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THE DOLBEAULT COMPLEX IN INFINITE DIMENSIONS I

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Les longs ouvrages me font peur.
Loin d'épuiser une matière,
On n'en doit prendre que la fleur. [La][1]

Introduction

Infinite dimensional complex analysis was a popular subject in the sixties and seventies, but in the last fifteen years enjoyed much less attention. Oddly enough, this very same period began to see the emergence of examples of infinite dimensional complex manifolds, in mathematical physics, representation theory, and geometry. Thus it appears to be a worthwhile undertaking to revisit infinite dimensional complex analysis, and, in particular, to clarify fundamental properties of infinite dimensional complex manifolds. The most fundamental questions at this point seem to be related to the solvability of the inhomogeneous Cauchy-Riemann, or $\bar\partial$, equations, and more generally to the study of the Dolbeault complex. (As an example, in [Le3] we use a result on the $\bar\partial$ equation to describe the Virasoro group as an infinite dimensional complex manifold.)

Up to now precious little has been known about the solvability of the infinite dimensional $\bar\partial$ equation; the available results pertain to solving the equation in domains in or over locally convex topological vector spaces, and almost exclusively on the level of $(0,1)$-forms. Postponing the precise definitions to sections 1, 2, below we discuss those results that are of immediate relevance to this paper; see [D2] for more. In [Li] Ligocka observes that Hörmander's proof of Ehrenpreis' theorem (see [E], [Hor]) can be extended to infinite dimensions to solve the equation

$$(*) \qquad \bar\partial u = f \qquad (\bar\partial f = 0)$$

where f is a given $(0,1)$-form with bounded support, and Mujica points out in [Mu] that — surprisingly — the same works for $(0,q)$-forms as well. Coeuré gives an example of a $(0,1)$-form f of class C^1 on an infinite dimensional Hilbert space for which (*) does not even admit local solutions; see [C], [Ma]. However, no such example is known with f of class C^2 of even $C^{1+\epsilon}$. We shall comment on the

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[1] We are in complete agreement with La Fontaine. We too are intimidated by long works, and prefer picking the flowers to exhausting a subject. Alas, the current state of our subject matter is such that we must labor before getting to the flowers. We encourage the reader to read on.

sharpness of Coeuré's example in section 9. Dineen's results in [D1] (see also [Ma]) amount to C^∞ counterexamples to the solvability of (*) on certain Fréchet spaces, and Meise and Vogt give more counterexamples in [MV].

Finally, in [R] Raboin considers (*) with f a $(0,1)$-form of class C^∞ and of "bounded type" on a pseudoconvex domain Ω in a separable Hilbert space H. Given a linear subspace $X \subset H$, he can solve (*) on $\Omega \cap X$, provided X is not too large in the sense that it is the range of a self adjoint operator $T : H \to H$ of trace class; see also [Ma]. A more special situation was considered earlier by Henrich in [He]. This then has some consequence about the solvability of (*) on nuclear Fréchet spaces and duals of such; see [R], [CP], [Hon].

In the present paper, which we intend to be the first installment of a series, we will study the Dolbeault complex in an algebraic setting. In sections 1 and 2 we introduce the basic concepts, and explain why we have chosen the definitions we will be working with over other possible ones. In sections 3 and 4 we do some ground work: we solve $\bar\partial$ on the formal level and also in a framework generalizing the result about compactly supported forms in $\mathbb{C}^n$. Sections 5 and 6 prepare section 7; in this latter we study the Dolbeault groups $H^q(P,L)$ of line bundles over projective spaces P, and we show that they vanish when $1 \le q < \dim P$. We also prove that line bundles over a projective space are classified by their degrees. In section 8 we draw some corollaries: we give a Chow-type theorem for hypersurfaces in projective spaces; prove that the Dolbeault groups of P are isomorphic to sheaf cohomology groups of the projective space P as an algebraic variety; prove that holomorphic vector bundles $E \to P$ of finite rank split into the sum of line bundles, when dim $P = \infty$; and finally describe the groups $H^{p,q}(P,E)$. (This all in varying degrees of generality.) Vanishing theorems on projective spaces imply vanishing theorems on more general projective manifolds; this we will discuss in a future publication.

Finally, in section 9 we solve the $\bar\partial$ equation with polynomial growth on affine spaces.

Our main concern will be manifolds modelled on separable Hilbert spaces. Not because the natural examples are necessarily such, on the contrary: many — such as loop groups — come in different flavors, modelled on function spaces of our choice; but some come only modelled on spaces of C^∞ functions, which are Fréchet spaces — such as diffeomorphism groups and certain loop spaces. The reason for focusing on Hilbert manifolds is that Hilbert spaces are the simplest infinite dimensional vector spaces, and therefore it is our primary task to understand analysis on them. Also, the non-Hilbert manifolds that arise in practice are modelled on spaces that are limits of Hilbert spaces. It may ultimately turn out that Hilbert spaces are not suitable for our purposes, and other types of spaces provide the natural arena for complex analysis, in which case our focus will have to shift to those spaces: but this circumstance must be brought out first by a careful study of Hilbert spaces.

This said, we shall nevertheless try to keep our discussion at the reasonably general level of locally convex spaces. At the same time we shall not hesitate to impose extra conditions on the model spaces that facilitate our analysis, as long as those conditions are met by separable Hilbert spaces (and also nuclear Fréchet spaces such as the space of smooth sections of finite dimensional vector bundles over compact manifolds).

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1. Classes of smoothness

Suppose V is a vector space over $\mathbb{R}$ or $\mathbb{C}$ endowed with a Hausdorff topology such that the vector space operations are continuous. If there exists a family $\mathcal{P}$ of seminorms, i.e. nonnegative subadditive functions $p : V \to \mathbb{R}$ that satisfy $p(\lambda x) = |\lambda| p(x)$, $\lambda \in \mathbb{R}$, $x \in V$, such that sets of the form

$$\{x \in V : p_j(x) < \epsilon_j,\ j = 1, \dots, n\}, \qquad n = 1, 2, \dots; \quad p_j \in \mathcal{P}; \quad \epsilon_j > 0,$$

constitute a neighborhood basis of $0 \in V$, then V is called a locally convex (topological vector) space (cf. [Sc]). In this paper we shall always assume without explicitly mentioning that all our locally convex spaces V are also sequentially complete, which means that if a sequence $\{x_n\} \subset V$ satisfies $p(x_n - x_m) \to 0$ as $n, m \to \infty$ for all $p \in \mathcal{P}$, then $\{x_n\}$ is convergent.

Given two such locally convex spaces V, W, an open set $\Omega \subset V$ and a mapping $u : \Omega \to W$, for any $x \in \Omega$ and $\xi \in V$ the directional derivative

$$du(x; \xi) = \lim (u(x + \lambda\xi) - u(x))/\lambda, \qquad \mathbb{R} \ni \lambda \to 0, \tag{1.1}$$

may or may not exist. If it does, for all x, ξ, and the differential $du : \Omega \times V \to W$ is continuous, u is said to be of class C^1, $u \in C^1(\Omega, W)$. In this case one easily shows that $du(x; \xi)$ depends $\mathbb{R}$-linearly on ξ. Indeed, linearity is just a property of restrictions of u to two dimensional subspaces, and so it follows from the fact that a function on $\mathbb{R}^2$ with continuous partials is differentiable.

Higher order differentiability is defined recursively, for example $u \in C^2(\Omega, W)$ if $du \in C^1(\Omega \times V, W)$, etc., and $C^\infty(\Omega, W) = \bigcap\limits_{k=1}^{\infty} C^k(\Omega, W)$. Also, higher order differentials can be defined; if $u \in C^1(\Omega, W)$ and for all $x \in \Omega$, $\xi, \eta \in V$ the directional derivatives

$$d^2u(x; \xi, \eta) = \lim (du(x + \lambda\eta; \xi) - du(x; \xi))/\lambda, \qquad \mathbb{R} \ni \lambda \to 0,$$

exist, then this formula defines the second differential of u. The mapping u is C^2 if and only if $d^2u : \Omega \times V \times V \to W$ is continuous. In this case $d^2u(x; \xi, \eta)$ is a symmetric bilinear (over $\mathbb{R}$) mapping in ξ, η. Similarly, if $u \in C^k(\Omega, W)$ we have k-linear symmetric differentials $d^k u(x; \xi_1, \dots, \xi_k)$, $k < \infty$. We say that such a u vanishes of order $k+1$ at $x \in \Omega$ if all differentials $d^j u$ vanish at x for all $j \le k$, including $d^0 u = u$.

Examples of mappings in $C^\infty(V, W)$, also called smooth mappings, come from multilinear maps $f : V \times V \times \dots V \to W$. Such maps are in abundance as a consequence of the Banach-Hahn theorem, and given such an f, $u(x) = f(x, \dots, x)$ defines a C^∞ mapping. Mappings that arise in this way are called homogeneous polynomials of degree k (if f is k-linear), and linear combinations of them are the polynomial mappings $V \to W$. If $u : \Omega \to W$ is $k < \infty$ times continuously differentiable in a neighborhood of $x_0 \in \Omega$, we can form its Taylor polynomial

$$T_k(x) = \sum_{j=0}^{k} \frac{1}{j!} d^j u(x_0; x - x_0, \dots, x - x_0),$$

and we find $u(x) = T_k(x) + R_k(x)$ with remainder R_k vanishing at x_0 of order $k+1$.

If the locally convex spaces V, W are over $\mathbb{C}$, a C^1 mapping $u : \Omega \to W$ is called holomorphic if the limit in (1.1) exists as $\mathbb{C} \ni \lambda \to 0$; or equivalently, if $du(x; \xi)$ is complex linear in ξ. In this case higher differentials exist and are also $\mathbb{C}$-multilinear.

Analyticity of mappings between *real* spaces V, W will be defined in terms of complexifications. We say that a mapping $u : \Omega \to W$ is real analytic and write $u \in C^\omega(\Omega, W)$ if $\Omega \subset V \subset \mathbb{C}\otimes V$ has a neighborhood $\widetilde{\Omega}$ in $\mathbb{C}\otimes V$ such that u extends to a holomorphic mapping $\widetilde{\Omega} \to \mathbb{C} \otimes W$. It is easy to show that if W is complex, then the analytic mapping u above admits a holomorphic extension $\widetilde{\Omega} \to W$. The notion of analyticity has other variants as well; with the target a Banach space, all these notions coincide. When $W = \mathbb{C}$ we will write $C^k(\Omega)$ instead of $C^k(\Omega, \mathbb{C})$.

We conclude this section by noting that other definitions of smoothness classes have also been around; a variety of differential calculi is investigated in [Ke]. For more on differential calculus in locally convex spaces and on manifolds, consult [D2], [Ha2], [Ma], [Mc], [Mi].

2. Manifolds

Given the notion of smooth maps, the definition of differentiable manifolds is immediate: If $k \geq 1$, a C^k-manifold is a Hausdorff space M with an open covering $M = \bigcup U_\alpha$, and homeomorphisms φ_α from U_α on open subsets of locally convex spaces such that all transition mappings $\varphi_\alpha \circ \varphi_\beta^{-1}$ are C^k where defined. If $k = \infty$, resp. ω, we talk about smooth, resp. real analytic manifolds. Vector bundles with locally convex fibers, in particular tangent bundles, differential forms, and the differential d are defined in the same way as in finite dimensions; see e.g. [Mc], [Mi], and for the theory of differential forms [AMR]. The Banach manifold framework of this latter carries over to ours with minor modifications; see also our treatment below of complex differential forms and $\bar\partial$. For vector bundles with normable fibers one can define dual and Hom bundles, but this is not practical when the fibers are general locally convex spaces. For example, given C^k vector bundles $E_1 \to M$, $E_2 \to M$, $\mathrm{Hom}(E_1, E_2)$ carries a natural structure of a C^k vector bundle only when E_1 has normable fibers (or rk $E_2 = 0$). This means that in general we cannot think of smooth differential forms as smooth sections of some vector bundle, which turns out to be only a minor inconvenience. We shall also consider locally trivial fiber bundles; these are given by a C^k mapping $\pi : M \to B$ between C^k manifolds such that each $b \in B$ has a neighborhood $U \subset B$ with the property that $\pi^{-1}(U)$ has a fiber preserving C^k diffeomorphism on a trivial bundle $U \times F \to U$. Here we assume that F is a C^k manifold. We shall denote the set of C^k sections of such a bundle by $C^k(B, M)$.

At this point we have to ask the reader to resist the temptation to define complex manifolds by requiring that the model spaces should be $\mathbb{C}$-vector spaces and by substituting "holomorphic" for "C^k" in the definition of real manifolds. We claim that the following is a more fruitful definition:

Definition 2.1. (a) An almost complex structure on a C^k manifold M is a splitting of the complexified tangent bundle

$$\mathbb{C} \otimes TM = T^{1,0} \oplus T^{0,1} \tag{2.1}$$

into the sum of two complex vector bundles of class C^{k-1}, $T^{0,1} = \overline{T^{1,0}}$.

(b) A complex manifold is a C^k manifold M ($k \geq 2$) endowed with an almost complex structure (2.1) that is integrable, i.e., for any open $U \subset M$ and $X, Y \in C^{k-1}(U, T^{1,0})$ the Lie bracket $[X, Y] \in C^{k-2}(U, T^{1,0})$.

(c) A C^1 mapping between almost complex manifolds $F : M \to N$ is (bi)holomorphic if (it is diffeomorphic and) its differential dF maps $T^{1,0}M$ into $T^{1,0}N$.

In what follows, unless otherwise mentioned, we shall always take $k = \infty$: manifolds, bundles, diffeomorphisms, almost complex structures, holomorphic maps will be smooth.

As examples, any locally convex complex vector space has a canonical complex manifold structure (see, e.g. [Le3]), and any finite dimensional complex manifold in the traditional sense of the word is such in the sense of the above definition. Conversely, any finite dimensional complex manifold according to our definition is locally biholomorphic to some $\mathbb{C}^n$; this is the content of the Newlander-Nirenberg theorem in [NN].

Why is Definition 2.1 preferable to the "obvious" one requiring roughly that each point p in the manifold should possess a neighborhood U that is biholomorphic to an open subset of some locally convex complex vector space? There are aesthetic reasons and practical ones as well. Firstly, in the "obvious" definition no mention is made of how large or small the neighborhood U should be. It can shrink as close to p as we please, and the definition would still yield the same notion of complex manifold. If this is so, we really should be prepared to go all the way, and shrink U to an *infinitesimal* neighborhood of p. Requiring that a first order infinitesimal neighborhood of $p \in M$ be isomorphic to a first order neighborhood of a point in a locally convex space amounts to endowing M with an almost complex structure, and the analogous condition with second order neighborhoods amounts to the integrability hypothesis. — In passing we note that the above does not apply to, say, projective algebraic manifolds, where the traditional definition prescribes very precisely how large the neighborhoods U which are isomorphic to the "models" must be.

Secondly, the natural setting for the machinery of the $\bar\partial$ operator is a complex manifold as in Definition 2.1 — we shall review this momentarily — and many successes of the theory of finite dimensional complex manifolds (Hodge-Kodaira theory, the results of Hörmander and Kohn) depend only on analytic properties of the $\bar\partial$ complex, but not on the existence of local coordinates. Indeed, the existence of local holomorphic functions is of no help at all in understanding global holomorphic functions (or sections). In this paper, too, we will demonstrate that a meaningful analysis of the $\bar\partial$ complex can be performed even in the infinite dimensional setting without ever (well, almost ever) referring to local holomorphic functions.

Of course, there is a blatant objection to the above line of thought: singular subvarieties, complex spaces, and above all, sheaf theory, can hardly be explained without local holomorphic functions. This is a serious objection and at this point we do not know how to fully counter it. Nevertheless, in section 6, we shall briefly show how to integrate possibly singular hypersurfaces and divisors into this theory. That this is feasible we take as an indication that a treatment of general subvarieties and complex spaces might also be possible within the given framework.

Our third point is halfway between aesthetic and practical, and concerns complex manifolds with boundary. Here even in finite dimensions the possible definitions become inequivalent (see [Hi]), but the one using integrable almost complex structures seems to be the most flexible, and it is certainly the one that has been used with success in deformation theory of complex manifolds with boundary and of CR manifolds; see e.g. [Ha1], [Ki], [Le1].

Fourthly, one can observe certain loop and related spaces in the world (see [Br], [Lb], [Le2]) that are complex manifolds but only if Definition 2.1 is adopted: indeed, they are not locally biholomorphic to locally convex spaces. These manifolds are modelled on Fréchet spaces, and we consider it one of the most important questions of the theory at this point to decide if similar examples can be constructed with Hilbert or Banach manifolds; or perhaps on the contrary it is true that a Hilbert, resp. Banach, complex manifold is locally biholomorphic to an open set in the model space. We note that the linearization of this problem asks for the solution of a vector-valued $\bar\partial$ equation.

A final point is that in theoretical physics Definition 2.1 seems to be in use, and with profit; see e.g. [BR].

This all said, complex manifolds with holomorphic charts will play a special role in the theory. A complex manifold will be called *rectifiable* if locally it is biholomorphic to open subsets of locally convex complex vector spaces.

We will define holomorphic fiber and vector bundles in the same spirit.

Definition 2.2. A holomorphic fiber bundle is a differentiable locally trivial fiber bundle $\pi : M \to B$, with M, B endowed with the structure of a complex manifold such that π is holomorphic.

We note that the fiber product of two holomorphic fiber bundles $M \to B$, $N \to B$ naturally carries the structure of a holomorphic fiber bundle. This fiber product will be denoted $M \times_B N \to B$.

Definition 2.3. A holomorphic vector bundle is a differentiable vector bundle $\pi : E \to M$ with fibers locally convex *complex* vector spaces; E, M are complex manifolds and π is holomorphic. In addition it is required that the vector space operations multiplication by a scalar and addition should be holomorphic maps from $\mathbb{C} \times E$, resp. $E \times_M E$, to E.

Thus we are not requiring holomorphic local triviality in either definition. For example, the question whether all holomorphic line bundles over open subsets of a certain manifold M are holomorphically locally trivial is tantamount to asking whether on M the $\bar\partial$ equation is locally solvable on the level of $(0,1)$-forms.

Next we shall discuss submanifolds. A closed subset N of a differentiable manifold M is a submanifold if the pair (M,N) is locally diffeomorphic to a pair (V,W) consisting of a locally convex space V and a closed subspace $W \subset V$. If W has a closed complement in V, N is called a split submanifold. For example the fibers of a fiber bundle are split submanifolds, as is the range of a section. When M is a complex manifold, a differentiable submanifold $N \subset M$ is called a complex submanifold if $T^{1,0}N = (\mathbb{C} \otimes TN) \cap T^{1,0}M$ defines an (almost, hence automatically integrable) complex structure on N. Again, this does not mean that the pair (M,N) at any $p \in N$ is locally biholomorphic to a pair (V,W) of locally convex complex vector spaces, $W \subset V$. If N does have this stronger property, we will say it is a *rectifiable* complex submanifold.

Finally we come to forms. Let $E \to M$ be a differentiable vector bundle, $r = 1, 2, \ldots$. A differential r-form on M with values in E is a mapping

$$f : \bigoplus^{r} TM \to E \tag{2.2}$$

that restricts, for any $x \in M$, to an alternating, continuous, real r-linear map $\bigoplus^{r} T_x M \to E_x$. A 0-form is just a section of E. Let us put it on record that f will be said to be of class C^k if the mapping (2.2) is. If E is a holomorphic vector bundle, such a form f can be uniquely extended to be complex multilinear on $\bigoplus^{r}(\mathbb{C}\otimes TM)$. Given nonnegative integers p, q, $p+q=r$, we say that f is a (p,q)-form if $f(\xi_1,\dots,\xi_r)=0$ whenever $\xi_i\in T^{1,0}_xM$, $\xi_j\in T^{0,1}_xM$ for $i=1,\dots,s$, $j=s+1,\dots,r$, and $s\neq p$. Thus a (p,q)-form is completely determined by its values on $\bigoplus^{p}T^{1,0}M\oplus\bigoplus^{q}T^{0,1}M$, which we will also denote by $T^{p,q}M$. One can thus think of a (p,q)-form as a mapping $T^{p,q}M\to E$ that restricts, for $x\in M$, to a multilinear mapping $T^{p,0}_xM\oplus T^{0,q}_xM\to E_x$, alternating on both summands. The complex vector space of (p,q)-forms of class C^k on an open set $U\subset M$ with values in E will be denoted $C^k_{p,q}(U,E)$, and by $C^k_{p,q}(U)$ when E is the trivial line bundle.

To define differential operators $\bar\partial=\bar\partial_E : C^k_{p,q}(M,E)\to C^{k-1}_{p,q+1}(M,E)$ we first look at a general C^k map $\Phi : N\to P$ between complex manifolds, $k\geq 1$. The complexified differential is a C^{k-1} bundle map

$$\Phi_* : T^{1,0}N\oplus T^{0,1}N\to T^{1,0}P\oplus T^{0,1}P.$$

Restricting Φ_* to $T^{0,1}N$ and projecting on $T^{1,0}P$ we obtain a C^{k-1} bundle map $\overline{D}\Phi : T^{0,1}N\to T^{1,0}P$ (sometimes, but not in this paper, denoted $\bar\partial\Phi$), which measures the deviation of Φ from holomorphic. Note that $\overline{D}$ respects holomorphic maps: for example, if $F : P\to Q$ is holomorphic, then $\overline{D}(F\circ\Phi)=F_*\overline{D}\Phi$.

Next let us look at a section $f\in C^k(M,E)$ of a holomorphic vector bundle $\pi : E\to M$. In particular, f is a map $M\to E$, and so to any $\xi\in T^{0,1}_xM$ there corresponds a vector $\overline{D}f(\xi)\in T^{1,0}_{f(\xi)}E$. In fact this vector is vertical: $\pi_*\overline{D}f(\xi)=\overline{D}(\pi\circ f)(\xi)=0$, since π is holomorphic and $\pi\circ f=\mathrm{id}_M$. On the other hand, vertical vectors in $T^{1,0}_vE$ can be canonically identified with vectors in $E_{\pi(v)}$; we denote the vector corresponding to $\overline{D}f(\xi)$ by $\bar\partial f(\xi)\in E_x$. This defines $\bar\partial f\in C^{k-1}_{0,1}(M,E)$.

Now consider a general $f\in C^k_{p,q}(M,E)$; we must define the value that $\bar\partial f$ assumes on a $p+q+1$ tuple $(\xi_0,\dots,\xi_{p+q})\in T^{p,q+1}_xM$. To this purpose, extend ξ_j to smooth sections X_j of $\mathbb{C}\otimes TM$ in a neighborhood of x (no type condition on $X_j(y)$ if $y\neq x$!). We will define

$$\begin{aligned}\bar\partial f(\xi_0,\dots,\xi_{p+q})&=\sum_{j=0}^{p+q}(-1)^j\bar\partial(f(X_0,\dots,\widehat{X}_j,\dots,X_{p+q}))(X_j)\\&\quad+\sum_{0\leq i<j\leq p+q}(-1)^{i+j}f([X_i,X_j],X_0,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_{p+q}),\end{aligned}\tag{2.3}$$

the right hand side evaluated at x. We have to check, however, that the expression $\mathcal{E}(X_0,\dots,X_{p+q})\in E_x$ on the right is independent of the particular choice of extensions X_j, and that it has the required smoothness and linearity properties. The latter one is obvious: e.g., if $a_j\in\mathbb{C}$, then $\mathcal{E}(a_0X_0,\dots)=\prod_j a_j\mathcal{E}(X_0,\dots)$. A little computation also shows that if a_j are smooth functions near $x\in M$, then

$$\mathcal{E}(a_0X_0,\dots)=\prod_j a_j(x)\mathcal{E}(X_0,\dots).\tag{2.4}$$

This will imply

Proposition 2.1. *$\mathcal{E}(X_0,\dots)=0$ as soon as one X_j vanishes at x.*

Accepting this for the moment, we conclude that, for general X_j, $\mathcal{E}(X_0,\dots)$ depends only on $X_j(x)=\xi_j$. Finally to check smoothness we can assume that M is smoothly (but not holomorphically) embedded in some locally convex space V

as an open subset. That identifies $\mathbb{C}\otimes TM$ with $\mathbb{C}\otimes TV|_M \cong M\times(\mathbb{C}\otimes V)$. Since any vector $\xi\in\mathbb{C}\otimes T_xV$ has a unique extension to a *constant* vector field on V, we can use such constant extensions X_j of ξ_j in (2.3), and the C^{k-1} dependence of the right hand side of (2.3) on $(\xi_j)\in\bigoplus^{p+q+1}(\mathbb{C}\otimes TM)$ becomes evident.

Proof of Proposition 2.1. We first observe the following. Let Y, Z be smooth vector fields in a neighborhood of x, and assume that Z is of type $(0,1)$ and vanishes at x. Then $[Y,Z]$ is also of type $(0,1)$ at x.

Indeed, suppose u is an arbitrary smooth function near x with $\overline{\partial}u(x)=0$. Noting that Zu vanishes to second order at x, we have

$$[Y,Z]u|_x = Y(Zu)-Z(Yu)|_x = 0,$$

and it is easy to check that this implies $[Y,Z]|_x$ is of type $(0,1)$.

Next decompose $X_j=X_j'+X_j''$ into $(1,0)$ and $(0,1)$ parts. In particular $X_j''(x)=0$ for $j<p$ and $X_j'(x)=0$ for $j\geq p$. From our observation above, and its analog for $(1,0)$ vector fields Z, it follows that

$$\mathcal{E}(X_0,\dots)=\mathcal{E}(X_0',\dots,X_{p-1}',X_p'',\dots X_{p+q}'').$$

Therefore it suffices to prove the proposition under the assumption we now make that the X_j are *everywhere* of type $(1,0)$, resp. $(0,1)$, for $j<p$, resp. $j\geq p$.

Construct a finite dimensional smooth submanifold $N\subset M$ such that $X_j(x)\in\mathbb{C}\otimes T_xN$, $j=0,\dots,p+q$, and let $t_1,\dots,t_n$ denote local coordinates on N, $t_\nu(x)=0$ for all ν. Suppose that for some h, $0\leq h\leq p+q$, $X_h(x)=0$. Then there are smooth vector fields $Y_\nu\in C^\infty(N,\mathbb{C}\otimes TM)$ such that

$$X_h=\sum_{\nu=1}^{n}t_\nu Y_\nu \qquad \text{on } N.$$

We can assume that Y_ν are of the same type as X_h, for we can always replace them by their $(1,0)$, resp. $(0,1)$, parts. Extend t_ν, Y_ν from N to smooth functions, resp. $(1,0)$ or $(0,1)$ vector fields, in a neighborhood of x in M. Using the same symbols for the extended objects, define $Z_j=X_j$ if $j\neq h$ and $Z_h=\sum_\nu t_\nu Y_\nu$. One checks that $[X_i,X_j](x)=[Z_i,Z_j](x)$ for all i,j. Using this, the definition and linearity of $\mathcal{E}$, and finally (2.4), we can write

$$\begin{aligned}\mathcal{E}(X_0,\dots,X_{p+q}) &= \mathcal{E}(Z_0,\dots,Z_{p+q})\\ &= \sum_\nu \mathcal{E}(Z_0,\dots,t_\nu Y_\nu,\dots,Z_{p+q})=0,\end{aligned}$$

as claimed.

As usual, forms f with $\bar\partial f=0$ we call closed, while the ones that can be represented $f=\bar\partial u$, exact. (2.3) allows one to check that $\bar\partial\bar\partial=0$ on $C^2_{p,q}(M,E)$ so that (ignoring precise differentiability) we can say that exact forms are closed. In particular, the collection $\{C^\infty_{0,q}(M,E),\bar\partial\}_{q\geq0}$ gives rise to a complex, the Dolbeault complex of E, which will be our principal object of study. We shall write $H^q(M,E)$ to denote the cohomology groups of the Dolbeault complex, and more generally, $H^{p,q}(M,E)$ to denote the cohomology groups of the complex $\{C^\infty_{p,q}(M,E),\bar\partial\}_{q\geq0}$. When $N\subset M$ is a complex submanifold, we shall abbreviate $H^{p,q}(N,E|_N)$ to $H^{p,q}(N,E)$.

3. The formal level

Complex manifolds etc. as defined in section 2 are isomorphic to model spaces (locally convex spaces) in second order neighborhoods of their points. As pointed out there, in finite dimensions that implies *local* isomorphism, while in infinite dimensions, in general, it does not. The general principle is, nevertheless, that even in infinite dimensions such isomorphisms hold in infinitesimal neighborhoods of arbitrary order, or even more generally: finite dimensional local results carry over to infinite dimensions, if only on the formal level — that is, on the level of not necessarily convergent series. In this paper we will need and prove only special cases (Lemma 3.1, Theorems 3.5 and 3.6), and return to other instances in another publication.

We shall start by solving the $\bar\partial$ equation to arbitrary order at a given point $x_0 \in V$ of a locally convex complex vector space, but a definition will be needed first. We have already explained the meaning of a vector-valued function U of class C^k defined on an open subset Ω of a locally convex space to vanish of order $k+1$ at a given point $x \in \Omega$ (here $k < \infty$). Clearly, this concept can be carried over to Ω an open subset of a differentiable manifold, and also to mappings of Ω into a smooth vector bundle $E \to M$: indeed, as locally $E = M \times W$ is a product, such a mapping $u : \Omega \to E$ can be projected to a mapping into the locally convex space W, and in this situation the notion of vanishing order has already been defined. Finally, if Ω itself is a fiber bundle over a base B, we say that $u : \Omega \to E$ vanishes of order $k+1$ at $x \in B$ if it does so at any point of the fiber over x. We shall denote this circumstance by $u = o(k)$, and if the need arises, we shall specify that this relation holds at x. For example $u = o(0)$ means u maps the fiber of x into the zero section of E. In particular, if f is an r-form of class C^k with values in the vector bundle $E \to M$, then $f = o(k)$ at x if the mapping (2.2) vanishes of order $k+1$ at any $v \in \bigoplus^r T_xM$. If E is a holomorphic vector bundle, then $f \in o(k)$ for an $f \in C^k_{p,q}(M,E)$ implies $\bar\partial f = o(k-1)$, at $x \in M$.

We shall make repeated use of the following simple

Lemma 3.1. *Suppose V, W are locally convex complex vector spaces, $\Omega \subset V$ is open, $k = 0, 1, \dots,$ $q = 1, 2, \dots,$ and $f \in C^k_{0,q}(\Omega, W)$ (i.e. f takes values in the holomorphically trivial bundle $E = \Omega \times W$). Assume that $\bar\partial f = o(k-1)$ at a certain $x_0 \in \Omega$ (condition vacuous if $k = 0$). Then there is a polynomial $u \in C^\infty_{0,q-1}(V, W)$ such that $\bar\partial u - f = o(k)$.*

Proof. Taylor expansion about x_0 gives a decomposition $f = f' + R$, where $f' : T^{0,q}\Omega \cong \Omega \times \bigoplus^q V \to W$ is polynomial of degree k in the variable $x \in \Omega$, and $R = o(k)$. It follows that $\bar\partial f' = o(k-1)$; as $\bar\partial f'$ is polynomial of degree $k-1$ in $x \in \Omega$, this implies $\bar\partial f' = 0$. Hence the lemma will be a consequence of

Proposition 3.2. *With V, W, Ω, q, x_0 as in Lemma 3.1, assume that $f \in C^\omega_{0,q}(\Omega, W)$ is closed. Then there are a neighborhood $\Omega_0 \subset \Omega$ of x_0 and $u \in C^\omega_{0,q-1}(\Omega_0, W)$ such that $\bar\partial u = f$ on Ω_0. If $\Omega = V$ and f is polynomial, we can choose $\Omega_0 = V$ and u as polynomial.*

Proof. The proof is by the time honored method of complexification. We shall assume $x_0 = 0$, and denote by J the linear transformation on V corresponding to multiplication by i; at this point we also forget the complex structure of V.

The complexification of V is $\mathbb{C}\otimes_{\mathbb{R}} V \cong V\oplus V$, on which $i\in\mathbb{C}$ acts by $i(x,y)=\mathcal{J}^{\mathbb{C}}(x,y)=(-y,x)$, $x,y\in V$. We shall identify TV, resp. $T(\mathbb{C}\otimes_{\mathbb{R}} V)$, with $V\times V$, resp. $(\mathbb{C}\otimes_{\mathbb{R}} V)\times(\mathbb{C}\otimes_{\mathbb{R}} V)$, and denote tangent vectors to V etc. by $(x;\xi)$, resp. $((x,y);(\xi,\eta))=(z;\zeta)$, $x,y,\xi,\eta\in V$. Also, remember that V is embedded into $\mathbb{C}\otimes_{\mathbb{R}} V$ as $\{(x,0): x\in V\}$.

Suppose ψ is a holomorphic W-valued function on some open neighborhood $\widetilde{\Omega}$ of $0\in\mathbb{C}\otimes_{\mathbb{R}} V$, and put $\varphi(x)=\psi(x,0)$, $x\in V\cap\widetilde{\Omega}$. In this case

$$d\psi(z;(\xi,-J\xi))=d\psi(z;(\xi,0))-d\psi(z;(0,J\xi)),$$

and, since ψ is holomorphic, the last term can be written $id\psi(z;(J\xi,0))$. Thus

$$d\psi((x,0);(\xi,-J\xi))=\bar\partial\varphi(x;\xi). \tag{3.1}$$

Returning to f, extend it to a holomorphic q-form g on some convex neighborhood $\widetilde{\Omega}\subset\mathbb{C}\otimes_{\mathbb{R}} V$ of 0. (If f is polynomial, we can take $\widetilde{\Omega}=\mathbb{C}\otimes_{\mathbb{R}} V$.) Using (3.1) we can compare the differentials

$$dg(z;\zeta_0,\dots,\zeta_q)=\sum_0^q(-1)^j d(g(x;\zeta_0,\dots,\widehat{\zeta_j},\dots,\zeta_q))(\zeta_j) \tag{3.2}$$

and

$$\bar\partial f(x;\xi_0,\dots,\xi_q)=\sum_0^q(-1)^j\bar\partial(f(x;\xi_0,\dots,\widehat{\xi_j},\dots,\xi_q))(\xi_j)=0 \tag{3.3}$$

(cf. (2.3)). We find $dg(z;\zeta_0,\dots,\zeta_q)=0$ if $z=(x,0)$, $\zeta_j=(\xi_j,-J\xi_j)$, $x,\xi_j\in V$. Since dg is holomorphic, the same holds for all $z\in\widetilde{\Omega}$. Thus, given $z_0\in\widetilde{\Omega}$ and the affine complex subspaces

$$V_{z_0}^{\pm}=\{z_0+(x,\pm Jx): x\in V\}\subset\mathbb{C}\otimes_{\mathbb{R}} V,$$

dg vanishes if restricted to $V_{z_0}^-\cap\widetilde{\Omega}$. Note that the manifolds V_z^- $(z\in V_0^+)$ foliate $\mathbb{C}\otimes_{\mathbb{R}} V$. By Poincaré's lemma on each such $V_z^-\cap\widetilde{\Omega}$ we can find $v_z\in C_{q-1}^\infty(V_z^-\cap\widetilde{\Omega})$ with $dv_z=g|_{V_z^-}$. In fact the formula that constructs v_z is quite explicit (see e.g. [AMR]), and gives a canonical such v_z once an "origin" is distinguished in $V_z^-\cap\widetilde{\Omega}$. If we choose this origin $z\in V_z^-\cap\widetilde{\Omega}$, it is easy to check that the forms v_z patch together to give a holomorphic W-valued q-form v on $\widetilde{\Omega}$, with Ker v containing tangent vectors to V_z^+. v is polynomial if f is; we have

$$dv|_{V_z^-\cap\Omega}=g|_{V_z^-\cap\Omega}. \tag{3.4}$$

The condition on Ker v implies $u=v|_{V\cap\widetilde{\Omega}}$ is an analytic $(0,q)$-form. Replacing g, f, and q with v, u, and $q-1$ in (3.2), (3.3), a comparison of the resulting formulas with (3.4) gives $\bar\partial u=f$, as required.

The proof of Lemma 3.1 provides an actual construction of the solution u, and it is easy to check that the solution operator $R: f\to u$ thus obtained is linear. In the sequel we shall need smoothness properties of R. Note, however, that the vector spaces between which the solution operator acts do not carry natural topologies. For this reason we must first define what we mean by smoothness in this situation: for operators acting between spaces of differential forms.

Thus, let $F \to N$ be a smooth vector bundle, U an open set in a locally convex space S, and $r = 0, 1, \dots;\ l = 0, 1, \dots, \infty, \omega$. We say that a family $\alpha_s \in C^l_r(N, F)$, $s \in U$, is of class C^l if the mapping

$$U \times \bigoplus^r TM \ni (s, \xi) \mapsto \alpha_s(\xi) \in F$$

is C^l.

Now suppose that we are given an additional vector bundle $F' \to N'$ and vector subspaces $A \subset C^l_r(N, F)$, $A' \subset C^l_{r'}(N', F')$. We say that an operator $Q : A \to A'$ is of class C^l if it carries C^l families $\{\alpha_s\} \subset A$ to C^l families $\{Q\alpha_s\} \subset A'$. It is immediate to verify that the solution operator $f \mapsto Rf$ constructed above is smooth:

Proposition 3.3. *In the situation of Lemma 3.1 the solution operator $f \mapsto u = Rf$ constructed in the proof is linear. Moreover, for any $l = k+1, \dots, \infty, \omega$ the restriction of R to C^l-forms is of class C^l.*

Observe that a W-valued (p, q)-form α of class C^l can be thought of as a C^l-multilinear alternating family α_ξ of W-valued $(0, q)$-forms, parametrized by $\xi \in \bigoplus^p W$. Also, $(\bar\partial\alpha)_\xi = \bar\partial(\alpha_\xi)$. Hence Proposition 3.3 implies

Theorem 3.4. *Suppose V, W are locally convex complex vector spaces, $\Omega \subset V$ is open, $k = 0, 1, \dots,$ $l = k+1, \dots, \infty, \omega,$ $p = 0, 1, \dots,$ $q = 1, 2, \dots,$ and $f \in C^l_{p,q}(\Omega, W)$. If $\bar\partial f = o(k-1)$ at a certain $x_0 \in \Omega$, then there is a $u \in C^l_{p,q-1}(\Omega, W)$ such that $\bar\partial u - f = o(k)$.*

Moreover, there is a linear solution operator $f \mapsto u$ of class C^l.

Our next result can be interpreted as asserting that formally, holomorphic vector bundles are trivial. If $\pi_i : E_i \to M$, $i = 1, 2$, are holomorphic vector bundles and $F : E_1 \to E_2$ is a smooth bundle homomorphism, the quantity that measures the deviation of F from holomorphic is

$$\overline{D}F : T^{0,1}E_1 \to T^{1,0}E_2,$$

introduced in section 2. π_2 and $\pi_2 \circ F$ being holomorphic we see that the image $\overline{D}F(\xi)$ of any $\xi \in T^{0,1}E_1$ is vertical, hence can be identified with a vector in E_2. Further, if $\xi \in T^{0,1}E_1$ itself is vertical, then $\overline{D}F(\xi) = 0$, since F is holomorphic, even complex linear, along the fibers of π_1. This suggests that we introduce the bundle $H^{0,1}E_1$ of horizontal vectors: this is the quotient of $T^{0,1}E_1$ by the subbundle of vertical vectors. Alternatively, $H^{0,1}E_1 \to E_1$ will be induced from the bundle $T^{0,1}M \to M$ by the mapping $\pi_1 : E_1 \to M$. Either way, $\overline{D}F$ factors through $H^{0,1}E_1$, and together with the identification between vertical vectors in $T^{1,0}E_2$ and vectors in E_2, we obtain a map

$$\bar\partial F : H^{0,1}E_1 \to E_2.$$

Note that $H^{0,1}E_1$ is a fiber bundle over M, so it makes sense to consider the order of vanishing of $\bar\partial F$ at a point $x \in M$, and this will be the same as the order of vanishing of $\overline{D}F$ at x.

Below, when dealing with different bundles, we shall use $\bar\partial_E$ to denote $\bar\partial$ operators associated with a holomorphic vector bundle E; when the bundle is trivial, we shall drop E.

Theorem 3.5. *Suppose $E \to \Omega$ is a smoothly trivializable holomorphic vector bundle with fiber isomorphic to a locally convex complex vector space W and $\Omega \subset V$ open in some locally convex complex vector space V. Given $x \in \Omega$ and $k = 0, 1, \dots$ there are smooth bundle homomorphisms $F : \Omega \times W \to E$ and $G : E \to \Omega \times W$ such that*

(1) $\bar\partial F = o(k), \quad \bar\partial_E G = o(k)$;
(2) $F \circ G - \mathrm{id}_E = o(k+1)$, $G \circ F - \mathrm{id}_{\Omega\times W} = o(k+1)$, *at* x.

(2) implies that $F_x : W \to E_x$ is an isomorphism, and if W is a Banach space, then it also follows that F, G are smooth bundle *isomorphisms* in a neighborhood of x. For more general fibers we see no reason why one should be able to choose F, G as isomorphisms. Nevertheless, the pair F, G as in the theorem is a perfect replacement for a bundle isomorphism; indeed, tensors can be transplanted between E and $\Omega\times W$ by F in one direction and by G in the other. For example an extension of Theorem 3.4 to bundle-valued forms is an immediate consequence:

Theorem 3.6. *Let $E \to \Omega$ be as in Theorem 3.5, $k = 0, 1, \dots$, $l = k+1, \dots, \infty$, $p = 0, 1, \dots$, $q = 1, 2, \dots$, and $f \in C^l_{p,q}(\Omega, E)$. If $\bar\partial f = o(k-1)$ at a certain $x_0 \in \Omega$ then there is a $u \in C^l_{p,q-1}(\Omega, E)$ such that $\bar\partial u - f = o(k)$. Again, there is a linear solution operator $f \mapsto u$ of class C^l.*

In fact, we can arrange $u \in C^\infty_{p,q-1}(\Omega, E)$, even if $l < \infty$; for this purpose we replace u supplied by the theorem by its Taylor polynomial of order $k+1$ at x_0 (after a smooth identification of E with $\Omega \times W$).

To prepare the proof of Theorem 3.5, choose a smooth bundle isomorphism $\Phi : \Omega\times W \to E$. Given a section $u \in C^1(\Omega, \Omega\times W)$ we want to compute $\bar\partial_E(\Phi\circ u)$. By the chain rule

$$(\Phi \circ u)_*\xi = \Phi_* u_*\xi = \Phi_*(u_*\xi)^{1,0} + \Phi_*(u_*\xi)^{0,1}, \tag{3.5}$$

where $\xi \in T^{0,1}_y\Omega$ and superscripts $1,0$, resp. $0,1$, indicate the components of a vector in $T^{1,0}(\Omega \times W) \oplus T^{0,1}(\Omega \times W)$. Observe that $(u_*\xi)^{1,0}$ is a vertical vector, and as such corresponds to $\bar\partial u(\xi) \in \{y\} \times W$; as Φ is complex linear on fibers, it follows that the first term on the right of (3.5) is also vertical and corresponds to $\Phi(\bar\partial u(\xi)) \in E_y$. Next, the vector $(u_*\xi)^{0,1}$ projects to the horizontal vector

$$\xi \in H^{0,1}_{u(y)}(\Omega \times W) \cong (T^{0,1}_y\Omega) \times W,$$

so that taking into account the definitions, (3.5) implies "Leibniz's formula"

$$\bar\partial_E(\Phi \circ u)(\xi) = \Phi(\bar\partial u(\xi)) + \bar\partial\Phi(u(y); \xi), \quad \text{or} \tag{3.6}$$

$$\bar\partial_E(\Phi \circ u) = \Phi \circ \bar\partial u + (\bar\partial\Phi) \circ u. \tag{3.7}$$

Assume now that u is C^2, put $f = \Phi^{-1} \circ \bar\partial_E(\Phi \circ u)$, and given a constant $(0,1)$ vector *field* ξ on Ω, apply (3.7) with u replaced by $f(\xi)$ to get

$$\Phi^{-1} \circ \bar\partial_E(\Phi \circ f(\xi)) = \bar\partial(f(\xi)) + \bar\partial\Phi \circ f(\xi).$$

In view of (2.3) we can therefore represent the difference between $\Phi^{-1}\circ\bar\partial_E(\Phi\circ f) = 0$ and $\bar\partial f$ as a linear expression in f, involving no differentiation on f. Denoting this expression $\bar\partial\Phi \wedge f$ we therefore find

$$\bar\partial(\Phi^{-1} \circ \bar\partial_E(\Phi \circ u)) = -\bar\partial\Phi \wedge (\Phi^{-1} \circ \bar\partial_E(\Phi \circ u)). \tag{3.8}$$

Now we are ready to verify Theorem 3.5.

Proof of Theorem 3.5. Assume $x = 0$. With Φ as above and $k = -1, 0, \dots$ we claim that given $w \in W$ we can find $u = u_w \in C^\infty(\Omega, \Omega \times W)$ such that $u(0) = (0, w)$ and $\bar\partial_E(\Phi \circ u) = o(k)$ at $x = 0$. Moreover, the family u_w, $w \in W$, will be smooth and linear.

When $k = -1$, $u_w(y) = (y, w)$ does it. Supposing we have found $u = u_w$ corresponding to a certain k, let us try to find $U = U_w$ corresponding to $k+1$ in the form $U = u + v$ with $v = o(k+1)$. The condition on U is that $\bar\partial_E(\Phi \circ U) = o(k+1)$, or, by (3.7) and collecting terms vanishing to order $k+2$,

$$\bar\partial v = -\Phi^{-1}\bar\partial_E(\Phi \circ u) + o(k+1). \tag{3.9}$$

In view of (3.8) and the inductive hypothesis, $\bar\partial$ of the right hand side is $o(k)$, and so by Proposition 3.3 (3.9) admits a solution $v = v_w$, depending smoothly and linearly on w. Since the right hand side of (3.9) is $o(k)$, the homogeneous polynomial $\widetilde{v}(y) = \widetilde{v}_w(y) = d^{k+2}v(0; y, \dots, y)$ also solves (3.9), and then $U = u + \widetilde{v}$ will be a section with properties required.

Now, if u_w is as in the claim we can define a smooth bundle homomorphism $\varphi : \Omega \times W \to \Omega \times W$ by $\varphi(y, w) = u_w(y)$; then $F = \Phi \circ \varphi$ satisfies $\bar\partial F = o(k)$. Also φ restricted to the zero section as well as to the fiber above $0 \in \Omega$ is the identity, hence the differential $\varphi_*(0, 0)$ is invertible. We can construct an approximate inverse ψ to φ, i.e. a smooth, indeed polynomial map $\psi : \Omega \times W \to \Omega \times W$ such that $\varphi \circ \psi - \mathrm{id} = o(k+1)$, $\psi \circ \varphi - \mathrm{id} = o(k+1)$ at 0. For this purpose one simply determines successively the terms of the homogeneous expansion $\psi = \mathrm{id} + \sum_2^{k+1} \psi_j$. On $\Omega \times \{0\}$, ψ reduces to the identity. Should ψ fail to be linear on the fibers $\{y\} \times W$, on each fiber we can replace it by its (complex) linear part to get another polynomial map $\overline{\psi} : \Omega \times W \to \Omega \times W$ which is now a bundle homomorphism. As φ was already linear on the fibers, we still have $\varphi \circ \overline{\psi} - \mathrm{id} = o(k+1)$, $\overline{\psi} \circ \varphi - \mathrm{id} = o(k+1)$. Hence the smooth bundle homomorphism $G = \overline{\psi} \circ \Phi^{-1}$ satisfies (2) of the Theorem. $\bar\partial_E G = o(k)$ now follows from $\bar\partial F = o(k)$ and (2), and the proof is complete.

4. The $\bar\partial$ equation in the whole space

Here we shall extend the simple result of Ehrenpreis that the equation $\bar\partial u = f$ with $f \in C^\infty_{0,1}(\mathbb{C}^n)$ closed and of bounded support has a solution u with bounded support if $n > 1$; see [E]. In the context of Banach spaces and $(0,1)$-forms such an immediate extension was published by Ligocka in [Li], and even the generalization to locally convex spaces is straightforward. However, the condition on the support being bounded must be modified, for in spaces that are not normable bounded sets have no interior points, and so continuous functions with bounded support vanish identically. We therefore introduce a new class, that of the *narrow* sets, which is broader than the class of bounded sets. Roughly, the property that we need for a set $S \subset V$ to be narrow is that S intersected with finite dimensional subspaces $A \subset V$ should be bounded, and uniformly so when A is slightly perturbed. This notion could be made precise in a mildly complicated definition; however, we have opted for a simpler if slightly more restrictive concept, which will do just as well for later applications we have in mind. This is meaningful in the context of locally convex spaces V that admit a continuous norm $p : V \to \mathbb{R}^+$ (as opposed to continuous seminorms, which always exist by definition). If M is a finite dimensional compact manifold, then $C^\infty(M) = V$ is of this species, with $p(f) = \max|f|$, while $C^\infty(M)$

for a noncompact M never supports a continuous norm. — Below we shall give two definitions for a set to be narrow: one for subsets of a locally convex space, the other, a relative concept, and of convenience only, for subsets of a trivial bundle $B \times V$ with finite dimensional fibers V.

Definition 4.1. (a) A subset S of a locally convex space V is narrow if V admits a continuous norm that is bounded on S.

(b) If B is a manifold and V a finite dimensional vector space, a subset S of the trivial bundle $B \times V \to B$ is narrow if $S \subset B \times K$ with some compact $K \subset V$.

To avoid confusion we put on record that a manifold M might be represented as a trivial bundle in different ways and whether a set $S \subset M$ is narrow depends on the representation $\pi : M = B \times V \to B$. For this reason we shall use the term "narrow for π" when necessary. — Now we can state the following result about global solvability.

Theorem 4.2. *Suppose $E \to V$ is a holomorphic vector bundle over a locally convex complex vector space V, $p = 0, 1, \ldots,$ $1 \le q < \dim V$, $k = 2, 3, \ldots, \infty$, and $f \in C^k_{p,q}(V, E)$ is a closed form. If f has narrow support, then there is a $u \in C^k_{p,q-1}(V, E)$ with narrow support such that $\bar\partial u = f$, provided*

(i) *$q = 1$, E is of finite rank, and $k = \infty$; or*

(ii) *E is trivial.*

Moreover, in these cases if we fix a narrow set $N \subset V$, then on the space of closed forms f supported in N there is a linear solution operator $f \mapsto u$ of class C^k, whose values u are also supported in some fixed narrow set N' (that depends on N).

As mentioned, case (i) of the theorem is due to Ehrenpreis and Ligocka, when E is the trivial line bundle (and $p = 0$). Mujica then solved $\bar\partial u = f$ for $f \in C^\infty_{0,q}(V)$ with bounded support, V a Banach space, but he did not get u with bounded support; see [Mu]. When $\dim_{\mathbb{C}} V$, rk $E < \infty$, the theorem also follows from Serre duality and Cartan's Theorem B.

Theorem 4.2 will be proved by induction on q, but in order to get the induction step right we shall need to consider a more general situation as follows:

Theorem 4.3. *Suppose B is a complex manifold, V a finite dimensional complex vector space, $E \to B \times V$ a holomorphic vector bundle, $1 \le q < \dim V$, $k = 2, 3, \ldots, \infty$, and $f \in C^k_{0,q}(B \times V, E)$ a closed form. If supp f, as a subset of the bundle $\pi : B \times V \to B$, is narrow, then there is a $u \in C^k_{0,q-1}(B \times V, E)$ with narrow support* supp $u \subset \pi^{-1}\pi(\text{supp } f)$ *such that $\bar\partial u = f$, provided*

(i) *$q = 1$, E is of finite rank, and $k = \infty$; or*

(ii) *E is trivial.*

Moreover, in these cases, if we fix a narrow subset N of $B \times V \to B$, on the space of closed forms f supported in N there is a linear solution operator $f \mapsto u$ of class C^k, whose values u are also supported in some fixed narrow set N'.

In the proof, as well as later, we shall make use of the following simple

Proposition 4.4. *Suppose $\pi : M \to B$ is a holomorphic fiber bundle, $E \to M$ a holomorphic vector bundle, and $g \in C^1_{0,q}(M, E)$ a closed form such that the tangent bundle of the fibers $F_b = \pi^{-1}(b)$, $b \in B$, is contained in the kernel* Ker g. *If ξ is* *a not necessarily continuous section of* $T^{0,q}M$ *along* F_b *for some* $b \in B$ *such that* $\pi_*\xi$ *is constant, then* $g(\xi)$ *is holomorphic along* F_b*.*

Proof. First of all, when $x \in F_b$, the value of $g(\xi(x))$ is completely determined if $\pi_*\xi(x)$ is known, since g vanishes on q-tuples that contain vertical vectors. Therefore to check $g(\xi)$ is holomorphic near a certain point $y \in F_b$ we can assume that $\xi = (\xi_j)$ is indeed a smooth section of $T^{0,q}M$ in a neighborhood $\Omega \subset M$ of y and $\pi_*\xi$ is constant along the fibers. We have to show that $\bar\partial(g(\xi))(\eta) = 0$ for any smooth section η of $T^{0,1}F_b$ near y. If Ω is sufficiently small, we can extend η to a smooth vertical section of $T^{0,1}\Omega$. This implies $[\xi_j, \eta]$ are vertical. Indeed,

$$\pi_*([\xi_j, \eta](x)) = [\pi_*\xi_j, \pi_*\eta](\pi(x)) = 0$$

when $x \in \Omega$. Hence by the hypothesis that g is closed and by (2.3)

$$0 = \bar\partial g(\eta, \xi_1, \ldots, \xi_q) = \bar\partial(g(\xi_1, \ldots, \xi_q))(\eta)$$

holds on $F_b \cap \Omega$, i.e. $\bar\partial(g(\xi_1, \ldots, \xi_q))|_{F_b} = 0$, as claimed.

Proof of Theorem 4.3. Case $q = 1$. Even when E is not assumed to be trivial, a theorem of Grauert in [G] implies that the restricted bundles $E|_{\{b\}\times V}$ are trivial, for $b \in B$, and so a (vector-valued variant of) Ehrenpreis' theorem gives a unique $u_b \in C^k(\{b\} \times V, E)$ with compact support such that $\bar\partial u_b = f|_{\{b\}\times V}$. If we define $u(b, x) = u_b(x)$, $x \in V$, then clearly $\operatorname{supp} u \subset \pi^{-1}\pi(\operatorname{supp} f)$. *Assuming* $u \in C^k(B \times V, E)$ and putting $g = \bar\partial u - f \in C^1_{0,q}(B \times V, E)$, Proposition 4.4 gives that for any section ξ of $T^{0,1}(B \times V)$ along $\{b\} \times V$ the function $g(\xi)$ is holomorphic, provided $\pi_*\xi$ is constant. Since $g(\xi)$ is compactly supported, too, $g \equiv 0$ and $\bar\partial u = f$ follows. Also, u is holomorphic outside $\operatorname{supp} f$, and this easily implies that $\operatorname{supp} u$ is contained in the fiberwise convex hull of $\operatorname{supp} f$, in particular it is narrow. To conclude the proof we have to verify $u \in C^k(B \times V, E)$. When $E = (B \times V) \times W$ is trivial, this is obvious from the integral formula — a convolution — that represents u_b. If $\operatorname{rk} E < \infty$ and $k = \infty$, we have to use deformation theory as follows.

For a given point $b_0 \in B$ choose a neighborhood $B_0 \subset B$ and a bounded open convex set $0 \in V_0 \subset V$ such that

$$(B_0 \times V) \cap \operatorname{supp} f, \quad (B_0 \times V) \cap \operatorname{supp} u \subset B_0 \times V_0.$$

If B_0 is sufficiently small we can construct a smooth connection on $E|_{B_0\times 2V_0}$. This will involve a partition of unity, but only in V. Assuming, as we may, that B_0 is (diffeomorphic to) a convex neighborhood of the origin in some locally convex space T, horizontal lifts of straight lines through $0 \in T \times V$ will define isomorphisms between the fiber $E_{(0,0)}$ and the fibers $E_{(b,x)}$, $(b, x) \in B_0 \times 2V_0$. Thus a trivialization of $E|_{B_0\times 2V_0}$ is obtained, which is smooth off $(0, 0)$. If we make sure that $b_0 \neq 0$, we have a smooth trivialization of $E|_{B'\times 2V_0}$, with $B' \subset B$ a neighborhood of b_0.

This trivialization will be used to smoothly identify each bundle $E|_{\{b\}\times \bar V_0}$ with $E|_{\{b_0\}\times \bar V_0}$, or with some trivial bundle over $\overline{V}_0$. Assuming $V = \mathbb{C}^n$, it will be advantageous to think of this latter trivial bundle as a trivial holomorphic bundle Θ over $\mathbb{CP}_n \supset V \supset \overline{V}_0$. Thus we have smooth bundle isomorphisms $\Phi_b : E|_{\{b\}\times \bar V_0} \to \Theta|_{\bar V_0}$, and $\Phi = \{\Phi_b\} : E|_{B'\times \bar V_0} \to \Theta$ is also smooth. We can assume Φ_{b_0} is holomorphic.

If $H \to \mathbb{CP}_n$ denotes the hyperplane section bundle and σ a holomorphic section whose divisor is disjoint from $\overline{V}_0$, then we can also set up bundle isomorphisms $\Psi_b : E|_{\{b\}\times \bar V_0} \to H^* \otimes \Theta|_{\bar V_0}$ by putting $\Psi_b(b, x) = \sigma^{-1}(x) \otimes \Phi(b, x)$. These Ψ_b will let us transplant $E|_{\{b\}\times \bar V_0}$-valued forms to $H^* \otimes \Theta|_{\bar V_0}$-valued forms, and the

$\bar\partial$ operator from the former bundle to a smooth family D_b of elliptic operators $C^\infty(H^*\otimes\Theta|_{\bar V_0})\to C^\infty_{0,1}(H^*\otimes\Theta|_{\bar V_0})$. In fact, we can extend D_b to a smooth family of elliptic operators

$$C^\infty(H^*\otimes\Theta)\to C^\infty_{0,1}(H^*\otimes\Theta)$$

(also denoted D_b), making sure that $D_{b_0}=\bar\partial_{H^*\otimes\Theta}$. Endow $H^*\otimes\Theta$ and $\mathbb{CP}_n$ with hermitian metrics, introduce the corresponding L^2 scalar products on the spaces (4.1), and form the adjoints D_b^* of D_b and the Laplacians

$$\Box_b=D_b^*D_b: C^\infty(H^*\otimes\Theta)\to C^\infty(H^*\otimes\Theta).$$

Above all note that Ker $\Box_{b_0}=$ Ker $D_{b_0}=(0)$, since the only holomorphic section of $H^*\otimes\Theta$ is the zero section. Deformation theory — as expounded e.g. in [Ko, Chapter 7] — in this situation provides a smooth family G_b of bounded (in all Sobolev spaces) linear operators, inverse to $\Box_b$, defined for b in some neighborhood $B''\subset B'$ of b_0.[2]

As said above, Ψ_b lets us transplant u_b and $f|_{\{b\}\times V}$ to $H^*\otimes\Theta|_{\bar V_0}$, and we extend the resulting sections, resp. forms, by 0 to get $\nu_b\in C^\infty(H^*\otimes\Theta)$, $\varphi_b\in C^\infty_{0,1}(H^*\otimes\Theta)$. Thus φ_b is a smooth family, and $D_b\nu_b=\varphi_b$. It follows that $\nu_b=G_bD_b^*\varphi_b$ is also a smooth family, and this implies u is indeed C^∞.

Case $q>1$. We are talking about forms with values in a locally convex complex vector space W. Assuming we know the theorem for $q-1$, we will prove it for q. We start with Mujica's idea in [Mu]. Write $V=V_1\oplus\mathbb{C}$ and $B_1=B\times V_1$; then $B\times V=B_1\times\mathbb{C}\xrightarrow{\pi_1}B_1$ is a trivial line bundle. Write $\mathbb{C}_b$ for $\mathbb{C}\times\{b\}$, $b\in B_1$. Given $\Xi=(\xi_2,\dots,\xi_q)\in T_b^{0,q-1}B_1$, define a compactly supported $f_\Xi\in C^k_{0,1}(\mathbb{C}_b,W)$ by

$$f_\Xi(\xi)=f(\xi,\xi_2',\dots,\xi_q'),\quad \xi\in T_x^{0,1}(\mathbb{C}_b),\ x\in\mathbb{C}_b,$$

where $\xi_j'\in T_x^{0,1}(B_1\times\mathbb{C})$ satisfy $\pi_{1*}\xi_j'=\xi_j$. Compactify $\mathbb{C}_b$ to Riemann spheres $\mathbb{P}_b$. Convolution of f_Ξ with the Cauchy kernel on $\mathbb{C}_b$ constructs the unique $u_\Xi\in C^k(\mathbb{P}_b,W)$ vanishing at $\infty\in\mathbb{P}_b$ such that $\bar\partial u_\Xi=f_\Xi$. The family u_Ξ is clearly alternating and linear in Ξ, moreover, depends smoothly on b,Ξ, hence gives rise to a form $u\in C^k_{0,q-1}(B_1\times\mathbb{CP}_1)$ such that

$$u(\xi_2',\dots,\xi_q')=u_\Xi(x),\qquad \xi_j'\in T_x^{0,1}(B_1\times\mathbb{CP}_1),\pi_*\xi_j'=\xi_j.$$

This means the fibers $\mathbb{P}_b$ are tangent to Ker u. A comparison of $\bar\partial u_\Xi=f_\Xi$ with the definition (2.3), mutatis mutandis, shows that $\mathbb{P}_b$ is also tangent to the kernel of $g=\bar\partial u-f$. Let ξ_j', $j=1,\dots,q$, denote arbitrary sections of $T^{0,1}(B_1\times\mathbb{CP}_1)$ along $\mathbb{P}_b$ such that the $\pi_*\xi_j'$ are constant. Proposition 4.4 implies that $h=g(\xi_1',\dots,\xi_q')$ is holomorphic, hence constant on $\mathbb{P}_b$. If, in addition we make sure that the ξ_j' are tangent to $B_1\times\{\infty\}$, then we see $h\equiv0$. Thus $g=0$ and so $\bar\partial u=f$.

However, u need not be narrowly supported on $\pi:B\times V\to B$. To study its support, introduce $B_2=B\times\mathbb{C}$, the projection $\pi_2:B\times V=B_2\times V_1\to B_2$, and also the notation $[S]_\pi=\pi^{-1}\pi(S)$ for $S\subset B\times V$, with $[S]_{\pi_1}$, $[S]_{\pi_2}$ defined analogously.

Since supp f was narrow for $\pi:B\times V\to B$, $[\text{supp } f]_{\pi_1}$ is narrow in the bundle $\pi_2:B_2\times V_1\to B_2$. The way u was constructed (by convolutions along the fibers of π_1) implies

$$[\text{supp } u]_{\pi_1}\subset[\text{supp } f]_{\pi_1}, \tag{4.2}$$

[2]Traditionally, the parameter space for deformation theory is finite dimensional. However, the results immediately carry over to infinite dimensional parameter spaces such as B'.

in particular supp u is also narrow for π_2. Since $\bar\partial u = 0$ holds on the trivial bundle

$$M = (B_2 \times V_1) \setminus [\text{supp } f]_{\pi_2} \xrightarrow{\pi_2} B_2 \setminus \pi_2(\text{supp } f),$$

and $\dim_{\mathbb{C}} V_1 > q - 1$, according to the inductive hypothesis there is a narrowly supported $v \in C^k_{0,q-2}(M, W)$ with $\bar\partial v = u|_M$, supp $v \subset [\text{supp } u]_{\pi_2}$.

Further, $[\text{supp } f]_{\pi_2}$ is narrow for π_1, whence it follows that there is a cutoff function $\chi \in C^\infty(B_1 \times V_2)$ with narrow support for π_1 and equal to 1 in a neighborhood of $[\text{supp } f]_{\pi_2}$. (We obtain such a χ by pulling back an appropriate cut-off function on V.) Obviously $U = u - \bar\partial((1-\chi)v) \in C^k_{0,q-1}(B \times V)$ solves $\bar\partial U = f$; we claim it is narrowly supported for π. Indeed, supp $U \subset$ supp χ is narrow for π_1, further supp $U \subset$ supp $u \cup$ supp v shows it is narrow for π_2 as well; and a subset of $B \times V$ narrow for both π_1, π_2 is narrow for π. The latter inclusion above, the support condition on v, and (4.2) also imply supp $U \subset [\text{supp } f]_\pi$, as required.

We leave it to the reader to check that the solution operators constructed above are linear and C^k.

Proof of Theorem 4.2. Choose a continuous norm $\| \ \|$ on V, bounded on supp f. Take a subspace $\mathbb{C}^{q+1} \subset V$, and using the Banach-Hahn theorem, extend $\mathrm{id}_{\mathbb{C}^{q+1}}$ to a continuous linear transformation $\varphi : (V, \| \ \|) \to \mathbb{C}^{q+1}$. Choose a compact $K \subset \mathbb{C}^{q+1}$ that contains $\varphi(\text{supp } f)$. Put $B = \text{Ker } \varphi$, a closed subspace of V; then $V = B \times \mathbb{C}^{q+1}$, and supp $f \subset B \times K$, so that f is narrowly supported for $\pi : B \times \mathbb{C}^{q+1} \to B$. Furthermore if $x \in V$, then $\|\pi(x)\| \le \|x\| + \|\varphi(x)\|$; this shows that $\pi(\text{supp } f) \subset B$ is a narrow set.

Hence Theorem 4.3 applies and in the case $p = 0$ gives a solution $u \in C^k_{0,q-1}(V, E)$ with narrow support for π; also supp $u \subset \pi^{-1}\pi(\text{supp } f)$, whence supp u is narrow in V. The case $p > 0$ as well as linearity and smoothness of the solution operator follow from the last statement of Theorem 4.3, as in section 3.

5. The $\bar\partial$ equation on fiber bundles

This section will prepare the study of the Dolbeault cohomology groups of vector bundles over projective spaces. Our main tools will be simple propositions that relate the cohomology of fiber bundles to cohomology of the fibers and of a cross section, somewhat in the spirit of Leray's spectral sequence. Indeed, most of what follows can be generalized and recast as a spectral sequence in Dolbeault cohomology. However, as such a generalization offers no immediate return, we shall not discuss it here.

We fix a holomorphic fiber bundle $\pi : M \to B$ with fibers F_b finite dimensional connected compact complex manifolds. Below we shall investigate the solvability of the equation $\bar\partial u = f$ on M under various conditions.

Proposition 5.1. *Suppose that $E \to M$ is a holomorphic vector bundle of finite rank, and that for certain p, q the dimension of the Dolbeault cohomology groups $H^{p,q-1}(F_b, E)$ is independent of $b \in B$. If all restrictions $f|_{F_b}$ of a form $f \in C^\infty_{p,q}(M, E)$ are exact, then for $b_0 \in B$ there are a neighborhood $b_0 \in B_0 \subset B$ and a form $v \in C^\infty_{p,q-1}(\pi^{-1}(B_0), E)$ such that $\bar\partial v|_{F_b} = f|_{F_b}$, $b \in B_0$. Moreover, we can even prescribe any initial value $v|_{F_{b_0}} = v_0$ for such a v, as long as it satisfies $\bar\partial v_0 = f|_{F_{b_0}}$. Finally, we can arrange that the operator $(f, v_0) \mapsto v$ is smooth and linear.*

Proof. This is quite standard, and goes along the same lines as the corresponding part of the proof of Theorem 4.3. As there we can choose B_0 so that the bundle $\pi : M \to B$ is smoothly trivial over B_0, hence $M|_{\pi^{-1}(B_0)}$ and $E|_{\pi^{-1}(B_0)}$ admit smooth hermitian metrics and the latter a smooth connection as well. On the one hand this lets us introduce L^2 inner products on the spaces $C^\infty_{r,s}(F_b, E)$, on the other it lets us construct $w \in C^\infty_{p,q-1}(\pi^{-1}(B_0), E)$ such that $w|_{F_{b_0}} = v_0$. Deformation theory (see [Ko], especially Theorem 7.10) then implies that the solution $u_b \in C^\infty_{p,q-1}(F_b, E)$ of the equation $\bar\partial u_b = (f - \bar\partial w)|_{F_b}$ that is orthogonal to the space of closed forms depends smoothly on b, and so the forms u_b, put together, give rise to a smooth, albeit relative form u' on $\pi^{-1}(B_0)$; relative in the sense that u' is only defined on vectors tangential to F_b. Nevertheless, the trivialization of π over B_0 lets us extend u' to a form $u \in C^\infty_{p,q-1}(\pi^{-1}(B_0), E)$; thus $\bar\partial u|_{F_b} = (f - \bar\partial w)_{F_b}$. Also note that $u|_{F_{b_0}} = 0$. It follows that $v = u + w$ has the required properties, and is a smooth linear function of (f, v_0).

Proposition 5.2. *Let $E \to M$ be a holomorphic vector bundle and $g \in C^1_{0,q}(M, E)$ a closed form such that* Ker g *contains all vertical vectors $\xi \in TF_b$, $b \in B$. Then $g = 0$, provided*

(i) *$H^0(F_b, E) = 0$ for all $b \in B$; or*

(ii) *$E|_{F_b}$ is trivial and g vanishes somewhere on F_b, for all $b \in B$.*

Proof. This is a straightforward consequence of Proposition 4.4. With an arbitrary section ξ of $T^{0,q}M$ such that $\pi_*\xi$ is constant along each F_b, that proposition shows $g(\xi)$ is holomorphic along the fibers. Now either of (i) and (ii) implies this function is 0.

Proposition 5.3. *Let $E \to M$ be a holomorphic vector bundle of finite rank such that $H^0(F_b, E) = 0$, $b \in B$. Then a closed form $f \in C^\infty_{0,1}(M, E)$ is exact if and only if $f|_{F_b}$ is exact for all $b \in B$. Moreover, on the space of fiberwise exact $f \in C^\infty_{0,1}(M, E)$ there is a smooth linear operator $R : f \mapsto u \in C^\infty(M, E)$ such that $u = Rf$ solves $\bar\partial u = f$.*

Proof. Only one implication needs to be proved: assume $f|_{F_b}$ are exact. Proposition 5.1 gives a covering of B by open sets B' and $u_{B'} \in C^\infty(\pi^{-1}(B'), E)$ such that $\bar\partial u_{B'}|_{F_b} = f|_{F_b}$, $b \in B'$. $H^0(F_b; E) = 0$ implies that these functions are compatible, and patch together to give $u \in C^\infty(M, E)$ with $\bar\partial u|_{F_b} = f|_{F_b}$. By virtue of Proposition 5.2 $g = \bar\partial u - f = 0$, i.e. f is exact. Also, the operator $f \mapsto u$ is smooth and linear.

A variant of the same idea gives:

Theorem 5.4. *Suppose $\pi : M \to B$ has a holomorphic section $\sigma : B \to M$, and let $E \to M$ be a holomorphic vector bundle of finite rank, trivial on all fibers F_b.*

A closed form $f \in C^\infty_{0,1}(M, E)$ is exact if and only if

(a) *$f|_{F_b}$ is exact for all $b \in B$; and*

(b) *$\sigma^* f$ is exact.*

Furthermore, suppose we are given a subspace $A \subset C^\infty_{0,1}(M, E)$ of fiberwise exact forms and a smooth linear operator $Q : A \to C^\infty(B, \sigma^* E)$ such that $\bar\partial(Qf) = \sigma^* f$ for $f \in A$. Then there is a smooth linear operator $R : A \to C^\infty(M, E)$ such that $\bar\partial(Rf) = f$, $f \in A$.

Proof. To prove the "if" part, first find a $w \in C^\infty(B, \sigma^* E)$ with $\bar\partial w = \sigma^* f$, and note that $f' = f - \bar\partial(w \circ \pi)$ is still closed, restricts to exact forms on F_b, but this time $f'|_{\sigma(B)} = 0$; and all we need is to show f' is exact. Next for $b \in B$ find the unique $u_b \in C^\infty(F_b, E)$ such that $\bar\partial u_b = f'|_{F_b}$ and $u_b(\sigma(b)) = 0$.

We claim that $u(x) = u_b(x)$ if $x \in F_b$ defines a smooth section of E. To check this near an arbitrary fiber F_{b_0}, recall that by Proposition 5.1 there are a neighborhood $B_0 \ni b_0$ and a $v \in C^\infty(\pi^{-1}(B_0), E)$ such that $\bar\partial v|_{F_b} = f'|_{F_b}$. This implies $v - v \circ \sigma \circ \pi = u|_{\pi^{-1}(B_0)}$, whence this latter is indeed smooth.

At this point we can conclude as before: $g = \bar\partial u - f' = 0$ by virtue of Proposition 5.2, case (ii), i.e., f' is exact. Furthermore, the above construction yields a solution operator R as claimed.

For $(0,q)$-forms, $q > 1$, we must be satisfied with weaker results. First of all, we shall have to assume that the fibers of $\pi : M \to B$ are one dimensional. If $f \in C^\infty_{0,q}(M, E)$ and $\Xi = (\xi_2, \dots, \xi_q) \in T^{0,q-1}_b B$ as in section 4 we define $f_\Xi \in C^\infty_{0,1}(F_b, E)$ by

$$f_\Xi(\xi) = f(\xi, \xi'_2, \dots, \xi'_q), \tag{5.1}$$

where $\xi \in T^{0,1}_x F_b$, $\xi'_j \in T^{0,1}_x M$, $x \in F_b$, and $\pi_* \xi'_j = \xi_j$; the value in (5.1) is independent of the particular choice of ξ'_j. When $q = 1$, we interpret (5.1) for "$\Xi \in T^{0,0}M$" (i.e. $\Xi = 0$) as $f_\Xi = f|_{F_b}$.

There is one concept we have to introduce before turning our attention to $(0,q)$-forms on fiber bundles. Let $E \to M$ be a holomorphic vector bundle, $S \subset M$ a subset, and $f \in C^\infty_{p,q}(M, E)$. We say that f is exact at S to order $k+1$ if there is a $u \in C^\infty_{p,q-1}(M, E)$ such that $\bar\partial u - f = o(k)$ at all points $x \in S$. As an illustration, Theorem 3.6 implies that for $E \to \Omega$ as there, any closed $f \in C^\infty_{p,q}(\Omega, E)$ is exact to arbitrary order at $x \in \Omega$. Also, we shall say that a subspace $A \subset C^\infty_{p,q}(M, E)$ has a smooth solution operator of order $k+1$ at S if there is a smooth linear operator $Q : A \to C^\infty_{p,q}(M, E)$ such that $u = Qf$ satisfies $\bar\partial u - f = o(k)$ at all points of S, $f \in A$.

Proposition 5.5. *Suppose the fibers of $\pi : M \to B$ are one dimensional, π has a holomorphic section $\sigma : B \to M$, and let $E \to M$ be a holomorphic vector bundle of finite rank. A closed form $f \in C^\infty_{0,q}(M, E)$, $q \geq 1$, will be exact provided*

(i) *$H^0(F_b, E) = 0$ and f_Ξ are exact for all $b \in B$, $\Xi \in T^{0,q-1}B$; or*
(ii) *$E|_{F_b}$ are trivial, f_Ξ are exact for all $b \in B$, $\Xi \in T^{0,q-1}B$, and $\sigma^* f$ is exact; or*
(iii) rk *$E = 1$, the fibers F_b are of genus 0, $E|_{F_b} \geq 0$, and f is exact to arbitrary order at $\sigma(B)$.*

Proof. (i), (ii). In case (i) for each $\Xi \in T^{0,q-1}B$ choose $u_\Xi \in C^\infty(F_b, E)$ so that $\bar\partial u_\Xi = f|_{F_b}$. In case (ii), first reduce to the situation when $f|_{\sigma(B)} = 0$ as in the proof of Theorem 5.4, then choose $u_\Xi \in C^\infty(F_b, E)$ so that $\bar\partial u_\Xi = f_\Xi$ and $u_\Xi(\sigma(b)) = 0$. In both cases, u_Ξ is uniquely determined, and its dependence on $\Xi \in T^{0,q-1}_b B$ is alternating multilinear. Further, an argument as in the proof of Theorem 5.4 gives that

$$u(\xi_1, \dots, \xi_{q-1}) = u_{\pi_*\xi_1, \dots, \pi_*\xi_{q-1}}(x), \qquad \xi_1, \dots, \xi_{q-1} \in T^{0,q}_x M, \tag{5.2}$$

defines a form $u \in C^\infty_{0,q-1}(M, E)$. Note that all vertical vectors are in Ker u.

We want to compare $\bar\partial u$ with f. Construct smooth sections $\xi, \xi_2, \dots, \xi_q$ of $T^{0,1}M$ in a neighborhood of x such that ξ is vertical and $\pi_*\xi_j$ are constant along the fibers. As in the proof of Proposition 4.4, $[\xi, \xi_j]$ are vertical. Hence (2.3) gives

$$\begin{aligned}\bar\partial u(\xi, \xi_2, \dots, \xi_q) &= \bar\partial(u(\xi_2, \dots, \xi_q))(\xi)\\ &= \bar\partial u_{\pi_*\xi_2, \dots, \pi_*\xi_q}(\xi) = f(\xi, \xi_2, \dots, \xi_q),\end{aligned}$$

i.e. the kernel of the closed form $g = \bar\partial u - f$ contains all vertical vectors. By virtue of Proposition 5.2 $g = 0$, and we are done.

(iii) We can safely assume that B is connected, whence all line bundles $E|_{F_b}$ have the same degree $d \geq 0$. Choose $w \in C^\infty_{0,q-1}(M, E)$ so that

$$f' = f - \bar\partial w = o(d) \tag{5.3}$$

at points of $\sigma(B)$. It will suffice to show f' is exact. Since $H^1(E|_{F_b}) = 0$, for each $\Xi \in T_b^{0,q-1}B$ there are $u_\Xi \in C^\infty_{0,q-1}(F_b, E)$ such that $\bar\partial u_\Xi = f'_\Xi$. (5.3) implies that the d-order jet (Taylor polynomial) $j_d\,(u_\Xi; \sigma(b))$ of u_Ξ at $\sigma(b) \in F_b$ is holomorphic. On the other hand, holomorphic sections of $E|_{F_b}$ can produce any holomorphic d-order jets. Thus, by adding on an appropriate holomorphic section of $E|_{F_b}$ we can arrange that $u_\Xi = o(d)$ at $\sigma(b)$; furthermore this initial condition uniquely specifies the solution u_Ξ. At this point we can conclude in the same way as in the first half of the proof that (5.2) defines $u \in C^\infty_{0,q-1}(M, E)$ which satisfies $\bar\partial u = f'$; with the only difference that instead of simply evoking Proposition 5.2 we note that $\bar\partial u - f'$, when evaluated on sections (ξ_j) of $T^{0,q+1}M$ such that $\pi_*\xi_j$ is constant along F_b, gives a holomorphic section h of $E|_{F_b}$ — this by Proposition 4.4 — and so $h = o(d)$ at $\sigma(b)$ implies $h = 0$. Thus $\overline{\partial} u = f'$, and f is exact as claimed.

The proof above in fact constructs a solution operator for the $\bar\partial$ equation as follows.

Proposition 5.6. *In the situation of Proposition 5.5, on a subspace $A \subset C^\infty_{0,q}(M, E)$ there is a smooth linear operator $R : A \to C^\infty_{0,q-1}(M, E)$ such that $u = Rf$ solves $\bar\partial u = f$, $f \in A$, provided*

(i) *$H^0(F_b, E) = 0$ and f_Ξ are exact for all $b \in B$, $\Xi \in T^{0,q-1}B$, $f \in A$; or*

(ii) *$E|_{F_b}$ are trivial and f_Ξ are exact for all $b \in B$, $\Xi \in T^{0,q-1}B$, $f \in A$, and A has a smooth solution operator of order 1 at $\sigma(B)$; or*

(iii) rk $E = 1$, *the fibers F_b are of genus 0, $E|_{F_b} \geq 0$ for all $b \in B$, and A has smooth solution operators of arbitrary order at $\sigma(B)$.*

Lastly in this section we shall discuss the construction of direct image sheaves in a very special case that will be needed later on. Suppose that $\pi : M \to B$ is locally holomorphically trivial and the fibers F_b are curves of genus zero. Fix integers $r, d \geq 1$. For each $b \in B$ consider the space of holomorphic germs $(F_b, \sigma(b)) \to \mathbb{C}^r$, and let $I_b = I_b(r, d)$ denote the the quotient of this space by the subspace of those germs that are $o(d)$ at $\sigma(b)$. Thus I_b is a space of holomorphic d-jets, and dim $I_b = r(d+1)$. Furthermore $I = \bigcup_b I_b$ has a natural structure of a locally trivial holomorphic vector bundle on B. Indeed, given any $b_0 \in B$ we can find a neighborhood $B_0 \subset B$ of b_0 and $r(d{+}1)$ holomorphic germs $(\pi^{-1}(B_0), \sigma(B_0)) \to \mathbb{C}^r$ whose d-jets restricted to F_b, $b \in B_0$, are linearly independent. These d-jets then define a local holomorphic trivialization of $I \to B$; and it is straightforward that the local trivializations that can be thus obtained are compatible.

If $v_b : (F_b, \sigma(b)) \to \mathbb{C}^r$ is a holomorphic germ, let $j(v_b) \in I_b$ denote its d-jet. Suppose next that $v : (M, \sigma(B)) \to \mathbb{C}^r$ is a smooth germ, holomorphic on all fibers of π. Then one verifies that $j(v)$ defines a smooth section of I. If in addition ξ is a $(0,1)$ vector field along some fiber F_b such that $\pi_*\xi \in T_b^{0,1}B$ is constant, then by Proposition 4.4 $\bar\partial v(\xi)$ is a holomorphic germ, and it is easy to check that

$$j(\bar\partial v(\xi)) = \bar\partial(j(v))(\eta), \qquad \eta = \pi_*\xi. \tag{5.4}$$

Here on the right hand side $\bar\partial$ stands for the Cauchy–Riemann operator of the bundle $I \to B$.

Next assume that, more generally, $E \to M$ is a holomorphic vector bundle of rank $r < \infty$ that is trivial to arbitrary order at $\sigma(B)$. By this we mean that for any $k = 1, 2, \dots$ $\sigma(B)$ has a neighborhood $U \subset M$ and there is a smooth vector bundle homomorphism $\Phi : E|_U \to U \times \mathbb{C}^r$ such that $\Phi|_{\sigma(M)}$ is an isomorphism and $\bar\partial\Phi = o(k)$ at all points of $\sigma(M)$. If $k > d$ and Φ is fixed, with any holomorphic germ v_b of a section of $E|_{F_b}$ at $\sigma(b)$ we can associate the d-jet of $\Phi \circ v_b$ at $\sigma(b)$. As this will be a holomorphic jet, we obtain a jet in I_b, again denoted $j(v_b)$. As before, if v is the germ of a smooth section of E at $\sigma(B)$ that is holomorphic along the fibers of π, and ξ is a $(0,1)$ vector field along some fiber F_b such that $\pi_*\xi$ is constant, then $v(\xi)$ is holomorphic, and (5.4) holds.

For an arbitrary $b \in B$ let $R_b = R_b(d) \subset I_b(d, r)$ denote the image of $H^0(F_b, E)$ under j, and put $R = \bigcup R_b \subset I$.

Proposition 5.7. *In addition to the assumptions above, let us suppose that all bundles $E|_{F_b}$, $b \in B$, are isomorphic. Then there is a d_0 such that for $d \ge d_0$ j is injective on $H^0(F_b, E)$ and R is a locally trivial holomorphic subbundle of I. Furthermore, with such d, if $E' \subset E$ is a subbundle, trivial to arbitrary order at $\sigma(B)$, such that $E'|_{F_b}$ are all isomorphic, then $R'_b = jH^0(F_b, E')$ form a locally holomorphically trivial subbundle $R' = \bigcup R'_b$ of R.*

Proof. Since dim $H^0(F_b, E) < \infty$, we can find d_0 that makes j injective for $d \ge d_0$. We shall show that this implies the rest of the claim. First we check that R is a smooth subbundle of I; for this we must prove that any $\alpha_0 \in R_{b_0}$ can be extended to a local section α of R that is smooth as a section of $I \supset R$. Now if $\alpha_0 = j(v_0)$, $v_0 \in H^0(F_{b_0}, E)$, then by Proposition 5.1 we can extend v_0 to a section v of E in some neighborhood $\pi^{-1}(B_0)$ of F_{b_0} so that $v|_{F_b}$ is holomorphic, $b \in B_0$. It follows that $\alpha = j(v)$ will do.

Similarly, to show that R is a locally trivial holomorphic subbundle we must prove that any $\alpha_0 \in R_{b_0}$ can be extended to a section of R that is holomorphic as a section of I. First note that there are a neighborhood $B_0 \subset B$ of b_0 and a locally trivial holomorphic subbundle $Q \subset I|_{B_0}$, complementary to R. It follows that I/Q is also locally holomorphically trivial, and by shrinking B_0 we can arrange that it is trivial. Let $\rho : I \to I/Q$ denote the natural projection. Then $\gamma_0 = \rho(\alpha_0) \in (I/Q)_{b_0}$ can be extended to a holomorphic section γ of I/Q, and γ can be lifted to a unique smooth section α of R, $\alpha(b_0) = \alpha_0$. In fact, α is holomorphic. To verify this, for $b \in B_0$ denote by u_b the unique element of $H^0(F_b, E)$ such that $j(u_b) = \alpha(b)$. An argument as in the proof of Theorem 5.4 gives that $u(x) = u_b(x)$, $x \in F_b$, defines a smooth section u of $E|_{B_0}$. Take an arbitrary $(0,1)$ vector field ξ along some F_b such that $\pi_*\xi = \eta \in T_b^{0,1}B$ is constant. By virtue of (5.4)

$$\rho j(\bar\partial u(\xi)) = \rho\bar\partial(j(u))(\xi) = \rho(\bar\partial\alpha)(\xi) = \bar\partial\gamma(\xi) = 0.$$

Thus $j(\bar{\partial}u(\xi)) \in Q_b$. On the other hand $j(\bar{\partial}u(\xi)) \in R_b$ by the definition of R, so that $j(\bar{\partial}u(\xi)) = 0$, hence $\bar{\partial}u(\xi) = 0$. Therefore u is holomorphic and so also is $\alpha = j(u)$.

The same proof with R' replacing R gives that R' is a locally holomorphically trivial holomorphic subbundle.

6. Divisors

Before analyzing projective spaces, divisors will have to be discussed briefly. Let M be an arbitrary complex manifold, and consider holomorphic line bundles $L \to M$, $L' \to M$ with holomorphic sections $\sigma \in H^0(M, L)$, $\sigma' \in H^0(M, L')$ whose zero sets have no interior points. We will say that the pair (L, σ) is equivalent to the pair (L', σ') if there is a holomorphic (in particular C^∞, by our convention) isomorphism $\Phi : L \to L'$ such that $\Phi \circ \sigma = \sigma'$. Observe that if such a Φ exists it is unique, being uniquely determined on the dense set where $\sigma \neq 0$. We define an *effective divisor* on M to be an equivalence class of pairs (L, σ). Effective divisors on M form a commutative semigroup, with zero element, the operation coming from the operation

$$(L, \sigma) \otimes (L', \sigma') = (L \otimes L', \sigma \otimes \sigma').$$

The group ("Grothendieck group") canonically associated with this semigroup is called the group of divisors $\mathcal{D}(M)$. Since for effective divisors the cancellation rule holds, the semigroup $\mathcal{D}_{\text{eff}}(M)$ of effective divisors is embedded in $\mathcal{D}(M)$. In fact, it is easy to show that there are sheaves $\mathcal{D}_{\text{eff}} \subset \mathcal{D}$ of commutative (semi)groups on M whose sections over an open $U \subset M$ are precisely the (effective) divisors on U.

All this immediately leads to the notion of a hypersurface in M: a nowhere dense closed set $S \subset M$ is a hypersurface if for every point $x \in S$ there are a neighborhood $U \subset M$, a line bundle $L \to U$, and $\sigma \in H^0(U, L)$ such that $S \cap U = \{\sigma = 0\}$. For example, if $D \subset \mathcal{D}_{\text{eff}}(M)$ is an effective divisor represented by (L, σ), then the set $|D| = \{x \in M : \sigma(x) = 0\}$ is a hypersurface.

Next we shall briefly address the question of how to associate a divisor to a complex submanifold $N \subset M$ of codimension one. We say that a divisor represented by $L \to M$, $\sigma \in H^0(M, L)$, corresponds to N if N is the zero set of σ, and $d\sigma \neq 0$ on N ($d\sigma$ computed in some smooth local trivialization). We will show elsewhere that if such a divisor exists, it is unique; here we shall discuss a special case only. Clearly, if M is an open set in a locally convex complex vector space V, and $W \subset V$ is a closed subspace of codimension one, then $N = M \cap W$ determines a divisor: take L trivial and σ the restriction to M of a linear form l on W with Ker $l = W$. Moreover, it is easy to see that at least among divisors represented by *locally holomorphically trivial* bundles this is the only one corresponding to N. Indeed, if (L', σ') also represents N with L' locally trivial, then $\sigma' \otimes \sigma^{-1}$ is a nonvanishing holomorphic section of $L' \otimes L^{-1} \simeq \text{Hom}(L, L')$ over $M \setminus N$, which is also locally bounded on M. It follows that $\sigma' \otimes \sigma^{-1}$ and its inverse as well extend holomorphically to the whole of M so that we obtain an isomorphism $\Phi : L \to L'$ such that $\Phi \circ \sigma = \sigma'$.

Hence more generally, any rectifiable codimension one submanifold (cf. section 2) of a complex manifold M gives rise to a divisor. Now it is easy to see that in a rectifiable manifold any complex submanifold of finite codimension is itself rectifiable. In particular, in such a complex manifold M any complex submanifold of codimension one gives rise to a divisor, which we shall denote $[N]$. In the broader context of general complex manifolds M and complex submanifolds N of codimension one it still seems that $[N]$ exists, but, again, a full discussion of this must be left to a subsequent publication.

7. Projective spaces

Out of a locally convex complex vector space V one can construct a complex manifold $P = \mathbb{P}V$, the projectivization of V, as follows. As a topological space, P is the quotient of $V \setminus \{0\}$ by the equivalence relation $x \sim y$ if $x = \lambda y$ with some $\lambda \in \mathbb{C}$. The class of $x \in V$ will be denoted $[x]$. Equivalently, P is the space of one dimensional subspaces (lines) of V. Since V is regular in the sense of topology, one easily concludes that P is Hausdorff. The manifold structure on P will be modelled on hyperplanes in V: if l is a nonzero linear form on V, the mapping $[x] \to x/l(x)$ defines a homeomorphism between $P(l) = \{[x] \in P : l(x) \neq 0\}$ and the affine hyperplane $\{y \in V : l(y) = 1\}$, which then can be composed with a translation to get a homeomorphism $P(l) \to \operatorname{Ker} l$. The transition mappings being biholomorphic, we have thus defined a rectifiable complex manifold structure on P. Such manifolds will be called projective spaces.

So far everything works out as in finite dimensions, except that $\mathbb{P}V$ may fail to be a regular topological space. Indeed, $\mathbb{P}V$ is regular precisely when V admits a continuous norm. For suppose $P = \mathbb{P}V$ is regular. Then a point $y \in P(l)$ can be separated from the hyperplane $H(l) = \{[x] \in P : l(x) = 0\}$ by disjoint open sets $U_1 \ni y$, $U_2 \supset H(l)$, and it can be assumed that $U_1 \subset P(l)$ is balanced convex (when $P(l)$ is identified with $\operatorname{Ker} l$ so that y becomes $0 \in \operatorname{Ker} l$, as explained above). Since U_1 cannot contain a line, its Minkowski function defines a continuous norm on $\operatorname{Ker} l$, whence a continuous norm on V is easily obtained.

Conversely, if V admits a continuous norm, whose unit sphere is denoted S, then a closed $F \subset \mathbb{P}V$ and $y \in \mathbb{P}V \setminus F$ can even be separated by a function as follows. Denote the canonical projection by $\pi : V \setminus \{0\} \to \mathbb{P}V$, and, using an appropriate seminorm, construct a continuous function $f \geq 0$ on S that vanishes on the closed set $S \cap \pi^{-1}(F)$ and is positive at some point of $S \cap \pi^{-1}(y)$. Then $g(x) = \int_0^{2\pi} f(e^{it}x)dt$ is a continuous function on S, constant on fibers of π, vanishes on $S \cap \pi^{-1}(F)$ and is positive on $S \cap \pi^{-1}(y)$. Hence g can be pushed forward by π to produce a continuous function on $\mathbb{P}V$, positive at y, vanishing on $F = \pi(S \cap \pi^{-1}(F))$. — Note that this last equality is true only because S is the unit sphere of a norm rather than seminorm.

Given this, it will come as no surprise that we find it easier to do analysis on projective spaces modelled on locally convex spaces with continuous norms than on general projective spaces. Indeed, only the former set up supplies us with cut-off functions supported in arbitrarily small neighborhoods of points.

Another construction we will be using is blowing up a point of a complex manifold M. As we shall need only a very special case, the notion will be explained only when M is rectifiable, and then it is quite straightforward. Start with an open set $\Omega \subset V$ in a locally convex complex vector space: its blow up at $0 \in \Omega$ is

$$\mathrm{Bl}_0\Omega = \{(x, e) \in \Omega \times \mathbb{P}V : x \in e\} \tag{7.1}$$

(here we think of $\mathbb{P}V$ as lines in V). One verifies that $\mathrm{Bl}_0\Omega$ is a rectifiable complex submanifold of $\Omega \times \mathbb{P}V$.

Using this construction as a local model, the rectifiable complex manifold $\mathrm{Bl}_x M$ can be defined for an arbitrary rectifiable complex manifold M and $x \in M$. The blow up of a projective space P, $\mathrm{Bl}_x P$, can be defined directly, too. First choose a projective hyperplane $P' \subset P$, $x \notin P'$, and note that projective lines $p \subset P$ through x can be identified with points of P': $p \leftrightarrow p \cap P'$. Then

$$\mathrm{Bl}_x P = \{(y,p) \in P \times P' : \text{the line } p \text{ contains } y\}.$$

Projection on P' exhibits $\mathrm{Bl}_x P$ as a holomorphic fiber bundle with fibers projective lines. $\pi : \mathrm{Bl}_x P \to P'$ has a holomorphic section σ that associates with any $p \in P'$ the point

$$\sigma(p) = (x,p) \in \mathrm{Bl}_x P. \tag{7.2}$$

There is also another projection $\rho : \mathrm{Bl}_x P \to P$, $\rho(y,p) = y$, which is one-to-one outside the codimension 1 complex submanifold $\rho^{-1}(x) = \sigma(P')$.

Now we turn our attention to holomorphic line bundles on projective spaces. Fix a hyperplane $P' \subset P$, and construct a holomorphic line bundle H, the hyperplane section bundle, corresponding to the divisor $[P']$, as explained in section 6. As usual, H^n will denote the $|n|$'th tensor power of H or of its dual H^* depending on whether n is positive or not. These line bundles are even topologically nonisomorphic, as their restrictions to projective lines have different degrees. Any line bundle L when restricted to a projective line $p \subset P$ becomes isomorphic to $H^n|_p$, $n = \deg L|_p$, independent of p. We shall prove

Theorem 7.1. *Any holomorphic line bundle $L \to P$ is isomorphic to some H^n.*

In Theorem 7.3 we shall prove that the Dolbeault cohomology group $H^1(P)$ with values in the trivial line bundle vanishes. This implies that holomorphic line bundles over P are classified by their Chern classes $c_1(L) \in \check{H}^2(P;\mathbb{Z})$ in Čech cohomology, and so one might attempt to prove Theorem 7.1 by showing that $\check{H}^2(P;\mathbb{Z}) \approx \mathbb{Z}$ and is generated by $c_1(H)$. However, we have not been able to give a topological proof of this latter fact that would work quite generally (when P is not paracompact, nor even regular): therefore, we prove Theorem 7.1 using complex analysis. In fact we shall derive Theorem 7.1 as a special case of

Proposition 7.2. *If $E \to P$ is a holomorphic vector bundle of rank $r < \infty$ and $E|_p$ is isomorphic to $\bigoplus^r H^n|_p$ for any projective line $p \subset P$, then E is isomorphic to $\bigoplus^r H^n$.*

In the finite dimensional context this was first observed by Van de Ven in [V]. The infinite dimensional situation can be dealt with similarly.

Proof. Upon replacing E by $E \otimes H^{-n}$, we can reduce the theorem to the statement that a holomorphic vector bundle $E \to P$ that is trivial on all lines must be globally trivial. So we shall suppose E is trivial on all lines.

With an arbitrary $x \in P$ and hyperplane $P' \subset P$, $x \notin P'$, consider the blow up $M = \mathrm{Bl}_x P$ as a holomorphic fiber bundle $\pi : M \to P'$ with fibers curves of genus 0. The projection $\rho : M \to P$ induces a holomorphic vector bundle $\widetilde{E} = \rho^* E \to M$, trivial on the fibers F_p of π. Pick r independent vectors $\zeta_0^j \in E_x$, and pull them back by ρ to produce holomorphic sections $\zeta^j \in H^0(\sigma(P'), \widetilde{E})$, with σ given in (7.2). For any $p \in P'$ denote by $s_p^j \in H^0(F_p, \widetilde{E})$ the unique holomorphic section such that $s_p^j(\sigma(p)) = \zeta^j(\sigma(p))$. For fixed j the sections s_p^j taken all together define a section s^j of $\widetilde{E}$, and the s^j will be everywhere independent. Our first business will be to show $s^j \in C^\infty(M, \widetilde{E})$.

Suppose we want to verify s^j is smooth in the neighborhood of a fiber F_{p_0}. We will apply Proposition 5.1 with $f \equiv 0 \in C^\infty_{0,1}(M, \widetilde{E})$, and get a neighborhood $B_0 \subset P'$ of p_0 plus $v^j \in C^\infty(\pi^{-1}(B_0), \widetilde{E})$ with $\bar\partial v^j|_{F_p} = 0$, $p \in B_0$, and $v^j|_{F_{p_0}} = s^j_{p_0}$. On $\pi^{-1}(B_0)$ we can write $v^j = \sum_k \lambda_{jk} s^k|_{\pi^{-1}(B_0)}$, with fiberwise holomorphic, hence fiberwise constant, functions λ_{jk}, $\lambda_{jk} = \delta_{jk}$ on F_{p_0}. In fact, looking at the restrictions to $\sigma(P')$ we find $\lambda_{jk} \in C^\infty(\pi^{-1}(B_0))$, and so s^j are smooth near F_{p_0}.

Once we know $s^j \in C^\infty(M, \widetilde{E})$ we can apply Proposition 5.2 (ii) to conclude $\bar\partial s^j = g = 0$. (For this we have to note that $\bar\partial s^j$ vanishes in points of $\sigma(P')$, both on vertical vectors and on vectors tangential to $\sigma(P')$.) Hence the s^j holomorphically trivialize $\widetilde{E}$, and upon pushing them down to $P \setminus \{x\}$ we find everywhere independent $t^j \in H^0(P \setminus \{x\}, E)$. In particular, E is locally trivial on $P \setminus \{x\}$. By varying x we find E is locally trivial on P. Now we invoke Hartogs' theorem (proved using one dimensional Cauchy integrals) to conclude that the sections t^j above extend holomorphically to x. Since $t^j(x) = \zeta^j_0$ are also independent, E is indeed trivial.

In the case of a line bundle L we shall accordingly say $L > 0$ or $L \geq 0$, etc., if L is isomorphic to H^n with $n > 0$, $n \geq 0$, etc. We also put $n = \deg L$.

To formulate our vanishing theorem on projective spaces we need to introduce the following concept. A differentiable manifold M will be called localizing (or we say: M localizes) if for any nonempty open set $U \subset M$ there is a not identically zero function in $C^\infty(M)$ supported in U. This property obviously implies that M is a (completely) regular topological space; on the other hand, if a manifold M modelled on a locally convex space V is known to be regular, then M is localizing precisely when V is. Examples of localizing spaces are Hilbert spaces and some other Banach spaces as well as nuclear spaces such as $C^\infty(X)$ with X a finite dimensional manifold; cf. [BF], [DGZ]. I am indebted to A. Pełczyński for the latter reference.

The localizing property of a projective space has to do with hermitian metrics. If $E \to M$ is a complex vector bundle, a smooth hermitian metric on E is a smooth function $h : E \to \mathbb{R}$ whose restrictions to fibers E_x, $x \in M$, are positive definite hermitian quadratic forms. Now if $P = \mathbb{P}V$ localizes, then the hyperplane section bundle $H \to P$ — and so all line bundles on P — carry smooth hermitian metrics. This can be seen as follows. If P localizes, then it is regular, so that V admits a continuous norm, and it also localizes. Choose $\chi \in C^\infty(V)$ supported in the unit ball of a norm, such that $\chi(0) \neq 0$. Define

$$g(x) = i \int_{\mathbb{C}} |\chi(\lambda x)|^2 d\lambda \wedge d\bar\lambda, \qquad x \in V \setminus \{0\}.$$

This is a smooth positive function and

$$g(\mu x) = |\mu|^{-2} g(x), \qquad \mu \in \mathbb{C}.$$

Now look at the blow up $\mathrm{Bl}_0 V$. Using notation (7.1), the projection $(x, e) \mapsto e$ exhibits $\mathrm{Bl}_0 V$ as a line bundle over $\mathbb{P}V$; one checks this line bundle is isomorphic to H^*. As g^{-1} pulls back to $\mathrm{Bl}_0 V \approx H^*$ as a smooth hermitian metric, it follows that H also admits a smooth hermitian metric.

Theorem 7.3. *Let $L \to P$ be a holomorphic line bundle over a projective space P, $1 \le q < \dim P$. Then $H^q(P,L) = 0$, provided*

(i) *P localizes; or*
(ii) *$L \le 0$ and $q = 1$.*

Proof. As before, with an arbitrary $x \in P$ and hyperplane $P' \subset P$, $x \notin P'$, consider the blow up $M = \mathrm{Bl}_x P$ as a holomorphic fiber bundle $\pi : M \to P'$. The projection $\rho : M \to P$ induces a holomorphic line bundle $\widetilde{L} = \rho^* L \to M$. We must show that any closed $f \in C^\infty_{0,q}(P,L)$ is also exact. This we will do by pulling back f to $\widetilde{f} = \rho^* f \in C^\infty_{0,q}(M,\widetilde{L})$, and studying the $\bar\partial$ equation on M.

Case (i). We claim that with $\sigma : P' \to M$ the section given in (7.2), $\widetilde{f}$ is exact to arbitrary order at $\sigma(P')$. For choose a neighborhood $\Omega \subset P$ of x that is biholomorphic to an open set in a locally convex space. By Theorem 3.6 for any $k = 1, 2, \dots$ there is $v_k \in C^\infty_{0,q-1}(\Omega, L)$ such that $\bar\partial v_k - f = o(k)$ at x. With a cut-off function $\chi \in C^\infty(P)$ supported in Ω and equal to 1 in a neighborhood of x, we get $\bar\partial(\chi v_k) - f = o(k)$; since χv_k can be extended smoothly to the whole of P, the pullbacks $\rho^*(\chi v_k)$ demonstrate that $\widetilde{f}$ is indeed exact to arbitrary order.

If $L \ge 0$, then Proposition 5.5(iii) implies there is a $\widetilde{v} \in C^\infty_{0,q-1}(M,\widetilde{L})$ with $\bar\partial\widetilde{v} = \widetilde{f}$, and $(\rho^{-1})^*\widetilde{v}$ defines a form $v \in C^\infty_{0,q-1}(P \setminus \{x\}, L)$ such that

$$\bar\partial v = f|_{P\setminus\{x\}}. \tag{7.3}$$

Such a v can be obtained for negative L as well, at the price of a little complication. As explained in section 6, $\sigma(P')$ determines a divisor on M: let the pair (Λ, s) represent this divisor. Now Λ, restricted to the fibers F_p of $\pi : M \to P'$, is of degree 1, so that with $d = -\deg L > 0$, $\Lambda^d \otimes L$ becomes trivial on all F_p. Look at the closed form $f' = s^d \otimes \widetilde{f} \in C^\infty_{0,q}(M, \Lambda^d \otimes L)$, and notice that $f'|_{\sigma(P')} = 0$. As $H^1(F_p, L \otimes \Lambda^d) = 0$, Proposition 5.5(ii) implies there is a $v' \in C^\infty_{0,q-1}(M, \Lambda^d \otimes L)$ with $\bar\partial v' = f'$. Hence $s^{-d} \otimes v' = \widetilde{v} \in C^\infty_{0,q-1}(M \setminus \sigma(P'); L)$ satisfies $\bar\partial\widetilde{v} = \widetilde{f}$. Pulling back by ρ^{-1} again gives v as in (7.3).

The domain $P \setminus P'$ can be identified with the locally convex space on which P is modelled. As this space has a continuous norm, x will have a neighborhood $\Omega \subset P \setminus P'$ that is narrow. With a cut-off function $\chi \in C^\infty(P)$ as above, supported in Ω, observe that the closed form $g = f - \bar\partial((1-\chi)v) \in C^\infty_{0,q-1}(P \setminus P', L)$ has narrow support. Also observe that $L \approx H^{\deg L}$ is trivial on $P \setminus P'$. Hence by Theorem 4.2 there is a $w \in C^\infty_{0,q-1}(P \setminus P', L)$ with narrow support in $P \setminus P'$, such that $\bar\partial w = g|_{P\setminus P'}$. The support condition implies that w in fact smoothly extends to all of P, whence $f = \bar\partial((1-\chi)v + w)$, as required.

Case (ii). The crucial observation is that the restriction of f to any projective plane $P_2 \subset P$ is exact. This follows from case (i) applied to P_2, but of course has long been known: it is an instance of the Kodaira vanishing theorem. In particular $\widetilde{f}$ restricted to any fiber F_p is exact. Also, $\rho^*\widetilde{f}|_{\sigma(P')} = 0$ since $\rho(\sigma(P')) = \{x\}$. Hence we can apply Proposition 5.3 when $L < 0$ and Theorem 5.4 when L is trivial to conclude there is a $\widetilde{v} \in C^\infty(M,\widetilde{L})$ such that $\bar\partial\widetilde{v} = \widetilde{f}$. Pushing this section down to P shows that *for any $x \in P$ there is $v_x \in C^\infty(P \setminus \{x\}, L)$ such that $\bar\partial v_x = f|_{P\setminus\{x\}}$.*

It is again easy to remove the singularity. Indeed, with $y \ne x$ construct a corresponding v_y. Since $h = v_x - v_y \in H^0(P \setminus \{x,y\}, L)$, and L is trivial in a neighborhood of x, Hartogs' theorem applies and gives that h holomorphically extends to x. Hence v_x smoothly extends to x, and this concludes the proof.

To study the Dolbeault groups $H^{p,q}$ we shall need an extension of Theorem 7.3, which follows the same way as above, taking into account the existence of smooth solution operators at each step:

Theorem 7.4. *With P, L, q as in Theorem 7.3 there is a smooth linear operator R from the space of closed forms $f \in C^\infty_{0,q}(P, L)$ to $C^\infty_{0,q-1}(P, L)$ that gives a solution $u = Rf$ of the equation $\overline{\partial} u = f$.*

8. More on projective spaces

The theorems of section 7 have consequences, some of them quite straightforward, that we shall discuss below.

Theorem 8.1. *A complex submanifold of a projective space $\mathbb{P}V$ of codimension one is an algebraic submanifold, i.e. it can be defined as the zero set of a homogeneous polynomial on V with simple zeros.*

Proof. If $M \subset \mathbb{P}V$ is such a submanifold, it determines a divisor and so a line bundle $L \to \mathbb{P}V$ and $\sigma \in H^0(\mathbb{P}V, L)$ with first order zeros along M. By Theorem 7.1 we can assume that L is some power of the hyperplane section bundle, $L = H^n$. On the other hand any $\sigma \in H^0(\mathbb{P}V, H^n)$ gives rise to a holomorphic function $\widetilde{Q}$ on the total space of the dual hyperplane bundle H^*, homogeneous of degree n on the fibers. Now the bundle $H^* \to \mathbb{P}V$ is isomorphic to the bundle $\mathrm{Bl}_0 V \to \mathbb{P}V$ (cf. (7.1)), given by $(x, e) \mapsto e$. Hence the holomorphic function $\widetilde{Q}$ descends to $V \setminus \{0\}$, and in fact to a $Q \in H^0(V)$. As Q is homogeneous of degree n on lines through 0, it must be a homogeneous polynomial; clearly $M = \{Q = 0\}$.

Similarly, the notion of algebraic divisors can be introduced, and then one can prove that on a projective space all divisors are algebraic.

Next we turn to sheaf cohomology of projective spaces P. We are only able to deal with P as an algebraic variety. Thus, we introduce the Zariski topology on P, a basis of open sets for which consists of sets of the form $U_D = P \setminus |D|$, $D \in \mathcal{D}_{\text{eff}}(P)$. Here, as in section 6, if an effective divisor D is represented by (L, σ), $|D|$ denotes the set $\{\sigma = 0\}$. For such D, call a holomorphic function h on U_D regular if $\sigma^n \otimes h \in H^0(U_D, L^n)$ extends holomorphically to P for some $n > 0$. Denoting the ring of regular functions on U_D by $\mathcal{O}(U_D)$, one verifies that there is indeed a sheaf $\mathcal{O}$ of rings on P whose sections over U_D constitute $\mathcal{O}(U_D)$.

Similarly, with a holomorphic line bundle $\Lambda \to P$ one associates an invertible sheaf $\mathcal{L}$ of modules over $\mathcal{O}$. The sections of $\mathcal{L}$ over U_D are those $s \in H^0(U_D, \Lambda)$ for which $\sigma^n \otimes s$ extends to P for some $n > 0$. We shall denote by $H^q(P, \mathcal{L})$ the sheaf cohomology groups, defined by flabby resolutions, and by $\check{H}^q(P, \mathcal{L})$ the Čech cohomology groups.

We shall say that a manifold M admits smooth partitions of unity if for any open cover $\mathfrak{U} = \{U\}$ of M there are $\chi_U \in C^\infty(M)$, supported in U, such that $\sum_U \chi_U = 1$, the sum being locally finite. Hilbert and separable nuclear spaces are examples of such manifolds, and paracompact manifolds modelled on spaces that admit smooth partitions of unity are further examples. Consult [AMR], [BF], [DGZ], [T] and [Ma, Appendix 3] on this matter. Let us emphasize that for simplicity in this definition one always uses the locally convex topology on M, even if M might be endowed with other topologies such as the Zariski topology when M is a projective space.

Theorem 8.2. *If a projective space P admits smooth partitions of unity and $\Lambda \to P$ is a holomorphic line bundle with associated sheaf $\mathcal{L}$, then $H^q(P,\mathcal{L}) \approx \check{H}^q(P,\mathcal{L}) \approx H^q(P,\Lambda)$, $q = 0, 1, \dots$. In particular, $H^q(P,\mathcal{L}) = \check{H}^q(P,\mathcal{L}) = 0$ if $1 \le q < \dim P$.*

To prove the theorem we will construct a resolution $0 \to \mathcal{L} \to \mathcal{E}_0 \to \mathcal{E}_1 \to \dots$. This will work for all localizing projective spaces.

If $D \in \mathcal{D}_{\text{eff}}(P)$ is represented by (L,σ), define $\mathcal{E}_q(U_D)$ as the space of those $f \in C^\infty_{0,q}(U_D,\Lambda)$ for which $\sigma^n \otimes f$ extends to a form in $C^\infty_{0,q}(P, L^n \otimes \Lambda)$ for some n. Let $\mathcal{E}_q$ stand for the corresponding sheaf. The $\bar\partial$ operator on L induces homomorphisms $\bar\partial_q : \mathcal{E}_q \to \mathcal{E}_{q+1}$, and there is also an inclusion $i : \mathcal{L} \to \mathcal{E}_0$. Theorem 7.3 implies

Lemma 8.3. *The complex*

$$0 \longrightarrow \mathcal{L} \xrightarrow{\ i\ } \mathcal{E}_0 \xrightarrow{\bar\partial_0} \mathcal{E}_1 \xrightarrow{\bar\partial_1} \dots$$

is a resolution, if P is localizing.

(Rigorously speaking Theorem 7.3 implies this only when $\dim P = \infty$; the finite dimensional case has of course been well known.)

When P admits smooth partitions of unity, the sheaves $\mathcal{E}_q$ are fine, and the sheaf cohomology part of Theorem 8.2 follows from Lemma 8.3. More generally, Theorem 7.3 implies that any covering of P by sets of the form U_D is a Leray covering, so that Čech and sheaf cohomology groups are isomorphic.

In fact all vanishing theorems above can be extended to finite rank vector bundles. For example we have

Theorem 8.4. *If $E \to P$ is a holomorphic vector bundle of finite rank over a localizing infinite dimensional projective space, then $H^q(P,E) = 0$, $q \ge 1$.*

This follows from Theorem 7.3 and

Theorem 8.5. *Any holomorphic vector bundle $E \to P$ of finite rank over a localizing infinite dimensional projective space is isomorphic to the sum of line bundles.*

Theorem 8.5 will be a consequence of Theorem 7.3 and a theorem of Barth–Van de Ven, Sato, and Tjurin on the splitting of infinitely extendible vector bundles on finite dimensional projective spaces; see [BV], [Sa], [Tj]. In their splitting theorem a finite dimensional holomorphic vector bundle $F \to \mathbb{P}_n$ is considered, and it is assumed that for an arbitrary $m > n$ there is a holomorphic vector bundle $F_m \to \mathbb{P}_m$ that restricts to F on a linearly embedded $\mathbb{P}_n \subset \mathbb{P}_m$. The theorem then claims that such a vector bundle splits into the sum of line bundles. We note that Tjurin in [Tj] proves a splitting theorem in an "infinite dimensional" setting; however, his "infinite" projective algebraic manifolds, given by certain sequences of finite dimensional manifolds, are rather different from our infinite dimensional manifolds.

Proof of Theorem 8.5. We shall prove the theorem in two stages: in one stage with the assumption that E has a smooth hermitian metric, in the other without this hypothesis. Both proofs run parallel up to a point, and for this reason we shall discuss the first part of the proofs in one.

Because of the splitting theorem discussed above, the restriction of E to any finite dimensional projective subspace $\Pi \subset P$ splits into the sum of μ_1 copies of

$H^{n_1}|_\Pi$, μ_2 copies of $H^{n_2}|_\Pi$, etc., $n_1 > n_2 > \dots$. If another finite dimensional $\Pi' \subset P$ is chosen, the corresponding numbers n_i', μ_i' must be the same if $\Pi' \supset \Pi$; hence it follows that n_i, μ_i are altogether independent of Π.

We will prove by induction on $r = \text{rk } E = \sum \mu_i$ that E is isomorphic to $\bigoplus_i \bigoplus^{\mu_i} H^{n_i}$. When $r = 1$, or more generally when $r = \mu_1$, this follows from Theorem 7.1, resp. Proposition 7.2. Assume the claim is true for ranks less than r, and also that $\mu_1 < r$. By tensoring E with a line bundle we can assume $n_1 = 0$, and so $\dim H^0(\Pi, E) = \mu_1$ for any finite dimensional subspace $\Pi \subset P$. For an arbitrary finite dimensional subspace $\Pi \subset P$ through $y \in P$ there is a μ_1 dimensional subspace $W_{\Pi,y} \subset E_y$ that is spanned by global sections of $E|_\Pi$. Again $\Pi' \supset \Pi$ implies $W_{\Pi',y} = W_{\Pi,y}$, so $W_{\Pi,y} = W_y$ is independent of Π. We will show that $\Theta = \bigcup_{y\in P} W_y$ is a holomorphically trivial subbundle of E.

To this end, as before, blow up P at some $x \in P$ to obtain a holomorphic fiber bundle $\pi : M = \text{Bl}_x P \to P'$, and from E induce a vector bundle $\widetilde{E} \to M$ by the projection $\rho : M \to P$. Choose linearly independent vectors $\zeta_0^j \in W_x$, $1 \le j \le \mu_1$, pull them back by ρ to sections $\zeta^j \in H^0(\sigma(P'), \widetilde{E})$, σ given in (7.2). As in the proof of Proposition 7.2 we can extend ζ^j to everywhere independent sections $s^j \in H^0(M, \widetilde{E})$, which can be pushed down to everywhere independent sections $t^j \in H^0(P \setminus \{x\}, E)$; the t^j are also sections of $\Theta|_{P\setminus\{x\}}$. By varying x we first find that Θ is a locally trivial holomorphic subbundle of E, second, by Hartogs' theorem that the t^j extend to independent holomorphic sections of Θ, so that Θ is indeed globally trivial.

By the inductive hypothesis E/Θ is isomorphic to $E^0 = \bigoplus_{i\ge 2} \bigoplus^{\mu_i} H^{n_i}$. (Note also that if E admits a smooth hermitian metric, so does E/Θ.) We claim that any isomorphism $\phi : E^0 \to E/\Theta$ can be lifted to a holomorphic homomorphism $\psi : E^0 \to E$, whence it will indeed follow that $E = \Theta \oplus \psi(E^0)$ is the sum of line bundles.

At this point we first treat the hermitian case. Then Θ has a smooth complementary subbundle in E, and so ϕ can be lifted to a *smooth* homomorphism $\Psi : E^0 \to E$. We can find a smooth homomorphism $U : E^0 \to \Theta$ such that $\psi = \Psi + U$ is holomorphic by solving a $\bar\partial$ equation with values in the split bundle $\text{Hom}(E^0, \Theta)$, so that the existence of ψ follows from Theorem 7.3. Thus the splitting theorem is proved for all hermitian $E \to P$.

Without a smooth hermitian metric on E we shall proceed as follows. Returning to the fiber bundle $\pi : M = \text{Bl}_x P \to P'$, with an integer $d \ge 1$ perform the construction of the "jet" bundle $I = I(r(r-\mu_1), d) \to P'$ as in the latter part of section 5. If H' denotes the hyperplane section bundle on P', one checks that

$$I \approx \bigoplus^{r(r-\mu_1)} \bigoplus_{n=0}^{d} H'^n;$$

in particular, I carries a hermitian metric (cf. section 7).

Pull back the bundles $E^0 \to P$, $\Theta \to P$ to bundles $\widetilde{E}^0 \to M$, $\widetilde{\Theta} \to M$, and note that the bundle $\text{Hom}(\widetilde{E}^0, \widetilde{E})$ is trivial to arbitrary order at $\sigma(P')$, since by Theorem 3.5 $\text{Hom}(E^0, E)$ is trivial to arbitrary order at x. With the mapping $j : H^0(F_b, \text{Hom}(\widetilde{E}^0, \widetilde{E})) \to I_b$ as in section 5, choose d so that Proposition 5.7 applies; thus j is injective and $jH^0(F_b, \text{Hom}(\widetilde{E}^0, \widetilde{E})) = R_b$ form a holomorphic subbundle $R = \bigcup R_b$ of I. Further, with $R'_b = jH^0(F_b, \mathrm{Hom}(\widetilde{E}^0, \widetilde{\Theta}))$, $R' = \bigcup R'_b$ is a holomorphic subbundle of $R \subset I$. Now R' has a complementary holomorphic subbundle Q. Such a bundle is constructed by lifting the identinty map $R/R' \to R/R'$ to a homomorphism $R/R' \to R$, which can be done as a similar lifting in the first stage of this proof. Indeed, all that is needed is that $R \subset I$ is hermitian and $H^1(P', \mathrm{Hom}(R/R', R')) = 0$, this latter because $\mathrm{Hom}(R/R', R')$ is also hermitian, hence splits into line bundles.

The splitting $R = Q \oplus R'$ will be used to construct the required lifting $\psi : E^0 \to E$ of $\phi : E^0 \to E/\Theta$. Let $\widetilde{\phi} : \widetilde{E}^0 \to \widetilde{E}/\widetilde{\Theta}$ denote the isomorphism induced by ϕ. For each b there is a $v_b \in H^0(F_b, \mathrm{Hom}(\widetilde{E}^0, \widetilde{E}))$ that covers $\widetilde{\phi}|_{F_b}$; moreover, this v_b will be unique if we require in addition that $j(v_b) \in Q_b$. As in the proof of Theorem 5.4 we find that the v_b patch together to a smooth section v of $\mathrm{Hom}(\widetilde{E}^0, \widetilde{E})$. Let ξ be a section of $T^{0,1}M$ along some F_b such that $\pi_* \xi = \eta$ is constant. Then by virtue of (5.4)

$$j(\bar\partial v(\xi)) = \bar\partial(j(v))(\eta) \in Q_b.$$

On the other hand also

$$j(\bar\partial v(\xi)) \in R'_b = jH^0(F_b, \mathrm{Hom}(\widetilde{E}^0, \widetilde{\Theta})),$$

since v is the lift of $\widetilde{\phi}$, which is holomorphic. It follows that $j(\bar\partial v(\xi)) = 0$ and even $\bar\partial v(\xi) = 0$, as j is injective. Therefore v is holomorphic, and descends to a lift $\psi \in H^0(P \setminus \{x\}, \mathrm{Hom}(E^0, E))$ of ϕ. The existence of this lift implies E is locally trivial on $P \setminus \{x\}$; varying x then shows E is locally trivial on P. Hence Hartogs' theorem applies and gives that ψ extends to x. As we have seen, the construction of this lift ψ implies the induction step, and so the proof is complete.

Last we turn to the Dolbeault groups $H^{p,q}(P, E)$ when $p > 0$. Denote the trivial line bundle over P by $\mathbf{1}$, the trivial vector bundle over P with fiber V by $\underline{V}$, and observe that $T^{1,0}P$ carries the structure of a holomorphic vector bundle[3] and can be included in an exact sequence

$$0 \longrightarrow \mathbf{1} \xrightarrow{\alpha} H \otimes \underline{V} \xrightarrow{\beta} T^{1,0}P \longrightarrow 0. \tag{8.1}$$

Here α is the composition of $\mathbf{1} \approx H \otimes H^*$ and the inclusion $H \otimes H^* \hookrightarrow H \otimes \underline{V}$ (note that H^* is the tautological line bundle, so $H^* \subset \underline{V}$). To define β take $e \in P$, which we think of as a line $e \subset V$. An arbitrary element $m \in (H \otimes \underline{V})_e \approx \mathrm{Hom}(H^*_e, \underline{V}_e) = \mathrm{Hom}(e, V)$ gives rise to a linear mapping $l : e \to V$, and induces a one-parameter family $l_t = \mathrm{id}_e + tl$ $(t \in \mathbb{C})$ of linear mappings $e \to V$. The ranges of these l_t define a holomorphic mapping $\lambda : t \mapsto l_t(e)$ of some neighborhood of $t = 0 \in \mathbb{C}$ into P, and we put $\beta(m) = \lambda_*(0)\partial/\partial t \in T^{1,0}_e P$. It is then easy to verify that (8.1) is exact (as in the finite dimensional case, cf. [Da], where the dual of (8.1) is used to compute $H^{p,q}(P)$, $\dim P < \infty$).

We shall be able to deal with the groups $H^{p,q}(P, E)$ when (8.1) smoothly splits, i.e., there is a smooth homomorphism $\omega : H \otimes \underline{V} \to \mathbf{1}$ such that $\omega\alpha = \mathrm{id}_{\mathbf{1}}$. This will be so if V admits a positive definite hermitian form, or if P admits smooth partitions of unity. In the former case an ω can be defined by specifying that its kernel at $e \in P$ is the orthogonal complement of $e \subset V$; in the latter case local splittings can be fused into a global one.

[3]For an arbitrary complex manifold M, $T^{1,0}M \to M$ has a natural structure of a holomorphic vector bundle. When M is rectifiable, this structure is defined as in finite dimensions.

For any holomorphic vector bundle $E \to P$ and $p = 0, 1, \dots$ we define two cochain complexes $\mathfrak{C}^p(E) = \{\mathfrak{C}^{p,q}(E), \overline{\partial}\}_{q \geq 0}$ and $\mathfrak{A}^p(E) = \{\mathfrak{A}^{p,q}(E), \overline{\partial}\}_{q \geq 0}$. $\mathfrak{C}^{p,q}(E)$ is simply $C^\infty_{p,q}(P, E)$ with $\overline{\partial}$ the Cauchy-Riemann operator introduced in section 2. On the other hand an element g of $\mathfrak{A}^{p,q}(E)$ is a smooth, multilinear, alternating family $\{g_v\}$ of forms $g_v \in C^\infty_{0,q}(P, E)$, parametrized by $v \in \bigoplus^p V$, and $\overline{\partial}$ acts on each component: $\overline{\partial}\{g_v\} = \{\overline{\partial} g_v\}$.

For clarity we shall denote $H^k \otimes E$ by $E(k)$, k integer.

Proposition 8.6. *Assuming (8.1) smoothly splits, for $p = 1, 2, \dots$ there is an exact sequence of cochain complexes*

$$0 \longrightarrow \mathfrak{C}^p(E) \xrightarrow{\varphi} \mathfrak{A}^p(E(-p)) \xrightarrow{\psi} \mathfrak{C}^{p-1}(E) \longrightarrow 0. \tag{8.2}$$

Proof. To define φ, suppose $f \in C^\infty_{p,q}(P, E)$. If $v = (v_1, \dots, v_p) \in \bigoplus^p V$ and $\xi \in T^{0,q}_x P$, with an arbitrary nonzero $\delta \in H_x$ put

$$g_v(\xi) = \delta^{-p} \otimes f(\beta(x, \delta \otimes v_1), \dots, \beta(x, \delta \otimes v_p), \xi) \in (E(-p))_x.$$

This is independent of the choice of δ, and determines $\varphi(f) = \{g_v\} \in \mathfrak{A}^{p,q}(E(-p))$. Next, suppose $g = \{g_v\} \in \mathfrak{A}^{p,q}(E(-p))$. We define a form $\psi(g) = h \in C^\infty_{p-1,q}(E)$ as follows. Suppose $\xi \in T^{0,q}_x P$, $\eta = (\eta_1, \dots, \eta_{p-1}) \in T^{p-1,0}_x P$. Find $v_j \in V$ such that $\beta(x, \delta \otimes v_j) = \eta_j \quad (j = 1, \dots, p-1)$, where $0 \neq \delta \in H_x$ is again arbitrary, and let

$$h(\eta_1, \dots, \eta_{p-1}, \xi) = \delta^p \otimes g_{(v_1, \dots, v_{p-1}, \delta^{-1} \otimes \alpha(x,1))}(\xi) \in E_x. \tag{8.3}$$

Although even for fixed δ the choice of v_j is not unique, the right hand side of (8.3) is independent of such a choice: this follows from the alternating property of $\{g_v\}$ combined with the exactness of (8.1). Also, the right hand side of (8.3) is independent of δ, and indeed defines $h \in C^\infty_{p-1,q}(E)$.

We shall mostly leave it to the reader to verify that with φ, ψ thus constructed (8.2) is indeed an exact sequence of cochain complexes, restricting ourselves to showing exactness at $\mathfrak{C}^{p-1}(E)$ (the only place where the splitting of (8.1) is used). Thus, let $h \in C^\infty_{p-1,q}(P, E)$. Think of the values of the splitting homomorphism ω as complex numbers, and with $v_j \in V$, $\xi \in T^{0,q}_x P$, $0 \neq \delta \in H_x$ define

$$g'_{v_1, \dots, v_p}(\xi) = \omega(x, \delta \otimes v_p) \delta^{-p} \otimes h(\beta(x, \delta \otimes v_1), \dots, \beta(x, \delta \otimes v_{p-1}), \xi).$$

This again is independent of the choice of δ, and defines a smooth, multilinear family $\{g'_v\}$ of $E(-p)$-valued $(0,q)$-forms, which is alternating in $v_1, \dots, v_{p-1}$. Antisymmetrization then yields a $g = \{g_v\} \in \mathfrak{A}^p(E(-p))$ such that $\psi(g) = h$.

Since the cohomology groups of the complex $\mathfrak{C}^p(E)$ are the Dolbeault groups $H^{p,q}(P, E)$, the cohomological sequence associated with (8.2) gives

Theorem 8.7. *Assuming (8.1) smoothly splits, there is an exact sequence*

$$\begin{aligned} \cdots \longrightarrow H^{q-1}(\mathfrak{A}^p(E(-p))) \longrightarrow H^{p-1,q-1}(P, E) \\ \longrightarrow H^{p,q}(P, E) \longrightarrow H^q(\mathfrak{A}^p(E(-p))) \longrightarrow \dots \ . \end{aligned}$$

When P is infinite dimensional and localizes, and rk $E < \infty$, Theorems 7.4 and 8.5 imply $H^q(\mathfrak{A}^p(E(-p))) = 0$, $q \geq 1$, so that Theorem 8.7 lets us compute the groups $H^{p,q}(P, E)$ inductively. For example we have the following corollary.

Corollary 8.8. *If* $\dim P = \infty$, *P admits smooth partitions of unity, and* $L \to P$ *is a holomorphic line bundle, then*

(a) $H^{p,q}(P, L) \approx H^{p-1,q-1}(P, L)$, $q > 1$;

(b) *assuming also* $\deg L < p$, $H^{p,1}(P, L) \approx H^{p-1,0}(P, L)$ *and* $H^{p,0}(P, L) = 0$.

In particular, if $\deg L < 0$, *then all groups* $H^{p,q}(P, L) = 0$; *when L is trivial we obtain* $H^{p,q}(P) = 0$ *for* $p \neq q$ *and* $H^{p,p}(P) \approx \mathbb{C}$.

9. The $\bar\partial$ equation with polynomial growth

In this last section we shall prove a result on the $\bar\partial$ equation in a locally convex space, whose nature is half way between algebraic and analytical. It sheds some light on Coeuré's example [C], [Ma] and other peculiarities that occur in infinite dimensions. As our purpose here is illustrative only, we shall discuss only scalar-valued $(0, 1)$-forms, although slightly weaker results can be obtained in the same spirit for vector-valued (p, q)-forms.

Let V be a locally convex complex vector space and $f = f(x; \xi)$ a q-form on it, $x \in V$, $\xi \in \bigoplus^{q} V$. (Thus we identify TV with $V \times V$.) If d is any positive number, we shall say that f is of order d (along lines) if the following is true: given $x_1, x_2 \in V$, $\xi \in \bigoplus^{q} V$, there are neighborhoods $U_i \subset V$, $U \subset \bigoplus^{q} V$ of x_i, ξ, and a number A such that whenever $x_i' \in U_i$, $\xi' \in U$, and $\lambda \in \mathbb{C}$ we have

$$|f(x_1' + \lambda x_2'; \xi')| \leq A(1 + |\lambda|)^d.$$

Theorem 9.1. *Suppose* $f \in C^1_{0,1}(V)$ *is closed and of order* $k - \epsilon$, $k = 1, 2, \dots$ *and* $\epsilon > 0$. *If* $f|_\Omega \in C^k_{0,1}(\Omega)$ *for some open set* $\Omega \neq \emptyset$, *then the equation* $\bar\partial u = f$ *has a solution* $u \in C^1(V)$.

We find this theorem most perplexing. Indeed, we know of no other instance in analysis where high differentiability near one point has global solvability as a consequence. Even more surprisingly, this theorem is sharp for all k. When $k = 1$, Coeuré's counterexample in [C], [Ma] can be arranged so that $f \in C^1_{0,1}(l_2)$ is of order 1. For this all one has to do is define his form $f = \sum_{n=1}^{\infty} \beta(x_n) d\bar x_n$, with *compactly supported* $\beta \in C^1(\mathbb{C})$ exhibiting a certain irregular behavior at 0, as in the original construction. Similar examples with higher regularity in other l^p spaces can also be given to show that Theorem 9.1 would be false if $k - \epsilon$ were replaced by k. We shall discuss this issue in a sequel to this paper.

Now Coeuré's f is such that $\bar\partial u = f$ is not even locally solvable. This is no accident. It turns out that global nonsolvability quite often implies local nonsolvability.

Theorem 9.2. *Suppose V has a continuous norm and localizes. If for a closed* $f \in C^1_{0,1}(V)$ *of some finite order the equation* $\bar\partial u = f$ *is solvable on some nonempty open set, then it is solvable on V.*

Proof. If there is a local solution, then using a suitable cut-off function we can produce a $v \in C^1(V)$ with narrow support such that $\bar\partial v = f$ on some open $\Omega \neq \emptyset$. Then $f' = f - \bar\partial v$ is still of finite order, and C^∞ on Ω. Hence by Theorem 9.1 $f' = \bar\partial w$ with $w \in C^1(V)$, and so indeed $f = \bar\partial(v + w)$.

We remark that the same would hold without any condition on V, but to prove this a slightly stronger formulation of Theorem 9.1 would be required.

Theorem 9.1 should be contrasted with Henrich's theorem [He], the first about solving $\bar\partial$ in a Hilbert space. He too considers $(0,1)$-forms of polynomial growth, of class C^1, and solves the equation with u defined only on certain dense subspaces.

The basic idea in proving our theorem will be the same as in section 7: integrating the equation $\bar\partial u = f$ along lines, and making sure that the slicewise solutions thus obtained define a solution on the entire space. To implement this approach we need some preparation.

Proposition 9.3. *Suppose $\Omega \subset V$ is an open set in a locally convex complex vector space, $l = 1, 2, \dots,$ $v \in C(\Omega)$, $g \in C^l_{0,1}(\Omega)$. If for any affine line $e \subset V$ we have $\bar\partial v|_e = g|_e$ in the weak sense, then $v \in C^l(\Omega)$ and $\bar\partial v = g$ on Ω.*

Proof. To say that $\bar\partial v|_e = g|_e$ in the weak sense means that with any $\omega \in C^\infty_{1,0}(e)$ compactly supported in $\Omega \cap e$

$$\int_{\Omega\cap e} g \wedge \omega = -\int_{\Omega\cap e} v\bar\partial\omega. \tag{9.1}$$

Basic elliptic regularity theory then tells you that in this case $v|_e \in C^l(\Omega\cap e)$. Also, Fubini's theorem implies that (9.1) holds for e any d-dimensional affine subspace, $d < \infty$, with ω a $(d, d-1)$-form; and again v is C^l on this subspace. Now the proof can be concluded along the lines as Lemma 5 in [Ma, Appendix 3].

Next we shall consider the $\bar\partial$ equation on a domain $M = \Omega \times \mathbb{C}$, with $\Omega \subset V$ a domain. Given a form $\psi \in C^k_{0,q}(\Omega \times \mathbb{C})$ it makes sense to say $\psi \in o(k)$ at $M_0 = \Omega \times \{0\}$; cf. section 3. We will also introduce a concept that measures growth at infinity. We shall write $\psi = O(d)$ at M_∞ if for any $x \in \Omega$, $\xi \in \bigoplus^q (V \times \mathbb{C})$ there are neighborhoods $U \subset \Omega$, $\widetilde U \subset \bigoplus^q (V \times \mathbb{C})$ and a number A such that with $y \in U$, $\eta \in \widetilde U$, and $\lambda \in \mathbb{C}$

$$|\psi((y,\lambda);\eta)| \le A(1+|\lambda|)^d.$$

Proposition 9.4. *Let $\varphi \in C^1_{0,1}(M)$ be closed. Assume that φ is C^k on a neighborhood of M_0, and*

(i) *$\varphi = o(k)$ at M_0;*
(ii) *$\varphi = O(k-\epsilon)$ at M_∞, with some $\epsilon > 0$.*

Then there is a unique $v \in C^1(M)$ with $\bar\partial v = \varphi$, of class C^k in a neighborhood of M_0, such that

(iii) *$v = o(k)$ at M_0; and*
(iv) *$v = O(k+1-\epsilon/2)$ at M_∞.*

Proof. Uniqueness is easy: if v, v' are two solutions, then $h = v - v'$ is holomorphic on M. By (iv) h must restrict to polynomials of degree $\le k$ on each line $\{x\} \times \mathbb{C} = \mathbb{C}_x$, and by (iii) these polynomials must be zero.

To show existence, introduce a modified Cauchy kernel on $\mathbb{C}$:

$$C(\lambda,\mu) = \left(\frac{1}{\mu-\lambda} - \sum_{j=0}^{k} \frac{\lambda^j}{\mu^{j+1}}\right) d\mu = \frac{\lambda^{k+1} d\mu}{\mu^{k+1}(\mu-\lambda)}, \tag{9.2}$$

which we also think of as a $(1,0)$-form on M. Put

$$\Delta_x(R) = \{(x,\mu) \in \Omega \times \mathbb{C} : 1/R < |\mu| < R\}; \quad x \in \Omega, 1 < R \le \infty,$$

and define

$$v_R(x,\lambda)=\frac{i}{2\pi}\int_{\Delta_x(R)}\varphi\wedge C,\quad v=v_\infty. \tag{9.3}$$

On the right of (9.3) the variables of φ are x, μ. One easily verifies that $v_R\in C(M)$,

$$\begin{aligned}&\lim_{(x,\lambda)\to(y,0)}\lambda^{-k}v_R(x,\lambda)=0,\\ &v_R=O(k+1-\frac{\epsilon}{2})\quad \text{at } M_\infty,\end{aligned} \tag{9.4}$$

all this for $1<R\le\infty$. Further

$$\lim_{R\to\infty} v_R=v \qquad \text{locally uniformly.} \tag{9.5}$$

In addition, when $1<R<\infty$, v_R is C^1 on the set where $|\lambda|\ne R,1/R$. For such R we have

$$\frac{\partial v_R(x,\lambda)}{\partial\bar\lambda}\,d\overline{\lambda}=\begin{cases}\varphi|_{\mathbb{C}_x} & \text{on } \Delta_x(R),\\ 0 & \text{on } \mathbb{C}_x\setminus\overline{\Delta_x(R)};\end{cases} \tag{9.6}$$

cf. [Hor, Theorem 1.2.2], for instance.

To compute $\bar\partial v_R$ on vectors of the form $(\xi,0)$, $\xi\in T^{0,1}\Omega$, denote by $L_{(\xi,0)}$ the Lie derivative along the constant vector field $(\xi,0)$. By Cartan's formula, for $1/R<|\lambda|<R$

$$\begin{aligned}\bar\partial v_R((x,\lambda);(\xi,0))&=\frac{i}{2\pi}\int_{\Delta_x(R)}L_{(\xi,0)}\varphi\wedge C\\ &=\frac{i}{2\pi}\int_{\Delta_x(R)}\bar\partial(\varphi(\cdot;(\xi,0)))\wedge C\\ &=\varphi((x,\lambda);(\xi,0))+\frac{i}{2\pi}\int_{\partial\Delta_x(R)}\varphi(\cdot;(\xi,0))\wedge C,\end{aligned}$$

with $\partial\Delta_x(R)=\{(x,\mu):|\mu|=1/R \text{ or } |\mu|=R\}$, and the last line follows from Pompeiu's formula; cf. [Hor, Theorem 1.2.1].[4] When $|\lambda|<1/R$ or $|\lambda|>R$ the same formula holds except in the last line only the integral remains. This and (9.6) imply $\bar\partial v_R\to\varphi$ weakly on each line $e\subset V\times\mathbb{C}\supset M$ as $R\to\infty$ so that in view of (9.5) for each such line $\bar\partial v|_{e\cap M}=\varphi|_{e\cap M}$ in the weak sense. Hence $v\in C^1(M)$, in fact v is C^k near M_0 by Proposition 9.3, and $\bar\partial v=\varphi$. (9.4) now implies $v=o(k)$ at M_0, and the proof is complete.

Proof of Theorem 9.1. Fix $x\in\Omega$ and construct a polynomial w on V such that $f'=f-\bar\partial w=o(k)$ at x; cf. Lemma 3.1. Next blow up x to get Bl_xV, which is the total space of a line bundle $\mathrm{Bl}_xV\to\mathbb{P}V$. Pull back f' to $\widetilde{f}\in C^1_{0,1}(\mathrm{Bl}_xV)$. Since Bl_xV can be covered by open sets M that are trivial line bundles over open sets $\Omega\subset\mathbb{P}V$, Ω biholomorphic to domains in a hyperplane in V, Proposition 9.4 applies and provides us with $\widetilde{u}\in C^1(\mathrm{Bl}_xV)$ such that $\bar\partial\widetilde{u}=\widetilde{f}$. Pushing $\widetilde{u}$ down to V we obtain $u_x\in C^1(V\setminus\{x\})$ with $\bar\partial u_x=f'$ off x. We get rid of the singularity at x as before: with some $y\in\Omega\setminus\{x\}$ construct $u_y\in C^1(V\setminus\{y\})$, $\bar\partial u_y=f'$ off y. Then u_x-u_y is holomorphic in a punctured neighborhood of x, hence extends

[4] Hörmander in [Hor] uses the name "Cauchy formula", and the formula in question is known under other names as well. However, in [MS] the authors argue convincingly that the name that the actual facts justify is "Pompeiu's formula".

holomorphically to x (unless dim $V = 1$, in which case the theorem is well known). It follows that u_x is smooth even in x, and so $u = u_x + w$ is the required solution.

As Bo Berndtsson pointed out to me, the above proof has a kinship with [Sk] that treats $\bar\partial u = f$ in $\mathbb{C}^n$, with f of polynomial growth. Morally, that proof first reduces the problem, with the help of a local solution, to the case when f vanishes to high order at the origin, and then applies an explicit integral formula (on $\mathbb{C}^n$) to produce u. This, by the way, was the approach that Henrich took up in [He], too.

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