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A BILINEAR APPROACH TO THE RESTRICTION AND KAKEYA CONJECTURES

TERENCE TAO, ANA VARGAS, AND LUIS VEGA

Contents

1. Introduction

The purpose of this paper is to investigate bilinear variants of the restriction and Kakeya conjectures, to relate them to the standard formulations of these conjectures, and to give applications of this bilinear approach to existing conjectures. The methods used are based on several observations and results of Bourgain (see [2]-[6]), together with some refinements by Moyua, Vargas, and Vega [17, 18].

This paper is organized as follows. In the first section we discuss bilinear restriction estimates, and show how one can pass back and forth between these estimates and the standard restriction estimates. We also generalize the 12/7 bilinear restriction estimate of [18] to higher dimensions.

In the second section we give analogues of the above results for the Kakeya operator. In particular we give a bilinear improvement to Wolff's Kakeya theorem in arbitrary dimension.

In the third section we give applications of these bilinear estimates in three dimensions. For example, we are able to improve the 42/11 exponent in Wolff's restriction theorem to 34/9. We are also able to prove a sharp (L^p, L^q) restriction theorem which improves on the classical (L^2, L^4) Tomas-Stein theorem, and also give some concrete progress on a bilinear restriction conjecture of Klainerman and Machedon. We also give a non-bilinear approach to these estimates, which gives weaker results but is more direct and probably has a wider range of application.

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Finally, we collect some standard harmonic analysis estimates in an Appendix for easy reference.

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2. Bilinear restriction estimates

Fix[1] $n \geq 2$ and $A > 0$, and let Q be the cube $[-1,1]^{n-1}$ in $\mathbf{R}^{n-1}$. Let $\Phi : Q \to \mathbf{R}$ be a phase function satisfying the following conditions:

- $\|\partial^\alpha \Phi\|_\infty \leq A$ for all $0 \leq |\alpha| \leq N$, where N is a large constant.
- $\Phi(0) = \nabla\Phi(0) = 0$.
- For all $x \in Q$, the eigenvalues of the Hessian $\Phi_{x_i x_j}(x)$ all lie in $[1-\epsilon_0, 1+\epsilon_0]$, where $\epsilon_0 > 0$ is a small constant.

We will call such a phase *elliptic*. The model example of an elliptic phase function is of course the quadratic phase $\Phi(x) = \frac{1}{2}|x|^2$, but any smooth compact convex surface with non-vanishing curvature can be decomposed into finitely many graphs whose graphing function (after an affine transformation) obeys the above properties. In particular, the unit sphere can be decomposed in this manner.

We will consider linear and bilinear bounds for the operator $\Re^* : L^1(Q) \to L^\infty(\mathbf{R}^n)$ defined by

$$\Re^* f(\underline{x}, x_n) = \int_Q e^{-2\pi i(\underline{x}\cdot y + x_n \Phi(y))} f(y)\, dy.$$

This operator can be thought of as an adjoint restriction operator associated to the surface $\{(y, \Phi(y)) : y \in Q\}$. For $0 < p, q \leq \infty$, we use $R^*(p \to q)$ to denote the estimate

$$\|\Re^* f\|_q \lesssim \|f\|_p$$

for all test functions f, with the constant depending only on n and A. Similarly, we use $R^*(p_1 \times p_2 \to q)$ to denote the estimate

$$\|\Re^* f \Re^* g\|_q \lesssim \|f\|_{p_2} \|g\|_{p_1}$$

for all test functions f, g supported on Q_1, Q_2 respectively, where Q_1, Q_2 are any subcubes of Q whose size and separation are comparable to 1. (We will call such cubes $O(1)$-separated in the sequel.)

Estimates of the form $R^*(p \to q)$ are adjoint restriction estimates and have attracted wide interest. The (sharp) restriction conjecture states that

Conjecture 2.1. *$R^*(p \to q)$ holds whenever $q > \frac{2n}{n-1}$ and $p' \leq \frac{n-1}{n+1} q$.*

These conditions are well known to be best possible (see e.g. [26]). This conjecture has been verified for $n = 2$ [7], but remains open in higher dimensions. The main difficulty lies in making the q exponent as low as possible; the estimate is trivial for $q = \infty$, Hölder's inequality can be used to raise p, and in certain cases factorization theory can be used to lower p. When (p, q) lie on the sharp line

$$p' = \frac{n-1}{n+1} q \tag{1}$$

we abbreviate the estimate $R^*(p \to q)$ to $R^*_s(q)$.

[1] All constants in this section are assumed to depend only on n and A.

TABLE 1. Known restriction theorems for $n = 3$. ε denotes an arbitrary positive number

1.	$R^*(1 \to \infty) = R^*_s(\infty)$	Riemann-Lebesgue
2.	$R^*(2 \to 6)$	Stein, 1967 [10]
3.	$R^*(2 \to 4 + \varepsilon)$	Tomas, 1975 [26]
4.	$R^*(2 \to 4) = R^*_s(4)$	Stein, 1975 (For $n = 3$: Sjölin, 1972)
5.	$R^*(4 - \frac{2}{15} + \varepsilon \to 4 - \frac{2}{15} + \varepsilon)$	Bourgain, 1991 [2]
6.	$R^*(4 - \frac{2}{11} + \varepsilon \to 4 - \frac{2}{11} + \varepsilon)$	Wolff, 1995 [27]
7.	$R^*(\frac{7}{3} + \varepsilon \to 4 - \frac{2}{11} + \varepsilon)$	Moyua, Vargas, Vega, 1995 [17, 18]
8.	$R^*(\frac{170}{77} + \varepsilon \to 4 - \frac{2}{9} + \varepsilon)$	Theorem 4.1
9.	$R^*_s(4 - \frac{5}{27} + \varepsilon)$	Theorem 4.1
?.	$R^*_s(3 + \varepsilon)$?	(critical value)

We summarize[2] the known results in $n = 3$ in Table 1. The classical theorem of Tomas and Stein states that $R^*(2 \to \frac{2(n+1)}{n-1}) = R^*_s(\frac{2(n+1)}{n-1})$ for any $n \geq 2$. Later improvements have been made on this result ([2], [6], [27]); in particular, Moyua, Vargas, and Vega [17, 18] have recently observed that one has the estimate $R^*(\frac{7}{3}+\varepsilon \to \frac{42}{11}+\varepsilon)$ in three dimensions. However, none of these improvements to the Tomas-Stein theorem lies on the sharp line $p' = \frac{n-1}{n+1}q$. As one of the applications of this paper we will prove a new restriction theorem on this sharp line.

Our improvements will be based on the earlier defined bilinear restriction estimates $R^*(p \times p \to q)$, which we will now discuss. These estimates have appeared implicitly in many works (e.g. [5], [18]), and are closely related to null form estimates for the wave equation (see [14]-[16]; related ideas also appear in [1]), but do not appear to have been explicitly studied until very recently.

When $(p, 2q)$ lie in the range predicted by Conjecture 2.1, then $R^*(p, 2q)$ and $R^*(p \times p \to q)$ are almost equivalent. Indeed, in Section 2.5 we will prove

Theorem 2.2. *Let $n \geq 2$ and $1 < p, q < \infty$ be such that $2q > \frac{2n}{n-1}$ and $p' \leq \frac{n-1}{n+1}2q$.*

Then $R^*(p, 2q)$ implies $R^*(p \times p \to q)$. Furthermore, if $R^*(\tilde{p} \times \tilde{p} \to \tilde{q})$ holds for all $(\frac{1}{\tilde{p}}, \frac{1}{\tilde{q}})$ in a neighbourhood of $(\frac{1}{p}, \frac{1}{q})$, then $R^*(p, 2q)$ holds.

However, the bilinear estimate $R^*(p \times p \to q)$ can hold for exponents which are not covered by the above theorem. For instance, when $n = 2$ an easy computation using Plancherel's theorem and a change of variables shows that $R^*(2 \times 2 \to 2)$ holds, even though the Knapp example shows that $R^*(2, 4)$ fails completely. Thus one expects the range of exponents for the bilinear restriction estimate to be larger than that of Conjecture 2.1. For $n = 3$ the first results in this direction were by Bourgain [5] (although the theorem $R^*(\frac{16}{9} \times \frac{16}{9} \to 2)$ implicitly appeared in [3]); more recently, Moyua, Vargas, and Vega [18] showed that

$$R^*(\frac{12}{7} \times \frac{12}{7} \to 2) \tag{2}$$

for $n = 3$. We modestly generalize this result to higher dimensions as

Theorem 2.3. *Suppose that $n \geq 2$. Then*

$$R^*(p \times p \to 2)$$

holds if and only if $p \geq \frac{4n}{3n-2}$.

[2]Some of the earlier results were not stated for arbitrary elliptic phase functions.

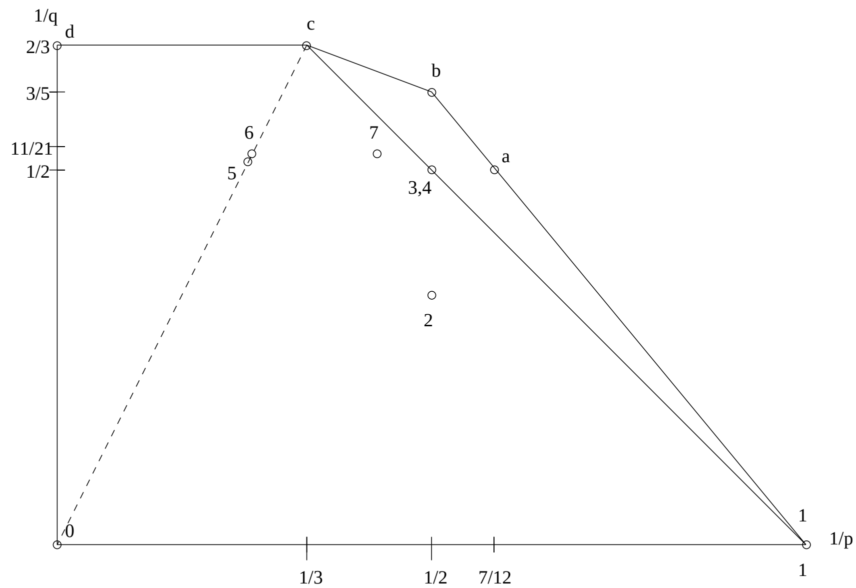

FIGURE 1. Prior status of $R^*(p \times p \to q)$ and $R^*(p \to 2q)$ for $n = 3$.

Recently[3] Klainerman and Machedon conjectured that

$$R^*(2 \times 2 \to \frac{n+2}{n}) \tag{3}$$

for all $n \geq 2$. By interpolating (3) with what is implied by Conjecture 2.1, one is led to the following

Conjecture 2.4. *If $n \geq 2$, then $R^*(p \times p \to q)$ holds whenever*

$$q \geq \frac{n}{n-1}, \tag{4}$$

$$\frac{n+2}{2q} + \frac{n}{p} \leq n, \tag{5}$$

$$\frac{n+2}{2q} + \frac{n-2}{p} \leq n-1. \tag{6}$$

By Theorem 2.3 and interpolation the conjecture is verified for $q \geq 2$ (and thus for $n = 2$). The exponents in the above conjecture are best possible; we will sketch the proof of this statement in Section 2.7. From Theorem 2.2 we see that Conjecture 2.4 implies Conjecture 2.1.

We depict the conjectured ranges for the estimates $R^*(p \to 2q)$ and $R^*(p \times p \to q)$ in Figure 1. The restriction conjecture states that $R^*(p \to 2q)$ holds for all (p, q) in the trapezoidal region bounded by 1, c, d, and 0, except for the upper line between c and d inclusive; by the above Theorem, this is almost equivalent to $R^*(p \times p \to q)$ holding in this region. Klainerman's conjecture asserts that $R^*(p \times p \to q)$ holds at the endpoint b. The combined Conjecture 2.4 states that $R^*(p \times p \to q)$ holds in the pentagonal region bounded by 1, b, c, d, and 0, including the upper line mentioned previously; this region is best possible.

[3]Workshop in Harmonic Analysis and PDE, MSRI, July 1997.

By Theorem 2.2 the standard restriction estimate $R^*(p \to 2q)$ and the bilinear estimate $R^*(p \times p \to q)$ are essentially equivalent in the line between c and 1. The points 1–7 correspond to the standard restriction results, while the point a corresponds to the bilinear restriction theorem (2).

From Theorem 2.2 and bilinear interpolation it is possible to obtain new linear and bilinear restriction theorems; for instance, by interpolating between the bilinear form[4] of the result in [17] and (2) and using Theorem 2.2, one may obtain the sharp restriction theorem $R^*_S(q)$ for all $q > 4 - \frac{2}{17}$. We will improve on these results in Section 4.

2.5. **Proof of Theorem 2.2.** The first implication is a trivial consequence of Hölder's inequality, so we concentrate on the latter. In view of the known results for $n = 2$ we may take $n \geq 3$. From the Tomas-Stein theorem (see e.g. [23]) and the necessity of (4) it suffices to consider the case $\frac{2n}{n-1} < 2q < \frac{2(n+1)}{n-1}$. In particular we may assume that $1 < q < 2$.

The bilinear hypothesis $R^*(\tilde{p} \times \tilde{p} \to \tilde{q})$ allows us to control $\mathfrak{R}^* f \mathfrak{R}^* g$ if f and g have $O(1)$-separated supports. By a parabolic rescaling argument this will imply a similar estimate when f and g have $O(2^{-j})$-separated supports for any $j > 0$. Piecing these estimates together one may obtain an estimate on $\mathfrak{R}^* f \mathfrak{R}^* g$ for arbitrary f, g, from which the conclusion $R^*(p \to q)$ will follow.

We now turn to the details. Assume that the hypotheses of Theorem 2.2 hold. We have to show that

$$\|\mathfrak{R}^* f\|_{2q} \lesssim \|f\|_p.$$

By Marcinkeiwicz interpolation it suffices to show the restricted estimate

$$\|\mathfrak{R}^* \chi_\Omega\|_{2q} \lesssim |\Omega|^{1/p}$$

for a slightly better value of $(p, 2q)$, where Ω is some arbitrary subset of Q.

Let j_0 be the positive integer such that $|\Omega| \sim 2^{-j_0(n-1)}$. Then by squaring the above estimate, we reduce to

$$2^{\frac{2(n-1)}{p} j_0} \|\mathfrak{R}^* \chi_\Omega \mathfrak{R}^* \chi_\Omega\|_q \lesssim 1. \tag{7}$$

The next step is a Whitney decomposition. For each $j > 0$, we dyadically decompose Q into $\sim 2^{(n-1)j}$ dyadic subcubes τ_k^j of sidelength 2^{-j} in the usual manner. If τ_k^j, $\tau_{k'}^j$ are two cubes with the same sidelength which are not adjacent but have adjacent parents, we say that these cubes are *close* and write $\tau_k^j \sim \tau_{k'}^j$. For almost every $x, y \in Q$ there exists a unique pair of close cubes τ_k^j, $\tau_{k'}^j$ containing x and y respectively. Thus we have

$$\mathfrak{R}^* \chi_\Omega \mathfrak{R}^* \chi_\Omega = \sum_j \sum_{k,k' : \tau_k^j \sim \tau_{k'}^j} \mathfrak{R}^* \chi_{\Omega \cap \tau_k^j} \mathfrak{R}^* \chi_{\Omega \cap \tau_{k'}^j}.$$

Thus to prove (7) it suffices to show that

$$2^{\frac{2(n-1)}{p} j_0} \Big\| \sum_{k,k' : \tau_k^j \sim \tau_{k'}^j} \mathfrak{R}^* \chi_{\Omega \cap \tau_k^j} \mathfrak{R}^* \chi_{\Omega \cap \tau_{k'}^j} \Big\|_q \lesssim 2^{-\varepsilon |j - j_0|} \tag{8}$$

for all $j > 0$ and some $\varepsilon > 0$, since (7) follows from the triangle inequality. Informally, the above estimate asserts that the most significant separation scale is of the order of $2^{-j_0} = |\Omega|^{1/(n-1)}$; this is already evident from the Knapp example.

[4] I.e. $R^*(\frac{7}{3} \times \frac{7}{3}, \frac{21}{11})$.

Our next reduction will be to exploit some quasi-orthogonality between the functions $\mathfrak{R}^* \chi_{\Omega \cap \tau_k^j} \mathfrak{R}^* \chi_{\Omega \cap \tau_{k'}^j}$. From the definition of $\mathfrak{R}^*$ we see that the Fourier transform $\mathfrak{R}^* \chi_{\Omega \cap \tau_k^j}$ is supported on the infinite tube $\tau_k^j \times \mathbf{R}$. Thus, the Fourier transform of $\mathfrak{R}^* \chi_{\Omega \cap \tau_k^j} \mathfrak{R}^* \chi_{\Omega \cap \tau_{k'}^j}$ is supported in the tube

$$T_{j,k} = \tilde{\tau}_k^j \times \mathbf{R},$$

where $\tilde{\tau}_k^j$ is a cube of sidelength $C 2^{-j}$ whose center is twice that of τ_k^j. From Lemma 6.1 in the Appendix and the assumption $q < 2$, we thus have

$$\|\mathfrak{R}^* \chi_{\Omega \cap \tau_k^j} \mathfrak{R}^* \chi_{\Omega \cap \tau_{k'}^j}\|_q \lesssim \Big(\sum_k \|\mathfrak{R}^* \chi_{\Omega \cap \tau_k^j} \mathfrak{R}^* \chi_{\Omega \cap \tau_{k'}^j}\|_q^q\Big)^{1/q}.$$

Thus (8) will be proven if we can show that

$$2^{\frac{2(n-1)q}{p} j_0} \sum_k \Big\| \sum_{k' : \tau_k^j \sim \tau_{k'}^j} \mathfrak{R}^* \chi_{\Omega \cap \tau_k^j} \mathfrak{R}^* \chi_{\Omega \cap \tau_{k'}^j} \Big\|_q^q \lesssim 2^{-\varepsilon q |j - j_0|}. \tag{9}$$

This will follow from the following estimate.

Proposition 2.6. *For all $\tilde{p}$ in a neighbourhood of p, we have*

$$\|\mathfrak{R}^* \chi_{\Omega \cap \tau_k^j} \mathfrak{R}^* \chi_{\Omega \cap \tau_{k'}^j}\|_q \lesssim 2^{-\frac{2(n-1)}{\tilde{p}'} j} 2^{\frac{n+1}{q} j} |\Omega \cap \tau_k^j|^{1/\tilde{p}} |\Omega \cap \tau_{k'}^j|^{1/\tilde{p}}. \tag{10}$$

Proof. This will be accomplished by a parabolic rescaling argument. By translating Φ and subtracting a harmless affine factor[5] we may assume that τ_k^j is centered at the origin. We now observe that since Φ is an elliptic phase, the function

$$\tilde{\Phi}(x) = 2^{2j} \Phi(2^{-j} x)$$

is also elliptic. Since $R^*(\tilde{p} \times \tilde{p} \to q)$ holds for all $\tilde{p}$ in a neighbourhood of p by assumption, we have

$$\|\tilde{\mathfrak{R}}^* f \tilde{\mathfrak{R}}^* g\|_q \lesssim \|f\|_{\tilde{p}} \|g\|_{\tilde{p}}$$

whenever f and g are supported on disjoint $O(1)$-separated cubes, where $\tilde{\mathfrak{R}}^*$ is the adjoint restriction operator corresponding to $\tilde{\Phi}$. Applying a parabolic scaling $(\underline{x}, x_n) \to (2^j \underline{x}, 2^{2j} x_n)$ to this estimate one obtains

$$\|\mathfrak{R}^* f \mathfrak{R}^* g\|_q \lesssim 2^{-\frac{2(n-1)}{\tilde{p}'} j} 2^{\frac{n+1}{q} j} \|f\|_{\tilde{p}} \|g\|_{\tilde{p}}$$

whenever f and g are supported on τ_k^j and $\tau_{\tilde{k}}^j$ respectively, and (10) follows. □

It remains to obtain (9) from the proposition. Let $\tilde{p} < p$ be such that (10) holds. If we apply (10) and the triangle inequality, we see that (9) reduces to

$$2^{\frac{2(n-1)q}{p} j_0} \sum_k \Big(\sum_{k' : \tau_k^j \sim \tau_{k'}^j} 2^{-\frac{2(n-1)}{\tilde{p}'} j} 2^{\frac{n+1}{q} j} |\Omega \cap \tau_k^j|^{1/\tilde{p}} |\Omega \cap \tau_{k'}^j|^{1/\tilde{p}} \Big)^q \lesssim 2^{-\varepsilon q |j - j_0|}.$$

By polarization and the fact that for each k there are only finitely many cubes $\tau_{k'}^j$ close to τ_k^j, this in turn reduces to

$$2^{\frac{2(n-1)q}{p} j_0} 2^{-\frac{2(n-1)q}{\tilde{p}'} j} 2^{(n+1)j} \sum_k |\Omega \cap \tau_k^j|^{\frac{2q}{\tilde{p}}} \lesssim 2^{-\varepsilon q |j - j_0|}. \tag{11}$$

[5] Strictly speaking, one may need to increase A by a constant factor to do this; we will gloss over this technicality.

We divide into two cases: $\tilde{p} \leq 2q$ and $\tilde{p} > 2q$. If $\tilde{p} \leq 2q$, then we use (66) from Lemma 6.2 in the Appendix with $\alpha = 1$ to obtain

$$\sum_k |\Omega \cap \tau_k^j|^{\frac{2q}{\tilde{p}}} \lesssim 2^{-(n-1)j_0} 2^{-(n-1)\max(j,j_0)(\frac{2q}{\tilde{p}}-1)}.$$

Thus (11) reduces to

$$\begin{aligned} \frac{2(n-1)q}{p} j_0 - \frac{2(n-1)q}{\tilde{p}'} j + (n+1)j - (n-1)j_0 \\ - (n-1)\max(j,j_0)(\frac{2q}{\tilde{p}} - 1) \leq -\varepsilon q |j - j_0|. \end{aligned} \tag{12}$$

By convexity it suffices to verify this inequality for the values $j = 0$, $j = j_0$, and $j_0 = 0$. When $j = 0$ (12) becomes

$$2(n-1)qj_0(\frac{1}{p} - \frac{1}{\tilde{p}}) \leq -\varepsilon q j_0, \tag{13}$$

which is true for some $\varepsilon > 0$ since $\tilde{p} < p$. When $j = j_0$ (12) becomes

$$2(n-1)q(\frac{1}{p} - 1 + \frac{n+1}{2(n-1)q}) \leq 0,$$

which holds since $p' \leq \frac{n-1}{n+1}2q$. Finally, when $j_0 = 0$ (12) becomes

$$(2n - 2(n-1)q)j \leq -\varepsilon q j, \tag{14}$$

which holds for some $\varepsilon > 0$ since $2q > \frac{2n}{n-1}$.

It remains to treat the case $\tilde{p} > 2q$. By repeating the above procedure but with (66) replaced by (67), we see that (11) reduces to

$$\begin{aligned} \frac{2(n-1)q}{p} j_0 - \frac{2(n-1)q}{\tilde{p}'} j + (n+1)j - (n-1)\frac{2q}{\tilde{p}} j_0 \\ + (n-1)j(1 - \frac{2q}{\tilde{p}}) \leq -\varepsilon q |j - j_0|. \end{aligned} \tag{15}$$

Since the left-hand side is completely linear it suffices to verify this when $j = 0$ and when $j_0 = 0$. But in these two cases (15) reduces (13), (14) as before, and so the argument proceeds as in the previous case. $\square$

The fact that Theorem 2.2 requires knowledge of $R^*(p \times p \to q)$ for all elliptic phase functions is a defect of the argument. When restricted to the quadratic phase $\Phi(x) = \frac{1}{2}|x|^2$ however, no other phase functions are required in the proof, due to the algebraic properties of Φ. The quadratic phase is the simplest of all the elliptic phases; indeed, a parabolic scaling and limiting argument shows that any sharp restriction theorem for an elliptic phase implies the corresponding estimate for the quadratic phase (see [24]).

2.7. **Necessity of** (4)-(6)**.** In this section we sketch the proof of the assertion that the conditions in Conjecture 2.4 are necessary. For simplicity we take Φ to be a graphing function for a small portion of a sphere; one can easily modify the arguments below for more general phases. The estimate $R^*(p \times p \to q)$ can then be rewritten as

$$\|\widehat{fd\sigma}\widehat{gd\sigma}\|_q \lesssim \|f\|_p \|g\|_p, \tag{16}$$

where $d\sigma$ is surface measure on the unit sphere S^{n-1}, and f and g are functions on fixed disjoint caps C_1, C_2 in S^{n-1} whose size and separation are comparable to a small quantity $\epsilon = \epsilon_n$.

To prove (4), we take $f(w) = 1$ on C_1, and $g(w) = e^{-2\pi i x_0 \cdot w}$ on C_2, where $x_0 \in \mathbf{R}^n$ is a point to be determined later. From standard stationary phase estimates, we see that for any $R \gg 1$ one can find a cube Q of sidelength R such that $|\widehat{fd\sigma}(x)| \sim R^{-\frac{n-1}{2}}$ on Q. By choosing x_0 appropriately, one can also arrange matters so that $|\widehat{gd\sigma}(x)| \sim R^{-\frac{n-1}{2}}$ on the same cube Q. By inserting these estimates into (16) one obtains

$$R^{-\frac{n-1}{2}} R^{-\frac{n-1}{2}} |Q|^{1/q} \lesssim 1.$$

If one now uses the fact that $|Q| \sim R^n$ and takes $R \to \infty$, then condition (4) follows.

The proof of the necessity of (5) and (6) is based on modifications of the standard Knapp example. We note in passing that without modification the Knapp example only gives the weaker condition

$$\frac{n}{2q} + \frac{n-1}{p} \leq n - 1.$$

To prove (5), we will take f and g to be "squashed caps".[6] We factor $\mathbf{R}^n$ as $\mathbf{R}^2 \times \mathbf{R}^{n-2}$, and use S^1 to denote the great circle $S^1 = S^{n-1} \cap (\mathbf{R}^2 \times \{0\})$. We may assume that S^1 intersects C_1 and C_2. Fix $0 < \delta \ll 1$. We take f and g to be the characteristic functions of the sets

$$C_i \cap (B_2(w_i, \delta^2) \times B_{n-2}(0, \delta)), \quad i = 1, 2,$$

respectively, where $B_k(x, R)$ denotes the ball in $\mathbf{R}^k$ of radius R centered at x, and w_i are arbitrary elements of $S^1 \cap C_i$ for $i = 1, 2$. Then the Fourier transforms of $fd\sigma$, $gd\sigma$ exhibit essentially no cancellation on the box

$$B_2(0, \frac{1}{C\delta^2}) \times B_{n-2}(0, \frac{1}{C\delta}). \tag{17}$$

Indeed, we have $|\widehat{fd\sigma}(x)| \sim |\widehat{gd\sigma}(x)| \sim \delta^n$ on this set. Inserting this estimate into (16) one obtains

$$\delta^n \delta^n (\delta^{-n-2})^{1/q} \lesssim \delta^{\frac{n}{p}} \delta^{\frac{n}{p}},$$

and by taking $\delta \to 0$ one obtains (5).

The estimate (6) is proven by taking f and g to be "stretched caps". With the notation as before we take f and g to be the characteristic functions of

$$C_i \cap (\mathbf{R}^2 \times B_{n-2}(0, \delta)), \quad i = 1, 2,$$

respectively, to begin with, although we will later need to multiply f and g by a phase as in the proof of (4).

When restricted to the slab $\mathbf{R}^2 \times B_{n-2}(0, \frac{1}{C\delta})$, the functions $\widehat{fd\sigma}, \widehat{gd\sigma}$ behave essentially like Fourier transforms of measures on S^1. Indeed, a stationary phase computation shows that

$$|\widehat{fd\sigma}(x)| \sim \delta^{n-2} |x|^{-\frac{1}{2}}$$

on a large portion of this slab, and similarly for $\widehat{gd\sigma}$. Thus, multiplying by a phase to translate $\widehat{fd\sigma}$ and $\widehat{gd\sigma}$ as necessary, one can arrange matters so that

$$|\widehat{fd\sigma}(x)| \sim |\widehat{gd\sigma}(x)| \sim \delta^{n-2} (\delta^{-2})^{-\frac{1}{2}} = \delta^{n-1}$$

[6]This example was discovered independently by the authors and Sergiu Klainerman.

on the box (17). Inserting this into (16) one obtains

$$\delta^{n-1}\delta^{n-1}(\delta^{-n-2})^{1/q} \lesssim \delta^{\frac{n-2}{p}}\delta^{\frac{n-2}{p}},$$

and (6) follows by taking $\delta \to 0$. $\square$

Unlike the situation with the disc multiplier problem [11], it appears that the Besicovitch set construction does not give any further restrictions on p, q. Indeed, for $n = 2$ the conditions (4)-(6) are sufficient as well as necessary.

2.8. **Proof of Theorem 2.3.** Our argument will be a routine modification of the one in [18].

The necessity of the condition on p follows from Section 2.7, so we will only show the sufficiency of this condition. By Hölder's inequality it suffices to show that

$$R^*(\frac{4n}{3n-2} \times \frac{4n}{3n-2} \to 2).$$

By symmetry and interpolation this will follow from

$$R^*(2 \times \frac{n}{n-1} \to 2).$$

It suffices to show that

(18)
$$\int \Re^* f_1(x)\Re^* g_1(x)\overline{\Re^* f_2(x)\Re^* g_2(x)}\ dx \lesssim \|f_1\|_1\|g_1\|_{\frac{n}{n-1}}\|f_2\|_\infty\|g_2\|_{\frac{n}{n-1}}$$

for all f_1, f_2, g_1, g_2, supported on Q_1, Q_1, Q_2, Q_2 respectively, where Q_1 and Q_2 are $O(1)$-separated cubes. Indeed, by applying the symmetry $f_1 \leftrightarrow f_2$, $g_1 \leftrightarrow g_2$ to (18) and applying multi-linear interpolation one obtains

$$\int \Re^* f_1(x)\Re^* g_1(x)\overline{\Re^* f_2(x)\Re^* g_2(x)}\ dx \lesssim \|f_1\|_2\|g_1\|_{\frac{n}{n-1}}\|f_2\|_2\|g_2\|_{\frac{n}{n-1}},$$

and the desired estimate follows from substituting $f_1 = f_2 = f$, $g_1 = g_2 = g$.

It remains to prove (18). By Plancherel's theorem the left-hand side can be written as

$$\int f_1(x)g_1(y)\overline{f_2(z)g_2(w)}\delta(\Phi(x)+\Phi(y)-\Phi(z)-\Phi(w))\delta(x+y-z-w)\ dxdydzdw,$$

where δ is the Dirac distribution. From the positivity of the kernel in the above expression we may reduce (18) to

$$\begin{aligned}\int f_1(x)g_1(y)f_2(z)g_2(w)\chi_1(x)\chi_2(y)\chi_1(z)\chi_2(w)&\\ \delta(\Phi(x)+\Phi(y)-\Phi(z)-\Phi(w))\delta(x+y-z-w)\ dxdydzdw&\\ \lesssim \|f_1\|_1\|g_1\|_{\frac{n}{n-1}}\|f_2\|_\infty\|g_2\|_{\frac{n}{n-1}}&\end{aligned}$$

for arbitrary functions f_1, f_2, g_1, g_2 on $\mathbf{R}^{n-1}$, where χ_1 and χ_2 are smooth cutoff functions adapted to (a slight thickening of) Q_1 and Q_2 respectively. Since f_1 and f_2 are controlled in L^1 and L^∞ respectively, we may assume that $f_1(x) = \delta(x-x_0)$ and $f_2 \equiv 1$ for some x_0; we may take $x_0 = 0$ by translating Φ and subtracting off a harmless affine factor. In particular, we may assume that 0 is in (a slight thickening of) Q_1. The estimate (18) thus reduces to

$$\int g_1(y)g_2(w)\chi_2(y)\chi_1(y-w)\chi_2(w)\delta(\Phi(y)-\Phi(y-w)-\Phi(w))\ dydw \lesssim \|g_1\|_{\frac{n}{n-1}}\|g_2\|_{\frac{n}{n-1}},$$

which by duality becomes

$$\|Tg\|_n \lesssim \|g\|_{\frac{n}{n-1}}, \tag{19}$$

where T is the averaging operator

$$Tg(y) = \int g(w)\chi_2(y)\chi_1(y-w)\chi_2(w)\delta(\Phi(y)-\Phi(y-w)-\Phi(w))\ dw.$$

It is well known (see below) that the estimate (19) will hold if the defining function $\phi(y,w) = \Phi(y) - \Phi(y-w) - \Phi(w)$ satisfies the rotational curvature condition

$$\left|\det\begin{pmatrix} \phi & \phi_y \\ \phi_w & \phi_{yw} \end{pmatrix}\right| > 0 \quad \text{when } \phi = 0 \tag{20}$$

uniformly on the support of $\chi_2(y)\chi_1(y-w)\chi_2(w)$.

However, since Φ is elliptic, we have the estimates

$$\Phi_{y_iy_j} = \delta_{ij} + O(\epsilon_0), \quad \Phi_{y_i}(y) = y_i + O(\epsilon_0|y|), \quad \Phi(y) = \frac{1}{2}|y|^2 + O(\epsilon_0|y|^2),$$

where δ_{ij} is the Kronecker delta. Inserting these estimates into the definition of ϕ, one can estimate the above determinant as

$$\left|\det\begin{pmatrix} \phi & \phi_y \\ \phi_w & \phi_{yw} \end{pmatrix}\right| = |w|^2 + O(\epsilon_0(|y|^2+|w|^2)) \quad \text{when } \phi = 0.$$

However, from the support of $\chi_2(y)\chi_2(w)$ and the assumption that 0 is in a thickening of Q_1 we see that $|w|, |y| \sim 1$. Thus (20) follows, if ϵ_0 is sufficiently small. This finishes the proof. $\square$

The above proof shows that there exist asymmetrical bilinear restriction theorems in addition to the symmetrical ones. In particular, one may conjecture that

$$R^*(\frac{n+2}{n} \times \frac{n+2}{2} \to \frac{n+2}{n}),$$

which is a strengthening of (3). Non-symmetrical versions of the counterexamples in the previous section show that this conjecture is best possible.

For $n \leq 3$ Theorem 2.3 is an improvement on the classical Tomas-Stein theorem. However for $n > 3$ the two estimates are not directly comparable. Because of this, we have no significant improvements to Wolff's restriction theorem [27] in four and higher dimensions.

For completeness we sketch a proof of the following standard fact which was used in the above proof.

Lemma 2.9. *If ϕ satisfies the rotational curvature condition* (20) *on the support of a cutoff function $\psi(y,w)$, then the operator*

$$Tf(y) = \int_{\mathbf{R}^{n-1}} f(w)\psi(y,w)\delta(\phi(y,w))\ dw$$

obeys (19).

Proof. We imbed this operator in the analytic family T_ζ defined by

$$T_\zeta(y) = \int f(w)\psi(y,w)a_\zeta(\phi(y,w))\ dw,$$

where a_ζ is defined for $Re(\zeta) > 0$ by

$$a_\zeta(t) = e^{\zeta^2}\frac{t_+^{\zeta-1}}{\Gamma(\zeta)}\varphi(t)$$

and φ is a cutoff function adapted to $[-\varepsilon,\varepsilon]$ for some small $\varepsilon > 0$; for $Re(\zeta) \le 0$ a_ζ (and thus T_ζ) is defined by analytic continuation. Since $T = T_0$, (19) will follow from complex interpolation between the estimates

$$\|T_{1+it}f\|_\infty \lesssim \|f\|_1,$$
$$\|T_{-\frac{n-2}{2}+it}f\|_2 \lesssim \|f\|_2$$

for all real t and some fixed $N > 0$. The former estimate follows immediately from the observation that the kernel of T_{1+it} is uniformly bounded in t (indeed, the e^{ζ^2} term makes it rapidly decreasing in t). To prove the latter estimate, it suffices to show that $T_{-\frac{n-2}{2}+it}$ is a Fourier integral operator of order 0 uniformly in t (see e.g. [12]). Accordingly, we write $T_{-\frac{n-2}{2}+it}$ as

$$T_{-\frac{n-2}{2}+it}f(y) = \int e^{2\pi i\phi(y,z)\cdot\xi}f(w)\psi(y,w)\hat{a}_{-\frac{n-2}{2}+it}(\xi)\ dwd\xi,$$

where ξ ranges over $\mathbf{R}$. From the rotational curvature hypothesis (20) we see that the phase is non-degenerate in the sense of [12]. Since the amplitude is a symbol of order $-\frac{n-2}{2}$, y, w range over an $(n-1)$-dimensional space, and ξ ranges over a 1-dimensional space, the reduction-of-variables theorem (see e.g. [12]) states that $T_{-\frac{n-2}{2}+it}$ will be a Fourier integral operator of order 0, as desired; the uniformity in t follows from the rapid decrease of $a_{-\frac{n-2}{2}+it}$ with respect to t, caused by the e^{ζ^2} factor. □

3. Bilinear Kakeya estimates

We now begin the second part of this paper, in which we give analogues of the previous results for the Kakeya operator.

Throughout this section $0 < \delta \ll 1$ will be a small parameter, and we will use $A \lesssim B$ to denote the estimate $A \le C_\varepsilon \delta^{-\varepsilon} B$ for all $\varepsilon > 0$, otherwise we write $A \gg B$. We say that a quantity A has *logarithmic size* if $1 \lesssim |A| \lesssim 1$, while we say it has *polynomial size* if $\delta^C \lesssim |A| \lesssim \delta^{-C}$ for some constant C. Finally, all functions and quantities in this section are assumed to be non-negative.

Let $\mathcal{E}$ be a δ-net of the unit cube Q in $\mathbf{R}^{n-1}$. We give two measures on $\mathcal{E}$; the counting measure di and the normalized counting measure $d\omega = \delta^{n-1}di$. For $\omega, i \in \mathcal{E}$, define the $\delta \times 1$ tube T_ω^i by

$$T_\omega^i = \{(\underline{y}, y_n) \in \mathbf{R}^n : |y_n| \le 1, |\underline{y} - y_n\omega - i| \le \delta\};$$

we will call ω and i the direction and base of T_ω^i respectively. Note that for fixed ω the tubes T_ω^i essentially form a partition of the unit ball $B(0,1)$. This discretization is not essential to the statements and estimates, but it allows for some technical simplification to the argument.

For any function f on $\mathbf{R}^n$, define the discretized x-ray transform $Xf = X_\delta f$ on $\mathcal{E} \times \mathcal{E}$ by

$$Xf(\omega, i) = \delta^{1-n} \int_{T_\omega^i} f(x)\, dx.$$

For $1 \leq p, q \leq \infty$, let $K(p \to q)$ denote the estimate

$$\|Xf\|_{L^q_\omega L^\infty_i} \lesssim \delta^{-\frac{n}{p}+1} \|f\|_p,$$

where $\mathcal{E} \times \mathcal{E}$ is understood to be endowed with the measure $d\omega di$. By taking f to be the characteristic function of a δ-ball we see that the factor $\delta^{-\frac{n}{p}+1}$ is best possible.

The Kakeya conjecture asserts that $K(p \to q)$ holds if and only if $1 \leq p \leq n$ and $q \leq (n-1)p'$. In particular, it is conjectured that $K(n \to n)$ holds. It is easy to see that these conditions on p, q are necessary. The conjecture is trivial for $p = 1$; the difficulty is in making p (and to a lesser extent q) as large as possible. So far the best result on this conjecture is due to Wolff [27], who showed that

$$K(\frac{n+2}{2} \to \frac{(n-1)(n+2)}{n}).$$

In particular, for $n = 3$ we have $K(\frac{5}{2} \to \frac{10}{3})$. This estimate is sharp in the sense that the $\frac{10}{3}$ exponent cannot be raised without decreasing the $\frac{5}{2}$ exponent.

The adjoint estimate

$$\|X^* g\|_{p'} \lesssim \delta^{-\frac{n}{p}+1} \|g\|_{L^{q'}_\omega L^1_i}$$

to $K(p \to q)$ will be denoted $K^*(q' \to p')$; note that

$$X^* g(x) = \delta^{1-n} \int\int g(\omega, i) \chi_{T_\omega^i}(x)\, d\omega di = \sum_\omega \sum_i g(\omega, i) \chi_{T_\omega^i}(x). \tag{21}$$

Following the philosophy of the previous sections, we define the bilinear version[7] $K^*(q' \times q' \to \frac{p'}{2})$ of the above estimate by

$$\|X^* f X^* g\|_{p'/2} \lesssim \delta^{-\frac{2n}{p}+2} \|f\|_{L^{q'}_\omega L^1_i} \|g\|_{L^{q'}_\omega L^1_i}$$

for all f, g supported on $\mathcal{E}_1 \times \mathcal{E}$, $\mathcal{E}_2 \times \mathcal{E}$, where $\mathcal{E}_1$ and $\mathcal{E}_2$ are $O(1)$-separated subsets of $\mathcal{E}$.

We have the following analogue of Theorem 2.2, which we will prove in Section 3.9. For technical reasons we will restrict ourselves to the case $p \leq q$, which is the case of most interest. It is likely that one can use factorization theory and affine invariance to extend these results to the case $p > q$.

Theorem 3.1. *Suppose that $1 \leq p \leq q \leq (n-1)p'$. Then the hypotheses $K(p \to q)$ and $K^*(q' \times q' \to \frac{p'}{2})$ are equivalent.*

As with the restriction conjecture, it is possible to have bilinear Kakeya estimates which are outside the range of the usual Kakeya conjecture. For instance, one has the easy estimate

Proposition 3.2. *For any $n \geq 2$ we have $K^*(1 \times 1 \to 1)$.*

[7] Note that the last exponent will usually be less than 1.

We defer the simple proof of this proposition to Section 3.5.

Interpolating this estimate with the estimate

$$K^*(\frac{n}{n-1} \times \frac{n}{n-1} \to \frac{n}{2(n-1)}),$$

which by Theorem 3.1 is the bilinear form of the Kakeya conjecture, we see that the Kakeya conjecture is equivalent to

Conjecture 3.3. *If $n \geq 2$ and $1 \leq p, q \leq \infty$, then $K^*(q' \times q' \to \frac{p'}{2})$ holds if and only if*

$$p \leq n, \tag{22}$$

$$\frac{n-2}{q} + \frac{2}{p} \geq 1. \tag{23}$$

We will show the necessity of (22) and (23) in Section 3.11. These two conditions correspond to (4) and (6) respectively; the analogue of (5) is the degenerate condition $q \leq \infty$.

Wolff's theorem [27] is equivalent to

$$K^*(\frac{(n-1)(n+2)}{n^2-2} \times \frac{(n-1)(n+2)}{n^2-2} \to \frac{n+2}{2n});$$

in particular, we have $K^*(\frac{10}{7} \times \frac{10}{7} \to \frac{5}{6})$ for $n = 3$. In Section 3.7 we improve the above estimate to

Theorem 3.4. *For all $n \geq 2$ we have*

$$K^*(\frac{n+2}{n+1} \times \frac{n+2}{n+1} \to \frac{n+2}{2n}).$$

In particular, we have

$$K^*(\frac{5}{4} \times \frac{5}{4} \to \frac{5}{6}) \tag{24}$$

in three dimensions. The result can be thought of as a bilinear version of the (false) estimate $K(\frac{n+2}{2} \to n+2)$, and is sharp in the sense that (23) is obeyed with equality.

We display the known Kakeya and bilinear Kakeya results in Figure 2. The trapezoidal region represents the conjectured range of (p, q) for which $K(p \to q)$ should hold, and the pentagonal enlargement represents the range on which the bilinear version $K^*(q' \times q' \to \frac{p'}{2})$ should hold. By Theorem 3.1 the two estimates are equivalent in the triangular region below the dashed line. The point 1 is the trivial $L^1 \to L^\infty$ estimate, while the point 2 represents the higher-dimensional analogue of Cordoba's argument ([8], [2]), and 3 is the "bush" argument as given by Bourgain [2] (see also [9], [8]). Proposition 3.2, the bilinear improvement to Cordoba's argument, is the point 4. The point 5 is Bourgain's Kakeya maximal theorem [2] (see also [21]), while 6 is Wolff's theorem [27], which we improve in Theorem 3.4 to the point 7. The region to the right of the dotted line thus represents the best results known to date (excepting the results in [28], which are not directly representable on this figure).

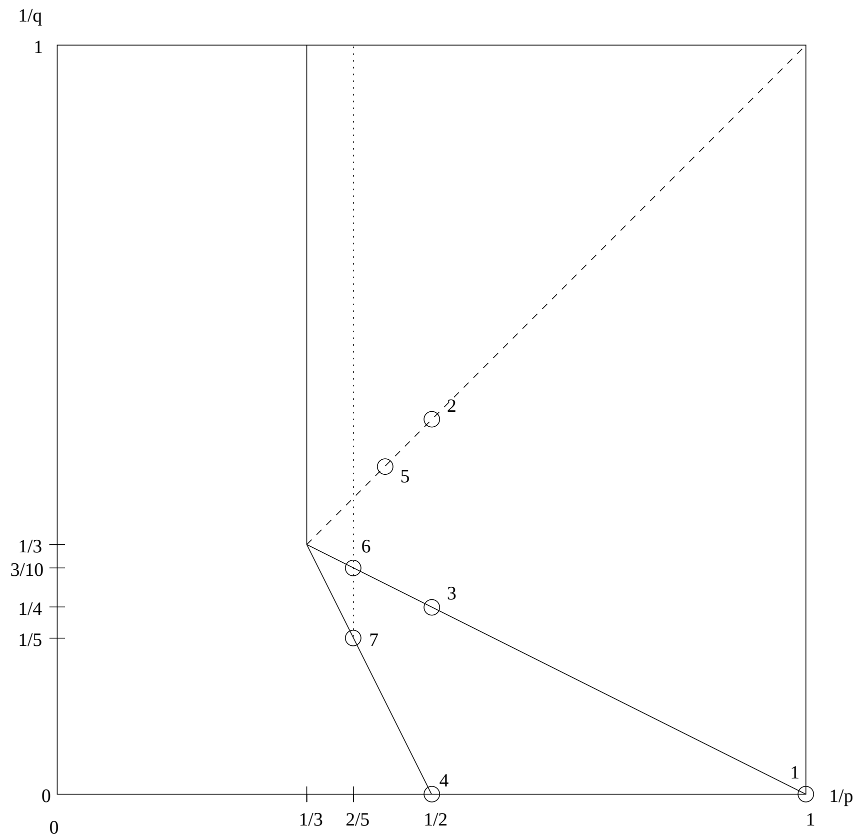

FIGURE 2. Status of $K(p \to q)$ and $K^*(q' \times q' \to \frac{p'}{2})$ for $n = 3$.

3.5. **Proof of Proposition 3.2.** We will need the following geometric observation of Cordoba:

Lemma 3.6. *For any* $\omega_1, \omega_2, i_1, i_2 \in \mathcal{E}$*, one has*

$$\langle \chi_{T^{i_1}_{\omega_1}}, \chi_{T^{i_2}_{\omega_2}} \rangle = |T^{i_1}_{\omega_1} \cap T^{i_2}_{\omega_2}| \lesssim \frac{\delta^n}{|\omega_1 - \omega_2| + \delta}.$$

In particular, if ω_1 and ω_2 have unit separation, then the intersection between the two tubes has measure at most δ^n. We leave the easy proof of this lemma to the reader.

From this observation we easily see that

$$\begin{aligned} \|X^* f X^* g\|_1 &= \langle X^*_\delta f, X^*_\delta g \rangle \\ &= \int\int\int\int \delta^{1-n}\delta^{1-n} \langle \chi_{T^{i_1}_{\omega_1}}, \chi_{T^{i_2}_{\omega_2}} \rangle f(\omega_1, i_1) g(\omega_2, i_2)\ d\omega_1 di_1 d\omega_2 di_2 \\ &\lesssim \int\int\int\int \delta^{1-n}\delta^{1-n}\delta^n f(\omega_1, i_1) g(\omega_2, i_2)\ d\omega_1 di_1 d\omega_2 di_2 \\ &= \delta^{2-n} \|f\|_{L^1_\omega L^1_i} \|g\|_{L^1_\omega L^1_i}, \end{aligned}$$

which is $K^*(1 \times 1 \to 1)$, as desired. $\square$

3.7. **Proof of Theorem 3.4.** Apart from several technical changes, this theorem will be proven using the geometric and combinatorial arguments of Wolff [27], namely Cordoba's observation (Lemma 3.6) and the "brush" argument. The bilinear setting allows for some simplification since the average angular separation σ between two tubes, as defined in [27], may be (heuristically at least) taken to be 1. In fact, this result informally follows by setting $\sigma = 1$ in Lemma 2.1 of [28], and removing the "two ends" condition as in that paper. We will also take advantage of some simplifications noted by later authors (notably [19, 20, 21, 22, 28]). Of course, due to the fact that we are in a bilinearized adjoint setting, there are some technical difficulties, most notably defining the analogue of the quantity λ in [27]. Also, since the target space $L^{(n+2)/2n}$ is not a Banach space, certain reductions and techniques (e.g. duality, elimination of the i_1, i_2 variables) become unavailable. In particular, the Lebesgue space approach of [13] becomes technically very difficult, and we will use restricted weak-type methods instead. In other words, we will use the pigeon-hole principle to reduce as many functions as possible to characteristic functions.

We first make the trivial observation that since X is discretized, the operator boundedness of X on Lebesgue spaces is automatic with some large power of δ^{-1}; the issue is to control the dependence on δ efficiently.

Let us normalize f and g so that

$$\|f\|_{L_\omega^{\frac{n+2}{n+1}} L_i^1} = \|g\|_{L_\omega^{\frac{n+2}{n+1}} L_i^1} = 1.$$

We have to show that

$$\|X^* f X^* g\|_{\frac{n+2}{2n}} \lesssim \delta^{-2\frac{n-2}{n+2}}.$$

It will suffice to show the weak-type bound

$$|\{X^* f X^* g \gtrsim \alpha\}| \lesssim \alpha^{-\frac{n+2}{2n}} \delta^{-\frac{n-2}{n}} \tag{25}$$

for all $\alpha > 0$, since the strong-type estimate can be recovered (with only a logarithmic loss) by integrating this over all α of polynomial size; the contribution of $\alpha \gg \delta^{-C}$ or $\alpha \ll \delta^C$ can be easily controlled using trivial estimates.

We now make the assumption that there exist sets $\Omega_j \subset \mathcal{E}_j$ of cardinality $M_j > 0$ for $j = 1, 2$ such that

(26)
$$\|f(\omega, \cdot)\|_{L_i^1} = (M_1 \delta^{n-1})^{-\frac{n+1}{n+2}} \chi_{\Omega_1}(\omega), \quad \|g(\omega, \cdot)\|_{L_i^1} = (M_2 \delta^{n-1})^{-\frac{n+1}{n+2}} \chi_{\Omega_2}(\omega).$$

This assumption is justified as any $L^{(n+2)/(n+1)}$-normalized f, g can be majorized by a sum of at most logarithmically many functions of this type. We may assume that the M_j have polynomial size.

From the pigeon-hole principle (25) will follow from the estimate

$$|E| \lesssim (\alpha_1 \alpha_2)^{-\frac{n+2}{2n}} \delta^{-\frac{n-2}{n}}, \tag{27}$$

where E is any set such that

$$X^* f \geq \alpha_1, X^* g \geq \alpha_2 \text{ on } E, \tag{28}$$

and $\alpha_1, \alpha_2 > 0$ are arbitrary. We may assume that $|E|$, α_1, α_2 have polynomial size, since this estimate is easily obtainable (with a large gain) otherwise.

The α_j, $j = 1, 2$, represent a normalized multiplicity of the tubes in the supports of f and g; roughly speaking, they are related to the quantity N defined in [27] by the informal relationship

$$\alpha_j \approx (M_j \delta^{n-1})^{-\frac{n+1}{n+2}} N_j.$$

Define the quantity A by

$$|E| = A(\alpha_1 \alpha_2)^{-\frac{n+2}{2n}} \delta^{-\frac{n-2}{n}}. \tag{29}$$

We have to show that $A \lesssim 1$. We may assume without loss of generality that A is essentially minimal in the sense that

$$|E| \lesssim A(\alpha_1 \alpha_2)^{-\frac{n+2}{2n}} \delta^{-\frac{n-2}{n}} \tag{30}$$

for all α_1, α_2, E, f, g which obey (26), (28).

To copy Wolff's argument in [27] we will need some control on the quantity $|E \cap T_\omega^i|$ if T_ω^i is a "typical" tube in a direction in $\mathcal{E}_1$. (In [27] such control is automatic as one is not working in the adjoint setting.) From (28) and (21) we have the pointwise estimate

$$\delta^{1-n} \int \int f(\omega, i) \chi_{T_\omega^i}(x) \; d\omega di \geq \alpha_1 \chi_E(x).$$

Integrating this on E we obtain

$$\int \int f(\omega, i) |T_\omega^i \cap E| \; d\omega di \geq \lambda_1 \delta^{n-1} (M_1 \delta^{n-1})^{\frac{1}{n+2}}, \tag{31}$$

where λ_j is defined for $j = 1, 2$ by

$$\lambda_j = \frac{\alpha_j |E|}{(M_j \delta^{n-1})^{\frac{1}{n+2}}}. \tag{32}$$

From our assumptions we see that the λ_j are of polynomial size.

The λ_j are the analogues of the quantity λ in [27]. Indeed, from (26) and (31) we expect $|T_\omega^i \cap E| \sim \lambda_1 \delta^{n-1}$ on the average. In fact, because we are considering only an extremal configuration, a more precise statement is possible. We say that a tube T_ω^i is *good* if $|T_\omega^i \cap E| \geq \frac{1}{4} \lambda_1 \delta^{n-1}$. Let G be the set of all (ω, i) in the support of f associated to good tubes. The following improvement of (31) states that most tubes are good.

Proposition 3.8. *We have*

$$\int \int_G f(\omega, i) \; d\omega di \sim (M_1 \delta^{n-1})^{\frac{1}{n+2}}.$$

In particular, we have that G is non-empty, so that $\lambda_1 \lesssim 1$.

Proof. The upper bound follows immediately from (31), so it suffices to show the lower bound. Let $c > 0$ be a small number of logarithmic size to be chosen later. If the lower bound failed, then we would have

$$\int \int_G f(\omega, i) \; d\omega di \leq c (M_1 \delta^{n-1})^{\frac{1}{n+2}}.$$

The idea is to then replace f by $\tilde{f} = f \chi_G$, and contradict the extremality of A in (30).

Of course, we must modify $\tilde{f}$ further, as well as E, α_1 and M_1, in order to retain (26) and (28). We replace E by

$$\tilde{E} = \{x \in E : X^*(f - \tilde{f}) < \frac{1}{2}\alpha_1\};$$

note that $\tilde{f}$ obeys (28) if E is replaced by $\tilde{E}$ and α_1 is replaced by $\frac{1}{2}\alpha_1$.

The next step is to show that $\tilde{E}$ is comparable to E in size. From the definition of $\tilde{E}$ we see that

$$\int_E X^*(f - \tilde{f}) \geq \frac{1}{2}\alpha_1 |E \backslash \tilde{E}|.$$

However, we have from (21) and the definition of $\tilde{f}$ that

$$\int_E X^*(f - \tilde{f}) = \delta^{1-n} \int \int_{G^c} f(\omega, i)|T_\omega^i \cap E| \; d\omega di.$$

Combining the two estimates and using the definition of G we obtain

$$\frac{1}{2}\alpha_1 |E \backslash \tilde{E}| \leq \delta^{1-n} \int \int f(\omega, i) \frac{1}{4}\lambda_1 \delta^{n-1} \; d\omega di.$$

Using (26) and (32) this simplifies to

$$|E \backslash \tilde{E}| \leq \frac{1}{2}|E|,$$

so that $|\tilde{E}| \sim |E|$ as desired.

We now have to modify $\tilde{f}$, α_1, $\tilde{E}$, and M_1 further so that (26) is restored. From hypothesis we have

$$\int \|\tilde{f}(\omega, \cdot)\|_{L_i^1} \; d\omega = \int \int \tilde{f}(\omega, i) \; d\omega di < c(M_1 \delta^{n-1})^{\frac{1}{n+2}}.$$

However, from (26) we have

$$\|\tilde{f}(\omega, \cdot)\|_{L_i^1} \leq (M_1 \delta^{n-1})^{-\frac{n+1}{n+2}}.$$

Thus by Hölder's inequality this implies that

$$\|\tilde{f}\|_{L_\omega^{\frac{n+2}{n+1}} L_i^1} \leq c^{\frac{n+1}{n+2}}.$$

Thus, as before, we can find a logarithmic number of functions $\tilde{f}_k$ which each obey (26) for some M_1^k, and such that

$$\tilde{f} \lesssim c^{\frac{n+1}{n+2}} \sum_k \tilde{f}_k.$$

This implies that

$$\sum_k X^* \tilde{f}_k \gtrsim c^{-\frac{n+1}{n+2}} \alpha_1$$

on $\tilde{E}$. Thus, by reducing $\tilde{E}$ by a logarithmic factor one can find a k such that

$$X^* \tilde{f}_k \gtrsim c^{-\frac{n+1}{n+2}} \alpha_1$$

on the reduced set (which we will still call $\tilde{E}$).

Thus (28) is satisfied with f replaced by $\tilde{f}_k$, E replaced by $\tilde{E}$, and α_1 replaced by $\tilde{\alpha}_1 \sim c^{-\frac{n+1}{n+2}} \alpha_1$. But from the definition of A this implies that

$$|\tilde{E}| \lesssim A(\tilde{\alpha}_1 \alpha_2)^{-\frac{n+2}{2n}} \delta^{-\frac{n-2}{n}}.$$

Comparing this with (30) and our estimates for $\tilde{E}$ and $\tilde{\alpha}$ we thus obtain a contradiction, if c is sufficiently small. $\square$

From the above proposition, the definition of G and the identity

$$\int_E X^*(f\chi_G) = \delta^{1-n} \int\int_G f(\omega, i)|T_\omega^i \cap E| \; d\omega di$$

we obtain

$$\int_E X^*(f\chi_G) \gtrsim (M_1\delta^{n-1})^{\frac{1}{n+2}}\lambda_1.$$

From (32) this becomes

$$\int_E X^*(f\chi_G) \gtrsim \alpha_1|E|.$$

From (26) we thus have

$$\int X^*(f\chi_G)X^*g \gtrsim \alpha_1\alpha_2|E|.$$

Expanding out X^*g using (21) this becomes

$$\delta^{1-n} \int\int g(\omega, i)(\int_{T_\omega^i} X^*(f\chi_G)) \; d\omega di \gtrsim \alpha_1\alpha_2|E|.$$

On the other hand, from (26) we have

$$\int\int g(\omega, i) \; d\omega di = (M_2\delta^{n-1})^{\frac{1}{n+2}}.$$

Thus there must exist (ω_0, i_0) in the support of g such that

$$\int_{T_{\omega_0}^{i_0}} X^*(f\chi_G) \gtrsim \frac{\alpha_1\alpha_2\delta^{n-1}|E|}{(M_2\delta^{n-1})^{\frac{1}{n+2}}} = \alpha_1\lambda_2\delta^{n-1}. \tag{33}$$

The tube $T_{\omega_0}^{i_0}$ plays the role of the central tube of a "brush". Unlike Wolff's argument in [27] (which considered more general angular separations σ than the unit separation), we will be able to obtain our estimate using only a single brush. On the other hand, by utilizing the extremality hypothesis as in Proposition 3.8, one could certainly obtain a large number of brushes if desired.

By affine invariance we may take $\omega_0 = i_0 = 0$, so that the central tube is the vertical tube through the origin. In particular, 0 is in $\mathcal{E}_1$, so every ω in $\mathcal{E}_2$ has roughly unit separation from the origin.

Let $G_0 \subset G$ be the collection of all good (ω, i) in the support of f such that T_ω^i intersects the central tube T_0^0. Then expanding out $X^*(f\chi_G)$ in (33), we thus obtain

$$\int\int_{G_0} f(\omega, i)\delta^{1-n}|T_0^0 \cap T_\omega^i| \; d\omega di \gtrsim \alpha_1\lambda_2\delta^{n-1}.$$

From Lemma 3.6 we have $|T_0^0 \cap T_\omega^i| \lesssim \delta^n$, so that

$$\int\int_{G_0} f(\omega, i) \; d\omega di \gtrsim \alpha_1\lambda_2\delta^{n-2}.$$

Let $\Omega_0 \subset \Omega_1$ be the collection of all ω such that (ω, i) is in G_0 for at least one i. From (26) we see that

$$\int\int_{G_0} f(\omega, i) \; d\omega di \lesssim \#\Omega_0\delta^{n-1}(M_1\delta^{n-1})^{-\frac{n+1}{n+2}},$$

so that

$$\#\Omega_0 \gtrsim \frac{\alpha_1 \lambda_2 \delta^{-1}}{(M_1 \delta^{n-1})^{-\frac{n+1}{n+2}}}. \tag{34}$$

For each $\omega \in \Omega_0$ we choose a tube T_ω from G_0 which is in the direction of ω. These tubes form the "bristles" of the brush. From construction, $|\omega| \sim 1$, T_ω intersects T_0^0, and

$$|T_\omega \cap E| \gtrsim \lambda_1 \delta^{n-1}.$$

As in Wolff [27], we will use (36) to obtain a lower bound on the size of E. More precisely, we will show that

$$|E| \gtrsim \#\Omega_0 \lambda_1^n \delta^{n-1}. \tag{35}$$

Combining this with (34) yields

$$|E| \gtrsim \frac{\alpha_1 \lambda_1^n \lambda_2 \delta^{n-2}}{(M_1 \delta^{n-1})^{-\frac{n+1}{n+2}}}.$$

By a completely symmetrical argument one also has

$$|E| \gtrsim \frac{\alpha_2 \lambda_2^n \lambda_1 \delta^{n-2}}{(M_2 \delta^{n-1})^{-\frac{n+1}{n+2}}}.$$

Multiplying these estimates together one obtains

$$|E|^2 \gtrsim \frac{\alpha_1 \alpha_2 (\lambda_1 \lambda_2)^{n+1} \delta^{2n-4}}{(M_1 \delta^{n-1} M_2 \delta^{n-1})^{-\frac{n+1}{n+2}}}.$$

Applying (32) this reduces to

$$|E|^2 \gtrsim (\alpha_1 \alpha_2)^{n+2} |E|^{2n+2} \delta^{2n-4},$$

which simplifies to (27), as desired.

It remains to prove (35). We use the argument in [27]; we adopt the observation in [22] (see also [13]) that one does not need to utilize the "two ends" reduction in [27] to achieve (35).

We need some notation. For all dyadic numbers $\lambda_1 \lesssim \beta \lesssim 1$ let Γ_β be the cylindrical region

$$\Gamma_\beta = \{(\underline{y}, y_n) : |\underline{y}| \sim \beta\}.$$

From the properties of T_ω we see that

$$\sum_{\lambda_1 \lesssim \beta \lesssim 1} |T_\omega \cap E \cap \Gamma_\beta| \gtrsim \lambda_1 \delta^{n-1}$$

for all $\omega \in \Omega_0$. By the pigeonhole principle, one can refine Ω_0 by a logarithmic factor so that

$$|T_\omega \cap E \cap \Gamma_\beta| \gtrsim \lambda_1 \delta^{n-1} \tag{36}$$

for all ω in the refined Ω_0, and some $\lambda_1 \lesssim \beta \lesssim 1$ independent of the choice of ω; henceforth this β is considered fixed.

The directions in Ω_0 are δ-separated. It will be more convenient to work with a sparser set of directions, so we take $\tilde{\Omega}_0$ to be any δ/β-net of Ω_0. From the estimates $\#\tilde{\Omega}_0 \gtrsim \beta^{n-1} \#\Omega_0$ and $\beta \gtrsim \lambda_1$ we see that (35) will follow from

$$|E| \gtrsim \#\tilde{\Omega}_0 \frac{\lambda_1^2}{\beta} \delta^{n-1}. \tag{37}$$

Let Θ be a δ/β-net of the unit sphere S^{n-2} in $\mathbf{R}^{n-1}$. For each $\omega \in \tilde{\Omega}_0$, we associate an (essentially unique) element $\theta = \theta_\omega$ of Θ by requiring that

$$|\theta - \frac{\omega}{|\omega|}| \lesssim \delta/\beta;$$

recall that $|\omega| \sim 1$ for all $\omega \in \tilde{\Omega}_0$. Furthermore, from elementary geometry and the fact that T_ω intersects T_0^0 we see that $T_\omega \cap \Gamma_\beta$ is contained in the slab Π_θ given by

$$\Pi_\theta = \{(\underline{y}, y_n) : |\underline{y}| \sim \beta, |\frac{\underline{y}}{|\underline{y}|} - \theta| \lesssim \delta/\beta\}.$$

As the Π_θ are essentially disjoint, (37) will follow from the estimate

$$|E \cap \Pi_\theta| \gtrsim \#\tilde{\Omega}_{0,\theta} \frac{\lambda_1^2}{\beta} \delta^{n-1} \tag{38}$$

for all $\theta \in \Theta$, where

$$\tilde{\Omega}_{0,\theta} = \{\omega \in \tilde{\Omega}_0 : \theta_\omega = \theta\}.$$

For the remainder of the argument ω (and later $\tilde{\omega}$) are always assumed to range over $\tilde{\Omega}_{0,\theta}$.

We now estimate the quantity

$$Q = \int_{E\cap\Pi_\theta} \sum_\omega \chi_{T_\omega}$$

in two different ways. Firstly, from the above geometrical considerations and (36) we have

$$|T_\omega \cap E \cap \Pi_\theta| \gtrsim \lambda_1 \delta^{n-1}$$

for all ω in $\tilde{\Omega}_{0,\theta}$. Summing the above estimate we obtain

$$Q \gtrsim \#\tilde{\Omega}_{0,\theta} \lambda_1 \delta^{n-1}. \tag{39}$$

We now obtain a different estimate for Q. From the Cauchy-Schwarz inequality we have

$$Q \lesssim |E \cap \Pi_\theta|^{1/2} (\int_{|E\cap\Pi_\theta|} (\sum_\omega \chi_{T_\omega})^2)^{1/2}.$$

Squaring both sides and expanding out the integrand into the diagonal and off-diagonal term, this reduces to

$$\frac{Q^2}{|E \cap \Pi_\theta|} \lesssim (\int_{|E\cap\Pi_\theta|} \sum_\omega \chi_{T_\omega}) + \sum \sum_{\omega \neq \tilde{\omega}} |T_\omega \cap T_{\tilde{\omega}} \cap E \cap \Pi_\theta|. \tag{40}$$

The first term on the right-hand side is just Q. The second term we may estimate by Lemma 3.6. Thus (40) becomes

$$\frac{Q^2}{|E \cap \Pi_\theta|} \lesssim Q + \sum \sum_{\omega \neq \tilde{\omega}} \frac{\delta^n}{|\omega - \tilde{\omega}|}.$$

However, ω, $\tilde{\omega}$ range over a δ/β-separated set whose elements are within δ/β of the ray $\mathbf{R}^+\theta$. Thus for each ω, the number of $\tilde{\omega}$ such that $|\omega - \tilde{\omega}| \sim 2^{-j}$ is at most $1/(2^j\delta/\beta)$ for any j. Thus the above estimate reduces to

$$\frac{Q^2}{|E \cap \Pi_\theta|} \lesssim Q + \sum_{\delta/\beta \lesssim 2^j \lesssim 1} \#\tilde{\Omega}_{0,\theta} \frac{1}{2^j\delta/\beta} \frac{\delta^n}{2^{-j}}.$$

Since the number of such j is only logarithmic, we may simplify the above to

$$\frac{1}{|E \cap \Pi_\theta|} \lesssim \frac{1}{Q} + \frac{\#\tilde{\Omega}_{0,\theta}\beta\delta^{n-1}}{Q^2}.$$

Combining this with (39) and using the hypothesis $\beta \gtrsim \lambda_1$ we obtain

$$\frac{1}{|E \cap \Pi_\theta|} \lesssim \frac{\beta}{\#\tilde{\Omega}_{0,\theta}\lambda_1^2\delta^{n-1}},$$

which is (38). This finishes the proof. □

3.9. **Proof of Theorem 3.1.** The proof will be a reprise of the argument in Theorem 2.2. The main difference is that the quasi-orthogonality estimate is replaced by a quasi-triangle inequality, namely Lemma 6.3 in the Appendix. Also the argument is technically simpler as we allow a logarithmic loss in the estimates. The case $p = 1$ is trivial, so we will assume $p > 1$.

The implication of $K^*(q' \times q' \to \frac{p'}{2})$ from $K(p \to q)$ is immediate from duality and Hölder's inequality. Now suppose that $K^*(q' \times q' \to \frac{p'}{2})$ holds for some p, q obeying $1 \leq p \leq n$, $1 \leq q \leq (n-1)p'$. We have to show that $K^*(q' \to p')$ holds. Since the Kakeya conjecture is known to hold for $p \leq 2$ (see e.g. [2], [27]) we may assume that $p > 2$.

Let f be an arbitrary function on $\mathcal{E} \times \mathcal{E}$. We have to show that

$$\|X^* f X^* f\|_{\frac{p'}{2}} \lesssim \delta^{-\frac{2n}{p}+2}\|f\|^2_{L^{q'}_\omega L^1_i}. \tag{41}$$

For each integer $j > 0$ such that $\delta \lesssim 2^{-j}$, we divide $\mathcal{E}$ into $\sim 2^{(n-1)j}$ dyadic "subcubes" $\mathcal{E} \cap \tau^j_k$ of sidelength 2^{-j}, and define the notion of closeness $\tau^j_k \sim \tau^j_{k'}$ as in Section 2.5. We partition $X^* f X^* f$ as

$$X^* f X^* f = \sum_j \sum_{k,k' : \tau^j_k \sim \tau^j_{k'}} \sum_{m,m'} X^*(f\chi_{\tau^j_k} \otimes \chi_{\tau^j_m}) X^*(f\chi_{\tau^j_{k'}} \otimes \chi_{\tau^j_{m'}}).$$

We now observe the geometric fact that the summand in the above expression is only non-zero when τ^j_m and $\tau^j_{m'}$ are within $O(2^{-j})$ of each other; we will implicitly assume this in our summation. By inserting the above identity into (41) and applying Lemma 6.3 from the Appendix we reduce to

$$\sum_j \sum_{k,k' : \tau^j_k \sim \tau^j_{k'}} \sum_{m,m'} \|X^*(f\chi_{\tau^j_k} \otimes \chi_{\tau^j_m}) X^*(f\chi_{\tau^j_{k'}} \otimes \chi_{\tau^j_{m'}})\|^{\frac{p'}{2}}_{\frac{p'}{2}} \lesssim \delta^{\frac{p'}{2}(-\frac{2n}{p}+2)}\|f\|^{p'}_{L^{q'}_\omega L^1_i}. \tag{42}$$

This will follow from the following analogue of Proposition 2.6.

Proposition 3.10. *We have*

$$\|X^*(f\chi_{\tau^j_k} \otimes \chi_{\tau^j_m}) X^*(f\chi_{\tau^j_{k'}} \otimes \chi_{\tau^j_{m'}})\|^{\frac{p'}{2}}_{\frac{p'}{2}} \lesssim 2^{(1-\frac{(n-1)p'}{q})j}\delta^{\frac{p'}{2}(-\frac{2n}{p}+2)} \|f\chi_{\tau^j_k} \otimes \chi_{\tau^j_m}\|^{\frac{p'}{2}}_{L^{q'}_\omega L^1_i} \|f\chi_{\tau^j_{k'}} \otimes \chi_{\tau^j_{m'}}\|^{\frac{p'}{2}}_{L^{q'}_\omega L^1_i}.$$

Proof. By an affine transformation we may take τ^j_k, τ^j_m to be centered at the origin.

Applying the hypothesis $K^*(q' \times q' \to \frac{p'}{2})$ to tubes of eccentricity $2^j\delta$ we see that

$$\|X^*_{2^j\delta} f X^*_{2^j\delta} g\|_{p'/2} \lesssim (2^j\delta)^{-\frac{2n}{p}+2}\|f\|_{L^{q'}_\omega L^1_i}\|g\|_{L^{q'}_\omega L^1_i}$$

for all f, g whose ω-supports are on disjoint cubes. Applying the rescaling $(\underline{x}, x_n) \to (2^{-j}\underline{x}, x_n)$, $(\omega, i) \to (2^{-j}\omega, 2^{-j}i)$ to this estimate we obtain

$$\|X_\delta^* f X_\delta^* g\|_{p'/2} \lesssim 2^{j(n-1)} 2^{j(n-1)} 2^{-\frac{n-1}{p'/2}j} (2^j\delta)^{-\frac{2n}{p}+2} 2^{-\frac{n-1}{q}j} \|f\|_{L^{q'}_\omega L^1_i} 2^{-\frac{n-1}{q}j} \|g\|_{L^{q'}_\omega L^1_i}$$

whenever f and g are supported on $\tau_k^j \times \tau_m^j$ and $\tau_{\tilde{k}}^j \times \tau_{m'}^j$ respectively, and the proposition follows from substitution and some algebra. $\square$

From this proposition (42) reduces to

$$\sum_j \sum_{k,k': \tau_k^j \sim \tau_{k'}^j} \sum_{m,m'} 2^{(1-\frac{(n-1)p'}{q})j} \|f\chi_{\tau_k^j} \otimes \chi_{\tau_m^j}\|^{\frac{p'}{2}}_{L^{q'}_\omega L^1_i} \|f\chi_{\tau_{k'}^j} \otimes \chi_{\tau_{m'}^j}\|^{\frac{p'}{2}}_{L^{q'}_\omega L^1_i} \lesssim \|f\|^{p'}_{L^{q'}_\omega L^1_i}.$$

Since there are only logarithmically many j's it suffices to show this for a fixed j. By polarization it suffices to show that

$$\sum_k \sum_m 2^{(1-\frac{(n-1)p'}{q})j} \|f\chi_{\tau_k^j} \otimes \chi_{\tau_m^j}\|^{p'}_{L^{q'}_\omega L^1_i} \lesssim \|f\|^{p'}_{L^{q'}_\omega L^1_i},$$

which we rewrite as

$$2^{(\frac{1}{p'} - \frac{n-1}{q})j} \left(\sum_k \sum_m \|f_{k,m}\|^{p'}_{L^{q'}_\omega L^1_i} \right)^{1/p'} \lesssim \|f\|_{L^{q'}_\omega L^1_i}, \tag{43}$$

where $f_{k,m} = f\chi_{\tau_k^j} \otimes \chi_{\tau_m^j}$. It suffices to verify this for the case $p = 1$ and for the endpoint $(p, q) = (n, n)$, since the general case $1 \le p \le q \le (n-1)p'$ follows by interpolation. In these two cases (43) becomes

$$2^{-\frac{n-1}{q}j} \sup_{k,m} \|f_{k,m}\|_{L^q_\omega L^1_\omega} \lesssim \|f\|_{L^q_\omega L^1_\omega}, \tag{44}$$

$$\left(\sum_k \sum_m \|f_{k,m}\|^{n'}_{L^{n'}_\omega L^1_i} \right)^{1/n'} \lesssim \|f\|_{L^{n'}_\omega L^1_i}, \tag{45}$$

respectively. The estimate (44) is trivial, while (45) follows from a further interpolation between the trivial estimates

$$\sup_{k,m} \|f_{k,m}\|_{L^\infty_\omega L^1_i} \lesssim \|f\|_{L^\infty_\omega L^1_i},$$
$$\sum_k \sum_m \|f_{k,m}\|_{L^1_\omega L^1_i} \lesssim \|f\|_{L^1_\omega L^1_i}. \qquad \square$$

We note that if one inserts the result of Theorem 3.4 into the above line of reasoning, then one not only recovers Wolff's Kakeya estimate, but also the entropy estimate improvement proven in Lemma 2.1 of [28].

3.11. **Necessity of** (22)**-**(23)**.** We now show that the assumptions (22) and (23) in Conjecture 3.3 are necessary.

To show the necessity of (22), we take

$$f(\omega, i) = \chi_{\mathcal{E}_1}(\omega)\delta_{i,0}, \quad g(\omega, i) = \chi_{\mathcal{E}_2}(\omega)\delta_{i,i_0},$$

where $\delta_{i,j}$ denotes the Kronecker delta function and i_0 is a suitable point. A routine calculation using (21) shows that (if i_0 is chosen properly) $X^*f, X^*g \sim 1$ on a ball of radius ~ 1. Inserting this into $K^*(q' \times q' \to \frac{p'}{2})$ yields

$$1 \lesssim \delta^{-\frac{2n}{p}+2},$$

and by taking $\delta \to 0$ one obtains (22).

To show the necessity of (23), we adapt the "stretched caps" example used to show (6). We consider a tube $T = \mathbf{R} \times B_{n-2}(0,\delta)$ in $\mathbf{R}^{n-1}$, and take

$$f(\omega, i) = \chi_{\mathcal{E}_1 \cap T}(\omega)\delta_{i,i_1}, \quad g(\omega, i) = \chi_{\mathcal{E}_2 \cap T}(\omega)\delta_{i,i_2}.$$

If i_1, i_2 are chosen appropriately, then X^*f, X^*g are both comparable to 1 on a slab which looks roughly like $B_2(0,1) \times B_{n-2}(0,\delta)$. Inserting this into $K^*(q' \times q' \to \frac{p'}{2})$ yields

$$\delta^{\frac{n-2}{p'/2}} \lesssim \delta^{-\frac{2n}{p}+2}\delta^{\frac{n-2}{q'}}\delta^{\frac{n-2}{q'}},$$

and by taking $\delta \to 0$ one obtains (23).

4. Applications

In this section we use the bilinear estimates above to prove the following restriction theorems.

Theorem 4.1. *If $n = 3$, then $\mathbf{R}^*(p \to q)$ holds whenever $p > \frac{170}{77}$ and $q > \frac{34}{9}$. Furthermore, $\mathbf{R}^*_s(q)$ holds for all $q > 4 - \frac{5}{27}$.*

The proof of this theorem will be based on bilinear versions of certain arguments of Bourgain ([2], [6]; see also [17]). The first step will be to obtain localized linear and bilinear restriction theorems.

Definition 4.2. If $1 \le p, q \le \infty$ and $\alpha \ge 0$, then we use $\mathbf{R}^*(p \to q, \alpha)$ to denote the estimate

$$\|\Re^* f\|_{L^q(B_R)} \lesssim R^\alpha \|f\|_p,$$

and $\mathbf{R}^*(p \times p \to q, \alpha)$ to denote the estimate

$$\|\Re^* f \Re^* g\|_{L^q(B_R)} \lesssim R^\alpha \|f\|_p \|g\|_p,$$

where $f, g, \Re^*$ are as in Section 2 and B_R is a ball of radius R in $\mathbf{R}^n$ (the center of B_R is irrelevant by translation symmetry).

It will be convenient to recast these estimates as a restricted bilinear estimate on the Fourier transform.

Proposition 4.3. *$R^*(p \times p \to q, \alpha)$ is true if and only if one has*

$$\|\hat{f}\hat{g}\|_{L^p(B(x,R))} \lesssim R^\alpha R^{-1/p'}\|f\|_p R^{-1/p'}\|g\|_p \tag{46}$$

for all $R \gg 1$, $x \in \mathbf{R}^n$ and all functions f, g supported on A_1^R, A_2^R, where

$$A_i^R = \{(\underline{x}, \Phi(\underline{x}) + t) : \underline{x} \in Q_i, |t| \lesssim R^{-1}\}.$$

Proof. If $R^*(p \times p \to q, \alpha)$ holds, then (46) follows by translating Φ by $O(1/R)$ and averaging using Hölder's inequality. Now suppose that (46) holds. To show $R^*(p \times p \to q, \alpha)$ it suffices to show that

$$\|(\hat{\phi}_R \Re^* f)(\hat{\phi}_R \Re^* g)\|_q \lesssim R^\alpha \|f\|_p \|g\|_p,$$

where ϕ_R is a real radial L^1-normalized bump function adapted to $B(0, C/R)$, such that $\hat\phi_R$ is non-negative on $B(x, R)$. But this follows from (46), Young's inequality, and the identity

$$\hat\phi_R \mathfrak{R}^* f = \widehat{\tilde f d\sigma * \phi_R},$$

where $\tilde f(\underline{x}, \Phi(\underline{x})) = f(\underline{x})$ is the lift of f to the surface $\{(\underline{x}, \Phi(\underline{x})) : \underline{x} \in Q\}$. □

From this proposition and the trivial estimate

$$\|\hat f \hat g\|_1 \le \|\hat f\|_2 \|\hat g\|_2 = \|f\|_2 \|g\|_2$$

we obtain the bilinear trace lemma

$$\mathbf{R}^*(2 \times 2 \to 1, 1). \tag{47}$$

Thus by interpolating this with other estimates (such as (2)) we may obtain estimates of the form $\mathbf{R}^*(p \times p \to q, \alpha)$ with a large value of α. To lower the value of α we will use a bilinear form of an argument of Bourgain [2, 6] (see also [17]):

Lemma 4.4. *If $2 < p, q < \infty$ and $\alpha > 0$ are such that $K^*(\frac{p}{2} \times \frac{p}{2} \to \frac{q}{2})$ and $R^*(2 \times 2 \to q, \alpha)$ hold, then $R^*(p \times p \to q, \frac{\alpha}{2} + \varepsilon)$ holds for all $\varepsilon > 0$.*

Proof. From Proposition 4.3 it suffices to show that

$$\|\hat f \hat g\|_{L^q(B(0,R^2))} \lesssim R^{\alpha+\varepsilon} R^{-2/p'} \|f\|_p R^{-2/p'} \|g\|_p \tag{48}$$

for all f, g supported on $A_1^{R^2}$, $A_2^{R^2}$ respectively, and all $\varepsilon > 0$. (The implicit constants will depend on ε.)

Let ϕ_R be as in Proposition 4.3, and define $\phi_R^x(\xi) = e^{-2\pi i x \cdot \xi} \phi_R(\xi)$ for all $x \in \mathbf{R}^n$. Then from the hypothesis $R^*(2 \times 2 \to q, \alpha)$ and Proposition 4.3 we have

$$\|\widehat{\phi_R^x} \hat f \widehat{\phi_R^x} \hat g\|_{L^q(B(x,R))} \lesssim R^\alpha R^{-1/2} \|f * \phi_R^x\|_2 R^{-1/2} \|g * \phi_R^x\|_2$$

for all x. Averaging this over all $x \in B(0, R^2)$ we obtain

$$\|\hat f \hat g\|_{L^q(B(0,R^2))} \lesssim R^\alpha (R^{-n} \int_{B(0,R^2)} (R^{-1/2} \|f * \phi_R^x\|_2 R^{-1/2} \|g * \phi_R^x\|_2)^q \, dx)^{1/q}.$$

Thus to show (48) it suffices to show that

(49)

$$R^{-n} \int_{B(0,R^2)} (R^{-1/2} \|f * \phi_R^x\|_2 R^{-1/2} \|g * \phi_R^x\|_2)^q \, dx \lesssim R^\varepsilon (R^{-2/p'} \|f\|_p R^{-2/p'} \|g\|_p)^q.$$

This will be accomplished by repeated use of the uncertainty principle and Plancherel's theorem, together with the Kakeya hypothesis.

Let $\mathcal{E}$, $\mathcal{E}_1$, $\mathcal{E}_2$ be as in Section 3 with $\delta = \frac{1}{R}$. We partition the annuli $A_i^{R^2}$ into caps C_ω for $\omega \in \mathcal{E}_i$, defined by

$$C_\omega = \{(\underline{x}, x_n) \in A_i^{R^2} : -\nabla\Phi(x) = \omega + O(\frac{1}{R})\}.$$

From the ellipticity of Φ and some elementary geometry we see that the C_ω are essentially disks of diameter $1/R$ and thickness $1/R^2$ oriented in the direction $(\omega, 1)$, which form a finitely overlapping cover of $A_i^{R^2}$. We decompose

$$f = \sum_{\omega \in \mathcal{E}_1} f_\omega, \quad g = \sum_{\omega \in \mathcal{E}_2} g_\omega,$$

where f_ω, g_ω are adapted restrictions of f, g respectively to (a suitable dilate of) C_ω.

From the support conditions on f_ω, g_ω and ϕ_R^x we see that (49) reduces to

(50)
$$R^{-n}\int_{B(0,R^2)}(R^{-1/2}(\sum_{\omega\in\mathcal{E}_1}\|f_\omega*\phi_R^x\|_2^2)^{1/2}R^{-1/2}(\sum_{\omega\in\mathcal{E}_2}\|g_\omega*\phi_R^x\|_2^2)^{1/2})^q\ dx$$
$$\lesssim R^\varepsilon(R^{-2/p'}\|f\|_pR^{-2/p'}\|g\|_p)^q.$$

The function $\widehat{\phi_R^x}$ is rapidly decreasing outside of the ball $B(x,R)$. Thus by Plancherel's theorem the left-hand side of (50) is majorized by

(51)
$$R^{-n}\int_{B(0,R^2)}(R^{-1/2}(\sum_{\omega\in\mathcal{E}_1}\|\widehat{f_\omega}\|^2_{L^2(B(x,R))})^{1/2}R^{-1/2}(\sum_{\omega\in\mathcal{E}_2}\|\widehat{g_\omega}\|^2_{L^2(B(x,R))})^{1/2})^q\ dx,$$

since the portions of $\widehat{\phi_R^x}$ on translates of $B(x,R)$ can be handled by translation symmetry.

Let ψ_ω be a Schwarz function which is comparable to 1 on C_ω and rapidly decreasing away from this cap, and whose Fourier transform satisfies the pointwise estimate
$$|\hat\psi_\omega(x)|\lesssim R^{-n-1}\chi_{R^2\tilde T_0^\omega}(x),$$
where $\tilde T_0^\omega$ is a thickening of T_0^ω and $R^2\tilde T_0^\omega=\{R^2x : x\in\tilde T_0^\omega\}$.

If we define $\tilde f_\omega = f_\omega/\psi_\omega$, we have the estimate
$$|\widehat{f_\omega}(x)|=|\widehat{\tilde f_\omega}*\widehat{\psi_\omega}(x)|\lesssim R^{-n-1}\int_{x+R^2\tilde T_{\omega_0}}|\widehat{\tilde f_\omega}(y)|\ dy.$$

From Hölder's inequality and (21) we thus obtain
$$|\widehat{f_\omega}(x)|^2\lesssim R^{-n-1}\int_{x+R^2\tilde T_{\omega_0}}|\widehat{\tilde f_\omega}(y)|^2\ dy$$
$$\lesssim X^*F_\omega(\frac{x}{R^2}),$$
where
$$F_\omega(\omega',i)=\delta_{\omega,\omega'}R^{n-1}\int_{\tilde T_i^\omega}|\widehat{\tilde f_\omega}(R^2x)|^2\ dx$$
and $\delta_{\omega,\omega'}$ is the Kronecker delta.

Since X^*F_ω is essentially constant on balls of radius $1/R$ we essentially have
$$\|\widehat{f_\omega}\|^2_{L^2(B(x,R))}\lesssim R^nX^*F_\omega(\frac{x}{R^2}).$$

From this (and similar considerations for g) we see that (51) is majorized by
$$R^{-n}\int_{B(0,R^2)}(R^{-1/2}(\sum_{\omega\in\mathcal{E}_1}R^nX^*F_\omega(\frac{x}{R^2}))^{1/2}R^{-1/2}(\sum_{\omega\in\mathcal{E}_2}R^nX^*G_\omega(\frac{x}{R^2}))^{1/2})^q\ dx,$$
where G_ω is defined in analogy to F_ω. We simplify this as

(52)
$$R^{n+nq-q}\int_{B(0,1)}(X^*F(x)X^*G(x))^{q/2}\ dx,$$
where
$$F(\omega,i)=R^{n-1}\int_{\tilde T_i^\omega}|\widehat{\tilde f_\omega}(R^2x)|^2\ dx,\quad G(\omega,i)=R^{n-1}\int_{\tilde T_i^\omega}|\widehat{\tilde g_\omega}(R^2x)|^2\ dx.$$

On the other hand, from the definition of the hypothesis $K^*(\frac{p}{2} \times \frac{p}{2} \to \frac{q}{2})$ we have

$$\|X^*_{1/R} F X^*_{1/R} G\|_{q/2} \lesssim R^{\frac{2n}{q'}-2+\varepsilon} \|F\|_{L^{p/2}_\omega L^1_i} \|G\|_{L^{p/2}_\omega L^1_i}.$$

Comparing this with (50) and (52), we see that we will be done once we show that

$$R^{\frac{2}{q}(n+nq-q)} R^{\frac{2n}{q'}-2} \|F\|_{L^{p/2}_\omega L^1_i} \|G\|_{L^{p/2}_\omega L^1_i} \lesssim (R^{-2/p'} \|f\|_p R^{-2/p'} \|g\|_p)^2.$$

After some algebraic manipulation we see that it suffices to show that

$$R^{2n-\frac{4}{p}+2} \|F\|_{L^{p/2}_\omega L^1_i} \lesssim \|f\|_p^2,$$

together with the completely analogous estimate for g, G. From the definition of f_ω and the measure $d\omega$ we have

$$\|f\|_p^2 \sim R^{\frac{n-1}{p/2}} \|(\|f_\omega\|_p)^2\|_{L^{p/2}_\omega},$$

and so it suffices to show that

$$R^{2n-\frac{4}{p}+2} \|F(\omega,\cdot)\|_{L^1_i} \lesssim R^{\frac{2(n-1)}{p}} \|f_\omega\|_p^2$$

uniformly in ω. From Hölder's inequality,[8] the hypothesis $p \geq 2$ and the support conditions on f_ω we have

$$\|f_\omega\|_2 \lesssim R^{-(n+1)(\frac{1}{2}-\frac{1}{p})} \|f_\omega\|_p,$$

and so after some algebra we reduce to

$$\|F(\omega,\cdot)\|_{L^1_i} \lesssim R^{-n-1} \|f_\omega\|_2^2.$$

However, the left-hand side is majorized by

$$R^{n-1} \int_{B(0,C)} |\widehat{\tilde{f}_\omega}(R^2 x)|^2 \, dx \lesssim R^{-n-1} \|\widehat{\tilde{f}_\omega}\|_2^2,$$

and the claim follows from Plancherel's theorem and the pointwise comparability of f_ω and $\tilde{f}_\omega$. □

Applying Lemma 4.4 with $p = \frac{5}{2}$ and $q = \frac{5}{3}$ and using (24), we see that

$$\mathbf{R}^*(2 \times 2 \to \frac{5}{3}, \alpha) \implies \mathbf{R}^*(\frac{5}{2} \times \frac{5}{2} \to \frac{5}{3}, \frac{\alpha}{2} + \varepsilon)$$

for all $\alpha, \varepsilon > 0$. On the other hand, from interpolating (47) with (2) we obtain

$$\mathbf{R}^*(\frac{30}{17} \times \frac{30}{17} \to \frac{5}{3}, \frac{1}{5}),$$

so by another interpolation we obtain the implication

$$\mathbf{R}^*(\frac{5}{2} \times \frac{5}{2} \to \frac{5}{3}, \beta) \implies \mathbf{R}^*(2 \times 2 \to \frac{5}{3}, \frac{2}{5}\beta + \frac{3}{25})$$

for all $\beta > 0$. Combining these two implications we see that

$$\mathbf{R}^*(2 \times 2 \to \frac{5}{3}, \alpha) \implies \mathbf{R}^*(2 \times 2 \to \frac{5}{3}, \frac{1}{5}\alpha + \frac{3}{25} + \varepsilon).$$

The map $\alpha \to \frac{1}{5}\alpha + \frac{3}{25}$ is a contraction with fixed point $\alpha = \frac{3}{20}$. Since the estimate $\mathbf{R}^*(2 \times 2 \to \frac{5}{3}, \alpha)$ holds for at least one value of α, we thus see that

$$\mathbf{R}^*(2 \times 2 \to \frac{5}{3}, \frac{3}{20} + \varepsilon) \tag{53}$$

[8] This use of Hölder's inequality indicates some room for improvement in this Lemma. Indeed, one can replace the L^p norms on f, g by the B_p norms as used in [17], Lemma 2.2.

for all $\varepsilon > 0$. Applying Lemma 4.4 one more time, we obtain

$$\mathbf{R}^*(\frac{5}{2} \times \frac{5}{2} \to \frac{5}{3}, \frac{3}{40} + \varepsilon). \tag{54}$$

An inspection of the proof of Theorem 2.2 shows that the statement of the theorem still holds when $\mathbf{R}^*(p \times p \to q)$ and $\mathbf{R}^*(p \to 2q)$ are replaced by their local analogues $\mathbf{R}^*(p \times p \to q, \alpha)$ and $\mathbf{R}^*(p \to 2q, \alpha/2)$. Applying this to (54) we obtain

$$\mathbf{R}^*(\frac{5}{2} \to \frac{10}{3}, \frac{3}{80} + \varepsilon). \tag{55}$$

We now remove the α completely, borrowing the following argument of Bourgain [2, 6] (for the concrete case $n = 3$, $p > 20/7$, $q > 10/3$, $\alpha > 1/20$, $\tilde{p} > 7/3$, $\tilde{q} > 42/11$, see [17]):

Lemma 4.5 ([2, 6, 17])**.** *If p, q, α are such that $\frac{n+1}{2} > \alpha q$, then $\mathbf{R}^*(p \to q, \alpha)$ implies $\mathbf{R}^*(\tilde{p} \to \tilde{q})$ whenever*

$$\tilde{q} > 2 + \frac{q}{\frac{n+1}{2} - \alpha q}, \quad \frac{\tilde{q}}{\tilde{p}} < 1 + \frac{q}{p}\frac{1}{\frac{n+1}{2} - \alpha q}.$$

Applying this to (55) we obtain the first conclusion of Theorem 4.1. Using Theorem 2.2 to return to the bilinear setting, we thus obtain

$$\mathbf{R}^*(p \times p \to q) \text{ for } p > \frac{170}{77}, q > \frac{17}{9}.$$

Interpolating this with (2) and using Theorem 2.2, one obtains the second conclusion of Theorem 4.1.

We summarize the various estimates used in Figure 3 (on the next page), which is an expanded version of Figure 1. For comparison, the previously known results are also displayed. The dotted line thus represents the best global restriction theorems (both linear and bilinear) known to date. (It is possible to improve on these results slightly; see [25].)

By interpolating between the main result and (2) we also obtain some progress on Klainerman's conjecture for the sphere in $\mathbf{R}^3$:

Corollary 4.6. *If $n = 3$, then $R^*(2 \times 2 \to p)$ whenever $p > 2 - \frac{5}{69}$.*

These techniques are certainly not best possible. For instance, one can use the techniques in [5] to obtain better versions of Corollary 4.6. See [25].

The sharp restriction theorem $R_s^*(q)$ is scale-invariant under parabolic scaling. Thus, the compact support condition on Φ can be removed. In particular, one has a sharp restriction theorem for the entire paraboloid $\{(\underline{x}, \frac{1}{2}|\underline{x}|^2) : \underline{x} \in \mathbf{R}^{n-1}\}$ for $q > 4 - \frac{5}{27}$.

One can extend the above results to Bochner-Riesz multipliers, so that the Bochner-Riesz conjecture holds for $n = 3$ and $\max(p, p') \geq \frac{34}{9}$. We sketch the argument very briefly as follows. By the usual techniques of Carleson-Sjölin reduction and factorization theory (see [6]) it suffices to show that

$$\|Tf\|_{L^p(\mathbf{R}^n)} \lesssim \lambda^{-n/p}\|f\|_{L^\infty(Q)}$$

for all $p > 34/9$, $\lambda \gg 1$ and $f \in L^\infty(Q)$, where

$$Tf(x) = \int_Q e^{2\pi i\lambda|x-y|}a(x,y)f(y)\, dy,$$

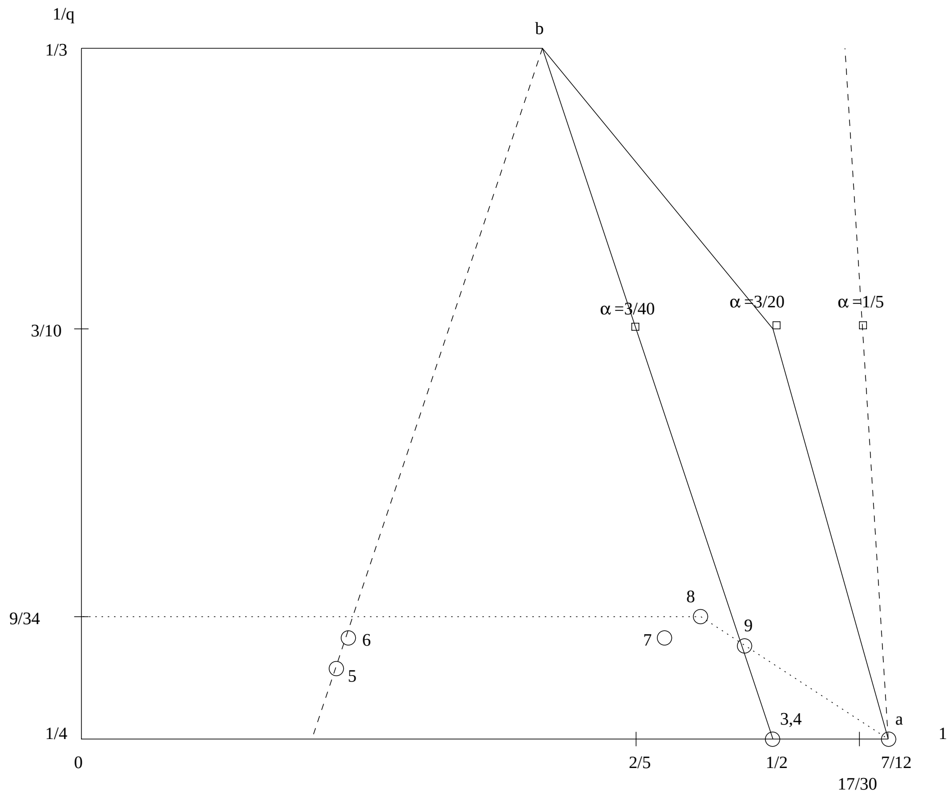

FIGURE 3. Estimates of the form $R^*(p \times p \to q, \alpha)$ and $R^*(p \to 2q, \alpha/2)$ for $n = 3$.

Q is thought of as imbedded in $\mathbf{R}^n$, and a is a bump function on $\mathbf{R}^n \times Q$ which is supported away from the diagonal $x = y$. By the analogue of Lemma 4.5 for Bochner-Riesz multipliers (see [6]) it suffices to show that

$$\|Tf\|_{10/3} \lesssim \lambda^{3/80+\varepsilon} \lambda^{-9/10} \|f\|_\infty$$

for all $\varepsilon > 0$. By a modification of Theorem 2.2 it suffices to show that

$$\|TfTg\|_{5/3} \lesssim \lambda^{3/40+\varepsilon} \lambda^{-9/10} \|f\|_\infty \lambda^{-9/10} \|g\|_\infty$$

for all f, g with $O(1)$ separated supports, together with variants of this estimate in which the phase function $|x-y|$ is replaced by a parabolically scaled (but essentially equivalent) version. However, from the analogue of Lemma 4.4 for Bochner-Riesz operators (which is proven by a bilinear modification of the arguments in [6]) this will follow from the restriction estimate (53) and the analogue of Theorem 3.4 for the Nikodym maximal operator (see e.g. [27]), which is proven similarly. The required Nikodym estimate also follows formally from the original formulation of Theorem 3.4; see the argument in [24].

In higher dimensions $n > 3$ Theorem 2.3 becomes too weak to be of much use, and we can only achieve a minor improvement on known results. By interpolating between (47) and the bilinear form $R^*(2 \times 2 \to \frac{n+1}{n-1})$ of the Tomas-Stein theorem, we obtain

$$R^*(2 \times 2 \to \frac{n+2}{n}, \frac{1}{n+2}).$$

Applying this and Theorem 3.4 to Lemma 4.4 we obtain

$$R^*(\frac{2(n+2)}{n+1} \times \frac{2(n+2)}{n+1} \to \frac{n+2}{n}, \frac{1}{2(n+1)} + \varepsilon).$$

Applying Theorem 2.2 this becomes

$$R^*(\frac{2(n+2)}{n+1} \to \frac{2(n+2)}{n}, \frac{1}{4(n+1)} + \varepsilon).$$

Applying Lemma 4.5 this becomes

$$R^*(p \to q) \text{ for } p > \frac{2n^2+6n+6}{n^2+3n+1}, \quad q > \frac{2n^2+6n+6}{n^2+n-1}.$$

This is only a slight improvement on the result in Wolff [27], which showed $R^*(q \to q)$ for the same range of q. For $n > 3$ the results obtained by interpolating these estimates with Theorem 2.3 are inferior to the Tomas-Stein theorem.

5. Further remarks

In the previous sections we obtained a non-trivial sharp restriction theorem $R^*_s(4-\varepsilon)$ from an ordinary restriction theorem $R^*(p \to q)$ (in this case $p > \frac{170}{77}, q > \frac{34}{9}$) and the bilinear estimate (2).

The original formulation of (2) in [17, 18] was stated in terms of X_r spaces. In this section we show how one can use these estimates instead of the bilinear estimate to obtain non-trivial sharp restriction theorems. Despite the fact that these estimates can be extended (for characteristic functions) from $r > \frac{12}{7}$ to $r \geq 4(\sqrt{2}-1)$, the methods we will use do not appear to be as efficient as the bilinear techniques. However, they seem to be more robust and applicable to a wider range of situations.

Proposition 5.1. *Let $n = 3$. Suppose that $R^*(p \to q)$ holds for some $2 < q < 4$, and suppose that the quantity $r = \frac{4p'}{q}$ satisfies $r > 4(\sqrt{2}-1)$. Then we have $R^*_s(w)$ for all $w > \frac{4+q}{2}$.*

Note that the points $(\frac{1}{p}, \frac{1}{q}), (1 - \frac{2}{w}, \frac{1}{w}), (\frac{1}{r}, \frac{1}{4})$ are collinear when $w = \frac{4+q}{2}$.

Proof. It suffices to show the restricted weak-type estimate

$$|\{|\Re^* \chi_\Omega| \gtrsim \lambda\}| \lesssim \frac{2^{-2(w-2)j_0}}{\lambda^w} \tag{56}$$

for all $\lambda > 0$ and $\Omega \subseteq Q$, where j_0 is the integer such that $|\Omega| \sim 2^{-2j_0}$. We may assume that $2^{-Cj_0} \lesssim \lambda \lesssim 1$ for some constant C, since (56) is trivial (by e.g. the Tomas-Stein restriction theorem) otherwise.

The idea of the proof will be to decompose Ω into a sparse set and a collection of sets concentrated on caps. On the sparse set the Tomas-Stein estimate $R^*_s(4)$ can be improved using the X_r estimates of [17, 18]. The sets on caps can be rescaled parabolically to become sets of measure comparable to 1, in which case the estimate $R^*(p \to q)$ is equivalent to the scale-invariant estimate $R^*_s(q)$. Combining the two estimates one expects to obtain $R^*_s(4-\varepsilon)$ for some $\varepsilon > 0$.

We now turn to the details. For each j let $0 < \alpha_j \leq 1$ be a quantity to be chosen later. By the usual Calderón-Zygmund stopping time arguments, we may partition Ω as

$$\Omega = \Omega_g \cup \bigcup_{(j,k) \in T} (\Omega \cap \tau_k^j),$$

where the "good" set Ω_g satisfies

$$|\Omega_g \cap \tau_k^j| \leq \alpha_j |\tau_k^j| \tag{57}$$

for all j, k, and $\{\tau_{j,k} : (j,k) \in T\}$ is a collection of disjoint dyadic cubes such that

$$\alpha_j |\tau_k^j| < |\Omega \cap \tau_k^j| \leq 4\alpha_{j-1} |\tau_k^j| \tag{58}$$

for all $(j,k) \in T$.

We decompose $\Re^* \chi_\Omega$ as

$$\Re^* \chi_\Omega = \Re^* \chi_{\Omega_g} + \sum_j \sum_{k:(j,k) \in T} \Re^* \chi_{\Omega \cap \tau_k^j}.$$

To control the contribution of $\Re^* \chi_{\Omega_g}$ we use the results of [17, 18] and the hypothesis $r > 4(\sqrt{2} - 1)$ to obtain

$$\| \Re^* \chi_{\Omega_g} \|_4 \lesssim \|\chi_{\Omega_g}\|_{X_r} = \Big(\sum_j \sum_k 2^{-4j} \Big(\frac{|\Omega_g \cap \tau_k^j|}{|\tau_k^j|} \Big)^{4/r} \Big)^{1/4}. \tag{59}$$

In order for $R^*(p \to q)$ to hold we must have $p' \leq \frac{1}{2} q$, so that $4/r \geq 2 > 1$. Applying (66) of Lemma 6.2 with $p = 4/r$ and $\alpha = \alpha_j$ we obtain (after some algebra)

$$\sum_k 2^{-4j} \Big(\frac{|\Omega_g \cap \tau_k^j|}{|\tau_k^j|} \Big)^{4/r} \lesssim \min(2^{(\frac{8}{r} - 4)(j - j_0)} 2^{-4j_0}, \alpha_j^{\frac{4}{r} - 1} 2^{2(j_0 - j)} 2^{-4j_0})$$

for each j; informally, this shows that the most significant scales occur when j is near j_0. Inserting these estimates into (59) we obtain

$$\| \Re^* \chi_{\Omega_g} \|_4^4 \lesssim \sum_j \min(2^{(\frac{8}{r} - 4)(j - j_0)}, \alpha_j^{\frac{4}{r} - 1} 2^{2(j_0 - j)}) 2^{-4j_0}. \tag{60}$$

Let m be a positive integer to be chosen later. We now choose α_j so that

$$\alpha^{\frac{4}{r} - 1} 2^{2(j_0 - j)} = 2^{-m} \tag{61}$$

when $-\frac{m}{\frac{8}{r} - 4} < j - j_0 < \frac{m}{2}$, and $\alpha_j = 1$ otherwise. The estimate (60) then becomes

$$\| \Re^* \chi_{\Omega_g} \|_4^4 \lesssim m 2^{-m} 2^{-4j_0}.$$

From Tchebyshev's inequality we thus obtain

$$|\{ \Re^* \chi_{\Omega_g} \gtrsim \lambda \}| \lesssim m 2^{-m} 2^{-4j_0} \lambda^{-4}. \tag{62}$$

It remains to estimate the quantity

$$\sum_{k:(j,k) \in T} \Re^* \chi_{\Omega \cap \tau_k^j} \tag{63}$$

for each j. Note that this quantity vanishes when $\alpha_j = 1$ by (58), so we may assume that $-\frac{m}{\frac{8}{r} - 4} < j - j_0 < \frac{m}{2}$.

We will estimate (63) in L^q norm. From Lemma 6.3 in the Appendix we have

$$\| \sum_{k:(j,k)\in T} \Re^* \chi_{\Omega\cap\tau_k^j} \|_q \lesssim (\sum_{k:(j,k)\in T} \| \Re^* \chi_{\Omega\cap\tau_k^j} \|_q^{q'})^{1/q'}. \tag{64}$$

On the other hand, by parabolically rescaling the hypothesis $R^*(p \to q)$ as in Proposition 2.6 we obtain

$$\| \Re^* \chi_{\Omega\cap\tau_k^j} \|_{L^q} \lesssim |\tau_k^j|^{1-\frac{2}{q}-\frac{1}{p}} |\Omega\cap\tau_k^j|^{1/p}.$$

Combining this with (64) and (58) we obtain

$$\| \sum_{k:(j,k)\in T} \Re^* \chi_{\Omega\cap\tau_k^j} \|_q \lesssim \big(\sum_{k:(j,k)\in T} (|\tau_k^j|^{1-\frac{2}{q}} (4\alpha_{j-1})^{1/p})^{q'} \big)^{1/q'}.$$

On the other hand, from (58) we have

$$\#\{k : (j,k) \in T\} \le 2^{2(j-j_0)} \alpha_j^{-1}.$$

Using this and the fact that $4\alpha_{j-1} \sim \alpha_j$ we obtain

$$\| \sum_{k:(j,k)\in T} \Re^* \chi_{\Omega\cap\tau_k^j} \|_q \lesssim |\tau_k^j|^{1-\frac{2}{q}} 2^{\frac{2}{q'}(j-j_0)} \alpha_j^{\frac{1}{p}-\frac{1}{q'}}. \tag{65}$$

Combining this with (61) and the definition of r one eventually obtains

$$\| \sum_{k:(j,k)\in T} \Re^* \chi_{\Omega\cap\tau_k^j} \|_q \lesssim 2^{-2(1-\frac{2}{q})j_0} 2^{\frac{m}{q}}$$

so by the triangle inequality and Tchebyshev's inequality we have

$$\{ | \sum_j \sum_{k:(j,k)\in T} \Re^* \chi_{\Omega\cap\tau_k^j} | \gtrsim \lambda \} \lesssim 2^{-2(q-2)j_0} m^q 2^m \lambda^{-q}.$$

Combining this with (62) we obtain

$$\{ |\Re^* \chi_\Omega| \gtrsim \lambda \} \lesssim m 2^{-m} 2^{-4j_0} \lambda^{-4} + 2^{-2(q-2)j_0} m^q 2^m \lambda^{-q}.$$

The claim (56) then follows by choosing $2^m = (2^{2j_0}\lambda)^{-\frac{4-q}{2}}$. $\square$

In practice Proposition 5.1 is inferior to the implications obtained by Theorem 2.2 and interpolation with (2). For instance, if we insert the first conclusion of Theorem 4.1 into Proposition 5.1 one obtains $R_s^*(w)$ for $w > 4 - \frac{1}{9}$, which is inferior to the second conclusion of Theorem 4.1.

6. Appendix: Some elementary harmonic analysis

In this section we state some elementary results which were used repeatedly in the paper.

We begin with a well-known quasi-orthogonality property of functions with disjoint frequency support. Define a *rectangle* to be the product of n (possibly half-infinite or infinite) intervals in $\mathbf{R}^n$.

Lemma 6.1. *Let R_k be a collection of rectangles in frequency space such that the dilates $2R_k$ are almost disjoint, and suppose that f_k are a collection of functions whose Fourier transforms are supported on R_k. Then for all $1 \le p \le \infty$ we have*

$$\| \sum_k f_k \|_p \lesssim (\sum_k \|f_k\|_p^{p^*})^{1/p^*},$$

where $p^* = \min(p, p')$.

Proof. Let P_k be a smooth Fourier multiplier adapted to $2R_k$ which equals 1 on R_k. We claim that

$$\| \sum_k P_k F_k \|_p \lesssim (\sum_k \|F_k\|_p^{p^*})^{1/p^*}$$

for arbitrary functions F_k; the lemma then follows by setting $F_k = P_k F_k = f_k$.

By interpolation it suffices to prove this estimate for $p = 1$, $p = 2$, and $p = \infty$. When $p = 2$ the estimate is immediate from Plancherel's theorem. When $p = 1$ or $p = \infty$ the lemma follows from the triangle inequality and the estimates

$$\|P_k F_k\|_1 \lesssim \|F_k\|_1, \quad \|P_k F_k\|_\infty \lesssim \|F_k\|_\infty,$$

which follow from Young's inequality and standard estimates on the kernel of P_k. □

The next lemma allows us to crudely estimate various X_r-type quantities.

Lemma 6.2. *Let $\Omega \subseteq Q$ be a set such that $|\Omega| \lesssim 2^{-(n-1)j_0}$ for some $j_0 \geq 0$, and let τ_k^j be defined as in Section 2.5. Let $0 \leq \alpha \leq 1$ be such that $|\Omega \cap \tau_k^j| \leq \alpha |\tau_k^j|$ for all k. If $p \geq 1$, then we have the estimate*

$$\sum_k |\Omega \cap \tau_k^j|^p \lesssim 2^{-(n-1)j_0} \min(\alpha 2^{-(n-1)j}, 2^{-(n-1)j_0})^{p-1}. \tag{66}$$

If $p \leq 1$, then we have the estimate

$$\sum_k |\Omega \cap \tau_k^j|^p \lesssim 2^{-(n-1)pj_0} 2^{(n-1)(1-p)j}. \tag{67}$$

Proof. By the log-convexity of L^p norms, $0 \leq p \leq \infty$, it suffices to prove the three bounds

$$\sup_k |\Omega \cap \tau_k^j| \lesssim \min(\alpha 2^{-(n-1)j}, 2^{-(n-1)j_0}), \tag{68}$$

$$\sum_k |\Omega \cap \tau_k^j| \lesssim 2^{-(n-1)j_0}, \tag{69}$$

$$\sum_k 1 \lesssim 2^{(n-1)j}. \tag{70}$$

But these bounds follow trivially from the estimates

$$|\Omega \cap \tau_k^j| \leq \min(|\Omega|, \alpha|\tau_k^j|) \lesssim \min(2^{-(n-1)j_0}, \alpha 2^{-(n-1)j}),$$

$$\sum_k |\Omega \cap \tau_k^j| = |\Omega| \lesssim 2^{-(n-1)j_0},$$

and the cardinality of the τ_k^j. □

We remark that without further information on Ω these bounds are best possible. In most cases we will set $\alpha = 1$.

Finally, we present a very easy inequality.

Lemma 6.3. *If $1 \leq p \leq \infty$, then*

$$(\sum_k |a_k|^p)^{1/p} \leq \sum_k |a_k|$$

for all sequences of numbers a_k. *Also, if* $0 < q \leq 1$, *then*

$$\| \sum_k f_k \|_q \leq (\sum_k \|f_k\|_q^q)^{1/q}.$$

Proof. The first estimate is trivial for $p = 1$ and $p = \infty$, and the general case follows by convexity. The second estimate follows by applying the first with $a_k = |f_k(x)|^q$, $p = 1/q$ and integrating. □

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Department of Mathematics, University of California–Los Angeles, Los Angeles, California 90024

E-mail address: tao@math.ucla.edu

Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain

E-mail address: ana.vargas@uam.es

Departamento de Matemáticas, Universidad del País Vasco, Apartado 644, 48080, Bilbao, Spain

E-mail address: mtpvegol@lg.ehu.es