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LOCAL RANKIN-SELBERG CONVOLUTIONS FOR GL_n: EXPLICIT CONDUCTOR FORMULA

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In this paper, F denotes a non-Archimedean local field with finite residue field of q elements. The discrete valuation ring of F is denoted $\mathfrak{o}$, and ψ is a non-trivial, continuous character of the additive group of F. We write $\mathfrak{p}$ for the maximal ideal of $\mathfrak{o}$ and $c(\psi)$ for the largest integer c such that $\mathfrak{p}^{-c} \subset \operatorname{Ker}\psi$.

For $i = 1, 2$, let n_i be a positive integer, write $G_i = \mathrm{GL}_{n_i}(F)$, and let π_i be an irreducible smooth representation of G_i. To the data π_1, π_2 and ψ, Jacquet, Piatetskii-Shapiro and Shalika [14] attach an *L-function* $L(\pi_1 \times \pi_2, s)$ and a *local constant* $\varepsilon(\pi_1 \times \pi_2, s, \psi)$, where s denotes a complex variable. The alternative approach of Shahidi [19] places these objects in a more general context; either way, they are absolutely central to the study of the local Langlands Conjecture [12].

The L-function has the form $L(\pi_1 \times \pi_2, s) = P(q^{-s})^{-1}$, where $P(X) \in \mathbb{C}[X]$ satisfies $P(0) = 1$. On the other hand,

$$\varepsilon(\pi_1 \times \pi_2, s, \psi) = \varepsilon(\pi_1 \times \pi_2, 0, \psi)\, q^{-f(\pi_1\times\pi_2,\psi)s},$$

for some integer $f(\pi_1 \times \pi_2, \psi)$. In fact, $f(\pi_1 \times \pi_2, \psi) = n_1 n_2 c(\psi) + f(\pi_1 \times \pi_2)$, where $f(\pi_1 \times \pi_2)$ is independent of ψ.

There is a full description of the function $L(\pi_1 \times \pi_2, s)$ in [14] which, at the same time, reduces the study of the local constant to the case where both π_i are supercuspidal. The aim of this paper is to give an *explicit formula for* $f(\pi_1 \times \pi_2)$ *when the* π_i *are supercuspidal.*

This formula (Theorem 6.5 below) contains substantial arithmetic information about the representations π_i and their relationship to each other, in terms of their description as induced representations [5]. As one of the consequences of this formula, we obtain sharp upper and lower bounds for $f(\pi_1 \times \pi_2)$, as we shall now explain.

For this purpose, we need to recall the local constant $\varepsilon(\pi, s, \psi)$ of a supercuspidal representation π of $\mathrm{GL}_n(F)$, in the sense of Godement and Jacquet [10]. This takes the form

$$\varepsilon(\pi, s, \psi) = \varepsilon(\pi, 0, \psi)\, q^{-f(\pi,\psi)s},$$

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where $f(\pi, \psi) = nc(\psi) + f(\pi)$, for an integer $f(\pi)$. One already has an explicit formula for $f(\pi)$ of the appropriate kind [2] or (6.1.2) below. This shows $f(\pi) \geqslant 0$. In fact, the case $f(\pi) = 0$ can only arise when $n = 1$ and π is an unramified quasicharacter of $F^\times \cong \mathrm{GL}_1(F)$; otherwise, we have $f(\pi) \geqslant n$. We say that π *has level zero* if $f(\pi) \leqslant n$. (An alternative characterization of such representations is given in 6.1 below.)

Another useful fact, derived from the Whittaker model description [13] of $f(\pi)$, relates the Godement-Jacquet conductor $f(\pi)$ to the one above:

$$f(\pi) = f(\pi \times 1_F),$$

where 1_F denotes the trivial representation of $\mathrm{GL}_1(F)$.

Let π_i be an irreducible supercuspidal representation of G_i, as above, and let $t(\pi_1, \pi_2)$ be the number of unramified quasicharacters χ of $F^\times$ such that $\pi_2 \otimes \chi \circ \det$ is equivalent to the contragredient $\check{\pi}_1$ of π_1. In the case $t(\pi_1, \pi_2) = 0$, we get

$$n_1 n_2 \leqslant f(\pi_1 \times \pi_2) \leqslant n_1 n_2 \max \{f(\pi_1)/n_1, f(\pi_2)/n_2\}.$$

Both bounds are best possible. Indeed, the stated upper bound is the "expected value" for randomly chosen π_i. One can specify precisely when the bound is achieved (6.5 again); for example, it is achieved if $f(\pi_1)/n_1 \neq f(\pi_2)/n_2$. The analysis of the general case relies on the fact that one can compare the representations π_i via their inducing data, even when $n_1 \neq n_2$: this matter is discussed in detail in [3], §8. The non-negative integer

$$n_1 n_2 \max \{f(\pi_1)/n_1, f(\pi_2)/n_2\} - f(\pi_1 \times \pi_2)$$

may reasonably be thought of as a measure of the closeness of π_2 to $\check{\pi}_1$. On the other hand, the lower bound on $f(\pi_1 \times \pi_2)$ is achieved when the π_i both have level zero, and essentially only then.

When $t = t(\pi_1, \pi_2) > 0$, we have $n_1 = n_2 \geqslant t$, $f(\pi_1) = f(\pi_2)$ and, excluding the trivial case where $n_1 = n_2 = 1$ and $f(\pi_1) = f(\pi_2) = 0$, we obtain

$$n_1^2 - n_1 \leqslant f(\pi_1 \times \pi_2) \leqslant n_1 f(\pi_1) - t.$$

Again, these bounds are sharp, being simultaneously attained when π_1 has level zero.

As an immediate consequence of the bounds for supercuspidal representations, we get

$$f(\pi_1 \times \pi_2) \geqslant 0$$

for all irreducible smooth (not necessarily supercuspidal) representations π_i. This bound is achieved when the π_i are unramified principal series representations, and essentially only then (Corollary 6.5). Sharp upper bounds for $f(\pi_1 \times \pi_2)$ in the general case are given in [4].

It is worthy of note that, in the supercuspidal case, the connection between the conductor $f(\pi_1 \times \pi_2)$ and the arithmetic inducing data for the π_i is quite close; if one fixes (π_1, n_1), one can often recover a large portion of the inducing data for π_1 from knowledge of $f(\pi_1 \times \pi_2)$ for certain (π_2, n_2) with n_2 dividing n_1 properly. Viewed in a different light, this indicates that, under a Langlands correspondence, the ramification structure of an irreducible Galois representation is intimately connected with the arithmetic structure of the inducing data for the corresponding supercuspidal representation of GL_n.

Our starting point is not the original definition given in [14] of the objects $L(\pi_1 \times \pi_2, s)$, $\boldsymbol{\varepsilon}(\pi_1 \times \pi_2, s, \psi)$, but rather the alternative and more general formulation of Shahidi [17], [18], [19] which relates L-functions and local constants to the theory of Plancherel measure and intertwining operators. When applied to GL_n in [18], this approach yields a formula relating L-functions and the conductor $f(\check{\pi}_1 \times \pi_2)$ to a certain composition of intertwining operators between representations of $G = \mathrm{GL}_{n_1+n_2}(F)$ parabolically induced from the Levi subgroup $G_1 \times G_2$. However, to extract concrete information, one has to be able to compute the composition. The general theory of types [6], and particularly the explicit existence theorems of [7], provide an effective method of doing this. The connection between conductors and the semisimple types [7] used to compute them is very close indeed; this suggests it will be difficult to avoid the use of types in situations like this.

The intertwining operators in question may be thought of as functions on a certain complex algebraic torus. They are in the first instance defined by integrals convergent only on some analytic open set, but it is known that they admit analytic continuation to meromorphic (in fact rational) functions on the whole torus. Our first observation (1.3) is that the analytically continued intertwining operator can be characterized algebraically, in a manner susceptible to analysis via the theory of types. For GL_n, this analysis breaks into two cases, distinguished by the triviality or otherwise of an associated L-function. In the first case, the existence of the relevant type allows one to compute the desired composition directly from the algebraic description of the operators. This method seems quite general and potentially of wide application; we have therefore given it a rather axiomatic treatment (2.4 below). The case with a non-trivial L-function is more involved. We have first to relate our algebraic description of the analytically continued intertwining operators to functorial structures attached to certain categories of representations (§3). Then we show (§4 below) that the intertwining operators can be transported across the Hecke algebra isomorphisms of [5]. This reduces the problem of computing them to a very special case in which, as it happens, no computation is necessary.

These reductions, combined with results from [18], give us $q^{f(\check{\pi}_1 \times \pi_2)}$ expressed as a quotient of (suitably normalized) volumes, one of a compact open subgroup of G and the other of a compact open subgroup of the Levi subgroup $G_1 \times G_2$ (Theorems 5.3, 5.4 below). These groups are described explicitly in [7] in terms of the inducing data for the π_i; we compute the quotient of volumes in §6, using the machinery of [5].

1. Analytic continuation of intertwining operators

This section is completely general; therefore, G here denotes the group of F-rational points of some connected reductive algebraic group defined over F. We fix a Levi subgroup L of G and a pair (P_u, P_ℓ) of mutually opposite parabolic subgroups of G with Levi component L. We write N_u, N_ℓ, respectively, for their unipotent radicals and fix Haar measures μ_u, μ_ℓ on N_u, N_ℓ, respectively. We will only actually use the case where the Levi subgroup L is maximal, but the extra generality costs nothing.

We use the symbol $\boldsymbol{\iota}$ to denote the functor of *normalized* parabolic induction [8], 3.1, and we abbreviate $\boldsymbol{\iota}_\mathrm{u} = \boldsymbol{\iota}_{P_\mathrm{u}}^G$; likewise $\boldsymbol{\iota}_\ell$. That is, if (σ, W) is a smooth representation of L, then, e.g., $\boldsymbol{\iota}_\mathrm{u}(W)$ will be the space $\mathcal{F}_\mathrm{u}(\sigma)$ of smooth functions

$f : G \to W$ such that

$$f(nmx) = \delta_{\mathrm{u}}^{1/2}\sigma(m)f(x), \quad n \in N_{\mathrm{u}},\ m \in L,\ x \in G,$$

where δ_{u} is the module of the action of L on N_{u}, i.e., $\delta_{P_{\mathrm{u}}}$ in the notation of [8]; the action of G on $\mathcal{F}_{\mathrm{u}}(\sigma)$ via $\boldsymbol{\iota}_{\mathrm{u}}(\sigma)$ is that of right translation.

The functor $\boldsymbol{\iota}$ has a well-known adjoint, namely, normalized Jacquet restriction which we denote here by $\boldsymbol{\rho}$. Thus, given smooth representations (π, V), (σ, W) of G and L respectively, we have an isomorphism

$$\mathrm{Hom}_G(V, \boldsymbol{\iota}_{\mathrm{u}}(W)) \cong \mathrm{Hom}_L(\boldsymbol{\rho}_{\mathrm{u}}(V), W), \tag{1.0.1}$$

which is natural in both V and W.

1.1. We write ${}^\circ L$ for the subgroup of L generated by all compact subgroups of L. Then ${}^\circ L$ is open and normal in L and we have $L/{}^\circ L \cong \mathbb{Z}^r$, for some integer $r \geqslant 0$ (which is in fact the rank of a maximal F-split torus in the centre of L). We set $\mathrm{X}(L) = \mathrm{Hom}(L/{}^\circ L, \mathbb{C}^\times)$, although we tend to regard the elements of this group as smooth homomorphisms $L \to \mathbb{C}^\times$. Thus $\mathrm{X}(L)$ is, in a natural way, a complex algebraic torus.

In this subsection we fix an *irreducible* smooth representation (σ, W) of L and consider the set of representations $(\sigma\chi, W)$ where χ varies over $\mathrm{X}(L)$. We will be especially interested in the family of operators

$$A = A_{\ell\mathrm{u}}(\sigma\chi, \mu_{\mathrm{u}}) : \mathcal{F}_\ell(\sigma\chi) \longrightarrow \mathcal{F}_{\mathrm{u}}(\sigma\chi)$$

defined initially by the formula

$$Af(g) = \int_{N_{\mathrm{u}}} f(ng)\,d\mu_{\mathrm{u}}(n), \quad f \in \mathcal{F}_\ell(\sigma\chi), \tag{1.1.1}$$

for those characters χ for which this integral converges.

For such χ, it is clear that $A \in \mathrm{Hom}_G(\mathcal{F}_\ell(\sigma\chi), \mathcal{F}_{\mathrm{u}}(\sigma\chi))$. Further, there is a certain sense in which the operator $A_{\ell\mathrm{u}}(\sigma\chi, \mu_{\mathrm{u}})$ varies analytically with χ; this will be explained in 1.3 below. Granting this, one then has (see [17], §2 and Theorem 2.2.2):

Proposition. *Let (σ, W) be an irreducible, unitary, smooth representation of L.*

(i) *The integral* (1.1.1) *converges, to a holomorphic function, on a non-empty analytic open set $\mathcal{D}$ in* $\mathrm{X}(L)$.
(ii) *The function $\chi \mapsto A_{\ell\mathrm{u}}(\sigma\chi, \mu_{\mathrm{u}})$ admits analytic continuation to a meromorphic function on the torus* $\mathrm{X}(L)$. *This function is holomorphic on a non-empty Zariski-open subset* Y_σ *of* $\mathrm{X}(L)$.

It is well known that, with σ as in the Proposition, the set of χ for which $\boldsymbol{\iota}_\ell(\sigma\chi)$ is irreducible contains a non-empty Zariski-open set (*cf.* [1], 2.11). This is undoubtedly true for general irreducible σ; indeed, a result of Waldspurger (see Sauvageot [16], Th. 3.2) shows immediately that the set in question is non-empty and the method of "algebraic families" [1] can then be used to prove it is Zariski-open. (We are indebted to an anonymous referee for this comment.) In any case, the result is easy for all representations σ considered here. To avoid a longer digression, we shall simply assume:

Assumption. *There exists a non-empty Zariski-open set Y'_σ of $\mathrm{X}(L)$ such that $\boldsymbol{\iota}_\ell(\sigma\chi)$ and $\boldsymbol{\iota}_{\mathrm{u}}(\sigma\chi)$ are both irreducible for $\chi \in \mathrm{Y}'_\sigma$.*

In the foregoing, we can interchange the roles of P_{u} and P_ℓ to define an operator $A_{\mathrm{u}\ell}(\sigma\chi, \mu_\ell)$, relative to a choice of Haar measure μ_ℓ on N_ℓ. Composing, we get

$$A_{\mathrm{u}\ell}(\sigma\chi, \mu_\ell) \circ A_{\ell\mathrm{u}}(\sigma\chi, \mu_{\mathrm{u}}) = \Phi(\sigma\chi, \mu_{\mathrm{u}} \otimes \mu_\ell)\, 1,\ \chi \in \mathrm{Y}'_\sigma, \tag{1.1.2}$$

where 1 is the identity operator on $\mathcal{F}_\ell(\sigma\chi)$ and Φ is a scalar-valued function which is defined and holomorphic on some non-empty Zariski-open set. In fact, Φ turns out to be a rational function of χ which is of substantial arithmetic interest in some generality [17], [18], [19]. In that context the choice of measure $\mu_{\mathrm{u}} \otimes \mu_\ell$ is significant: see, for example, §5 below.

Our aim, over the next few sections, is to compute the function Φ in some particularly important cases. To accomplish this, it will be necessary to give an algebraic characterization of the operators $A_{\mathrm{u}\ell}(\sigma\chi, \mu_\ell)$ which holds in the set Y'_σ. We turn to this now.

1.2. This subsection is quite general, so we take (σ, W) to be a smooth representation of L. We define a canonical map

$$\mathbf{e}_\sigma = \mathbf{e}_\sigma(\mu_{\mathrm{u}}) : W \longrightarrow \boldsymbol{\rho}_{\mathrm{u}}\boldsymbol{\iota}_\ell(W). \tag{1.2.1}$$

To do this, we choose a compact open subgroup K_{u} of N_{u} and for $w \in W$, we define a function $f_w = f_{w,K_{\mathrm{u}}} \in \mathcal{F}_\ell(\sigma)$ as follows: f_w is to have support $P_\ell K_{\mathrm{u}}$ and

$$f_w(xk) = \mu_{\mathrm{u}}(K_{\mathrm{u}})^{-1}\sigma(x)\delta_\ell(x)^{1/2}w, \quad x \in P_\ell,\ k \in K_{\mathrm{u}}. \tag{1.2.2}$$

It is obvious that $f_w \in \mathcal{F}_\ell(\sigma)$. We then define $\mathbf{e}_\sigma(w)$ to be the image of f_w in $\boldsymbol{\rho}_{\mathrm{u}}\boldsymbol{\iota}_\ell(W)$.

Proposition. *The above definition of $\mathbf{e}_\sigma$ is independent of the choice of K_{u}. Moreover, $\mathbf{e}_\sigma$ is an injective L-homomorphism $W \to \boldsymbol{\rho}_{\mathrm{u}}\boldsymbol{\iota}_\ell(W)$.*

Proof. Let K' denote some other compact open subgroup of N_{u}, and write $f'_w = f_{w,K'}$. We can find a compact open subgroup K'' of N_{u} containing both K_{u} and K'; we write $f''_w = f_{w,K''}$. By symmetry, it is enough to show that f_w, f''_w have the same image in the Jacquet module. However,

$$f''_w(g) = (K'' : K_{\mathrm{u}})^{-1} \sum_{x \in K_{\mathrm{u}}\backslash K''} f_w(gx)$$

for all $g \in G$, so these two functions have the same image, as required.

One checks that $\mathbf{e}_\sigma$ is an L-homomorphism. If f_w has zero image in the Jacquet module, there is a compact open subgroup K of N_{u} such that

$$f_w^K(g) = \int_K f_w(gk)\, d\mu_{\mathrm{u}}(k) = 0,$$

for all $g \in G$. This property is inherited by any compact open subgroup of N_{u} containing K, so we may assume that $K \supset K_{\mathrm{u}}$. However, we then have

$$f_w^K(1) = \mu_{\mathrm{u}}(K_{\mathrm{u}}) f_w(1) = w.$$

Hence $w = 0$, as required.

The map $\mathbf{e}_\sigma(\mu_{\mathrm{u}})$ does depend on the choice of Haar measure μ_{u} on N_{u}: indeed, we have the relation

$$\mathbf{e}_\sigma(c\mu_{\mathrm{u}}) = c^{-1}\mathbf{e}_\sigma(\mu_{\mathrm{u}}), \quad c > 0.$$

1.3. We return to the context of 1.1; in particular, (σ, W) is unitary, irreducible, and subject to Assumption 1.1. We now recall what it means for the operator-valued function $A_{\ell\mathrm{u}}(\sigma\chi, \mu_\mathrm{u})$ to be analytic in χ [11], [17], [20].

We fix a minimal F-Levi subgroup $L_0 \subset L$ and a special maximal compact subgroup K_0 of L_0 such that $P_0K_0 = G$, for every parabolic subgroup P_0 of G with Levi component L_0. In particular, we have $G = P_\ell K_0 = P_\mathrm{u} K_0$; thus a function $f \in \mathcal{F}_\ell(\sigma\chi)$ is determined by its restriction $f\,|_{K_0}$ to K_0. Moreover, the space

$$\mathcal{F}_\ell\,|_{K_0} = \{f\,|_{K_0} \colon f \in \mathcal{F}_\ell(\sigma\chi)\}$$

of W-valued functions on K_0 is *independent of* χ, and the restriction map $\mathcal{F}_\ell(\sigma\chi) \to \mathcal{F}_\ell\,|_{K_0}$ is a K_0-isomorphism. The same remarks apply to $\mathcal{F}_\mathrm{u}$. Thus, whenever $A_{\ell\mathrm{u}}(\sigma\chi)$ is defined, it gives us a K_0-homomorphism $\mathcal{F}_\ell\,|_{K_0} \to \mathcal{F}_\mathrm{u}\,|_{K_0}$. Now let K be some open subgroup of K_0. The space $\mathcal{F}_\ell^K\,|_{K_0}$ of K-fixed vectors in $\mathcal{F}_\ell\,|_{K_0}$ is then finite-dimensional, and $A_{\ell\mathrm{u}}(\sigma\chi)$ gives a linear map

$$A_{\ell\mathrm{u}}^K(\sigma\chi) \in \mathrm{Hom}_\mathbb{C}(\mathcal{F}_\ell^K\,|_{K_0}, \mathcal{F}_\mathrm{u}^K\,|_{K_0}).$$

To say that the function $\chi \mapsto A_{\ell\mathrm{u}}(\sigma\chi)$ is meromorphic then means that, for every open subgroup K of K_0, the function $\chi \mapsto A_{\ell\mathrm{u}}^K(\sigma\chi)$ is meromorphic in the usual sense.

Theorem. *Let (σ, W) be an irreducible smooth unitary representation of L satisfying the Assumption of* 1.1. *Let Y_σ be a Zariski-open set in $\mathrm{X}(L)$ on which $A_{\ell\mathrm{u}}(\sigma\chi, \mu_\mathrm{u})$ is holomorphic. Let K_u be some compact open subgroup of N_u, and set $f_w = f_{w,K_\mathrm{u}} \in \mathcal{F}_\ell(\sigma\chi)$, $w \in W$. We have*

$$A_{\ell\mathrm{u}}(\sigma\chi, \mu_\mathrm{u}) f_w(1) = w, \tag{1.3.1}$$

for every $\chi \in \mathrm{Y}_\sigma$ and every $w \in W$.

Proof. The assertion is independent of the choice of K_u used in the definition of the f_w. The identity (1.3.1) holds when χ lies in the domain $\mathcal{D}$ of convergence of the integral (1.1.1), by a straightforward computation. For general $\chi \in \mathrm{Y}_\sigma$ we proceed as follows. We recall (*cf.* [1], 3.5) that K_0 admits arbitrarily small open normal subgroups K with the property

$$K = K \cap N_\ell \cdot K \cap L \cdot K \cap N_\mathrm{u}.$$

In particular, given w we can choose K so that $K \cap L$ fixes w.

The process $f \mapsto f(1)$ gives a holomorphic map $\mathcal{F}_\mathrm{u}^K \to W^{K\cap L}$. Thus $\chi \mapsto A_{\ell\mathrm{u}}(\sigma\chi) f_w(1)$ is a meromorphic function on $\mathrm{X}(L)$; it takes the constant value w on $\mathcal{D}$ and hence on the whole of Y_σ.

By the Proposition and Assumption of 1.1, there is a non-empty Zariski-open set Z_σ of $\mathrm{X}(L)$ on which $A_{\ell\mathrm{u}}(\sigma\chi)$ is defined and $\boldsymbol{\iota}_\ell(\sigma\chi)$, $\boldsymbol{\iota}_\mathrm{u}(\sigma\chi)$ are irreducible. We return to the isomorphism (1.0.1),

$$\mathrm{Hom}_G(\boldsymbol{\iota}_\ell(\sigma\chi), \boldsymbol{\iota}_\mathrm{u}(\sigma\chi)) \longrightarrow \mathrm{Hom}_L(\boldsymbol{\rho}_\mathrm{u}\boldsymbol{\iota}_\ell(\sigma\chi), \sigma\chi),$$

and denote it $f \mapsto \bar{f}$. The identity (1.3.1) then implies

$$\bar{A}_{\ell\mathrm{u}}(\sigma\chi, \mu_\mathrm{u}) \circ \mathbf{e}_{\sigma\chi}(\mu_\mathrm{u}) = 1_W, \quad \chi \in \mathrm{Z}_\sigma. \tag{1.3.2}$$

Proposition. *The identity* (1.3.2) *determines $A_{\ell\mathrm{u}}(\sigma\chi)$ uniquely. That is, if $\chi \in \mathrm{Z}_\sigma$ and $f \in \mathrm{Hom}_G(\boldsymbol{\iota}_\ell(\sigma\chi), \boldsymbol{\iota}_\mathrm{u}(\sigma\chi))$ satisfies $\bar{f} \circ \mathbf{e}_{\sigma\chi} = 1_W$, then $f = A_{\ell\mathrm{u}}(\sigma\chi)$.*

It is this simple property on which some cases of our analysis of the intertwining operators A will be based.

2. Types and covers

We continue in the general situation of §1, and recall some basic concepts from [1] and [6]. This will enable us to compute the composite of intertwining operators (1.1.2) in one family of cases. We use the notation of [6], to which we refer for further details.

2.1. Let $\mathfrak{R}(G)$ denote the category of smooth complex representations of G. We consider the set of pairs (L,π), where L is an F-Levi subgroup of G and π is an irreducible supercuspidal representation of L, modulo the relation of *inertial equivalence:* two such pairs (L_i,π_i) are inertially equivalent (in G) if (L_2,π_2) is G-conjugate to $(L_1,\pi_1\otimes\chi)$, for some $\chi\in \mathrm{X}(L_1)$. We write $\mathcal{B}(G)$ for the set of inertial equivalence classes of these pairs.

An irreducible smooth representation σ of G determines an element of $\mathcal{B}(G)$ as follows. There is a parabolic subgroup P of G and an irreducible supercuspidal representation π of a Levi component L of P such that σ is equivalent to a composition factor of $\boldsymbol{\iota}_P^G(\pi)$. The inertial equivalence class of (L,π) is thereby uniquely determined, and called the *inertial support* of σ.

Fixing $\mathfrak{s}\in\mathcal{B}(G)$, we define a full sub-category $\mathfrak{R}^{\mathfrak{s}}(G)$ of $\mathfrak{R}(G)$ by demanding that its objects be those smooth representations of G whose irreducible sub-quotients all have inertial support $\mathfrak{s}$. The category $\mathfrak{R}(G)$ is then the direct product of the sub-categories $\mathfrak{R}^{\mathfrak{s}}(G)$, $\mathfrak{s}\in\mathcal{B}(G)$. (In the language of [1], the representation π defines an orbit $D_\pi=\{\pi\otimes\chi:\chi\in\mathrm{X}(L)\}$, and $\mathfrak{R}^{\mathfrak{s}}(G)$ is the category denoted $(\mathrm{Alg}\,G)(L,D_\pi)$ in [1].)

2.2. Now fix a Haar measure on G. Let K be a compact open subgroup of G and (τ,U) an irreducible smooth representation of K. We denote the contragredient of (τ,U) by $(\check{\tau},U^\vee)$. We write $\mathcal{H}(G,\tau)$ for the convolution algebra of $\mathrm{End}_{\mathbb{C}}(U^\vee)$-valued functions f on G which are compactly supported and which satisfy

$$f(hxk)=\check{\tau}(h)\,f(x)\,\check{\tau}(k),\qquad h,k\in K,\ x\in G.$$

Then, for $(\pi,V)\in\mathfrak{R}(G)$, the space

$$V_\tau=\mathrm{Hom}_K(U,V)$$

has a canonical left $\mathcal{H}(G,\tau)$-module structure [6], §2. Indeed, $V\mapsto V_\tau$ gives a functor from $\mathfrak{R}(G)$ to $\mathcal{H}(G,\tau)$-Mod.

Fix $\mathfrak{s}\in\mathcal{B}(G)$. The pair (K,τ) is *an* $\mathfrak{s}$*-type in* G if the irreducible representations of G which contain τ are exactly those with inertial support $\mathfrak{s}$. We then have [6], §4: *Let* $\mathfrak{s}\in\mathcal{B}(G)$ *and let* (K,τ) *be an* $\mathfrak{s}$*-type in* G. *A representation* $(\pi,V)\in\mathfrak{R}(G)$ *lies in* $\mathfrak{R}^{\mathfrak{s}}(G)$ *if and only if* V *is generated over* G *by its* τ*-isotypic space* V^τ. *The functor*

$$\begin{aligned}\mathfrak{R}^{\mathfrak{s}}(G)&\longrightarrow\mathcal{H}(G,\tau)\text{-Mod},\\(\pi,V)&\longmapsto V_\tau,\end{aligned}$$

is an equivalence of categories.

2.3. We recall some basic constructions described in [6], §8. Suppose now that L is a Levi subgroup of G, and we are given an element $\mathfrak{t}\in\mathcal{B}(L)$; this is the L-inertial equivalence class of some pair (M,π). The G-inertial equivalence class of (M,π) is then an element $\mathfrak{s}\in\mathcal{B}(G)$. We suppose given a $\mathfrak{t}$-type (K_L,τ_L) in L. In particular, $\mathfrak{R}^{\mathfrak{t}}(L)$ is canonically equivalent to $\mathcal{H}(L,\tau_L)$-Mod. Let (K,τ) be *a G-cover of* (K_L, τ_L), as defined in [6], 8.1. In particular, K is a compact open subgroup of G and τ is an irreducible smooth representation of G. The pair (K, τ) has certain properties, some of which we now recall:

(i) *Let P_u be a parabolic subgroup of G with Levi decomposition $P_\mathrm{u} = LN_\mathrm{u}$ and opposite $P_\ell = LN_\ell$. Then*

$$K = K \cap N_\ell \cdot K \cap L \cdot K \cap N_\mathrm{u} \quad \text{and} \quad K \cap L = K_L.$$

The representation τ is trivial on $K \cap N_\ell$ and $K \cap N_\mathrm{u}$, while $\tau \mid K_L \cong \tau_L$.

(ii) *The pair (K, τ) is an $\mathfrak{s}$-type in G.*

(iii) *There is a canonical algebra homomorphism*

$$\mathbf{j}_\mathrm{u} : \mathcal{H}(L, \tau_L) \longrightarrow \mathcal{H}(G, \tau)$$

which realizes the induction functor $\boldsymbol{\iota}_\mathrm{u} = \boldsymbol{\iota}_{P_\mathrm{u}}^G$. *That is, the diagram*

$$\begin{array}{ccc} \mathfrak{R}^{\mathfrak{t}}(L) & \xrightarrow{\boldsymbol{\iota}_\mathrm{u}} & \mathfrak{R}(G) \\ \approx\downarrow & & \downarrow\approx \\ \mathcal{H}(L, \tau_L)\text{-Mod} & \xrightarrow[(\mathbf{j}_\mathrm{u})_*]{} & \mathcal{H}(G, \tau)\text{-Mod} \end{array} \tag{2.3.1}$$

commutes, where

$$(\mathbf{j}_\mathrm{u})_* : M \longmapsto \mathrm{Hom}_{\mathcal{H}(L,\tau_L)}(\mathcal{H}(G, \tau), M).$$

In the definition of the module-theoretic induction functor $(\mathbf{j}_\mathrm{u})_*$, we view $\mathcal{H}(G, \tau)$ as a left $\mathcal{H}(L, \tau_L)$-module via $\mathbf{j}_\mathrm{u}$; the space $\mathrm{Hom}_{\mathcal{H}(L,\tau_L)}(\mathcal{H}(G, \tau), M)$ becomes a left $\mathcal{H}(G, \tau)$-module via the natural right action of $\mathcal{H}(G, \tau)$ on itself.

There is another useful relation here. The normalized Jacquet functor $\boldsymbol{\rho}_\mathrm{u}$ gives a functor

$$\mathfrak{R}^{\mathfrak{s}}(G) \longrightarrow \mathfrak{R}(L) \longrightarrow \mathfrak{R}^{\mathfrak{t}}(L),$$

where the second arrow is the canonical projection. We continue to denote this $\boldsymbol{\rho}_\mathrm{u}$. We get another commutative diagram

$$\begin{array}{ccc} \mathfrak{R}^{\mathfrak{s}}(G) & \xrightarrow{\boldsymbol{\rho}_\mathrm{u}} & \mathfrak{R}^{\mathfrak{t}}(L) \\ \approx\downarrow & & \downarrow\approx \\ \mathcal{H}(G, \tau)\text{-Mod} & \xrightarrow[\mathbf{j}_\mathrm{u}^*]{} & \mathcal{H}(L, \tau_L)\text{-Mod}, \end{array} \tag{2.3.2}$$

where $\mathbf{j}_\mathrm{u}^*$ denotes restriction along $\mathbf{j}_\mathrm{u}$.

2.4. We need a new concept.

Definition. Let $\mathfrak{t} \in \mathcal{B}(L)$, let (K_L, τ_L) be a $\mathfrak{t}$-type in L and (K, τ) a G-cover of (K_L, τ_L). We say that (K, τ) is a split cover if, for every choice of parabolic subgroup P_u with Levi L, the map $\mathbf{j}_\mathrm{u}$ is an isomorphism of algebras which preserves support of functions:

$$\mathrm{supp}\,(\mathbf{j}_\mathrm{u}(f)) = K \cdot \mathrm{supp}\, f \cdot K, \quad f \in \mathcal{H}(L, \tau_L).$$

In the presence of a split cover, it is very easy to compute the composition of intertwining operators (1.1.2). We now do this, using the notation of §1.

Proposition. *Let L be an F-Levi subgroup of G, let $\mathfrak{t} \in \mathcal{B}(L)$, and let (K_L, τ_L) be a $\mathfrak{t}$-type in L. Let P_{u} be a parabolic subgroup of G with Levi decomposition $P_{\mathrm{u}} = LN_{\mathrm{u}}$ and opposite $P_\ell = LN_\ell$. Suppose there exists a split G-cover (K, τ) of (K_L, τ_L). Then:*

(i) *Let $\sigma \in \mathfrak{R}^{\mathfrak{t}}(L)$ be irreducible. The induced representation $\boldsymbol{\iota}_{\mathrm{u}}(\sigma\chi)$ is irreducible, for every $\chi \in \mathrm{X}(L)$.*

(ii) *Let μ_{u}, μ_ℓ be the Haar measures on N_{u}, N_ℓ, respectively, such that $\mu_{\mathrm{u}}(K \cap N_{\mathrm{u}}) = \mu_\ell(K \cap N_\ell) = 1$. Then*

$$A_{\mathrm{u}\ell}(\sigma\chi, \mu_\ell) \circ A_{\ell\mathrm{u}}(\sigma\chi, \mu_{\mathrm{u}}) = 1,$$

for every $\chi \in \mathrm{X}(L)$.

Proof. Assertion (i) follows from (2.3.1) and the fact that $\mathbf{j}_{\mathrm{u}}$ is an isomorphism. Thus Assumption 1.1 is satisfied in this case.

As for (ii), we will need the following lemma. This is independent of χ, so we drop it from the notation. Write W for the representation space of σ.

Lemma. *For $w \in W$, write $f_w^\ell = f_{w, K\cap N_{\mathrm{u}}}$, in the notation of §1. The τ-isotypic subspace $\mathcal{F}_\ell(\sigma)^\tau$ of $\mathcal{F}_\ell(\sigma)$ then consists of the functions f_w^ℓ, for w ranging over W^{τ_L}. Moreover, if $f \in \mathcal{F}_\ell(\sigma)^\tau$ satisfies $f(1) = w$, then $f = f_w^\ell$.*

Proof. The first step is to show that the functions f_w^ℓ, for $w \in W^{\tau_L}$, lie in $\mathcal{F}_\ell(\sigma)^\tau$. To say that $w \in W$ lies in W^{τ_L} means that there is a K_L-homomorphism $\varphi : U \to W$ with w in the image of φ, $w = \varphi(u)$, say. One checks that, given such a φ, the map $u \mapsto f^\ell_{\varphi(u)}$ is a K_L-homomorphism $\varphi_\ell : U \to \mathcal{F}_\ell(\sigma)$. The function $f^\ell_{\varphi(u)}$ is fixed by the groups $\sigma(K \cap N_\ell)$, $\sigma(K \cap N_{\mathrm{u}})$, so φ_ℓ is in fact a K-homomorphism $U \to \mathcal{F}_\ell(\sigma)$ (by condition (i) in 2.3). Thus $f_w^\ell = f^\ell_{\varphi(u)} \in \mathcal{F}_\ell(\sigma)^\tau$, as desired.

Under the canonical map

$$\mathcal{F}_\ell(\sigma) \to \boldsymbol{\rho}_\ell \mathcal{F}_\ell(\sigma) \to \sigma,$$

we have $f_w^\ell \mapsto f_w^\ell(1) = w$. On the other hand, the canonical map $\mathcal{F}_\ell(\sigma) \to \boldsymbol{\rho}_\ell\mathcal{F}_\ell(\sigma)$ induces an isomorphism $\mathcal{F}_\ell(\sigma)^\tau \cong (\boldsymbol{\rho}_\ell \mathcal{F}_\ell(\sigma))^{\tau_L}$ [6], 7.9. However, since (K_L, τ_L) is a $\mathfrak{t}$-type, the L-subspace of $\boldsymbol{\rho}_\ell\mathcal{F}_\ell(\sigma)$ generated by its τ_L-isotypic space is the canonical projection of $\boldsymbol{\rho}_\ell\mathcal{F}_\ell(\sigma)$ into $\mathfrak{R}^{\mathfrak{t}}(L)$. Since (K, τ) is a split cover, this projection is just σ (2.3.2). Put another way, the τ_L-isotypic subspace of the Jacquet module is just W^{τ_L}. The lemma follows.

Returning to the proof of the proposition, let χ be such that $A = A_{\ell\mathrm{u}}(\sigma\chi)$ is defined. In particular, A is a G-homomorphism and so maps $\mathcal{F}_\ell(\sigma\chi)^\tau$ to $\mathcal{F}_{\mathrm{u}}(\sigma\chi)^\tau$. If we take $w \in W^{\tau_L}$, the function Af_w^ℓ therefore lies in $\mathcal{F}_{\mathrm{u}}(\sigma\chi)^\tau$; it satisfies $Af_w^\ell(1) = w$, by 1.3. By the lemma, interchanging the roles of P_ℓ and P_{u}, we have $A_{\ell\mathrm{u}}(\sigma\chi) f_w^\ell = f_w^{\mathrm{u}}$.

By symmetry, we get $A_{\mathrm{u}\ell}(\sigma\chi) f_w^{\mathrm{u}} = f_w^\ell$. Thus $A_{\mathrm{u}\ell}(\sigma\chi) \circ A_{\ell\mathrm{u}}(\sigma\chi)$ is the identity wherever it is defined, i.e., at least on a dense open set. The result now follows.

Comment. The choice of Haar measures in the proposition is made for convenience when working with covers. In arithmetic applications, the choice of measure is usually dictated by external considerations. In this light, the point of the proposition is that the composite of intertwining operators depends on the (normalized) volumes of the unipotent factors of the split G-cover attached to the representation σ.

3. Adjoint relations

This section is concerned with the relation between the functors $\boldsymbol{\iota}$, $\boldsymbol{\rho}$ and operations of contragredience. Its purpose is to give another description of the map $\mathbf{e}_\sigma$ of 1.2 in certain situations.

3.1. Taking G, $P_\ell = LN_\ell$, etc., as before, let X denote the representation space of $\boldsymbol{\iota}_\ell(\delta_\ell^{1/2})$. Thus X consists of the smooth functions $f : G \to \mathbb{C}$ which satisfy

$$f(nmx) = \delta_\ell(m)f(x), \quad n \in N_\ell,\ m \in L,\ x \in G.$$

Since $P_\ell \backslash G$ is compact, the space X admits a positive G-invariant functional I, unique up to positive scale; see, for example, [8], 2.4.2.

The space $C_c^\infty(N_{\mathrm{u}})$ of compactly supported smooth functions on N_{u} embeds in X, and the restriction of I to $C_c^\infty(N_{\mathrm{u}})$ is a Haar measure μ_{u} on N_{u}. Conversely, a Haar measure μ_{u} on N_{u} determines a functional I. We write $I = I(\mu_{\mathrm{u}})$ to emphasize this dependence when necessary.

3.2. Let (σ_i, W_i), $i = 1, 2$, be smooth representations of L, and suppose we have a non-degenerate, L-invariant bilinear pairing

$$\langle\, ,\, \rangle : W_1 \times W_2 \longrightarrow \mathbb{C}.$$

We use the notation $(\boldsymbol{\iota}_\ell(\sigma_i), \boldsymbol{\iota}_\ell(W_i))$ for the induced representation. Let $f_i \in \boldsymbol{\iota}_\ell(W_i)$. The function

$$x : g \longmapsto \langle f_1(g), f_2(g)\rangle, \qquad g \in G,$$

lies in X and one checks (*cf.* [8], 2.4.2) that the pairing

$$\langle f_1, f_2\rangle_{\boldsymbol{\iota}} = \langle f_1, f_2\rangle_{\boldsymbol{\iota},\mu_{\mathrm{u}}} = I(\mu_{\mathrm{u}})(x)$$

is a non-degenerate G-invariant pairing of $\boldsymbol{\iota}_\ell(W_1)$ with $\boldsymbol{\iota}_\ell(W_2)$. In particular, if we take $(\sigma_1, W_1) = (\sigma, W)$, $(\sigma_2, W_2) = (\check{\sigma}, W^\vee)$, and for $\langle\, ,\, \rangle$ the canonical pairing, then the associated pairing $\langle\, ,\, \rangle_{\boldsymbol{\iota}}$ induces an isomorphism

$$\boldsymbol{\iota}_\ell(W^\vee) \cong (\boldsymbol{\iota}_\ell(W))^\vee. \tag{3.2.1}$$

One checks that this isomorphism is natural in W.

3.3. We recall a parallel result. We fix *admissible* representations (π_i, V_i), $i = 1, 2$, of G and suppose we are given a non-degenerate G-invariant pairing $\langle\, ,\, \rangle$ between V_1 and V_2. This pairing determines a non-degenerate L-invariant pairing $\langle\, ,\, \rangle_{\boldsymbol{\rho}}$ of $\boldsymbol{\rho}_{\mathrm{u}}(V_1)$ with $\boldsymbol{\rho}_\ell(V_2)$ characterized as follows [8], 4.2.5. As in [8], 1.3, we fix a minimal parabolic subgroup P_0 of G contained in P_{u}; this gives us a semigroup $A^-(\epsilon)$ contained in the centre of L, for any real ϵ, $0 < \epsilon \leqslant 1$. The pairing $\langle\, ,\, \rangle_{\boldsymbol{\rho}}$ is then determined uniquely by the following property: *Let $v_i \in V_i$ and write $\bar{v}_1$, $\bar{v}_2$ for the images of v_1, v_2 in $\boldsymbol{\rho}_{\mathrm{u}}(V_1)$, $\boldsymbol{\rho}_\ell(V_2)$, respectively. There exists $\epsilon > 0$ (depending on the v_i) such that*

$$\langle \pi_1(a)v_1, v_2\rangle = \delta_{\mathrm{u}}^{1/2}(a)\, \langle \boldsymbol{\rho}_{\mathrm{u}}\pi_1(a)\bar{v}_1, \bar{v}_2\rangle_{\boldsymbol{\rho}}, \quad a \in A^-(\epsilon). \tag{3.3.1}$$

If (π, V) is an admissible representation of G, we hence obtain an isomorphism

$$\boldsymbol{\rho}_\ell(V^\vee) \cong \boldsymbol{\rho}_{\mathrm{u}}(V)^\vee, \tag{3.3.2}$$

which, as a further consequence of (3.3.1), is natural in V.

Remarks. (i) The factor $\delta_{\mathrm{u}}^{1/2}$ on the right hand side of (3.3.1) is explained by the fact that we are using *normalized* Jacquet restriction.

(ii) The isomorphism (3.3.1) holds for *smooth* representations (π, V), but the proof in [8] of the existence of the pairing $\langle\ ,\ \rangle_{\boldsymbol{\rho}}$ must be modified. We shall not explore this here.

3.4. We now consider the effect of the isomorphisms (3.2.1) and (3.3.2) on the adjoint pair $(\boldsymbol{\iota}_\ell, \boldsymbol{\rho}_\ell)$. We work in the categories $\mathcal{A}(G)$, $\mathcal{A}(L)$ of admissible representations of G, L, respectively; it is useful to note in this context that the functors $\boldsymbol{\iota}_\ell$, $\boldsymbol{\rho}_\ell$, $\boldsymbol{\iota}_{\mathrm{u}}$, $\boldsymbol{\rho}_{\mathrm{u}}$ all take admissible representations to admissible representations.

Proposition. *View $\boldsymbol{\rho}_{\mathrm{u}}$, $\boldsymbol{\rho}_\ell$ as functors from $\mathcal{A}(G)$ to $\mathcal{A}(L)$, and $\boldsymbol{\iota}_\ell$, $\boldsymbol{\iota}_{\mathrm{u}}$ as functors from $\mathcal{A}(L)$ to $\mathcal{A}(G)$. Then $(\boldsymbol{\iota}_\ell, \boldsymbol{\rho}_{\mathrm{u}})$ is an adjoint pair.*

Proof. Given admissible representations (π, V) of G and (σ, W) of L, we need to produce an isomorphism

$$\eta = \eta(W, V) : \mathrm{Hom}_G(\boldsymbol{\iota}_\ell(W), V) \cong \mathrm{Hom}_L(W, \boldsymbol{\rho}_{\mathrm{u}}(V))$$

in such a way that $\eta(W, V)$ is natural in both W and V. We proceed as follows. Let (π_i, V_i), $i = 1, 2$, be smooth representations of G; for $\phi \in \mathrm{Hom}_G(V_1, V_2^\vee)$, define $\check{\phi} \in \mathrm{Hom}_G(V_2, V_1^\vee)$ by $\check{\phi}(v_2) : v_1 \mapsto \phi(v_1)(v_2)$. The map $\phi \mapsto \check{\phi}$ is an isomorphism $\mathrm{Hom}_G(V_1, V_2^\vee) \cong \mathrm{Hom}_G(V_2, V_1^\vee)$, natural in V_1 and V_2. If the representations V_i are admissible, the canonical embedding $V_i \to V_i^{\vee\vee}$ is an isomorphism. It follows that we have a natural isomorphism

$$\mathrm{Hom}_G(V_1, V_2) \cong \mathrm{Hom}_G(V_2^\vee, V_1^\vee), \tag{3.4.1}$$

which we continue to denote $\phi \mapsto \check{\phi}$. Combining this with (3.2.1), we have a natural isomorphism

$$\mathrm{Hom}_G(\boldsymbol{\iota}_\ell(W), V) \cong \mathrm{Hom}_G(V^\vee, \boldsymbol{\iota}_\ell(W^\vee)). \tag{3.4.2}$$

On the other hand, if we combine (3.4.1) with (3.3.2) we obtain a natural isomorphism

$$\mathrm{Hom}_L(W, \boldsymbol{\rho}_{\mathrm{u}}(V)) \cong \mathrm{Hom}_L(\boldsymbol{\rho}_\ell(V^\vee), W^\vee). \tag{3.4.3}$$

Since $(\boldsymbol{\rho}_\ell, \boldsymbol{\iota}_\ell)$ is an adjoint pair (1.0.1), there is a natural isomorphism of the right hand side of (3.4.2) with that of (3.4.3); our result now follows.

We observe that the map η constructed in the last proof depends on the Haar measure μ_{u}.

3.5. We come to the critical observation. Given a smooth representation (σ, W) of L, we have the natural map

$$\begin{aligned} d(\sigma) : \boldsymbol{\rho}_\ell(\boldsymbol{\iota}_\ell(W)) &\longrightarrow W, \\ \boldsymbol{\rho}_\ell(f) &\longmapsto f(1), \quad f \in \boldsymbol{\iota}_\ell(W). \end{aligned}$$

If (σ, W) is admissible, then we may consider the map

$$d(\sigma)^\vee : W^\vee \to (\boldsymbol{\rho}_\ell(\boldsymbol{\iota}_\ell(W)))^\vee$$

and, using the isomorphisms of Proposition 3.4, we obtain a map

$$d(\sigma, \mu_{\mathrm{u}})^\vee : W^\vee \longrightarrow \boldsymbol{\rho}_{\mathrm{u}}(\boldsymbol{\iota}_\ell(W^\vee)).$$

We now prove:

Proposition. *Let (σ, W) be an admissible representation of L and fix a Haar measure μ_{u} on N_{u}. Then we have*

$$\mathbf{e}_\sigma(\mu_{\mathrm{u}}) = d(\check{\sigma}, \mu_{\mathrm{u}})^\vee,$$

that is, in the notation of 3.4, $\mathbf{e}_\sigma \in \mathrm{Hom}_L(W, \boldsymbol{\rho}_{\mathrm{u}}\boldsymbol{\iota}_\ell(W))$ *corresponds to the identity map* $1 \in \mathrm{Hom}_L(W, W)$ *under the map* $\eta(W, \boldsymbol{\iota}_\ell(W))$.

Proof. Let $\langle\,,\,\rangle_W$ be the canonical pairing of W with $W^\vee$. Set $\langle\,,\,\rangle = (\langle\,,\,\rangle_W)_{\boldsymbol{\iota}}$ (notation of 3.2) and put $\mathbf{e}(w) = \mathbf{e}_\sigma(\mu_{\mathrm{u}})(w)$. A straightforward diagram chase shows that the assertion $\mathbf{e}_\sigma(\mu_{\mathrm{u}}) = d(\check{\sigma}, \mu_{\mathrm{u}})^\vee$ is equivalent to saying that for all $w \in W$, $F \in \boldsymbol{\iota}_\ell(W^\vee)$ we have

$$\langle \mathbf{e}(w), \overline{F}\rangle_{\boldsymbol{\rho}} = F(1)(w),$$

where $\overline{F}$ is the image of F in $\boldsymbol{\rho}_\ell\boldsymbol{\iota}_\ell(W^\vee)$. This is what we will now prove.

To this end, we fix w, F as above and choose a compact open subgroup K of G such that

$$K = K \cap N_\ell \cdot K \cap L \cdot K \cap N_{\mathrm{u}},$$

with the further properties that $K \cap L$ fixes w and $K \cap N_{\mathrm{u}}$ fixes F. This is always possible, as in 1.3 above. Set $(\pi_1, V_1) = \boldsymbol{\iota}_\ell(\sigma, W)$, $(\pi_2, V_2) = \boldsymbol{\iota}_{\mathrm{u}}(\sigma, W)$ and abbreviate $K_{\mathrm{u}} = K \cap N_{\mathrm{u}}$. Then since V_1^K is finite-dimensional, we may pick $\epsilon > 0$ so that (3.3.1) holds for $v_2 = F$ and all $v_1 \in V_1^K$. If $a \in A^-(\epsilon)$, then a lies in the centre of L and we have that $\sigma(a^{-1})w$ is fixed by $K \cap L$. It then follows that $f_{\sigma(a^{-1})w, K_{\mathrm{u}}}$ (notation of §1) lies in V_1^K. Since $\mathbf{e}$ is an L-map, we have

$$\delta_{\mathrm{u}}^{1/2}(a)\langle \mathbf{e}(w), \boldsymbol{\rho}_\ell(F)\rangle_{\boldsymbol{\rho}} = \langle \pi_1(a) f_{\sigma(a^{-1})w, K_{\mathrm{u}}}, F\rangle.$$

On the other hand, a direct computation shows that

$$\pi_1(a) f_{\sigma(a^{-1})w, K_{\mathrm{u}}} = \delta_{\mathrm{u}}^{1/2}(a)\, f_{w, aK_{\mathrm{u}}a^{-1}}$$

and by [8], 1.4.3, we may pick ϵ', $0 < \epsilon' \leqslant \epsilon$, so that $aK_{\mathrm{u}}a^{-1} \subset K_{\mathrm{u}}$, $a \in A^-(\epsilon')$. Since F is invariant under K_{u} we see that

$$\langle \pi_1(a) f_{\sigma(a^{-1})w, K_{\mathrm{u}}}, F\rangle = \delta_{\mathrm{u}}^{1/2}(a)\, F(1)w, \quad a \in A^-(\epsilon'),$$

whence our result.

4. General linear groups

4.1. We now specialize to the case $G = \mathrm{GL}_n(F)$, for some $n \geqslant 1$. The group $\mathrm{X}(G)$ then consists of the homomorphisms

$$\chi_s : g \longmapsto \|\det g\|^s, \quad g \in G,$$

where $s \in \mathbb{C}$. Clearly, χ_s only depends on q^{-s}, where q is the size of the residue field of F. We use the notation

$$\pi[s] = \pi \otimes \chi_s, \quad s \in \mathbb{C},$$

where π is an irreducible smooth representation of G.

We now assume $n \geqslant 2$ and fix a maximal proper Levi subgroup L of G. We can identify L with $\mathrm{GL}_{n_1}(F) \times \mathrm{GL}_{n_2}(F)$, for positive integers n_1, n_2 such that $n_1 + n_2 = n$. We let $P_{\mathrm{u}} = LN_{\mathrm{u}}$, $P_\ell = LN_\ell$ denote the two parabolic subgroups of G with Levi component L. We abbreviate $G_i = \mathrm{GL}_{n_i}(F)$, $A_i = \mathbb{M}_{n_i}(F)$.

Let σ denote an irreducible *supercuspidal* representation of L; thus $\sigma = \sigma_1 \otimes \sigma_2$, where σ_i is an irreducible supercuspidal representation of G_i, $i = 1, 2$. We return to the problem of computing the composite of intertwining operators (1.1.2) in this particular case.

Proposition. *Let $\sigma = \sigma_1 \otimes \sigma_2$ be an irreducible supercuspidal representation of L. Let $\mathfrak{t} \in \mathcal{B}(L)$ be the inertial support of σ. There exists a $\mathfrak{t}$-type (K_L, τ_L) in L. The type (K_L, τ_L) may be chosen to admit a G-cover (K, τ). Moreover,*

(i) *if $n_1 = n_2$ and $\sigma_2 \cong \sigma_1[s]$ for some $s \in \mathbb{C}$, the cover (K, τ) is not split;*
(ii) *in all other cases, (K, τ) may be chosen split.*

Proof. We take $K_L = J_1 \times J_2$, $\tau_L = \lambda_1 \otimes \lambda_2$, where (J_i, λ_i) is a maximal simple type (see [5], 6.2) occurring in σ_i; the existence of such is guaranteed by *ibid.* 8.4.1. The existence of the covers is given by [7], 1.5; that the cover is split in case (ii) follows from [7], 8.2. In case (i), on the other hand, it is well known that the representation $\boldsymbol{\iota}_{\mathrm{u}}(\sigma_1 \otimes \sigma_1[1])$ is reducible; the cover cannot therefore be split (*cf.* (2.3.1)).

Remark. The last proposition is only concerned with the existence of the various covers. However, later on it will be essential to know the explicit construction of these as in [7].

4.2. In case (ii) of Proposition 4.1, the composition of intertwining operators has already been computed in 2.4. For the remaining case 4.1(i), we temporarily choose Haar measures μ_{u}, μ_ℓ on N_{u}, N_ℓ satisfying

$$\mu_{\mathrm{u}}(K \cap N_{\mathrm{u}}) = \mu_\ell(K \cap N_\ell) = 1, \tag{4.2.1}$$

where (K, τ) is the cover provided by 4.1. As in (1.1.2), we have a function $\Phi_F(\sigma_1, q^{-s_1}, q^{-s_2})$ such that

$$\Phi_F(\sigma_1, q^{-s_1}, q^{-s_2})\, 1 = A_{\mathrm{u}\ell}(\sigma_1[s_1] \otimes \sigma_1[s_2], \mu_\ell) \circ A_{\ell\mathrm{u}}(\sigma_1[s_1] \otimes \sigma_1[s_2], \mu_{\mathrm{u}}). \tag{4.2.2}$$

To compute this function, we need some fundamental concepts from [5]. First, we have to describe the maximal simple type (J_1, λ_1) appearing in the representation σ_1. Note that, since σ_2 is an unramified twist of σ_1, we can take $J_2 = J_1$, $\lambda_2 = \lambda_1$. Suppose first that σ_1 is not of level zero, i.e., does not admit a fixed vector for the first principal congruence subgroup of a maximal compact subgroup of G_1. There is then a simple stratum $[\mathfrak{A}_1, n_1, 0, \beta]$ in A_1 underlying λ_1. If σ_1 has level zero, we set $\beta = 0$, $\mathfrak{A}_1 = \mathbb{M}_{n_1}(\mathfrak{o})$. In either case, we write E for the field $F[\beta]$.

We next define an integer f by $[E{:}F]f = n_1$; there exists an unramified field extension $\mathbf{k}/E$ inside A_1, of degree f, such that $\mathbf{k}^\times$ normalizes $\mathfrak{A}_1$. We view $\mathbf{k}^\times$ as embedded in $L = G_1 \times G_2 \subset G$ on the diagonal; we let H denote the G-centralizer of $\mathbf{k}^\times$ and $M = L \cap H$. Thus $H \cong \mathrm{GL}_2(\mathbf{k})$ and $M \cong \mathbf{k}^\times \times \mathbf{k}^\times$. Let $1_{\mathbf{k}}$ denote the trivial character of $\mathbf{k}^\times$.

The data $(H, M, 1_{\mathbf{k}})$ can be treated in exactly the same way as (G, L, σ_1) to define a function $\Phi_{\mathbf{k}}(1_{\mathbf{k}}, q_{\mathbf{k}}^{-s_1}, q_{\mathbf{k}}^{-s_2})$, where $q_{\mathbf{k}}$ is the size of the residue field of $\mathbf{k}$. In particular, $Q_{\mathrm{u}} = P_{\mathrm{u}} \cap H$, $Q_\ell = P_\ell \cap H$ are the two parabolic subgroups of H with Levi M. The type in M corresponding to (K_L, τ_L) is the trivial character of the maximal compact subgroup of M; the cover corresponding to (K, τ) is the trivial character of the standard Iwahori subgroup $\mathcal{I}$ of $\mathrm{GL}_2(\mathbf{k})$. To define $\Phi_{\mathbf{k}}$, we choose Haar measures so that $\mathcal{I} \cap N_{\mathrm{u}}$, $\mathcal{I} \cap N_\ell$ have measure 1.

We now give the important reduction step in the calculation of Φ_F.

Theorem. *Let σ_1 be an irreducible supercuspidal representation of $\mathrm{GL}_{n_1}(F)$, and define $\Phi_F(\sigma_1, q^{-s_1}, q^{-s_2})$ by (4.2.2). Using the notation above, we have*

$$\Phi_F(\sigma_1, q^{-s_1}, q^{-s_2}) = \Phi_{\mathbf{k}}(1_{\mathbf{k}}, q_{\mathbf{k}}^{-s_1}, q_{\mathbf{k}}^{-s_2}).$$

Proof. We use the results of [5], Ch. 7, so we need some information concerning the cover (K, τ) provided by Proposition 4.1 in the case $\sigma_2 = \sigma_1$. This is straightforward: in the language of [5], there is a simple type (J, λ) with $J \supset K$ such that λ is induced by τ. (Indeed, the constructions of [7] at this point coincide with those of [5], 7.1, 7.2. See [7], sections 1 and 7, for more detail. In fact, τ is the representation denoted λ_P in [5].) In particular, $\mathcal{H}(G, \tau) \cong \mathcal{H}(G, \lambda)$, and $\mathcal{H}(G, \lambda)$ is an affine Hecke algebra [5], 5.6.6.

Let $\mathcal{A}_\tau(G)$ denote the category of admissible representations of G which are generated by their τ-isotypic spaces; thus $\mathcal{A}_\tau(G)$ is the sub-category of $\mathfrak{R}^{\mathfrak{s}}(G)$ consisting of admissible representations. (We confine attention to admissible representations here simply to avoid some irrelevant technical complications later in the proof.) In particular, $\mathcal{A}_\tau(G)$ is canonically equivalent to the category $\mathcal{H}(G, \tau)\text{-Mod}_{\mathrm{f}}$ of $\mathcal{H}(G, \tau)$-modules of finite complex dimension [5], 7.5.7, or [6], 4.3 (just as in 2.2). We similarly define $\mathcal{A}_{\tau_L}(L)$. Again, this is equivalent to $\mathcal{H}(L, \tau_L)\text{-Mod}_{\mathrm{f}}$.

Here, the normalized induction functor $\boldsymbol{\iota}_{\mathrm{u}}$ takes $\mathcal{A}_{\tau_L}(L)$ to $\mathcal{A}_\tau(G)$. Let $\mathbf{j}_{\mathrm{u}} : \mathcal{H}(L, \tau_L) \to \mathcal{H}(G, \tau)$ be the algebra homomorphism which realizes $\boldsymbol{\iota}_{\mathrm{u}}$ (see 2.3).

There is an exactly parallel set-up for the data $(H, M, \mathbf{1_k})$. We again denote by $\mathbf{j}_{\mathrm{u}}$ the algebra map $\mathcal{H}(M, 1_{\mathcal{I}\cap M}) \to \mathcal{H}(H, 1_{\mathcal{I}})$ which realizes the induction functor $\boldsymbol{\iota}_{\mathrm{u}} = \boldsymbol{\iota}_{Q_{\mathrm{u}}}^H$ from $\mathcal{A}_{1_{M\cap\mathcal{I}}}(M)$ to $\mathcal{A}_{1_{\mathcal{I}}}(H)$.

We similarly define maps $\mathbf{j}_\ell$ in the two cases.

There is a unique algebra isomorphism

$$\Psi_L : \mathcal{H}(M, 1_{M\cap\mathcal{I}}) \xrightarrow{\approx} \mathcal{H}(L, \tau_L)$$

such that, under the corresponding equivalence $\Psi_L^* : \mathcal{A}_{\tau_L}(L) \cong \mathcal{A}_{1_{M\cap\mathcal{I}}}(M)$, the representation $\sigma\chi$ corresponds to $\chi \mid M$ (*cf.* [5], 7.5.12). Now we appeal to [5], 7.6.20: there is a canonical algebra isomorphism

$$\Psi_G : \mathcal{H}(H, 1_{\mathcal{I}}) \xrightarrow{\approx} \mathcal{H}(G, \tau)$$

such that the diagram

$$\begin{array}{ccc} \mathcal{H}(M, 1_{M\cap\mathcal{I}}) & \xrightarrow{\Psi_L} & \mathcal{H}(L, \tau_L) \\ {\scriptstyle \mathbf{j}_{\mathrm{u}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{j}_{\mathrm{u}}} \\ \mathcal{H}(H, 1_{\mathcal{I}}) & \xrightarrow[\Psi_G]{} & \mathcal{H}(G, \tau) \end{array} \tag{4.2.3}$$

commutes. (The map Ψ_G is determined uniquely by this and certain subsidiary properties, recalled in the proof of the next lemma below.) Put another way, Ψ_G gives a natural isomorphism

$$\boldsymbol{\iota}_{\mathrm{u}}(\Psi_L^*(\pi)) \cong \Psi_G^*(\boldsymbol{\iota}_{\mathrm{u}}(\pi)),$$

for $\pi \in \mathcal{A}_{\tau_L}(L)$, where Ψ_G^* is the equivalence $\mathcal{A}_\tau(G) \cong \mathcal{A}_{1_{\mathcal{I}}}(H)$ induced by Ψ_G. Likewise, we have a natural isomorphism

$$\boldsymbol{\rho}_{\mathrm{u}}(\Psi_G^*(\eta)) \cong \Psi_L^*(\boldsymbol{\rho}_{\mathrm{u}}(\eta)), \quad \eta \in \mathcal{A}_\tau(G). \tag{4.2.4}$$

However, by 3.4, the functor $\boldsymbol{\rho}_{\mathrm{u}}$ has a co-adjoint, namely the induction functor $\boldsymbol{\iota}_\ell$. A co-adjoint is uniquely determined up to natural equivalence, so we have a natural isomorphism

$$\boldsymbol{\iota}_\ell(\Psi_L^*(\pi)) \cong \Psi_G^*(\boldsymbol{\iota}_\ell(\pi)), \quad \pi \in \mathcal{A}_{\tau_L}(L), \tag{4.2.5}$$

and a similar relation involving the Jacquet functor $\boldsymbol{\rho}_\ell$.

Lemma. *The diagram*

$$\begin{array}{ccc} \mathcal{H}(M, 1_{M\cap\mathcal{I}}) & \xrightarrow{\Psi_L} & \mathcal{H}(L, \tau_L) \\ {\scriptstyle \mathbf{j}_\ell}\downarrow & & \downarrow{\scriptstyle \mathbf{j}_\ell} \\ \mathcal{H}(H, 1_{\mathcal{I}}) & \xrightarrow[\Psi_G]{} & \mathcal{H}(G, \tau) \end{array}$$

commutes.

Proof. The algebra homomorphisms Ψ_L, Ψ_G preserve support of functions in the following sense. For example, take $f \in \mathcal{H}(H, 1_{\mathcal{I}})$; the function $\Psi_G(f)$ then has support $K \cdot \mathrm{supp}(f) \cdot K$. (This property, the choice of Ψ_L and the diagram (4.2.3) together determine Ψ_G uniquely.)

The maps $\mathbf{j}_\ell$ have a similar property; to explain this, take the case of the map $\mathbf{j}_\ell$ attached to G. The algebra $\mathcal{H}(L, \tau_L)$ has a sub-algebra $\mathcal{H}_\ell(L, \tau_L)$ of functions supported on elements of L which are "$K \cap N_\ell$-positive", in the sense of [6], §6. On these functions, $\mathbf{j}_\ell$ is given by an explicit formula and, for $f \in \mathcal{H}_\ell(L, \tau_L)$, $\mathbf{j}_\ell f$ has support $K \cdot \mathrm{supp}\, f \cdot K$. (See [6], §7 for details.)

Thus, if we take $f \in \mathcal{H}_\ell(M, 1_{\mathcal{I}\cap M})$, the functions $\mathbf{j}_\ell \Psi_L(f)$, $\Psi_G \mathbf{j}_\ell(f)$ both have support $K \cdot \mathrm{supp}(f) \cdot K$. A (K, K)-double coset in G supports at most a one-dimensional space of functions in $\mathcal{H}(G, \tau)$ ([5], 5.6.6) and the other three algebras in the picture have the analogous property. If this function f is supported on, say $m(M \cap \mathcal{I})$, we thus have a non-zero scalar $\phi(m)$ such that

$$\mathbf{j}_\ell \Psi_L(f) = \phi(m)\, \Psi_G \mathbf{j}_\ell(f).$$

The quantity $\phi(m)$ depends only on $m(M \cap \mathcal{I})$; by [6], 7.1, ϕ is multiplicative in the positive element m. In other words, ϕ extends (uniquely) to an unramified quasicharacter of M.

Given an unramified quasicharacter χ of M and $f \in \mathcal{H}(M, 1_{\mathcal{I}\cap M})$, define $f\chi$ by $f\chi : m \mapsto f(m)\chi(m)$. We have just shown that

$$\mathbf{j}_\ell \Psi_L(f) = \Psi_G \mathbf{j}_\ell(f\phi).$$

The relation (4.2.5) then forces $\phi = 1$.

By 3.5, we now have

$$\mathbf{e}_{\chi|M} = \Psi_L^*(\mathbf{e}_{\sigma\chi}), \quad \chi \in \mathrm{X}(L).$$

By Proposition 1.3, this gives us

$$A_{\ell\mathrm{u}}(\chi \mid M) = \Psi_G^*(A_{\ell\mathrm{u}}(\sigma\chi)).$$

However, by the lemma, we can interchange the roles of P_{u} and P_ℓ without changing Ψ_G; therefore

$$A_{\mathrm{u}\ell}(\chi \mid M) = \Psi_G^*(A_{\mathrm{u}\ell}(\sigma\chi)).$$

The theorem now follows.

5. The conductor-volume relation

We continue in the situation of §4. Thus $G = \mathrm{GL}_n(F)$ and L is a maximal proper Levi subgroup of G: $L = G_1 \times G_2$, $G_i = \mathrm{GL}_{n_i}(F)$. The parabolic subgroups P_{u}, P_ℓ are as before.

We take a non-trivial character ψ of the additive group of F with conductor $c(\psi)$: we recall that, by definition, $c(\psi)$ is the largest integer c such that $\mathfrak{p}^{-c} \subset \operatorname{Ker}\psi$.

5.1. *There is a Haar measure $\tilde\mu_L^\psi = \tilde\mu_{\mathrm{u}}^\psi \otimes \tilde\mu_\ell^\psi$ on $N_{\mathrm{u}} \times N_\ell$ with the following property. For $i = 1, 2$, let π_i be an irreducible, generic, unitary representation of G_i. Then*

$$A_{\mathrm{u}\ell}\big(\pi_1\left[\tfrac{s}{2}\right] \otimes \pi_2\left[\tfrac{-s}{2}\right], \tilde\mu_\ell^\psi\big) \circ A_{\ell\mathrm{u}}\big(\pi_1\left[\tfrac{-s}{2}\right] \otimes \pi_2\left[\tfrac{s}{2}\right], \tilde\mu_{\mathrm{u}}^\psi\big) = q^{-f(\check\pi_1 \times \pi_2, \psi)}\, \frac{L(\pi_1 \times \check\pi_2, s) L(\check\pi_1 \times \pi_2, -s)}{L(\pi_1 \times \check\pi_2, 1+s) L(\check\pi_1 \times \pi_2, 1-s)}\, 1.$$

For this, see [18], Introduction and Theorem 6.1 (or the more general and systematic account in [19]). For the notion of genericity, see [15].

Comments. If we replace $\tilde\mu_L^\psi$ by $k\,\tilde\mu_L^\psi$, for some $k > 0$, the left hand side of 5.1 gets multiplied by k. On the other hand, we have

$$f(\check\pi_1 \times \pi_2, \psi) = f(\check\pi_1 \times \pi_2) + n_1 n_2 c(\psi),$$

where $f(\check\pi_1 \times \pi_2)$ is independent of ψ. Thus, if we change ψ to ψ', the right hand side of 5.1 gets multiplied by $q^{n_1 n_2 (c(\psi') - c(\psi))}$. It follows that $\tilde\mu_L^\psi$ depends only on L and $c(\psi)$, and 5.1 is essentially independent of the choice of ψ.

5.2. The next step is to make the choice of measures more explicit. Let us write $N_{\mathrm{u}} = 1 + \mathfrak{n}_{\mathrm{u}}$, $N_\ell = 1 + \mathfrak{n}_\ell$, for vector spaces $\mathfrak{n}_{\mathrm{u}}$, $\mathfrak{n}_\ell$. In fact, $\mathfrak{n}_{\mathrm{u}} \cong N_{\mathrm{u}}$ as topological groups, and likewise for N_ℓ. The pairing

$$\begin{aligned} \mathfrak{n}_{\mathrm{u}} \times \mathfrak{n}_\ell &\longrightarrow \mathbb{C}, \\ (x, y) &\longmapsto \psi\left(\operatorname{tr}(xy)\right), \end{aligned}$$

identifies $\mathfrak{n}_\ell$ with the Pontrjagin dual $\hat{\mathfrak{n}}_{\mathrm{u}}$ of $\mathfrak{n}_{\mathrm{u}}$, so we have an identification of $\widehat{N}_{\mathrm{u}}$ with N_ℓ. We choose a Haar measure μ_{u}^ψ on N_{u} at random, and set

$$\mu_L^\psi = \mu_{\mathrm{u}}^\psi \otimes \hat\mu_{\mathrm{u}}^\psi, \tag{5.2.1}$$

where $\hat\mu_{\mathrm{u}}^\psi$ is the measure on N_ℓ dual to μ_{u}^ψ. This is independent of the original choice of μ_{u}^ψ.

5.3. We can now establish the first of our desired relations.

Theorem. (i) *We have $\mu_L^\psi = \tilde\mu_L^\psi$, for all non-trivial characters ψ of F.*

(ii) *For $i = 1, 2$, let σ_i be an irreducible supercuspidal representation of $G_i = \mathrm{GL}_{n_i}(F)$. In the case $n_1 = n_2$, assume that $\sigma_2 \not\cong \sigma_1[s]$ for any $s \in \mathbb{C}$. Let (K, τ) be as in* 4.1. *Then*

$$\mu_L^\psi(K \cap N_{\mathrm{u}} \times K \cap N_\ell) = q^{-f(\check\sigma_1 \times \sigma_2, \psi)}. \tag{5.3.1}$$

Comment. The measure appearing in (5.3.1) admits an illuminating interpretation. The G-cover (K, τ) of (K_L, τ_L) is attached to a hereditary $\mathfrak{o}$-order $\mathfrak{A}$ in $\mathbb{M}_n(F)$; in the notation of 4.1, this is the order defined by the direct sum ([7], 2.8) of the lattice chains underlying the factors (J_i, λ_i) of (K_L, τ_L). The unit group of this order admits a decomposition

$$\mathbf{U}(\mathfrak{A}) = \mathbf{U}(\mathfrak{A}) \cap N_\ell \cdot \mathbf{U}(\mathfrak{A}) \cap L \cdot \mathbf{U}(\mathfrak{A}) \cap N_{\mathrm{u}}.$$

If we take $c(\psi) = -1$, we get $\mu_L^\psi\,(\mathbf{U}(\mathfrak{A}) \cap N_\ell \times \mathbf{U}(\mathfrak{A}) \cap N_{\mathrm{u}}) = 1$, so

$$q^{f(\check{\sigma}_1 \times \sigma_2, \psi)} = (\mathbf{U}(\mathfrak{A}) \cap N_\ell : K \cap N_\ell) \cdot (\mathbf{U}(\mathfrak{A}) \cap N_{\mathrm{u}} : K \cap N_{\mathrm{u}}).$$

We now prove Theorem 5.3. First, we define $a(L, \psi) > 0$ by

$$\tilde{\mu}_L^\psi = a(L, \psi)\, \mu_L^\psi.$$

Next, twisting the σ_i by unramified quasicharacters has no effect on the conductor, so we may assume that the σ_i are both unitary. They are certainly generic [9], so we can use 5.1.

Under the hypotheses of 5.3(ii), we have

$$L(\check{\sigma}_1 \times \sigma_2, s) = L(\sigma_1 \times \check{\sigma}_2, s) = 1,$$

by [14], Proposition 8.1. Combining 5.1 with Propositions 4.1, 2.4, we get

$$\mu_L^\psi(K \cap N_{\mathrm{u}} \times K \cap N_\ell) = a(L, \psi)\, q^{-f(\check{\sigma}_1 \times \sigma_2, \psi)}. \tag{5.3.2}$$

We calculate the factor $a(L, \psi)$ relating the measures μ_L^ψ, $\tilde{\mu}_L^\psi$, via a rather different choice of representation σ. Initially, we let σ_1 be any irreducible unitary supercuspidal representation of G_1 except that, when $n_1 = 1$, we insist that σ_1, which is a quasicharacter of $F^\times$, is not unramified. For σ_2, we take the Steinberg representation $\mathrm{St}(n_2)$ of G_2. The representation σ admits a type (K_L, τ_L) as follows. We take the maximal simple type (J_1, λ_1) occurring in σ_1 (as in 4.1), and let $\mathcal{I}$ denote the standard Iwahori subgroup of G_2. We put $K_L = J_1 \times \mathcal{I}$, $\tau_L = \lambda_1 \otimes 1_{\mathcal{I}}$, where $1_{\mathcal{I}}$ is the trivial character of $\mathcal{I}$. This admits a split G-cover (K, τ). Assumption 1.1 is therefore valid in this case and the relation (5.3.2) holds in this situation, with the same value of $a(L, \psi)$.

The object of this argument is to prove that $a(L, \psi) = 1$, so we can make a convenient choice of σ_1. Consider first the case where $n_1 > 1$. Here, we assume that σ_1 is *of level zero;* this means it has a fixed vector for the first principal congruence subgroup of $\mathrm{GL}_{n_1}(\mathfrak{o})$. We then have [2]

$$f(\check{\sigma}_1, \psi) = f(\check{\sigma}_1 \times \mathbf{1}_F, \psi) = n_1(1 + c(\psi)).$$

However,

$$f(\check{\sigma}_1 \times \mathrm{St}(n_2), \psi) = n_2 f(\check{\sigma}_1 \times \mathbf{1}_F, \psi),$$

by [14], Theorems 3.1 and 8.2. On the other hand, the constructions in [7] give a group K of the form

$$K = \begin{pmatrix} \mathrm{GL}_{n_1}(\mathfrak{o}) & \mathfrak{o} \\ \mathfrak{p} & \mathcal{I} \end{pmatrix}.$$

Substituting in 5.3.2, we get $a(L, \psi) = 1$, as required.

In the case $n_1 = 1$, we simply insist that σ_1 is not unramified. The argument is straightforward and similar to the first case, so we omit the details. (Observe that the first argument is also valid when $n_1 = 1$ and $q \geqslant 3$: we can then take for σ_1 a tamely ramified quasicharacter which is not unramified.)

This proves both assertions of the theorem.

5.4. We now turn to the other case of 4.1, where σ_2 is an unramified twist of σ_1. Since twisting by unramified quasicharacters has no effect on the conductor, we may as well take $\sigma_2 = \sigma_1$.

Theorem. *Let σ_1 be an irreducible supercuspidal representation of $\mathrm{GL}_{n_1}(F)$. Let $t(\sigma_1)$ denote the number of unramified characters χ of $F^\times$ such that $\sigma_1 \otimes \chi \circ \det \cong \sigma_1$. Let (K, τ) be given by 4.1 (with $\sigma_2 = \sigma_1$), and define μ_L^ψ by (5.2.1). Then*

$$\mu_L^\psi(K \cap N_{\mathrm{u}} \times K \cap N_\ell) = q^{-(t(\sigma_1) + f(\check{\sigma}_1 \times \sigma_1, \psi))}.$$

Proof. If the assertion holds for one choice of ψ, it holds for all. To simplify the book-keeping, we take $c(\psi) = 0$. It will be convenient to start with the special case $n_1 = n_2 = 1$ and $\sigma_1 = 1_F$, the trivial character of $F^\times$. In this case, K is the standard Iwahori subgroup of $G = \mathrm{GL}_2(F)$. Thus

$$\mu_L^\psi(K \cap N_{\mathrm{u}} \times K \cap N_\ell) = q^{-1}. \tag{5.4.1}$$

For unramified quasicharacters χ_1, χ_2 of $F^\times$, we have

$$f(\chi_1 \times \chi_2) = f(\chi_1\chi_2 \times 1_F) = f(\chi_1\chi_2) = 0,$$

where the third f is the normalized Godement-Jacquet conductor. Thus 5.1 and Theorem 5.3(i) give

$$\begin{aligned} A_{\mathrm{u}\ell}(1_F \left[\tfrac{s}{2}\right] \otimes 1_F \left[\tfrac{-s}{2}\right], \tilde{\mu}_\ell^\psi) &\circ A_{\ell\mathrm{u}}(1_F \left[\tfrac{s}{2}\right] \otimes 1_F \left[\tfrac{-s}{2}\right], \tilde{\mu}_{\mathrm{u}}^\psi) \\ &= \frac{\zeta_F(s)\zeta_F(-s)}{\zeta_F(1+s)\zeta_F(1-s)} \, 1, \end{aligned}$$

where $\zeta_F(s) = (1 - q^{-s})^{-1}$. Now we return to the notation of 4.2; using $\mathbf{k}$ as base field in place of F, the special case just done gives

$$\Phi_{\mathbf{k}}(1_{\mathbf{k}}, q_{\mathbf{k}}^{-s/2}, q_{\mathbf{k}}^{s/2}) = q_{\mathbf{k}} \frac{\zeta_{\mathbf{k}}(s)\zeta_{\mathbf{k}}(-s)}{\zeta_{\mathbf{k}}(1+s)\zeta_{\mathbf{k}}(1-s)}.$$

However, by [14], Proposition 8.1, and [5], 6.2.5, we have $L(\check{\sigma}_1 \times \sigma_1, s) = \zeta_{\mathbf{k}}(s)$ and $q_{\mathbf{k}} = q^{t(\sigma_1)}$. Now we can apply Theorem 4.2 to 5.1 to get the result.

6. Explicit conductor formulæ

We proceed to compute the volumes occurring in the main theorems of §5. This is done in terms of the description of supercuspidal representations in [5]. We must therefore start by recalling a little of this. If E/F is a finite field extension, we write $\mathfrak{o}_E$ for the discrete valuation ring in E, $\mathfrak{p}_E$ for the maximal ideal of $\mathfrak{o}_E$, etc.

6.1. Let σ be an irreducible supercuspidal representation of $G = \mathrm{GL}_n(F)$. Via the description [5] of σ as an induced representation, we attach to σ a principal $\mathfrak{o}$-order $\mathfrak{A}$ in $A = \mathbb{M}_n(F)$; an integer $m \geqslant 0$ (called the *level of* σ); and an element $\beta \in A$ such that $E = F[\beta]$ is a field with $E^\times$ normalizing $\mathfrak{A}$.

First, suppose that σ admits a fixed vector for the group $1 + \mathfrak{p}\mathbb{M}_n(\mathfrak{o})$. In this case, we set

$$\mathfrak{A} = \mathbb{M}_n(\mathfrak{o}), \quad m = 0, \quad \beta = 0.$$

Otherwise, there is a maximal simple type occurring in σ and given by a simple stratum $[\mathfrak{A}, m, 0, \beta]$ in A [5], 8.4.1. This defines the required objects.

In all cases, e denotes the $\mathfrak{o}$-period of $\mathfrak{A}$ or, equivalently, the ramification index $e(E|F)$. Let $t(\sigma)$ denote the number of unramified quasicharacters χ of $F^\times$ such that $\sigma \otimes \chi \circ \det \cong \sigma$; we have [5], 6.2.5,

$$t(\sigma) = n/e. \tag{6.1.1}$$

We also recall [2] that the Godement-Jacquet conductor of σ is given by

$$f(\sigma,\psi) = \begin{cases} c(\psi) & \text{if } n = 1 \text{ and } \sigma \text{ is unramified,} \\ n(1 + c(\psi) + m/e) & \text{otherwise.} \end{cases} \tag{6.1.2}$$

Next, suppose that σ does not have level zero. In this case, the simple stratum $[\mathfrak{A}, m, 0, \beta]$ has a *characteristic polynomial* $\phi_0(X) \in \mathsf{k}[X]$ [5], p. 58, where $\mathsf{k} = \mathfrak{o}/\mathfrak{p}$. This is only defined relative to a choice of prime element of F: we assume this has been made once and for all. The polynomial $\phi_0(X)$ is a power of a monic irreducible polynomial $\phi_\sigma(X) \in \mathsf{k}[X]$, $\phi_\sigma(X) \neq X$, and ϕ_σ is an invariant of σ.

6.2. For $i = 1, 2$, let n_i be a positive integer and set $G_i = \mathrm{GL}_{n_i}(F)$. Let σ_i be an irreducible supercuspidal representation of G_i. We use the notation of 6.1, appending subscripts i as necessary.

We define integers e, m by

$$\begin{aligned} e &= \mathrm{lcm}\,(e_1, e_2), \\ m/e &= \max\,(m_1/e_1, m_2/e_2). \end{aligned} \tag{6.2.1}$$

We set $n = n_1 + n_2$, $G = \mathrm{GL}_n(F)$, $A = \mathbb{M}_n(F)$. We identify $G_1 \times G_2$ with the obvious Levi subgroup L of G. As before, $P_\mathrm{u} = LN_\mathrm{u}$, $P_\ell = LN_\ell$ are the parabolic subgroups of G with Levi component L. It will be convenient to regard L as the stabilizer of a decomposition of F^n of the form $F^n = V_1 \oplus V_2$.

Suppose first that the σ_i both have level zero. We say that the σ_i *are completely distinct* if one is not an unramified twist of the other.

On the other hand, suppose that σ_1, say, does not have level zero. We then say that the σ_i *are completely distinct* if either $m_1/e_1 \neq m_2/e_2$ or $m_1/e_1 = m_2/e_2$ but $\phi_{\sigma_1} \neq \phi_{\sigma_2}$.

Observe that, in any case, σ_2 cannot be an unramified twist of σ_1 when the σ_i are completely distinct.

6.3. We need some more notation in the case where the σ_i are not completely distinct, but do not have level zero. In the language of [7], §8, this means that the σ_i admit a *common approximation.* There is no need to recall the full definition of this concept, only enough to compute conductors.

First, the order $\mathfrak{A}_i$ is determined by a lattice chain $\mathcal{L}_i$ in V_i. We can form the direct sum $\Lambda = \mathcal{L}_1 \oplus \mathcal{L}_2$, in the sense of [7], 2.8. This is a lattice *sequence* in F^n, $n = n_1 + n_2$. In particular, Λ is a function $i \mapsto \Lambda(i)$ from $\mathbb{Z}$ to the set of $\mathfrak{o}$-lattices in F^n with the periodicity property $\Lambda(i+e) = \Lambda(i)$, where $e = \mathrm{lcm}(e_1, e_2)$ as above. In this case, we have $m_1/e_1 = m_2/e_2$, so $m = m_i e/e_i$.

A *common approximation* to the σ_i is then an object $([\Lambda, m, 0, \gamma], l, \vartheta)$ where l is an integer, $0 \leqslant l < m$, and γ is an element of the Levi L generating a field over F which stabilizes the lattice sequence Λ. The entry ϑ is a certain character of a compact group $H^{l+1}(\gamma, \Lambda)$ attached to the other data and which occurs in $\boldsymbol{\iota}_\ell(\sigma_1 \otimes \sigma_2)$.

We will only ever consider common approximations which are *best possible* in the following sense: if $([\Lambda, m, 0, \gamma'], l', \vartheta')$ is another common approximation, we have

$l \leqslant l'$. In this case, we can further assume that the stratum $[\Lambda, m, l, \gamma]$ is simple [7], 5.1. This means either $\gamma \in F$ or the integer $r = -k_0(\gamma, \Lambda)$ *loc. cit.* satisfies $l < r \leqslant m$. The quantities l and γ, which are the only ones to concern us, can then be specified explicitly in terms of the inducing data for the σ_i; for convenience, we have summarized this process in 6.15 below.

6.4. We need a sort of "generalized discriminant". We start with an elementary observation. Let $U \subset V$ be finite-dimensional F-vector spaces, and suppose we have linear maps $f : V \to V$, $g : V \to U$ such that the sequence

$$0 \longrightarrow U \longrightarrow V \xrightarrow{f} V \xrightarrow{g} U \longrightarrow 0$$

is exact. There is then a positive constant $C(f, g)$, depending only on the maps f and g, such that, for any $\mathfrak{o}$-lattices $\mathfrak{u}, \mathfrak{u}'$ in U, $\mathfrak{v}, \mathfrak{v}'$ in V, fitting into an exact sequence

$$0 \longrightarrow \mathfrak{u} \longrightarrow \mathfrak{v} \xrightarrow{f} \mathfrak{v}' \xrightarrow{g} \mathfrak{u}' \longrightarrow 0,$$

we have

$$\frac{\mu_U(\mathfrak{u})\,\mu_V(\mathfrak{v}')}{\mu_V(\mathfrak{v})\,\mu_U(\mathfrak{u}')} = C(f, g),$$

where μ_U, μ_V are Haar measures on U, V respectively. (See [5], 5.1.3, for a more general result of this kind.) The quantity $C(f, g)$ is clearly independent of the choice of measures.

We now assume given a finite field extension E/F and an element $\gamma \in E^\times$ such that $E = F[\gamma]$. Write $A(E) = \mathrm{End}_F(E)$. Let $s_\gamma : A(E) \to E$ be a tame co-restriction [5], 1.3, and write a_γ for the adjoint map $x \mapsto \gamma x - x\gamma$, $x \in A(E)$. We have an exact sequence

$$0 \longrightarrow E \longrightarrow A(E) \xrightarrow{a_\gamma} A(E) \xrightarrow{s_\gamma} E \longrightarrow 0.$$

We set

$$C(\gamma) = C(a_\gamma, s_\gamma) = q^{\mathfrak{c}(\gamma)}, \tag{6.4.1}$$

for some integer $\mathfrak{c}(\gamma)$. This number has an interesting arithmetic interpretation, which we give below in 6.13. At the moment, we simply need a rather coarse estimate.

Lemma. *With the notation above, suppose that $E \neq F$, that $n = -\nu_E(\gamma) > 0$, and that the stratum $[\mathfrak{A}(E), n, 0, \gamma]$ is simple, where $\mathfrak{A}(E)$ is the unique hereditary $\mathfrak{o}$-order in $A(E)$ normalized by $E^\times$. Then $\mathfrak{c}(\gamma) > 0$.*

Proof. We apply the above machinery to the exact sequence of [5], 3.1.16, setting the parameter m there equal to 0. This gives us an exact sequence

$$0 \longrightarrow \mathfrak{p}_E \longrightarrow \mathfrak{J}^1 \xrightarrow{a_\gamma} \left(\mathfrak{H}^1\right)^* \xrightarrow{s_\gamma} \mathfrak{o}_E \longrightarrow 0,$$

for $\mathfrak{o}$-lattices $\mathfrak{J}^1$, $(\mathfrak{H}^1)^*$ in $A(E)$ (whose definitions need not detain us) satisfying

$$\mathfrak{J}^1 \subset \mathfrak{P}(E) \subset \mathfrak{A}(E) \subset (\mathfrak{H}^1)^*,$$

where $\mathfrak{P}(E)$ is the radical of $\mathfrak{A}(E)$. Thus

$$\left((\mathfrak{H}^1)^* : \mathfrak{J}^1\right) \geqslant (\mathfrak{A}(E) : \mathfrak{P}(E)) > (\mathfrak{o}_E : \mathfrak{p}_E),$$

as required for the lemma.

In the case $\gamma \in F$, we of course get $\mathfrak{c}(\gamma) = 0$.

6.5. We can now give the desired conductor formulæ.

Theorem. *For $i = 1, 2$, let σ_i be an irreducible supercuspidal representation of* $\mathrm{GL}_{n_i}(F)$*. Define quantities m_i, e_i, β_i as in* 6.2*, and use the notation* (6.2.1), (6.4.1).

(i) *Suppose that $n_1 = n_2$ and $\sigma_2 \cong \sigma_1[s]$, for some $s \in \mathbb{C}$. Put $d = [F[\beta_1]:F]$. Then*

$$f(\check{\sigma}_1 \times \sigma_2) = n_1^2 \left(1 + \frac{\mathfrak{c}(\beta_1)}{d^2}\right) - \frac{n_1}{e_1}.$$

In particular, $f(\check{\sigma}_1 \times \sigma_2) \geqslant n_1^2 - n_1$, with equality if the σ_i have level zero. In the case $d > 1$, we also have

$$n_1^2 - n_1 < f(\check{\sigma}_1 \times \sigma_2) < n_1^2 \left(1 + \frac{m_1}{e_1}\right) - \frac{n_1}{e_1}.$$

(ii) *Suppose that the σ_i are completely distinct. Then*

$$f(\check{\sigma}_1 \times \sigma_2) = n_1 n_2 \left(1 + \frac{m}{e}\right).$$

In particular, $f(\check{\sigma}_1 \times \sigma_2) \geqslant n_1 n_2$, with equality if and only if the σ_i have level zero.

(iii) *Suppose that σ_2 is not equivalent to an unramified twist of σ_1 but that the σ_i are not completely distinct. Let $([\mathit{\Lambda}, m, 0, \gamma], l, \vartheta)$ be a best common approximation to the σ_i, and assume that the stratum $[\mathit{\Lambda}, m, l, \gamma]$ is simple. Put $d = [F[\gamma]:F]$. Then*

$$f(\check{\sigma}_1 \times \sigma_2) = n_1 n_2 \left(1 + \frac{\mathfrak{c}(\gamma)}{d^2} + \frac{l}{de}\right) < n_1 n_2 \left(1 + \frac{m}{e}\right).$$

If either $d > 1$ or $l > 0$, we also have

$$n_1 n_2 < f(\check{\sigma}_1 \times \sigma_2).$$

We remark that the conductor bounds given in the Introduction can be obtained from those of 6.5 via (6.1.2) and (6.2.1).

Corollary. *For $i = 1, 2$, let π_i be an irreducible smooth representation of the group* $\mathrm{GL}_{n_i}(F)$*. Then $f(\pi_1 \times \pi_2) \geqslant 0$. Indeed, the following conditions are equivalent:*

(a) $f(\pi_1 \times \pi_2) = 0$*;*

(b) *there is a quasicharacter χ of $F^\times$ such that both $\pi_1 \otimes \chi \circ \det$, $\pi_2 \otimes \chi^{-1} \circ \det$ are unramified principal series.*

When the π_i are supercuspidal, the corollary follows directly from the theorem; to get the general case, one uses the addition formulæ of [14], namely Theorems 3.1, 8.2 and 9.5, together with the observation

$$f\big((\pi_1 \otimes \chi \circ \det) \times (\pi_2 \otimes \chi^{-1} \circ \det)\big) = f(\pi_1 \times \pi_2). \tag{6.5.1}$$

The proof of the theorem will occupy most of the remainder of the section. Throughout, we work relative to a character ψ with $c(\psi) = -1$, since this gives the briefest form. Explicitly,

$$f(\check{\sigma}_1 \times \sigma_2, \psi) = f(\check{\sigma}_1 \times \sigma_2) - n_1 n_2.$$

We choose Haar measures μ_{u} on N_{u} and μ_ℓ on N_ℓ satisfying (5.2.1) relative to our choice of ψ. We take (K, τ) as in 4.1; we have to compute $\mu_{\mathrm{u}}(K \cap N_{\mathrm{u}})\mu_\ell(K \cap N_\ell)$. We use the other notation introduced in 6.1–6.4.

6.6. We start by dealing with the case where both σ_i *have level zero.* There is a maximal simple type in σ_i of the form (K_i, τ_i), where $K_i = \mathrm{GL}_{n_i}(\mathfrak{o})$ and τ_i is the inflation of an irreducible cuspidal representation of $\mathrm{GL}_{n_i}(\mathfrak{o}/\mathfrak{p})$. The group K underlying the semisimple type (K,τ) then has the form $K = \mathbf{U}(\mathfrak{A})$, where $\mathfrak{A}$ is the hereditary order of block matrices

$$\mathfrak{A} = \begin{pmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} \end{pmatrix},$$

in which the (i,j)-block has dimensions $n_i \times n_j$ [7], §7. From this, we get immediately that

$$\mu_{\mathrm{u}}(K \cap N_{\mathrm{u}})\, \mu_\ell(K \cap N_\ell) = 1.$$

In case (ii) of 6.5, Theorem 5.3(ii) gives us $f(\check{\sigma}_1 \times \sigma_2, \psi) = 0$; in case (i), we get $f(\check{\sigma}_1 \times \sigma_2, \psi) = -n_1$ from Theorem 5.4, as required.

6.7. We now prove the formula in 6.5(ii). Thus we assume that the σ_i are completely distinct. We can exclude the case where both have level zero, since it has been dealt with in 6.6.

We need the lattice sequence Λ introduced in 6.2. For $j \in \mathbb{Z}$, we write

$$\mathfrak{a}_j(\Lambda) = \{x \in A : x\Lambda(i) \subset \Lambda(i{+}j),\ i \in \mathbb{Z}\},$$

consistent with the notation of [7].

We assume, without loss, that $m_1/e_1 \geqslant m_2/e_2$. The construction of K in [7], §8, gives

$$K \cap N_{\mathrm{u}} = 1 + \mathfrak{a}_0(\Lambda) \cap \mathfrak{n}_{\mathrm{u}}, \qquad K \cap N_\ell = 1 + \mathfrak{a}_{m+1}(\Lambda) \cap \mathfrak{n}_\ell. \tag{6.7.1}$$

Now let us compute the product of volumes of the groups in (6.7.1). The orthogonal complement $(\mathfrak{a}_0 \cap \mathfrak{n}_{\mathrm{u}})^\perp$ of $\mathfrak{a}_0 \cap \mathfrak{n}_{\mathrm{u}}$ is $\mathfrak{a}_1 \cap \mathfrak{n}_\ell$, by [7], 2.10. Thus

$$\mu_{\mathrm{u}}(\mathfrak{a}_0 \cap \mathfrak{n}_{\mathrm{u}})\, \mu_\ell(\mathfrak{a}_1 \cap \mathfrak{n}_\ell) = 1.$$

We now observe that $\mathfrak{a}_{m+1} \cap \mathfrak{n}_\ell = \beta_1^{-1}\mathfrak{a}_1 \cap \mathfrak{n}_\ell$ (see [7], 3.7). The product of volumes we seek is therefore

$$(\mathfrak{a}_1 \cap \mathfrak{n}_\ell : \beta_1^{-1}\mathfrak{a}_1 \cap \mathfrak{n}_\ell) = \|\det \beta_1\|^{-1},$$

where we view β_1 as acting on $\mathfrak{n}_\ell$ by right multiplication. The valuation of β_1 in $F[\beta_1]$ is $-m_1$, so the required volume is $q^{-n_1 n_2 m_1/e_1}$. In particular, $f(\check{\sigma}_1 \times \sigma_2, \psi) \geqslant n_1 n_2$, with equality if and only if the σ_i both have level zero. This completes the proof of 6.5(ii).

6.8. We assume until further notice that σ_2 is not an unramified twist of σ_1, and that the σ_i are not completely distinct. Thus we are in case (iii) of Theorem 6.5.

Let us dispose of a trivial case. Let χ be a quasicharacter of $F^\times$, and set $\sigma_i' = \sigma_i \otimes \chi \circ \det$. Since we can transfer one-dimensional twists between variables in the conductor, we get

$$f(\check{\sigma}_1' \times \sigma_2', \psi) = f(\check{\sigma}_1 \times \sigma_2, \psi).$$

Lemma. *Let* $([\Lambda, m, 0, \gamma], l, \vartheta)$ *be a best common approximation to the* σ_i*. The following conditions are equivalent:*

(i) *The element* γ *lies in* F*.*

(ii) *There are completely distinct representations σ_i' and a quasicharacter χ of $F^\times$ such that*

$$\begin{aligned} \sigma_i &= \sigma_i' \otimes \chi \circ \det, \\ f(\sigma_i', \psi) &< f(\sigma_i, \psi), \end{aligned} \qquad i = 1, 2.$$

When these conditions are satisfied, we have

$$f(\sigma_i, \psi) = n_i f(\chi, \psi),$$
$$f(\check{\sigma}_1' \times \sigma_2', \psi) = f(\check{\sigma}_1 \times \sigma_2, \psi) = n_1 n_2 l/e.$$

Proof. The equivalence of (i) and (ii) follows easily from the definition of best common approximation [7], 8.3 (or use 6.15 below). The next assertion is implied by (6.1.2).

Finally, let m_i' be the level of the representation σ_i', as in 6.1, so that $f(\sigma_i', \psi) = n_i m_i'/e_i$. Here we have

$$l/e = \max(m_1'/e_1, m_2'/e_2),$$

and the last assertion of the lemma follows from 6.7.

This deals with the $d = 1$ case in part (iii) of the theorem.

6.9. Let $([\Lambda, m, 0, \gamma], l, \vartheta)$ be a best common approximation to the σ_i. We assume $\gamma \notin F$, and $l > 0$. We now put $E = F[\gamma]$ and write B for the A-centralizer of γ; thus $B \cong \mathbb{M}_{n/d}(E)$, where $d = [E{:}F]$. We recall that e denotes the F-period of the lattice sequence Λ as in 6.2.

In this situation, we have to prove:

$$q^{f(\check{\sigma}_1 \times \sigma_2, \psi)} = C(\gamma)^{n_1 n_2/d^2}\, q^{n_1 n_2 l/de}, \tag{6.9.1}$$

where $C(\gamma)$ is defined in 6.4. To do this, we choose a tame co-restriction s_γ on A relative to E/F, and write a_γ for the adjoint map $x \mapsto \gamma x - x\gamma$, $x \in A$. We have an exact sequence

$$0 \longrightarrow B \longrightarrow A \xrightarrow{a_\gamma} A \xrightarrow{s_\gamma} B \longrightarrow 0$$

which restricts (since $\gamma \in L$) to an exact sequence

$$0 \longrightarrow B \cap \mathfrak{n}_{\mathrm{u}} \longrightarrow A \cap \mathfrak{n}_{\mathrm{u}} \xrightarrow{a_\gamma} A \cap \mathfrak{n}_{\mathrm{u}} \xrightarrow{s_\gamma} B \cap \mathfrak{n}_{\mathrm{u}} \longrightarrow 0. \tag{6.9.2}$$

Lemma. *The quantity $C(a_\gamma, s_\gamma)$ given by the exact sequence* (6.9.2) *is*

$$C(a_\gamma, s_\gamma) = C(\gamma)^{n_1 n_2/d^2}.$$

We prove this at the end of the paragraph. Take the cover (K, τ) provided by 4.1 and set

$$K \cap N_{\mathrm{u}} = 1 + \mathfrak{k}_{\mathrm{u}}, \qquad K \cap N_\ell = 1 + \mathfrak{k}_\ell,$$

for lattices $\mathfrak{k}_{\mathrm{u}}$, $\mathfrak{k}_\ell$ in $\mathfrak{n}_{\mathrm{u}}$, $\mathfrak{n}_\ell$, respectively. Also write

$$\mathfrak{b}_j = \mathfrak{b}_j(\Lambda) = \mathfrak{a}_j(\Lambda) \cap B, \qquad j \in \mathbb{Z},$$

where the lattices $\mathfrak{a}_j$ are defined in 6.7. The group K is specified in [7], 8.6 (final display). In our present notation, the Lemma of [7], 6.3, says that the lattices $\mathfrak{k}_{\mathrm{u}}$, $\mathfrak{k}_\ell$ fit into an exact sequence

$$0 \longrightarrow \mathfrak{b}_0 \cap \mathfrak{n}_{\mathrm{u}} \longrightarrow \mathfrak{k}_{\mathrm{u}} \xrightarrow{a_\gamma} \mathfrak{k}_\ell^\perp \xrightarrow{s_\gamma} \mathfrak{b}_{-l} \cap \mathfrak{n}_{\mathrm{u}} \longrightarrow 0. \tag{6.9.3}$$

Applying 5.3, we have

$$q^{-f(\check{\sigma}_1\times\sigma_2,\psi)} = \mu_{\mathrm{u}}(\mathfrak{k}_{\mathrm{u}})/\mu_{\mathrm{u}}(\mathfrak{k}_\ell^\perp).$$

The considerations of 6.4 show this is equal to

$$C(\gamma)^{-n_1n_2/d^2}\,(\mathfrak{b}_0\cap\mathfrak{n}_{\mathrm{u}} : \mathfrak{b}_{-l}\cap\mathfrak{n}_{\mathrm{u}}).$$

One computes this index as in 6.7 to get (6.9.1).

This deals with the case $d>1$, $l>0$ in part (iii) of the theorem, once we have proved the lemma above. To do this, we use a device from [5], 1.2. Our underlying vector space F^n is an E-vector space; the choice of an E-basis $\mathcal{A}$ of F^n gives a decomposition $F^n = E\otimes_F V_0$, where V_0 is the F-linear span of $\mathcal{A}$. This gives an algebra isomorphism $A = A(E)\otimes_F \mathrm{End}_F(V_0)$ and hence an isomorphism $A = A(E)\otimes_E B$ of $(A(E),B)$-bimodules. Put another way, A is free as a left $A(E)$-module, and an E-basis $\mathcal{B}$ of B provides a left $A(E)$-basis of A, so that

$$A = \bigoplus_{b\in\mathcal{B}} A(E)\otimes b.$$

The tame co-restriction $s_\gamma : A\to B$ satisfies

$$s_\gamma : \sum_{b\in\mathcal{B}} x_b\otimes b \longmapsto \sum s_\gamma^0(x_b)b,$$

where the x_b are elements of $A(E)$ and s_γ^0 temporarily denotes a tame co-restriction $A(E)\to E$. Likewise for the adjoint map a_γ. However, B decomposes as the direct sum of its intersections with $\mathfrak{n}_{\mathrm{u}}$, $\mathfrak{n}_\ell$ and the F-algebra $\mathrm{End}_F(V_1)\oplus\mathrm{End}_F(V_2)$. Thus the sequence (6.9.2) is effectively a direct sum of n_1n_2/d^2 copies of the exact sequence

$$0\longrightarrow E\longrightarrow A(E)\xrightarrow{a_\gamma^0} A(E)\xrightarrow{s_\gamma^0} E\longrightarrow 0,$$

and the lemma follows.

6.10. We take the same notation as in 6.9, except that we now assume $l=0$ (and $\gamma\notin F$). The construction of K and the lattices $\mathfrak{k}_{\mathrm{u}}$, $\mathfrak{k}_\ell$ is slightly different in this case; see [7], §7. However, there is an exact sequence analogous to (6.9.3) from which we obtain

$$q^{f(\check{\sigma}_1\times\sigma_2,\psi)} = C(\gamma)^{n_1n_2/d^2}.$$

We have now established all the formulæ of part (iii) of the theorem.

6.11. We return to part (i) of the theorem; we may as well take $\sigma_2=\sigma_1$. Comparing with the situation of 6.10, we have $\gamma=\beta_1$ and the group K is the same as there. The explicit formula now follows from 5.4 and (6.1.1). The strict lower bound follows from Lemma 6.4.

6.12. The strict lower bound in (iii) likewise follows from Lemma 6.4. To complete the proof of the theorem, we have only to establish the strict upper bounds in parts (i) and (iii). We first treat (iii).

We return to the notation of 6.9, which is the only case in need of proof. In particular, $\gamma\notin F$. We need some notation. Set $E=F[\gamma]$ and $d=[E{:}F]$. There is a unique hereditary $\mathfrak{o}$-order $\mathfrak{A}(E)$ in $A(E)$ which is normalized by $E^\times$. We denote its radical by $\mathfrak{P}(E)$. We put $k=-\nu_E(\gamma)>0$ and $r=-k_0(\gamma,\mathfrak{A}(E))$ (notation of

[5], 1.4). We have $k \geqslant r > 0$, since we have excluded the possibility $\gamma \in F$. The integer k is given by

$$k = me(E|F)/e.$$

We also have $l < -k_0(\gamma, \Lambda) = re(E|F)/e$. It is therefore enough to prove:

$$\mathfrak{c}(\gamma) \leqslant \frac{kd^2}{e(\gamma)} - \frac{rd}{e(\gamma)}, \tag{6.12.1}$$

where we abbreviate $e(\gamma) = e(E|F)$.

We prove (6.12.1) by "induction along γ", in a manner reminiscent of many proofs in [5]. Suppose first that γ is minimal over F; we have excluded the case $E = F$, so we have $r = k$ [5], 1.4.15. We compute the quantity $C(\gamma)$ by taking the alternating product of volumes in the exact sequence *loc. cit.*

$$0 \longrightarrow \mathfrak{o}_E \longrightarrow \mathfrak{A}(E) \xrightarrow{a_\gamma} \mathfrak{P}(E)^{-k} \xrightarrow{s_\gamma} \mathfrak{p}_E^{-k} \longrightarrow 0.$$

That is,

$$C(\gamma) = q_E^{-k} q^{d^2k/e(\gamma)}.$$

The exponent of q here is $k(-d/e(\gamma) + d^2/e(\gamma))$, as required.

In the general case, i.e., where γ is not minimal over F, we consider the exact sequence [5], 1.4.7:

$$0 \longrightarrow \mathfrak{o}_E \longrightarrow \mathfrak{N}_{-r}(\gamma) \xrightarrow{a_\gamma} \mathfrak{P}(E)^{-r} \xrightarrow{s_\gamma} \mathfrak{p}_E^{-r} \longrightarrow 0.$$

Here, $\mathfrak{N}_{-r}(\gamma) = \mathfrak{N}_{-r}(\gamma, \mathfrak{A}(E))$, in the notation of [5], 1.4. We choose a simple stratum $[\mathfrak{A}(E), k, r, \delta]$ equivalent to $[\mathfrak{A}(E), k, r, \gamma]$ ([5], 2.4.1). In case $\delta \in F$, the element $\gamma - \delta$ is minimal over F, so $r = -\nu_E(\gamma - \delta) < k$ and $C(\gamma - \delta) = C(\gamma)$. The result is then given by the first case, so we assume $\delta \notin F$. Thus

$$k > -k_0(\delta, \mathfrak{A}(E)) > r.$$

We have $\mathfrak{N}_{-r}(\gamma) = \mathfrak{N}_{-r}(\delta, \mathfrak{A}(E))$ [5], 2.1.3, so we can compare with the exact sequence

$$0 \longrightarrow \mathfrak{B}_\delta \longrightarrow \mathfrak{N}_{-r}(\gamma) \xrightarrow{a_\delta} \mathfrak{P}(E)^{-r} \xrightarrow{s_\delta} \mathfrak{Q}_\delta^{-r} \longrightarrow 0.$$

Here, $\mathfrak{B}_\delta$ denotes the intersection of $\mathfrak{A}(E)$ with the $A(E)$-centralizer of δ and $\mathfrak{Q}_\delta$ the radical of $\mathfrak{B}_\delta$. We get (using μ for any Haar measure on the relevant vector space)

$$\begin{aligned} C(\gamma) &= \frac{\mu(\mathfrak{o}_E)}{\mu(\mathfrak{p}_E^{-r})} \, \frac{\mu(\mathfrak{P}(E)^{-r})}{\mu(\mathfrak{N}_{-r}(\delta))} \\ &= \frac{\mu(\mathfrak{o}_E)}{\mu(\mathfrak{p}_E^{-r})} \, \frac{\mu(\mathfrak{Q}_\delta^{-r})}{\mu(\mathfrak{B}_\delta)} \, C(\delta)^{d^2/d_\delta^2} \\ &= q^{-rd/e(\gamma)} q^{rd^2/d_\delta e(\gamma)} C(\delta)^{d^2/d_\delta^2}, \end{aligned}$$

where $d_\delta = [F[\delta]{:}F]$, etc. Now let $t = -k_0(\delta, \mathfrak{A}(F[\delta]))$, so that $k_0(\delta, \mathfrak{A}(E)) = -te(\gamma)/e(\delta)$. Note that $\nu_{F[\delta]}(\delta) = -ke(\delta)/e(\gamma)$. By inductive hypothesis therefore,

$$\mathfrak{c}(\delta) \leqslant kd_\delta^2/e(\gamma) - td_\delta/e(\delta).$$

In all,

$$\mathfrak{c}(\gamma) \leqslant \frac{kd^2}{e(\gamma)} - \frac{rd}{e(\gamma)} + \frac{rd^2}{d_\delta e(\gamma)} - \frac{td^2}{d_\delta e(\delta)}.$$

The result will follow when we show that $r/e(\gamma) < t/e(\delta)$. However,

$$te(\gamma)/e(\delta) = -k_0(\delta, \mathfrak{A}(E)) > r,$$

as required.

The strict upper bound in part (i) of the theorem also follows from (6.12.1), so we have now completed the proof.

6.13. We make some comments on the discriminant function $C(\gamma)$. First, we observe that the arguments of 6.12 can be refined to give a precise determination of $C(\gamma)$ in terms of the construction of γ from minimal elements as in [5], Ch. 2; see especially Theorem 2.4.1.

We next recall that γ is only determined up to a congruence in the algebra $A(E)$. Throughout the arguments above (starting with 6.9), we can replace γ by any element $\gamma' \in A(E)$ which is sufficiently close to γ, and nothing changes. We can therefore assume that the field extension $F[\gamma]/F$ *is separable.*

Abbreviating $E = F[\gamma]$ as before, we have a canonical projection $\mathrm{p} : A(E) \to E$, which is orthogonal with respect to the reduced trace pairing $A(E) \times A(E) \to F$. The orthogonal complement of E is $A'(E) = a_\gamma(A(E))$, and $A(E) = A'(E) \oplus E$. The adjoint a_γ acts on $A'(E)$ as an automorphism, and $s_\gamma = \alpha_0 \mathrm{p}$, for some $\alpha_0 \in E$ such that $\alpha_0 \mathfrak{o}_E = \mathfrak{p}\mathfrak{p}_E^{-1}\mathfrak{D}_{E/F}^{-1}$, where $\mathfrak{D}_{E/F}$ denotes the relative different (*cf.* [5], 1.3). Thus

$$C(\gamma) = (\mathfrak{p}_E^{e(E|F)-1} : \mathfrak{D}_{E/F})^{-1} \, \|\det{}_{A'(E)} \, a_\gamma\|^{-1}.$$

Of course, if γ_i, $1 \leqslant i \leqslant d$, are the conjugates of γ in some splitting field E' and we extend the absolute value $\| \ \|$ to one on E', we have

$$\| \det a_\gamma \| = \prod_{i \neq j} \|\gamma_i - \gamma_j\|.$$

6.14. We give a simple application of 6.5, for use in a subsequent paper. For an irreducible smooth representation π of $\mathrm{GL}_n(F)$ and a quasicharacter χ of $F^\times$, we use the notation

$$\pi \cdot \chi = \pi \otimes \chi \circ \det .$$

Proposition. *For $i = 1, 2$, let π_i be an irreducible supercuspidal representation of $\mathrm{GL}_{n_i}(F)$ with the property*

$$\pi_i \not\cong \pi_i \cdot \chi \tag{6.14.1}$$

for any unramified quasicharacter χ of $F^\times$.

(i) *Suppose that we have $\pi_1 \cdot \chi \not\cong \check{\pi}_2$ for any tamely ramified quasicharacter χ of $F^\times$. Then*

$$f(\pi_1 \cdot \chi \times \pi_2) = f(\pi_1 \times \pi_2)$$

for every tamely ramified quasicharacter χ of $F^\times$.

(ii) *Suppose that $\pi_1 \cong \check{\pi}_2$ and that $n_1 = n_2$ is a power of the residual characteristic p of F. If χ denotes a tamely ramified quasicharacter of $F^\times$, we have:*

$$f(\pi_1 \cdot \chi \times \pi_2) = f(\pi_1 \times \pi_2) \qquad \text{if } \chi \text{ is unramified,}$$
$$f(\pi_1 \cdot \chi \times \pi_2) = f(\pi_1 \times \pi_2) + 1 \qquad \text{if } \chi \text{ is ramified.}$$

Proof. In part (i), the representation π_2 cannot be an unramified twist of $\check{\pi}_1$. Consequently, the pair $(\sigma_1, \sigma_2) = (\check{\pi}_1, \pi_2)$ must satisfy the hypotheses of either 6.5(ii) or 6.5(iii). If both π_i have level zero, they are completely distinct, and the representations $\pi_1 \cdot \chi$, $\check{\pi}_2$ have the the same properties. Thus $f(\pi_1 \cdot \chi \times \pi_2) = f(\pi_1 \times \pi_2) = n_1 n_2$ in this case.

We may therefore assume that at least one of the π_i has positive level and, say, $f(\pi_1)/n_1 \leqslant f(\pi_2)/n_2$. If we are in case 6.5(iii), a best common approximation to $(\check{\pi}_1, \pi_2)$ is again a best common approximation to $(\check{\pi}_1 \cdot \chi, \pi_2)$, and the result follows. In case 6.5(ii), the representations $\check{\pi}_1 \cdot \chi$, π_2 are still completely distinct and $f(\pi_1 \cdot \chi \times \pi_2) = f(\pi_1 \times \pi_2) = n_2 f(\pi_2)$.

The first assertion in (ii) is easy: twisting with unramified characters does not change the conductor. The second assertion is trivial if $n_1 = n_2 = 1$. We exclude this and assume $n_1 = n_2 > 1$. If π_1 contains a maximal simple type with underlying simple stratum $[\mathfrak{A}, m, 0, \beta]$, say, our hypotheses (6.14.1) and (6.1.1) together imply $e(F[\beta]|F) = n_1$. So $F[\beta]/F$ is totally ramified of degree n_1 which, we recall, is a power of p. It follows that $\pi_1 \cdot \chi \not\cong \pi_1$, for any non-trivial tame quasicharacter χ (*cf.* [5], 5.2.2(iii)). The result then follows on comparing 6.5(i) with 6.5(iii).

6.15. As promised in 6.3, we give a method for finding the quantities l and γ appearing in the best common approximation to a given pair of (not completely distinct) supercuspidal representations. The description is simpler than in [7], §8, since we are here only concerned with a special case. The results of [3], §8, show that the outline here is equivalent to that of [7].

We recapitulate the notation. For $i = 1, 2$, σ_i is an irreducible supercuspidal representation of $\mathrm{GL}_{n_i}(F)$; this contains a maximal simple type with underlying simple stratum $[\mathfrak{A}_i, m_i, 0, \beta_i]$ in $\mathbb{M}_{n_i}(F)$. We have $e_i = e(\mathfrak{A}_i|\mathfrak{o}) = e(F[\beta_i]|F)$; we put $e = \mathrm{lcm}\,(e_1, e_2)$. The simple type in σ_i contains a simple character $\theta_i \in \mathcal{C}(\mathfrak{A}_i, 0, \beta_i)$, to use the notation of [5], Ch. 3.

Let f_i denote the residual degree $f(F[\beta_i]|F)$, and set $f = \mathrm{lcm}\,(f_1, f_2)$. Let V_0 be an F-vector space of dimension ef and let $\mathfrak{A}_0$ be a principal $\mathfrak{o}$-order in $\mathrm{End}_F(V_0)$ of period e. There are then F-algebra embeddings φ_i of the fields $F[\beta_i]$ in $\mathrm{End}_F(V_0)$ such that $\mathfrak{A}_0$ is normalized by both $\varphi_i(F[\beta_i])^\times$. The strata $[\mathfrak{A}_0, m, 0, \varphi_i(\beta_i)]$ are both simple.

Next, we use the canonical bijections $\mathcal{C}(\mathfrak{A}_i, 0, \beta_i) \to \mathcal{C}(\mathfrak{A}_0, 0, \varphi_i(\beta_i))$ of [5], 3.6.14, induced by the embeddings φ_i. Write θ_i' for the image of θ_i under this map, $i = 1, 2$. We obtain l as the least non-negative integer k such that the characters $\theta_i' \mid H^{k+1}(\varphi(\beta_i), \mathfrak{A}_0)$ intertwine in $\mathrm{Aut}_F(V_0)$. (This implies they are conjugate in $\mathrm{Aut}_F(V_0)$, by [5], 3.5.11.) We then choose a simple stratum $[\mathfrak{A}_0, m, l, \gamma_0]$ equivalent to, say, $[\mathfrak{A}_0, m, l, \varphi_1(\beta_1)]$. The element γ may then be taken as $\varphi_0(\gamma_0)$, for some embedding φ_0 of $F[\gamma_0]$ in A such that $\varphi_0(F[\gamma_0])^\times$ normalizes $\mathit{\Lambda}$ and is contained in L.

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