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C^* TENSOR CATEGORIES FROM QUANTUM GROUPS

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In a recent paper [Ki], Kirillov, Jr., defined a $*$-operation on the morphisms of the representation category of a quantum group $U_q\mathfrak{g}$, with $|q| = 1$, using constructions going back to Andersen, Kashiwara and Lusztig. He then conjectured that for any two morphisms $f, g : V \to W$, the form $(f, g) = Tr_q(g^*f)$ is positive semidefinite for certain roots of unity q, where Tr_q is the categorical q-trace. As a consequence, the morphisms in a certain quotient category, usually referred to as fusion category, become Hilbert spaces, and one obtains C^* categories. We prove this conjecture in this paper for all Lie types. In particular, one obtains for any object in the fusion category a subfactor whose index is the square of its categorical dimension. If the object is simple, the corresponding subfactor is irreducible.

We consider the tensor category $\mathcal{T}$ of tilting modules of a quantum group $U_q\mathfrak{g}$, where $\mathfrak{g}$ is a simple Lie algebra and $q = e^{\pi i/ld}$, with $d \in \{1, 2, 3\}$ being the ratio of the square lengths of a long and a short root and l being larger than the dual Coxeter number of $\mathfrak{g}$. Here, and in the following, we use the notations of Lusztig's book [Lu]. It is shown in [A], [AP] that $\mathcal{T}$ contains a semisimple quotient category $\mathcal{F}$ whose simple objects are labelled by the dominant weights λ which satisfy $\theta \cdot (\lambda + \rho) < ld$, where θ is the highest root (see Section 1 for notation). Any tilting module T can be written as $T = C_T \oplus C'_T$, where C_T and C'_T are direct sums of indecomposable tilting modules such that the q-dimension of each of them is not equal to 0 (for C_T), resp. equal to 0 (for C'_T). The quotient construction can be described by assigning the object $T \in \mathcal{T}$ to the object $C_T \in \mathcal{F}$, and the morphism $f : V \to W$ to the restriction $f_C : C_V \to C_W$. The problem is to define an antilinear operation $*$ on morphisms which is functorial, i.e. such that $(f_1 \otimes f_2)^* = f_1^* \otimes f_2^*$, where $f_i : V_i \to W_i$, and positive, i.e. $Tr_q(f^*f) \geq 0$ for any morphism f in $\mathcal{F}$. Kirillov defined a $*$-operation which is functorial; the remaining problem is to prove positivity.

The approach in this paper can be sketched in the following way: Assume that we have bases of W_1 and W_2 with respect to which the generators of the quantum group act via matrices whose coefficients are polynomials in q. Moreover, assume that we can find an idempotent p whose coefficients depend continuously on q such that $p(q)$ commutes with $U_q\mathfrak{g}$ for q in a neighborhood of 1 containing $q_0 = e^{\pi i/dl}$, and such that $p(q_0)$ projects onto $C_{W_1 \otimes W_2}$, with kernel $C'_{W_1 \otimes W_2}$. Moreover, assume we have defined a Hermitian form on $W = W_1 \otimes W_2$ also depending continuously on q which is nondegenerate on $p(q)W$ for all q near 1, and such that it is positive definite on $p(1)W$. Then it follows by continuity that it has to be positive definite

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on $p(q)W$ for all q for which $p(q)$ is well-defined, including $q_0 = e^{\pi i/dl}$. From this one can deduce the positivity property for all morphisms in $\mathcal{F}$.

The difficulty consists of constructing such p's for tensor products $W = W_1 \otimes W_2$ of tilting modules. If one takes arbitrary tensor products, it is not clear how to choose p in general, as the decomposition $W = C_W \oplus C'_W$ is not canonical. However, it is possible to find, for any Lie type, a simple generating tilting module V such that the decomposition of $V \otimes T_\lambda$ is canonical (or almost canonical for Lie types F_4 and E_8) due to the fact that all simple components (except possibly one) in the tensor product appear with multiplicity 1. Hence the continuity argument sketched in the previous paragraph works (in a slightly more complicated version for types F_4 and E_8). Such idempotents have already appeared before in [TW1, Theorem 5.1.2] for tensor powers of the vector representation of a classical Lie algebra, and we prove it here in general, using the canonical bases of Kashiwara and Lusztig. The positive definiteness of the Hermitian form for tensor products of arbitrary tilting modules can be deduced from this, as V generates the tensor category under consideration.

As an application, we construct examples of subfactors of the hyperfinite II_1 von Neumann factor for any object in $\mathcal{F}$, and we compute their indices and higher relative commutants. Among them are, as special cases, the Jones subfactors and subfactors constructed from Hecke algebras, and several series of subfactors constructed from the Kauffman polynomial.

Here is a more detailed account of the contents of this paper. In Section 1 we recall definitions of quantum groups and define a $*$-structure on them, i.e. we define an antilinear antimorphism $*$ for a quantum group. In Section 2 we define sesquilinear and Hermitian forms on Weyl modules and tensor products of them. The results of these two sections (except perhaps the explicit correction factor for getting Hermitian forms) are basically all known and they are contained, resp. reviewed, in Kirillov's paper. In Section 3 we define tilting modules and recall Andersen's results. We then prove the crucial positivity result for $T_\lambda \otimes V$ mentioned earlier in the introduction. We deduce from this the positivity result for morphisms. Section 4 contains the construction of subfactors and the computation of their indices and higher relative commutants.

Notes. 1. While writing up the first version of this paper, the author was made aware of the preprint [X] by Feng Xu which deals with the construction of subfactors from quantum groups of all Lie types except F_4 and E_8 (see Remark 4.6 for more details).

2. I would like to thank the referee for his detailed remarks on an earlier version, which led to an improved presentation of the material.

1. Quantum groups

In this section we fix notation for quantum groups. We recall various results and definitions, some of them with some alterations.

1.1. Definition. Let Φ be the root system corresponding to a semisimple Lie algebra $\mathfrak{g}$, with Cartan matrix A. Let $\cdot$ be the invariant form on the root lattice normalized so that $\alpha_i \cdot \alpha_i = 2$ for all short roots α_i (see [Lu, Chapter 1]). Let X be the weight lattice of $\mathfrak{g}$ which we assume to be embedded in the $\mathbf{Q}$ span of the root lattice, and extend the dot product to X by linearity. Let $\alpha_i,\ i = 1, 2, \dots, m$, be the simple roots of Φ. In the following we define $d_i = \alpha_i \cdot \alpha_i/2$ and $d = d_i$ if

α_i is a long root (or, equivalently, d is the quotient between the square lengths of a long and a short root). We assume x to be a variable, and we define $x_i = x^{d_i}$. Here the quantum group $U_x\mathfrak{g}$ is defined to be an algebra over the field of rational functions $\mathbf{R}(x)$ with generators $E_i,\ F_i,\ k_i,\ k_i^{-1}\quad (1 \le i \le m)$. Later, we shall also encounter formulas involving a d-th root of x, so we shall embed $\mathbf{R}(x)$ into $\mathbf{R}(x^{1/d})$, where $x^{1/d}$ is a formal variable whose d-th power is equal to x. We shall also use the elements $\tilde{K}_i = k_i^{d_i}$, and if $\mu = \sum m_i\alpha_i$, we define $\tilde{K}_\mu = \prod \tilde{K}_i^{m_i}$. The defining relations of the algebra $U_x\mathfrak{g}$ are

$$k_ik_j = k_jk_i, \quad k_ik_i^{-1} = k_i^{-1}k_i = 1,$$

$$k_iE_jk_i^{-1} = x^{a_{ij}}E_j, \quad k_iF_jk_i^{-1} = x^{-a_{ij}}F_j,$$

$$[E_i, F_j] = \delta_{ij}\frac{\tilde{K}_i - \tilde{K}_i^{-1}}{x_i - x_i^{-1}},$$

$$\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \frac{[1-a_{ij}]_{x_i}!}{[\nu]_{x_i}![1-a_{ij}-\nu]_{x_i}!}\ E_i^{1-a_{ij}-\nu}E_jE_i^\nu = 0, \quad i \ne j,$$

where

$$[n]_x!\ =\ \prod_{j=1}^{n}\frac{x^j - x^{-j}}{x - x^{-1}}.$$

A similar identity holds if we replace E_i and E_j by F_i and F_j, respectively, in the second to last equation. The algebra $U_x\mathfrak{g}$ has a Hopf algebra structure with comultiplication Δ defined by

$$\Delta(E_i) = \tilde{K}_i \otimes E_i + E_i \otimes 1, \quad \Delta(F_i) = 1 \otimes F_i + F_i \otimes \tilde{K}_i^{-1} \quad \text{and} \quad \Delta(k_i) = k_i \otimes k_i.$$

The antipode S is defined on the generators by

$$S(k_i) = k_i^{-1}, \quad S(E_i) = -\tilde{K}_i^{-1}E_i \quad \text{and} \quad S(F_i) = -F_i\tilde{K}_i,$$

and the counit ϵ is given by

$$\epsilon(E_i) = 0, \quad \epsilon(F_i) = 0 \quad \text{and} \quad \epsilon(k_i) = 1.$$

1.2. We define an involution $^-$ on $\mathbf{R}(x)$ by $\bar{r} = r$ for $r \in \mathbf{R}$ and by $\bar{x} = x^{-1}$. We denote the ring $\mathbf{R}[x, x^{-1}] \subset \mathbf{R}(x)$ by A, and by A_0 the subring $\{f \in A, \bar{f} = f\}$ of A. A map $\phi : V \to W$ between $\mathbf{R}(x)$ vector spaces is called antilinear if $\phi(fv) = \bar{f}\phi(v)$ for all $v \in V$ and $f \in \mathbf{R}(x)$. We define $\bar{E}_i = E_i,\ \bar{F}_i = F_i$ and $\bar{k}_i = k_i^{-1}$ for $i = 1, 2, \ldots, r$. It is easy to see that this extends to an antilinear algebra automorphism from $U_x\mathfrak{g}$ onto itself.

We also define an operation * on elements in $U_x\mathfrak{g}$. It is defined on generators by

$$E_i^* = F_i, \quad F_i^* = E_i \quad \text{and} \quad k_i^* = k_i^{-1}.$$

1.3. Lemma. *The map * defined in 1.2 induces an antilinear algebra antimorphism from $U_x\mathfrak{g}$ into itself which is also a coalgebra antimorphism. The induced antimorphism is also denoted by *.*

Proof. Let $U_x\mathfrak{g}^{opp}$ be the algebra which coincides as a set with $U_x\mathfrak{g}$, with opposite multiplication and antilinear scalar multiplication. It is easy to check that * defines an isomorphism between these two algebras. Moreover, it is easy to check for the generators that $\Delta(u)^* = \Delta'(u^*)$, where Δ' is the opposite coproduct. $\square$

1.4. ***R*-matrix.** R-matrices in connection with quantum groups were introduced by Drinfeld [D1]. Here we follow the presentation in [Lu, Chapter 32]; see also [LS], [KR]. Denote by U^+ the subalgebra of $U_x\mathfrak{g}$ generated by the E_i's, and by U^- the subalgebra generated by the F_i's. Both subalgebras have an obvious gradation via elements $\nu = (\nu_i) \in \mathbf{Z}^k$ as follows: The subspace U^+_ν is the span of all products of E_i's in which a given E_j occurs exactly ν_j times. One defines U^-_ν similarly. There exists a canonical element Θ in $\bigoplus_\nu U^-_\nu \otimes U^+_\nu$, uniquely determined by the properties $\Theta_0 = 1 \otimes 1$, where Θ_0 is the summand in $U^-_0 \otimes U^+_0$, and by

$$\bar{\Delta}(u) = \Theta\Delta(u)\Theta^{-1} \quad \text{for all } u \in U_x\mathfrak{g}.$$

It is easy to see that the map $(\lambda, \mu) \in X \times X \to -\lambda \cdot \mu$ satisfies the properties of the map f in [Lu, Chapter 32]. On any tensor product $V \otimes W$ of $U_x\mathfrak{g}$ modules define the operator Π by $\Pi(v \otimes w) = x^{-\lambda\cdot\mu} v \otimes w$, where v, w are weight vectors belonging to the weights λ and μ (see Remark 1.4.3 for an interpretation of this operator). Also observe that the exponent of X only takes values in $\frac{1}{d}\mathbf{Z}$, hence the expression is well-defined in the field $\mathbf{R}(x^{1/d})$ (see Section 1.1). Moreover, also define $R = \Pi^{-1}\bar{\Theta}$. We have the following

1.4.1. Lemma. (a) *$\Delta'(u) = R\Delta(u)R^{-1}$, where Δ' is the opposite coproduct.*
(b) *$(\Delta(u))^* = \Delta'(u^*)$ for all $u \in U_x\mathfrak{g}$.*
(c) *$\Delta'(u) = R^*\Delta(u)(R^{-1})^*$.*
(d) *$\bar{\Theta}^* = \Theta_{21}$, where $\Theta_{21} = P_U\Theta$ and $P_U(u_1 \otimes u_2) = u_2 \otimes u_1$ for $u_1, u_2 \in U_x\mathfrak{g}$.*
(e) *$R^* = R^{-1}_{21}$, where $R_{21} = \Pi^{-1}\bar{\Theta}_{21}$.*

Proof. Part (a) was proved in [LS] and [KR], going back to ideas of Drinfeld; for a proof in this notation see [Lu, Chapter 32] (where our R is $\mathcal{R}_f^{-1}$ in [Lu]). Part (b) is easily checked for the generators. The general claim follows as both $u \mapsto (\Delta(u))^*$ as well as $u \mapsto \Delta'(u^*)$ are antilinear algebra antimorphisms. Observe that $\Delta(u^*)^* = (R^{-1}\Delta'(u^*)R)^*$, by (a). Simplifying both sides, using (b), we get statement (c). For part (d) observe that also $(\bar{\Theta}^*)_{21} = P_U(\bar{\Theta}^*)$ satisfies the same properties as Θ. Indeed, we have for any $u \in U_x\mathfrak{g}$

$$(\bar{\Theta}^*)_{21}\Delta(u)(\bar{\Theta}^*)^{-1}_{21} = P_U(\bar{\Theta}^*\Delta'(u)\Theta^*) = P_U(\Theta\Delta(u^*)\bar{\Theta})^*$$

$$= P_U(\bar{\Delta}(u^*)^*) = P_U(\bar{\Delta}'(u)) = \bar{\Delta}(u);$$

moreover, one checks easily that $(\bar{\Theta})^*_{21}$ is in $\bigoplus_\nu U^-_\nu \otimes U^+_\nu$ and that $((\bar{\Theta})^*_{21})_0 = 1 \otimes 1$. Hence $(\bar{\Theta}^*)_{21} = \Theta$, which is equivalent to claim (d). Claim (e) follows from this and the definition of R. $\square$

1.4.2. Lemma (see [D2]). *Let V_λ, V_μ be the highest weight modules of $U_x\mathfrak{g}$. Define for any weight ν the quantity $G(\nu) = (\nu + 2\rho)\cdot\nu$. Then the element $R_{21}R$ commutes with the action of $U_x\mathfrak{g}$ on $V_\lambda \otimes V_\mu$, and it acts on any highest weight submodule $V_\nu \subset V_\lambda \otimes V_\mu$ via the scalar*

$$\alpha_\nu^{(\lambda,\mu)} = x^{G(\nu)-G(\lambda)-G(\mu)}.$$

1.4.3. Remark. It follows from results by Drinfeld ([D2]) that the operator Π lies in a certain completion of $U_x\mathfrak{g}$. Similarly, there exists an operator Γ (the quantum Casimir) in a completion of $U_x\mathfrak{g}$ which acts via the scalar $x^{G(\lambda)}$ on any irreducible module V_λ with highest weight λ.

1.5. Roots of unity. Consider the setting of Section 1.1. Define the elements $E_i^{(p)} = E_i^p/[p]_{x_i}!$ and $F_i^{(p)} = F_i^p/[p]_{x_i}!$ (see [Lu]). It can be shown that the algebraic relations among the $E_i^{(p)}$ as well as among the $F_i^{(p)}$ only involve Laurent polynomials in x.

Now let q be a complex number. Then we define the quantum group $U_q\mathfrak{g}$ to be the complex Hopf algebra with generators $E_i^{(p)}$, $F_i^{(p)}$ and $k_i^{\pm 1}$, where i runs through the simple roots of $\mathfrak{g}$, and $p \in \mathbf{N}$, and with the relations induced from the ones stated in 1.1. It can be shown that this defines a Hopf algebra with a highly nontrivial structure when q is a root of unity. It has well-defined actions on Weyl modules, to be discussed in the next section.

Observe that the definition of Π and R involves a choice of a d-th root of q. Unless otherwise stated, we will choose the root $e^{\pi i t/d}$ for $q = e^{\pi i t}$, with $|t| \leq 1$.

2. Weyl modules and sesquilinear forms

2.1. Weyl modules. Let V_λ be an irreducible highest weight module of $U_x\mathfrak{g}$. It follows from [Ka, Theorem 7] or [Lu] that there exists a basis of V_λ of elements of the form $u\eta_\lambda$, with η_λ the highest weight vector of V_λ and $u \in U_x^-\mathfrak{g}$, with respect to which the F_i's act via matrices with coefficients in $A = \mathbf{R}[x, x^{-1}]$. These basis vectors are weight vectors, so also the k_i's act via matrices with coefficients in A. More importantly, also the action of the $E_i^{(p)}$'s and $F_i^{(p)}$'s is given by matrices defined over A with respect to this basis (see [Lu]). So, in particular, we obtain a well-defined action of $U_q\mathfrak{g}$ on V_λ for any number $q \in \mathbf{C}\backslash\{0\}$. We call V_λ supplied with this canonical basis a Weyl module. If we want to distinguish the Weyl module over A from the one over $\mathbf{C}$ for $x = q$, we denote the latter by $V_\lambda(q)$. Similarly, if $v \in V_\lambda$, it can be written as a linear combination of the canonical basis with coefficients in $\mathbf{R}(x)$, and we denote by $v(q)$ the vector in $V_\lambda(q)$ obtained by evaluating the coefficients of that linear combination at $x = q$. We call a vector $v \in V_\lambda$ *regular* at q if the vector $v(q)$ is well-defined and nonzero. Similar conventions apply for tensor products of Weyl modules, where we use the obvious basis (tensor products of basis vectors of the factors) as the canonical basis (unlike the canonical bases defined for tensor products in [Lu]).

2.2. Lemma. *Let V_λ be a Weyl module with highest weight vector η_λ. Then there exists a sesquilinear form $(\ ,\ )$ on V_λ uniquely determined by $(\eta_\lambda, \eta_\lambda) = 1$ and by*

$$(a\eta_\lambda, b\eta_\lambda) = (b^* a\eta_\lambda, \eta_\lambda) = (\eta_\lambda, a^* b\eta_\lambda). \tag{2.2.1}$$

Proof. This is a variation of the inner product defined by Kashiwara; this variation appeared in [Ki] first, with slightly different notations. We give some details for the reader's convenience. The uniqueness is clear. To prove the existence, recall that V_λ^* is a $U_x\mathfrak{g}$ module, with the action defined by

$$\langle \xi, a\phi \rangle = \langle S^{-1}(a)\xi, \phi \rangle \quad \text{for} \quad \phi \in V_\lambda^*,\ \xi \in V_\lambda.$$

V_λ^* has highest weight $-w_0(\lambda)$, where w_0 is the longest element in the Weyl group. Define $\phi_\lambda \in V_\lambda^*$ by $\phi_\lambda(\eta_\lambda) = 1$ and $\phi_\lambda(\xi) = 0$ for any weight vector not belonging to the weight λ. It is easy to see that ϕ_λ is a lowest weight vector in V_λ^*.

Let $\omega_c : U_x\mathfrak{g} \to U_x\mathfrak{g},\ \omega_c(a) = S^{-1}(a^*)$. Then ω_c is an antilinear algebra homomorphism. Let $\bar{V}_\lambda$ be the vector space V_λ with antilinear scalar multiplication. Then $\bar{V}_\lambda$ becomes a $U_x\mathfrak{g}$ module with the action defined by $a.v = \omega_c(a)v$ which has highest weight $-w_0(\lambda)$, and lowest weight vector η_λ. Hence there exists an antilinear vector space isomorphism $\Phi : \bar{V}_\lambda \to V_\lambda^*$ such that $\Phi(\eta_\lambda) = \phi_\lambda$ and such that $\Phi(a\xi) = \omega_c(a)\Phi(\xi)$ for all $\xi \in \bar{V}_\lambda$. It is now easy to check that

$$(\eta, \xi) = \langle \eta, \Phi(\xi)\rangle, \quad \xi, \eta \in V_\lambda,$$

is well-defined. The last claim follows from

$$(a\eta, \xi) = \langle a\eta, \Phi(\xi)\rangle = \langle \eta, S^{-1}(a)\Phi(\xi)\rangle = \langle \eta, \Phi(a^*\xi)\rangle = (\eta, a^*\xi).$$

□

2.3. Proposition. *The sesquilinear form (,) defined in 2.2 for an irreducible highest weight module V_λ is Hermitian over A, i.e.* $(\xi, \eta) = \overline{(\eta, \xi)}$.

Proof. Let a_1, a_2 both be products of F_i's. We claim that $(a_1\eta_\lambda, a_2\eta_\lambda) \in A_0$. This is proved by induction on the number of factors. Assume $a_1 = F_r a_1'$, with a_1' being a subproduct of a_1. Then $(a_1\eta_\lambda, a_2\eta_\lambda) = (a_1'\eta_\lambda, E_r a_2\eta_\lambda)$. It suffices to show that $E_r a_2 \eta_\lambda = \sum_j f_j b_j \eta_\lambda$, with each b_j a product of F_i's with less factors than a_2, and $f_j \in A_0$. This is done by induction on the number of factors F_r in a_2. If a_2 does not contain a factor F_r, $E_r a_2\eta_\lambda = a_2 E_r \eta_\lambda = 0$. Otherwise, we can write $a_2 = a_2' F_r a_2''$, where a_2' is a product of elements of $\{F_i,\ i \neq r\}$. Then

$$E_r a_2 \eta_r = a_2'([\langle\alpha_r, \mu\rangle]_x + F_r E_r) a_2'' \eta_\lambda,$$

where μ is the weight of the weight vector $a_2''\eta_\lambda$. The element $[\langle\alpha_r, \mu\rangle]_x$ is in A_0, and $E_r a_2''$ can be written as a linear combination as stated, by the induction assumption. This finishes the proof of the claim

The claim of this proposition follows from the fact that the module V_λ is spanned by elements of the form $a_1\eta_\lambda$, $a_2\eta_\lambda$, as in Lemma 1.6, and the sesquilinearity of the form (,) (i.e. $f(v, w) = (fv, w) = (v, \bar{f}w)$ for all $f \in A$, $v, w \in V_\lambda$). □

2.4. Linkage principle. Assume that $q \in \mathbf{C}$ such that q^2 is a primitive dl-th root of unity, where d is the ratio of the square lengths of a long and a short root (see Section 1.1). So $d = 1$ in cases ADE, $d = 2$ in cases BC and F_4, and $d = 3$ for G_2. We include the case of q not being a root of unity by saying that it is a root of unity of order ∞. If we replace x by a number q, the Weyl modules $V_\lambda(q)$ are no longer simple in general. This can be made more precise as follows:

Let θ be the highest root of Φ (see e.g. the tables in [Kc, §6.7], [OV] or [H]), and let θ_s be the highest short root. In the notation of [Kc, §6.7], θ_s is equal to v_1 in type B, $(1/\sqrt{2})(v_1 + v_2)$ in type C, $(1/\sqrt{3})(v_1 - v_3)$ in type G_2 and v_1 in type F_4. Observe that with these identifications, the bilinear form $\cdot$ is d times the usual scalar product on $\mathbf{R}^n$. Using the already mentioned tables, it is easy to check the following statements for any dominant weight λ:

$$d|(\lambda \cdot \theta) \quad \text{and} \quad (\lambda \cdot \theta) \leq d(\lambda \cdot \theta_s). \tag{2.4.1}$$

Let E be the $\mathbf{R}$ vector space spanned by the roots. If q^2 is a primitive dl-th root of unity, we define the reflection group $\mathcal{W}$ to be the group generated by the Weyl group of $\mathfrak{g}$ and by the affine reflection in the hyperplane in $H_{\theta, dl}$ if d divides l; if q^2 is a primitive l-th root of unity such that d does *not* divide l, $\mathcal{W}$ is defined by adjoining to the Weyl group the affine reflection in the hyperplane $H_{\theta_s, l}$. Here $H_{\alpha,m}$ is defined for any root α and any integer m by

$$H_{\alpha,m} = \{x \in E, x \cdot \alpha = m\}.$$

The *linkage principle* states that there exists a nonzero homomorphism between a submodule of the Weyl module $V_\lambda(q)$ and a quotient module of the Weyl module $V_\mu(q)$ only if $w(\lambda+\rho) = \mu+\rho$ for some $w \in \mathcal{W}$; here ρ is half the sum of all positive roots (see [AP], [A]). We denote by $A_+ = A_+(q)$ the fundamental alcove, i.e. the open region which is bounded by the hyperplanes of simple reflections in the Weyl group and by $H_{\theta,ld}$ (resp. $H_{\theta_s,l}$) depending on whether the degree of the root of unity q^2 is divisible by d or not (i.e. l is not divisible by d in the second case); the degree may be ∞, in which case $A_+(q)$ is the whole dominant Weyl chamber. The closure of A_+ is denoted by $\bar{A}_+$. We also denote by $\Lambda = \Lambda(q)$ the set of weights λ such that $\lambda + \rho \in A_+$; similarly, $\bar{\Lambda}$ is the set of all dominant weights λ for which $\lambda + \rho \in \bar{A}_+$. Then we have the following

Proposition. *Let I denote the set of numbers $q = e^{\pi i t}$, with $|t| < 1/(m-d)$, where $m = (\lambda + \rho) \cdot \theta$. Then the Weyl module $V_\lambda(q)$ is simple, and the Hermitian form $(\ ,\ )$ is positive definite for all $q \in I$.*

Proof. Let $q \in I$. By our assumptions, q^2 is a root of unity of degree $s \geq m-d+1$. It is easy to check that in this case $\lambda+\rho$ is in $\bar{A}_+(q)$, using (2.4.1) and the fact that m is divisible by d. If $V_\lambda(q)$ had a submodule with highest weight μ, $\mu+\rho$ would also have to be in A_+, contradicting the linkage principle. Hence $V_\lambda(q)$ does not have a proper nontrivial submodule.

By Proposition 2.3 the form $(\ ,\ )$ becomes a Hermitian form in the classical sense whenever $x = q$ for any complex number q such that $|q| = 1$. It has to be nondegenerate for $q \in I$, as otherwise its nilspace would be a nontrivial submodule, by invariance (2.2.1). Let $(v_i)_i$ be the canonical basis of V, and let $C = (\ (v_i, v_j)\ )_{ij}$. Obviously the coefficients, and hence also the eigenvalues of C depend continuously on q. They are all real, positive for $q = 1$ and nonzero for any $q \in I$. Hence they have to be positive for all $q \in I$, by continuity, which shows the second claim. $\square$

2.5. Tensor products. Now let V_λ, V_μ be irreducible highest weight modules for $U_x\mathfrak{g}$, with Hermitian forms $(\ ,\ )_\lambda$ and $(\ ,\ )_\mu$. We define $(\ ,\ )_p$ on $V_\lambda \otimes V_\mu$ by $(v_1 \otimes v_1', v_2 \otimes v_2')_p = (v_1, v_2)_\lambda (v_1', v_2')_\mu$. This form does not satisfy (2.2.1) anymore, due to Lemma 1.4.1(b). We define $(\ ,\ )'$ on $V_\lambda \otimes V_\mu$ by

$$(v, v')' = (v, Rv')_p \quad \text{for } v, v' \in V_\lambda \otimes V_\mu.$$

As $\Delta'(a) = R\Delta(a)R^{-1}$, we obtain that

$$(av, v')' = (v, a^*v')' \quad \text{for } v, v' \in V_\lambda \otimes V_\mu. \tag{2.5.1}$$

$(\ ,\ )'$ is a sesquilinear form on $V_\lambda \otimes V_\mu$ which, however, is not Hermitian in general (as R is not given by a Hermitian matrix with respect to an orthonormal basis of $V_\lambda \otimes V_\mu$ (with respect to $(\ ,\ )_p$)). However, the form is nondegenerate, as R is invertible, and its values are in A for the standard basis, as the coefficients of R are in A (see [Lu]). Also observe that $(\ ,\ )'$ is well-defined for $x = q \in \mathbf{C}$, for any value of $q \neq 0$, and that it is nondegenerate (as the R-matrix is invertible also for $x = q$).

Finally, let V_{λ_i} be a Weyl module with highest weight λ_i, for $i = 1, 2, \ldots, s$. We define the operator $\Xi = \Xi^{(s)}$ on $\bigotimes_{i=1}^s V_{\lambda_i}$ by

$$\Xi^{(s)} v = x^{(-G(\nu) + \sum G(\lambda_i))/2} v$$

whenever v is in a highest weight module $V_\nu \subset \bigotimes_{i=1}^s V_{\lambda_i}$ with highest weight ν; $G(\nu)$, $G(\lambda_i)$ are defined as in Section 1.4.2. As $U_x\mathfrak{g}$ is semisimple, $\Xi^{(s)}$ is well-defined. We shall see that $\Xi^{(s)}$ is still well-defined for $x = q$ if q is sufficiently close to 1.

2.6. Theorem. *Let V_λ, V_μ be highest weight modules and let $v, v' \in V_\lambda \otimes V_\mu$. Then the form $(\ , \ )$, defined by*

$$(v, v') = (v, R\Xi v')_p = (v, \Xi v')',$$

is Hermitian, nondegenerate and invariant, the last condition meaning that it satisfies $(uv, v') = (v, u^* v')$ for all $u \in U_x\mathfrak{g}$.

Proof. Using Lemma 1.4.1(c), we see that (2.5.1) also holds if we replace R by R^* in the definition of $(\ , \ )'$. Hence the form $(\ , \ )''$ defined by $(v, v')'' = (v, (R + R^*)v')_p$ also satisfies $(uv, v')'' = (v, u^* v')''$, for all $u \in U_x\mathfrak{g}$. As $R + R^*$ is a Hermitian operator with respect to the Hermitian form $(\ , \ )_p$, the form $(\ , \ )''$ is also Hermitian.

By definition, the operator Ξ commutes with the action of $U_x\mathfrak{g}$. The invariance condition for $(\ , \ )$ follows from this and (2.5.1). Recall that $R^{-1}R^* = (R_{21}R)^{-1}$, by Lemma 1.4.1(e). Moreover, $R_{21}R$ commutes with the action of $U_x\mathfrak{g}$ and acts on any highest weight module $V_\nu \subset V_\lambda \otimes V_\mu$ with highest weight ν via the scalar $\alpha_\nu^{(\lambda,\mu)}$ (see 1.4.2). Hence its action on $V_\lambda \otimes V_\mu$ coincides with the one of $(\Xi^{(2)})^{-2}$. So if $v \in V_\nu$, then $(R + R^*)v = R(1 + R^{-1}R^*)v = (1 + (\alpha_\nu^{(\lambda,\mu)})^{-1})Rv$. Therefore we get

$$(v, v) = (v, R\Xi v)_p = (\alpha_\nu^{(\lambda,\mu)})^{1/2}(v, Rv)_p = \frac{(\alpha_\nu^{(\lambda,\mu)})^{1/2}}{1 + (\alpha_\nu^{(\lambda,\mu)})} \ (v, v)'' \ \in A_0.$$

Hence $(\ , \ )$ is Hermitian. □

We extend the definition of $(\ , \ )'$ to tensor products $\bigotimes_{i=1}^s V_i$ of Weyl modules inductively by decomposing the tensor product of the first $(s-1)$ factors into a direct sum $\bigoplus_j M_j$ of Weyl modules. We then define $(\ , \ )_p$ on $M_j \otimes V_s$ as before, using as sesquilinear form on M_j the restriction of $(\ , \ )'$ defined on $\bigotimes_{i=1}^{s-1} V_i$ (which usually is not Hermitian). In order to distinguish the various sesquilinear forms, we may also write $(\ , \ )^{(s)}$ for the form $(\ , \ )'$ defined on a tensor product of s Weyl modules.

A perhaps more conceptual, alternative way of defining $(\ , \ )'$ would be to perturb the product sesquilinear form by the canonical element $R^{(s)}$ which intertwines the s-th coproduct with the s-th opposite coproduct (see e.g. [TW1, Section 1.1]); for $s = 2$ we get the usual R-matrix. The equivalence of these two definitions should become clear from the proof of the following corollary.

Corollary. *The definition of $(\ , \ )'$ does not depend on the order in which $W = \bigotimes_{i=1}^s V_i$ is decomposed into a direct sum of simple modules. Moreover, we obtain an invariant Hermitian form $(\ , \ )$ on W by $(w, w') = (w, \Xi^{(s)} w')'$.*

Proof. Let us show the claim for $s = 3$. There are two ways to define $(\ , \ )'$ on $\bigotimes_{i=1}^3 V_i$, by first decomposing $V_1 \otimes V_2$ into irreducibles, or by first decomposing $V_2 \otimes V_3$ into irreducibles. Recall that as $U_x\mathfrak{g}$ is a quasitriangular Hopf algebra, we have

$$(\Delta \otimes id)(R) = R_{13}R_{23} \quad \text{and} \quad (id \otimes \Delta)(R) = R_{13}R_{12};$$

see [D1, Section 10] or [Ks]. By definition of $(\ ,\ )'$, we have to perturb $(\ ,\ )_p$ by the operator $R_{12}(\Delta\otimes id)(R) = R_{12}R_{13}R_{23}$ in the first case, and by $R_{23}(id\otimes\Delta)(R) = R_{23}R_{13}R_{12}$ in the second case. Equality of the two possible definitions for $(\ ,\ )'$ now follows from the fact that the R matrix satisfies the Yang-Baxter equation. The case $s>3$ follows from this by induction.

To show that multiplication by $\Xi^{(s)}$ turns $(\ ,\ )'$ into a Hermitian form, consider a highest weight vector v in $M_j\otimes V_s$, with weight ν. Let μ be the highest weight of M_j and let λ_i be the highest weight of V_i. By the induction assumption, we can find a $U_x\mathfrak{g}$-isomorphism $\psi: M_j\to V_\mu$, with V_μ a Weyl module with highest weight μ and canonical Hermitian form $(\ ,\ )$ such that

$$x^{(-G(\mu)+\sum_{i=1}^{s-1}G(\lambda_i))/2}(m,m')^{(s-1)} = (\psi(m),\psi(m'))\quad \text{for all } m,m'\in M_j.$$

Let $\Psi=\psi\otimes 1: M_j\otimes V_s\to V_\mu\otimes V_s$. Then, by Theorem 2.6, we have

$$\begin{aligned}(v,\Xi^{(s)}v)^{(s)} &= x^{(-G(\mu)+\sum_{i=1}^{s-1}G(\lambda_i))/2}(v,(\alpha_\nu^{(\lambda_s,\mu)})^{-1/2}v)^{(s)}\\ &= (\Psi(v),\Xi^{(2)}\Psi(v))^{(2)}\in A_0.\end{aligned}$$

The general claim follows from Proposition 2.3 and the semisimplicity of $U_x\mathfrak{g}$. $\square$

2.7. Proposition (Positivity)**.** *Assume that $p\in End_U(V\otimes W)$ is an idempotent such that*

- $p(q)$ is well-defined for all $q=e^{\pi it}$, with t in some interval (a,b);

- $(\ ,\ )_{p(q)V(q)\otimes W(q)}$ is well-defined and nondegenerate whenever $t\in(a,b)$, and positive definite for some $c\in(a,b)$.

Then $(\ ,\ )_{p(q)V(q)\otimes W(q)}$ is positive definite for all $t\in(a,b)$.

Proof. As in Proposition 2.4. $\square$

2.8. $*$-structures. We consider the collection $\mathcal{C}$ of tensor products of Weyl modules. We fix an invariant Hermitian form for each of these Weyl modules, as in 2.2. By Theorem 2.6 and its corollary, this induces a unique nondegenerate sesquilinear form $(\ ,\ )'$ and a unique Hermitian form $(\ ,\ )$ for each object M in $\mathcal{C}$. Let M_1, M_2 be objects in $\mathcal{C}$. We define a $*$-operation for any $f\in Hom(M_1,M_2)$ by

$$(f^*v,w)' = (v,fw)'\quad \text{for all } v\in M_1,\ w\in M_2.$$

We denote the subset of $U_x\mathfrak{g}$-intertwiners between M_1 and M_2 by $Hom_U(M_1,M_2)$. Then we have

Proposition. (a) *If $f\in Hom_U(M_1,M_2)$, then $f^*\in Hom_U(M_2,M_1)$.*

(b) *If $f\in Hom_U(M_1,M_2)$, then $(f^*v,w)=(v,fw)$ for all $v\in M_1,\ w\in M_2$.*

(c) *If $f_1\in Hom_U(V_1,V_2)$ and $f_2\in Hom_U(W_1,W_2)$, then $(f_1\otimes f_2)^* = f_1^*\otimes f_2^*$.*

Proof. Statement (a) follows from the invariance of $(\ ,\ )'$, property (2.2.1). For (b) one observes that f also commutes with Ξ. Statement (c) follows from the fact that $f_1\otimes f_2$ commutes with Θ and Π, hence also with R (see 1.4). $\square$

3. Tilting modules

3.1. Definitions and properties. In the following we consider modules over $U_q\mathfrak{g}$, with q^2 a primitive dl-th root of unity. By the linkage principle (see Section 2.4) the Weyl module $V_\lambda(q)$ is simple if $(\lambda+\rho)\in A_+$. We shall denote the set of dominant weights λ with $\lambda+\rho\in A_+$ by Λ. In the following we always assume l to be sufficiently large such that at least $\kappa+\rho\in A_+$, where κ is the highest weight of the fundamental module(s) V of $\mathfrak{g}$; for types A and C these are the vector representations, while for types B and D one takes the spin representations as fundamental modules. More information about fundamental modules, also for the exceptional cases, can be found in Section 3.5.

A $U_q\mathfrak{g}$-module T is called a tilting module if it is a direct summand of some tensor power of the fundamental module V, or if it is a direct sum of such modules. The fact that this definition is equivalent to Andersen's can be easily deduced from [A, Cor. 2.6] and properties of V (see 3.5). Tilting modules satisfy the following properties (see [A], [AP]):

(a) Any tensor product of tilting modules is again a tilting module.

(b) Any tilting module T admits a Weyl series, i.e. a sequence of modules $0 \subset V_1 \subset V_2 \subset \cdots \subset V_r = T$ such that V_{i+1}/V_i is isomorphic to a Weyl module. If T is obtained from a $U_x\mathfrak{g}$ module T' with a suitable basis by setting $x = q$, the number of factors isomorphic to $V_\lambda(q)$ in a Weyl series of T is equal to the multiplicity of V_λ in T' (this can, e.g., be seen by looking at the character of T').

(c) For each dominant weight μ there exists a unique up to isomorphism indecomposable tilting module T_μ with highest weight μ.

(d) Any tilting module is isomorphic to a direct sum of indecomposable tilting modules T_μ.

(e) $T_\lambda = V_\lambda$ whenever $\lambda \in \bar{\Lambda}$, i.e. if $\lambda + \rho \in \bar{A}_+$.

(f) A Weyl module V_λ appears as a factor in a Weyl series of T_μ only if λ and μ are linked via the linkage principle (see 2.4).

3.2. Categorical trace. Let W be a $U_q\mathfrak{g}$ module. We define the q-trace Tr_q on $End(W)$ by

$$Tr_q(f) = Tr(f\tilde{K}_{-2\rho}),$$

where 2ρ is the sum of positive roots of $\mathfrak{g}$, $\tilde{K}_{-2\rho}$ is as in [Lu] (see Section 1.1 for notation), and Tr is the usual trace on $End(W)$. Tr_q does define a trace on $End_U(W)$, as $\tilde{K}_{2\rho}$ is in $U_q\mathfrak{g}$. The q-dimension of W is defined to be $Tr_q(1_W)$, where 1_W is the identity on W. For a Weyl module V_λ we get the formula

$$dim_q V_\lambda = \prod_{\alpha\in\Phi_+} \frac{[(\lambda+\rho)\cdot\alpha]}{[\rho\cdot\alpha]} = \prod_{\alpha\in\Phi_+} \frac{q^{(\lambda+\rho)\cdot\alpha} - q^{-(\lambda+\rho)\cdot\alpha}}{q^{\rho\cdot\alpha} - q^{-\rho\cdot\alpha}}, \tag{3.2.1}$$

where Φ_+ is the set of positive roots. This formula appeared first in Weyl's derivation of the dimension formula. For explicit values in classical cases see e.g. [W1], [W2]. The q-trace can also be defined in the following categorical way (see e.g. [JS], and in a C^* setting [LR]): Let W^* be the dual of W. It can be made into a $U_q\mathfrak{g}$ module by defining the action via the antipode S by

$$\langle a\phi, w\rangle = \langle \phi, S(a)w\rangle, \quad a \in U_q\mathfrak{g},\ w \in W,\ \phi \in W_l^*. \tag{3.2.2}$$

Let (w_i) and (ϕ_i) be dual bases. Then we define $U_q\mathfrak{g}$ morphisms between the trivial module $\mathbf{1}$ and $W^* \otimes W$ by

$$b_W : 1 \in \mathbf{1} \mapsto \sum_i \phi_i \otimes \tilde{K}_{-2\rho} v_i \in W^* \otimes W \quad \text{and} \quad d_W : \phi \otimes w \mapsto \langle \phi, w\rangle.$$

Then we also have

$$Tr_q(f) = d_W \circ (1 \otimes f) \circ b_W, \quad f \in End_U(W).$$

The map $E_{W_2} : End_U(W_1 \otimes W_2) \to End_U(W_2)$ is defined by

$$E_{W_2}(f) = (d_{W_1} \otimes 1) \circ (1 \otimes f) \circ (b_{W_1} \otimes 1). \tag{3.2.3}$$

E_{W_2} is usually called a contraction. It obviously satisfies the bimodule property

$$E_{W_2}((1 \otimes f_1)f(1 \otimes f_2)) = f_1 E_{W_2}(f) f_2,$$

for $f_i \in End_U(W_2)$, $i = 1, 2$, and $f \in End_U(W_1 \otimes W_2)$, and the trace preserving property

$$Tr_q(E_{W_2}(f)) = Tr_q(f), \quad f \in End_U(W_1 \otimes W_2).$$

The maps $\tilde{b}_W : \mathbf{1} \to W \otimes W^*$ and $\tilde{d}_W : W \otimes W^* \to \mathbf{1}$ defined by

$$\tilde{b}_W : 1 \in \mathbf{1} \mapsto \sum_i v_i \otimes \phi_i \quad \text{and} \quad \tilde{d}_W : w \otimes \phi \mapsto \langle \phi, \tilde{K}_{-2\rho} w \rangle$$

are again $U_q\mathfrak{g}$-module maps. We can now define

$$\tilde{E}_{W_1} : End_U(W_1 \otimes W_2) \to End_U(W_1)$$

in analogy to (3.2.3), by embedding $End_U(W_1 \otimes W_2)$ into $End_U(W_1 \otimes W_2 \otimes W_2^*)$ and using the operators $\tilde{b}_{W_2}$ and $\tilde{d}_{W_2}$. With these definitions, also $\tilde{E}_{W_1}$ satisfies the bimodule property, and is trace preserving.

A morphism $f : V \to W$ is called *negligible* if $Tr_q(gf) = 0$ for all morphisms $g : W \to V$. The vector space of all negligible morphisms from V to W is denoted by $Neg(V, W)$.

3.3. Fusion categories. We describe here what we mean by fusion categories, by which we will just understand a certain quotient in the category of tilting modules. For more on categorical definitions see e.g. [JS], [T], [KL, Part IV]. In the following we assume q^2 to be a primitive dl-th root of unity. The properties listed in 3.1(a) and (d) imply that any tilting module can be written as a direct sum of indecomposable tilting modules. Moreover, we can write any tilting module T as a direct sum $T = C_T \oplus C'_T$, where C_T is a direct sum of indecomposable tilting modules T_μ with $\mu \in \Lambda$, and C'_T is a direct sum of indecomposable tilting modules T_μ such that $\mu \notin \Lambda$. Then we have the following results (see [A], [AP]):

(a) The module T_μ has nonzero q-dimension if and only if $\mu \in \Lambda$; in this case it is isomorphic to V_μ, and its q-dimension is positive for $q = e^{\pi i/dl}$.

(b) The modules T_μ with $\mu \notin \Lambda$ generate a tensor ideal $\mathcal{I}$ in which each object has q-dimension 0.

(c) The quotient map with respect to the tensor ideal $\mathcal{I}$ from $\mathcal{T}$ to $\mathcal{F} = \mathcal{T}/\mathcal{I}$ is described as follows: it assigns to the tilting module T the tilting module C_T, and to the morphism $f : T_1 \to T_2$ the morphism $p_2 f p_1$, where p_i is the projection from T_i onto C_{T_i} with kernel C'_{T_i}, $i = 1, 2$.

(d) The quotient construction can be described on the level of morphisms as follows: For any tilting modules T_1, T_2 there exists an isomorphism

$$Hom_{\mathcal{F}}(C_{T_1}, C_{T_2}) \cong Hom_{\mathcal{T}}(T_1, T_2)/Neg(T_1, T_2)$$

which is compatible with tensor products and compositions of morphisms; here $Neg(T_1, T_2)$ denotes the vector space of negligible morphisms. In particular, for any morphisms $f : T_1 \to T_2$ and $g : T_2 \to T_1$ we have $Tr_q(gf) = Tr_q(p_1 g p_2 f p_1)$, with p_1, p_2 as in (c).

(e) The category $\mathcal{F}$ is a semisimple, rigid, braided category. Its simple objects are labelled by the dominant weights in Λ, i.e. by the set of dominant weights λ which satisfy $(\lambda+\rho)\cdot\theta < ld$ (see 2.4).

3.4. Lemma. *Let W be a $U_q\mathfrak{g}$ module which is isomorphic to a direct sum of simple Weyl modules V_λ with $\lambda \in \Lambda$. Then both $e_W = (1/dim_q W)(b_W \circ d_W)$ and $\tilde{e}_W = (1/dim_q W)(\tilde{b}_W \circ \tilde{d}_W)$ are selfadjoint projections with, respect to the Hermitian form defined on $W^* \otimes W$, resp. $W \otimes W^*$.*

Proof. Let us assume first that W itself is simple. Then e_W is the orthogonal projection onto $\mathbf{1} \subset W^* \otimes W$, which appears with multiplicity 1. Hence e_W is a central idempotent in $End_U(W \otimes W^*)$. As $*$ is an algebra antimorphism, e_W^* is a central idempotent, which coincides with e_W for $q = 1$. Hence it has to coincide with e_W in general, by continuity.

For the general case, we define a basis (v_{i_j}) for each V_i. This gives us a dual basis (ϕ_{i_j}) which spans a module $V_i^* \subset W^*$ for each i. As R leaves each $V_r^* \otimes V_s$ invariant, it is easy to check that the restriction of e_W to $V_r^* \otimes V_s$ coincides with $\delta_{r,s} e_{V_r}$. Moreover, for $q = e^{\pi i/l}$, $dim_q V_\lambda > 0$ for all $\lambda \in \Lambda$, and hence also $dim_q W$. One deduces from this that $dim_q W \neq 0$ for arbitrary roots of unity by a simple Galois argument. The claim now follows from the special case treated in the previous paragraph. The proof for $\tilde{e}_W$ goes similarly. $\square$

3.5. Fundamental modules. In the following we define certain *fundamental modules* for each Lie type. We have added their highest weights, denoted by κ in the exceptional cases, using notation of the tables in [Kc, 6.7] for the reader's convenience (see also the tables in [OV]). If one represents roots α, β as vectors in $\mathbf{R}^n$ with respect to these tables, one has $\alpha \cdot \beta = d\langle \alpha, \beta\rangle$, where $\langle\ ,\ \rangle$ is the usual scalar product in $\mathbf{R}^n$.

- the vector representations for Lie types A and C,
- the spinor representations for Lie types B and D (there are two for type D),
- the 7-dimensional module for Lie type G_2, with highest weight $\kappa = \frac{1}{\sqrt{3}}(1,0,-1)$,
- the 26-dimensional module for Lie type F_4, with highest weight $\kappa = (1,0,0,0)$,
- the representations belonging to the dual weights corresponding to the vertices farthest from the triple point of the Dynkin diagrams E_6 (corresponding to the highest weights $\kappa = \frac{1}{6}(5,-1,-1,-1,-1,-1,3\sqrt{2})$ and $\kappa = \frac{1}{6}(1,1,1,1,1,-5,3\sqrt{2})$), E_7 (with $\kappa = \frac{1}{4}(-3,1,1,1,1,1,1,-3)$) and E_8, where $\kappa = \theta = (1,1,0,0,0,0,0,0)$.

Theorem. *We consider the generic case of $U_x\mathfrak{g}$.*

(a) *Any irreducible $U_x\mathfrak{g}$-representation appears in some tensor power of fundamental representations.*

(b) *For any Weyl module V_λ we have $V_\lambda \otimes V \cong \bigoplus V_\mu \otimes \mathbf{R}(x)^{m_\mu}$, with the multiplicity m_μ being equal to* 0 *or* 1 *except possibly in type F_4 or E_8. In the latter case $m_\mu > 1$ only if $\mu = \lambda$.*

(c) *Assume $(\lambda+\rho)\cdot\theta < dl$. Then $(\mu+\rho)\cdot\theta \leq ld$ for all μ as in* (b) *with $m_\mu \geq 1$ except possibly $\mu = \lambda + \kappa$ in Lie type E_8, where κ is the highest weight of V.*

Proof. Statement (a) is well-known and can be found in standard works on Lie algebras and Lie groups. Statement (b) can be shown as follows: All weights of V have multiplicity 1, except for F_4 and E_8; in the latter case only the weight 0 has multiplicity > 1. From this one derives that V_μ is a direct summand in $V_\lambda \otimes V$

only if $\mu = \lambda + \gamma$, with γ a weight of V, and that its multiplicity is at most the multiplicity of γ in V (see e.g. [Ka], [Li]).

To prove statement (c), one only has to observe that d divides $\theta \cdot \gamma$ for any weight γ and that $\theta \cdot \gamma \le d$ for all weights γ of V, except for $\gamma = \kappa$ in type E_8. This is easy to check using the tables in [OV]. $\square$

3.6. Positivity. The sesquilinear forms defined in Section 2 naturally define sesquilinear forms (over $\mathbf{C}$) for tensor products of the form $V_\lambda(q) \otimes V_\mu(q)$, except possibly for the form $(\ ,\ )$, where we needed the fact that we can decompose the tensor product into a direct sum of simple modules.

3.6.1. Lemma. *Let $q = e^{\pi i t}$, with $|t| \le 1/dl$. Let V_λ, V_μ be Weyl modules, and let v be a regular (see 2.1) highest weight vector in $V_\lambda \otimes V_\mu$ with weight γ such that $(\gamma + \rho) \cdot \theta \le dl$.*

(a) *The form $(\ ,\ )'$ is well-defined and nondegenerate on tensor products $\bigotimes_{i=1}^{s} V_{\lambda_i}(q)$ (as in Section 3.6) if $q = e^{\pi i t}$, with $|t| \le 1/ld$ and $(\lambda_i + \rho) \cdot \theta < ld$.*

(b) *If $(v(q), v(q))' \ne 0$, then the orthogonal projection p onto $U_x\mathfrak{g}v$ (with respect to $(\ ,\ )$) is also well-defined for $x = q$.*

Proof. Statement (a) was shown for $s = 1$ in Proposition 2.4. For $s = 2$, it follows from that proposition that $(\ ,\ )_p$ is nondegenerate. As R is well-defined and invertible for all $q \ne 0$, $(\ ,\ )'$ is nondegenerate. The claim for $s > 2$ can be shown similarly, using the element $R^{(s)}$ (see alternate definition of $(\ ,\ )$ before Corollary 2.6).

For proving (b), let η_γ be the highest weight vector of the Weyl module V_γ, with $(\eta_\gamma, \eta_\gamma) = 1$. Then we have an isomorphism of $U_x\mathfrak{g}$-modules given by $a\eta_\gamma \in V_\gamma \mapsto av \in U_x\mathfrak{g}v$, for all $a \in U_x\mathfrak{g}$, such that $(a_1 v, a_2 v)' = c(a_1\eta_\gamma, a_2\eta_\gamma)$, where $c = (v, v)'$, and $a_1, a_2 \in U_x\mathfrak{g}$. As the form $(\ ,\ )$ is nondegenerate on $V_\gamma(q)$ (Proposition 2.4), the form $(\ ,\ )'$ is nondegenerate on $U_q\mathfrak{g}v(q)$, by our assumptions for $v(q)$, and (2.2.1). Hence we can find an orthonormal basis (e_i) of $U_x\mathfrak{g}v$ which is still well-defined for $x = q$. Hence so is $p(q)$, as $p(q)w(q) = \sum c(q)^{-1}(w(q), e_i(q))' e_i(q)$ for any $w(q) \in (V_\lambda \otimes V_\mu)(q)$. $\square$

3.6.2. Lemma. *Let $q_0 = e^{\pi i/ld}$ and let $\lambda \in \Lambda = \Lambda(q_0)$. Let $W = V_\lambda \otimes V$, with the obvious basis derived from the canonical bases of V_λ and V. Assume that $W = \bigoplus V_\mu \otimes \mathbf{R}(x)^{\otimes m_\mu}$, and let p_γ be the projection onto the direct summand $V_\gamma \otimes \mathbf{R}(x)^{m_\gamma}$, with kernel being the direct summands labelled by $\mu \ne \gamma$.*

(a) *If $q = e^{\pi i t}$ with $|t| < dl$, then $(\ ,\ )$ is well-defined in $W(q)$ and positive definite.*

(b) *If $\gamma \in \bar{\Lambda}$ is a dominant weight with $m_\gamma = 1$ and such that $w(\gamma + \rho) \ne \mu + \rho$ for any μ with $m_\mu \ne 0$, and for any $w \in \mathcal{W}$, then p_γ is well-defined also at q_0.*

Proof. As usual, if S is a subset of a vector space V with form $(\ ,\ )$, we denote by $S^\perp$ the subspace $\{v \in V,\ (s, v) = 0 \text{ for all } s \in S\}$. Let v_γ be a regular (see 2.1) highest weight vector in W with weight γ as in statement (b). Assume $(v_\gamma(q_0), v_\gamma(q_0))' = 0$. Then $(\ ,\ )'$ induces the 0-form on $W_\gamma = U_{q_0}\mathfrak{g}v_\gamma(q_0)$, which is isomorphic to the simple module $V_\gamma(q_0)$. Hence $W_\gamma \subset W_\gamma^\perp$, and both $W/W_\gamma^\perp$ and W_γ are isomorphic to $V_\gamma(q_0)$. As only one factor in a Weyl series of $W(q_0)$ can be isomorphic to $V_\gamma(q_0)$ (see 3.1(b)), another factor would have to contain a proper submodule isomorphic to $V_\gamma(q_0)$. But this would contradict the linkage principle (see Section 2.4 and 3.1(f)). Hence $(v_\gamma(q_0), v_\gamma(q_0))' \neq 0$. Statement (b) follows from this and Lemma 3.6.1(b).

For statement (a), observe that q^2 has to be a root of unity of degree $> ld$. Let γ be a dominant weight such that $m_\gamma \neq 0$. As $(\gamma + \rho) \cdot \theta \leq d(l+1)$ by Theorem 3.5(c), $V_\gamma(q)$ is a simple module (see Proposition 2.4). One shows as in the previous paragraph that the restriction of (,) to $V_\gamma(q)$ has to be nondegenerate if $m_\gamma = 1$. This implies $(v_\gamma(q), v_\gamma(q)) \neq 0$, and hence the well-definedness of $p_\gamma(q)$, by Lemma 3.6.1. As the only possibly remaining p_γ with $m_\gamma > 1$ is the complement of the sum of p_γ's with $m_\gamma = 1$, it is also well-defined at $x = q$. Hence (,) is well-defined for any q as in the statement, and depends continuously on q. It is therefore positive definite, by Proposition 2.7. □

Proposition. *Let q_0, λ and W be as in Lemma 3.6.2.*

(a) *Then there exists a selfadjoint idempotent $p \in End_{U_x\mathfrak{g}}(W)$ such that $p(q)$ is well-defined in a neighborhood N of q_0, and such that $p(q_0)$ has image $C_{W(q_0)}$ and kernel $C'_{W(q_0)}$ for a decomposition $W(q_0) = C_{W(q_0)} \oplus C'_{W(q_0)}$ as in Section 3.3.*

(b) *The form (,) is well-defined and positive definite on $C_{W(q_0)}$.*

Proof. We are going to construct a decomposition

$$(*) \qquad V_\lambda(q_0) \otimes V(q_0) \ = \ \bigoplus_\mu T_\mu(q_0) \otimes \mathbf{C}^{\tilde m_\mu}$$

of $V_\lambda(q_0) \otimes V(q_0)$ into a direct sum of indecomposable tilting modules with the following property: for each label μ occurring in this decomposition for which $\mu \in \Lambda$ there exists a selfadjoint idempotent p_μ in $End_{U_x\mathfrak{g}}(W)$, well-defined for $x = q$ with q in a neighborhood N of q_0 such that $p_\mu(q_0)$ is the well-defined projection onto the summand $T_\mu(q_0) \otimes \mathbf{C}^{\tilde m_\mu}$, with the kernel being the remaining direct summands.

Let us first show that this claim implies statement (a), where we can take for p the idempotent $p = \sum_{\mu \in \Lambda} p_\mu$. The operator Ξ is well-defined on $p(q)W(q)$ for all q's for which the $p_\mu(q)$'s are well-defined; indeed the restriction of Ξ to pW is equal to $\sum_\mu x^{-G(\mu)+G(\lambda)+G(\kappa)} p_\mu$. Hence we can define the form (,) on $p(q)W(q)$ also for $q \in N$. As $p(q)$ is selfadjoint, (,) is nondegenerate on $p(q)W(q)$ for all $q \in N$. As it is positive definite for $q = e^{\pi i t}$ with $|t| < 1/ld$, by Lemma 3.6.2, it is so also for all $q \in N$ (the argument goes as in the proof of Proposition 2.4). Hence it suffices to prove the claim in the previous paragraph.

Case 1: Lie type $\neq E_8$. Fix a dominant weight γ such that $\tilde m_\gamma \neq 0$ in $(*)$ and such that $\gamma \in \Lambda(q_0)$. By Lemma 3.6.2 and Theorem 3.5(b) and (c), p_γ is well-defined at q_0 for all γ, except possibly for $\gamma = \lambda$ or $\gamma = \lambda + \kappa$ in Lie types F_4 and E_8. For F_4, also $\lambda + \kappa \in \bar\Lambda$ and hence $p_\lambda = 1 - \sum_{\gamma \neq \lambda} p_\gamma$ is also well-defined at q_0. Hence all idempotents p_γ are also well-defined in a neighborhood of q_0 in these cases.

Case 2: Lie type E_8. In this case, it suffices to consider the module $M = (p_\lambda + p_{\lambda+\kappa})W$ (the sum $p_\lambda + p_{\lambda+\kappa}$ is well-defined at $x = q_0$, being the complement of the well-defined idempotent $\sum_{\gamma \neq \lambda, \lambda+\kappa} p_\gamma$). The tilting module $M(q_0)$ contains the highest weight vector $v_\lambda \otimes v_\kappa$, which generates a Weyl module isomorphic to, and denoted by, $V_{\lambda+\kappa}(q_0)$. If the restriction of (,)$'$ to $V_{\lambda+\kappa}(q_0)$ were nondegenerate, it would be a direct summand, and hence an indecomposable tilting module. But this would contradict the fact that the q-dimension of $T_{\lambda+\kappa}$ is equal to 0, as $dim_q V_{\lambda+\kappa} \neq 0$ for $(\lambda + \kappa + \rho) \cdot \theta = (l+1)d$.

So we can assume that $(\ ,\ )'$ is degenerate on $V_{\lambda+\kappa}(q_0)$ with nilmodule $N(q_0)$. As $(\ ,\ )'$ is nondegenerate on $M(q_0)$, $N(q_0) \cong M(q_0)/N(q_0)^\perp$, which is a quotient of the module $M(q_0)/V_{\lambda+\kappa}(q_0)$. But the latter quotient is isomorphic to a direct sum of Weyl modules $V_\lambda(q_0)$ (see 3.1(b)) and hence semisimple. So the quotient $M(q_0)/N(q_0)^\perp$ splits modulo $V_{\lambda+\kappa}(q_0)$, and we obtain a module $T(q_0)$ containing $V_{\lambda+\kappa}(q_0)$ such that $T(q_0)/V_{\lambda+\kappa}(q_0) \cong N(q_0)$ and such that $(\ ,\ )'$ is nondegenerate on $T(q_0)$.

We show that $T(q_0)$ is indecomposable. Let v be a highest weight vector in the quotient $T/V_{\lambda+\kappa}(q_0)$. By construction of $T(q_0)$, there exists a vector $v' \in N(q_0) \subset V_{\lambda+\kappa}(q_0)$ such that $(v, v') \neq 0$. By construction of the Weyl module there exists an element $u \in U_q\mathfrak{g}^-$, the subalgebra of $U_q\mathfrak{g}$ generated by the F_i's, such that $v' = uv_{\lambda+\kappa}$. But then

$$0 \neq (v, uv_{\lambda+\kappa}) = (u^*v, v_{\lambda+\kappa}),$$

which implies $u^*v \neq 0$. But as v is a highest weight vector in the quotient, and $u^* \in U_q\mathfrak{g}^+$, $u^*v \in V_{\lambda+\kappa}(q_0)$. Hence $T(q_0)$ is indecomposable and therefore isomorphic to $T_{\lambda+\kappa}$.

As $(\ ,\ )'$ is nondegenerate on $T(q_0)$, $M(q_0)$ is the direct sum of $T(q_0)$ and its orthogonal complement $M_\lambda(q_0)$. Again by 3.1(b), the module $M_\lambda(q_0)$ is isomorphic to a direct sum of Weyl modules each of which is isomorphic to $V_\lambda(q_0) \cong T_\lambda(q_0)$. The form $(\ ,\ )'$ is nondegenerate on $M_\lambda(q_0)$ (as it is so on $M(q_0)$ and $T(q_0)$), and we can obviously renormalize it to the Hermitian form $(\ ,\ )$ on this particular subspace. The desired projection p_λ can now be explicitly constructed as follows: By invariance, $(\ ,\ )$ also has to be nondegenerate on the weight space $M_\lambda[\lambda](q_0)$ belonging to the highest weight of $M_\lambda(q_0)$. Fix an orthogonal basis $(v_i(q_0))$ of that space such that $(v_i(q_0), v_i(q_0)) \neq 0$ for any index i. It is easy to find weight vectors $v_i \in W$ with respect to $U_x\mathfrak{g}$ whose values at $x = q$ coincide with $v_i(q)$ and which are orthogonal to each other over $\mathbf{R}(x)$ (the second condition can easily be achieved by applying Gram-Schmid to an arbitrary collection of vectors who have the desired values at $x = q_0$). Using these vectors, the projection p_λ can now be defined explicitly as a sum of projections as defined in Lemma 3.6.1. It is well-defined at $x = q_0$, and hence also in a neighborhood of q_0. This finishes the proof. □

Corollary 1. *Decompose $V(q_0)^{\otimes n}$ into a direct sum of indecomposable tilting modules, and let C_n be the submodules consisting of all such direct summands T_μ for which $\mu \in \Lambda$, with complement C_n'. Then $(\ ,\ )$ is well-defined and positive definite on C_n.*

Corollary 2. *Using notation of Corollary 1, let p_r be the orthogonal projection onto C_r, with kernel C_r', for any $r \in \mathbf{N}$. Let $f : V(q_0)^{\otimes n} \to V(q_0)^{\otimes m}$ be linear. Then $(p_m f p_n)^* p_m f p_n(q_0)$ has only nonnegative eigenvalues.*

Proof. One shows by induction on n that the restriction of the form $(\ ,\ )$ on $V(q_0)^{\otimes n}$ (as defined in 2.8) to C_n is well-defined and positive definite also for $x = q_0$. Corollary 2 is now obvious. □

3.7. Theorem. *Let $q = e^{\pi i/dl}$. The fusion category $\mathcal{F}$ has the structure of a C^* category. This means there exists an antilinear $*$-operation on the morphisms of $\mathcal{F}$ satisfying*

(a) *functoriality, i.e. $(f_1 \otimes f_2)^* = f_1^* \otimes f_2^*$ for all morphisms f_1, f_2 of $\mathcal{F}$, and*

(b) *positivity, i.e.* $Tr_q(f^*f) \geq 0$ *for all morphisms* f *of* $\mathcal{F}$*, with equality only if* f *is a zero morphism. Here* Tr_q *is the categorical trace for the source of* f*.*

Proof. The functoriality property (a) was shown in 2.8. To show positivity, we first show semipositivity for the category $\mathcal{T}$ of all tilting modules. By our definition of that category, it suffices to show this for any object in $\mathcal{C}$, as defined in 2.8. Any such object is a direct summand of tensor powers of the fundamental module V. It therefore suffices to prove $Tr_q(f^*f) \geq 0$ for any morphism between $V^{\otimes n_1}$ and $V^{\otimes n_2}$. We decompose $V^{\otimes n_i}$ as a direct sum $C_{n_i} \oplus C'_{n_i}$, for $i = 1, 2$, and define the projections p_{n_i} as in Corollary 1. By 3.3(d), we have $Tr_q(f^*f) = Tr_q(p_{n_1} f^* p_{n_2} f p_{n_1})$. Observe that for any idempotent p in $End_U(C_{n_1})$ the value of $Tr_q(p)$ is equal to the q-dimension of its image, which is positive by 3.3(a). Hence $Tr_q(f^*f) \geq 0$, by Corollary 2. As both C_{n_1} and C_{n_2} are Hilbert spaces, and as the q-dimension of any submodule of C_{n_1} is positive, $Tr_q(f^*f) = 0$ only if $p_2 f p_1 \neq 0$. □

3.8. Unitary braid representations. Let W be an object in $\mathcal{F}$. We define the operator $\check{R}^W \in \mathrm{End}(W^{\otimes 2})$ by $\check{R}^W(w_1 \otimes w_2) = P_W R(w_1 \otimes w_2)$, where P_W interchanges the factors in $W^{\otimes 2}$. It follows from the definition of R-matrix that $\check{R}^W$ is in $\mathrm{End}_{\mathcal{F}} V^{\otimes 2}$. Moreover, we define, for $n \in \mathbf{N}$, elements $\check{R}_i^W \in \mathrm{End}(W^{\otimes n})$ by

$$\check{R}_i^W = 1_{i-1} \otimes \check{R}^W \otimes 1_{n-1-i},$$

where 1_j is the identity on $W^{\otimes j}$. Let $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ be the generators of Artin's braid group B_n (see e.g. [Ks]). Then we have the following simple

Corollary. *There exists a unitary representation of* B_n *on the Hilbert space* $End_{\mathcal{F}}(W^{\otimes n})$*, defined by* $\sigma_i.f = \check{R}_i^W f$ *for all* $f \in End_{\mathcal{F}}(W^{\otimes n})$ *and* $i = 1, 2, \ldots, n-1$*.*

Proof. It essentially follows from the definition of quasitriangular Hopf algebra that the map $\sigma_i \mapsto \check{R}_i^W$ induces a representation of B_n (see e.g. [Ks]). To prove the unitarity statement, let us consider $n = 2$ first, with $v, v' \in W^{\otimes 2}$. Using the definition of $(\ , \ )$ (Theorem 2.6), Lemma 1.4.1(e) and the fact that $\check{R}^W \in \mathrm{End}_{\mathcal{F}}(W^{\otimes 2})$ commutes with the action of Ξ, we get

$$(\check{R}^W v, \check{R}^W v') = (P_W R v, R \Xi P_W R v')_p = (Rv, P_W R P_W R \Xi v')_p$$

$$= (v, R_{21}^{-1} R_{21} R \Xi v')_p = (v, v').$$

Hence $(\check{R}^W)^* = (\check{R}^W)^{-1}$. Using functoriality of the $*$-operation, Theorem 3.7(a), we also get $(\check{R}_i^W)^* = (\check{R}_i^W)^{-1}$ for $n > 2$ and any $i = 1, 2, \ldots, n-1$. The claim now follows easily from the definition of the inner product on $\mathrm{End}_{\mathcal{F}}(W^{\otimes n})$. □

4. Construction of subfactors

After having constructed a C^* tensor category, the construction of subfactors is parallel to the examples of subfactors obtained from product type group actions (see [Wa1]); the following can be considered a generalization of the construction in [W2].

4.1. Notations. Let $A_1 \subset A_2 \subset \cdots$ and $B_1 \subset B_2 \subset \cdots$ be ascending sequences of finite-dimensional C^* algebras such that $A_n \subset B_n$. Moreover, we assume that all these algebras have the same identity, and that there exists a well-defined finite trace function tr on the inductive limit of the B_n's such that $tr(1) = 1$, and such that the restriction of tr to any A_n or B_n is faithful (i.e. $tr(a^*a) = 0$ only if $a = 0$ for any $a \in A_n$ or $a \in B_n$, $n \in \mathbf{N}$). We say that these sequences satisfy the *commuting square condition* if

$$tr(ab) = tr(a\varepsilon_{A_n}(b)) \quad \text{for } a \in A_{n+1},\ b \in B_n;$$

here ε_{A_n} is the trace preserving conditional expectation, which is uniquely defined by

$$tr(\varepsilon_{A_n}(b)a) = tr(ba) \quad \text{for } b \in B_n, a \in A_n.$$

It is well-known that if tr is a factor trace on both the inductive limits of the A_n's and B_n's, one obtains a pair of II_1 factors $A \subset B$. For details, see e.g. [GHJ], [PP] or [W1].

Jones' basic construction is defined to be the von Neumann algebra $B^{(1)}$ generated by B, acting on (a Hilbert space completion of) itself via left regular representation together with the orthogonal projection e_A onto the subspace generated by A. We have the relation

$$e_A b e_A = \varepsilon_A(b)e_A, \quad \text{for all } b \in B.$$

The algebra $B^{(1)}$ is either a II_∞ factor (in which case the index $[B : A]$ is defined to be equal to ∞) or it is a II_1 factor with unique trace tr normalized so that $tr(1) = 1$. In this case the index is defined to be equal to

$$[B : A] = \frac{1}{tr(e_A)}.$$

In case of finite index, the construction can be iterated, and one obtains a sequence of II_1 factors $A \subset B \subset B^{(1)} \subset B^{(2)} \subset \cdots$. The centralizers of A in $B^{(i)}$ and of B in $B^{(i)}$, denoted by $A' \cap B^{(i)}$ and $B' \cap B^{(i)}$, respectively, are called the higher relative commutants of $A \subset B$.

We say that the sequences $(A_n) \subset (B_n)$ are periodic with period d if there exists $N \in \mathbf{N}$ such that for any $n > N$ there exist bijections between the centers of A_n and A_{n+d} as well as of B_n and B_{n+d} compatible with the inclusions between the various algebras (see [W1, p. 357] for more details). It will turn out that for our examples this will be a consequence of the fact that we only have finitely many simple objects in our C^* tensor category (see below).

4.2. We fix an object W in $\mathcal{F}$. All the following definitions will be in terms of this fixed object W. We may also consider W as a tilting module such that $W = C_W$, with the notations of Section 3.3, i.e. W is isomorphic to a direct sum of Weyl modules V_λ with $\lambda \in \Lambda$. Then we define $\tilde{B}_n = End_{\mathcal{T}}(W^{\otimes n})$ and $B_n = End_{\mathcal{F}}(W^{\otimes n}) = \tilde{B}_n/Neg(W^{\otimes n}, W^{\otimes n})$. Similarly, we define $\tilde{A}_n = 1 \otimes \tilde{B}_{n-1} \subset \tilde{B}_n$ and $A_n = \tilde{A}_n/Neg(1 \otimes W^{\otimes n-1}, 1 \otimes W^{\otimes n-1})$.

4.3. Proposition. (a) *We have well-defined inclusions of* $A_n \subset B_n$, $A_n \subset A_{n+1}$ *and* $B_n \subset B_{n+1}$.

(b) *The algebras* A_n *and* B_n *have a* C^**-structure which is compatible with the inclusions in* (a).

(c) *The sequence of algebras* (A_n) *and* (B_n) *satisfy the commuting square condition.*

(d) *The sequence of algebras* (A_n) *and* (B_n) *is periodic.*

Proof. Statements (a) and (c) are well-known, and are already essentially contained in [A, Lemma 3.6] (similar arguments already appeared for special cases in [W2]). We recall some of the arguments (which are essentially the arguments showing that the quotient category is well-defined), for the reader's convenience: For (a), one has to show that the negligible morphisms in $\tilde{A}_n$ are also negligible in $\tilde{B}_n$. But if $a \in \tilde{A}_n$ is negligible and $b \in \tilde{B}_n$ is arbitrary, we also have

$$Tr_q(ab) = Tr_q(aE_{W^{\otimes n-1}}(b)) = 0,$$

using the properties of $E_{W^{\otimes n-1}}$ (see 3.2). The well-definedness of the other inclusions is shown similarly. This shows (a). By Theorem 3.7, the pairing $(b_1, b_2) = Tr_q(b_2^* b_1)$ is positive definite on B_n. Hence B_n, with the given $*$-structure is a C^* algebra. The compatibility of the $*$ structure with the inclusions follows from the functoriality of $*$, Theorem 3.7 (a). To prove (c), observe that

$$\varepsilon_{A_n}(b) = \frac{1}{dim_q W}\ 1 \otimes E_{W^{\otimes n-1}}(b) \quad mod\ Neg(W^{\otimes n}, W^{\otimes n}),$$

which can be checked easily using the definition of ε_{A_n} and the properties of $E_{W^{\otimes n-1}}$ (see Section 3.2). Statement (c) follows from this. Statement (d) is a consequence of the fact that $\mathcal{F}$ only has finitely many objects. So if the trivial object $\mathbf{1}$ appears in $W^{\otimes d}$, the module $W^{\otimes n} \cong W^{\otimes n} \otimes \mathbf{1}$ is isomorphic to a submodule of $W^{\otimes n+d}$. If p is an orthogonal projection onto a simple summand of $W^{\otimes d}$ which is isomorphic to $\mathbf{1}$, the map $z_\lambda \to z_\lambda \otimes p$ defines a bijection between the centers of B_n and $(1 \otimes p)B_{n+d}(1 \otimes p)$ as well as a bijection between the centers of A_n and $(1 \otimes p)A_{n+d}(1 \otimes p)$. The decomposition of a minimal idempotent in $A_n z_\lambda$ as a sum of minimal idempotents in B_n corresponds to the decomposition of the tensor product of $V_\lambda \otimes W$ into a direct sum of irreducibles. This is obviously compatible with our identification of the centers. □

Remark. In the following we shall not distinguish in notation between an object in $\mathcal{F}$ and a corresponding tilting module W for which $W = C_W$ coincides with the given object.

4.4. Theorem. *Let notations be as in 4.2 and 4.3. Then there exists a pair of hyperfinite* II_1 *factors* $A \subset B$ *such that*

(a) *its index* $[B : A]$ *is equal to* $(dim_q W)^2$,

(b) *its higher relative commutants are given by* $A' \cap B^{(i)} \cong End_{\mathcal{F}}(\cdots \otimes W^* \otimes W)$ *($i+1$ factors) and by* $B' \cap B_i \cong End_{\mathcal{F}}(\cdots \otimes W \otimes W^*)$ *(i factors).*

Proof. In order to compute the index, we embed B_n into $\tilde{B}_n^{(1)} = End_{\mathcal{F}}(W^* \otimes W^{\otimes n})$ in the obvious way. Observe that $e_1 = e_W \otimes 1_{W^{\otimes n-1}}$ is a selfadjoint idempotent in $\tilde{B}_n^{(1)}$ which commutes with A_n and satisfies $e_1 b e_1 = \varepsilon_{B_n}(b) e_1$ (see Lemma 3.4 and the proof of Proposition 4.3(c)). Hence the algebra $B_n^{(1)}$ generated by B_n and e_1 has to contain a direct summand which is isomorphic to Jones' basic construction for $A_n \subset B_n$ (see [W1, Theorem 1.1]). Moreover, as $(A_n) \subset (B_n)$ satisfy the periodicity condition, $B_n^{(1)}$ actually becomes isomorphic to this direct summand for n sufficiently large, by [W1, Theorem 1.5]. In particular, $(\bigcup_{n \in \mathbf{N}} B_n) \cup \{e_1\}$ generates a factor which is isomorphic to Jones' basic construction $B^{(1)} = \langle B, e_A \rangle$ with the isomorphism given by $b \in B \mapsto b$ and $e_1 \mapsto e_A$. In particular, the subfactor $A \subset B$ has the index $1/tr(e_1)$. As e_1 is a rank 1 idempotent in $W \otimes W^*$ acting in the weight space corresponding to the weight 0, and as $dim_q W = dim_q W^*$, we get

$$tr(e_1) = Tr_q(e_1)/dim_q(W)^2 = 1/dim_q(W)^2.$$

By the same argument, one shows that the inclusion of von Neumann factors (where the closures are taken with respect to the weak topology)

$$B = \left(\bigcup_n B_n\right)^- \subset \left(\bigcup_n \tilde{B}_n^{(1)}\right)^- = \tilde{B}^{(1)}$$

has the same index: one embeds these algebras into $End_{\mathcal{F}}(W \otimes W^* \otimes W^{\otimes n})$, and computes the trace of the idempotent $e_2 = (1/dim_q(W)^2)\tilde{b}_W \circ \tilde{d}_W$ as before. Hence $[\tilde{B}^{(1)} : B] = [B^{(1)} : B]$, and as obviously $B^{(1)} \subset \tilde{B}^{(1)}$, we get equality. By the same argument, using induction on i, one shows that the i-th extension $B^{(i)}$ via Jones' basic construction is equal to the von Neumann algebra generated by the union of $\tilde{B}_n^{(i)} = End_{\mathcal{F}}((\cdots W^* \otimes W \otimes W^*) \otimes W^{\otimes n})$, with i factors in the brackets, and the usual embedding of $\tilde{B}_n^{(i)}$ into $\tilde{B}_{n+1}^{(i)}$.

Obviously, $A' \cap B^{(i)}$ contains an algebra isomorphic to

$$C_i = End_{\mathcal{F}}(\cdots\ W^* \otimes W \otimes W^*) \quad (i \text{ factors}).$$

On the other hand, we can find for arbitrarily large N a number $n \geq N$ such that $W^{\otimes n}$ contains the trivial object $\mathbf{1}$ as a summand. We have the injective $\mathcal{F}$-morphisms

$$(\cdots W^* \otimes W \otimes W^*) \cong (\cdots W^* \otimes W \otimes W^*) \otimes \mathbf{1} \to (\cdots W^* \otimes W \otimes W^*) \otimes W^{\otimes n},$$

where we have i factors in the brackets, as usual. Now let p be a minimal idempotent projecting onto $\mathbf{1} \subset W^{\otimes n}$. Then the $\mathcal{F}$-morphisms above induce an isomorphism between C_i and $p\tilde{B}_n^{(i)}p$; in particular these algebras have the same dimensions. This implies the equality as stated, by [W1, Theorem 1.6]. The proof for $B' \cap B^{(i)}$ goes similarly. □

4.5. Principal graph. Assume that W is a simple highest weight module V_λ with $\lambda \in \Lambda$. The principal graph, which describes the inclusion of higher relative commutants, can be easily expressed as follows. Let $N_{\lambda\mu}^\nu$ be the multiplicity of V_ν in $V_\lambda \otimes V_\mu$ in the fusion category (we assume that both V_ν and V_μ are nonzero objects in $\mathcal{F}$). Moreover, let d be the smallest number such that $W^{\otimes d}$ contains a summand isomorphic to $\mathbf{1}$. Fix s large enough such that the periodicity assumption holds for A_{sd}, B_{sd} and B_{sd+1}. Then the principal graph has vertices labelled by the elements $\lambda \in \Lambda$ for which V_λ occurs in $W^{\otimes sd}$ or $W^{\otimes sd+1}$, and two labels μ and ν are connected by $N_{\lambda\mu}^\nu$ edges. The numbers $N_{\lambda\mu}^\nu$ are referred to as Verlinde coefficients [V]. They can be easily reduced to multiplicity coefficients in the classical case for the module V. See e.g. [GW] (for type A) or [Kc, Exercises 13.34, 35] for more precise and more general formulas.

Example. Let us consider the examples of subfactors obtained from $U_q sl_k$ for $q = e^{\pi i/l}$ (these are the same subfactors as the ones constructed from the (k,l)-quotient of the Hecke algebras of type A, with $q = e^{2\pi i/l}$ (see [W1])). Here the principal graph can be described as follows: Recall that a (k,l) diagram is a Young diagram with at most k rows such that the number of boxes in its first row and in its k-th row differ by at most $l-k$ boxes. Fix n such that it is divisible by k and such that it is large enough so that there exists a 1-1 bijection between all (k,l) diagrams with n boxes and with $n+k$ boxes. Then the principal graph has vertices labelled by all (k,l) diagrams with n boxes and with $n+1$ boxes, and the vertices are connected by a single edge if and only if they differ by exactly one box. The trivial vertex (also labelled by $*$) is the one labelled by the rectangular diagram with sidelengths k and n/k (which corresponds to the trivial representation of $U_q sl_k$). These results have previously been obtained by this author, and announced without proof in a talk in 1989. They can also be found in the paper [EK] by Evans and Kawahigashi, who use a different method of proof.

4.6. Remarks. 1. A statement similar to our Theorem 4.4 can also be found in the preprint [X] by Feng Xu, but only for W being one of the fundamental modules V (as described in 3.4), for all Lie types except F_4 and E_8. In his preprint as well as in the current paper, it is exploited that the multiplicity coefficients for tensor products involving such modules are small (see Theorem 3.5(b) and, in [X], property $(*)$ in the introduction). Otherwise, the two approaches seem to be different. Instead of tensor categories, Xu uses Popa's formalism of standard λ-lattices; see [Po2]. Although it is not hard to construct additional subfactors by considering the higher relative commutants of his subfactors, it does not seem to be possible to construct a subfactor for every object in the fusion category (unlike in the approach in this paper).

2. Among the examples of subfactors constructed in Theorem 4.4, one obtains the Jones subfactors from the fundamental representation of $U_q sl_2$, the examples constructed in [W1] for Lie type A, and in [W2] for Lie types C and presumably B. This statement follows from the fact that $End_{\mathcal{F}}(V^{\otimes n})$ is generated by the image of the braid group via the representation induced by the R-matrices. The examples in [W2] corresponding to the full even-dimensional orthogonal groups cannot be obtained via the method in this paper without additional work. C^*-tensor categories for these cases have been constructed in [TW2] by different methods.

3. If q is real, one can define commuting squares as well. Here one can just take the usual matrix $*$-operation with respect to the Lusztig or Kashiwara canonical bases, i.e. all the subtleties with defining Hermitian forms for tensor products are not necessary. The special case of $U_q sl_2$ was essentially already observed in [PP]. For the rest, one essentially just copies the constructions in Sections 3.2 and in this section. The constructions in [X] in this setting would also go through for the additional cases we proved here.

4. As an example, we give explicit formulas for the indices of the series of subfactors obtained from the simplest representation V of type F_4, which are equal to $(dim_q V)^2$ with

$$\begin{aligned} dim_q V &= \frac{[13][18][8][3]}{[9][6][4][1]} \\ &= \frac{2\sin(13\pi/l)\cos(9\pi/l)\cos(4\pi/l)}{\sin(\pi/l)\cos(3\pi/l)}; \end{aligned}$$

the last equality holds for $q = e^{\pi i/2l}$. The principal graphs of these subfactors become quite complicated. For large l, most of the vertices will have 26 edges, among them one double edge (which is counted twice).

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