composition

$$\Lambda^2 H_1(\mathfrak{t}_g) \rightarrow \bigoplus_{1 \leq i < j \leq r+n} \mathbb{Q}(1)$$

of c_0 with the projection

$$e^{r+n} : \mathfrak{t}_{g,r}^n(2)^{Sp_g} \rightarrow \bigoplus_{1 \leq i < j \leq r+n} \mathbb{Q}(1).$$

To do this, it suffices to compute e^2 in the case of \mathfrak{t}_g^2 , for then the map e^{r+n} is simply the sum of the e^2 's over all diagonals.

In order to compute

$$e^2 : \mathfrak{t}_{g,r}^n(2)^{Sp_g} \rightarrow \mathbb{Q}(1)$$

we use the fact that a punctured tubular neighbourhood of the diagonal Δ in $C \times C$ is homeomorphic to the frame bundle of the tangent bundle of C . In this way, we obtain homomorphisms

$$\mathfrak{t}_{g,1} \rightarrow \mathfrak{t}_g^2 \text{ and } \mathfrak{p}_{g,1} \rightarrow \mathfrak{p}_g^2.$$

In particular, we have a map

$$(26) \quad \mathfrak{p}_{g,1}(2)^{Sp_g} \rightarrow \mathfrak{p}_g^2(2)^{Sp_g}.$$

Using the fact that $\text{Gr}_\bullet^W \mathfrak{p}_{g,1} \rightarrow \text{Gr}_\bullet^W \mathfrak{p}_g^2$ is a homomorphism and that on H_1 it is the diagonal map $V \rightarrow V \oplus V$, we see that the map (26) takes the generator $\sum_k [A_k, B_k]$ of $\mathfrak{p}_{g,1}(2)^{Sp_g}$ to

$$\begin{aligned} \sum_{k=1}^g \left([A_k^{(1)}, B_k^{(1)}] + [A_k^{(1)}, B_k^{(2)}] + [A_k^{(2)}, B_k^{(1)}] + [A_k^{(2)}, B_k^{(2)}] \right) \\ = 2 \sum_{k=1}^g \left([A_k^{(1)}, B_k^{(2)}] + [A_k^{(2)}, B_k^{(1)}] \right). \end{aligned}$$

It follows that in \mathfrak{t}_g^2 , the map c_0 is given by

$$c_0[u, v] = \langle u, v \rangle \gamma - \frac{12 \langle u, v \rangle}{g(2g+1)} \sum_{k=1}^g \left([A_k^{(1)}, B_k^{(2)}] + [A_k^{(2)}, B_k^{(1)}] \right).$$

This completes the determination of c_0 in general and, with it, the descriptions of the $\mathfrak{t}_{g,r}^n$.

14. APPLICATIONS

14.1. Cup products and Massey products. We have shown that for all $g \geq 6$ and all $r, n \geq 0$, the Lie algebra $\text{Gr}_\bullet^W \mathfrak{t}_{g,r}^n$ has a presentation with only quadratic relations. This implies, using the short exact sequence in [49] for example, that the cup product

$$\Lambda^2 \text{Gr}_\bullet^W H_{\text{cts}}^1(\mathfrak{t}_{g,r}^n) \rightarrow \text{Gr}_\bullet^W H_{\text{cts}}^2(\mathfrak{t}_{g,r}^n)$$

is surjective. It follows that the cup product

$$\Lambda^2 H_{\text{cts}}^1(\mathfrak{t}_{g,r}^n) \rightarrow H_{\text{cts}}^2(\mathfrak{t}_{g,r}^n)$$

is also surjective.

Recall that the l -fold Massey products constructed from $H^1(A^\bullet)$, where A^\bullet is a d.g.a., are defined on a subspace D_l of $H^1(A^\bullet)^{\otimes l}$ and take values in $H^2(A^\bullet)/I_{l-1}$, where I_{l-1} denotes the lift to $H^2(A^\bullet)$ of the image of the Massey products of order $< l$. It follows that all Massey products in $H_{\text{cts}}^2(\mathfrak{t}_{g,r}^n)$ of order ≥ 3 vanish when $g \geq 6$ as the cup product (Massey products of order 2) map is surjective.

Since the natural map $H_{\text{cts}}^2(\mathfrak{t}_{g,r}^n) \rightarrow H^2(T_{g,r}^n, \mathbb{Q})$ is injective (cf. (5.1)) and preserves Massey products, we have:

Theorem 14.1. *For all $g \geq 6$, all Massey products of order ≥ 3 in $H^2(T_{g,r}, \mathbb{Q})$ vanish. \square*

Remark 14.2. It follows from the fact that there are non-trivial cubic relations and no quadratic relations in a minimal presentation of \mathfrak{t}_3 that the cup product

$$\Lambda^2 H^1(T_3, \mathbb{Q}) \rightarrow H^2(T_3, \mathbb{Q})$$

vanishes, and that the Massey triple product map

$$H^1(T_3, \mathbb{Q})^{\otimes 3} \rightarrow H^2(T_3, \mathbb{Q})$$

is non-trivial.

It follows from (10.1) and (10.2) that for all $g \geq 6$, $H_{\text{cts}}^2(\mathfrak{t}_g)$ has highest weight decomposition

$$H_{\text{cts}}^2(\mathfrak{t}_g) \cong V(\lambda_6) + V(\lambda_4) + V(\lambda_2) + V(\lambda_2 + \lambda_4).$$

Theorem 14.3. *For all $g \geq 3$, there is an (unnatural) isomorphism*

$$H_{\text{cts}}^2(\mathfrak{t}_{g,r}^n) \cong H_{\text{cts}}^2(\mathfrak{t}_g) \oplus (H_{\text{cts}}^1(\mathfrak{p}_{g,r}^n) \otimes H_{\text{cts}}^1(\mathfrak{t}_g)) \oplus H_{\text{cts}}^2(\mathfrak{p}_{g,r}^n)$$

of Sp_g modules.

Proof. Choose a base point of $\mathcal{M}_{g,r}^n$. Then

$$0 \rightarrow \mathfrak{p}_{g,r}^n \rightarrow \mathfrak{t}_{g,r}^n \rightarrow \mathfrak{t}_g \rightarrow 0$$

is an exact sequence of mixed Hodge structures, and the corresponding spectral sequence

$$E_2^{s,t} = H_{\text{cts}}^s(\mathfrak{t}_g, H_{\text{cts}}^t(\mathfrak{p}_{g,r}^n)) \implies H_{\text{cts}}^{s+t}(\mathfrak{t}_{g,r}^n)$$

is a spectral sequence in the category of mixed Hodge structures. Since \mathfrak{t}_g has negative weights, the weights on $H_{\text{cts}}^k(\mathfrak{t}_g)$ are $\geq k$. This and the fact that $H^k(\mathfrak{p}_{g,r}^n)$ is a trivial \mathfrak{t}_g module when $k = 1$ and 2 imply that $E_\infty^{s,t} = E_2^{s,t}$ when $s + t = 2$. The result follows. \square

14.2. **Johnson’s conjecture.** In [23], Johnson constructed maps

$$\phi_k : H_k(T_g) \rightarrow H_{k+2}(\text{Jac } S)/[S] \times H_k(\text{Jac } S),$$

which generalize the classical Johnson homomorphism, which is the case $k = 1$. He conjectured that these homomorphisms are isomorphisms for all k and sufficiently large g .

The following result is an improvement of some unpublished computations of Morita (cf. [36, §4]).

Theorem 14.4. *For all $g \geq 3$, the map ϕ_2 is not injective.*

Proof. It is not difficult to see that each ϕ_k is $Sp_g(\mathbb{Z})$ equivariant. Consider its adjoint

$$\phi_k^t : H^{k+2}(\text{Jac } S, \mathbb{Q})/\omega \wedge H^k(\text{Jac } S, \mathbb{Q}) \rightarrow H^k(T_g, \mathbb{Q}).$$

This is also $Sp_g(\mathbb{Z})$ equivariant. The domain of ϕ_2^t is the primitive cohomology group $PH^4(\text{Jac } S, \mathbb{Q})$. This is the restriction to $Sp_g(\mathbb{Z})$ of the rational representation of Sp_g with highest weight λ_4 . Since this is an irreducible $Sp_g(\mathbb{Z})$ module, the image of ϕ_2^t is either isomorphic to $V(\lambda_4)$ or trivial. But $H^2(T_g, \mathbb{Q})$ contains the rational representation $H_{\text{cts}}^2(\mathfrak{t}_g)$. It follows from the results in §14.1 that

$$H_{\text{cts}}^2(\mathfrak{t}_g)/\text{im } \phi_2^t \cap H_{\text{cts}}^2(\mathfrak{t}_g)$$

is non-trivial as it contains $V(\lambda_6) + V(\lambda_2 + \lambda_4)$ when $g \geq 6$; $V(\lambda_2 + \lambda_4)$ when $g = 4, 5$; and $V(\lambda_3)$ when $g = 3$. The result follows. \square

14.3. **Filtrations of T_g^1 .** There is a filtration

$$(27) \quad T_g = L^1 T_g^1 \supseteq L^2 T_g^1 \supseteq L^3 T_g^1 \supseteq \dots$$

of T_g^1 , where $K^k T_g^1$ is defined to be

$$\{\phi \in T_g^1 : \phi_* : \pi_1(S, x) \rightarrow \pi_1(S, x) \text{ is congruent to the identity mod } \Gamma^{k+1}\}.$$

It is quite common in the literature for this filtration to be called the *relative weight filtration*, as it is in [2] and [43]. In view of (4.10) and (14.6), I feel that this terminology is likely to result in confusion.

Proposition 14.5. *This filtration is a descending central series of T_g^1 with torsion free quotients and has the property that*

$$\bigcap_{k=1}^{\infty} L^k T_g^1$$

is trivial.

Proof. This follows directly from the fact that the fundamental group of a compact Riemann surface is residually nilpotent [4], and the fact that the graded quotients of the lower central series of a surface group are torsion free [32]. \square

The most rapidly descending series with torsion free quotients of a group G is the series

$$G = D^1 G \supseteq D^2 G \supseteq D^3 G \supseteq \dots,$$

where

$$D^k G = \{g \in G : \text{there is an integer } n > 0 \text{ such that } g^n \in \Gamma^k G\}.$$

This filtration has the property that $D^k G/D^{k+1}$ is the k th term of the lower central series of G mod torsion. Proofs of these assertions can be found in [44].

In the current situation, we have

$$D^k T_g^1 \subseteq L^k T_g^1.$$

Johnson’s Theorem [22] implies that $D^2 T_g^1 = L^2 T_g^1$ when $g \geq 3$. The computations (9.6) and (10.1) imply that the kernel of $D^2 T_g^1/D^3 \rightarrow L^2 T_g^1/L^3$ is isomorphic to \mathbb{Z} . Morita was aware of the fact that the kernel was at least this big — cf. his work on the Casson invariant [37], and asked whether there is a k such that $D^3 T_g^1 \supseteq L^k T_g^1$. That is, whether the kernel of $D^2 T_g^1/D^3 \rightarrow L^2 T_g^1/L^3$ can be detected by the action of T_g^1 on the quotients of π_g by the terms of its lower central series.

More generally, one can ask if the topologies on T_g^1 determined by the filtrations D^\bullet and L^\bullet are equivalent. (Both are separated.) That is, for each $k \in \mathbb{N}$, can one find a positive integer $n(k)$ such that $L^{n(k)} T_g^1 \subseteq D^k T_g^1$?

Since the groups T_g^1/D^k and T_g^1/L^k are torsion free nilpotent, they imbed as a Zariski dense subgroup of a unipotent group defined over \mathbb{Q} . One obtains two inverse systems of unipotent groups. It is clear that the first pronilpotent group is the Malcev completion T_g^1 of T_g^1 , and the second is the pronilpotent group associated to the pronilpotent Lie algebra $\mathfrak{h}_g := \text{im}\{\mathfrak{t}_g^1 \rightarrow \mathfrak{d}_g\}$, where \mathfrak{d}_g is the pronilpotent Lie algebra defined in §9. The two topologies on T_g^1 are equivalent if and only if the natural map $\mathfrak{t}_g^1 \rightarrow \mathfrak{h}_g$ is an isomorphism. Equivalently, they are equivalent if and only if $\mathfrak{t}_g^1 \rightarrow \mathfrak{d}_g$ is injective. It is also clear that the filtration L^\bullet of \mathfrak{t}_g^1 induced from that of T_g^1 is the pullback of the weight filtration of \mathfrak{h}_g^1 , so that

$$(L^k T_g^1/L^{k+1}) \otimes \mathbb{Q} \cong \text{Gr}_L^k \mathfrak{t}_g^1 \cong \text{Gr}_{-k}^W \mathfrak{h}_g^1$$

and that $L^k \mathfrak{t}_g^1 \supseteq \mathbb{G}_a$ for all $k \geq 1$ — cf. (3.4).

Theorem 14.6. *For all $g \geq 3$ and all $k \geq 1$, $L^k \mathfrak{t}_g^1 \supseteq \mathbb{G}_a$ so that the natural representation $\mathfrak{t}_g^1 \rightarrow \mathfrak{d}_g$ is not injective as its kernel contains \mathbb{G}_a . In particular, there is no $k \geq 1$ such that $W_{-3} \mathfrak{t}_g^1 \supseteq L^k \mathfrak{t}_g^1$. \square*

One can define a filtration L^\bullet of T_g by defining $L^k T_g$ to be the image of $L^k T_g^1$. Using similar arguments, one can prove that the filtrations L^\bullet and D^\bullet of T_g do not define equivalent topologies.

14.4. A question of Asada and Nakamura. There is an issue raised by Asada and Nakamura in [2, (4.5)] which is closely related to Morita’s question. Denote by $\pi_{g,1}$ the fundamental group $\pi_1(S, v)$ of S with respect to the tangent vector v . It is naturally isomorphic to the fundamental group of the punctured surface S minus the anchor point x of v . Note that $T_{g,1}$ acts on $\pi_{g,1}$. Asada and Nakamura define a filtration M^\bullet of T_g^1 as follows: First define a filtration L^\bullet of $T_{g,1}$ as in the previous section: ϕ is in $L^k T_{g,1}$ if and only if ϕ induces the identity on $\pi_{g,1}$ modulo the $(k + 1)$ st term of its lower central series. Define $M^k T_g^1$ to be the image of $L^k T_{g,1}$ in T_g^1 . They then ask whether, after tensoring with \mathbb{Q} , the sequence

$$0 \rightarrow \text{Gr}_{\bullet}^W \pi_g \rightarrow \text{Gr}_{\bullet}^M T_g^1 \rightarrow \text{Gr}_{\bullet}^L T_g \rightarrow 0$$

is exact. (Recall from (4.7) that the lower central series of \mathfrak{p}_g agrees with its weight filtration.) We now give a proof that this is indeed the case. We continue with the notation of the previous section.

The filtration M^\bullet induces a filtration of \mathfrak{t}_g^1 . Their question then becomes: is the sequence

$$0 \rightarrow \text{Gr}_\bullet^W \mathfrak{p}_g \rightarrow \text{Gr}_\bullet^M \mathfrak{t}_g^1 \rightarrow \text{Gr}_\bullet^L \mathfrak{t}_g \rightarrow 0$$

exact? Fix a base point of $\mathcal{M}_{g,1}$ so that $\mathfrak{t}_{g,1}, \mathfrak{t}_g^1, \pi_{g,1}, \mathfrak{d}_g, \mathfrak{p}_g$, etc. all have compatible MHSs; the MHS on $\mathfrak{p}_{g,1}$ is the limit MHS on $\pi_1(S - \{x\}, x_o)$ associated to the “degeneration”, where x_o approaches x from the direction of v . Denote the image of \mathfrak{t}_g^1 in \mathfrak{d}_g by \mathfrak{h}_g^1 , and the image of \mathfrak{t}_g in \mathfrak{o}_g by \mathfrak{h}_g . These have canonical mixed Hodge structures determined by the choice of the base point. Since the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{p}_g & \longrightarrow & \mathfrak{t}_g^1 & \longrightarrow & \mathfrak{t}_g & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{p}_g & \longrightarrow & \mathfrak{h}_g^1 & \longrightarrow & \mathfrak{h}_g & \longrightarrow & 0 \end{array}$$

commutes and since the top row is exact, it follows that the bottom row is exact. Since Gr_\bullet^W is an exact functor, and since $\text{Gr}_k^W \mathfrak{h}_g^n \cong \text{Gr}_k^L \mathfrak{t}_g^n$ when $n = 0, 1$, this implies that the sequence

$$0 \rightarrow \text{Gr}_\bullet^W \mathfrak{p}_g \rightarrow \text{Gr}_\bullet^L \mathfrak{t}_g^1 \rightarrow \text{Gr}_\bullet^L \mathfrak{t}_g \rightarrow 0$$

is exact. To complete the proof, we show that the filtrations L^\bullet and M^\bullet of \mathfrak{t}_g^1 are equal.

Denote by b_o the element of $\pi_{g,1}$ that corresponds to rotating the tangent vector once about x — this is a “Dehn twist about the boundary of $S - \{x\}$ ”. The action of $T_{g,1}$ on $\pi_{g,1}$ fixes b_o , and therefore induces a homomorphism $T_{g,1} \rightarrow \text{Aut}(\pi_{g,1}, b_o)$ into the automorphisms of $\pi_{g,1}$ that fix b_o . Set $w_o = \log b_o$. This we interpret as an element of $\mathfrak{p}_{g,1}$. The homomorphism above induces a homomorphism $T_{g,1} \rightarrow \text{Aut}(\mathfrak{p}_{g,1}, w_o)$, and therefore a Lie algebra homomorphism

$$\mathfrak{t}_{g,1} \rightarrow \text{Der}(\mathfrak{p}_{g,1}, w_o)$$

into the derivations of $\mathfrak{p}_{g,1}$ that annihilate w_o . It follows from standard properties of limit MHSs that w_o spans a copy of $\mathbb{Q}(1)$ in $\mathfrak{p}_{g,1}$. But $\text{Der}(\mathfrak{p}_{g,1}, w_o)$ is the kernel of the map $\text{Der} \mathfrak{p}_{g,1} \rightarrow \mathfrak{p}_{g,1}$ that takes ϕ to $\phi(w_o) - w_o$. Since w_o is a Hodge class, this is a morphism of MHS. It follows that $\text{Der}(\mathfrak{p}_{g,1}, w_o)$ has a natural MHS.

Since \mathfrak{p}_g is the quotient of $\mathfrak{p}_{g,1}$ by the ideal generated by w_o as MHS, the homomorphism

$$\text{Der}(\mathfrak{p}_{g,1}, w_o) \rightarrow \text{Der} \mathfrak{p}_g$$

is a morphism of MHS. The filtration L^\bullet of $\mathfrak{t}_{g,1}$ is the inverse image of the weight filtration under the homomorphism $\mathfrak{t}_{g,1} \rightarrow \text{Der}(\mathfrak{p}_{g,1}, w_o)$. The equality of the filtrations L^\bullet and M^\bullet of \mathfrak{t}_g^1 now follows from the strictness properties of the weight filtration as the diagram

$$\begin{array}{ccc} \mathfrak{t}_{g,1} & \longrightarrow & \mathfrak{t}_g^1 \\ \downarrow & & \downarrow \\ \text{Der}(\mathfrak{p}_{g,1}, w_o) & \longrightarrow & \text{Der} \mathfrak{p}_g \end{array}$$

commutes and all arrows are morphisms of MHS.

14.5. **Cohomology of \mathfrak{t}_g and vanishing differentials.** Since $\mathfrak{t}_{g,r}^n = W_{-1}\mathfrak{t}_{g,r}^n$, it follows that

$$W_{k-1}H_{\text{cts}}^k(\mathfrak{t}_{g,r}^n) = 0$$

for all $k \geq 0$. The *lowest weight subring* of $H_{\text{cts}}^\bullet(\mathfrak{t}_{g,r}^n)$ is defined to be the subring

$$\bigoplus_{k \geq 0} W_k H_{\text{cts}}^k(\mathfrak{t}_{g,r}^n).$$

By [15, (9.2)], this is a quadratic algebra generated by $H_{\text{cts}}^1(\mathfrak{t}_{g,r}^n)$ and where the relations are dual to the second graded quotient of the lower central series of $T_{g,r}^n$.

Theorem 14.7. *For each irreducible representation $V(\lambda)$ of Sp_g , the image of the natural homomorphism*

$$[W_k H_{\text{cts}}^k(\mathfrak{t}_{g,r}^n) \otimes V(\lambda)]^{Sp_g} \rightarrow H^0(Sp_g(\mathbb{Z}), H^k(T_{g,r}^n) \otimes V(\lambda)) = E_2^{0,k}$$

is contained in

$$E_\infty^{0,k} = \text{im} \{ H^k(\Gamma_{g,r}^n, V(\lambda)) \rightarrow H^k(T_{g,r}^n) \otimes V(\lambda) \}.$$

Proof. Fix a base point of $\mathcal{M}_{g,r}^n$ so that $\mathfrak{t}_{g,r}^n, \mathfrak{u}_{g,r}^n$, etc. all have compatible MHSs. Since the extension

$$0 \rightarrow \mathbb{G}_a \rightarrow \mathfrak{t}_{g,r}^n \rightarrow \mathfrak{u}_{g,r}^n \rightarrow 0$$

is central with kernel isomorphic to $\mathbb{Q}(1)$, it follows from the Gysin sequence that the induced map

$$\bigoplus_{k \geq 0} W_k H_{\text{cts}}^k(\mathfrak{u}_{g,r}^n) \rightarrow \bigoplus_{k \geq 0} W_k H_{\text{cts}}^k(\mathfrak{t}_{g,r}^n)$$

is surjective, with kernel the ideal generated by the cohomology class in $W_2 H_{\text{cts}}^2(\mathfrak{u}_{g,r}^n)$ corresponding to the extension above.

By (7.3), there is a canonical map

$$[H_{\text{cts}}^k(\mathfrak{u}_{g,r}^n) \otimes V(\lambda)]^{Sp_g} \longrightarrow H^k(\Gamma_{g,r}^n, \mathbb{V}(\lambda)).$$

The result follows because the diagram

$$\begin{array}{ccc} [W_k H_{\text{cts}}^k(\mathfrak{u}_{g,r}^n) \otimes V(\lambda)]^{Sp_g} & \longrightarrow & H^\bullet(\Gamma_{g,r}^n, \mathbb{V}(\lambda)) \\ \downarrow & & \downarrow \\ [W_k H_{\text{cts}}^k(\mathfrak{t}_{g,r}^n) \otimes V(\lambda)]^{Sp_g} & \longrightarrow & H^0(Sp_g(\mathbb{Z}), H^k(T_{g,r}^n) \otimes V(\lambda)) \end{array}$$

commutes, and because the left hand vertical map is surjective. □

14.6. **Morita’s conjecture.** We now prove a result which is, in some sense, an affirmation of Morita’s conjecture [39, 2.7]. Our result is an analogue of his theorem [39, 6.2] which is a solution to the conjecture in the first non-trivial case. He also informs me that he has proved the second non-trivial case of the conjecture over $\frac{1}{24}\mathbb{Z}$.

Suppose that $g \geq 3$. Denote the k th term of the lower central series of π_g by $\pi^{(k)}$. Set

$$\pi_{(k)} = \pi_g / \pi^{(k+1)}.$$

We know from Labute’s theorem [32] that this is a torsion free nilpotent group. For each $k \geq 1$, there is a natural representation

$$\rho_k : \Gamma_g^1 \rightarrow \text{Aut } \pi_{(k)}.$$

The first is simply the standard representation $\Gamma_g^1 \rightarrow Sp_g(\mathbb{Z})$. Denote the \mathbb{Q} -form of the unipotent completion of $\pi_{(k)}$ by $\mathcal{P}_{(k)}$. Since $\pi_{(k)}$ is torsion free, the canonical map $\pi_{(k)} \rightarrow \mathcal{P}_{(k)}$ is injective. By the universal mapping property of unipotent completion, we see that each ρ_k extends to a homomorphism

$$\tilde{\rho}_k : \Gamma_g^1 \rightarrow \text{Aut } \mathcal{P}_{(k)}.$$

Denote the Lie algebra of $\mathcal{P}_{(k)}$ by $\mathfrak{p}_{(k)}$. Then $\text{Aut } \mathcal{P}_{(k)} \cong \text{Aut } \mathfrak{p}_{(k)}$. It follows that $\text{Aut } \mathcal{P}_{(k)}$ is a linear algebraic group. Denote the Zariski closure of the image of $\tilde{\rho}_k$ in this by G_k . It is easy to see that G_k is an extension of $Sp_g(\mathbb{Q})$ by a unipotent group:

$$1 \rightarrow U_k \rightarrow G_k \rightarrow Sp_g(\mathbb{Q}) \rightarrow 1.$$

This extension is split exact, so that

$$G_k \cong Sp_g(\mathbb{Q}) \times U_k.$$

By the universal mapping property of the relative completion of Γ_g^1 , there is a homomorphism $\mathcal{G}_g^1 \rightarrow G_k$ which commutes with the projections to Sp_g . The following result follows directly from the fact (3.4) that the natural map $\mathcal{T}_g^1 \rightarrow \mathcal{U}_g^1$ is surjective.¹²

Lemma 14.8. *For each $k \geq 2$, $\tilde{\rho}(T_g^1)$ is Zariski dense in U_k . □*

Proposition 14.9. *For each $k \geq 2$, the image of $\tilde{\rho}_k$ is a discrete subgroup of $G_k(\mathbb{R})$, and the quotient $\text{im } \rho_k \backslash G_k(\mathbb{R})$ has finite volume with respect to any left invariant metric on $G_k(\mathbb{R})$.*

Proof. Since every finitely generated subgroup of the \mathbb{Q} points of a unipotent group U is discrete in $U(\mathbb{R})$, it follows that $\tilde{\rho}_k(T_g^1)$ is a discrete subgroup of $U_k(\mathbb{R})$. Since it is also Zariski dense, it is cocompact. The result now follows as the image of Γ_g^1 in $Sp_g(\mathbb{R})$ is $Sp_g(\mathbb{Z})$, which is discrete and of finite covolume. □

We should note that Morita works with $\Gamma_{g,1}$ rather than with Γ_g^1 as we do. Our arguments work equally well in his case; we chose to work with Γ_g^1 as it seems more natural. In conclusion, we remark that the Lie algebra of U_k is simply the image $\mathfrak{h}_g^1/W_{-k-1}$ of \mathfrak{u}_g^1 in $\text{Der } \mathfrak{p}_{(k)}$. It follows that the Lie algebra of U_k has a MHS, and is therefore isomorphic to its associated graded after tensoring with \mathbb{C} .

15. THE UNIVERSAL CONNECTION

In this section we construct a universal connection form

$$\tilde{\omega} \in E^\bullet(\text{Torelli space}) \hat{\otimes} \text{Gr}^\bullet \mathfrak{t}_{g,r}^n$$

with “scalar curvature” on Torelli space when $g \geq 3$. Here $E^\bullet(X)$ denotes the C^∞ de Rham complex of a smooth manifold X , and $\hat{\otimes}$ the completed tensor product.¹³

¹²A direct proof of the lemma can be given — cf. the proof of [17, (4.6)].

¹³The completed tensor product $E^\bullet(X) \hat{\otimes} \text{Gr}^\bullet \mathfrak{g}$ is defined to be the inverse limit

$$\varprojlim E^\bullet(X) \otimes \text{Gr}^\bullet \mathfrak{g} / \oplus_{l \geq m} \text{Gr}^l \mathfrak{g}.$$

This is the analogue of the universal connection

$$\sum_{ij} d \log(z_i - z_j) X_{ij} \in E^\bullet(X_n) \otimes \text{Gr}^\bullet \mathfrak{p}_n$$

for the braid group P_n . Here X_n denotes the classifying space

$$\mathbb{C}^n - \{\text{fat diagonal}\}$$

of the pure braid group, (z_1, \dots, z_n) its coordinates, and \mathfrak{p}_n the Malcev Lie algebra associated to P_n . A reasonably precise dictionary between the case of braid groups and the absolute mapping class groups Γ_g is given in Table 1.

Question 15.1. The Lie algebra $\text{Gr}^\bullet \mathfrak{p}_n$ has interesting finite-dimensional representations; namely those associated to Hecke algebras. Are there analogous representations of $\text{Gr}^\bullet \mathfrak{t}_{g,r}^n$ where the canonical central \mathbb{G}_a acts via scalar transformations? These should lead to interesting projective representations of $\Gamma_{g,r}^n$.

We now give the construction of the connection. First recall that Torelli space $\mathcal{H}_{g,r}^n$ is the quotient of the Teichmüller space associated to $\Gamma_{g,r}^n$ by $T_{g,r}^n$. It is the moduli space of isomorphism classes of $(n + r + 2g + 1)$ -tuples

$$(C; x_1, \dots, x_n; v_1, \dots, v_r; a_1, \dots, a_g, b_1, \dots, b_g),$$

where C is a compact Riemann surface of genus g , x_1, \dots, x_n are n marked points, v_1, \dots, v_r are r marked tangent vectors, and a_1, \dots, b_g is a symplectic basis of $H_1(C, \mathbb{Z})$. Since $T_{g,r}^n$ is torsion free and Teichmüller space is contractible, $\mathcal{H}_{g,r}^n$ is the classifying space of $T_{g,r}^n$.

The bulk of the work needed for the construction of the connection has already been done in [19, §14]. Fix a point $x \in \mathcal{M}_{g,r}^n$. It follows from [19, §14.2] that there is a 1-form

$$\omega \in E^\bullet(\mathcal{H}_{g,r}^n) \hat{\otimes} \text{Gr}^\bullet \mathfrak{u}_{g,r}^n$$

which is integrable and is $Sp_g(\mathbb{Z})$ invariant in that

$$(28) \quad s^* \omega = Ad(s) \omega$$

for all $s \in Sp_g(\mathbb{Z})$. That is, if

$$\omega = \sum_I w_I X_I, \text{ where } w_I \in E^1(\mathcal{H}_{g,r}^n) \text{ and } X_I \in \text{Gr}^\bullet \mathfrak{u}_{g,r}^n,$$

then for all $s \in Sp_g(\mathbb{Z})$,

$$\sum_I (s^* w_I) X_I = \sum_I w_I (X_I \cdot s^{-1}),$$

where $X \cdot s$ denotes the canonical action of $s \in S$ on $X \in \mathfrak{u}_{g,r}^n$. This should be compared with the case of braids where the corresponding formula is easily verified — cf. [19, (14.6)].

Since there is a canonical isomorphism

$$\mathfrak{u}_{g,r}^n(x) \cong \prod_{l \geq 1} \text{Gr}_{-l}^W \mathfrak{u}_{g,r}^n(x),$$

this form gives rise to a flat connection on the trivial right $\mathcal{U}_{g,r}^n$ principal bundle

$$\mathcal{H}_{g,r}^n \times \mathcal{U}_{g,r}^n \rightarrow \mathcal{H}_{g,r}^n.$$

TABLE 1

Braid Groups	Mapping Class Groups	Comments
B_n	Γ_g	the group of interest
Σ_n	$Sp_g(\mathbb{C})$	a semi-simple algebraic group G
$\rho : B_n \rightarrow \Sigma_n$	$\rho : \Gamma_g \rightarrow Sp_g(\mathbb{C})$	homomorphism to G with dense image
Σ_n	$Sp_g(\mathbb{Z})$	the image of ρ , an arithmetic group
P_n	T_g	the kernel of ρ , a residually torsion free nilpotent group
\mathcal{P}_n	\mathcal{U}_g	prounipotent radical of the relative completion
$B_n \rightarrow \Sigma_n \times \mathcal{P}_n$	$\Gamma_g \rightarrow \mathcal{G}_g \cong Sp_g(\mathbb{C}) \times \mathcal{U}_g$	the relative completion
\mathcal{P}_n	\mathcal{T}_g	the unipotent completion of the kernel of ρ
$\text{id} : \mathcal{P}_n \rightarrow \mathcal{P}_n$	$\mathcal{T}_g \rightarrow \mathcal{U}_g$	the homomorphism to the prounipotent radical
\mathfrak{p}_n	\mathfrak{t}_g	the pronilpotent Lie algebra corresponding to $\ker \rho$
$\text{Gr}^\bullet \mathfrak{p}_n = \mathbb{L}(H_1(P_n))/R$	$\text{Gr}^\bullet \mathfrak{t}_g = \mathbb{L}(H_1(T_g))/R$	quadratic presentations as graded Lie algebras in the category of representations of G
$\{[X_{ij}, X_{kl}], [X_{ij}, X_{ik} + X_{jk}]\}$	$V(\lambda_6), V(\lambda_4), V(\lambda_2), V(\lambda_2 + \lambda_4)$	the quadratic relations
$X_n := \mathbb{C}^n - \{\text{fat diagonal}\}$	$\mathcal{H}_g := \text{Torelli space}$	the classifying space of the kernel of $\ker \rho$
$Y_n := \Sigma_n \setminus X_n$	$\mathcal{M}_g = Sp_g(\mathbb{Z}) \backslash \mathcal{H}_g$	the classifying space of the group of interest
$\sum_{ij} w_{ij} X_{ij} \in E^\bullet(X_n) \otimes \text{Gr}^\bullet \mathfrak{p}_n$	$\omega \in E^\bullet(\mathcal{H}_g) \hat{\otimes} \text{Gr}^\bullet \mathfrak{t}_g$	the “universal (projectively) flat connection” on the classifying space of $\ker \rho$

Note that $Sp_g(\mathbb{Z})$ acts on this bundle via the diagonal action. The composite

$$\mathcal{U}_{g,r}^n \rightarrow \mathcal{H}_{g,r}^n \rightarrow \mathcal{M}_{g,r}^n$$

is a left principal $Sp_g(\mathbb{Z}) \times \mathcal{U}_{g,r}^n$ bundle (in the orbifold sense.) The invariance condition (28) means that the connection defined by ω is invariant under the $Sp_g(\mathbb{Z}) \times \mathcal{U}_{g,r}^n$ action. The monodromy yields a representation

$$\Gamma_{g,r}^n \rightarrow Sp_g(\mathbb{C}) \times \mathcal{U}_{g,r}^n(x).$$

As proved in [19, §14.2], this is the \mathbb{C} form of the completion of $\Gamma_{g,r}^n$ with respect to the canonical homomorphism $\Gamma_{g,r}^n \rightarrow Sp_g(\mathbb{C})$.

Since the sequence

$$0 \rightarrow \mathbb{G}_a \rightarrow \mathfrak{t}_{g,r}^n \rightarrow \mathfrak{u}_{g,r}^n \rightarrow 0$$

splits canonically over \mathbb{C} (given the choice of the base point x), ω has a canonical lift

$$\tilde{\omega} \in E^\bullet(\mathcal{H}_{g,r}^n) \hat{\otimes} \text{Gr}_\bullet^W \mathfrak{t}_{g,r}^n$$

to $\text{Gr}_\bullet^W \mathfrak{t}_{g,r}^n$. This form is not integrable, but since ω is integrable, the curvature of $\tilde{\omega}$ takes values in the central \mathbb{G}_a . It also has the invariance property (28).

We will say that a representation $\phi : \text{Gr}_\bullet^W \mathfrak{t}_{g,r}^n \rightarrow \text{End}(V)$ is *projective* if the image of \mathbb{G}_a consists of scalar matrices. If V is an Sp_g module and ϕ is Sp_g equivariant, then ϕ should integrate to a homomorphism

$$\Gamma_{g,r}^n \rightarrow PGL(V),$$

at least when ϕ is “sufficiently small”, since, in this case, the composite

$$\omega_\phi \in E^\bullet(X_n) \otimes \text{End}(V)/\text{scalars},$$

an infinite sum, should converge to an integrable 1-form. The equivariance of ϕ implies that ω_ϕ has the invariance property (28), leading to a projectively flat bundle over $\mathcal{M}_{g,r}^n$ with fiber V over the base point x .

APPENDIX A. INDEX OF NOTATION

This is an index of principal notation. Some notational conventions appear after the index.

$\Gamma_{g,r}^n$	mapping class group — genus g , n points, r tangents	p. 598
$T_{g,r}^n$	Torelli group, genus g , n points, r tangents	p. 598
$\mathbb{L}(V)$	free Lie algebra generated by a vector space V	p. 598
$F_{g,r}^n$	configuration space of points and tangents on a genus g surface	p. 601
$\pi_{g,r}^n$	$\pi_1(F_{g,r}^n, *)$ — pure braid group, genus g	p. 601
$\Gamma_{g,r}^n[l]$	level l subgroup of $\Gamma_{g,r}^n$	p. 604
$\mathcal{G}_{g,r}^n$	relative completion of $\Gamma_{g,r}^n$	p. 604
$\mathcal{U}_{g,r}^n$	prounipotent radical of $\mathcal{G}_{g,r}^n$	p. 604
$\mathfrak{u}_{g,r}^n$	Lie algebra of $\mathcal{U}_{g,r}^n$	p. 604
$\mathcal{T}_{g,r}^n$	unipotent completion of $T_{g,r}^n$	p. 605
$\mathfrak{t}_{g,r}^n$	Lie algebra of $\mathcal{T}_{g,r}^n$	p. 605
$\mathcal{P}_{g,r}^n$	unipotent completion of $\pi_{g,r}^n$	p. 606
$\mathfrak{p}_{g,r}^n$	Lie algebra of $\mathcal{P}_{g,r}^n$	p. 606
$\mathcal{M}_{g,r}^n[l]$	moduli space of decorated genus g curves and level l structure	p. 608
$H_{\text{cts}}^\bullet(\pi, \mathbb{Q})$	continuous cohomology of a group π	p. 613
$H_{\text{cts}}^\bullet(\mathfrak{g})$	continuous cohomology of a Lie algebra \mathfrak{g}	p. 614
\mathfrak{sp}_g	symplectic Lie algebra of rank g	p. 616
$R(\mathfrak{sp}_g)$	representation ring of \mathfrak{sp}_g	p. 616

λ_j	j th fundamental weight of \mathfrak{sp}_g	p. 617
$V(\lambda)$	\mathfrak{sp}_g module with highest weight λ	p. 617
$ \lambda $	the “size” of $V(\lambda)$	p. 617
V_λ	the λ -isotypical part of an \mathfrak{sp}_g module V	p. 621
\mathfrak{p}_g	shorthand for \mathfrak{p}_g^1	p. 622
π_g	shorthand for π_g^1	p. 622
$\mathfrak{g}(l)$	l th graded quotient of the weight filtration of \mathfrak{g}	p. 622
\mathfrak{d}_g	the image of \mathfrak{t}_g^1 in $\text{Der } \mathfrak{p}_g$	p. 623
\mathfrak{o}_g	quotient of \mathfrak{d}_g by the inner derivations	p. 623

In notation of the form $Y_{g,r}^n$, the decorations r and n are omitted when they are zero. So, for example, $F_{g,0}^1$ is written F_g^1 . In the case of $\pi_{g,r}^n$ and $\mathfrak{p}_{g,r}^n$ this is carried one step further; we denote π_g^1 by π_g , and \mathfrak{p}_g^1 by \mathfrak{p}_g . In notation of the form $Y[l]$, the level l is omitted when it is one. So, for example, $\mathcal{M}_{g,0}^2(1)$ is written \mathcal{M}_g^2 .

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REFERENCES

- [1] M. Asada, M. Kaneko: *On the automorphism group of some pro- l fundamental groups*, in *Galois Representations and Arithmetic Algebraic Geometry*, editor Y. Ihara, Advanced Studies in Pure Mathematics 12 (1987), 137–159. MR **89h**:20053
- [2] M. Asada, H. Nakamura: *On graded quotient modules of mapping class groups of surfaces*, Israel J. Math. 90 (1995), 93–113. MR **96j**:57015
- [3] D. Bar-Natan: *On the Vassiliev knot invariants*, Topology 34 (1995), 423–472. MR **97d**:57004
- [4] G. Baumslag: *On generalized free products*, Math. Zeit. 78 (1962), 423–438. MR **25**:3980
- [5] J. Birman: *Braids, Links, and Mapping Class Groups*, Annals of Math. Studies 82, Princeton University Press, Princeton, 1975. MR **54**:13894
- [6] S. Bloch, I. Kriz: *Mixed Tate Motives*, Ann. of Math. 140 (1994), 557–605. MR **96c**:11062
- [7] A. Borel: *Stable real cohomology of arithmetic groups*, Ann. Sci. Ecole Norm. Sup. 7 (1974), 235–272. MR **52**:8338
- [8] A. Borel: *Stable real cohomology of arithmetic groups II*, Manifolds and Groups, Papers in Honor of Yozo Matsushima, Progress in Mathematics 14, Birkhauser, Boston, 1981, 21–55. MR **83h**:22023
- [9] A. Borel: *Private communication*, November 1994.
- [10] J. Carlson, R. Hain: *Extensions of variations of mixed Hodge structure*, Théorie de Hodge, Luminy, Juin 1987, Astérisque 179–180 (1989), 39–65. MR **91c**:14016
- [11] P. Cartier: *Construction combinatoire des invariants de Vassiliev-Kontsevich des noeuds*, C. R. Acad. Sci. Paris, t. 316, Série I (1993), 1205–1210. MR **94c**:57004
- [12] K.-T. Chen: *Extension of C^∞ function algebra and Malcev completion of π_1* , Adv. in Math. 23 (1977), 181–210.
- [13] W. Fulton, J. Harris: *Representation Theory*, GTM 129, Springer-Verlag, 1991. MR **93a**:20069

- [14] R. Hain: *The de Rham homotopy theory of complex algebraic varieties I*, *K-Theory* 1 (1987), 271–324. MR **88h**:14029
- [15] R. Hain: *Algebraic cycles and extensions of variations of mixed Hodge structure*, *Proc. Symp. Pure Math.* 53 (1991), 175–221. MR **92j**:14008
- [16] R. Hain: *Letter to Looijenga*, June 4, 1991.
- [17] R. Hain: *Completions of mapping class groups and the cycle $C - C^-$* , in *Mapping Class Groups and Moduli Spaces of Riemann Surfaces*, C.-F. Bödigheimer and R. Hain, editors, *Contemp. Math.* 150 (1993), 75–105. MR **95e**:14018
- [18] R. Hain: *Torelli groups and Geometry of Moduli Spaces of Curves*, in *Current Topics in Complex Algebraic Geometry* (C. H. Clemens and J. Kollar, eds.) MSRI publications no. 28, Cambridge University Press, 1995, 97–143. MR **97d**:14036
- [19] R. Hain: *The Hodge de Rham Theory of Relative Malcev Completion*, preprint, 1995.
- [20] D. Johnson: *An abelian quotient of the mapping class group \mathcal{I}_g* , *Math. Ann.* 249 (1980), 225–242. MR **87a**:57008
- [21] D. Johnson: *The structure of the Torelli group I: A finite set of generators for \mathcal{I}* , *Ann. of Math.* 118 (1983), 423–442. MR **85a**:57005
- [22] D. Johnson: *The structure of the Torelli group—III: The abelianization of \mathcal{I}* , *Topology* 24 (1985), 127–144. MR **87a**:57016
- [23] D. Johnson: *A survey of the Torelli group*, *Contemp. Math.* 20 (1983), 165–179. MR **85d**:57009
- [24] A. Kabanov: *The second cohomology of moduli spaces of Riemann surfaces with twisted coefficients*, Thesis, Duke University, 1995.
- [25] A. Kabanov: *The second cohomology with symplectic coefficients of the moduli space of smooth projective curves*, *Compositio Math.*, to appear.
- [26] A. Kabanov: *Stability of Schur functors*, *J. Algebra*, to appear.
- [27] C. Kassel: *Quantum Groups*, GTM 155, Springer-Verlag, 1995. MR **96e**:17041
- [28] T. Kohno: *Série de Poincaré-Koszul associée aux groupes de tresses pures*, *Invent. Math.* 82 (1985), 57–75. MR **87e**:32015a
- [29] T. Kohno: *Monodromy representations of braid groups and Yang-Baxter equations*, *Ann. Inst. Fourier, Grenoble*, 37 (1987), 139–160. MR **89h**:17030
- [30] T. Kohno, T. Oda: *The lower central series of the pure braid group of an algebraic curve*, in *Galois Representations and Arithmetic Algebraic Geometry*, editor: Y. Ihara, *Advanced Studies in Pure Mathematics* 12 (1987), 201–219. MR **90a**:14026
- [31] M. Kontsevich: *Vassiliev’s knot invariants*, *Advances in Soviet Math.* 16, part 2 (1995), 137–150. MR **94k**:57014
- [32] J. Labute: *On the descending central series of groups with a single defining relation*, *J. Algebra* 14 (1970), 16–23. MR **40**:4342
- [33] A. Malcev: *On a class of homogeneous spaces*, *Izv. Akad. Nauk SSSR Ser. Mat.* 13 (1949), 9–32; English translation, *Amer. Math. Soc. Transl.* 39 (1962). MR **10**:507e
- [34] G. Mess: *The Torelli groups for genus 2 and 3 surfaces*, *Topology* 31 (1992), 775–790. MR **93k**:57003
- [35] J. Morgan: *The algebraic topology of smooth algebraic varieties*, *Publ. Math. IHES* 48 (1978), 137–204; correction, *Publ. Math. IHES* 64 (1986), 185. MR **80e**:55020
- [36] S. Morita: *On the structure and the homology of the Torelli group*, *Proc. Japan Acad.* 65 (1989), 147–150. MR **90h**:57015
- [37] S. Morita: *Casson’s invariant for homology 3-spheres and characteristic classes of surface bundles I*, *Topology* 28 (1989), 305–323. MR **90h**:57020
- [38] S. Morita: *On the structure of the Torelli group and the Casson invariant*, *Topology* 30 (1991), 603–621. MR **92i**:57016
- [39] S. Morita: *The extension of Johnson’s homomorphism from the Torelli group to the mapping class group*, *Invent. Math.* 111 (1993), 197–224. MR **93j**:57001
- [40] S. Morita: *Abelian quotients of subgroups of the mapping class group of surfaces*, *Duke Math. Journal* 70 (1993), 699–726. MR **94d**:57003
- [41] H. Nakamura, N. Takao, R. Ueno: *Some stability properties of Teichmüller modular function fields with pro- l weight structures*, *Math. Ann.* 302 (1995), 197–213. MR **96h**:14041
- [42] K. Nomizu: *On the cohomology of compact homogeneous spaces of nilpotent Lie groups*, *Ann. of Math.* 59 (1954), 167–189. MR **16**:219e

- [43] T. Oda: *A lower bound for the weight graded quotients associated with the relative weight filtration on the Teichmüller group*, preprint, 1992.
- [44] D. Passman: *The Algebraic Structure of Group Rings*, John Wiley and Sons, New York, 1977. MR **81d**:16001
- [45] M. Pikaart: *An orbifold partition of \overline{M}_g^n* , in *The moduli space of curves*, R. Dijkgraaf, C. Faber, G. van der Geer, editors, Birkhauser, 1995, 467–482. MR **96m**:17002
- [46] C. Reutenauer: *Free Lie Algebras*, Oxford University Press, 1993. MR **94j**:17002
- [47] M. Saito: *Mixed Hodge modules and admissible variations*, C. R. Acad. Sci. Paris, t. 309 (1989), Série I, 351–356. MR **91i**:32029
- [48] J.-P. Serre: *Lie Algebras and Lie Groups*, Benjamin, New York, 1965. MR **36**:1582
- [49] D. Sullivan: *On the intersection ring of compact 3-manifolds*, Topology 14 (1975), 275–277. MR **52**:4296
- [50] D. Sullivan: *Infinitesimal computations in topology*, Publ. Math. IHES 47 (1977), 269–331. MR **58**:31119
- [51] S. Zucker: *Hodge theory with degenerating coefficients: L_2 cohomology in the Poincaré metric*, Ann. of Math. 109 (1979), 415–476. MR **81a**:14002

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