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A NEW PROOF OF FEDERER'S STRUCTURE THEOREM FOR k-DIMENSIONAL SUBSETS OF $\mathbf{R}^N$

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1. Introduction

If X is a subset of $\mathbf{R}^N$, let $\mathcal{H}^k(X)$ denote the k-dimensional hausdorff measure of X. We write $\mathcal{I}^k(X) = 0$ if

$$\mathcal{H}^k(\pi_K X) = 0$$

for almost every k-plane K in $\mathbf{R}^N$, where $\pi_K : \mathbf{R}^N \to K$ is orthogonal projection. Otherwise we write $\mathcal{I}^k(X) > 0$.

This paper gives a proof of the following theorem.

1.1. Structure Theorem. *Let X be a set in $\mathbf{R}^N$ with $\mathcal{H}^k(X) < \infty$ and $\mathcal{I}^k(X) > 0$. Then there is a k-dimensional C^1 submanifold M with $\mathcal{H}^k(M \cap X) > 0$.*

Let $\mathcal{C}$ be the class of countable unions of k-dimensional C^1 submanifolds of $\mathbf{R}^N$. Since $\mathcal{C}$ is closed under countable unions, there is an $S \in \mathcal{C}$ that maximizes $\mathcal{H}^k(S \cap X)$. Letting $Y = S \cap X$ and applying the structure theorem to $Z = X \setminus Y$, we get

1.2. Corollary. *Suppose $X \subset \mathbf{R}^N$ and $\mathcal{H}^k(X) < \infty$. Then $X = Y \cup Z$, where Y is the portion of X contained in a countable union of k-dimensional C^1 submanifolds and where*

$$\mathcal{H}^k(\pi_K Z) = 0$$

for almost every k-plane $K \subset \mathbf{R}^N$.

The structure theorem was first proved in 1939 by Besicovitch [B] in case $k = 1$ and $N = 2$. Actually Besicovitch proved the existence of a lipschitz curve M rather than a C^1 curve, but the C^1 result follows easily; cf. [S, 11.1] or [Fe2, 3.2.29]. In 1947, Federer [Fe1] proved the structure theorem for all k and N. This paper deduces Federer's result from Besicovitch's special case. The proof is rather simple. In particular, it uses neither estimates nor covering theorems.

The paper is organized as follows. Section 2 contains preliminary remarks about the structure theorem. Section 3 shows that if the structure theorem holds for 1-dimensional sets in spaces of dimension $< N$, then it holds for 1-dimensional sets in $\mathbf{R}^N$. In particular, since we know (thanks to Besicovitch) that it holds for 1-dimensional sets in $\mathbf{R}^2$, it must hold for 1-dimensional sets in any euclidean space.

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Section 4 defines the approximate derivatives of functions and gives some of their basic properties. Section 5 shows how the structure theorem for sets of dimension k follows from the case $k = 1$ proved in §3.

The idea of the proof is the following. If $X \subset \mathbf{R}^N$, the projection $\pi_H X$ of X to a typical subspace H has the same dimension but lower codimension, so by induction we can apply the structure theorem to $\pi_H X$. Similarly, the intersection of X with a typical subspace H will be a set $X \cap H$ in H of lower dimension (and the same codimension), so again we can apply the structure theorem to $X \cap H$. But (as shown below) the validity of the structure theorem for such projections and intersections of X implies its validity for X.

For a proof of Besicovitch's theorem, see [Fa]. Besicovitch's original papers [B] are also very enjoyable. For a general account of the measure theoretic properties of k-dimensional subsets of $\mathbf{R}^N$, see Mattila's recent book [M] or Federer's encyclopedic book [Fe2]. The first chapter of [S] also proves most of the properties of hausdorff measures used in this paper.

Peter Jones recently informed me that he, Nets Katz, and Ana Vargas had also proved, for the case of hypersurfaces, that Federer's theorem follows from Besicovitch's. See [J].

2. Preliminary remarks

2.1. It suffices to prove the structure theorem for borel sets. For if X is any set with $\mathcal{H}^k(X) < \infty$, then there is a borel set $X' \supset X$ with $\mathcal{H}^k(X') = \mathcal{H}^k(X)$. If $\mathcal{I}^k(X) > 0$, then clearly $\mathcal{I}^k(X') > 0$. Thus the borel set case of the structure theorem implies that there is a k-dimensional C^1 manifold M with $\mathcal{H}^k(M \cap X') > 0$. Now since M is $\mathcal{H}^k$-measurable and $\mathcal{H}^k(X') = \mathcal{H}^k(X)$, it follows that $\mathcal{H}^k(X \cap M) = \mathcal{H}^k(X' \cap M) > 0$, which proves the theorem for X.

2.2. Let $\mathcal{S} = \mathcal{S}(k, N)$ be the class of borel sets $S \subset \mathbf{R}^N$ such that $\mathcal{I}^k(S) = 0$. If $X \subset \mathbf{R}^N$, let

$$\mathcal{Z}^k(X) = \sup\{\mathcal{H}^k(X \cap S) : S \in \mathcal{S}\}. \tag{$*$}$$

Note that the hypothesis of the structure theorem:

$$0 < \mathcal{I}^k(X) \text{ and } \mathcal{H}^k(X) < \infty \tag{h1}$$

is equivalent to

$$\mathcal{Z}^k(X) < \mathcal{H}^k(X) < \infty. \tag{h2}$$

Since the class $\mathcal{S}$ above is closed under countable unions, the supremum in $(*)$ is attained by a set $S \in \mathcal{S}$. Let $\tilde{X} = X \setminus S$. Then of course $\mathcal{Z}^k(\tilde{X}) = 0$. Note that if X satisfies the hypotheses of the structure theorem, then so does $\tilde{X}$, and if $\tilde{X}$ satisfies the conclusion, then so does X. Consequently, it suffices to prove the structure theorem for borel sets X such that

$$0 < \mathcal{H}^k(X) < \infty \text{ and } \mathcal{Z}^k(X) = 0. \tag{h3}$$

This hypothesis is very convenient to work with because it is inherited by any subset of X of positive $\mathcal{H}^k$-measure. (This is not the case with the weaker hypothesis (h1) or (h2).)

2.3. In proving the structure theorem, we can also assume that X is compact. For if X is a borel set with $0 < \mathcal{H}^k(X) < \infty$, then X contains a compact subset X' with the same property. And if $\mathcal{Z}^k(X) = 0$, then clearly $\mathcal{Z}^k(X') = 0$.

3. The proof for one-dimensional sets

3.1. Definition. Let $X \subset \mathbf{R}^N$ be a set with $0 < \mathcal{H}^k(X) < \infty$, and let V be a linear subspace of $\mathbf{R}^N$. We say that X **projects to** V **nicely** if the following holds: whenever $S \subset X$ has positive $\mathcal{H}^k$-measure, then $\pi_V S$ also has positive $\mathcal{H}^k$-measure.

3.2. Lemma. *Let C be a C^1 curve of finite length in $\mathbf{R}^N$. Then for almost every line L in $\mathbf{R}^N$, the curve C projects to L nicely.*

Proof. Orient C, and let $v(x)$ be the unit tangent vector to C at x. Let μ be the measure on $\mathbf{R}^N$ given by

$$\mu(S) = \mathcal{H}^1\{x \in C : v(x) \in S\}.$$

For $i = 1, 2, \dots, N-1$, define S_i to be the set of i-dimensional subspaces V of $\mathbf{R}^N$ such that

$$\mu(V) > 0$$

and such that no proper subspace of V has positive μ-measure.

By definition, any two distinct elements of S_i intersect in a set of μ-measure 0. Thus each S_i contains at most countably many elements. For any i-plane V (with $1 \le i < N-1$), the set of lines L in $\mathbf{R}^N$ perpendicular to V has measure 0. Thus for almost every line L in $\mathbf{R}^N$, L will not be perpendicular to any element of any S_i. But it is easily checked that C projects nicely to any such L. □

3.3. Lemma. *Suppose the structure theorem holds for 1-dimensional sets in euclidean spaces of dimension $< N$. Let X be a borel set in $\mathbf{R}^N$ with $0 < \mathcal{H}^1(X) < \infty$ and $\mathcal{I}^1(X) > 0$. Let $1 \le k < N$. Then there is a set $\mathcal{S}_k(X)$ of k-planes in $\mathbf{R}^N$ such that*

1. *for each $K \in \mathcal{S}_k(X)$, there is a C^1 curve $C = C_K \subset K$ such that $\pi_K(X) \cap C$ has positive $\mathcal{H}^1$-measure, and*
2. *$\mathcal{S}_k(X)$ is a positive-measure borel subset of the grassmannian $G(k, \mathbf{R}^N)$ of all k-planes in $\mathbf{R}^N$.*

Of course "positive-measure" in (2) refers to the natural volume measure on $G(k, \mathbf{R}^N)$.

Proof. We may assume X is compact. (The general case reduces to the compact case by the discussion in §2.2 and §2.3.) Note that

$$(*) \qquad \int_{L \in G(1,\mathbf{R}^N)} \mathcal{H}^1(\pi_L X)\, dL = c \int_{K \in G(k,\mathbf{R}^N)} \int_{L \in G(1,K)} \mathcal{H}^1(\pi_L X)\, dL\, dK$$

where the integrations are with respect to the natural volume measures on $G(1,\mathbf{R}^N)$, $G(k, \mathbf{R}^N)$, and $G(1, K)$, respectively. (See the appendix A.1 for a proof that $L \mapsto \mathcal{H}^1(\pi_L X)$ is a borel function.)

By hypothesis, $(*)$ is positive. Thus $\mathcal{S}$ has positive measure, where $\mathcal{S}$ is the set of k-planes K such that

$$(\dagger) \qquad \int_{L \in K} \mathcal{H}^1(\pi_L X)\, dL > 0.$$

Now for $L \subset K$,

$$\pi_L X = \pi_L(X')$$

where $X' = \pi_K X$. Thus (†) asserts that $\mathcal{I}^k(X') > 0$ (regarding X' as a subset of the euclidean space K). Of course $\mathcal{H}^k(X') \le \mathcal{H}^k(X) < \infty$.

Thus by the inductive hypothesis, the structure theorem applies to $X' = \pi_K X$, so there is a C^1 curve C in K such that $\mathcal{H}^1(X' \cap C) > 0$. □

3.4. Theorem. *The structure theorem holds for* 1*-dimensional sets in* $\mathbf{R}^N$*. That is, if* X *is a subset of* $\mathbf{R}^N$ *with* $\mathcal{H}^1(X) < \infty$ *and* $\mathcal{I}^1(X) > 0$*, then there is a* C^1 *curve* C *such that*

$$\mathcal{H}^1(X \cap C) > 0.$$

Proof. For $N = 2$, this is Besicovitch's theorem. Thus suppose $N > 2$. We may suppose by induction that the structure theorem holds in euclidean spaces of dimension $< N$. By §2.2 and §2.3, we may also assume that X is a compact set and that

$$\mathcal{Z}^1(X) = 0.$$

By Lemma 3.2, there is a hyperplane $H \subset \mathbf{R}^N$ and a compact C^1 curve C in H such that

$$\mathcal{H}^1(\pi_H X \cap C) > 0.$$

Of course this implies that $\mathcal{H}^1(X \cap M) > 0$ where $M = \pi_H^{-1} C$. Thus we may assume that $X \subset M$; otherwise replace X by $X \cap M$ in the argument below.

Similarly, we may assume that X projects to H nicely (in the sense of Definition 3.1). For otherwise let $Z \subset X$ maximize $\mathcal{H}^1(Z)$ among sets Z such that $\mathcal{H}^1(\pi_H Z) = 0$, and then replace X by $X \setminus Z$ in the rest of the proof below.

Note, by Lemma 3.2, that for almost every line $L \in G(1, H)$,

$$C \text{ projects to } L \text{ nicely.} \tag{1}$$

Hence for almost every 2-plane $V \in G(2, \mathbf{R}^N)$,

$$V \cap H = L \tag{2}$$

for some line $L \in G(1, H)$ satisfying (1).

Now by Lemma 3.3, we can choose such a 2-plane V so that V contains a C^1 curve C' with $\mathcal{H}^1(\pi_V X \cap C') > 0$. This implies that

$$\mathcal{H}^1(X \cap M') > 0$$

where $M' = \pi_V^{-1} C'$.

Let $Y = \{y \in M : v(\pi_H y) \perp L\}$ where $v(\cdot)$ is the unit tangent vectorfield on the curve C. Then by (1), $\pi_H Y$ has $\mathcal{H}^1$-measure 0. Since X projects to H nicely, this implies that $X \cap Y$ has $\mathcal{H}^1$-measure 0. Thus

$$(X \setminus Y) \cap M' = X \cap (M \setminus Y) \cap M'$$

has positive $\mathcal{H}^1$-measure. But M and M' intersect transversely except along Y, so $(M \setminus Y) \cap M'$ is a C^1 curve. □

4. Approximately differentiable functions

Let $f : S \to \mathbf{R}^m$ be a function defined on a subset S of $\mathbf{R}^k$. We say that f is approximately differentiable at $x_0 \in S$ if there is a function h defined in a neighborhood of x_0 such that

(1) h is differentiable at x_0,
(2) $h(x_0) = f(x_0)$, and
(3) the set $\{x : x \notin S \text{ or } f(x) \neq h(x)\}$ has density 0 at x_0.

The derivative $Dh(x_0)$ is then called the approximate derivative of f at x_0. The approximate partial derivative $\mathrm{ap}\mathbf{D}_i\, f(x)$ is the approximate derivative of

$$t \mapsto f(x + te_i)$$

at $t = 0$, provided this approximate derivative exists.

We will use the following two theorems about approximate partial derivatives:

4.1. Theorem. *Let $S \subset \mathbf{R}^k$ be a Lebesgue measurable set, and let $f : S \to \mathbf{R}^m$ be a Lebesgue measurable function. Then the following are equivalent:*

(1) *f is approximately differentiable almost everywhere in S.*
(2) *The approximate partial derivatives $\mathrm{ap}\mathbf{D}_i\, f$ $(1 \leq i \leq k)$ exist almost everywhere in S.*
(3) *There exist measurable subsets S_i of S such that $S \setminus \bigcup_i S_i$ has measure 0 and such that the restriction of f to each f_i is lipschitz.*

See [Fe2, 3.1.4, 3.1.8] for the proof (which is closely related to the proof of Rademacher's theorem that lipschitz functions are differentiable almost everywhere).

4.2. Theorem. *Let X be a compact subset of euclidean space V, and let $\pi : V \to L$ be orthogonal projection onto a line. Suppose $X = Y \cup Z$ where Y is contained in a countable union of C^1 embedded curves and $\pi(Z)$ has measure 0. Suppose*

$$f : \pi(X) \subset L \to L^{\perp}$$

is a measurable function whose graph lies in X:

$$\{(x, f(x)) : x \in \pi(X)\} \subset X.$$

Then f is approximately differentiable almost everywhere.

Proof. Let $C_1, C_2, \dots$ be the countable collection of C^1 curves. Let

$$Z' = \bigcup_i \{z \in C_i : \text{the tangent line to } C_i \text{ at } z \text{ is perpendicular to } L\}.$$

Then $\mathcal{H}^1(\pi Z') = 0$. Thus we can assume $Z' = \emptyset$ (otherwise include $Z' \cap X$ in Z). We can also assume that each C_i is connected and therefore a graph over an interval in L.

Let

$$U_i = \{x \in \pi(X) \setminus \pi(Z) : (x, f(x)) \in C_i\}.$$

Then for each i, f agrees with a differentiable function on U_i. Thus f is approximately differentiable almost everywhere in U_i. Also, $\bigcup U_i$ covers almost all of $\pi(X)$. Therefore f is approximately differentiable almost everywhere in $\pi(X)$. $\square$

5. Reduction to one-dimensional sets

Let X be a compact set in $\mathbf{R}^N$ with $0 < \mathcal{H}^k(X) < \infty$. Consider pairs (V, L) where V is an affine subspace of $\mathbf{R}^N$ of dimension $N - k + 1$ and where L is an affine line in V. We say such a pair (V, L) is **good** for X provided:

(1) $\mathcal{H}^1(X \cap V) < \infty$, and
(2) $X \cap V = Y \cup Z$ where Y is contained in a countable union of C^1 curves and

$$\mathcal{H}^1(\pi_L Z) = 0.$$

The set of (V, L) that are good for X forms a borel subset of the set of all pairs (V, L) (by appendix A.4).

5.1. Lemma. *If $X \subset \mathbf{R}^N$ is a compact set with $0 < \mathcal{H}^k(X) < \infty$, then almost every pair (V, L) is good for X.*

Proof. Fix any $(k-1)$-dimensional subspace W of $\mathbf{R}^N$. Note

$$\int_{w \in W} \mathcal{H}^1(X \cap {\pi_W}^{-1} w)\, dw \le \mathcal{H}^k(X) < \infty,$$

so for almost every $w \in W$, $\mathcal{H}^1(X \cap {\pi_W}^{-1} w) < \infty$. This proves that (1) holds for almost every $(N - k + 1)$-plane V.

Now fix a V such that (1) holds. From §3 we know that the structure theorem applies to the 1-dimensional set $X \cap V$. By Corollary 1.2 to the structure theorem, $X \cap V = Y \cup Z$ where Y is contained in a countable union of C^1 curves and

$$\mathcal{H}^1(\pi_L Z) = 0$$

for almost every line $L \subset V$. This proves (2). $\square$

In the next lemma, we use the lexicographical ordering of $\mathbf{R}^m$, according to which $x < y$ if for some ℓ, $x_\ell < y_\ell$ and $x_i = y_i$ for $i < \ell$. Note that every nonempty compact set has a least element according to this ordering.

5.2. Lemma. *Let X be a compact subset of $\mathbf{R}^N$ with finite $\mathcal{H}^k$-measure. Let $\Pi : \mathbf{R}^N \to \mathbf{R}^k$ be orthogonal projection to the first k-coordinates. If $\rho : \mathbf{R}^N \to \mathbf{R}^N$ is a rotation, let*

$$\begin{aligned} &f_\rho : \Pi(\rho X) \subset \mathbf{R}^k \to \mathbf{R}^{N-k}, \\ &f_\rho(x) = \min\{y \in \mathbf{R}^{N-k} : (x, y) \in \rho X\} \end{aligned}$$

where the minimum is with respect to the lexicographical ordering of $\mathbf{R}^k$.

Then for almost every ρ, the function f_ρ is approximately differentiable almost everywhere.

Proof. Let $\pi : \mathbf{R}^N \to \mathbf{R}^{k-1}$ be orthogonal projection to the first $(k-1)$-coordinates. For $a \in \mathbf{R}^{k-1}$, let

$$\begin{aligned} V_a &= \pi^{-1} a, \\ L_a &= \{x \in V_a : x_{k+1} = x_{k+2} = \cdots = x_N = 0\} \\ &= V_a \cap (\mathbf{R}^k \times [0]^{N-k}). \end{aligned}$$

By Lemma 5.1, for almost every rotation ρ, the following holds: for almost every $a \in \mathbf{R}^{k-1}$, the pair (V_a, L_a) is good for ρX. For such a ρ and a, Theorem 4.2

implies that the function

$$t \mapsto f_\rho(a,t)$$

is approximately differentiable almost everywhere.

Thus we have shown that for almost every ρ, the approximate partial derivative $\mathrm{ap}\mathbf{D}_k\, f_\rho$ exists almost everywhere. Likewise (for almost every ρ) the other approximate partial derivatives $\mathrm{ap}\mathbf{D}_i\, f_\rho$ exist almost everywhere. Hence (§3.1) for such a ρ, f_ρ is approximately differentiable almost everywhere. □

5.3. Theorem. *The structure theorem holds for k-dimensional sets in $\mathbf{R}^N$.*

Proof. Let X be a compact set in $\mathbf{R}^N$ with $0 < \mathcal{I}^k(X)$ and $\mathcal{H}^k(X) < \infty$.

Since $\mathcal{I}^k(X) > 0$, there must be a positive-measure set of rotations $\rho : \mathbf{R}^N \to \mathbf{R}^N$ such that

$$(*) \qquad \mathcal{H}^k(\Pi\rho X) > 0$$

where $\Pi : \mathbf{R}^N \to \mathbf{R}^k$ is orthogonal projection to the first k-coordinates. By Lemma 5.2, we can choose such a ρ so that f_ρ is approximately differentiable almost everywhere. Hence (by §3.1) there is a measurable subset $S \subset \Pi X$ of positive measure such that f_ρ is lipschitz on S. (Note by $(*)$ that the domain of f has positive measure.) Hence ([S, 5.1, 5.3] or [Fe2, 3.1.16]) there exists a measurable subset S' of S and a C^1 function $h : \mathbf{R}^k \to \mathbf{R}^{N-k}$ such that $\mathcal{H}^k(S') > 0$ and such that $f = h$ on S'. Now let M be the graph of h. Then M is a C^1 submanifold and $M \cap \rho X$ has positive $\mathcal{H}^k$-measure since its projection under Π includes S'. □

Appendix

The proofs in §3 and §5 depend on the measurability of certain sets of rotations and certain subsets of grassmannians. Such measurability is a consequence of the following propositions.

Let E be a euclidean space, and let $L(E,E)$ be the space of linear maps from E to itself. Let $\mathcal{C}$ be the space of compact subsets of E, metrized by the hausdorff metric ([Fa, p.37] or [Fe2, 2.10.21]). Of course $\mathcal{C}$ is a complete, separable metric space.

A.1. Theorem. *The following are borel functions:*

(1) $\mathcal{H}^{k,\delta} : \mathcal{C} \to \mathbf{R}$.
(2) $\mathcal{H}^k : \mathcal{C} \to \mathbf{R}$.
(3) $G : \mathcal{C} \times L(E,E) \to \mathcal{C}$ *where*

$$G(X,\lambda) = \lambda(X).$$

(4) $H : \mathcal{C} \times \{$*affine subspaces of* $E\} \to \mathcal{C}$ *where* $H(X,V) = X \cap V$.

Here $\mathcal{H}^{k,\delta}(S)$ is the infimum of sums

$$(*) \qquad \sum_i \omega_k (\operatorname{diam} S_i)^k$$

corresponding to all countable coverings $S \subset \bigcup S_i$ of S by sets S_i each of diameter $< \delta$. (The number ω_k is the volume of a ball in $\mathbf{R}^k$ of diameter 1.)

Proof. If $\mathcal{H}^{k,\delta}(X) < a$, then we can find a covering so that $(*)$ is $< a$. Note that we can (by expanding each S_i slightly) find such a covering by open sets. Thus $\bigcup_i X_i$ will be open, so, for $X' \in \mathcal{C}$ sufficiently close to X, the X_i's will also cover X'. That is, $\mathcal{H}^{k,\delta}(X') < a$. This proves that $\mathcal{H}^{k,\delta} : \mathcal{C} \to \mathbf{R}$ is upper-semicontinuous and therefore borel.

Since $\mathcal{H}^k(X) = \lim_{j\to\infty} \mathcal{H}^{k,1/j}(X)$, (2) follows immediately.

The other assertions are trivial. $\square$

A.2. Theorem. *Let*

$$\phi(X) = \sup\{\mathcal{H}^1(X \cap S) : S \textit{ is rectifiable}\}.$$

Then $\phi : \mathcal{C} \to \mathbf{R}$ is borel.

Proof. Let $\mathcal{C}(n)$ be the class of images of maps

$$f : [0, n] \to E$$

such that $f(0) = 0$ and such that f is lipschitz with lipschitz constant 1. Then $\mathcal{C}(n)$ is compact by the Arzela-Ascoli theorem.

Note that

$$\begin{aligned} \phi(X) &= \sup_n \sup_{S\in\mathcal{C}(n)} \mathcal{H}^1(X \cap S) \\ &= \sup_n \sup_{S\in\mathcal{C}(n)} \sup_\delta \mathcal{H}^{1,\delta}(X \cap S) \\ &= \sup_n \sup_\delta \sup_{S\in\mathcal{C}(n)} \mathcal{H}^{1,\delta}(X \cap S). \qquad (\dagger) \end{aligned}$$

Since $\mathcal{H}^{1,\delta} : \mathcal{C} \to \mathbf{R}$ is upper-semicontinuous and $\mathcal{C}(n)$ is compact,

$$\sup_{S\in\mathcal{C}(n)} \mathcal{H}^{1,\delta}(X \cap S)$$

is an upper-semicontinuous function of $X \in \mathcal{C}$. But now in $(\dagger)$, it suffices to take the supremum over a countable set of δ's.

Thus ϕ is the supremum of a countable collection of borel functions, and is therefore borel. $\square$

A.3. Theorem. *Let $\mathcal{L}(E)$ be the space of lines in euclidean space E. If $L \in \mathcal{L}(E)$ and $S \subset E$, let $\mu_L(S)$ be the length of the projection of S to L, counting multiplicities. Let $\nu_L(X)$ be the μ_L-measure of the rectifiable part of X:*

$$\nu_L(X) = \sup\{\mu_L(X \cap S) : S \textit{ is rectifiable}\}.$$

Then $(L, X) \mapsto \mu_L(X)$ and $(L, X) \mapsto \nu_L(X)$ are borel functions on $\mathcal{L}(E) \times \mathcal{C}$.

Proof. It suffices to prove that these are borel functions of X for a fixed L, because if L' is any other line, then

$$\mu_{L'}(X) = \mu_L(\rho_{L',L} X)$$

where $\rho_{L',L}$ is an isometry of E taking L' to L. Thus we fix an L and let $\mu = \mu_L$, $\nu = \nu_L$, and $\pi = \pi_L$.

Note

$$\mu(X) = \lim_{j\to\infty} \mu^{1/j}(X)$$

where

$$\mu^\delta(S) = \inf\left\{\sum_i \operatorname{diam}(\pi S_i) : S \subset \bigcup_i S_i, \quad \operatorname{diam} S_i < \delta \quad \forall i\right\}.$$

Then $\mu^\delta(X)$ is upper-semicontinuous (just as $\mathcal{H}^{1,\delta}$ was) and therefore $\mu : \mathcal{C} \to \mathbf{R}$ is borel.

The proof that $\nu : \mathcal{C} \to \mathbf{R}$ is borel is exactly like the proof of Theorem A.2. $\square$

A.4. Corollary. *If X is compact and $\mathcal{H}^k(X) < \infty$, then the set of those (V, L) that are good for X (see §5) is a borel set.*

Proof. Note that (V, L) is good for X if and only if

(1) $\mathcal{H}^1(X \cap V) < \infty$, and

(2) $\mu_L(X \cap V) = \nu_L(X \cap V)$.

By A.1 and A.3, these are borel conditions. $\square$

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