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L-SERIES WITH NONZERO CENTRAL CRITICAL VALUE

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1. Introduction

Suppose that $f = \sum_{n\geq 1} a_n(f)q^n$ is a cusp form of weight $2k$ ($k \in \mathbb{N}$). We denote by $L(f,s)$ the L-function of f. For $\mathrm{Re}(s)$ sufficiently large, the value of $L(f,s)$ is given by $L(f,s) = \sum_{n\geq 1} \frac{a_n(f)}{n^s}$, and one can show that $L(f,s)$ has analytic continuation to the entire complex plane. The value of $L(f,s)$ at $s = k$ will be of particular interest to us, and we will refer to this value as the *central critical value* of $L(f,s)$.

Let χ_D denote the Dirichlet character associated to the extension $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$, that is, $\chi_D(n) = \left(\frac{\Delta_D}{n}\right)$, where Δ_D denotes the discriminant of $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$. Define the D^{th} *quadratic twist of* f to be $f_{\chi_D} = \sum_{n\geq 1} a_n(f)\chi_D(n)q^n$. For any integer D, the L-function of f_{χ_D} is the D^{th} quadratic twist of $L(f,s)$, that is, $L(f_{\chi_D}, s) = \sum_{n\geq 1} \frac{a_n(f)\chi_D(n)}{n^s}$. We will be interested in determining how often $L(f_{\chi_D}, s)$ has nonzero central critical value as D varies over all integers. Since $\chi_{Dm^2} = \chi_D$, we will restrict our attention to the square-free integers D. We expect that as we let D vary over all of the square-free integers, a positive proportion of the L-functions $L(f_{\chi_D}, s)$ will have nonzero central critical value. Indeed, Goldfeld [7] conjectures that for newforms f of weight 2, $L(f_{\chi_D}, 1) \neq 0$ for $\frac{1}{2}$ of the square-free integers.

Given an elliptic curve $E : y^2 = x^3+Ax^2+Bx+C$ ($A, B, C \in \mathbb{Z}$) with conductor N_E and an integer D, we define the D^{th} *quadratic twist of* E to be the curve $E_D : y^2 = x^3+ADx^2+BD^2x+CD^3$. Let $L(E_D, s)$ denote the L-function associated to E_D. For square-free D coprime to $2N_E$, $L(E_D, s)$ is simply the D^{th} quadratic twist of $L(E_1, s)$.

If $f \in S_2(N)$ is a newform with integer coefficients, we know via the theory of Eichler and Shimura that there is an elliptic curve E over $\mathbb{Q}$ having conductor N so that $L(E,s) = L(f,s)$. Thus if D is coprime to $2N$, then $L(E_D, s) = L(f_{\chi_D}, s)$. Also, one knows from the work of Kolyvagin [13], as supplemented by the work of Murty and Murty [17] or that of Bump, Friedberg and Hoffstein [3] (see also [10] for a shorter proof), that if E is a modular elliptic curve and if $L(E,1) \neq 0$, then the rank of E is 0. Thus, if f has the property that a positive proportion of the twists of $L(f,s)$ have nonzero central critical value, then this implies that a positive density of the quadratic twists E_D have rank 0.

There have been many papers which have proved results in this direction. For example, in [2], [3], [6], [10], [16], [17], [19], [28] one can find general theorems on

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the vanishing and nonvanishing of the quadratic twists of a given L-function. These theorems ensure that an infinite number of the quadratic twists of an L-function associated to a cusp form will have nonzero central critical value.

In [20], Ono has shown several examples of cusp forms f associated to elliptic curves such that for a positive density of the primes p, the p^{th} quadratic twist of $L(f, s)$ will have nonzero central critical value. Ono also proves a theorem which gives sufficient conditions under which a cusp form associated to an elliptic curve will have this property. Using methods similar to those of Ono, the author [11] was able to prove that the elliptic curve $E_p : y^2 = x^3 - 32p^3$ has rank 0 for at least 1/3 of the primes p.

Subsequently, Ono and Skinner [22] used the theory of Galois representations to extend Ono's theorem to all even weight eigenforms satisfying a very mild hypothesis. In fact they verify that this hypothesis is satisfied for all modular elliptic curves of conductor less than or equal to 100.

In a series of two papers [8], [9], Heath-Brown has done an extensive investigation of the behavior of the 2-Selmer groups associated to the quadratic twists of the congruent number curve: $y^2 = x^3 - x$. He states as a corollary to one of his theorems that at least 5/16 of these quadratic twists have rank 0. This result along with the Birch and Swinnerton-Dyer conjecture implies that at least 5/16 of the quadratic twists of the L-function $L(E, s)$ associated to the congruent number curve should have nonzero central critical value.

Using ideas developed by Frey [5] along with the Davenport-Heilbronn theorem [18], Wong [27] has shown the existence of an infinite family of nonisomorphic elliptic curves such that a positive proportion of the quadratic twists of each curve have rank 0. Thus if we assume the Birch and Swinnerton-Dyer and Shimura-Taniyama conjectures, Wong's result would then imply the existence of an infinite family of weight 2 cusp forms $\{f_i\}$ such that a positive proportion of the twists of each $L(f_i, s)$ have nonzero central critical value.

In section 3 of this paper we exhibit weight 2 newforms F such that $L(F_{\chi_D}, 1) \neq 0$ for all D in a subset of the square-free natural numbers having positive lower density. We now describe the first of those results. Let E denote the elliptic curve with equation $y^2 = x^3 - x^2 + 72x + 368$. Then E is a modular curve (it is the -1 twist of $X_0(14)$). We let F denote the weight 2 cusp form whose Mellin transform is $L(E, s)$. We then prove unconditionally:

Theorem 1. *$L(F_{\chi_D}, 1) \neq 0$ for at least 7/64 of the square-free natural numbers D.*

In light of Kolyvagin's work, we have as a corollary to Theorem 1

Corollary 2. *For at least 7/64 of the square-free natural numbers D, E_D has rank 0.*

Our proof differs from those of Heath-Brown and Wong in that while they work directly with the Selmer groups of elliptic curves, our proof uses the theory of modular forms developed by Shimura and Waldspurger to gain information about the central critical values of the L-functions associated to elliptic curves. An outline of the proof of Theorem 1 is as follows. Using ideas of Schoeneberg [23] and Siegel [25], we construct a weight 3/2 cusp form f as the difference of the theta functions associated to two inequivalent ternary quadratic forms Q_1 and Q_2 which together make up a genus of ternary forms. This f will be an eigenform for all of the

Hecke operators and will lift through the Shimura correspondence to $F_{\chi_{-1}}$. By a theorem of Waldspurger [26] we will be able to equate the vanishing of the central critical values of the quadratic twists of $L(F, s)$ to the vanishing of certain Fourier coefficients of f. Since our ternary forms Q_1 and Q_2 are the only forms in a certain genus of ternary forms, we are able to study the automorph structure of these forms to show that the Fourier coefficients of f are related modulo 3 to certain class numbers of imaginary quadratic number fields. We will then use the Davenport-Heilbronn theorem (see [18]) to show that at least $7/64$ of these class numbers are not divisible by 3, and hence, the associated Fourier coefficients of f are nonzero. It will then follow that at least $7/64$ of the quadratic twists of $L(F, s)$ have nonzero central critical value.

The key ingredient in the above argument is that our ternary forms Q_1 and Q_2 are a complete set of representatives for a genus of forms having the correct automorph structure. In particular, if A_i denotes the number of automorphs of Q_i $(i = 1, 2)$, then we need that $A_1 + A_2 \equiv 0$ modulo 3 while $3 \nmid A_1A_2$. This congruence modulo 3 is what allows the use of the Davenport-Heilbronn Theorem. It is important to note that there are examples of genera of ternary quadratic forms which contain exactly two equivalence classes but whose automorph structure does not satisfy the above congruence modulo 3 (see the tables of ternary forms in [14]) and, for such examples, the Davenport-Heilbronn Theorem is of no consequence.

2. Background

The theory developed by Waldspurger in [26] provides a tool for obtaining information about the central critical values of the L-series $L(f_{\chi_n}, s)$ associated to the quadratic twists of a particular integral weight newform f. Before stating his results we need to introduce one more bit of notation. If f is a newform of weight $2k$ and if χ is a Dirichlet character, then f_χ is an eigenform for all of the Hecke operators. Hence, by the theory of newforms developed in [1] and [15], there exists a newform of weight $2k$ which, following Waldspurger, we will denote $f \cdot \chi$ with the same eigenvalues as f_χ for all but finitely many of the Hecke operators. In fact it is the central critical values of the $L(f \cdot \chi_n, s)$ which Waldspurger's theorem allows us to relate to the Fourier coefficients of a half-integral weight form. Since f_{χ_n} and $f \cdot \chi_n$ have the same eigenvalues for all but a finite number of the Hecke operators, it follows that $L(f \cdot \chi_n, s)$ and $L(f_{\chi_n}, s)$ differ only by a finite number of Euler factors. Thus, $L(f_{\chi_n}, k) = 0$ if and only if $L(f \cdot \chi_n, k) = 0$.

Now, we are ready to state a special case of the main theorem in [26].

Theorem 2.1. *Let $k \geq 3$ be an odd integer, $N \in 4\mathbb{N}$, χ a Dirichlet character modulo N, and M some divisor of N so that χ^2 is a Dirichlet character modulo M. Suppose $F \in S_{k-1}(M, \chi^2)$ is a newform with Hecke eigenvalues $\lambda_p(F)$. Suppose also that there exists a cusp form $f \in S_{k/2}(N, \chi)$ having the property that for all but finitely many primes p, $T_p f = \lambda_p(F) f$. Finally suppose that the Dirichlet character ν defined by $\nu(n) = \chi(n)(\frac{-1}{n})^{\frac{k-1}{2}}$ has conductor divisible by 4. Let $\mathbb{N}^{\mathrm{sf}}$ denote the square-free natural numbers. Then there is a function $\mathbb{A} : \mathbb{N}^{\mathrm{sf}} \to \mathbb{C}$, depending only on F and satisfying the condition*

$$(\mathbb{A}(t))^2 = L(F \cdot \nu^{-1}\chi_t, \frac{k-1}{2}) \cdot \epsilon(\nu^{-1}\chi_t, 1/2),$$

where $\epsilon(\psi, s)$ is chosen so that if $L(\psi, s)$ is the Dirichlet L-function for the Dirichlet character ψ and if

$$\Lambda(\psi, s) = \begin{cases} \pi^{-s/2}\Gamma(\frac{s}{2})L(\psi, s) & \text{if } \psi(-1) = 1, \\ \pi^{-(s+1)/2}\Gamma(\frac{s+1}{2})L(\psi, s) & \text{if } \psi(-1) = -1, \end{cases}$$

then

$$\Lambda(\psi^{-1}, 1 - s) = \epsilon(\psi, s)\Lambda(\psi, s).$$

Moreover f can be written as a finite $\mathbb{C}$-linear combination of Hecke eigenforms f_i such that $a_t(f_i) = c(t^{\mathrm{sf}}, F)\mathbb{A}(t)$, where t^{sf} denotes the square-free part of t and $c(t^{\mathrm{sf}}, F) \in \mathbb{C}$.

In particular, we can deduce from Theorem 2.1 that if $a_t(f) \neq 0$, then

$$L(F \cdot \nu^{-1}\chi_t, \frac{k-1}{2}) \neq 0.$$

Also, we will use the following theorem which is an immediate corollary of a theorem of Davenport and Heilbronn [4] as improved by Nakagawa and Horie [18].

Theorem 2.2. *Suppose that m and N satisfy:*

1. *If p is an odd prime dividing (N, m), then $p^2 \mid N$ and $p^2 \nmid m$, and*
2. *If N is even, then either $4 \mid N$ and $m \equiv 1$ modulo 4 or $16 \mid N$ and $m \equiv 8$ or 12 modulo 16.*

Let T denote the set of discriminants Δ of imaginary quadratic extensions of $\mathbb{Q}$ in the arithmetic progression $\Delta \equiv m$ modulo N. Then there is a subset S of T having lower density at least $\frac{1}{2}$ in T such that if $\Delta \in S$, then $3 \nmid h(\Delta)$.

3. Nonvanishing theorems

If Q is a primitive positive definite ternary quadratic form, then we will denote the discriminant of Q by d_Q. We also define $\theta_Q(\tau) = \sum_{x,y,z\in\mathbb{Z}} q^{Q(x,y,z)}$ $(q = e^{2\pi i\tau})$. It is well known (see [24]) that θ_Q is a modular form of weight 3/2, and we have a theorem of Siegel [25] which states that if Q_1 and Q_2 are two primitive positive definite ternary quadratic forms belonging to the same genus, then $(\theta_{Q_1} - \theta_{Q_2})$ is a cusp form.

Proposition 3.1. *Suppose that Q_1 and Q_2 are even integral primitive positive definite ternary quadratic forms and that Q_1 and Q_2 are the only forms in a genus of forms. Let A_i denote the number of automorphs of Q_i $(i = 1, 2)$. Assume that $3 \nmid A_1A_2$ but $3 \mid A_1 + A_2$. Suppose also that $f = (\theta_{Q_1} - \theta_{Q_2}) \in S_{3/2}(N, \chi_q)$ is a Hecke-eigenform which lifts through the Shimura correspondence to a cusp form $F \in S_2(N/2)$. Let G denote the unique weight 2 newform of trivial character having $\lambda_p(F) = \lambda_p(G)$ for all but finitely many of the primes p, and let N_G denote the level of G. Put*

(1)

$$R = \left\{a \in (\mathbb{Z}/4d_{Q_1}^{\mathrm{sf}}\mathbb{Z})^* : \exists \text{ a square-free } n \equiv a \pmod{8W} \text{ with } 3 \nmid a_n(f)\right\} \quad \text{and,}$$

$$\delta = \frac{\#R}{8d_{Q_1}^{\mathrm{sf}} \prod_{p \mid d_{Q_1}^{\mathrm{sf}}}(1 - \frac{1}{p^2})}.$$

Then, the set of square-free natural numbers n such that $L(G \cdot \chi_{-qn}, 1) \neq 0$ has lower density at least δ in the square-free natural numbers.

Proof. Let $R(Q_1, m)$ denote the number of essentially distinct primitive representations of m by the genus of ternary forms containing Q_1 and let $r_i(m)$ denote the total number of representations of m by Q_i $(i = 1, 2)$. Then for sufficiently large square-free natural numbers m, we have $R(Q_1, m) = \frac{r_1(m)}{A_1} + \frac{r_2(m)}{A_2}$.

Now, suppose that $a \in R$. Then there exists $n \equiv a$ modulo $4d^{\mathrm{sf}}_{Q_1}$ such that $3 \nmid a_n(f)$; hence, it follows from the construction of f that $R(Q_1, n) \neq 0$. Thus, $R(Q_1, m) \neq 0$ for all natural numbers $m \equiv a$ modulo $4d^{\mathrm{sf}}_{Q_1}$. Applying a theorem of Gauss (see [12, Theorem 86]) and recalling the relationship of the class number of an order in an imaginary quadratic field to the class number of the ring of integers in the same field, we have that for all square-free natural numbers $m \equiv a$ modulo $4d^{\mathrm{sf}}_{Q_1}$,

$$R(Q_1, m) = \rho h(\Delta_{-m}), \tag{2}$$

where $\rho \in \mathbb{Q}$. Since $3 \nmid A_1A_2$ and $3 \mid A_1 + A_2$, we have that $A_1A_2R(Q_1, m) = A_2r_1(m)+A_1r_2(m) \equiv A_2(r_1(m)-r_2(m))$ modulo 3. From our construction of f, we have that $a_m(f) = r_1(m) - r_2(m)$. Therefore, $3 \mid a_m(f)$ if and only if $3 \mid \rho h(\Delta_{-m})$. Recall that $3 \nmid a_n(f)$ and $n \equiv a$ modulo $4d^{\mathrm{sf}}_{Q_1}$, and therefore $3 \nmid \rho h(\Delta_{-n})$. Thus, we see that $\mathrm{ord}_3(\rho) \leq 0$. By the Davenport-Heilbronn Theorem (Theorem 2.2), we have for at least half of the square-free natural numbers $m \equiv a$ modulo $4d^{\mathrm{sf}}_{Q_1}$ that $3 \nmid h(\Delta_{-m})$. Therefore, we have that $\mathrm{ord}_3(\rho) = 0$ and hence it follows for all square-free natural numbers $m \equiv a$ modulo $4d^{\mathrm{sf}}_{Q_1}$ that $3 \mid a_m(f)$ if and only if $3 \mid h(\Delta_{-m})$. Now, applying Theorem 2.2 again, we see for each $a \in R$ that $a_m(G) \neq 0$ for at least $1/2$ of the square-free natural numbers $m \equiv a$ modulo $4d^{\mathrm{sf}}_{Q_1}$, and hence by Theorem 2.1 that $L(G \cdot \chi_{-qm}, 1) \neq 0$. We note that each $a \in R$ gives rise to $d^{\mathrm{sf}}_{Q_1}$ arithmetic progressions modulo $4(d^{\mathrm{sf}}_{Q_1})^2$, and that the total number of arithmetic progressions modulo $4(d^{\mathrm{sf}}_{Q_1})^2$ in which square-free numbers reside is $4(d^{\mathrm{sf}}_{Q_1})^2 \prod_{p \mid d^{\mathrm{sf}}_{Q_1}} (1 - \frac{1}{p^2})$. Thus the density of square-free natural numbers m which are congruent modulo $4d^{\mathrm{sf}}_{Q_1}$ to some $a \in R$ is $\frac{\#R \cdot d^{\mathrm{sf}}_{Q_1}}{4(d^{\mathrm{sf}}_{Q_1})^2 \prod_{p \mid d^{\mathrm{sf}}_{Q_1}} (1 - \frac{1}{p^2})}$. The proposition now follows from Theorem 2.2. ☐

Example 3.1. Let

$$\begin{aligned} Q_1(x, y, z) &= x^2 + 7y^2 + 7z^2, \text{ and} \\ Q_2(x, y, z) &= 2x^2 + 4y^2 + 7z^2 - 2xy. \end{aligned} \tag{3}$$

Then one can easily check that Q_1 and Q_2 both have discriminant 196 and that they are both in the same genus. The numbers of automorphs of Q_1 and Q_2 are 8 and 4 respectively. Also, we can calculate (see [14] or [24]) that $\theta_{Q_1}, \theta_{Q_2} \in M_{3/2}(28)$, and therefore by [25] we have that $f = (\theta_{Q_1} - \theta_{Q_2}) \in S_{3/2}(28)$. We checked computationally that f is an eigenform for all of the Hecke operators and that f lifts through the Shimura correspondence to twice the weight 2 newform F of level 14 associated to the elliptic curve $E : y^2 = x^3 + x^2 + 72x - 368$ of conductor 14, that is, $L(F, s) = L(E, s)$. Thus, f satisfies the hypotheses of Proposition 3.1. In this case, we have $d^{\mathrm{sf}}_{Q_1} = 14$, and by computing the first 200 coefficients of f, we see that 1, 9, 15, 23, 25, 29, 37, 39 and 53 are all elements of R. Therefore, $\delta = 7/64$, and Theorem 1 now follows from Proposition 3.1.

We were also able to obtain positive density nonvanishing results as in Corollary 2 for the quadratic twists of nine other elliptic curves. Since the calculations involved in the verification of the hypotheses of Proposition 3.1 are completely analogous to the calculations discussed in Example 3.1, we omit them and simply present the results in the following table. We list for each curve E a Weierstrauss equation for E, the conductor N_E of E, and the lower bound δ_E on the lower density of square-free natural numbers d such that $L(E_{-d}, 1) \neq 0$.

E	N_E	δ_E
$y^2 = x^3 + x^2 + 72x - 368$	14	7/64
$y^2 = x^3 + 8$	576	1/4
$y^2 = x^3 + 1$	36	5/24
$y^2 = x^3 + 4x^2 - 144x - 944$	19	19/240
$y^2 = x^3 + x^2 + 4x + 4$	20	5/72
$y^2 = x^3 + x^2 - 72x - 496$	26	13/112
$y^2 = x^3 + x^2 + 24x + 144$	30	5/128
$y^2 = x^3 + x^2 - 48x + 64$	34	17/144
$y^2 = x^3 + x^2 + 3x - 1$	44	11/144
$y^2 = x^3 + 5x^2 - 200x - 14000$	50	5/24

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