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POLISH GROUP ACTIONS: DICHOTOMIES AND GENERALIZED ELEMENTARY EMBEDDINGS

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1. Introduction

The results in this paper involve two different topics in the descriptive theory of Polish group actions. The book Becker-Kechris [6] is an introduction to that theory. Our two topics—and two collections of theorems—are rather unrelated, but the proofs for both topics are essentially the same.

Locally compact Polish groups, i.e., second countable locally compact groups, are the traditional objects of study in the field known as "topological groups". More recently, there has been interest in nonlocally compact Polish groups, such as S_∞, the symmetric group on $\mathbb{N}$, topologized as a subspace of $\mathbb{N}^\mathbb{N}$. Much of this paper is concerned with a proper subclass of the class of Polish groups, namely those Polish groups which admit a complete left-invariant metric. We call these *cli groups*. The class of cli groups includes all locally compact groups (Proposition 3.C. 2 (d)), but also much more. For example, all solvable groups are cli (Hjorth-Solecki; see Proposition 3.C.2 (f)), and a closed subgroup G of S_∞ is cli iff G is closed in $\mathbb{N}^\mathbb{N}$ (see Example 3.C.3 (a)).

We study continuous actions by Polish groups on Polish spaces and the associated orbit equivalence relation. For locally compact groups, all of our results are known, or trivial, or both. But some of our results can be viewed as generalizations of known theorems about locally compact groups to the larger class of cli groups.

One of the two topics considered in this paper is dichotomy theorems for equivalence relations. These are theorems which assert that either the quotient space is "small" or else it contains a copy of a specific "large" set. We prove some dichotomy theorems for the orbit space of actions by cli groups. There are two types of dichotomies, which we call the Silver-Vaught Dichotomy and the Glimm-Effros Dichotomy.

The *Silver-Vaught Dichotomy* asserts that either there are only countably many equivalence classes or else there is a perfect set of pairwise inequivalent elements. Assuming the negation of the continuum hypothesis, the Silver-Vaught Dichotomy is equivalent to the proposition that there are either countably many equivalence classes or $2^{\aleph_0}$ equivalence classes. The *Topological Vaught Conjecture*, which is open, is: for any continuous action by a Polish group G on a Polish space X, the orbit equivalence relation satisfies the Silver-Vaught Dichotomy. This conjecture implies—and was motivated by—the Vaught Conjecture in model theory. Indeed,

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the Topological Vaught Conjecture for S_∞ is equivalent to the Vaught Conjecture for $L_{\omega_1\omega}$ (Becker-Kechris [6, 6.2.7]). In this paper, we prove the Topological Vaught Conjecture for all cli groups (Corollary 5.C.6), and, in fact, prove a topological version of the Martin Conjecture for cli groups (Theorem 5.C.5). We also prove that (assuming a weak large cardinal axiom) a Borel-measurable action of a cli group on an analytic space satisfies the Silver-Vaught Dichotomy (Corollary 6.D.2). This latter fact is false for S_∞; there are well known model theoretic counterexamples. Thus this result demonstrates a difference between cli groups and arbitrary Polish groups. Some special cases of these theorems—including the case of locally compact groups—were known. We discuss special cases, history, etc., at the end of §5.C.

The *Glimm-Effros Dichotomy* for an equivalence relation R asserts that either R contains a copy of the Vitali equivalence relation E_0 (equivalently, there exists a nonatomic ergodic measure for R) or else there is a countable Borel separating family for R. (For definitions and details, see §5.B.) This dichotomy implies the Silver-Vaught Dichotomy. This dichotomy originates with work of Glimm and Effros that was motivated by some questions about operator algebras. Glimm [16] proved that the orbit space of a Polish G-action satisfies the Glimm-Effros Dichotomy if G is locally compact. Later, Effros [12] proved it for any Polish group G, provided the orbit equivalence relation is F_σ. Still later, Harrington-Kechris-Louveau [18] studied this dichotomy in the abstract (and named it). There exist Polish S_∞-spaces which violate the Glimm-Effros Dichotomy; thus there is no conjecture for this dichotomy analogous to the Topological Vaught Conjecture. In this paper we prove that for any cli group G, the orbit equivalence relation of a Polish G-action satisfies the Glimm-Effros Dichotomy (Theorem 5.B.2). It is an open question whether there exists *any* non-cli G which satisfies the Glimm-Effros Dichotomy. Again, some special cases were known. In all previously known special cases, a strong form of the dichotomy held: either the orbit space contains a copy of E_0 or else every orbit is G_δ. Hjorth-Solecki [22, 4.1] proved that not all cli groups satisfy this strong form of the Glimm-Effros Dichotomy. What is proved in this paper (Theorem 5.B.2) is: either the orbit space contains a copy of E_0 or else every orbit is $\mathbf{\Pi}^0_\omega$.

The *logic actions* are particular continuous actions by S_∞ on codes for countable L-structures, where the orbit equivalence relation is isomorphism. (These actions are discussed in detail in §2.D.) Much of the past work on Polish group actions has been concerned with *generalized model theory*, that is, generalizing concepts and theorems from the logic actions to other actions by other Polish groups. This includes a lot of the research in invariant descriptive set theory and in matters related to the Topological Vaught Conjecture. For example, Suzuki [44] and Miller [33] generalized the concept of "atomic model" (see Proposition 2.E.3).

The other topic considered in this paper is generalized model theory. We generalize one concept and two theorems. The concept is that of *elementary embeddability.* That is, for any Polish G-space, we can define a notion of one orbit $G{\cdot}x$ being "elementarily embeddable" into another orbit $G{\cdot}y$ (see Definition 3.D.1 and Theorem 3.D.2). One of the theorems being generalized is Vaught's Theorem that countable atomic models are prime models, that is, if $\mathfrak{A}$ omits all nonprincipal types and $\mathfrak{A} \equiv \mathfrak{B}$, then $\mathfrak{A}$ is elementarily embeddable into $\mathfrak{B}$. We generalize this theorem to arbitrary Polish groups G and *enhanced* Polish G-spaces (Theorem 5.A.1). The enhanced Polish G-spaces are a special kind of G-space—with a very technical definition that is given in 2.C.3. We also prove a weak form of the generalized Vaught

Theorem applicable to arbitrary Polish G-spaces (see Theorem 4.A.2). The other theorem of generalized model theory is a generalized version of the following theorem of Steel [42, 3.2]: assume for all $r \in 2^{\mathbb{N}}$, $r^{\#}$ exists; let $\mathcal{S}$ be any pseudo-$L_{\omega_1\omega}$ class of countable L-structures which violates the Silver-Vaught Dichotomy; then there exists an ω_1-sequence $\langle \mathfrak{A}_\alpha : \alpha < \omega_1 \rangle$ of nonisomorphic members of $\mathcal{S}$ such that for all $\alpha < \beta < \omega_1$, $\mathfrak{A}_\alpha$ is elementarily embeddable into $\mathfrak{A}_\beta$. (See Theorem 6.A.2.) These two results constitute the beginning of the author's work in generalized model theory; more will appear in Becker [4].

In the case of cli groups, the concept of "elementary embeddability" trivializes: if $G\cdot x$ is "elementarily embeddable" into $G\cdot y$, then $G\cdot x = G\cdot y$. That is what enables us to prove the dichotomy theorems for cli groups. And that is why there is absolutely no hope of proving the model theoretic Vaught Conjecture by the methods of this paper.

The Main Theorem of this paper is stated and proved in §4. In §§5, 6 we derive consequences of the Main Theorem: both the dichotomy theorems and the theorems of generalized model theory. The consequences for Polish and Borel G-spaces are in §5, those for analytic G-spaces in §6. In §§2, 3 there is a large amount of preliminary material. In particular, in §§3.C, 3.D we define and study the two new concepts introduced in this paper, cli groups and generalized elementary embeddings, respectively.

2. Polish group actions

2.A. Descriptive set theory. A *Polish space* is a separable completely metrizable topological space.

We assume the reader is familiar with the elementary descriptive set theory of Polish spaces. Two references for this are Kechris [28] and Moschovakis [36]. In §6—the last section of this paper—we use *effective* descriptive set theory. For information on this topic, see Moschovakis [36].

The pointclass of Borel sets ramifies into a transfinite hierarchy of Borel classes $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, $1 \le \alpha < \omega_1$, defined as follows:

$$\mathbf{\Sigma}^0_1 = \text{open},\ \ \mathbf{\Pi}^0_1 = \text{closed};$$

$$\mathbf{\Sigma}^0_\alpha = \{\textstyle\bigcup_{n\in\mathbb{N}} A_n : A_n \in \mathbf{\Pi}^0_{\beta_n} \text{ for some } \beta_n < \alpha\};$$

$$\mathbf{\Pi}^0_\alpha = \text{the class of complements of } \mathbf{\Sigma}^0_\alpha \text{ sets.}$$

We denote by $\mathbf{\Sigma}^1_1$ the class of *analytic* sets, i.e., the continuous images of Borel sets. We denote by $\mathbf{\Pi}^1_1$ the class of *coanalytic* sets, i.e., the complements of analytic sets. We define the projective classes $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, $\mathbf{\Delta}^1_n$ by induction, as follows:

$$\mathbf{\Sigma}^1_{n+1} = \text{the class of continuous images of } \mathbf{\Pi}^1_n \text{ sets;}$$

$$\mathbf{\Pi}^1_{n+1} = \text{the class of complements of } \mathbf{\Sigma}^1_{n+1} \text{ sets;}$$

$$\mathbf{\Delta}^1_n = \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n.$$

Thus, by Souslin's Theorem, $\mathbf{\Delta}^1_1 = \text{Borel}$. All of the above pointclasses have corresponding effective, or "lightface", pointclasses, which are used in §6.

An equivalence relation R on a Polish space Y is always viewed as a pointset in $Y \times Y$. R is an *analytic* equivalence relation if it is an analytic set in $Y \times Y$ (with the product topology); and similarly for other pointclasses.

Throughout this paper, we will be considering a set Y together with two (or more) different topologies on Y. Suppose τ and t are topologies on Y. Then t is *finer* than τ if every τ-open set is also t-open.

2.A.1. Definition. A topological space is a *Baire space* if a countable intersection of dense open sets is dense, that is, the space satisfies the Baire Category Theorem. A topological space is an *analytic space* if it is homeomorphic to an analytic set in a Polish space. A topological space $\langle Z, t\rangle$ is called *acceptable* if the following three properties hold.

(a) $\langle Z, t\rangle$ is second countable.

(b) $\langle Z, t\rangle$ is a strong Choquet space (see Becker-Kechris [6, 5.2] or Kechris [28, 8.D] for the definition).

(c) There exists a topology τ on Z such that t is finer than τ and such that $\langle Z, \tau\rangle$ is an analytic space.

There are two relevant cases where a topological space is acceptable. The first case is where $\langle Z, t\rangle$ is a Polish space (letting $\tau = t$). Except for §6 of this paper, this is really the only case that need concern us, and the reader who is not interested in §6 can regard "acceptable topological space" as being a synonym for "Polish space". The second case, which comes up in §6, involves variants of the Gandy-Harrington topology (see Harrington-Kechris-Louveau [18]). That is, given a Polish space $\langle Y, \tau\rangle$ and an analytic set $Z \subset Y$, there exists a topology t on Z such that t is second countable and strong Choquet and such that t is finer than τ; in this situation, $\langle Z, t\rangle$ is, of course, acceptable. (Remark: These spaces are not regular.)

We do not use the strong Choquet property directly. The following proposition contains all that we need regarding acceptable spaces.

2.A.2. Proposition. *Let $\langle Y, t\rangle$ be an acceptable topological space.*

(a) *$\langle Y, t\rangle$ is a Baire space.*

(b) *Any G_δ subspace of $\langle Y, t\rangle$ is acceptable.*

(c) *Let R be an equivalence relation on Y such that R has the property of Baire with respect to $\langle Y, t\rangle \times \langle Y, t\rangle$. At least one of the following two cases must hold.*

(i) *There exists a continuous function $f : 2^{\mathbb{N}} \to Y$ such that for any two distinct points $\mathbf{p}$, $\mathbf{q} \in 2^{\mathbb{N}}$, $f(\mathbf{p})$ and $f(\mathbf{q})$ are in different R-equivalence classes.*

(ii) *There exists a nonmeager R-equivalence class.*

(d) *Let τ be a topology on Y such that t is finer than τ and such that $\langle Y, \tau\rangle$ is an analytic space.*

(i) *If $S \subset Y$ is $\mathbf{\Sigma}^1_1$ with respect to $\langle Y, \tau\rangle$, then S has the property of Baire with respect to $\langle Y, t\rangle$; moreover, S has the property of Baire with respect to any subspace of $\langle Y, t\rangle$ which includes S.*

(ii) *If $Q \subset Y \times Y$ is $\mathbf{\Sigma}^1_1$ with respect to $\langle Y, \tau\rangle \times \langle Y, \tau\rangle$, then Q has the property of Baire with respect to $\langle Y, t\rangle \times \langle Y, t\rangle$.*

For a proof of parts (a) and (b) of Proposition 2.A.2, see Kechris [28, 8.15, 8.16 and 25.19]. For a proof of parts (c) and (d), see Martin-Kechris [32, 9.4].

2.B. Polish groups. A *Polish group* is a topological group whose topology is Polish. We, of course, regard two topological groups as being the same if there exists a bijection between them which is both a group isomorphism and a homeomorphism.

2.B.1. Examples. (a) Every second countable locally compact group is Polish. This includes all countable discrete groups, $(\mathbb{R}^n, +)$, $GL(n, \mathbb{C})$ and $U(n)$ (where $U(n)$ denotes the compact subgroup of $GL(n, \mathbb{C})$ consisting of unitary matrices).

(b) $(X, +)$, where X is a separable infinite-dimensional Banach space, i.e., the infinite-dimensional analog of $(\mathbb{R}^n, +)$.

(c) The unitary group $U(\mathcal{H})$ of a separable infinite-dimensional complex Hilbert space $\mathcal{H}$, with the strong (or equivalently, weak) operator topology, i.e., the infinite-dimensional analog of $U(n)$. See Kechris [28, 9.B] for definitions and details.

(d) S_∞, the symmetric group on $\mathbb{N}$, with the following topology t. If b is a bijection between finite subsets of $\mathbb{N}$, let $N^b = \{g \in S_\infty : g \text{ extends } b\}$; then $\{N^b\}$ is a basis for t. We call this the *canonical basis* for S_∞. It is easy to see that t is a Polish topology, and that, in fact, $\langle S_\infty, t\rangle$ is homeomorphic to the Baire space, $\mathbb{N}^\mathbb{N}$. The canonical basis is closed under multiplication by group elements, that is, for any $g \in S_\infty$, gN^b and $N^b g$ are in the basis. (In general, Polish groups do not have a countable basis with this property; see Becker-Kechris [6, 1.5.1].) The elements of S_∞ are functions from $\mathbb{N}$ to $\mathbb{N}$ (permutations), that is, elements of $\mathbb{N}^\mathbb{N}$. Thus S_∞ is a G_δ subspace of $\mathbb{N}^\mathbb{N}$; t is the topology on S_∞ which it inherits as a subspace. Viewing elements of S_∞ as functions from $\mathbb{N}$ to $\mathbb{N}$, t is the topology of pointwise convergence.

(e) The group $H(K)$ of homeomorphisms of a compact metrizable space K, with the topology it inherits as a G_δ subspace of $C(K, K)$, that is, the topology of uniform convergence.

More examples will be given in 3.C.2.

If G is a Polish group and H is a closed subgroup of G, then H is Polish; if H is a normal subgroup, then G/H (with the quotient topology) is also Polish (see Becker-Kechris [6, 1.2.3]). The class of Polish groups is also closed under finite or countable products. This last operation allows us to build nonlocally compact Polish groups out of locally compact groups, e.g., $\mathbb{Z}^\mathbb{N}$ and $\mathbb{R}^\mathbb{N}$.

In this paper, *group* means Polish group (but we sometimes put in the word "Polish" for emphasis). G always denotes a group and g and h—sometimes with indices—always denote elements of a group. M and N—sometimes with indices—always denote open subsets of a group.

2.B.2. Definition. Let G be a group. An *enhanced basis* $\mathcal{N}^*$ for G is a pair $\langle\{N_i : i \in \mathbb{N}\}, \{g_k^{\langle i,j\rangle} : i,\ j,\ k \in \mathbb{N}\}\rangle$, satisfying the following five conditions.

(a) $\mathcal{N} = \{N_i : i \in \mathbb{N}\}$ is a countable basis for G, with $G \in \mathcal{N}$.

(b) For all $i,\ j,\ k \in \mathbb{N},\ g_k^{\langle i,j\rangle} \in G$.

(c) For all $i,\ j,\ k \in \mathbb{N},\ g_k^{\langle i,j\rangle} N_j \in \mathcal{N}$.

(d) For all $i,\ j,\ k \in \mathbb{N}$, if $N_i \neq \emptyset$ and $N_j \neq \emptyset$, then $(g_k^{\langle i,j\rangle} N_j) \cap N_i \neq \emptyset$.

(e) For all $i,\ j \in \mathbb{N}$, for all $M \in \mathcal{N}$ with $M \subset N_j$:

$$\bigcup\{gM : g \in G \ \&\ gM \cap N_i \neq \emptyset\} \subset \bigcup_{k\in\mathbb{N}} (g_k^{\langle i,j\rangle} M).$$

2.B.3. Proposition. *For any group G, there exists an enhanced basis for G.*

Proof. Let A be a countable dense subset of G. Let $\mathcal{N}$ be a countable basis for G containing G and closed under left multiplication by members of A. Let $\{N_i\}$ be an enumeration of $\mathcal{N}$. For any $i,\ j$ such that $N_i \neq \emptyset$ and $N_j \neq \emptyset$, let $\{g_k^{\langle i,j\rangle} : k \in \mathbb{N}\}$ be an enumeration of the following set: $\{g \in A : (gN_j) \cap N_i \neq \emptyset\}$. □

2.B.4. Example. Let $\mathcal{N}$ be the canonical basis for S_∞ (2.B.1 (d)) and let $\{N_i : i \in \mathbb{N}\}$ be an enumeration of $\mathcal{N}$. As shown in the proof of 2.B.3, there exist $g_k^{\langle i,j\rangle} \in S_\infty$ such that $\langle \{N_i\}, \{g_k^{\langle i,j\rangle}\}\rangle$ is an enhanced basis for S_∞.

2.B.5. Proposition. *Let G be a group and let $\langle \{N_i\}, \{g_k^{\langle i,j\rangle}\}\rangle$ be an enhanced basis for G.*

(a) *For any $i,\ j \in \mathbb{N}$, $\{g_k^{\langle i,j\rangle} : k \in \mathbb{N}\}$ is dense in the open neighborhood $\{g \in G : gN_j \cap N_i \neq \emptyset\}$.*

(b) *There exist $i,\ j \in \mathbb{N}$ such that $\{g_k^{\langle i,j\rangle} : k \in \mathbb{N}\}$ is dense in G.*

Proof. (a) This follows from part (e) of the definition of "enhanced basis".

(b) By part (a) of that definition, there exist $i,\ j$ such that $N_i = N_j = G$. $\square$

2.C. Actions of Polish groups. Let G be a group and X a set. An *action* of G on X is a function $a : G \times X \to X$ (where $a(g,x)$ is written $g\cdot x$), such that $e\cdot x = x$ and $g\cdot(h\cdot x) = gh\cdot x$. We call (X,a)—or just X when a is understood—a *G-space*. We use the terms "G-space" and "action of G" interchangeably. If X is a topological space and a is continuous, then (X,a) is a *topological G-space*; if, in addition, X is a Polish space, then (X,a) is a *Polish G-space*, and if X is an acceptable topological space (see Definition 2.A.1), then (X,a) is an *acceptable G-space*.

2.C.1. Examples. We give a number of examples of Polish G-spaces, for various G.

(a) Let H be a Polish group and let G be a closed subgroup of H. H is a Polish G-space under any of the following three actions: the *left action*, $g\cdot h = gh$; the *right action*, $g\cdot h = hg^{-1}$; the *conjugation action*, $g\cdot h = ghg^{-1}$. In the first two of these, the orbit equivalence relation is the coset decomposition, and all orbits are closed. In the third of these, the orbits are, in general, more complicated pointsets. In fact, the orbit structure is complex enough that the topic of this paper is nontrivial for conjugation actions (see 3.D.6 (c), below).

(b) If G is a group of permutations of X, then G acts on X in the obvious way: $g\cdot x = g(x)$. If K is compact metrizable and G is a closed subgroup of $H(K)$ (see 2.B.1 (e)), this action of G on K is continuous, so K is a Polish G-space. We call this the *evaluation action*.

(c) The left, right and conjugation actions of S_∞ on itself extend to actions of S_∞ on $\mathbb{N}^\mathbb{N}$: $g\cdot f = g \circ f$; $g\cdot f = f \circ g^{-1}$; $g\cdot f = g \circ f \circ g^{-1}$. All three are continuous. Similarly, for compact K, there are continuous left, right and conjugation actions of $H(K)$ on $C(K,K)$.

(d) The logic actions are a collection of S_∞-actions which arise in model theory. These actions will be discussed in §2.D.

(e) Let $U(n)$, $\mathcal{H}$ and $U(\mathcal{H})$ be as in 2.B.1 (a) and (c). $U(n)$ and $U(\mathcal{H})$ are groups of operators acting on $\mathbb{C}^n$ and $\mathcal{H}$, respectively. Two points in $\mathbb{C}^n$ or $\mathcal{H}$ are in the same orbit of this action iff they have the same norm. So the unit ball B of $\mathcal{H}$ is an invariant set. $\mathbb{C}^n$ is a Polish $U(n)$-space. If $\mathcal{H}$ is given the norm topology, then $\mathcal{H}$ (and hence B) is a Polish $U(\mathcal{H})$-space. In this case, all orbits are closed. If $\mathcal{H}$ is given the weak (or equivalently, weak*) topology w (which, of course, is not Polish or even metrizable), then the action is not continuous; that is, $\langle \mathcal{H}, w\rangle$ is not a topological $U(\mathcal{H})$-space. But $\langle B, w\rangle$ is a Polish $U(\mathcal{H})$-space. The orbits of $\langle B, w\rangle$ are G_δ, but not closed. For the above Polish spaces, the continuity of the action can be established as follows. It is easy to see that, with respect to either topology on $\mathcal{H}$, the action of $U(\mathcal{H})$ on $\mathcal{H}$ is continuous in each variable separately. And for a Polish space, separate continuity implies joint continuity (see Kechris [28, 9.14]).

(f) For any group G, let $\mathcal{F}(G)$ denote the collection of closed subsets of G, and let a denote the action of G on $\mathcal{F}(G)$ by left multiplication. Clearly $(\mathcal{F}(G), a)$ is a G-space, but it does not seem to be a topological G-space in any obvious way. However there is a Polish topology t on $\mathcal{F}(G)$ which turns $(\mathcal{F}(G), a)$ into a Polish G-space; moreover, the Borel structure of t is the Effros Borel structure on $\mathcal{F}(G)$. See Becker-Kechris [6, 2.4 (ii) and 5.2.1] for details.

One can, of course, form subspaces of G-spaces and products of G-spaces. A finite or countable product of Polish G-spaces is a Polish G-space.

Becker-Kechris [6] is our basic reference for actions of Polish groups, and we follow this book in notation and terminology. We assume the reader is reasonably familiar with the material in this book, e.g., with the properties of Vaught transforms, and with dichotomies for the orbit space.

Throughout this paper, there is essentially one fixed (but arbitrary) action $a : G \times X \to X$ under consideration, and there is never any occasion to consider two actions at the same time. Therefore we always use the dot notation $g{\cdot}x$ rather than $a(g, x)$, and we refer to the G-space and the associated orbit equivalence relation as X and E, respectively, rather than as (X, a) and E_a. The letters X and E are reserved for precisely this purpose; x, y, z—sometimes with indices—always denote elements of X; and C, D—sometimes with indices—always denote open subsets of X. While we never change the G-space X, we do change the topology on X. The action will be continuous with respect to all these topologies. With the exception of three corollaries (5.B.3, 5.C.7, 6.D.2), we never consider discontinuous actions. We also sometimes consider the action restricted to various invariant subsets Y of X; in this case, $(Y, a \restriction Y)$ is, of course, itself a G-space, but we prefer not to think of it that way, but rather to view Y as a pointset in the ambient G-space X.

Given a Polish G-space X, the orbit equivalence relation E is analytic, but in general is not Borel (see Becker-Kechris [6, §7] and Kechris [28, 27.D]). Nevertheless, by a theorem of Ryll-Nardzewski, every orbit is a Borel set (see Becker-Kechris [6, 2.3.4]).

2.C.2. Proposition. *For any Polish G-space X, there exists a unique partition of X—$X = \bigcup\{Y_u : u \in U\}$, with the Y_u's nonempty and pairwise disjoint—satisfying the following three properties.*

(a) *For all u, Y_u is invariant.*
(b) *For all u, Y_u is a G_δ.*
(c) *For all u, every orbit in Y_u is dense in Y_u.*

Proof. Let $\{C_j\}$ be a countable basis for X. For $u \in 2^{\mathbb{N}}$, define

$$Y_u = [\bigcap\{G{\cdot}C_j : u(j) = 1\}] \cap [\bigcap\{X \setminus G{\cdot}C_j : u(j) = 0\}].$$

Let $U = \{u \in 2^{\mathbb{N}} : Y_u \neq \emptyset\}$. Clearly (a)–(c) hold. To show uniqueness, suppose $\{Z_v : v \in V\}$ is another such partition. Fix $u \in U$. Then $\{Z_v \cap Y_u : v \in V\}$ partitions Y_u into dense G_δ sets, so this partition has only one member; that is, there is a v such that $Y_u \subset Z_v$. Similarly, $Z_v \subset Y_u$. □

We call this the *canonical partition* of X, and call the Y_u's the *pieces* of the canonical partition.

2.C.3. Definition. Let G be a group. A 5-tuple $(\mathcal{N}^*, X, a, t, \mathcal{C})$ satisfying the following four conditions is an *enhanced Polish G-space.*

(a) $\mathcal{N}^* = \langle \mathcal{N}, \{g_k^{\langle i,j\rangle} : i,\ j,\ k \in \mathbb{N}\}\rangle$ is an enhanced basis for G.
(b) X is a Polish space with topology t and $\mathcal{C}$ is a countable basis for t.
(c) $a : G \times X \to X$ is an action which is continuous with respect to t.
(d) For any $i,\ j \in \mathbb{N}$ and for any $C,\ D \in \mathcal{C}$, the following set is t-open:

$$\left[C \setminus \left(\bigcup_{k\in\mathbb{N}} (g_k^{\langle i,j\rangle} \cdot D)\right)\right]^{\Delta}.$$

When $\mathcal{N}^*,\ a,\ t$ and $\mathcal{C}$ are understood—or irrelevant—we refer to "the enhanced Polish G-space X". Some concrete examples of enhanced Polish G-spaces will be given in 2.D.4, below.

2.C.4. Proposition. *Let G be a group and let $(\mathcal{N}^*, X, a, t, \mathcal{C})$ be an enhanced Polish G-space.*

(a) *For any $D \in \mathcal{C}$, $G{\cdot}D$ is clopen.*
(b) *All of the pieces of the canonical partition of X are closed.*

Proof. (b) follows easily from (a) and the definition of the pieces Y_u in the proof of 2.C.2. It is trivial that $G{\cdot}D$ is open. So we need only show that $G{\cdot}D$ is closed for $D \in \mathcal{C}$. Fix D. We use the notation: $\mathcal{N}^* = \langle \mathcal{N}, \{g_k^{\langle i,j\rangle}\}\rangle$ and $\mathcal{N} = \{N_i\}$.

Claim. If $\{g_k^{\langle i,j\rangle} : k \in \mathbb{N}\}$ is dense in G, then for any $C \in \mathcal{C}$,

$$\left[C \setminus \left(\bigcup_{k\in\mathbb{N}} (g_k^{\langle i,j\rangle} \cdot D)\right)\right]^{\Delta} = G{\cdot}C \setminus G{\cdot}D.$$

By Proposition 2.B.5 (b), such an $i,\ j$ exist. So assuming the Claim, for any $C \in \mathcal{C},\ G{\cdot}C \setminus G{\cdot}D$ is open. Hence $X \setminus G{\cdot}D$ is a union of open sets, and this proposition is proved.

To prove the Claim, assume it is false. Clearly $A = \left[C \setminus \left(\bigcup_{k\in\mathbb{N}} (g_k^{\langle i,j\rangle} \cdot D)\right)\right]^{\Delta}$ is an invariant set, $A \subset G{\cdot}C$ and $G{\cdot}C \setminus G{\cdot}D \subset A$. So $A \cap G{\cdot}D \neq \emptyset$. Let $x \in A \cap D$. Let

$$Q = \left\{h \in G : h{\cdot}x \in \bigcup_{k\in\mathbb{N}} (g_k^{\langle i,j\rangle} \cdot D)\right\}.$$

For all $k,\ g_k^{\langle i,j\rangle} \in Q$. So Q is dense open. Thus for a comeager set of h's in $G,\ h{\cdot}x \in \bigcup_{k\in\mathbb{N}} (g_k^{\langle i,j\rangle} \cdot D)$. Hence by definition of $A,\ x \notin A$, a contradiction. $\square$

2.C.5. Proposition. *Let G be a group, let $\mathcal{N}^*$ be an enhanced basis for G, let (X, a) be a Polish G-space, and let τ be the topology on X. There exists a topology t on X with basis $\mathcal{C}$ such that:*

(a) *t is finer than τ;*
(b) *for all $C \in \mathcal{C}$ there exists an $n \in \mathbb{N}$ such that C is $\mathbf{\Sigma}_n^0$ (with respect to τ);*
(c) *$(\mathcal{N}^*, X, a, t, \mathcal{C})$ is an enhanced Polish G-space.*

Proof. Let $G,\ \mathcal{N}^*,\ X,\ a,\ \tau$ satisfy the hypothesis. Let $\mathcal{N}^* = \langle \mathcal{N}, \{g_k^{\langle i,j\rangle}\}\rangle$. Fix a countable basis $\mathcal{D}$ for $\langle X, \tau\rangle$. Let $\mathcal{B}$ be a countable Boolean algebra of subsets of X

satisfying the following five conditions.

(i) For all $B \in \mathcal{B}$ and all $i,\ j \in \mathbb{N}$, $\bigcup_{k\in\mathbb{N}}(g_k^{\langle i,j\rangle}\cdot B) \in \mathcal{B}$.

(ii) For all $B \in \mathcal{B}$ and all $N \in \mathcal{N}$, $B^{\Delta N} \in \mathcal{B}$.

(iii) The topology t' generated by the basis $\mathcal{B}$ is Polish.

(iv) $\mathcal{D} \subset \mathcal{B}$.

(v) For all $B \in \mathcal{B}$, there is an $n \in \mathbb{N}$ such that B is $\mathbf{\Sigma}^0_n$ (with respect to τ).

Such a $\mathcal{B}$ exists by Becker-Kechris [6, 5.1.3]. Let $\mathcal{S} = \{B^{\Delta N} : B \in \mathcal{B},\ N \in \mathcal{N}\}$, let t be the topology on X generated by the subbasis $\mathcal{S}^* = \mathcal{S} \cup \mathcal{D}$, and let $\mathcal{C}$ be all finite intersections of elements of $\mathcal{S}^*$.

Clearly 2.C.5 (a) and (b) hold. The proof of Becker-Kechris [6, 5.1.8] shows that t is Polish and a is continuous with respect to t; that is, (b) and (c) in the definition of "enhanced Polish G-space" hold. All that remains to be proved is that part (d) of that definition holds. To prove this, fix $i,\ j \in \mathbb{N}$ and $C,\ D \in \mathcal{C}$. Note that $\mathcal{C} \subset \mathcal{B}$, so $C,\ D \in \mathcal{B}$. So by (i) in the definition of $\mathcal{B}$, $A = C \setminus \left(\bigcup_{k\in\mathbb{N}}(g_k^{\langle i,j\rangle}\cdot D)\right)$ is in $\mathcal{B}$. Hence $A^{\Delta} \in \mathcal{S}$, and therefore is t-open. $\square$

2.D. The logic actions. Let L denote a nonempty countable relational language: $L = (R_i)_{i\in I}$, where I is a nonempty countable index set and R_i is an n_i-ary relation symbol. Denote by X_L the space

$$X_L = \prod_{i\in I} 2^{(\mathbb{N}^{n_i})}.$$

Thus X_L is the product of countably many 2-point spaces. We call the product topology on X_L the *usual topology*; this topology makes X_L homeomorphic to the Cantor space. We view X_L as the space of countably infinite L-structures, identifying every $x = (x_i)_{i\in I} \in X_L$ with the structure

$$\mathfrak{A}_x = \left(\mathbb{N}, R_i^{\mathfrak{A}_x}\right)_{i\in I},$$

where $R_i^{\mathfrak{A}_x}(\overline{s}) \Leftrightarrow x_i(\overline{s}) = 1$, for $\overline{s} \in \mathbb{N}^{n_i}$. We call x the *code* for the L-structure $\mathfrak{A}_x$. The *logic action* J_L of S_∞ on X_L is defined as

$$J_L(g,x) = y \text{ iff } (\forall i \in I)[y_i(s_1,\dots,s_{n_i}) = 1 \Leftrightarrow x_i(g^{-1}(s_1),\dots,g^{-1}(s_{n_i})) = 1];$$

that is, $J_L(g,x) = y$ iff the permutation $g : \mathbb{N} \to \mathbb{N}$ is an isomorphism of $\mathfrak{A}_x$ onto $\mathfrak{A}_y$. This is a continuous action with respect to the usual topology on X_L. The orbit equivalence relation is clearly isomorphism (of the L-structures encoded by elements of X_L).

It turns out that the usual topology on X_L is not the interesting one. We discuss some other topologies, below.

$L_{\omega_1\omega}$ is an infinitary language. There is a countably infinite set of *variables*. The *atomic formulas* are those of the form $v_1 = v_2$ or $R_i(v_1, v_2, \dots, v_{n_i})$, where R_i is a relation symbol of L and the v_j's are arbitrary variables. The *formulas* of $L_{\omega_1\omega}$ are built up from these atomic formulas using negation, (first-order) quantifiers, and finite or countably infinite conjunctions and disjunctions. Of course, the *sentences* of $L_{\omega_1\omega}$ are those formulas with no free variables. For any $L_{\omega_1\omega}$ sentence φ and any L-structure $\mathfrak{A}$, $\mathfrak{A} \models \varphi$ means that φ is true for the structure $\mathfrak{A}$. Note that there are uncountably many $L_{\omega_1\omega}$ sentences. In this paper we consider only relational languages; as is well known, this entails no loss of generality.

Model theory is the branch of mathematical logic which studies languages, structures, and the truth relation $\models$. For an introduction to model theory, see Hodges [23]. For purposes of this paper, the term "model theory" refers only to the model theory of $L_{\omega_1\omega}$, and only to countable structures. For information on $L_{\omega_1\omega}$, see Keisler [30].

A *fragment* F of $L_{\omega_1\omega}$ is a set of formulas of $L_{\omega_1\omega}$ containing all atomic formulas and closed under subformulas, negation, quantifiers and finite conjunctions and disjunctions. The smallest fragment is the usual, finitary, first-order logic. For any fragment F of $L_{\omega_1\omega}$ and any L-structures $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{A} \equiv_F \mathfrak{B}$ means that for any sentence φ in F: $\mathfrak{A} \models \varphi$ iff $\mathfrak{B} \models \varphi$. If F is finitary logic, then $\equiv_F$ is *elementary equivalence*, denoted simply as $\equiv$.

Countable fragments of $L_{\omega_1\omega}$ can be thought of as being a generalization of finitary first-order languages ($L_{\omega\omega}$). For example, much of the theory of atomic, homogeneous and saturated models can be generalized. (However the Compactness Theorem does not generalize.)

2.D.1. Definition. For φ $(= \varphi(\overline{v}))$ a formula of $L_{\omega_1\omega}$ and $\overline{s}$ a finite sequence from $\mathbb{N}$ of the appropriate length, let

$$\mathrm{Mod}(\varphi, \overline{s}) = \{x \in X_L : \mathfrak{A}_x \models \varphi[\overline{s}]\},$$

where $\varphi[\overline{s}]$ denotes the sentence obtained from the formula $\varphi(\overline{v})$ by substituting $\overline{s}$ for the free variables. (If φ is a sentence, we write $\mathrm{Mod}(\varphi)$ for $\mathrm{Mod}(\varphi, (\ ))$.) For F a fragment of $L_{\omega_1\omega}$, let

$$\mathcal{B}_F = \{\mathrm{Mod}(\varphi, \overline{s}) : \varphi \in F,\ \overline{s} \in \mathbb{N}^{<\mathbb{N}}\},$$

and let t_F denote the topology on X_L generated by the basis $\mathcal{B}_F$.

2.D.2. Proposition. *Let L be a countable relational language and let F be a countable fragment of $L_{\omega_1\omega}$.*

(a) *t_F is finer than the usual topology on X_L.*
(b) *t_F is a Polish topology on X_L.*
(c) *The action $J_L : S_\infty \times X_L \to X_L$ is continuous with respect to t_F.*

For a proof of 2.D.2, see Sami [38, 1.1 and 4.2].

2.D.3. Proposition. *Let L be a countable relational language and let F be a countable fragment of $L_{\omega_1\omega}$. Let $x,\ y \in X_L$. The following are equivalent.*

(a) *x and y are in the same piece of the canonical partition of $\langle X_L, t_F\rangle$.*
(b) *$\mathfrak{A}_x \equiv_F \mathfrak{A}_y$.*

The canonical partition is defined in Proposition 2.C.2 ff. The proof of 2.D.3 is easy and is left to the reader. In light of 2.D.3, the pieces of the canonical partition can be viewed as being a generalization of the concept of a *complete theory.*

Parts (b) and (c) of Proposition 2.D.2 say that $(\langle X_L, t_F\rangle, J_L)$ is a Polish S_∞-space. In fact, more is true: it is an enhanced Polish S_∞-space.

2.D.4. Theorem. *Let L be a countable relational language and let F be a countable fragment of $L_{\omega_1\omega}$. Let $\mathcal{N}^* = \langle\{N_i\}, \{g_k^{\langle i,j\rangle}\}\rangle$ be an enhanced basis for S_∞ such that $\{N_i\}$ is an enumeration of the canonical basis for S_∞* (see 2.B.1 (d) and 2.B.4). *Then the 5-tuple $(\mathcal{N}^*, X_L, J_L, t_F, \mathcal{B}_F)$ is an enhanced Polish S_∞-space.*

Proof. Conditions (a)–(c) in the definition of "enhanced Polish G-space" (2.C.3) are clear, so all that needs to be verified is condition (d) of that definition. To prove (d), it will suffice to show that for any $i,\ j \in \mathbb{N}$ and for any D in the basis $\mathcal{B}_F$, $\widehat{D} = \bigcup_{k\in\mathbb{N}}(g_k^{\langle i,j\rangle} \cdot D)$ is t_F-closed. Since F is closed under negation, $\mathcal{B}_F$ is a clopen basis, hence it will suffice to show that $\widehat{D} \in \mathcal{B}_F$. Toward proving this, fix $i,\ j \in \mathbb{N}$ and $D \in \mathcal{B}_F$.

Claim 1. There exists an $n \in \mathbb{N}$, an n-tuple $(s_1, \dots, s_n) \in \mathbb{N}^n$, and a formula $\varphi(v_1, \dots, v_n) \in F$ with n free variables such that for all $h \in S_\infty$ and for all $x \in X_L$:

$$x \in h\cdot D \iff \mathfrak{A}_x \models \varphi[h(s_1), \dots, h(s_n)].$$

Proof of Claim 1. By definition of $\mathcal{B}_F$, there exist $n,\ s_1, \dots, s_n$ and $\varphi \in F$ such that $D = \{x \in X_L : \mathfrak{A}_x \models \varphi[s_1, \dots, s_n]\}$. We now have:

$$\begin{aligned} x \in h\cdot D &\iff h^{-1}\cdot x \in D \\ &\iff \mathfrak{A}_{(h^{-1}\cdot x)} \models \varphi[s_1, \dots, s_n] \\ &\iff \mathfrak{A}_x \models \varphi[h(s_1), \dots, h(s_n)]. \end{aligned}$$

This proves Claim 1.

Fix $n, s_1, \dots, s_n$ and φ satisfying Claim 1.

An L-formula is called *simple* if it is built out of atomic formulas of the form $v_1 = v_2$ ($v_1,\ v_2$ arbitrary variables) by using negation and finite conjunctions and disjunctions. A simple formula is in F. When members of $\mathbb{N}$ are substituted for the variables of a simple formula, we obtain a sentence about $\mathbb{N}$ which is either true or false (with respect to $(\mathbb{N}, =)$); if true, it is true for $\mathfrak{A}_x$ for all $x \in X_L$; if false, it is false for $\mathfrak{A}_x$ for all $x \in X_L$.

Claim 2. There exists an $m \in \mathbb{N}$, an m-tuple $(r_1, \dots, r_m) \in \mathbb{N}^m$ and a simple formula $\psi(u_1, \dots, u_m, v_1, \dots, v_n, w_1, \dots, w_n)$ with $m + 2n$ free variables such that for all $t_1, \dots, t_n \in \mathbb{N}^n$:

$$(\exists k \in \mathbb{N})(g_k^{\langle i,j\rangle}(s_1) = t_1 \wedge \dots \wedge g_k^{\langle i,j\rangle}(s_n) = t_n)$$

$$\iff (\psi[r_1, \dots, r_m, s_1, \dots, s_n, t_1, \dots, t_n] \text{ is true}).$$

Proof of Claim 2. As $\{N_i\}$ is the canonical basis, there exist bijections c and d between finite subsets of $\mathbb{N}$ such that $N_i = N^c$ and $N_j = N^d$. A moment's thought about the canonical basis should convince the reader that there exists a finite sequence of natural numbers—$r^1_1, \dots, r^1_{m'}, r^2_1, \dots, r^2_{m'}, r^3_1, \dots, r^3_{m''}, r^4_1, \dots, r^4_{m''}$—such that for any $g \in S_\infty,\ gN^d \cap N^c \neq \emptyset$ iff:

$$(*) \qquad \left(\bigwedge_{1\le k\le m'} g(r^1_k) = r^2_k\right) \bigwedge \left(\bigwedge_{1\le k\le m''} g(r^3_k) \neq r^4_k\right).$$

Let $m = 2m' + 2m''$; the above r's form the m-tuple whose existence is claimed. Let $\widehat{N}$ be the open set of all $g \in S_\infty$ which satisfy (*), that is, all $g \in S_\infty$ such that $gN_j \cap N_i \neq \emptyset$. By (d) in the definition of "enhanced basis" (2.B.2), for all $k \in \mathbb{N}$, $g_k^{\langle i,j\rangle} \in \widehat{N}$. And by Proposition 2.B.5 (a), $\{g_k^{\langle i,j\rangle} : k \in \mathbb{N}\}$ is dense in $\widehat{N}$. So for any $t_1, \dots, t_n \in \mathbb{N}^n$:

$$(\exists k \in \mathbb{N})(g_k^{\langle i,j\rangle}(s_1) = t_1 \wedge \dots \wedge g_k^{\langle i,j\rangle}(s_n) = t_n) \text{ iff}$$

$$(\exists g \in \widehat{N})(g(s_1) = t_1 \wedge \dots \wedge g(s_n) = t_n).$$

Let $\psi[\overline{r}, s_1, \dots, s_n, w_1, \dots, w_n]$ be the following formula:

$$\left(\bigwedge_{1 \le k < l \le n} (s_k = s_l \text{ iff } w_k = w_l) \right) \bigwedge$$

$$\left(\bigwedge_{\substack{1 \le l \le n \\ 1 \le k \le m'}} (s_l = r^1_k \text{ iff } w_l = r^2_k) \right) \bigwedge \left(\bigwedge_{\substack{1 \le l \le n \\ 1 \le k \le m''}} (\text{if } s_l = r^3_k \text{ then } w_l \ne r^4_k) \right).$$

This works, which proves Claim 2.

Fix m, $r_1, \dots, r_m$ and ψ satisfying Claim 2. Now using both claims, we can complete the proof.

$$\begin{aligned}
x \in \widehat{D} &\iff x \in \bigcup_{k \in \mathbb{N}} (g^{\langle i,j \rangle}_k \cdot D) \\
&\iff (\exists k \in \mathbb{N})(x \in g^{\langle i,j \rangle}_k \cdot D) \\
&\iff (\exists k \in \mathbb{N})(\mathfrak{A}_x \models \varphi[g^{\langle i,j \rangle}_k(s_1), \dots, g^{\langle i,j \rangle}_k(s_n)]) \\
&\iff (\exists (t_1, \dots, t_n) \in \mathbb{N}^n)[(\exists k \in \mathbb{N}) \\
&\quad (g^{\langle i,j \rangle}_k(s_1) = t_1 \wedge \dots \wedge g^{\langle i,j \rangle}_k(s_n) = t_n) \text{ and } \mathfrak{A}_x \models \varphi[t_1, \dots, t_n]] \\
&\iff (\exists (t_1, \dots, t_n) \in \mathbb{N}^n) \\
&\quad [\psi[r_1, \dots, r_m, s_1, \dots, s_n, t_1, \dots, t_n] \text{ and } \mathfrak{A}_x \models \varphi[t_1, \dots, t_n]] \\
&\iff (\exists (t_1, \dots, t_n) \in \mathbb{N}^n) \\
&\quad (\mathfrak{A}_x \models \psi[r_1, \dots, r_m, s_1, \dots, s_n, t_1, \dots, t_n] \text{ and } \mathfrak{A}_x \models \varphi[t_1, \dots, t_n]) \\
&\iff \mathfrak{A}_x \models (\exists w_1 \dots \exists w_n) \\
&\quad (\psi[r_1, \dots, r_m, s_1, \dots, s_n, w_1, \dots, w_n] \wedge \varphi(w_1, \dots, w_n)).
\end{aligned}$$

So $\widehat{D} = \mathrm{Mod}(\theta, \overline{q})$, where $\overline{q} = (r_1, \dots, r_m, s_1, \dots, s_n)$ and $\theta(u_1, \dots, u_m, v_1, \dots, v_n)$ is the following formula:

$$(\exists w_1 \dots \exists w_n)(\psi(u_1, \dots, u_m, v_1, \dots, v_n, w_1, \dots, w_n) \wedge \varphi(w_1, \dots, w_n)).$$

Since $\theta \in F$, $\widehat{D} \in \mathcal{B}_F$. $\square$

The definition of "enhanced Polish G-space" is rather technical and hard to motivate. (Of course, this definition makes the proof of the dichotomy theorems work, which may be motivation enough.) This concept can perhaps be motivated as follows: it is an attempt to generalize properties of the t_F-topology on X_L from the logic actions to other actions. Since F is a fragment, the basis $\mathcal{B}_F$ has very strong closure properties. Given any group G and any G-space X, one would like a topology t on X with a basis $\mathcal{C}$ which has similar strong closure properties. But that is impossible: the fact that F is closed under negation means that $\mathcal{B}_F$ is closed under complement; in general, one cannot get a basis closed under complement. (For example, if G is connected, then in a continuous action every orbit is connected, so if there are nontrivial orbits in X, the topology on X cannot have a clopen basis.) The closure property of the basis in part (d) of the definition of "enhanced Polish G-space" is, as shown above, a weak consequence of the stronger closure properties of $\mathcal{B}_F$. Thus the technical concept of an "enhanced Polish G-space" can be viewed as an attempt to get a basis for an arbitrary G-space with $\mathcal{B}_F$-like closure properties which are as strong as possible—but not stronger. (Note also that Proposition 2.C.4 generalizes well known properties of the logic actions with the t_F-topology.)

The logic actions are an important class of examples, since this paper is concerned with generalizing a concept from model theory, *elementary embeddings* (see §3.D), and generalizing some theorems of model theory (see 5.A.1 and 6.A.2). That is, we are generalizing a known concept and known theorems from logic actions to other actions. In the sequel, when introducing various ideas we will frequently refer back to the logic actions as concrete examples of these ideas. (While the logic actions are important for motivation, they are not used in the proofs of the results in this paper. In the formal sense, they are irrelevant.)

2.E. The G_δ orbits. In the case of Polish G-spaces, the following theorem is due to Effros [11, 2.1]. Another proof of Effros's Theorem can be found in Kechris [27, 3.1].

2.E.1. Theorem. *Let X be an acceptable G-space and let $x \in X$. If $G\cdot x$ is a G_δ subset of X, then the canonical map $g \mapsto g\cdot x$, from G onto $G\cdot x$, is open.*

Proof. The proof of Effros's Theorem (2.E.1 for *Polish* G-spaces) which is given in Kechris [27, 3.1] does not actually use the fact that X is Polish. That proof that the canonical map is open uses only the following three facts.

(a) $G\cdot x$ is a second countable Baire space.

(b) The action is continuous.

(c) The image of every open subset of G, under the canonical map, has the property of Baire with respect to the topological space $G\cdot x$.

Now (a) follows from parts (a) and (b) of 2.A.2, (b) is obvious, and (c) follows from 2.A.2 (d) (i). Hence that proof establishes 2.E.1. □

When X is Polish, the converse of 2.E.1 holds: if the canonical map from G onto $G\cdot x$ is open, then $G\cdot x$ is a G_δ.

Open Question. *Is the converse of 2.E.1 true?*

It can be shown that for any acceptable G-space X and any $x \in X$, the canonical map from G onto $G\cdot x$ is open iff $G\cdot x$ is a Polish space iff $G\cdot x$ is a Baire space iff $G\cdot x$ is not meager in itself. In this paper, we will never have occasion to consider any orbits of this sort which are not already known to be G_δ orbits.

2.E.2 Proposition. *Let X be an acceptable G-space. Any nonmeager orbit of X is a G_δ.*

Proof. Let $G\cdot x$ be a nonmeager orbit. By Proposition 2.A.2 (d) (i), $G\cdot x$ has the property of Baire, so there is an open neighborhood C such that $G\cdot x$ is comeager-in-C. Let $Y = G\cdot C$. Then $Y \neq \emptyset$, Y is an invariant open subset of X and $G\cdot x$ is comeager in Y. So there exists a set $Q \subset G\cdot x$ such that Q is a dense G_δ with respect to Y. Of course, Q is also a G_δ with respect to X, and, therefore, the Vaught transform Q^* is a G_δ with respect to X. So it will suffice to show that $Q^* = G\cdot x$. This follows from the Kuratowski-Ulam Theorem (as in the proof of Effros's Theorem given in Kechris [27, 3.1]). For this argument to work, we need to know the following two facts.

(a) Y is a second countable Baire space.

(b) The action is continuous.

Again, (a) follows from parts (a) and (b) of 2.A.2. □

The main theorem of this paper (4.A.2) is a theorem about G_δ orbits.

Recall the canonical partition of a Polish G-space X (Proposition 2.C.2). Clearly each piece of the partition contains at most one G_δ orbit. A piece—or indeed all of X—may contain no G_δ orbits; an example of this is the orbit equivalence relation known as E_0, which is defined in §5.B, below. Every orbit of X is a G_δ iff each piece of the canonical partition contains only one orbit.

We conclude §2.E by discussing the relationship between these group action notions and model theoretic notions. An L-structure $\mathfrak{A}$ is called *atomic* if every type realized in $\mathfrak{A}$ is principal. This definition of "atomic" can be generalized from finitary languages to countable fragments F of $L_{\omega_1\omega}$, giving us a concept of *F-atomic*.

2.E.3. Proposition (Suzuki [44], Miller [33]). *Let L be a countable relational language, let $x \in X_L$ and let F be a countable fragment of $L_{\omega_1\omega}$. The following are equivalent.*

(a) *$\mathfrak{A}_x$ is F-atomic.*

(b) *$S_\infty{\cdot}x$ is a G_δ orbit with respect to the t_F-topology on X_L.*

Recall that the t_F-topology (Definition 2.D.1) is Polish, hence acceptable. Thus the concept of a G_δ orbit is a way of generalizing the concept of an atomic model from the logic actions to arbitrary acceptable G-spaces. Indeed, one of the major theorems of this paper (Theorem 5.A.1) is a generalization of a known theorem about atomic models: atomic models are prime models.

The theorem of Effros (2.E.1) is also a version of a known theorem about atomic models: atomic models are homogeneous. To see this, consider a countable fragment F and an F-atomic model $\mathfrak{A}_x$, where $x \in X_L$. By Proposition 2.E.3, $S_\infty{\cdot}x$ is a t_F-G_δ orbit. So Theorem 2.E.1 asserts the following fact: for any open set $M \subset S_\infty$, $M{\cdot}x$ is t_F-open in $S_\infty{\cdot}x$. This fact can be established model theoretically as follows. It will suffice to prove it for sets M in the canonical basis $\{N^b\}$ for S_∞ (2.B.1 (d)). Let b be a bijection between finite subsets of $\mathbb{N}$, say $b : \{s_1, \dots , s_n\} \to \{t_1, \dots , t_n\}$, where $b(s_i) = t_i$, and consider $M = N^b$. Since $\mathfrak{A}_x$ is F-atomic, there is an F-formula $\varphi(v_1, \dots , v_n)$ which generates the n-type satisfied by $(s_1, \dots , s_n)$ in $\mathfrak{A}_x$. Since $\mathfrak{A}_x$ if F-homogeneous,

$$N^b{\cdot}x = \mathrm{Mod}(\varphi, (t_1, \dots , t_n)) \cap S_\infty{\cdot}x,$$

which is, of course, t_F-open in $S_\infty{\cdot}x$. For more on this topic, see Becker [4].

3. Left-invariant metrics

3.A. Metrics on Polish groups. A metric d on a group G is called *left-invariant* (respectively, *right-invariant*) when for all $h,\ g_1,\ g_2 \in G,\ d(hg_1, hg_2) = d(g_1, g_2)$ (respectively, $d(g_1h, g_2h) = d(g_1, g_2)$). If d is both left-invariant and right-invariant, we call it *two-sided-invariant*.

3.A.1 Theorem (Birkhoff-Kakutani; see Montgomery-Zippin [34], 1.22). *For any Polish group G—in fact, for any metrizable group G—there exists a left-invariant metric d on G compatible with the topology.*

There is, of course, a complete symmetry between left and right. (Remark: This symmetry refers only to the group itself, not to actions of the group.) Therefore, a Polish group G also has a compatible right-invariant metric. In fact, there is a canonical way to go from one to the other.

3.A.2. Proposition. *Let G be a group, let d be a metric on G and let d^* be the metric $d^*(g,h) = d(g^{-1}, h^{-1})$.*

(a) *d is compatible with the topology of G iff d^* is.*
(b) *d is complete iff d^* is.*
(c) *d is left-invariant iff d^* is right-invariant.*
(d) *d is right-invariant iff d^* is left-invariant.*

In general, a Polish group does not have any compatible two-sided-invariant metric. But some Polish groups, e.g., abelian groups, do have such metrics. These groups form an interesting subclass of the class of all Polish groups, which will be discussed in §3.C.

Some concrete examples of left- and right-invariant metrics will be given in 3.C.3.

A Polish group has a compatible left-invariant metric and also has a compatible complete metric, but does not, in general, have one compatible metric which is both left-invariant and complete; those groups that do, form another interesting subclass to be discussed in §3.C.

3.A.3. Proposition (Becker-Kechris [6], 1.2.2)**.** *Let G be a Polish group. If d is a compatible metric on G which is two-sided-invariant, then d is complete.*

3.A.4. Proposition. *Let G be a group, let d be a compatible metric on G and let d' be the metric $d'(g,h) = \dfrac{d(g,h)}{1+d(g,h)}$. Then d' is also a compatible metric on G and for any $g,\ h \in G$, $d'(g,h) < 1$. Furthermore:*

(a) *if d is complete, so is d';*
(b) *if d is left-invariant, so is d';*
(c) *if d is right-invariant, so is d'.*

In this paper, a *metric* on G always means a metric compatible with the topology. The letter d—sometimes with indices—always denotes such a metric. In light of 3.A.4, we may always assume that our metrics give G diameter 1.

In this paper, whenever we have an action of G, we work with left-invariant, rather than right-invariant, metrics. Of course, there is a difference between left and right, since $g{\cdot}(h{\cdot}x) = gh{\cdot}x$, but, in general, $g{\cdot}(h{\cdot}x) \neq hg{\cdot}x$. The generalization of the concept of "elementary embedding" is defined using left-invariant metrics (Definition 3.D.1). One could, of course, give the same definition with "left" replaced by "right", but it would define something different, something about which we have nothing to say in this paper.

3.B. Cauchy sequences.

3.B.1. Proposition. *Let G be a group, let d and d' be compatible left-invariant metrics on G and let g_n be a sequence from G. The sequence g_n is d-Cauchy iff it is d'-Cauchy.*

Proof. Suppose g_n is d-Cauchy but not d'-Cauchy. By passing to a subsequence, we may assume that there is an $\epsilon > 0$ such that for all n, $d'(g_{n+1}, g_n) > \epsilon$. For all $n \in \mathbb{N}$, let $h_n = g_n^{-1} g_{n+1}$. Using the left-invariance of both metrics, we have that:

$$\begin{aligned}\lim_{n\to\infty} d(h_n, e) &= \lim_{n\to\infty} d(g_n^{-1} g_{n+1}, g_n^{-1} g_n)\\ &= \lim_{n\to\infty} d(g_{n+1}, g_n)\\ &= 0;\end{aligned}$$

but for all n,

$$\begin{aligned} d'(h_n, e) &= d'(g_n^{-1}g_{n+1}, g_n^{-1}g_n) \\ &= d'(g_{n+1}, g_n) \\ &> \epsilon. \end{aligned}$$

That is, h_n converges to e in the d-metric but not in the d'-metric. This is absurd, since both metrics give the same topology. $\square$

We call a sequence from G *ι-Cauchy* if it is d-Cauchy for some (equivalently, for any) left-invariant metric d on G.

3.B.2. Corollary. *Let G be a group. If G has a complete left-invariant metric, then every left-invariant metric on G is complete.*

3.B.3. Proposition. *Let G be a group, let g_n and h_n be two ι-Cauchy sequences from G, and for all $n \in \mathbb{N}$, let $k_n = g_n h_n$. Then the sequence k_n is also ι-Cauchy.*

Proof. Let d be a left-invariant metric on G. We first prove the following special case of this proposition.

Claim. Let g_n be an ι-Cauchy sequence from G and let $h \in G$. Then the sequence $g_n h$ is also ι-Cauchy.

Proof of Claim. Given $\epsilon > 0$; we must find a $q \in \mathbb{N}$ such that for all $m,\ n \geq q,\ d(g_m h, g_n h) < \epsilon$. To do this, first fix $\delta > 0$ such that for all $g \in G$:

$$d(g, e) < \delta \Rightarrow d(h^{-1}gh, e) < \epsilon.$$

Then let q be large enough so that for $m,\ n \geq q,\ d(g_m, g_n) < \delta$. We show that this q works. Fix $m,\ n \geq q$.

$$\begin{aligned} d(g_n^{-1}g_m, e) &= d(g_n g_n^{-1} g_m, g_n e) \\ &= d(g_m, g_n) \\ &< \delta. \end{aligned}$$

Therefore, $d(h^{-1}(g_n^{-1}g_m)h, e) < \epsilon$. So

$$\begin{aligned} d(g_m h, g_n h) &= d((g_n h)^{-1}(g_m h), (g_n h)^{-1}(g_n h)) \\ &= d(h^{-1}(g_n^{-1}g_m)h, e) \\ &< \epsilon. \end{aligned}$$

This proves the Claim.

To prove the general proposition, fix $\epsilon > 0$; we must find q such that for all $m,\ n \geq q,\ d(k_m, k_n) < \epsilon$. Let r be large enough so that for all $m,\ n \geq r,\ d(h_m, h_n) < \epsilon/4$. For all $n \in \mathbb{N}$, let $g'_n = g_n h_r$. By the Claim, g'_n is an ι-Cauchy sequence. So there exists a $q \geq r$ such that for all $m,\ n \geq q,\ d(g'_m, g'_n) < \epsilon/4$. We show that this q works. Fix $m,\ n \geq q$.

$$\begin{aligned} d(k_m, k_n) &= d(g_m h_m, g_n h_n) \\ &\leq d(g_m h_m, g_m h_r) + d(g_m h_r, g_q h_r) + d(g_q h_r, g_n h_r) + d(g_n h_r, g_n h_n) \\ &= d(h_m, h_r) + d(g'_m, g'_q) + d(g'_q, g'_n) + d(h_r, h_n) \\ &< \epsilon/4 + \epsilon/4 + \epsilon/4 + \epsilon/4 \\ &= \epsilon. \end{aligned}$$

$\square$

While the left-invariant metrics on G are not, in general, right-invariant, the Claim in the above proof shows that multiplication on the right cannot distort distances too badly. In constrast to this claim, it is not true that if g_n is an ι-Cauchy sequence then the sequence g_n^{-1} must also be ι-Cauchy. Indeed, this is equivalent to saying that the left-invariant metrics on G are complete (see Becker-Kechris [6, 1.2.2]).

3.C. The cli groups. In §3.C, we discuss various subclasses of the class of all Polish groups. There are those classes of groups which are algebraically nice: abelian, nilpotent, solvable. (For definitions, see, for example, Rotman [37].) There are those classes of groups which are topologically small: discrete (= countable), compact, locally compact. And finally, we consider two classes of groups which are defined by metric properties: the tsi groups and the cli groups. These are two classes which apparently have not been considered (or named) until quite recently. But recent work (e.g., Hjorth-Solecki [22] and Gao [15]) seems to indicate that they are very natural and important classes, which merit further study. A *tsi group* is a Polish group which has a two-sided-invariant metric. A *cli group* is a Polish group which has a complete left-invariant metric.

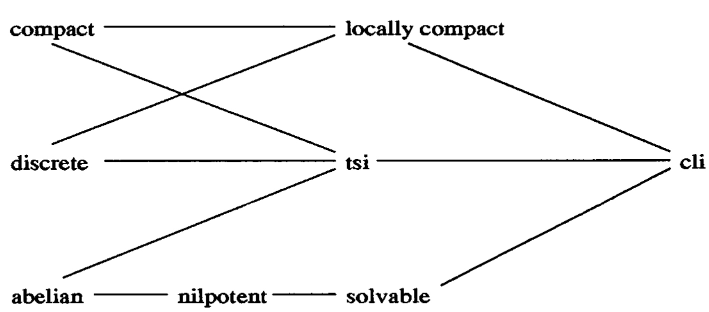
Our main interest is in the class of cli groups. Most of this section of the paper is concerned with the question of which Polish groups are cli and which are not.

We begin with closure properties. If G is a cli group and H is a closed subgroup of G, then H is cli. If G is a cli group and H is a closed normal subgroup of G, then G/H is cli; equivalently, a Polish group which is the continuous homomorphic image of a cli group is cli. And the class of cli groups is closed under finite or countable products. The first of the above closure properties is obvious, and the third closure property follows easily from Proposition 3.A.4. The second closure property is a portion of the following result.

3.C.1. Theorem (Gao [15]). *Let G be a Polish group and let H be a closed normal subgroup of G. The following are equivalent.*

(a) *G is cli.*
(b) *Both H and G/H are cli.*

The following diagram shows the inclusions between all of the classes of groups mentioned above.



A line connecting two classes of groups means that the class on the left is included in the class on the right. All inclusions shown are proper, and there are no other inclusions (even for intersections and unions of these classes). For the classes of groups other than the two new classes of tsi and cli groups, this is all well known. The next proposition justifies these facts for the new parts of the diagram.

3.C.2. Proposition. (a) *All compact groups are tsi.*

(b) *All discrete groups are tsi.*

(c) *All abelian groups are tsi.*

(d) *All locally compact groups are cli.*

(e) *All tsi groups are cli.*

(f) *All solvable groups are cli.*

(g) *There exists a group G_1 such that:*

(i) *G_1 is tsi;*

(ii) *G_1 is not locally compact;*

(iii) *G_1 is not solvable.*

(h) *There exists a group G_2 such that:*

(i) *G_2 is locally compact;*

(ii) *G_2 is nilpotent;*

(iii) *G_2 is not tsi.*

(i) *There exists a group G_3 such that:*

(i) *G_3 is cli;*

(ii) *G_3 is not locally compact;*

(iii) *G_3 is not tsi;*

(iv) *G_3 is not solvable.*

Proof. (a) Let G be compact. By Theorem 3.A.1, there exists a compatible left-invariant metric d on G. Define $\hat{d} : G \times G \to \mathbb{R}$ as follows:

$$\hat{d}(g_1, g_2) = \sup_{h \in G} d(g_1 h, g_2 h).$$

Compactness ensures that the supremum exists and is realized by some $h \in G$. Then $\hat{d}$ is a metric on G and is two-sided-invariant; the verification of this is straightforward. It remains to be shown that the $\hat{d}$-topology is the same as the d-topology. Since the d-topology is compact, it will suffice to show that the d-topology is finer than the $\hat{d}$-topology. Let N be the open ball about g_0 of radius ϵ, with respect to $\hat{d}$. $G \setminus N$ is the projection of the d-compact set $\{(g, h) \in G \times G : d(gh, g_0 h) \geq \epsilon\}$, hence is d-compact. So N is d-open.

(b) Let G be discrete. Let d be the trivial metric on G: $d(g, h) = 1$ if $g \neq h$. Then d is two-sided-invariant.

(c) Let G be abelian. Again using Theorem 3.A.1, there exists a left-invariant metric on G. Obviously, such a metric is two-sided-invariant.

(d) Let G be locally compact and, using 3.A.1 one more time, let d be a left-invariant metric on G. It will suffice to show that any d-Cauchy sequence from G has a convergent subsequence. Let g_n be a d-Cauchy sequence. There exists an $\epsilon > 0$ such that $K = \{h \in G : d(h, e) \leq \epsilon\}$ is compact. Let $m \in \mathbb{N}$ be such that for all $n > m$, $d(g_n, g_m) < \epsilon$. Let $K' = g_m K$. For all $n \geq m$:

$$\begin{aligned} d(g_m^{-1} g_n, e) &= d(g_m g_m^{-1} g_n, g_m e) \\ &= d(g_n, g_m) \\ &< \epsilon; \end{aligned}$$

hence $g_m^{-1} g_n \in K$ and, therefore, $g_n \in g_m K = K'$. As K' is compact, the sequence g_n has a convergent subsequence.

(e) Proposition 3.A.3.

(f) By (c) and (e), all abelian groups are cli. Now an easy induction on the length of the derived series, using this fact and Theorem 3.C.1, shows that any solvable group is cli. (Remark: Part (f) of 3.C.2 is due to Hjorth and Solecki, who proved this result prior to Gao's discovery of Theorem 3.C.1.)

(g) There is a fixed tsi group—which we call G_1—which is *surjectively universal* for the class of tsi groups. That is, a Polish group is a tsi group iff it is the continuous homomorphic image of G_1. G_1 is obtained as follows: start with the free group F on continuum many generators, with the Graev metric; G_1 is the completion of that metric group. For details, see Kechris [27, 2.11]. By (c), there exist tsi groups which are not σ-compact, e.g., $\mathbb{Z}^{\mathbb{N}}$; as these groups are the continuous image of G_1, G_1 is not σ-compact, hence not locally compact. A subgroup of a solvable group is solvable, and so is a homomorphic image of a solvable group (see Rotman [37, 6.11 and 6.12]). Therefore, F is not solvable, and since G_1 contains a copy of F, neither is G_1.

(h) Let G_2 be the group of all 3×3 matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix},$$

where a, b, $c \in \mathbb{R}$. G_2 is a nilpotent group of class 2. (This can be seen by direct computation. The commutator subgroup of G_2 consists of all matrices of the form

$$\begin{pmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $d \in \mathbb{R}$, which is abelian.) G_2 is obviously locally compact. It is easy to see that if G is any tsi group and $g_n,\ h_n$ are any two sequences from G, then:

$$\lim_{n\to\infty} g_n h_n = e \Rightarrow \lim_{n\to\infty} h_n g_n = e.$$

So to prove that G_2 is not tsi, we need only find $g_n,\ h_n \in G_2$ violating the above property. Let

$$g_n = \begin{pmatrix} 1 & n & 0 \\ 0 & 1 & 1/n \\ 0 & 0 & 1 \end{pmatrix} \text{ and } h_n = \begin{pmatrix} 1 & -n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(i) Let $G_3 = G_1 \oplus G_2$. □

The above facts make the class of cli groups appear to be a very large class. Next we present those facts which make the class of cli groups appear to be a very small class. Recall Corollary 3.B.2 and Proposition 3.A.2, which give us an easy way to show that a particular group G is not cli: to show that G is not cli, we need only find one metric d on G such that d is either left-invariant or right-invariant but d is not complete.

3.C.3. Examples. We give some examples of cli and non-cli groups.

(a) Let d be the usual complete metric on $\mathbb{N}^{\mathbb{N}}$:

$$d(\mathbf{p}, \mathbf{q}) = \frac{1}{1 + (\text{least } n \text{ such that } \mathbf{p}(n) \neq \mathbf{q}(n))}.$$

Recall (2.B.1 (d)) that S_∞ is topologized as a G_δ-subspace of $\mathbb{N}^\mathbb{N}$. Thus we obtain a compatible metric on S_∞ (or any subgroup of S_∞) by restricting d. We also denote this restriction by d, and call it the *canonical metric* on S_∞. It is easy to see that d is left-invariant. A sequence g_n from S_∞ is d-Cauchy iff it converges *in the space* $\mathbb{N}^\mathbb{N}$. The limit of a convergent (in $\mathbb{N}^\mathbb{N}$, i.e., pointwise convergent) sequence of permutations must be a one-one function, but it need not be onto. Such a sequence from S_∞ is d-Cauchy but not d-convergent *in the subspace* S_∞. Therefore, S_∞ is not cli. This same argument serves to characterize which closed subgroups of S_∞ are cli: a subgroup G of S_∞ is cli iff G is closed in $\mathbb{N}^\mathbb{N}$. There is also a model theoretic characterization, due to Gao and Hjorth (see Gao [15]), proved using the above characterization. For any countable language L and any countably infinite L-structure $\mathfrak{A}$, the automorphism group $\mathrm{Aut}(\mathfrak{A})$ is a closed subgroup of S_∞. Moreover, every closed subgroup of S_∞ is of the form $\mathrm{Aut}(\mathfrak{A})$, for some such $\mathfrak{A}$ (which is not unique); see Becker-Kechris [6, 1.5]. The Gao-Hjorth Theorem is: for any countably infinite L-structure $\mathfrak{A}$, $\mathrm{Aut}(\mathfrak{A})$ is cli iff the Scott sentence of $\mathfrak{A}$ does not have an uncountable model.

(b) Let K be a compact metrizable space, and consider the group $H(K)$ of homeomorphisms of K (2.B.1 (e)). $H(K)$ is topologized as a subspace of $C(K,K)$. Let d be the usual complete metric on $C(K,K)$, that is, the sup-norm metric:

$$d(f_1,f_2)=\sup\{\hat{d}(f_1(\mathbf{p}),f_2(\mathbf{p})):\mathbf{p}\in K\},$$

where $\hat{d}$ is a complete metric on K. The restriction of d to $H(K)$ is right-invariant. Thus we obtain the following characterization theorem of Gao [15]: for any compact metrizable space K, and any subgroup G of $H(K)$, G is cli iff G is closed in $C(K,K)$. $H(K)$ itself may or may not be cli, depending on what K is. Consider the case where K is the Hilbert cube, $[0,1]^\mathbb{N}$. By a theorem of Uspenskii (see Kechris [28, 9.18]), $H([0,1]^\mathbb{N})$ is a *universal* Polish group. That is, every Polish group is a closed subgroup of $H([0,1]^\mathbb{N})$. The theorems of Uspenskii and Gao, put together, give us a way of characterizing, for arbitrary Polish groups G, whether G is cli: G is cli iff G is a subgroup of $H([0,1]^\mathbb{N})$ that is closed in $C([0,1]^\mathbb{N},[0,1]^\mathbb{N})$. (Remark: Uspenskii's Theorem, together with the fact that d is a right-invariant metric on $H([0,1]^\mathbb{N})$, gives us another proof of the Birkhoff-Kakutani Theorem—Theorem 3.A.1—for Polish groups.)

(c) Consider the unitary groups $U(n)$ and $U(\mathcal{H})$ (see 2.B.1 (a) and (c)). $U(n)$ is compact, hence cli. $U(\mathcal{H})$ contains a closed copy of S_∞, hence it is not cli.

Open Questions. (a) *Is there a group which is universal for the class of cli groups?*

(b) *Is there a group which is surjectively universal for the class of cli groups?*

The terms in the above questions are defined in 3.C.3 (b) and in the proof of 3.C.2 (g). The class of all Polish groups has a universal element, namely $H([0,1]^\mathbb{N})$, and the class of groups admitting a countable basis closed under multiplication also has a universal element, namely S_∞ (Becker-Kechris [6, 1.5.1]). Question (a) asks whether there is a similar characterization theorem for cli groups. The class of tsi groups has a surjectively universal element, namely G_1, and the class of abelian groups also has a surjectively universal element, namely G_1/H, where H is the closure of the commutator subgroup of G_1 (see Kechris [27, 2.11]). It is also open whether the class of all Polish groups has a surjectively universal element.

3.D. Generalized elementary embeddings. Given a countable relational language L, a countable fragment F of $L_{\omega_1\omega}$, and two (not necessarily countable) L-structures $\mathfrak{A} = \langle A, \dots \rangle$ and $\mathfrak{B} = \langle B, \dots \rangle$ (that is, $\mathfrak{A}$ is a collection of relations on the underlying set A, and similarly for $\mathfrak{B}$ and B), an *F-embedding* of $\mathfrak{A}$ into $\mathfrak{B}$ is a one-one function $I : A \to B$ such that for every $m \in \mathbb{N}$, for every formula $\varphi(v_1, \dots, v_m)$ of F with m free variables, and for every $(a_1, \dots, a_m) \in A^m$,

$$\mathfrak{A} \models \varphi[a_1, \dots, a_m] \text{ iff } \mathfrak{B} \models \varphi[I(a_1), \dots, I(a_m)].$$

(See §2.D for notation and terminology.) In case F is finitary logic, an F-embedding is called an *elementary embedding*. $\mathfrak{A}$ is *F-embeddable* (respectively, *elementarily embeddable*) into $\mathfrak{B}$ if there exists an F-embedding (respectively, elementary embedding) of $\mathfrak{A}$ into $\mathfrak{B}$.

Elementary embeddability is one of the fundamental concepts of model theory (or rather, of finitary, first-order model theory, which is the most important part of model theory). If one identifies branches of mathematics with categories, then model theory is about the category whose objects are L-structures and whose morphisms are elementary embeddings. F-embeddability is a generalization of elementary embeddability. We wish to further generalize the concept: from logic actions with the t_F-topology to other topological G-spaces. (The t_F-topology is defined in 2.D.1.) We first give the definition of "$\imath$-embeddability", and then show (3.D.2, below) that what we have defined is, indeed, a generalization of "elementary embeddability".

3.D.1. Definition. Let G be a Polish group, let X be a topological G-space, let $x,\ y \in X$ and let g_n be a sequence of elements of G. We call the sequence g_n an *$\imath$-embedding of x into y* if g_n is $\imath$-Cauchy and $g_n \cdot x$ converges to y. We say that *x is $\imath$-embeddable into y* if there exists an $\imath$-embedding of x into y.

This definition obviously depends on the topology of X.

3.D.2. Theorem. *Let L be a countable relational language, let F be a countable fragment of $L_{\omega_1\omega}$ and let $x,\ y \in X_L$. The following are equivalent.*

(a) *x is $\imath$-embeddable into y with respect to the t_F-topology on X_L.*

(b) *$\mathfrak{A}_x$ is F-embeddable into $\mathfrak{A}_y$.*

Proof. (a) $\Longrightarrow$ (b). Let g_n be a sequence from S_∞ which is $\imath$-Cauchy and such that $g_n \cdot x$ t_F-converges to y. Let d be the canonical metric on S_∞ (3.C.3 (a)). As d is left-invariant, g_n is d-Cauchy. By definition of d, this means that for every fixed $j \in \mathbb{N},\ g_n(j)$ is constant for sufficiently large n. Define $I : \mathbb{N} \to \mathbb{N}$ by

$$I(j) = g_n(j), \text{ for large } n.$$

Clearly, I is a one-one function from the underlying set of $\mathfrak{A}_x$ into the underlying set of $\mathfrak{A}_y$. All that remains to be shown is that formulas of F are preserved in going from $\mathfrak{A}_x$ to $\mathfrak{A}_y$ by I.

So fix a formula $\varphi(v_1, \dots, v_m)$ of F with m free variables, and fix $s_1, \dots, s_m$ in $\mathbb{N}$. We must show that:

$$\mathfrak{A}_x \models \varphi[s_1, \dots, s_m] \Longrightarrow \mathfrak{A}_y \models \varphi[I(s_1), \dots, I(s_m)].$$

Let $P = \{z \in X_L : \mathfrak{A}_z \models \varphi[I(s_1), \dots, I(s_m)]\}$. Note that P is t_F-closed. We now have:

$$\begin{aligned}
\mathfrak{A}_x \models \varphi[s_1, \dots, s_m] &\Longrightarrow \text{ for all } g \in S_\infty,\ \mathfrak{A}_{(g\cdot x)} \models \varphi[g(s_1), \dots, g(s_m)] \\
&\Longrightarrow \text{ for all } n,\ \mathfrak{A}_{(g_n\cdot x)} \models \varphi[g_n(s_1), \dots, g_n(s_m)] \\
&\Longrightarrow \text{ for all sufficiently large } n,\ \mathfrak{A}_{(g_n\cdot x)} \models \varphi[I(s_1), \dots, I(s_m)] \\
&\Longrightarrow \text{ for all sufficiently large } n,\ g_n\cdot x \in P \\
&\Longrightarrow y \in P \\
&\Longrightarrow \mathfrak{A}_y \models \varphi[I(s_1), \dots, I(s_m)].
\end{aligned}$$

(b) $\Longrightarrow$ (a). Let $I : \mathbb{N} \to \mathbb{N}$ be an F-embedding of $\mathfrak{A}_x$ into $\mathfrak{A}_y$. Let $C_0, C_1, C_2, \dots$ be a decreasing sequence of t_F-open sets such that $\bigcap_n C_n = \{y\}$.

Claim. For all $n \in \mathbb{N}$, there exists a $g_n \in S_\infty$ such that:

(i) for all $k < n$, $g_n(k) = I(k)$;

(ii) $g_n\cdot x \in C_n$.

Proof of Claim. Fix n. By definition of the t_F-topology (2.D.1), there is a formula $\varphi(v_1, \dots, v_m, w_1, \dots, w_n)$ in $m + n$ free variables (for some $m \in \mathbb{N}$), with $\varphi \in F$, and there are m *distinct* numbers $s_1, \dots, s_m$ in $\mathbb{N} \setminus \{I(0), \dots, I(n-1)\}$ such that if

$$C'_n = \{z \in X_L : \mathfrak{A}_z \models \varphi[s_1, \dots, s_m, I(0), \dots, I(n-1)]\},$$

then $y \in C'_n \subset C_n$. Since $y \in C'_n$, $\mathfrak{A}_y \models \varphi[s_1, \dots, s_m, I(0), \dots, I(n-1)]$. Therefore, $\mathfrak{A}_y \models \psi[I(0), \dots, I(n-1)]$, where $\psi(w_1, \dots, w_n)$ is the following formula:

$$(\exists v_1 \dots \exists v_m)\Bigg[\Bigg(\bigwedge_{1\le i<j\le m} v_i \neq v_j\Bigg) \bigwedge \Bigg(\bigwedge_{\substack{1\le i\le m\\ 1\le j\le n}} v_i \neq w_j\Bigg) \bigwedge \varphi(v_1, \dots, v_m, w_1, \dots, w_n)\Bigg].$$

Since $\varphi \in F$ and F is a fragment, $\psi \in F$; and $I : \mathbb{N} \to \mathbb{N}$ is an F-embedding of $\mathfrak{A}_x$ into $\mathfrak{A}_y$; hence $\mathfrak{A}_x \models \psi[0, \dots, n-1]$. Therefore, by definition of ψ, there exist m *distinct* numbers $t_1, \dots, t_m$ in $\mathbb{N}\setminus\{0, \dots, n-1\}$ such that $\mathfrak{A}_x \models \varphi[t_1, \dots, t_m, 0, \dots, n-1]$. Clearly there exists a $g_n \in S_\infty$ such that

$$g_n \restriction \{0, \dots, n-1\} = I \restriction \{0, \dots, n-1\}$$

and for $j \in \{1, \dots, m\}$, $g_n(t_j) = s_j$. Fix such a g_n. Then (i) of the Claim is obvious. To prove (ii) of the Claim, note that:

$$\begin{aligned}
&\mathfrak{A}_x \models \varphi[t_1, \dots, t_m, 0, \dots, n-1] \\
&\quad\Longrightarrow \mathfrak{A}_{(g_n\cdot x)} \models \varphi[g_n(t_1), \dots, g_n(t_m), g_n(0), \dots, g_n(n-1)] \\
&\quad\Longrightarrow \mathfrak{A}_{(g_n\cdot x)} \models \varphi[s_1, \dots, s_m, I(0), \dots, I(n-1)] \\
&\quad\Longrightarrow g_n\cdot x \in C'_n \\
&\quad\Longrightarrow g_n\cdot x \in C_n.
\end{aligned}$$

This proves the Claim.

By part (i) of the Claim, g_n is d-Cauchy, where d is the canonical metric on S_∞; therefore, g_n is $\imath$-Cauchy. By part (ii) of the Claim and the definition of the C_n's, $g_n\cdot x$ t_F-converges to y. So g_n is an $\imath$-embedding of x into y. □

If F is finitary logic, then Theorem 3.D.2 states that x is ι-embeddable into y with respect to t_F iff $\mathfrak{A}_x$ is elementarily embeddable into $\mathfrak{A}_y$. Thus the concept of ι-embeddability is a generalization of the concept of elementary embeddability, as promised. We are not claiming that the concept of ι-embedding is a generalization of the concept of elementary embedding. There are some serious flaws with that analogy. For one thing, there does not seem to be any way to compose ι-embeddings (but see 3.D.4 (b), below); they cannot be regarded as morphisms.

Other concrete examples of ι-embeddability will be given in 3.D.6. The terminology "ι-embeddability" is motivated by the special case of logic actions (Theorem 3.D.2), and may be inappropriate in general. As some of these other examples will show, there is, in general, nothing actually being "embedded". Before giving these interesting concrete examples, we consider the very uninteresting example of Proposition 3.D.3, below, and then consider the properties of ι-embeddability. Note that if, in the definition of "x is ι-embeddable into y" (3.D.1), "ι-Cauchy" had been replaced by "convergent", we would have defined the following: *x is in the same orbit as y.* For cli groups, ι-Cauchy sequences are convergent, which proves the following.

3.D.3. Proposition. *Let G be a cli group, let X be a topological G-space, and let x, $y \in X$. The following are equivalent.*

(a) *x is ι-embeddable into y.*

(b) *x is in the same orbit as y.*

3.D.4. Proposition. *Let G be a group and let X be an acceptable G-space.*

(a) *For all x, $y \in X$, if $y \in G\cdot x$, then x is ι-embeddable into y.*

(b) *For all x, y, $z \in X$, if x is ι-embeddable into y and y is ι-embeddable into z, then x is ι-embeddable into z.*

(c) *For all x, y, x', $y' \in X$, if x is ι-embeddable into y, $x' \in G\cdot x$ and $y' \in G\cdot y$, then x' is ι-embeddable into y'.*

Proof. (a) Let $y = h\cdot x$. For all $n \in \mathbb{N}$, let $g_n = h$. Then the constant sequence g_n is an ι-embedding of x into y.

(b) Let the sequences h_n and g_n be ι-embeddings of x into y and of y into z, respectively. Let C_0, C_1, $C_2, \dots$ be a decreasing sequence of open subsets of X such that $\bigcap_n C_n = \{z\}$. Let $\alpha : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function such that for all n, $g_{\alpha(n)}\cdot y \in C_n$.

Since the action is continuous, for each $n \in \mathbb{N}$, there is an open neighborhood D_n of y such that $g_{\alpha(n)}\cdot D_n \subset C_n$. Let $\beta : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function such that for all n, $h_{\beta(n)}\cdot x \in D_n$.

For $n \in \mathbb{N}$, let $k_n = g_{\alpha(n)}h_{\beta(n)} \in G$. Now $g_{\alpha(n)}$ and $h_{\beta(n)}$ are subsequences of the ι-Cauchy sequences g_n and h_n, so clearly they are ι-Cauchy. By Proposition 3.B.3, the sequence k_n is also ι-Cauchy. Since

$$\begin{aligned} k_n\cdot x &= (g_{\alpha(n)}h_{\beta(n)})\cdot x \\ &= g_{\alpha(n)}\cdot(h_{\beta(n)}\cdot x) \\ &\in g_{\alpha(n)}\cdot D_n \\ &\subset C_n, \end{aligned}$$

clearly $k_n\cdot x$ converges to z. So the sequence k_n is an ι-embedding of x into z.

(c) This follows from (a) and (b). ∎

In light of 3.D.4 (c), the property of being ι-embeddable is really a property of the orbits, not of the points. So we can speak of one orbit being *ι-embeddable* into another orbit. Thus ι-embeddability is a binary relation on orbits, a relation which, by 3.D.4, is reflexive and transitive. It is not, in general, either symmetric or antisymmetric, as can be seen in the special case of elementary embeddings.

3.D.5. Proposition. *Let X be a Polish G-space, and let $G \cdot x$ and $G \cdot y$ be two orbits of X. If there exists a G_δ set $Y \subset X$ such that $G{\cdot}y \subset Y$ and $G{\cdot}x \cap Y = \emptyset$, then $G{\cdot}x$ is not ι-embeddable into $G{\cdot}y$. Therefore:*

(a) *if $G{\cdot}x$ and $G{\cdot}y$ are in different pieces of the canonical partition of X, then $G{\cdot}x$ is not ι-embeddable into $G{\cdot}y$;*

(b) *if $G{\cdot}y$ is a G_δ orbit and $G{\cdot}x \neq G{\cdot}y$, then $G{\cdot}x$ is not ι-embeddable into $G{\cdot}y$.*

Proof. Let $Y = \bigcap_{i\in\mathbb{N}} C_i$, each C_i open. For some fixed $i \in \mathbb{N}$—call it i_0—the closed set $Q = \{g \in G : g{\cdot}x \notin C_{i_0}\}$ is nonmeager. Let N be a nonempty open subset of Q. Let $h \in N$. Let d be a left-invariant metric on G. Let $\epsilon > 0$ be such that for all $g \in G$, if $d(g, h) < \epsilon$ then $g \in N$.

Assume, towards a contradiction, that g_n is an ι-embedding of x into y. Then g_n is ι-Cauchy and $g_n{\cdot}x$ converges to y. Fix $m \in \mathbb{N}$ such that for all $n \geq m$, $d(g_n, g_m) < \epsilon$. Let $h' = hg_m^{-1}$ and for all $n \in \mathbb{N}$, let $g'_n = h'g_n$. Then g'_n is also an ι-Cauchy sequence, and

$$g'_n{\cdot}x = (h'g_n){\cdot}x = h'{\cdot}(g_n{\cdot}x),$$

so the sequence $g'_n{\cdot}x$ converges to $h'{\cdot}y$. As $h'{\cdot}y \in G{\cdot}y \subset Y \subset C_{i_0}$, for all sufficiently large n, $g'_n{\cdot}x \in C_{i_0}$. But for $n \geq m$:

$$\begin{aligned} d(g'_n, h) &= d(h'g_n, h) \\ &= d((hg_m^{-1})g_n, h) \\ &= d((hg_m^{-1})^{-1}(hg_m^{-1})g_n, (hg_m^{-1})^{-1}h) \\ &= d(g_n, g_m) \\ &< \epsilon; \end{aligned}$$

so $g'_n \in N \subset Q$, hence $g'_n{\cdot}x \notin C_{i_0}$. □

In the case of logic actions, 3.D.5 (a) is the following trivial fact: if $\mathfrak{A}_x \not\equiv \mathfrak{A}_y$, then $\mathfrak{A}_x$ is not elementarily embeddable into $\mathfrak{A}_y$. And 3.D.5 (b) is the well known fact that if $\mathfrak{A}_y$ is atomic and $\mathfrak{A}_x$ is elementarily embeddable into $\mathfrak{A}_y$, then $\mathfrak{A}_x \cong \mathfrak{A}_y$. (Similarly for countable fragments F.)

3.D.6. Examples. We consider a number of specific examples of G-spaces, and examine the question of ι-embeddability for their orbits.

(a) Let L be a language, and consider the S_∞-space X_L. For any $x, y \in X_L$, x is ι-embeddable into y with respect to the usual topology on X_L iff $\mathfrak{A}_x$ is Σ_1-embeddable into $\mathfrak{A}_y$. The proof is similar to the proof of Theorem 3.D.2.

(b) Fix a countable language L_0 and an L_0-structure $\mathfrak{A}_0$ with underlying set $\mathbb{N}$. The group $\mathrm{Aut}(\mathfrak{A}_0)$ is a closed subgroup of S_∞. Let L be a language disjoint from L_0 and let

$$Y = \{x \in X_{L_0\cup L} : \text{the } L_0\text{-reduct of } \mathfrak{A}_x \text{ is equal to } \mathfrak{A}_0\}.$$

The *relativized logic action* $J : \mathrm{Aut}(\mathfrak{A}_0) \times Y \to Y$ is obtained by restricting the domain of the logic action

$$J_{L_0 \cup L} : S_\infty \times X_{L_0 \cup L} \to X_{L_0 \cup L}.$$

Then for any $(L_0 \cup L)$-fragment F, (Y, J) is a Polish $\mathrm{Aut}(\mathfrak{A}_0)$-space with respect to the t_F-topology on Y. For x, $y \in Y$, x and y are in the same J-orbit iff there is an isomorphism from $\mathfrak{A}_x$ to $\mathfrak{A}_y$ which is an automorphism of their L_0-reducts $\mathfrak{A}_0$. For any countable $(L_0 \cup L)$-fragment F containing the Scott sentence of $\mathfrak{A}_0$, and for any x, $y \in Y$, the following are equivalent:

(i) x is ι-embeddable into y with respect to the $\mathrm{Aut}(\mathfrak{A}_0)$-space $(\langle Y, t_F\rangle, J)$;

(ii) $\mathfrak{A}_x$ is F-embeddable into $\mathfrak{A}_y$.

The proof is similar to the proof of Theorem 3.D.2, with one extra detail—in the Claim, g_n must be in $\mathrm{Aut}(\mathfrak{A}_0)$. That such a g_n can be found uses the Scott analysis (see, e.g., Barwise [1, VII, 6]): for any k-tuple $\overline{q}$ from $\mathfrak{A}_0$, there is an L_0-formula θ of F in k free variables such that if $\overline{r}$ is any k-tuple from $\mathfrak{A}_0$ which satisfies θ, then there is an automorphism of $\mathfrak{A}_0$ taking $\overline{q}$ to $\overline{r}$. This example is related to work of Gao [15]. If the fragment F does not contain the Scott sentence of $\mathfrak{A}_0$, then the meaning of "ι-embeddability" with respect to t_F is obscure.

(c) Consider S_∞ acting on itself by conjugation (see 2.C.1 (a) and (c)). The ι-embeddability relation on the orbits of this action is nontrivial. To see this, view S_∞ as the group of permutations of $\mathbb{Z}$ (rather than $\mathbb{N}$), and let h_1, h_2 be the following elements of $S_\infty : h_1(m) = m+1; h_2(m) = m+2$. Then h_1 is not in the same orbit as h_2. For any $n \in \mathbb{N}$, let g_n be some permutation of $\mathbb{Z}$ satisfying the following two properties.

(i) For all i with $-n \le i \le n,\ g_n(i) = 2i$.

(ii) For all j with $1 \le j \le 2n,\ g_n(n+j) = -2n+2j-1$.

The sequence g_n is an ι-embedding of h_1 into h_2.

(d) Suppose X is a Polish G-space and all of the orbits of X are G_δ; equivalently, each piece of the canonical partition contains only one orbit. Then by Proposition 3.D.5, the ι-embeddability relation on the orbits of X is trivial. This shows that there are many actions by non-cli groups with no nontrivial ι-embeddability, for example:

(i) the evaluation action of $H([0,1])$ on $[0,1]$ (see 2.C.1 (b) and 3.C.3 (b));

(ii) $U(\mathcal{H})$ acting on $\mathcal{H}$, where $\mathcal{H}$ has the norm topology;

(iii) $U(\mathcal{H})$ acting on the unit ball of $\mathcal{H}$ with the weak topology (see 2.C.1 (e) and 3.C.3 (c), for both (ii) and (iii)).

(e) Let $Y = \{\mathbf{p} \in 2^{\mathbb{N}} :$ for infinitely many $n \in \mathbb{N},\ \mathbf{p}(n) = 1\}$, and for all points $\mathbf{q} \in 2^{\mathbb{N}} \setminus Y$, let $m_{\mathbf{q}}$ denote the least m such that for all $n \ge m,\ \mathbf{q}(n) = 0$. Consider the following compact subset K of $2^{\mathbb{N}} \times [0,1]$:

$$(Y \times \{0\}) \cup \{(\mathbf{q}, r) : \mathbf{q} \notin Y \text{ and } r \le \frac{1}{1+m_{\mathbf{q}}}\}.$$

Consider the evaluation action of $H(K)$ on K. This action has exactly four orbits:

$$Y \times \{0\},\ (2^{\mathbb{N}} \setminus Y) \times \{0\},\ \{(\mathbf{q}, r) : 0 < r < \frac{1}{1+m_{\mathbf{q}}}\} \text{ and } \{(\mathbf{q}, r) : r = \frac{1}{1+m_{\mathbf{q}}}\}.$$

The first of these is G_δ, the second F_σ, the third open and the fourth a difference of two open sets. So Proposition 3.D.5 rules out all nontrivial ι-embeddability with one possible exception: perhaps $Y \times \{0\}$ is ι-embeddable into $(2^{\mathbb{N}} \setminus Y) \times \{0\}$. In fact, it is. Let $\mathbf{p} \in Y$ and let $\mathbf{q} \in 2^{\mathbb{N}} \setminus Y$; we sketch the proof that $(\mathbf{p}, 0)$ is ι-embeddable into

$(\mathbf{q}, 0)$. Let $\{I_n\}$ and $\{J_n\}$ be clopen neighborhood bases of $\mathbf{p}$ and $\mathbf{q}$, respectively. Recursively choose $h_n \in H(K)$ satisfying the following two properties.

(i) For all $(\mathbf{s}, r) \in K$, let $(\mathbf{s}', r')$ denote $h_n(\mathbf{s}, r)$; then for all $(\mathbf{s}, r) \in K$, $\mathbf{s} \in J_n \Leftrightarrow \mathbf{s}' \in I_n$.

(ii) $h_{n+1} \restriction (K \setminus (J_n \times [0,1])) = h_n \restriction (K \setminus (J_n \times [0,1]))$.

Since $\mathbf{p} \in Y$, (i) and (ii) imply that the sequence h_n is uniformly convergent. So if $g_n = h_n^{-1}$, then the sequence g_n is ι-Cauchy (see 3.A.2 (c) and 3.C.3 (b)). And (i) implies that $g_n \cdot (\mathbf{p}, 0) \in J_n \times \{0\}$, hence the sequence $g_n \cdot (\mathbf{p}, 0)$ converges to $(\mathbf{q}, 0)$.

(f) Let K be compact, let G be a closed subgroup of $H(K)$ and consider the evaluation action $a : G \times K \to K$. (Examples (a), (d) (i) and (e), above, are special cases of this.) For any $x \in K$, let $F(x) \subset C(K, K) \times K$ denote the closure in $C(K, K) \times K$ of $a^{-1}(x) \subset G \times K$. For any x, $y \in K$, the following are equivalent:

(i) x is ι-embeddable into y;

(ii) y is in the projection of $F(x)$.

This characterization follows easily from the ideas in 3.A.2 and 3.C.3 (b).

(g) Suppose we have a sequence of Polish groups, G_n, and for each n, we have a Polish G_n-space (X_n, a_n). Let $a : (\prod_n G_n) \times (\prod_n X_n) \to (\prod_n X_n)$ be the product action. Let $\overline{x} = (x_0, x_1, \dots)$ and $\overline{y} = (y_0, y_1, \dots)$ be points in this $\prod_n G_n$-space. Then $\overline{x}$ is ι-embeddable into $\overline{y}$ with respect to a iff for all n, x_n is ι-embeddable into y_n with respect to a_n. (See Proposition 3.A.4.)

(h) Let $a : G \times X \to X$ be an action. Suppose that τ and t are two topologies on X such that t is finer than τ and such that both $(\langle X, \tau\rangle, a)$ and $(\langle X, t\rangle, a)$ are topological G-spaces. Let $G \cdot x$ and $G \cdot y$ be two orbits of a. If $G \cdot x$ is ι-embeddable into $G \cdot y$ with respect to t, then $G \cdot x$ is also ι-embeddable into $G \cdot y$ with respect to τ.

For X a Polish G-space and $x \in X$, define

$$\mathbf{Emb}(x) = \{y \in X : x \text{ is } \iota\text{-embeddable into } y\}.$$

Clearly $\mathbf{Emb}(x)$ is an invariant subset of X. It is also clear from the definition that $\mathbf{Emb}(x)$ is a $\mathbf{\Sigma}^1_1$ set (cf. 3.D.6 (f)). The following example shows that $\mathbf{Emb}(x)$ need not be a Borel set.

3.D.7. Example. Let L be the language with one binary relation symbol, and consider the S_∞-space X_L. Let the fragment F be finitary logic, and consider the t_F-topology on X_L. In this situation, for any $x \in X_L$,

$$\mathbf{Emb}(x) = \{y \in X_L : \mathfrak{A}_x \text{ is elementarily embeddable into } \mathfrak{A}_y\}.$$

Let $x \in X_L$ be such that $\mathfrak{A}_x$ is a linear ordering of order-type $\omega_1^{CK}(1+\eta)$, where ω_1^{CK} denotes the least admissible ordinal and η denotes the order-type of the rational numbers. Then $\mathbf{Emb}(x)$ is not a Borel set. To see this, let $Y_1 \subset X_L$ be the set of (codes for) all admissible ordinals and let $Y_2 \subset X_L$ be the set of (codes for) all linear orderings of the form $\alpha(1 + \eta)$, for α an admissible ordinal. Friedman [13] proved that $Y_1 \cup Y_2$ is $\mathbf{\Sigma}^1_1$. By the Boundedness Theorem, Y_1 is not $\mathbf{\Sigma}^1_1$. Therefore, there is no Borel set $B \subset X_L$ such that $Y_1 \cap B = \emptyset$ and $Y_2 \subset B$. But $Y_1 \cap \mathbf{Emb}(x) = \emptyset$ and $Y_2 \subset \mathbf{Emb}(x)$. The former fact is obvious. The latter fact can be proved using Ehrenfeucht-Fraïssé games (see Hodges [23, 3.2]); or using Steel forcing (see Steel [41] or Harrington [17]—or, for a forcing-free, strictly topological approach to Steel forcing, see Becker-Dougherty [5]).

4. The Main Theorem

4.A. The theorem on the existence of $\imath$-embeddings.

4.A.1. Definition. Let G be a group and let $\mathcal{N}^* = \langle \{N_i\}, \{g_k^{\langle i,j\rangle}\}\rangle$ be an enhanced basis for G (see Definition 2.B.2). Let X be an acceptable G-space. For any open sets C, $D \subset X$ and any i, $j \in \mathbb{N}$, let $A(C, D, i, j)$ denote the following subset of X:

$$\left[C \setminus \left(\bigcup_{k\in\mathbb{N}} (g_k^{\langle i,j\rangle} \cdot D)\right)\right]^{\Delta}.$$

Let $\mathcal{C}$ be a basis for X. Let $G{\cdot}x$ and $G{\cdot}y$ be two orbits of X. We say that the pair $(\mathcal{N}^*, \mathcal{C})$ *differentiates* $G{\cdot}y$ *from* $G{\cdot}x$ if there exist sets C, $D \in \mathcal{C}$ and i, $j \in \mathbb{N}$ such that:

(a) $G{\cdot}y \subset A(C, D, i, j)$;
(b) $G{\cdot}x \cap A(C, D, i, j) = \emptyset$.

The above definition is not symmetric: it is possible that $(\mathcal{N}^*, \mathcal{C})$ differentiates $G{\cdot}y$ from $G{\cdot}x$ but does not differentiate $G{\cdot}x$ from $G{\cdot}y$.

4.A.2. Theorem. *Let G be a Polish group, let $\mathcal{N}^*$ be an enhanced basis for G, let X be an acceptable G-space and let $\mathcal{C}$ be a basis for X. Let $G{\cdot}x$ be a G_δ orbit of X and let $G{\cdot}y$ be an arbitrary orbit of X. If y is in the closure of $G{\cdot}x$ and $(\mathcal{N}^*, \mathcal{C})$ does not differentiate $G{\cdot}y$ from $G{\cdot}x$, then $G{\cdot}x$ is $\imath$-embeddable into $G{\cdot}y$.*

Theorem 4.A.2 is the main theorem of this paper. It will be proved in parts B and C of §4.

The most important special case of 4.A.2 is the case when X is a Polish G-space. In this case, for any C, $D \in \mathcal{C}$ and any i, $j \in \mathbb{N}$, $A(C, D, i, j)$ is F_σ. Now if $G{\cdot}x$ is any G_δ orbit and $G{\cdot}y$ is any other orbit, then trivially $G{\cdot}y$ can be separated from $G{\cdot}x$ by an invariant F_σ set: $X \setminus G{\cdot}x$. Theorem 4.A.2 states that if $y \in \text{closure}(G{\cdot}x)$ but $G{\cdot}x$ is not $\imath$-embeddable into $G{\cdot}y$, then $G{\cdot}y$ can be separated from $G{\cdot}x$ by a very special kind of invariant F_σ set: a set of the form $A(C, D, i, j)$.

In §§5, 6 we discuss applications of Theorem 4.A.2. This theorem has two types of consequences: generalizations of theorems of model theory, and dichotomy theorems for cli groups. In §5, we consider the case of Polish and Borel G-spaces and in §6, the case of analytic G-spaces. The material used in the proof of Theorem 4.A.2 is not needed to read §§5, 6.

4.B. Distinguishing an orbit from a given G_δ orbit. The proof of Theorem 4.A.2 breaks into two parts. In this section of the paper we describe a method of separating $G \cdot y$ from $G \cdot x$—called “distinguishing” the orbits—which is, a priori, different from differentiating the orbits. In §4.C we prove a statement (Lemma 4.C.1) which is essentially Theorem 4.A.2 with the hypothesis that $G{\cdot}y$ cannot be *differentiated* from $G{\cdot}x$ replaced by the hypothesis that it cannot be *distinguished*. In this section we prove (Lemma 4.B.7) that if it cannot be differentiated then it cannot be distinguished. Thus Theorem 4.A.2 follows immediately from Lemmas 4.B.7 and 4.C.1.

4.B.1. Definition. Let X be an acceptable G-space and let $G{\cdot}x$ be an orbit of X. If $A \subset G{\cdot}x$ is an open subset of $G{\cdot}x$, let $\overbrace{A}$ denote the following subset of X:

$$\overbrace{A} = \bigcup\{D : D \subset X \text{ is open and } D \cap G{\cdot}x \subset A\}.$$

4.B.2. Proposition. *Let X be an acceptable G-space, let $G\cdot x$ be an orbit of X and let A and B be open subsets of $G\cdot x$.*

(a) $\overbrace{A}$ *is the maximal open subset of X whose intersection with $G\cdot x$ is A.*

(b) *If $A \subset B$, then $\overbrace{A} \subset \overbrace{B}$.*

(c) *For any $g \in G$, $g\cdot \overbrace{A} = \overbrace{g\cdot A}$.*

Recall that by Theorem 2.E.1, if $G\cdot x$ is a G_δ orbit, then for any open $N \subset G$, $N\cdot x$ is open in $G\cdot x$; therefore, $\overbrace{N\cdot x}$ is well defined.

4.B.3. Example. Consider the logic actions (§2.D): L is a language, F is a countable fragment of $L_{\omega_1\omega}$, and we have a Polish S_∞-space X_L, where X_L has the t_F-topology. Fix $x \in X_L$. A basic open subset A of $S_\infty \cdot x$ has the form $[\mathrm{Mod}(\varphi, \overline{s}) \cap S_\infty \cdot x]$ (see 2.D.1); in this case,

$$\overbrace{A} = \mathrm{Mod}(\varphi, \overline{s}) \cup \{y \in X_L : \mathfrak{A}_y \not\equiv_F \mathfrak{A}_x\}.$$

Now suppose $S_\infty \cdot x$ is a G_δ orbit. By 2.E.3, $\mathfrak{A}_x$ is an F-atomic model. Let N be a member of the canonical basis for S_∞ (2.B.1 (d)). As $\mathfrak{A}_x$ is atomic, $\mathfrak{A}_x$ is homogeneous and all types realized in $\mathfrak{A}_x$ are principal. Therefore, as shown at the end of §2.E, $N \cdot x$ is $[\mathrm{Mod}(\varphi, \overline{s}) \cap S_\infty \cdot x]$, for some φ and $\overline{s}$. Of course, $\overbrace{N\cdot x}$ is the set $\overbrace{A}$, described above.

4.B.4. Definition. Let G be a group and let $\mathcal{N}^* = \langle \{N_i\}, \{g_k^{\langle i,j\rangle}\}\rangle$ be an enhanced basis for G. Let X be an acceptable G-space and let $G\cdot x$ be a G_δ orbit of X. For any i, $j \in \mathbb{N}$, define $Q_j^i(x)$ to be the following subset of X:

$$Q_j^i(x) = \left[\overbrace{N_i\cdot x} \setminus \left(\bigcup_{k\in\mathbb{N}} \overbrace{(g_k^{\langle i,j\rangle} N_j)\cdot x}\right)\right]^{\Delta}.$$

4.B.5. Proposition. *For any group G, any enhanced basis $\mathcal{N}^*$ for G, any acceptable G-space X, any G_δ orbit $G\cdot x$ of X, and any i, $j \in \mathbb{N}$:*

(a) *$Q_j^i(x)$ is invariant;*

(b) *if $N_j \neq \emptyset$, then $G\cdot x \cap Q_j^i(x) = \emptyset$.*

Proof. (a) Obvious.

(b) Suppose this is false. By (a), $x \in Q_j^i(x)$. So there exists an $h \in G$ such that

$$(*) \qquad h\cdot x \in \left[\overbrace{N_i\cdot x} \setminus \left(\bigcup_{k\in\mathbb{N}} \overbrace{(g_k^{\langle i,j\rangle} N_j)\cdot x}\right)\right].$$

Now (*) implies that $h\cdot x \in \overbrace{N_i\cdot x}$, hence $h\cdot x \in N_i\cdot x$. Let $h' \in N_i$ be such that $h\cdot x = h'\cdot x$. Since $N_j \neq \emptyset$, there is a $g \in G$ such that $h' \in gN_j$. By part (e) of the definition of "enhanced basis" (2.B.2)—taking $M = N_j$—there exists a $k \in \mathbb{N}$ such that $h' \in g_k^{\langle i,j\rangle} N_j$. By (*), $h\cdot x \notin \overbrace{(g_k^{\langle i,j\rangle} N_j)\cdot x}$, hence $h\cdot x \notin (g_k^{\langle i,j\rangle} N_j)\cdot x$. But $h' \in g_k^{\langle i,j\rangle} N_j$, so $h\cdot x \neq h'\cdot x$, a contradiction. □

4.B.6. Definition. Let G be a group with enhanced basis $\mathcal{N}^* = \langle \{N_i\}, \{g_k^{\langle i,j\rangle}\}\rangle$ and let X be an acceptable G-space. Let $G{\cdot}x$ be a G_δ orbit and $G{\cdot}y$ be an arbitrary orbit. For i, $j \in \mathbb{N}$, we say that the pair (i, j) *distinguishes $G\cdot y$ from x with respect to $\mathcal{N}^*$* if $N_j \neq \emptyset$ and $G\cdot y \subset Q^i_j(x)$. We say that $\mathcal{N}^*$ *distinguishes $G\cdot y$ from x* if there exist i, $j \in \mathbb{N}$ such that (i, j) distinguishes $G\cdot y$ from x with respect to $\mathcal{N}^*$.

4.B.7. Lemma. *Let G be a group with enhanced basis $\mathcal{N}^* = \langle\{N_i\}, \{g_k^{\langle i,j\rangle}\}\rangle$ and let X be an acceptable G-space. Let $G\cdot x$ be a G_δ orbit and let $G\cdot y$ be an arbitrary orbit. If $\mathcal{N}^*$ distinguishes $G\cdot y$ from x, then for any basis $\mathcal{C}$ for X, the pair $(\mathcal{N}^*, \mathcal{C})$ differentiates $G\cdot y$ from $G\cdot x$.*

The definitions of both "distinguish" (4.B.6) and "differentiate" (4.A.1) assert that $G \cdot y$ can be separated from $G \cdot x$ by a set of a particular form: $Q^i_j(x)$ and $A(C, D, i, j)$, respectively. Note that if, in the definition of "differentiate", we had replaced "C, $D \in \mathcal{C}$" by "C, D open", then Lemma 4.B.7 would be trivial—it is essentially the definition of "distinguish" plus 4.B.5 (b) and 4.B.2 (c). The content of Lemma 4.B.7 is that the open sets C, D can be taken to be members of the basis.

Proof of Lemma 4.B.7. Suppose we are given G, $\mathcal{N}^*$, X, x, y and $\mathcal{C}$ satisfying the hypothesis of 4.B.7. Thus the enhanced basis $\mathcal{N}^*$ distinguishes $G\cdot y$ from x. Let i, $j \in \mathbb{N}$ be such that the pair (i, j) distinguishes $G\cdot y$ from x with respect to $\mathcal{N}^*$. Then $y \in Q^i_j(x)$. By definition of the set $Q^i_j(x)$ (4.B.4), there is a nonmeager set $P_0 \subset G$ such that for all $h \in P_0$:

(a) $h\cdot y \in \overbrace{N_i\cdot x}$;

(b) for all $k \in \mathbb{N}$, $h\cdot y \notin \overbrace{(g_k^{\langle i,j\rangle} N_j)\cdot x}$.

Now $\overbrace{N_i\cdot x}$ is open, so there is a set C in the basis $\mathcal{C}$ such that $C \subset \overbrace{N_i\cdot x}$ and there is a nonmeager set $P_1 \subset P_0$ such that for all $h \in P_1$:

(a′) $h\cdot y \in C$.

By definition of "distinguish", $N_j \neq \emptyset$. Hence $\overbrace{N_j\cdot x}$ is an open set and $N_j\cdot x \neq \emptyset$. Obviously $N_j\cdot x \subset \overbrace{N_j\cdot x}$, so there is a set D in the basis $\mathcal{C}$ such that $D \subset \overbrace{N_j\cdot x}$ and $D \cap N_j\cdot x \neq \emptyset$. For any fixed $k \in \mathbb{N}$:

$$\begin{aligned} g_k^{\langle i,j\rangle}\cdot D \cap G\cdot x &\subset g_k^{\langle i,j\rangle}\cdot\overbrace{(N_j\cdot x)}\cap G\cdot x \\ &= \overbrace{g_k^{\langle i,j\rangle}\cdot(N_j\cdot x)}\cap G\cdot x \quad \text{(by 4.B.2 (c))} \\ &= \overbrace{(g_k^{\langle i,j\rangle} N_j)\cdot x}\cap G\cdot x \\ &= (g_k^{\langle i,j\rangle} N_j)\cdot x. \end{aligned}$$

Therefore, $g_k^{\langle i,j\rangle}\cdot D$ is an open set whose intersection with $G \cdot x$ is a subset of $(g_k^{\langle i,j\rangle} N_j)\cdot x$. So by definition of the function $B \mapsto \overbrace{B}$ (4.B.1), $g_k^{\langle i,j\rangle}\cdot D$ is a subset of $\overbrace{(g_k^{\langle i,j\rangle} N_j)\cdot x}$. Thus we have shown that for all $h \in P_1$:

(b′) for all $k \in \mathbb{N}$, $h\cdot y \notin g_k^{\langle i,j\rangle}\cdot D$.

We have now chosen C, $D \in \mathcal{C}$ and i, $j \in \mathbb{N}$. We prove that C, D, i and j witness the fact that $(\mathcal{N}^*, \mathcal{C})$ differentiates $G\cdot y$ from $G\cdot x$. Let $A = A(C, D, i, j)$ (defined in 4.A.1). By (a′) and (b′) above, $y \in A$. Clearly A is invariant, so $G\cdot y \subset A$. That is, condition (a) in the definition of "differentiates" holds. To complete the proof, we need only show that condition (b) of that definition holds: $A \cap G\cdot x = \emptyset$.

Assume, towards a contradiction, that $A \cap G{\cdot}x \neq \emptyset$. Then there is an x' in

$$\left[C \setminus \left(\bigcup_{k\in\mathbb{N}} (g_k^{\langle i,j\rangle}{\cdot}D)\right)\right] \cap G{\cdot}x.$$

We now have:

$$x' \in C \cap G{\cdot}x \subset \widehat{N_i{\cdot}x} \cap G{\cdot}x = N_i{\cdot}x.$$

Fix $h' \in N_i$ such that $h'{\cdot}x = x'$. Then we know the following fact:

(c) For all $k \in \mathbb{N}$, $h'{\cdot}x \notin g_k^{\langle i,j\rangle}{\cdot}D$.

Let $M_D = \{g \in N_j : g{\cdot}x \in D\}$. As the action is continuous, M_D is open. The set D was chosen so that $D \cap N_j{\cdot}x \neq \emptyset$. Therefore, $M_D \neq \emptyset$. So there exists a nonempty $M \in \mathcal{N}$ such that $M \subset M_D$. Then for any $k \in \mathbb{N}$, $g_k^{\langle i,j\rangle}M \subset g_k^{\langle i,j\rangle}M_D$; hence:

$$\begin{aligned}(g_k^{\langle i,j\rangle}M){\cdot}x &\subset (g_k^{\langle i,j\rangle}M_D){\cdot}x\\ &= g_k^{\langle i,j\rangle}{\cdot}(M_D{\cdot}x)\\ &\subset g_k^{\langle i,j\rangle}{\cdot}D.\end{aligned}$$

Therefore, (c) implies the following:

(c′) For all $k \in \mathbb{N}$, $h'{\cdot}x \notin (g_k^{\langle i,j\rangle}M){\cdot}x$.

Now $M \subset M_D \subset N_j$. So part (e) in the definition of "enhanced basis" (2.B.2) says:

$$\bigcup\{gM : g \in G \ \&\ gM \cap N_i \neq \emptyset\} \subset \bigcup_{k\in\mathbb{N}} (g_k^{\langle i,j\rangle}M).$$

Therefore, (c′) implies the following:

(c″) For all g such that $gM \cap N_i \neq \emptyset$, $h'{\cdot}x \notin (gM){\cdot}x$.

Since $M \neq \emptyset$, there is a $g' \in G$ such that $h' \in g'M$. Clearly $h'{\cdot}x \in (g'M){\cdot}x$.

But h' was chosen to be a member of N_i. So $g'M \cap N_i \neq \emptyset$. That is, g' satisfies the hypothesis of (c″) above. So by (c″), $h'{\cdot}x \notin (g'M){\cdot}x$, a contradiction. □

4.C. Distinguishability and $\imath$-embeddings.

4.C.1. Lemma. *Let G be a group, let $\mathcal{N}^*$ be an enhanced basis for G and let X be an acceptable G-space. Let $G{\cdot}x$ be a G_δ orbit of X and let $G{\cdot}y$ be an arbitrary orbit of X. If $y \in$ closure$(G{\cdot}x)$ but $\mathcal{N}^*$ does not distinguish $G{\cdot}y$ from x, then $G{\cdot}x$ is $\imath$-embeddable into $G{\cdot}y$.*

Proof. Let G, $\mathcal{N}^*$, X, x and y satisfy the hypothesis. We use the notation

$$\mathcal{N}^* = \langle\{N_i\}, \{g_k^{\langle i,j\rangle}\}\rangle$$

and $\mathcal{N} = \{N_i\}$. Let d and d' be two compatible metrics on G such that d is left-invariant and d' is complete. Recall that since $G{\cdot}x$ is a G_δ orbit, by 2.E.1, for all open $M \subset G$, $\widehat{M{\cdot}x}$ is well defined.

We first construct, for each $n \in \mathbb{N}$, a nonempty set $M_n \in \mathcal{N}$ and an element h_n of G satisfying the following four properties.

(a) d-diam$(M_n) < 1/2^n$ (for $n \geq 1$).

(b) $M_{n+1} \cap M_n \neq \emptyset$.

(c) $d'(h_{n+1}, h_n) < 1/2^n$.

(d) $h_n\cdot y \in \overbrace{M_n\cdot x}$.

The construction is by induction on n. Let $M_0 = G$ and $h_0 = e$. Then (a)–(c) are vacuous and (d) says that $y \in \overbrace{G\cdot x}$. By Proposition 4.B.2 (a), $\overbrace{G\cdot x} = X$, hence (d) holds.

Suppose the construction has been carried out up to stage n: we have M_n and h_n satisfying (a)–(d). We now construct M_{n+1} and h_{n+1}.

Let $i \in \mathbb{N}$ be such that $M_n = N_i$. Let $j \in \mathbb{N}$ be such that $N_j \neq \emptyset$ and d-diam$(N_j) < 1/2^{n+1}$. Since $\mathcal{N}^*$ does not distinguish $G\cdot y$ from x, the pair (i,j) does not distinguish $G\cdot y$ from x with respect to $\mathcal{N}^*$. By definition of "distinguish" (4.B.6), this means that it is not the case that $G\cdot y \subset Q^i_j(x)$. By Proposition 4.B.5 (a), $h_n\cdot y \notin Q^i_j(x)$. By definition of $Q^i_j(x)$ (4.B.4), this means that there is a comeager set $P \subset G$ such that for all $h \in P$:

$$h\cdot(h_n\cdot y) \notin \left[\overbrace{N_i\cdot x} \setminus \left(\bigcup_{k\in\mathbb{N}} \overbrace{(g_k^{\langle i,j\rangle}N_j)\cdot x}\right)\right]. \tag{*}$$

Let

$$R = \{h \in G : h\cdot(h_n\cdot y) \in \overbrace{N_i\cdot x} \ \&\ d'(hh_n, h_n) < 1/2^n\}.$$

The action is continuous and $\overbrace{N_i\cdot x}$ is, by definition, open. So R is open. Since $M_n = N_i$, by (d) of the induction hypothesis, $e \in R$; so $R \neq \emptyset$. As P is comeager, $P \cap R \neq \emptyset$. Let $h' \in P \cap R$. Let $h_{n+1} = h'h_n$. Clearly (c) holds.

Now $h' \in P$ and $h'\cdot(h_n\cdot y) \in \overbrace{N_i\cdot x}$. So (*) implies that there is a $k \in \mathbb{N}$ such that

$$h'\cdot(h_n\cdot y) \in \overbrace{(g_k^{\langle i,j\rangle}N_j)\cdot x}\,.$$

Fix such a k. Let $M_{n+1} = g_k^{\langle i,j\rangle}N_j$. By part (c) of the definition of "enhanced basis" (2.B.2), $M_{n+1} \in \mathcal{N}$. Notice that:

$$\begin{aligned} h_{n+1}\cdot y &= (h'h_n)\cdot y \\ &= h'\cdot(h_n\cdot y) \\ &\in \overbrace{(g_k^{\langle i,j\rangle}N_j)\cdot x} \\ &= \overbrace{M_{n+1}\cdot x}\,. \end{aligned}$$

Therefore, (d) holds.

The set N_j was chosen so that

$$d\text{-diam}(N_j) < 1/2^{n+1}.$$

Since d is left-invariant, d-diam$(g_k^{\langle i,j\rangle}N_j) = d$-diam$(N_j)$. And $M_{n+1} = g_k^{\langle i,j\rangle}N_j$. Therefore, d-diam$(M_{n+1}) < 1/2^{n+1}$, hence (a) holds.

Now $M_n = N_i$, $M_{n+1} = g_k^{\langle i,j\rangle}N_j$, $N_i \neq \emptyset$ and $N_j \neq \emptyset$. So part (d) of the definition of "enhanced basis" ensures that M_{n+1} intersects M_n. That is, (b) holds. This completes stage $n+1$ of the construction.

Thus we have M_n's and h_n's satisfying (a)–(d), above.

By (c), the sequence h_n is d'-Cauchy. And d' is a complete metric on G. So there is some $h \in G$ such that h_n converges to h. Let $y' = h\cdot y$. Since the action is continuous, $h_n \cdot y$ converges to y'. By definition of "acceptable" topology (2.A.1), the topological space X has a countable basis. So there exists a sequence C_n of open subsets of X such that:

(i) for all $n \in \mathbb{N}$, $C_{n+1} \subset C_n$;

(ii) $\{C_n\}$ is a neighborhood basis for y';

(iii) for all $n \in \mathbb{N}$, $h_n \cdot y \in C_n$.

Claim. For all $n \in \mathbb{N}$, there exists a point $x_n \in M_n \cdot x$ such that $x_n \in C_n$.

Proof of Claim. Fix n. Let $A = \overbrace{M_n \cdot x} \cap C_n$. Then A is open. By (d) and (iii), $h_n \cdot y \in A$. Thus $G \cdot A$ is an open set containing y. By hypothesis, $y \in \text{closure}(G \cdot x)$, so $G \cdot A \cap G \cdot x \neq \emptyset$. Therefore, $A \cap G \cdot x \neq \emptyset$. Let $x_n \in A \cap G \cdot x$. Clearly $x_n \in C_n$, and

$$x_n \in A \cap G \cdot x \subset \overbrace{M_n \cdot x} \cap G \cdot x = M_n \cdot x.$$

This proves the claim.

Fix a sequence x_n satisfying the claim. For all $n \in \mathbb{N}$, let $g_n \in M_n$ be such that $x_n = g_n \cdot x$.

We now complete the proof of Lemma 4.C.1. We must show that $G \cdot x$ is ι-embeddable into $G \cdot y$ $(= G \cdot y')$. To do this, it will suffice to show that the point x is ι-embeddable into the point y' (see Proposition 3.D.4 and remark following). We show this by proving that the sequence g_n is an ι-embedding of x into y' (as defined in 3.D.1). There are thus two facts which must be proved: first, that the sequence g_n is ι-Cauchy; second, that the sequence $g_n \cdot x$ converges to y'.

For all n, $g_n \in M_n$. So by (a) and (b), the sequence g_n is d-Cauchy. And d is a left-invariant metric on G. So g_n is ι-Cauchy. For all n, $g_n \cdot x = x_n$, so by the Claim, $g_n \cdot x \in C_n$. Hence, by (i) and (ii), the sequence $g_n \cdot x$ converges to y'. □

Proof of Theorem 4.A.2. Lemmas 4.B.7 and 4.C.1. □

5. Consequences for Polish and Borel G-spaces

5.A. Generalized model theory: Vaught's Theorem. In §5 we derive some consequences of the Main Theorem (4.A.2). In §5.A we generalize a theorem of model theory, and in §§5.B, 5.C—using this generalized model theory of §5.A—we prove dichotomy theorems for cli groups. What is being generalized here is Vaught's Theorem that countable atomic models are prime models; that is, if $\mathfrak{A}$ is countable atomic and $\mathfrak{A} \equiv \mathfrak{B}$, then $\mathfrak{A}$ is elementarily embeddable into $\mathfrak{B}$. This generalized model theory is applicable to enhanced Polish G-spaces, a concept which was defined in 2.C.3.

5.A.1. Theorem. *Let G be a Polish group and let X be an enhanced Polish G-space. Let $G \cdot x$ be a G_δ orbit of X and let $G \cdot y$ be an arbitrary orbit of X. The following are equivalent.*

(a) *$G \cdot x$ and $G \cdot y$ are in the same piece of the canonical partition of X.*

(b) $G \cdot y \subset$ closure$(G \cdot x)$.

(c) *$G \cdot x$ is ι-embeddable into $G \cdot y$.*

Proof. (a) $\Longrightarrow$ (b). Proposition 2.C.2 (c).

(b) $\Longrightarrow$ (c). First of all, note that X is an acceptable G-space. So Theorem 4.A.2 is applicable. Secondly, note that we have abused the language in calling X an "enhanced Polish G-space"; actually, the enhanced Polish G-space is a 5-tuple $(\mathcal{N}^*, X, a, t, \mathcal{C})$, where $\mathcal{N}^* = \langle \mathcal{N}, \{g_k^{\langle i,j \rangle}\} \rangle$ is an enhanced basis for G and $\mathcal{C}$ is a basis for the Polish space $\langle X, t\rangle$. Now suppose (c) is false: $G{\cdot}x$ is not $\imath$-embeddable into $G{\cdot}y$. By Theorem 4.A.2, $(\mathcal{N}^*, \mathcal{C})$ differentiates $G{\cdot}y$ from $G{\cdot}x$. This means that there exist sets C, $D \in \mathcal{C}$ and there exist $i, j \in \mathbb{N}$, such that if we set

$$A(C, D, i, j) = \left[C \setminus \left(\bigcup_{k\in\mathbb{N}} (g_k^{\langle i,j\rangle} \cdot D)\right)\right]^{\Delta},$$

then $G{\cdot}y \subset A(C, D, i, j)$ and $G{\cdot}x \cap A(C, D, i, j) = \emptyset$. Since $G{\cdot}y \subset \text{closure}(G{\cdot}x)$, the set $A(C, D, i, j)$ cannot be open. But by part (d) of the definition of "enhanced Polish G-space" (2.C.3), $A(C, D, i, j)$ is open.

(c) $\Longrightarrow$ (a). Proposition 3.D.5 (a). ☐

Clearly 5.A.1 and 2.C.4 (b) imply that for any enhanced Polish G-space X and any $x \in X$: if the orbit $G{\cdot}x$ is G_δ, then $\mathbf{Emb}(x)$ is a closed set. Recall from 2.D.4 and 3.D.7 that for arbitrary orbits $G{\cdot}x$ of an enhanced Polish G-space X, $\mathbf{Emb}(x)$ need not even be a Borel set. Hence the hypothesis of 5.A.1 that $G{\cdot}x$ be a G_δ orbit is necessary.

5.A.2. Corollary. *Let L be a countable language, let F be a countable fragment of $L_{\omega_1\omega}$, let $\mathfrak{A}$ be a countable L-structure which is F-atomic, and let $\mathfrak{B}$ be an arbitrary L-structure. The following are equivalent.*

(a) $\mathfrak{A} \equiv_F \mathfrak{B}$.

(b) $\mathfrak{A}$ *is F-embeddable into* $\mathfrak{B}$.

Proof. By the Löwenheim-Skolem Theorem, it will suffice to prove this for countable $\mathfrak{B}$. As is well known, it will also suffice to prove it for relational languages. If L is empty or either $\mathfrak{A}$ or $\mathfrak{B}$ is finite, the result is trivial. So we can assume that both $\mathfrak{A}$ and $\mathfrak{B}$ have underlying set $\mathbb{N}$. Thus to prove 5.A.2, we need only consider the situation that corresponds to the logic actions: we have the S_∞-space X_L and two points x, $y \in X_L$ such that $\mathfrak{A} = \mathfrak{A}_x$ and $\mathfrak{B} = \mathfrak{A}_y$.

For this situation, Corollary 5.A.2 is a special case of the (a) $\Longleftrightarrow$ (c) part of Theorem 5.A.1. This is so because of the following four facts. First, by Theorem 2.D.4, the space X_L, with the t_F-topology, is an enhanced Polish S_∞-space. Second, by Proposition 2.E.3, $S_\infty \cdot x$ is a G_δ orbit. Third, by Proposition 2.D.3, $S_\infty{\cdot}x$ and $S_\infty \cdot y$ are in the same piece of the canonical partition of X_L iff $\mathfrak{A}_x \equiv_F \mathfrak{A}_y$. Fourth, by Theorem 3.D.2, $S_\infty \cdot x$ is $\imath$-embeddable into $S_\infty{\cdot}y$ iff $\mathfrak{A}_x$ is F-embeddable into $\mathfrak{A}_y$. ☐

Corollary 5.A.2 is a well known theorem of model theory. In the case of finitary logic (that is, the case in which F is the smallest fragment), Corollary 5.A.2 is due to Vaught [45]. Vaught's proof easily yields the general version of 5.A.2—for arbitrary F—and, indeed, this is the usual proof which is found in books on model theory, e.g., Keisler [30]. The proof of 5.A.2 given above seems to be a new proof of this result; it is not the usual proof translated into the language of group actions.

Thus Theorem 5.A.1 is a generalization of a theorem of model theory, as promised. More generalized model theory can be found in §6.A of this paper and in the forthcoming paper Becker [4], which will contain a generalized version of much of the theory of atomic, homogeneous and saturated models, for example, the theorem that atomic models are homogeneous.

We remark that the generalized model theory of Theorem 5.A.1, while applicable to arbitrary Polish groups G, is not applicable to arbitrary Polish G-spaces, but only to enhanced Polish G-spaces. Proposition 2.C.5 allows us to change the topology and turn any Polish G-space into an enhanced Polish G-space. But changing the topology changes the set of G_δ orbits, changes the canonical partition, changes the closure of $G\cdot x$, and also changes the ι-embeddability of $G\cdot x$ into $G\cdot y$. It will be shown following Theorem 5.C.10, below, that the "enhanced" hypothesis in 5.A.1 is necessary. If we remove the "enhanced" hypothesis, we can only get a weak form of the generalized Vaught Theorem, namely Theorem 4.A.2.

5.B. The Glimm-Effros Dichotomy for cli groups. Throughout §§5.B, 5.C, let Y denote a Polish space and let R denote an arbitrary $\mathbf{\Sigma}^1_1$ equivalence relation on Y. The equivalence relation R may or may not be induced by an action. Also recall our convention from §2.C: the letter X is reserved for G-spaces and the letter E is reserved for the orbit equivalence relation on X. For any Polish G-space, the orbit equivalence relation is $\mathbf{\Sigma}^1_1$, but in general, is not Borel (see Becker–Kechris [6, Chapter 7]). Nevertheless, every orbit is a Borel set (Ryll–Nardzewski; see Becker–Kechris [6, 2.3.4]). There exist $\mathbf{\Sigma}^1_1$ equivalence relations with no Borel equivalence classes (see Becker [3, proof of 4.2]); so clearly not every $\mathbf{\Sigma}^1_1$ equivalence relation is induced by a continuous action of a Polish group. In fact, something much stronger is true: it is a theorem of Kechris–Louveau [29, 4.2] that there exists a Borel equivalence relation which is not induced by any continuous (or even any "definable") action of a Polish group. Orbit equivalence relations are very special, and it is therefore possible that they have properties not shared by arbitrary $\mathbf{\Sigma}^1_1$ equivalence relations. We consider such possibilities in §5.C. And, of course, cli groups are a very special type of Polish group. Their orbit equivalence relations have properties not shared by arbitrary orbit equivalence relations, as will be shown in this section and in §6.D.

Let E_0 denote the following equivalence relation on $2^{\mathbb{N}}$:

$$xE_0y \iff \exists n \forall m \geq n\,(x(m) = y(m))\,.$$

(Remark: This is an orbit equivalence relation of a $\mathbb{Z}$-action.) We say that R *contains a copy of* E_0 if there exists a one-one continuous function $f : 2^{\mathbb{N}} \to Y$ such that for all $x, y \in 2^{\mathbb{N}}$:

$$xE_0y \iff f(x)Rf(y).$$

If R is an orbit equivalence relation, this definition has two equivalent forms. R contains a copy of E_0 iff there exists a universally measurable (as opposed to continuous) f as above. And R contains a copy of E_0 iff R admits a nonatomic ergodic measure. (A measure is *ergodic* if every R-invariant measurable set has either measure 0 or measure 1.) Assuming a weak large cardinal axiom, these three propositions are equivalent for an arbitrary $\mathbf{\Sigma}^1_1$ equivalence relation R; it is open whether that equivalence is provable in ZFC.

An equivalence relation R on Y is called *smooth* if there is a Borel-measurable function $f : Y \to 2^{\mathbb{N}}$ such that for all $\mathbf{p}$, $\mathbf{q} \in Y$:

$$\mathbf{p}R\mathbf{q} \iff f(\mathbf{p}) = f(\mathbf{q}).$$

This is equivalent to saying that R admits a countable Borel separating family, i.e., a family $\{B_n\}_{n\in\mathbb{N}}$ of Borel subsets of Y such that for all $\mathbf{p}, \mathbf{q} \in Y$:

$$\mathbf{p}R\mathbf{q} \text{ iff } \forall n(\mathbf{p} \in B_n \iff \mathbf{q} \in B_n).$$

If R is smooth, then it is a Borel equivalence relation.

A smooth equivalence relation cannot contain a copy of E_0. We say that the equivalence relation R *satisfies the Glimm–Effros Dichotomy* if either R is smooth or else R contains a copy of E_0. In some sense, this dichotomy says that either the quotient space Y/R is "small" or else it contains a copy of a specific "large" set. We say that an action or a G-space *satisfies the Glimm–Effros Dichotomy* if the associated orbit equivalence relation does.

This dichotomy originates with work of Glimm and Effros which was motivated by some questions about operator algebras. First Glimm [16] proved that if G is a locally compact Polish group, then any Polish G-space satisfies the Glimm–Effros Dichotomy. Later Effros [12] proved a stronger theorem: for any Polish group G and any Polish G-space, if the associated orbit equivalence relation E is F_σ, then E satisfies the dichotomy. Still later, Harrington–Kechris–Louveau [18] (who introduced the name "Glimm–Effros Dichotomy") proved an even stronger theorem: for any Borel equivalence relation R, R satisfies the Glimm–Effros Dichotomy. More historical remarks are given at the end of §5.C. For more information on the dichotomy, see Glimm [16], Effros [11], [12], Harrington–Kechris–Louveau [18], Becker–Kechris [6, 3.4], Hjorth–Solecki [22] and Hjorth [21].

There exist Polish S_∞-actions which violate the Glimm–Effros Dichotomy. One example is the logic action for the language of groups, restricted to the invariant G_δ set of (codes for) abelian p-groups. Countable abelian p-groups are classified up to isomorphism by the Ulm invariants. Using this classification, Friedman–Stanley [14, 2.1.4] proved that the dichotomy is violated.

5.B.1. Proposition. *Let G be a group and let X be a Polish G-space. The following are equivalent.*

(a) *The G-space X satisfies the Glimm-Effros Dichotomy.*

(b) *One of the following holds.*

  (i) *The orbit equivalence relation of X contains a copy of E_0.*

  (ii) *There exists an $\alpha < \omega_1$ such that every orbit of X is $\mathbf{\Pi}^0_\alpha$.*

Proof. (a) $\Longrightarrow$ (b). If the equivalence relation is smooth, then it is Borel, so (ii) holds.

(b) $\Longrightarrow$ (a). Trivially, (b) (i) implies (a). So assume (b) (ii) holds. This implies that the orbit equivalence relation is Borel (see Sami [38, 3.7]). So by the aforementioned theorem of Harrington-Kechris-Louveau [18], this equivalence relation satisfies the Glimm-Effros Dichotomy. □

It is possible that both (b) (i) and (b) (ii) of 5.B.1 hold. An example of this is the equivalence relation E_0 itself.

5.B.2. Theorem. *Let G be a cli group. For any Polish G-space X, one of the following holds.*

(a) *The orbit equivalence relation of X contains a copy of E_0.*

(b) *Every orbit of X is $\mathbf{\Pi}^0_\omega$.*

Thus for all cli groups G, every Polish G-space satisfies the Glimm-Effros Dichotomy.

Open Question. *Does there exist any non-cli Polish group G such that every Polish G-space satisfies the Glimm-Effros Dichotomy?*

Theorem 5.B.2 will be proved below, via a sequence of lemmas. Before proving it, we give two corollaries.

5.B.3. Corollary. *Let G be a cli group. Let X be a standard Borel G-space. Then the orbit equivalence relation of X satisfies the Glimm-Effros Dichotomy.*

Proof. By Becker-Kechris [6, 5.2.1], every standard Borel G-space is Borel-isomorphic to a Polish G-space, and the Glimm-Effros Dichotomy is preserved under Borel-isomorphism. □

For the definition of "standard Borel G-space", see Becker-Kechris [6, 2.3]. A particular instance of a standard Borel G-space is the restriction of a Polish G-space to an invariant Borel set. This proves the following corollary.

5.B.4. Corollary. *Let G be a cli group, let X be a Polish G-space, let $B \subset X$ be an invariant Borel set and let R denote the orbit equivalence relation on B. Then R satisfies the Glimm-Effros Dichotomy.*

In the situation of 5.B.4, if there is no copy of E_0, then by Proposition 5.B.1, the orbits of B are of bounded Borel rank; but they are certainly not, in general, $\mathbf{\Pi}^0_\omega$. For example, one could take B to be a single orbit that is not $\mathbf{\Pi}^0_\omega$. Such orbits exist; see Example 5.C.9, below.

We now begin the proof of Theorem 5.B.2.

5.B.5. Definition. A G-space X *satisfies the Strong Glimm-Effros Dichotomy* if one of the following holds.

(a) The orbit equivalence relation of X contains a copy of E_0.
(b) Every orbit of X is G_δ.

By Proposition 5.B.1, this terminology is appropriate. Also note that by Proposition 2.C.2, condition (b) of 5.B.5 is equivalent to the following: each piece of the canonical partition of X has only one orbit. By Harrington-Kechris-Louveau [18, 1.2], (b) implies that the orbit equivalence relation is smooth; hence (a) and (b) cannot both hold. Example 3.D.6 (e) is a (non-cli) Polish group G and a Polish G-space which satisfies the Glimm-Effros Dichotomy but violates the Strong Glimm-Effros Dichotomy.

5.B.6. Lemma. *Let X be a Polish G-space. If every G_δ orbit of X is also F_σ, then X satisfies the Strong Glimm-Effros Dichotomy.*

Proof. By a theorem of Becker-Kechris [6, 3.4.6], for any Polish G-space, either the orbit equivalence relation contains a copy of E_0 or else there is at least one G_δ orbit. Now let E be the orbit equivalence relation of X and suppose that this lemma is false. Then there is a piece Y of the canonical partition of X such that Y has at least two orbits and $E \restriction Y$ does not contain a copy of E_0. Y is G_δ (see 2.C.2 (b)), hence Y, itself, is a Polish G-space. We apply the above theorem to the Polish G-space Y: Y must contain a G_δ orbit $G\cdot y$. Let $Y' = Y \setminus G\cdot y$. By hypothesis, $G\cdot y$ is F_σ, so Y' is a nonempty G_δ. We again apply the above theorem, this time to the Polish G-space Y': Y' contains a G_δ orbit. So Y contains at least two G_δ orbits. Clearly they cannot both be dense in Y, which contradicts 2.C.2 (c). □

Lemma 5.B.6 is a strong form of the theorem of Effros [12] that states that if the orbit equivalence relation is F_σ, then the action satisfies the Strong Glimm-Effros Dichotomy.

5.B.7. Lemma. *Let G be a cli group and let X be an enhanced Polish G-space. Every G_δ orbit of X is closed.*

Proof. Suppose $G\cdot x$ is a G_δ orbit. Let $y \in \text{closure}(G\cdot x)$; we must show that $G\cdot x = G\cdot y$. By Theorem 5.A.1, $G\cdot x$ is ι-embeddable into $G\cdot y$. But G is a cli group, so by Proposition 3.D.3, the ι-embeddability relation is trivial. Therefore, $G\cdot x = G\cdot y$. □

5.B.8. Lemma. *Let G be a cli group and let X be an enhanced Polish G-space. One of the following holds.*

(a) *The orbit equivalence relation of X contains a copy of E_0.*
(b) *Every orbit of X is closed.*

Proof. Lemmas 5.B.6 and 5.B.7, together, imply that X satisfies the Strong Glimm-Effros Dichotomy. That is, either (a) holds, or else every orbit is G_δ. Again using 5.B.7, these G_δ orbits are all closed. □

Proof of Theorem 5.B.2. We are given a cli group G and a Polish G-space X. Let τ denote the Polish topology on X. By Proposition 2.C.5, there exists a new topology t on X such that $\langle X, t\rangle$ and $\langle X, \tau\rangle$ are Borel-isomorphic, $\langle X, t\rangle$ is an enhanced Polish G-space, and such that there exists a countable basis $\mathcal{C}$ for t with the following property:

(*) for all $C \in \mathcal{C}$ there exists an $n \in \mathbb{N}$ such that C is $\mathbf{\Sigma}^0_n$ (with respect to τ).

Applying Lemma 5.B.8 to the enhanced Polish G-space $\langle X, t\rangle$, we see that one of the following holds.

(a′) The orbit equivalence relation of $\langle X, t\rangle$ contains a copy of E_0.
(b′) Every orbit of X is t-closed.

The property of containing a copy of E_0 is preserved by Borel-isomorphism, hence (a) follows from (a′). And (*) implies that every t-closed set is $\mathbf{\Pi}^0_\omega$ with respect to τ, hence (b) follows from (b′). □

5.C. The Silver-Vaught Dichotomy for cli groups. Let R be a $\mathbf{\Sigma}^1_1$ equivalence relation on a Polish space Y. We say that R has *perfectly many* equivalence classes if there exists a nonempty perfect set $P \subset Y$ such that any two distinct points of P are in different R-equivalence classes. We say that R *satisfies the Silver-Vaught Dichotomy* if either R has only countably many equivalence classes or else R has perfectly many equivalence classes. We say that an action or a G-space *satisfies the Silver-Vaught Dichotomy* if the associated orbit equivalence relation does.

Silver [39] proved that all Borel (or even $\mathbf{\Pi}^1_1$) equivalence relations satisfy the Silver-Vaught Dichotomy. (Remark: There is a proof of Silver's Theorem, due to Harrington, which is easier than the original proof, and which can be found in Martin-Kechris [32]; in this proof, Silver's Theorem is deduced from Proposition 2.A.2 (c), using the Gandy-Harrington topology.) There are $\mathbf{\Sigma}^1_1$ equivalence relations which violate it—see Example 5.C.2 (c), below.

5.C.1. Theorem (Burgess [8])**.** *Let R be a $\mathbf{\Sigma}^1_1$ equivalence relation on a Polish space Y. One of the following three cases holds.*

(a) *R has countably many equivalence classes.*
(b) *R has $\aleph_1$ and not perfectly many equivalence classes.*
(c) *R has perfectly many equivalence classes.*

Thus $\mathbf{\Sigma}^1_1$ equivalence relations come in three types. Assuming that the continuum hypothesis (CH) is false, the three types are just three cardinalities for the set of equivalence classes: $\aleph_0$, $\aleph_1$, $2^{\aleph_0}$. But if CH is true, we need a different way of distinguishing case (b) from case (c), and that is where the concept "perfectly many" comes in. The Silver-Vaught Dichotomy says that case (b) does not occur; assuming ¬CH, this dichotomy says that the set of equivalence classes is not a counterexample to CH. (The size of the continuum is irrelevant to any of the results in this paper.)

This dichotomy also asserts that either the quotient space Y/R is "small" or else it contains a copy of a specific "large" set. The two dichotomies are thus similar in form; they differ in the meaning of "small" and "large". The Glimm-Effros Dichotomy implies the Silver-Vaught Dichotomy, but not conversely. There are perfectly many isomorphism-types of countable abelian p-groups, hence this example satisfies the latter dichotomy and violates the former.

The *Topological Vaught Conjecture* is the following statement (which is, of course, an open question): for any Polish group G and for any Polish G-space X, X satisfies the Silver-Vaught Dichotomy.

The Topological Vaught Conjecture implies—and was motivated by—a famous conjecture in model theory, known as the Vaught Conjecture. The original *Vaught Conjecture* is the following conjecture about finitary first-order logic: any first-order theory in a countable language has either countably many or perfectly many isomorphism-types of countable models. This is open, as is the stronger *Vaught Conjecture for* $L_{\omega_1\omega}$, which is the same assertion for $L_{\omega_1\omega}$ sentences rather than first-order theories. "Perfectly many" refers to the orbit equivalence relation of the logic actions (see §2.D). The Vaught Conjecture for $L_{\omega_1\omega}$ is thus the special case of the Topological Vaught Conjecture corresponding to the Polish S_∞-actions $J'_L : S_\infty \times Y \to Y$, where: L is a countable language and $J_L : S_\infty \times X_L \to X_L$ is the corresponding logic action; φ is a sentence of $L_{\omega_1\omega}$ and F is a countable fragment of $L_{\omega_1\omega}$ with $\varphi \in F$; $Y \subset X_L$ is the invariant set $\mathrm{Mod}(\varphi)$, where X_L and hence Y have the t_F-topology, in which Y is clopen, hence Polish; and the action J'_L is the restriction of J_L to Y. (The above terminology is defined in 2.D.1.)

By Becker-Kechris [6, 5.2.1], a standard Borel G-space is Borel-isomorphic to a Polish G-space. Hence the Topological Vaught Conjecture implies that for all G, all standard Borel G-spaces satisfy the Silver-Vaught Dichotomy (cf. Corollary 5.B.3). This, in turn, implies that for all G, for any Polish G-space X, and for any invariant Borel set $B \subset X$, the orbit equivalence relation on B satisfies the Silver-Vaught Dichotomy (cf. Corollary 5.B.4). The hypothesis in the previous sentence that B is Borel is necessary, as shown by the following examples.

5.C.2. Examples. Let L be the language with one binary relation symbol and consider the logic action for L.

(a) There exists an invariant $\mathbf{\Pi}^1_1$ set $P \subset X_L$ which violates the Silver-Vaught Dichotomy, namely the (set of codes for the) countable ordinals.

(b) There also exist invariant $\mathbf{\Sigma}^1_1$ sets $S \subset X_L$ which violate the dichotomy. One example of such an S, due to Friedman [13], was given in Example 3.D.7 of this paper. A simpler example is:

$$S = \{x \in X_L : \mathfrak{A}_x \text{ is a countable linear ordering which is the order-type of an ordered abelian group}\}.$$

Morel [35] proved that there are exactly $\aleph_1$—and not perfectly many—isomorphism-types in S.

(c) Let P be as in Example (a), above, and let R be the following equivalence relation on X_L:

$$xRy \iff [\mathfrak{A}_x \cong \mathfrak{A}_y \text{ or } (x \notin P \text{ and } y \notin P)].$$

Then R is a $\mathbf{\Sigma}^1_1$ equivalence relation on a Polish space which violates the dichotomy.

Many special cases of the Topological Vaught Conjecture are known to hold. We discuss some of these in the historical remarks at the end of this section of the paper. For more information on the Topological Vaught Conjecture, see Sami [38], Becker [2], Becker-Kechris [6, Chapter 6], Hjorth-Solecki [22] and Hjorth [21]; on Silver-Vaught-type dichotomies, see Martin-Kechris [32], Stern [43], Harrington-Sami [19] and Harrington-Shelah [20]; on the abstract concept of *dichotomies*, see Kechris [26] and Kechris-Louveau [29].

5.C.3. Proposition. *Let G be a group and let X be a Polish G-space. The following are equivalent.*

(a) *The G-space X satisfies the Silver-Vaught Dichotomy.*

(b) *One of the following holds.*

  (i) *X has perfectly many orbits.*

  (ii) *There exists an $\alpha < \omega_1$ such that every orbit of X is $\mathbf{\Pi}^0_\alpha$.*

Proof. (a) $\Longrightarrow$ (b). Trivial.

(b) $\Longrightarrow$ (a). Trivially (b) (i) implies (a). So assume (b) (ii) holds. This implies that the orbit equivalence relation is Borel (see Sami [38, 3.7]). So by Silver's Theorem, this equivalence relation satisfies the Silver-Vaught Dichotomy. □

In the case of logic actions—restricted to $\mathrm{Mod}(\varphi)$, for some $\varphi \in L_{\omega_1\omega}$—(b) (ii) is equivalent to saying that the Scott ranks of the countable models of φ are bounded below ω_1 (Becker-Kechris [6, 7.1.4]). It is possible that both (b) (i) and (b) (ii) of 5.C.3 hold. Again, E_0 itself is an example. And there are many model theoretic examples.

There is a strong form of the model theoretic Vaught Conjecture, a rather technical statement known as the *Martin Conjecture* (see Wagner [46] or Bouscaren [7]). The Martin Conjecture implies that for any countable fragment F and any $L_{\omega_1\omega}$ sentence $\varphi \in F$, either φ has perfectly many countable models or else all orbits in $\mathrm{Mod}(\varphi)$ are $\mathbf{\Pi}^0_\omega$ with respect to the t_F-topology. We can generalize this idea from logic actions to arbitrary Polish G-spaces, as follows.

5.C.4. Definition. A G-space X satisfies the *Topological Martin Conjecture* if one of the following holds.

(a) X has perfectly many orbits.

(b) Every orbit of X is $\mathbf{\Pi}^0_\omega$.

By Proposition 5.C.3, the Topological Martin Conjecture implies the Topological Vaught Conjecture. The Topological Martin Conjecture—which originates with this paper—is also open. But for cli groups G, all G-spaces satisfy it; this is the next theorem.

5.C.5 Theorem. *Let G be a cli group. For any Polish G-space X, one of the following holds.*

(a) *X has perfectly many orbits.*

(b) *Every orbit of X is $\mathbf{\Pi}^0_\omega$.*

Proof. If X contains a copy of E_0, then it contains perfectly many orbits. So this theorem follows from Theorem 5.B.2. □

5.C.6. Corollary. *Let G be a cli group. Let X be a Polish G-space. Then X satisfies the Topological Vaught Conjecture.*

5.C.7. Corollary. *Let G be a cli group. Let X be a standard Borel G-space. Then the orbit equivalence relation of X satisfies the Silver-Vaught Dichotomy.*

5.C.8. Corollary. *Let G be a cli group, let X be a Polish G-space, let $B \subset X$ be an invariant Borel set and let R denote the orbit equivalence relation on B. Then R satisfies the Silver-Vaught Dichotomy.*

A Polish group G *satisfies the Topological Vaught Conjecture* (or *satisfies the Glimm-Effros Dichotomy*) if every Polish G-space does. We have, of course, proved that all cli groups G satisfy both (5.B.3 and 5.C.6). A number of special cases of these facts were known previously. We close §5.C with a discussion of these special cases and related matters.

The Topological Vaught Conjecture was motivated by the model theoretic Vaught Conjecture, and a number of special cases of the model theoretic conjecture have been proved; see Becker-Kechris [6, page 86] and the references therein. The model theoretic Vaught Conjecture for $L_{\omega_1\omega}$ is equivalent to the Topological Vaught Conjecture for the group S_∞ (Becker-Kechris [6, 6.2.7]), which is, of course, open.

Which Polish groups G are known to satisfy the Topological Vaught Conjecture? It is implicit in Glimm [16] that locally compact groups do. Later, in the late 1970's, Sami proved that abelian groups do (published in Sami [38]). And that is all that was known for about 15 years, until recent work of Solecki [40] and Hjorth-Solecki [22], which we now describe.

A group G is called *tame* if for every Polish G-space X, the orbit equivalence relation of X is Borel. In light of the theorem of Silver [39], all tame groups satisfy the Topological Vaught Conjecture. Locally compact groups are tame, which reproves the above result of Glimm. By a theorem of Mackey (see Becker-Kechris [6, 2.3.5]), a closed subgroup of a tame group is tame. And clearly the continuous homomorphic image of a tame group is tame. Solecki [40] has studied the tameness question for groups of the form $G = \prod_{n\in\mathbb{N}} H_n$, where the H_n's are countable discrete groups. He showed that there are a number of tame nonlocally compact groups of this form, e.g., $G = (\mathbb{Z}(p^\infty))^{\mathbb{N}} \oplus H$, where $\mathbb{Z}(p^\infty)$ is the quasicyclic p-group, that is, the multiplicative group of all complex p^n-th roots of unity for some n, and H is an arbitrary discrete group. As H need not be abelian, he thus produced new examples of groups known to satisfy the Topological Vaught Conjecture. He also showed that certain groups of this form are not tame. One of his examples follows.

5.C.9. Example (Solecki [40])**.** $\mathbb{Z}^{\mathbb{N}}$ is not tame. (By Sami [38, 3.7], this is equivalent to saying that there is a Polish $\mathbb{Z}^{\mathbb{N}}$-space X such that the Borel ranks of the orbits of X are unbounded.) Since $\mathbb{Z}^{\mathbb{N}}$ is abelian, there are nontame groups which satisfy the Topological Vaught Conjecture.

Hjorth-Solecki [22] then strengthened Sami's theorem on abelian groups in two separate ways: they proved the Topological Vaught Conjecture for nilpotent groups and for tsi groups. (See the diagram of inclusions following Theorem 3.C.1.) A countable product of tsi groups is tsi. In particular, those groups considered by Solecki [40]—countable products of discrete groups—are tsi groups.

By Proposition 3.C.2, any group which is locally compact or tsi or nilpotent is cli. Therefore, Corollary 5.C.6—the Topological Vaught Conjecture for cli groups—covered all the previously known cases of the Topological Vaught Conjecture. (And by Proposition 3.C.2 (i), it covers some additional cases as well.) Indeed, at the time Corollary 5.C.6 was proved it was an open question whether there is *any* non-cli group which satisfies the Topological Vaught Conjecture. Hjorth [21] later found an example of a non-cli group that does. His example is a closed subgroup of S_∞, the group of automorphisms of a structure which has certain technical model theoretic properties.

As previously mentioned, the Glimm-Effros Dichotomy implies the Silver-Vaught Dichotomy. However, the two dichotomies arose in different branches of mathematics, operator algebras and model theory, respectively, and it appears that until rather recently, the mathematicians interested in one of the dichotomies were completely unaware of the other. Harrington-Kechris-Louveau [18] seems to be the first paper that considers both.

Which Polish groups G satisfy the Glimm-Effros Dichotomy? As was pointed out in §5.B, S_∞ does not. There is thus no conjecture analogous to the Topological Vaught Conjecture for this dichotomy. Glimm [16] proved that locally compact groups do satisfy it. And that was all that was known until the work of Solecki [40] and Hjorth-Solecki [22]. (It was not known that abelian groups satisfy it.)

Harrington-Kechris-Louveau [18] proved that Borel equivalence relations satisfy the Glimm-Effros Dichotomy, hence all tame groups G do. This includes the examples of Solecki [40], such as $(\mathbb{Z}(p^\infty))^{\mathbb{N}}$. Hjorth-Solecki [22] proved the Glimm-Effros Dichotomy for nilpotent groups and for tsi groups (hence for abelian groups). So there are nontame groups, such as $\mathbb{Z}^{\mathbb{N}}$ (Example 5.C.9), which satisfy this dichotomy. Again, Corollary 5.B.3—the Glimm-Effros Dichotomy for cli groups—covered all the previously known cases of groups satisfying the Glimm-Effros Dichotomy. As mentioned in §5.B, it is still an open question whether there is *any* non-cli group which satisfies the Glimm-Effros Dichotomy.

But Theorem 5.B.2 and Corollary 5.B.3 do not actually subsume the previous work, because the previous work established something more than the Glimm-Effros Dichotomy, namely the Strong Glimm-Effros Dichotomy (defined in 5.B.5). Specifically, Glimm [16] proved the Strong Glimm-Effros Dichotomy for locally compact groups, and Hjorth-Solecki [22] proved it for nilpotent groups and tsi groups.

5.C.10. Theorem (Hjorth-Solecki [22], 4.1)**.** *There exists a Polish group G such that G is solvable of rank two, and there exists a Polish G-space X such that X has exactly two orbits, both of which are dense.*

Clearly the G-space X of 5.C.10 violates the Strong Glimm-Effros Dichotomy. By Proposition 3.C.2 (f), all solvable groups are cli. Thus actions of tsi groups are better behaved than those of arbitrary cli groups, as witnessed by the Strong Glimm-Effros Dichotomy; and, of course, actions of cli groups are better behaved than those of arbitrary Polish groups, as witnessed by the Glimm-Effros Dichotomy. By Lemma 5.B.8, if G is a cli group and X is an enhanced Polish G-space, then X does satisfy the Strong Glimm-Effros Dichotomy. This shows that enhanced Polish G-spaces are better behaved than arbitrary Polish G-spaces, and also shows that the “enhanced” hypothesis in 5.B.8 (and hence in 5.A.1) is necessary. While 5.C.10

shows that in Theorems 5.B.2 and 5.C.5, $\mathbf{\Pi}^0_\omega$ cannot be strengthened to $\mathbf{\Pi}^0_2(= G_\delta)$, it is not known that $\mathbf{\Pi}^0_\omega$ is the best possible bound.

Open Question. *Does there exist a cli group G, a Polish G-space X, and an orbit $G{\cdot}x$ of X such that the following two properties are satisfied?*
(a) *X has only countably many orbits.*
(b) *$G{\cdot}x$ is not $\mathbf{\Sigma}^0_\omega$.*

The Strong Glimm-Effros Dichotomy for tsi groups also yields a strong version of the Topological Martin Conjecture (5.C.4) for tsi groups: either X has perfectly many orbits or else every orbit is G_δ. While the Topological Martin Conjecture may be true for all Polish groups, this strong version of the Topological Martin Conjecture is false for the logic actions by S_∞. Translated into model theoretic language, this strong version says: for any first-order theory T, either T has perfectly many models or else every complete theory extending T is $\aleph_0$-categorical. That is, of course, absurd. The best possible bound for the model theoretic Martin Conjecture is $\mathbf{\Pi}^0_\omega$. It is remarkable that for cli groups we obtain exactly the "right" bound—right from the model theoretic point of view. But it also seems that we obtain the right bound for the wrong reason.

6. Consequences for analytic G-spaces

6.A. Generalized model theory: Steel's Theorem. In §6, we derive some other consequences of the Main Theorem (4.A.2). First we generalize a theorem of model theory from the logic actions to arbitrary Polish G-spaces. This result is stated in §6.A, and proved in §§6.B, 6.C. This result has several applications: one is given in §6.D; others will appear in Becker [4]. In §6.D, we prove another dichotomy theorem for cli groups. The proof in §6.D can be read independently of §§6.B, 6.C.

The proofs in §6 all involve a large cardinal axiom. The axiom which will be used is denoted #, and is the following statement:

$$\text{for all } r \in 2^{\mathbb{N}},\ r^{\#} \text{ exists.}$$

The axiom # is equivalent to $\mathbf{\Pi}^1_1$-determinacy, and also equivalent to several other interesting propositions. The existence of a measurable cardinal implies # (but is far stronger). For more information on large cardinal axioms, see Kanamori [25].

The analog of the Topological Vaught Conjecture for invariant $\mathbf{\Sigma}^1_1$ (as opposed to Borel) sets is false.

6.A.1. Proposition. *Let G be any Polish group such that S_∞ is a closed subgroup of G. There exists a Polish G-space X and an invariant $\mathbf{\Sigma}^1_1$ set $S \subset X$ such that S has uncountably many but not perfectly many orbits.*

Proof. For $G = S_\infty$, such an X and S are described in Example 5.C.2 (b). By a theorem of Mackey (see Becker-Kechris [6, 2.3.5]), if S_∞ is a closed subgroup of G, then this S_∞-action extends to a G-action satisfying Proposition 6.A.1. □

6.A.2. Theorem. *Assume #. Let G be a Polish group, let X be a Polish G-space and let $S \subset X$ be an invariant $\mathbf{\Sigma}^1_1$ set such that S has uncountably many but not perfectly many orbits. There exists an ω_1-sequence $\langle \mathcal{O}_\alpha : \alpha < \omega_1 \rangle$ of distinct orbits of S such that the following property holds. For any countable ordinals α and β, if $\alpha < \beta$, then $\mathcal{O}_\alpha$ is $\imath$-embeddable into $\mathcal{O}_\beta$.*

Theorem 6.A.2 will be proved in parts B and C of §6.

Actually, a result stronger than Theorem 6.A.2 will be proved (see Lemma 6.C.4, below). It is stronger in two ways: we can explicitly identify the orbits $\mathcal{O}_\alpha$; and we can get ι-embeddings with respect to topologies on X which are finer than the original topology. This stronger result is very technical, so we do not state it here.

Open Question. *Is Theorem 6.A.2 provable in ZFC?*

Let L be a countable language. A collection $\mathcal{S}$ of countably infinite L-structures is called a *pseudo-$L_{\omega_1\omega}$ class* if there exists a countable language L' with $L \subset L'$ and there exists an $L'_{\omega_1\omega}$ sentence φ such that

$$\mathcal{S} = \{\mathfrak{A} : \mathfrak{A} \text{ is a countably infinite } L\text{-structure} \\ \text{and } \mathfrak{A} \text{ is the } L\text{-reduct of a model of } \varphi\}.$$

6.A.3. Corollary (Steel [42], 3.2)**.** *Assume #. Let L be a countable language, let F be a countable fragment of $L_{\omega_1\omega}$ and let $\mathcal{S}$ be a pseudo-$L_{\omega_1\omega}$ class of L-structures such that $\mathcal{S}$ contains uncountably many but not perfectly many L-structures. There exists an ω_1-sequence $\langle \mathfrak{A}_\alpha : \alpha < \omega_1\rangle$ of nonisomorphic members of $\mathcal{S}$ such that the following property holds. For any countable ordinals α and β, if $\alpha < \beta$, then $\mathfrak{A}_\alpha$ is F-embeddable into $\mathfrak{A}_\beta$.*

Proof. Without loss of generality, L is a relational language. L must be nonempty, since there is more than one countably infinite L-structure. That is, to prove 6.A.3, we need only consider the situation that corresponds to the logic actions.

For this situation, Corollary 6.A.3 is a special case of Theorem 6.A.2. This is so because of the following two facts. First, by Steel [42, 1.3.1], for any set $A \subset X_L$, A is an invariant $\mathbf{\Sigma}^1_1$ set iff $\{\mathfrak{A}_x : x \in A\}$ is a pseudo-$L_{\omega_1\omega}$ class. Second, by Theorem 3.D.2, for x, $y \in X_L$, x is ι-embeddable into y with respect to the t_F-topology on X_L iff $\mathfrak{A}_x$ is F-embeddable into $\mathfrak{A}_y$. □

Thus Theorem 6.A.2 is a generalization of a theorem of model theory. Or to be more precise, it is a generalization of part of a theorem, since Steel [42, 3.2] actually states some additional properties of the ω_1-sequence $\langle \mathfrak{A}_\alpha : \alpha < \omega_1\rangle$ not stated in Corollary 6.A.3. We do not know whether other parts of Steel's result generalize from logic actions to arbitrary Polish G-spaces. The proof of Corollary 6.A.3 given in this paper is new; it is not Steel's proof translated into the language of group actions.

6.B. Descriptive set theoretic lemmas. In §§6.B, 6.C, we prove Theorem 6.A.2. Throughout these two sections we have a fixed group G, a fixed Polish G-space (X, a) and a fixed invariant $\mathbf{\Sigma}^1_1$ set $S \subset X$ with uncountably many but not perfectly many orbits. Let τ denote the Polish topology on X. By Proposition 2.B.3, there exists an enhanced basis $\mathcal{N}^* = \langle \mathcal{N}, \{g_k^{\langle i,j\rangle}\}\rangle$ for G; we fix such an $\mathcal{N}^*$. Finally, we fix a countable basis $\mathcal{D}$ for $\langle X, \tau\rangle$. All definitions in the sequel depend on the above, but we regard them as fixed, and hence leave them out of the notation.

Reading §§6.B, 6.C requires more background in set theory than is required for reading the rest of this paper. We assume the reader is familiar with the basic results of descriptive set theory, including effective descriptive set theory and including the second level of the projective hierarchy. This material can be found in Moschovakis [36]. We also assume familiarity with models of set theory, including: the Shoenfield Absoluteness Theorem; the models $L[r]$, for $r \in 2^{\mathbb{N}}$; the theory of

Silver indiscernibles; and very elementary forcing (specifically, collapsing a cardinal and "making it countable"). All of this can be found in Jech [24]; all except the forcing can be found in Moschovakis [36]; a kinder and gentler introduction to forcing can be found in Kunen [31].

In this section we give preliminary definitions and prove some lemmas. Much of the material in this section involves calculating the complexity of various pointsets with respect to the analytical hierarchy. None of this involves models of set theory; that material is in §6.C.

Throughout §§6.B, 6.C, we do effective descriptive set theory. The letter r denotes a "real", i.e., an element of $2^{\mathbb{N}}$. The terminology *effective-in-r* for a group G, a Polish G-space, an enhanced basis, etc., has the obvious meaning (cf. Becker-Kechris [6, 7.2] and Becker [2, 0.E]).

We encode countable ordinals by elements of $2^{\mathbb{N}}$ in the usual way, and use the usual terminology, for example, a *condition* $p \in \alpha^{<\mathbb{N}}$. Let $w \in WO$ and let $\alpha = |w|$. For any $\gamma < \alpha$, there is a canonical way to obtain a code for γ from w: take the appropriate initial segment of $\leq_w$, where $\leq_w$ is the wellordering of a subset of $\mathbb{N}$ represented by w. Let $w \restriction \gamma$ denote this code for γ.

6.B.1. Proposition. (a) *Let $\gamma_1 < \gamma_2 < \alpha < \omega_1$. For any code w for α, $w \restriction \gamma_1$ is recursive-in-$(w \restriction \gamma_2)$.*

(b) *Let $\gamma < \alpha < \beta < \omega_1$, where α and β are limit ordinals. Let W_α and W_β be the Polish spaces of codes for α and β, respectively. Let A_α and A_β be comeager subsets of W_α and W_β, respectively. Let $p \in \alpha^{<\mathbb{N}}$. There exist u, v such that:*

(i) *$u \in A_\alpha$ and $v \in A_\beta$;*

(ii) *u and v are both consistent with the condition p;*

(iii) *$u \restriction \gamma = v \restriction \gamma$.*

6.B.2. Definition. Let $r \in 2^{\mathbb{N}}$ be such that $G, \mathcal{N}^*, X, \tau, \mathcal{D}$ and a are all effective-in-r. For any countable limit ordinal α, and for any code w for α, we define a new topology—denoted $t(r, \alpha, w)$—on X, as follows. Let

$$\mathcal{B} = \{B \subset X : (\exists \gamma < \alpha)(B \text{ is } \Sigma^1_1(r, w \restriction \gamma))\}.$$

Let $\mathcal{S} = \{B^{\Delta N} : B \in \mathcal{B},\ N \in \mathcal{N}\}$, and let $t(r, \alpha, w)$ be the topology on X generated by the subbasis $\mathcal{S} \cup \mathcal{D}$.

In the sequel, all references to pointclasses for pointsets in X, e.g., $\Sigma^1_1(r)$, refer to the original Polish topology τ on X. For $\gamma < \omega_1$, $\Sigma^1_1(r, \gamma)$ denotes the pointclass of sets which are uniformly $\Sigma^1_1(r, w)$ for all codes w for γ.

6.B.3. Lemma. *Let r be as in Definition 6.B.2. Let $\alpha < \omega_1$ be a limit ordinal and let w be a code for α.*

(a) *$(\langle X, t(r, \alpha, w)\rangle, a)$ is an acceptable G-space. The $t(r, \alpha, w)$ topology is finer than τ.*

(b) *For any $\gamma < \alpha$, any invariant $\Sigma^1_1(r, \gamma)$ subset of X is open with respect to the topology $t(r, \alpha, w)$.*

Proof. (a) By Proposition 6.B.1 (a), the set $\mathcal{B}$ in Definition 6.B.2 is closed under finite unions and intersections. Therefore, the proof in Becker-Kechris [6, 5.4.5] demonstrates: that a is continuous with respect to $t(r, \alpha, w)$, hence we have a topological G-space; and that $\langle X, t(r, \alpha, w)\rangle$ is a strong Choquet space. It is obvious that $t(r, \alpha, w)$ is second countable and finer than τ. So $t(r, \alpha, w)$ is acceptable, by definition (2.A.1).

(b) Obvious. $\square$

6.B.4. Lemma. *Let $r \in 2^{\mathbb{N}}$ be such that G and (X, a) are effective-in-r.*

(a) *Let $\mathcal{F}(G)$ denote the effective-in-r Borel space of closed subsets of G, with the Effros Borel structure (as defined in Kechris* [28, 12.C]*). The following subset of $X \times \mathcal{F}(G)$ is $\Pi^1_1(r)$:*

$$\{(x, H) : H \text{ is the stabilizer subgroup of } x\}.$$

(b) *Let $x \in X$ and let H be the stabilizer subgroup of x. The orbit $G \cdot x$ is $\Delta^1_1(r, x, H)$, uniformly-in-(x, H).*

Proof. (a) Obvious.

(b) This is an effectivized version of a theorem of Dixmier [10]. The Dixmier proof—which can also be found in Kechris [28, 12.17]—establishes 6.B.4 (b). $\square$

6.B.5. Definition. Let r be as in Definition 6.B.2. Define $W^r \subset 2^{\mathbb{N}} \times X$ as follows:

$$\begin{aligned} W^r(w, x) \iff\ & w \in WO \ \&\ |w| \text{ is a limit ordinal} \\ & \&\ (\text{the orbit } G \cdot x \text{ is } G_\delta \text{ with respect to} \\ & \qquad\qquad \text{the topology } t(r, |w|, w)). \end{aligned}$$

6.B.6. Lemma. *Let r be as in Definition 6.B.2. Then W^r is $\Sigma^1_2(r)$.*

Proof. Fix r. For $u \in 2^{\mathbb{N}}$, let $\mathcal{B}^u = \{B \subset X : B \text{ is } \Sigma^1_1(r, u)\}$, let $\mathcal{S}^u = \{B^{\Delta N} : B \in \mathcal{B}^u,\ N \in \mathcal{N}\}$, and let $\mathcal{C}^u$ be all finite intersections of members of $\mathcal{S}^u \cup \mathcal{D}$. We can easily get an enumeration C^u_i of $\mathcal{C}^u$ such that

$$\{(u, i, x) \in 2^{\mathbb{N}} \times \mathbb{N} \times X : x \in C^u_i\}$$

is $\Sigma^1_1(r)$. By Proposition 6.B.1 (a), the basis for $t(r, |w|, w)$ is

$$\{C^u_i : i \in \mathbb{N} \ \&\ (\exists \gamma < |w|)(u = w \restriction \gamma)\}.$$

Now $G \cdot x$ is G_δ iff there exists a G_δ set $Q \subset G \cdot x$ such that $G \cdot x = Q^*$. Hence,

$$\begin{aligned} W^r(w, x) \iff\ & w \in WO \ \&\ |w| \text{ is a limit ordinal} \\ & \&\ (\exists \sigma_1 \in \mathbb{N}^{(\mathbb{N}\times\mathbb{N})})(\exists \sigma_2 \in (2^{\mathbb{N}})^{(\mathbb{N}\times\mathbb{N})}) \\ & (\exists H \in \mathcal{F}(G))[(\forall m \forall n)(\exists \gamma < |w| \text{ such that } \sigma_2(m, n) = w \restriction \gamma) \ \& \\ & (\text{for a comeager set of } g \in G)(\forall m \exists n \text{ such that } g \cdot x \in C^{\sigma_2(m,n)}_{\sigma_1(m,n)}) \ \& \\ & [H \text{ is the stabilizer subgroup of } x \ \&\ (\forall y \in X)(\text{if } y \notin G \cdot x \text{ then} \\ & \quad \exists m \forall n,\ y \notin C^{\sigma_2(m,n)}_{\sigma_1(m,n)})]]. \end{aligned}$$

Using Lemma 6.B.4, the above formula demonstrates that W^r is $\Sigma^1_2(r)$. $\square$

Assume #. Fix $s \in 2^{\mathbb{N}}$ such that G, $\mathcal{N}^*$, X, τ, $\mathcal{D}$ and a are all effective-in-s and S is $\Sigma^1_1(s)$. Fix r such that $s^\#$ is recursive-in-r. Note that this r satisfies the hypothesis of 6.B.2–6.B.6.

6.B.7. Lemma. *Assume #. There exist two prewellorderings of S, $\psi : S \to \omega_1$ and $\widetilde{\psi} : S \to \omega_1$, satisfying the following six properties.*

(a) *If $y \in G \cdot x$, then $\psi(y) = \psi(x)$.*

(b) *For all $\alpha < \omega_1$, $\{x \in S : \psi(x) < \alpha\}$ contains only countably many orbits.*

(c) *For all $\alpha < \omega_1$, the set $\psi^{\geq \alpha} = \{x \in S : \psi(x) \geq \alpha\}$ is $\Sigma^1_1(r, \alpha)$, uniformly-in-α.*

(d) *The levels of the prewellordering* $\widetilde{\psi}$ *are the orbits of* S.
(e) $\widetilde{\psi}$ *is a* $\Delta^1_2(r)$ *prewellordering of* S.
(f) *If* $\psi(x) < \psi(y)$, *then* $\widetilde{\psi}(x) < \widetilde{\psi}(y)$.

Proof. First, we must get a ψ satisfying (a)–(c). In the case where X is an effective Polish G-space, this is proved in Becker [2, 3.1 and 3.3], taking $\psi(x) = \omega_1^{G\cdot x}$. In general, relativize. (Remark: This does not require #; $\psi^{\geq\alpha}$ is $\Sigma^1_1(s,\alpha)$, uniformly.)

By a theorem of Burgess [9], assuming #, there is a $\Delta^1_2(s^{\#})$ prewellordering φ of S such that the levels of φ are the orbits of S. Let $\widetilde{\psi}$ be obtained by prewellordering each ψ-equivalence class according to φ. Clearly (d)–(f) hold. □

Fix ψ and $\widetilde{\psi}$ satisfying Lemma 6.B.7.

6.B.8. Lemma. *Assume #. For any countable limit ordinal* α *and any code* w *for* α, *there exists an orbit* $G\cdot x$ *such that:*
(a) $G\cdot x \subset S$;
(b) $\psi(x) \geq \alpha$;
(c) $G\cdot x$ *is a* G_δ *orbit of* X *with respect to the topology* $t(r,\alpha,w)$.

Proof. Fix α and w. Let $Y = S \cap [\bigcap_{\gamma<\alpha} \psi^{\geq\gamma}]$. By Lemma 6.B.7 (c), $\psi^{\geq\gamma}$ is $\Sigma^1_1(r,\gamma)$; by 6.B.7 (a), it is invariant; hence by Lemma 6.B.3 (b), $\psi^{\geq\gamma}$ is open with respect to the topology $t(r,\alpha,w)$. Similarly, S is $t(r,\alpha,w)$-open. Therefore, Y is a $t(r,\alpha,w)$-G_δ subset of X. Lemma 6.B.7 (b) implies that $Y \neq \emptyset$. By Lemma 6.B.3 (a), $\langle X, t(r,\alpha,w)\rangle$ is an acceptable topological space. So by Proposition 2.A.2 (b), $\langle Y, t(r,\alpha,w)\rangle$ is itself acceptable. Of course, Y is $\mathbf{\Sigma}^1_1$ with respect to the original Polish topology τ; hence $\langle Y,\tau\rangle$ is an analytic space. The orbit equivalence relation E on Y is a $\mathbf{\Sigma}^1_1$ subset of $\langle Y,\tau\rangle \times \langle Y,\tau\rangle$. So parts (c) and (d) of Proposition 2.A.2 allow us to conclude that one of the following must hold.

(i) There exists a continuous function $f : 2^{\mathbb{N}} \to \langle Y, t(r,\alpha,w)\rangle$ such that for any two distinct points $\mathbf{p}$, $\mathbf{q} \in 2^{\mathbb{N}}$, $f(\mathbf{p})$ and $f(\mathbf{q})$ are in different orbits.

(ii) There exists an orbit $G\cdot x \subset Y$ which is not meager in $\langle Y, t(r,\alpha,w)\rangle$.

If case (i) occurs, then since $t(r,\alpha,w)$ is finer than τ, f is also continuous with respect to τ; hence Y contains τ-perfectly many orbits. But $Y \subset S$ and, by hypothesis, S does not contain perfectly many orbits.

So case (ii) must hold. Let $G\cdot x$ be an orbit satisfying case (ii). Trivially, (a) and (b) of 6.B.8 are satisfied. Now $G\cdot x$ is not meager in the acceptable topological space $\langle Y, t(r,\alpha,w)\rangle$. So by Proposition 2.E.2, $G\cdot x$ is a G_δ subset of Y with respect to $t(r,\alpha,w)$. As Y is a $t(r,\alpha,w)$-G_δ subset of X, (c) is also satisfied. □

6.B.9. Definition. For any countable limit ordinal α and any code w for α, let $\mathcal{O}(\alpha,w)$ denote the $\widetilde{\psi}$-least orbit satisfying Lemma 6.B.8 for w. Define $I \subset 2^{\mathbb{N}} \times X$ as follows:

$$I = \{(w,x) : w \in \mathrm{WO} \ \&\ |w| \text{ is a limit ordinal } \&\ G\cdot x = \mathcal{O}(|w|,w)\}.$$

6.B.10. Lemma. *Assume #.*

(a) *For any countable limit ordinal* α *and any code* w *for* α, *there is an* $x \in \mathcal{O}(\alpha,w)$ *such that* x *is* $\Delta^1_2(r,w)$.

(b) I *is in the effective-in-*r*-*σ*-algebra generated by* $\Pi^1_2(r)$.

(c) *For any countable limit ordinal* α, *the function* $w \mapsto \mathcal{O}(\alpha,w)$ *is constant on a nonmeager set of codes for* α.

Proof. Let Q be the following subset of $2^{\mathbb{N}} \times X$:

$$\{(w,x) : w \in \mathrm{WO} \ \&\ |w| \text{ is a limit ordinal} \\ \&\ \text{the orbit } G{\cdot}x \text{ satisfies Lemma 6.B.8 for } w\}.$$

That is, $(w,x) \in Q$ iff: $G{\cdot}x \subset S$, $\psi(x) \geq |w|$ and $W^r(w,x)$. (W^r is defined in 6.B.5.) So by Lemmas 6.B.6 and 6.B.7 (c), Q is $\Sigma^1_2(r)$.

(a) Fix α and w. Let $Z = \{z \in X : z \in \Delta^1_2(r,w) \ \&\ Q(w,z)\}$. By the Kondo-Addison Theorem, $Z \neq \emptyset$. Fix a $z \in Z$ which is $\widetilde{\psi}$-minimal among members of Z. It will suffice to show that $G{\cdot}z = \mathcal{O}(\alpha,w)$. Suppose not. Let $Y = \{y \in X : Q(w,y) \ \&\ \widetilde{\psi}(y) < \widetilde{\psi}(z)\}$. Then by definition of $\mathcal{O}(\alpha,w)$ (6.B.9), $Y \neq \emptyset$. Now Q is $\Sigma^1_2(r)$ and by Lemma 6.B.7(e), $\widetilde{\psi}$ is $\Delta^1_2(r)$. Hence Y is $\Sigma^1_2(r,w)$. Again using the Kondo-Addison Theorem, Y contains a $\Delta^1_2(r,w)$ element, contradicting the minimality of z.

(b) By part (a), above:

$$(w,x) \in I \iff [\exists z \in \Delta^1_2(r,w)][Q(w,z) \ \&\ \forall y[\text{ if } (y \in S \ \&\ \widetilde{\psi}(y) < \widetilde{\psi}(z)), \\ \text{then } \neg Q(w,y)] \ \&\ x \in G{\cdot}z].$$

As previously mentioned, Q is $\Sigma^1_2(r)$ and $\widetilde{\psi}$ is $\Delta^1_2(r)$.

(c) Let W be the Polish space of codes for α. Let R be the following equivalence relation on W:

$$wRw' \text{ iff } \mathcal{O}(\alpha,w) = \mathcal{O}(\alpha,w').$$

By (a), above:

$$wRw' \Leftrightarrow [\exists x \in \Delta^1_2(r,w)][(w,x) \in I \ \&\ (w',x) \in I].$$

By (b), above, this formula demonstrates that R is in the σ-algebra generated by $\mathbf{\Pi}^1_2$. The axiom # implies that $\mathbf{\Pi}^1_2$ sets have the property of Baire. Therefore R has the property of Baire. As W is a Polish space, hence acceptable, Proposition 2.A.2 (c) shows that either R has perfectly many equivalence classes or else R has a nonmeager equivalence class. In the latter case, (c) is proved. Suppose the former case holds: $P \subset W$ is a perfect set of pairwise R-inequivalent points. By (a) and (b), above, there is a choice function $F : w \mapsto F(w)$—which for each $w \in W$ chooses an $F(w)$ in $\mathcal{O}(\alpha,w)$—such that F is $\mathcal{M}$-measurable, where $\mathcal{M}$ is the σ-algebra generated by $\mathbf{\Pi}^1_2$. Hence $F \restriction P$ has the property of Baire. So there is a Cantor set $P' \subset P$ such that $F \restriction P'$ is continuous. Recall that $\mathcal{O}(\alpha,w) \subset S$. Hence $F[P']$ is a perfect subset of S. By definition of R, no two points in $F[P']$ are in the same orbit. So S contains perfectly many orbits, contrary to assumption. $\square$

6.B.11. Definition. For any countable limit ordinal α, let $\mathcal{O}(\alpha)$ denote the $\widetilde{\psi}$-least orbit $G{\cdot}x$ such that for a nonmeager set of codes w for α, $G{\cdot}x = \mathcal{O}(\alpha,w)$. Define $J \subset 2^{\mathbb{N}} \times X$ as follows:

$$J = \{(w,x) : w \in \mathrm{WO} \ \&\ |w| \text{ is a limit ordinal} \ \&\ G{\cdot}x = \mathcal{O}(|w|)\}.$$

Thus for each countable limit ordinal α, we have picked out a canonical orbit: $\mathcal{O}(\alpha)$. By 6.B.7 (d) and 6.B.10 (c), $\mathcal{O}(\alpha)$ is well defined. We need one final descriptive set theoretic lemma, one involving these canonical orbits.

6.B.12. Lemma. *Assume #.*

(a) *For any countable limit ordinal* α *and any code* w *for* α*, there is an* $x \in \mathcal{O}(\alpha)$ *such that* x *is* $\Delta^1_2(r, w)$.

(b) J *is in the effective-in-*r*-*σ*-algebra generated by* $\Pi^1_2(r)$.

Proof. Similar to the proof of Lemma 6.B.10 (a), (b). □

6.C. Proof of the generalized Steel Theorem. As in §6.B, we have fixed G, $\mathcal{N}^* = \langle \mathcal{N}, \{g_k^{\langle i,j\rangle}\}\rangle$, X, τ, $\mathcal{D}$, a, S, r, ψ, $\widetilde{\psi}$. And for any countable limit ordinal α and any code w for α, we have defined a topology $t(r, \alpha, w)$ on X (6.B.2), and have defined orbits $\mathcal{O}(\alpha, w)$ and $\mathcal{O}(\alpha)$ in S, and corresponding pointsets I and J in $2^{\mathbb{N}} \times X$ (6.B.9 and 6.B.11).

We now consider the model $L[r]$, and other models extending it, and their relationship with these canonical orbits. All of our models—including forcing extensions—are submodels of V. The axiom # implies that (in V): for any $t \in 2^{\mathbb{N}}$, and for any countable ordinal α, the set of codes for α which are generic over $L[t]$ is comeager.

6.C.1. Lemma. *Assume #.*

(a) *For any transitive model* M *of* ZFC *with* $L[r] \subset M \subset V$ *and for any countable limit ordinal* α*, if* $M \models$ *"*α *is countable", then there exists an* $x \in \mathcal{O}(\alpha)$ *such that* $x \in M$.

(b) *There is a formula* $\varphi(v_1, v_2, v_3)$ *of the language of set theory, with three free variables and no parameters, with the following property. Let* M *be any transitive model of* ZFC *with* $L[r] \subset M \subset V$*, let* α *be any limit ordinal which is countable in* M *and let* w *be any code for* α *which is in* M*. For any* $x \in X \cap M$:

$$x \in \mathcal{O}(\alpha) \text{ iff } M \models \varphi(r, w, x).$$

Proof. Lemma 6.B.12 and the Shoenfield Absoluteness Theorem. □

Fix the formula φ satisfying 6.C.1 (b).

6.C.2. Lemma. *Assume #. Let* $\gamma < \alpha < \beta < \omega_1$*, and suppose that* α *and* β *are Silver indiscernibles for* $L[r]$*. Let* w *be any code for* α *which is generic over* $L[r]$*. For any invariant* $\Sigma^1_2(r, w \upharpoonright \gamma)$ *set* $Z \subset X$*, if* $\mathcal{O}(\alpha) \subset Z$*, then* $\mathcal{O}(\beta) \subset Z$.

Proof. Suppose the lemma is false: $\mathcal{O}(\alpha) \subset Z$ and $\mathcal{O}(\beta) \subset X \setminus Z$.

Let $\varphi_Z(v_1, v_2, v_3)$ be a Σ^1_2-formula with three free variables such that for all $x \in X$: $\varphi_Z(r, w \upharpoonright \gamma, x)$ iff $x \in Z$. Define another formula $\widehat{\varphi}_Z(v_1, v_2, v_3)$ as follows. $\widehat{\varphi}_Z(v_1, v_2, v_3)$ means: $(\exists g \in G)\varphi_Z(v_1, v_2, g{\cdot}v_3)$. Then $\widehat{\varphi}_Z$ is also a Σ^1_2-formula. And $\widehat{\varphi}_Z$ has the following invariance property.

(a) For any t_1, $t_2 \in 2^{\mathbb{N}}$ and any y, $z \in X$, if $z \in G{\cdot}y$, then: $\widehat{\varphi}_Z(t_1, t_2, y)$ iff $\widehat{\varphi}_Z(t_1, t_2, z)$.

Moreover, for all $x \in X$:

(b) $\widehat{\varphi}_Z(r, w \upharpoonright \gamma, x)$ iff $x \in Z$.

Let $\mathbb{P}_\alpha$ and $\mathbb{P}_\beta$ denote the posets for collapsing α and β, respectively, let $\Vdash_\alpha$ and $\Vdash_\beta$ denote the forcing relations, and let $\dot{w}_\alpha$ and $\dot{w}_\beta$ be the names—in the appropriate forcing language—for the generic codes for α and β, respectively.

Claim. There is a condition $p \in \mathbb{P}_\alpha$ such that for any code w' for α which extends p and is generic over $L[r]$:

$$\begin{aligned} L[r, w'] \models \text{“}&(\forall x \in X)(\text{if } \varphi(r, w', x), \text{ then } \widehat{\varphi}_Z(r, w' \upharpoonright \gamma, x))\ \& \\ &\mathbb{P}_\beta \Vdash_\beta (\forall x \in X)(\text{if } \varphi(r, \dot{w}_\beta, x), \text{ then } \neg\widehat{\varphi}_Z(r, w' \upharpoonright \gamma, x))\text{”}. \end{aligned}$$

Proof of Claim. It will suffice to show that it is true for $w' = w$; since w is generic over $L[r]$, there is then a condition p which forces the statement in quotation marks, and we are done. We first show that the first conjunct of the statement in quotation marks holds in $L[r,w]$. Let $x \in X \cap L[r,w]$, and suppose that $L[r,w] \models \varphi(r,w,x)$. By Lemma 6.C.1 (b), $x \in \mathcal{O}(\alpha)$. Now $\mathcal{O}(\alpha) \subset Z$, so $x \in Z$, hence by (b), $\widehat{\varphi}_Z(r, w \restriction \gamma, x)$ is true (in V). By the Shoenfield Absoluteness Theorem, $L[r,w] \models \widehat{\varphi}_Z(r, w \restriction \gamma, x)$. Hence the first conjunct holds in $L[r,w]$. Now we prove that the second conjunct holds in that model. Let v be any code for β. Suppose $x \in X \cap L[r,w,v]$ and $L[r,w,v] \models \varphi(r,v,x)$; by 6.C.1 (b), $x \in \mathcal{O}(\beta)$; so $x \in X \setminus Z$ and hence by (b), $\neg\widehat{\varphi}_Z(r, w \restriction \gamma, x)$ is true; again using Shoenfield's Theorem, $L[r,w,v] \models \neg\widehat{\varphi}_Z(r, w \restriction \gamma, x)$. Thus for any code v for β:

$$L[r,w,v] \models (\forall x \in X)(\text{if } \varphi(r,v,x),\ \text{then } \neg\widehat{\varphi}_Z(r, w \restriction \gamma, x)).$$

In particular, for any code v for β which is generic over $L[r,w]$, the above is true; therefore (in $L[r,w]$), $\mathbb{P}_\beta$ forces it to be true. Hence the second conjunct holds in $L[r,w]$. This proves the claim.

Fix a condition p in $\mathbb{P}_\alpha$ satisfying the claim. Of course, p is also a condition in $\mathbb{P}_\beta$. By Proposition 6.B.1 (b), there exist codes u and v for α and β, respectively, both extending p and both generic over $L[r]$, such that $u \restriction \gamma = v \restriction \gamma$.

The claim tells us that:

$$L[r] \models \text{“}p \Vdash_\alpha (\forall x \in X)(\text{ if } \varphi(r, \dot{w}_\alpha, x),\ \text{then } \widehat{\varphi}_Z(r, \dot{w}_\alpha \restriction \gamma, x))\text{”}.$$

The statement which is forced by p is a statement, in the language of set theory, whose only parameters are r, γ, and the generic code for α, where $\gamma < \alpha$. Of course, p is a finite sequence of ordinals less than α. Since $\alpha < \beta$ and both are Silver indiscernibles for $L[r]$, we have:

$$L[r] \models \text{“}p \Vdash_\beta (\forall x \in X)(\text{if } \varphi(r, \dot{w}_\beta, x),\ \text{then } \widehat{\varphi}_Z(r, \dot{w}_\beta \restriction \gamma, x))\text{”}.$$

Since v is generic over $L[r]$:

$$L[r,v] \models (\forall x \in X)(\text{if } \varphi(r,v,x),\ \text{then } \widehat{\varphi}_Z(r, v \restriction \gamma, x)).$$

By Lemma 6.C.1 (a), there is a $y \in \mathcal{O}(\beta)$ such that $y \in L[r,v]$, and by 6.C.1 (b), $L[r,v] \models \varphi(r,v,y)$. Therefore, $L[r,v] \models \widehat{\varphi}_Z(r, v \restriction \gamma, y)$. Again, this is absolute, so $\widehat{\varphi}_Z(r, v \restriction \gamma, y)$ is true. Since $u \restriction \gamma = v \restriction \gamma$, $\widehat{\varphi}_Z(r, u \restriction \gamma, y)$ is true. Thus we have shown the following fact.

(c) There exists a $y \in \mathcal{O}(\beta)$ such that $\widehat{\varphi}_Z(r, u \restriction \gamma, y)$.

Let v' be a code for β which is generic over $L[r,u]$. The claim also tells us (letting $w' = u$) that:

$$L[r,u,v'] \models (\forall x \in X)(\text{if } \varphi(r,v',x),\ \text{then } \neg\widehat{\varphi}_Z(r, u \restriction \gamma, x)).$$

By Lemma 6.C.1 (a), there is a $z \in \mathcal{O}(\beta)$ such that $z \in L[r,u,v']$, and by 6.C.1 (b), $L[r,u,v'] \models \varphi(r,v',z)$. Therefore, $L[r,u,v'] \models \neg\widehat{\varphi}_Z(r, u \restriction \gamma, z)$. This is absolute, so $\neg\widehat{\varphi}_Z(r, u \restriction \gamma, z)$ is true. Thus we have shown the following fact.

(d) There exists a z in $\mathcal{O}(\beta)$ such that $\neg\widehat{\varphi}_Z(r, u \restriction \gamma, z)$.

Clearly (a), (c) and (d), together, are absurd. $\square$

6.C.3. Lemma. *Assume #. Let $\alpha < \beta < \omega_1$, and suppose that α and β are Silver indiscernibles for $L[r]$. Then $\mathcal{O}(\alpha) \neq \mathcal{O}(\beta)$.*

Proof. We use the same notation regarding forcing as in the proof of 6.C.2. By Lemma 6.C.1, for any countable limit ordinal $\delta > \alpha$, $\mathcal{O}(\alpha) = \mathcal{O}(\delta)$ iff:

$$L[r] \models \text{“}\mathbb{P}_\delta \Vdash_\delta (\forall x \in X)(\text{if } \varphi(r, \dot{w}_\delta, x), \text{ then } \varphi(r, \dot{w}_\delta \restriction \alpha, x))\text{”}.$$

The only parameters in the statement in quotation marks are δ, r and α. So if γ and γ' are two Silver indiscernibles for $L[r]$, both greater than α, then the statement in quotation marks is true in $L[r]$ for $\delta = \gamma$ iff it is true in $L[r]$ for $\delta = \gamma'$. That is, $\mathcal{O}(\alpha) = \mathcal{O}(\beta)$ iff for all countable Silver indiscernibles $\beta' > \alpha$, $\mathcal{O}(\alpha) = \mathcal{O}(\beta')$.

Recall that we have chosen $\psi : S \to \omega_1$ satisfying Lemma 6.B.7. Recall also that for any countable limit ordinal δ, $\mathcal{O}(\delta) \subset S$ and for all $x \in \mathcal{O}(\delta)$, $\psi(x) \geq \delta$ (Lemmas 6.B.7 (a) and 6.B.8 (a), (b) and Definitions 6.B.9 and 6.B.11). Now let $y \in \mathcal{O}(\alpha)$, and let β' be a countable indiscernible greater than $\psi(y)$ (hence greater than α). As shown above, it will suffice to prove that $\mathcal{O}(\beta') \neq \mathcal{O}(\alpha)$. Suppose $\mathcal{O}(\beta') = \mathcal{O}(\alpha)$. Then $y \in \mathcal{O}(\beta')$, hence $\psi(y) \geq \beta' > \psi(y)$. □

6.C.4. Lemma. *Assume #. Let $\alpha < \beta < \omega_1$, and suppose that α and β are Silver indiscernibles for $L[r]$. There exists a code w for α such that $\mathcal{O}(\alpha)$ is $\imath$-embeddable into $\mathcal{O}(\beta)$ with respect to the topology $t(r, \alpha, w)$.*

Proof. There exists a code w for α such that:
(a) $\mathcal{O}(\alpha) = \mathcal{O}(\alpha, w)$;
(b) w is generic over $L[r]$.

This is so because by definition of $\mathcal{O}(\alpha)$ (6.B.11), the set of codes satisfying (a) is nonmeager, and by #, the set of codes satisfying (b) is comeager. Fix a code w satisfying (a) and (b); we prove 6.C.4 for this w.

Let $\mathcal{C}$ denote the basis for $t(r, \alpha, w)$ that was defined in 6.B.2. That is, $C \in \mathcal{C}$ iff C is a finite intersection of sets in the τ-basis $\mathcal{D}$ and sets of the form $B^{\Delta N}$, where B is $\Sigma^1_1(r, w \restriction \gamma)$ for some $\gamma < \alpha$ and $N \in \mathcal{N}$.

Let $x \in \mathcal{O}(\alpha)$ and $y \in \mathcal{O}(\beta)$. First we show that y is in the $t(r, \alpha, w)$-closure of $G{\cdot}x$. For suppose not. Then there is a $C \in \mathcal{C}$ such that $y \in C$ and $C \cap G{\cdot}x = \emptyset$. By definition of $\mathcal{C}$, C is $\Sigma^1_1(r, w \restriction \gamma_1, \ldots, w \restriction \gamma_k)$ for some $\gamma_1, \ldots, \gamma_k < \alpha$. By Proposition 6.B.1 (a), there exists a $\gamma < \alpha$ such that C is $\Sigma^1_1(r, w \restriction \gamma)$. Let

$$Z = \{x \in X : (\forall g \in G)(g{\cdot}x \notin C)\}.$$

Then Z is $\Pi^1_1(r, w \restriction \gamma)$, hence $\Sigma^1_2(r, w \restriction \gamma)$, where $\gamma < \alpha$ and where, by (b), w is generic over $L[r]$; Z is invariant; $\mathcal{O}(\alpha) = G{\cdot}x \subset Z$; and $\mathcal{O}(\beta) = G{\cdot}y \subset X \setminus Z$. This contradicts Lemma 6.C.2.

For any C, $D \in \mathcal{C}$, and any i, $j \in \mathbb{N}$, let $A(C, D, i, j)$ denote the following subset of X:

$$\left[C \setminus \left(\bigcup_{k \in \mathbb{N}} (g_k^{\langle i,j \rangle} \cdot D) \right) \right]^{\Delta}.$$

(Here $g_k^{\langle i,j \rangle} \in G$, and the indices i, j, k refer to the enhanced basis $\mathcal{N}^*$ for G which was chosen at the beginning of §6.B. “Enhanced basis” is defined in 2.B.2.) For any C, $D \in \mathcal{C}$ and any i, $j \in \mathbb{N}$, $A(C, D, i, j)$ is invariant. By definition of $\mathcal{C}$ and Proposition 6.B.1 (a), for any C, $D \in \mathcal{C}$ and any i, $j \in \mathbb{N}$ there exists a $\gamma < \alpha$ such that $A(C, D, i, j)$ is $\Pi^1_2(r, w \restriction \gamma)$.

Next we show that $(\mathcal{N}^*, \mathcal{C})$ does not differentiate $G \cdot y$ from $G \cdot x$ (as defined in 4.A.1). For suppose it does. By definition of “differentiate”, there exist sets C, $D \in \mathcal{C}$ and there exist i, $j \in \mathbb{N}$ such that $G{\cdot}y \subset A(C, D, i, j)$ and $G{\cdot}x \cap A(C, D, i, j) = \emptyset$.

Let $Z = X \setminus A(C, D, i, j)$. Then Z is $\Sigma^1_2(r, w \restriction \gamma)$, where $\gamma < \alpha$ and where, by (b), w is generic over $L[r]$; Z is invariant; $\mathcal{O}(\alpha) = G{\cdot}x \subset Z$; and $\mathcal{O}(\beta) = G{\cdot}y \subset X \setminus Z$. This contradicts Lemma 6.C.2.

By Lemma 6.B.3 (a), $(\langle X, t(r, \alpha, w)\rangle, a)$ is an acceptable G-space. By (a), $\mathcal{O}(\alpha) = \mathcal{O}(\alpha, w)$. And $\mathcal{O}(\alpha, w)$ is a G_δ orbit with respect to the topology $t(r, \alpha, w)$ (Definition 6.B.9 and Proposition 6.B.8 (c)). Thus the hypothesis of the Main Theorem (4.A.2) is satisfied for the acceptable G-space $\langle X, t(r, \alpha, w)\rangle$: $\mathcal{O}(\alpha) = G{\cdot}x$ is a G_δ orbit, $\mathcal{O}(\beta) = G{\cdot}y$ is an arbitrary orbit, y is in the closure of $G{\cdot}x$ and $(\mathcal{N}^*, \mathcal{C})$ does not differentiate $G{\cdot}y$ from $G{\cdot}x$. So by Theorem 4.A.2, $\mathcal{O}(\alpha)$ is ι-embeddable into $\mathcal{O}(\beta)$. ☐

Proof of Theorem 6.A.2. Let $\langle \gamma_\alpha : \alpha < \omega_1\rangle$ be the ω_1-sequence of countable Silver indiscernibles for $L[r]$, in increasing order. For $\alpha < \omega_1$, let $\mathcal{O}_\alpha = \mathcal{O}(\gamma_\alpha)$. By Lemma 6.B.8 (a), and Definitions 6.B.9 and 6.B.11, $\mathcal{O}_\alpha \subset S$. By Lemma 6.C.3, if $\alpha < \beta$, then $\mathcal{O}_\alpha \neq \mathcal{O}_\beta$. Thus $\langle \mathcal{O}_\alpha : \alpha < \omega_1\rangle$ is, indeed, an ω_1-sequence of distinct orbits of S.

To complete the proof, fix $\alpha < \beta < \omega_1$; we must show that $\mathcal{O}_\alpha$ is ι-embeddable into $\mathcal{O}_\beta$ (with respect to the topology τ, i.e., with respect to the original Polish topology on X). By Lemma 6.C.4, there exists a code w for γ_α such that $\mathcal{O}_\alpha$ is ι-embeddable into $\mathcal{O}_\beta$ with respect to the topology $t(r, \gamma_\alpha, w)$. By Lemma 6.B.3 (a), $t(r, \gamma_\alpha, w)$ is finer than τ. Therefore (see Example 3.D.6 (h)), $\mathcal{O}_\alpha$ is also ι-embeddable into $\mathcal{O}_\beta$ with respect to the topology τ. ☐

6.D. The Silver-Vaught Dichotomy for cli groups and $\mathbf{\Sigma}^1_1$ sets.

6.D.1. Theorem. *Assume #. Let G be a cli group, let X be a Polish G-space, let $S \subset X$ be an invariant $\mathbf{\Sigma}^1_1$ set and let R denote the orbit equivalence relation on S. Then R satisfies the Silver-Vaught Dichotomy.*

Proof. Assume that R violates the Silver-Vaught Dichotomy. Then S satisfies the hypothesis of Theorem 6.A.2. That theorem asserts that the ι-embeddability relation on S is nontrivial: there exist two distinct orbits of S, $\mathcal{O}_0$ and $\mathcal{O}_1$, such that $\mathcal{O}_0$ is ι-embeddable into $\mathcal{O}_1$. But since G is a cli group, by Proposition 3.D.3, the ι-embeddability relation is trivial. ☐

Some special cases of Theorem 6.D.1 had been proved earlier (in ZFC): for G abelian this is due to Sami [38] and for G nilpotent this is due to Hjorth-Solecki [22]. Of course, by Silver's Theorem, it is true for tame groups.

An *analytic* G-space is a Borel-measurable action $a : G \times A \to A$, where A is an analytic Borel space, i.e., a measurable space which is Borel-isomorphic to an analytic ($\mathbf{\Sigma}^1_1$) set in a Polish space.

6.D.2. Corollary. *Assume #. Let G be a cli group. Let A be an analytic G-space. Then the orbit equivalence relation of A satisfies the Silver-Vaught Dichotomy.*

Proof. By Becker-Kechris [6, 2.6.1 and remark following], every analytic G-space is Borel-isomorphic to the restriction of a Polish G-space X to an invariant $\mathbf{\Sigma}^1_1$ set $S \subset X$. And the Silver-Vaught Dichotomy is preserved under Borel-isomorphism. ☐

Every Borel set is an analytic set and every standard Borel G-space is an analytic G-space. Therefore, Corollaries 5.C.7 and 5.C.8 are special cases of Corollary 6.D.2 and Theorem 6.D.1, respectively. In some sense, the analytic case in this section is more significant than the Borel case in §5.C: for all we know, 5.C.7 and 5.C.8 may be true for arbitrary Polish groups; but 6.D.1 and 6.D.2 demonstrate a property of cli groups that is not true for all Polish groups (see Example 5.C.2 (b)).

At the time these results were proved, it was an open question whether or not Theorem 6.D.1 was provable in ZFC. Since 6.D.1 was provably equivalent to a Π^1_3-statement (see Becker-Kechris [6, 7.2]), it seemed that it "should" be provable in ZFC, but the author was unable to prove it. Later Hjorth [21] proved Theorem 6.D.1 (and hence Corollary 6.D.2) in ZFC. Hjorth's proof is "more set theoretic" than the proof given here; that is, it uses considerably more forcing and inner model theory. Although Theorem 6.D.1 is essentially a result in classical descriptive set theory, there is no known classical proof, from any axioms.

It is still open whether Theorem 6.A.2—or even Corollary 6.A.3—is provable in ZFC. As far as is known, neither of them is provably equivalent to a Π^1_3-statement; therefore, there is no particular reason to believe that they "should" be provable in ZFC.

Hjorth [21] has also proved, under appropriate set theoretic assumptions, that 6.D.1 holds not merely for $\mathbf{\Sigma}^1_1$ sets $S \subset X$, but for any "definable" set $S \subset X$.

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