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INTEGRAL CRYSTALLINE COHOMOLOGY OVER VERY RAMIFIED VALUATION RINGS

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1. Introduction

In this paper we intend to show how Fontaine's comparison-theory ([Fo1], [Fo2]) between crystalline and p-adic étale cohomology can be extented to very ramified base rings. So far it had been developed either for $\mathbb{Q}_p$-coefficients, or $\mathbb{Z}_p$-coefficients if the base is unramified over the Witt-vectors (see [Fa2]). For example it states that for any p-adic discrete valuation-ring V there exists a ring $B(V)$ with the following property:

If X denotes a smooth and proper V-scheme, the p-adic étale cohomology of the generic fiber $X \otimes_V \bar{K}$ is related to the crystalline cohomology of X/V_0 (V_0 defined below) by an isomorphism

$$H^*_{et}(X \otimes_V \bar{K}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B(V) = H^*_{crys}(X/V_0) \otimes_{V_0} B(V).$$

One can recover the cohomologies from the above by either taking Frobenius-invariants in filtration degree 0, or Galois invariants. Thus crystalline and étale cohomology determine each other.

A more general theory might be possible, but so far the attempts to do that have not been entirely successful (see [Fa4]), as the theorem there does not seem to have any applicable consequences. However it turns out that for schemes for which the Hodge cohomology has no torsion, and also for p-divisible groups (in some sense an honorable member of the previous class), one can find meaningful results. A. Vasiu has applied them (Ph.D. Thesis, Princeton, 1994) to construct good models for certain Shimura-varieties.

The main new (for this purpose) idea is the use of crystalline cohomology over the PD-hull with respect to (V_0, p) of the base, instead of the simplified versions used by Fontaine. Thus the point we want to make here is that his ideas can yield stronger results than stated in the literature so far.

There is a perennial question of whether one should strive for maximal generality. The most important examples are abelian varieties, and for them we give a mostly self-contained treatment. On the other hand one can generalise to proper smooth schemes, or smooth schemes admitting a good compactification, or semistable schemes, etc. For these one needs the general comparison theorem proved in [Fa2] and [Fa4], and I see no good reason to repeat the arguments there in detail, since I cannot offer any improvement. However I give an overview, and

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for abelian varieties the comparison theorem is reproved here. The reader only interested in this special case can ignore all discussions about crystalline cohomology and log-structures, and the ring $B^+(R)$ for general base rings R.

As usual in this theory V denotes a complete discrete valuation-ring, π one of its uniformisers. We assume that the residue field $k = V/\pi \cdot V$ is perfect of characteristic $p > 0$, and denote by K the fraction field of V, which we assume to have characteristic zero. Furthermore $V_0 = W(k) \subset V$ is the ring of Witt-vectors, K_0 its field of fractions, and $e = [K : K_0]$ the ramification degree. Also φ denotes the Frobenius on V_0. We work with the crystalline site of PD-nilpotent embeddings that is Berthelot's nilpotent crystalline site. Unfortunately this excludes $p = 2$. For this prime one can use Berthelot's crystalline site. However as some basic finiteness results are not well documented in the literature we explain this variant only in an appendix. I have to thank the referee for pointing out this difficulty. So from now on until section 7 the prime p is always at least three. We add some explanatory material as our definitions are slightly different from [Fa2] and [Fo2]. The work was supported by the NSF, grant DMS-9303475. The referees and editors of Journal of the AMS have made a tremendous effort to make the presentation as clear as possible. If they have not succeeded this is entirely my fault. I thank them heartily.

2. Hodge cohomology and crystalline cohomology

Suppose X is a proper and smooth V-scheme. Then the crystalline cohomology of X should be a filtered V_0-crystal on the base $Spec(V)$. It is known that such crystals are not entirely determined by their value on $Spec(V)$, but one has to use a smooth PD-hull, as follows:

V is a totally ramified extension of V_0, and its uniformiser π has minimal equation $f(\pi) = 0$. Here

$$f(t) = T^e + \sum_{0 \leq i < e} a_i \cdot T^i$$

is an Eisenstein polynomial, that is, all a_i are divisible by p, and a_0/p is a unit. If $R = V_0[[T]]$ denotes the ring of formal power-series, then $V = R/f \cdot R$. The PD-hull R_V of V is the PD-completion of the ring obtained by adjoining to R divided powers $f^n/n!$. As (p) has already divided powers we might as well adjoin instead $T^{e \cdot n}/n!$, so that R_V depends indeed only on the ramification index e. Obviously R_V is contained in $K_0[[T]]$, and consists of power series $\sum a_n \cdot T^n$ such that $a_n \cdot [n/e]!$ is integral for all n. A decreasing filtration is defined on R_V by the rule that $F^q(R_V)$ is the closure of the ideal generated by divided powers $f^n/n!$ with $n \geq q$. This is just the usual PD-filtration, and it depends on V and not just e. If X is a smooth and proper V-scheme we can define the relative crystalline cohomology of X/R_V, as in [B], Ch. III, 1. It uses infinitesimal thickenings $\mathcal{U}$ of open subsets $U \subset X$ such that $\mathcal{U}$ is a scheme over R_V, and the ideal defining U in $\mathcal{U}$ has a nilpotent PD-structure compatible with that on $F^1(R_V)$ and (p). A sheaf $\mathcal{F}$ on this site associates to each $\mathcal{U}$ as above a Zariski sheaf $\mathcal{F}_{\mathcal{U}}$ on $\mathcal{U}$, and to each morphism $f : \mathcal{U}_1 \to \mathcal{U}_2$ of PD-thickenings a pullback map $f^*(\mathcal{F}_{\mathcal{U}_2}) \to \mathcal{F}_{\mathcal{U}_1}$ satisfying the usual compatabilities. For example the structure sheaves $\mathcal{O}_{\mathcal{U}}$ define a sheaf of rings $\mathcal{O}_{\mathcal{X}}/R_V$, which is filtered by the divided powers $F^q(\mathcal{O}_{\mathcal{X}/R_V})$ of the ideal defining X.

A crystal of vector bundles is a crystalline sheaf $\mathcal{E}$ such that each $\mathcal{E}_{\mathcal{U}}$ is locally free of finite rank over $\mathcal{O}_{\mathcal{U}}$, and such that the pullbacks f^* induce isomorphisms of vector bundles. For example if X lifts to a smooth formal scheme $\mathcal{X}$ over R_V, such a crystal corresponds to a vector bundle $\mathcal{E}_{\mathcal{X}}$ on $\mathcal{X}$ with an integrable connection $\bigtriangledown$. In general one can cover X by opens $U_i \subset X$ which embed into smooth formal R_V-schemes $\mathcal{U}_i$, and then one has to give the evaluations $\mathcal{E}_{\mathcal{U}_i}$ of $\mathcal{E}$ on the completed PD-hull of U_i in $\mathcal{U}_i$. These are vector bundles with integrable connection $\bigtriangledown$ as before. Furthermore the induced objects on the completed PD-hull of $U_i \cap U_j \subset \mathcal{U}_i \times_{R_V} \mathcal{U}_j$ should be isomorphic, with certain transitivity conditions.

The crystalline cohomology $H^*(X/R_V, \mathcal{E})$ of such a crystal is computable by de Rham complexes. If X embeds globally into a smooth formal R_V-scheme $\mathcal{X}$ one uses the hypercohomology of the de Rham complex for $\mathcal{E}_{\mathcal{X}}$ on the completed PD-hull of X in $\mathcal{X}$. In general one uses local embeddings to build a double-complex, as in [Fa2], IV, e). The result is a filtered complex which up to canonical filtered quasi-isomorphism is independent of all choices, and which as a complex without filtration depends only on the reduction of X modulo p ([B], IV, Th. 2.15 and VII, Prop. 1.1.9).

Thus crystalline cohomology is a well defined (up to canonical isomorphism) object in the filtered derived category. It is built from filtered complexes (and maps up to filtered homotopy) by inverting filtered quasi-isomorphisms (maps which induce isomorphisms on cohomology of the complex itself as well as of its associated graded). All our filtrations are decreasing and bounded above, but usually not finite.

More generally we can consider filtered crystals, that is, we require that each $\mathcal{E}_{\mathcal{U}}$ is filtered compatibly with the filtration on $\mathcal{O}_{\mathcal{U}}$, and that locally $\mathcal{E}_{\mathcal{U}}$ is filtered free over $\mathcal{O}_{\mathcal{U}}$. Here a "filtered free" module over a filtered ring $\mathcal{R}$ is a direct sum of copies of $\mathcal{R}$ with the filtration shifted by a constant amount. The associated graded then has a basis over $gr_F(R)$ consisting of homogeneous elements. Furthermore for pullback maps corresponding to $f : \mathcal{U}_1 \to \mathcal{U}_2$, $\mathcal{E}_{\mathcal{U}_2}$ should be the filtered pullback of $\mathcal{E}_{\mathcal{U}_1}$, that is,

$$F^q(\mathcal{E}_{\mathcal{U}_1}) = \sum_{a+b=q} F^a(\mathcal{O}_{\mathcal{U}_1}) \cdot f^*(F^b(\mathcal{E}_{\mathcal{U}_2})).$$

As usual this means that (if X embeds globally) the universal $\mathcal{E}_{\mathcal{X}}$ is filtered, and that the connection $\bigtriangledown$ maps $F^q(\mathcal{E}_{\mathcal{X}})$ to $F^{q-1}(\mathcal{E}_{\mathcal{X}}) \otimes \Omega^1_{\mathcal{X}/R_V}$, respectively the corresponding more local statements if we can only embed locally. The associated graded of the complex representing $H^*(X/R_V, \mathcal{E})$ is then a module over $gr_F(R_V)$. If we reduce modulo the augmentation ideal of this graded ring, that is, form the derived tensor product with $gr^0_F(R_V) = V$, we obtain the hypercohomology of X with values in the associated graded of the de Rham complex $\mathcal{E}_X \otimes \Omega^*_{X/V}$. The latter is called the Hodge cohomology of $\mathcal{E}_X$. Note that the differential in the associated graded of the de Rham complex is $\mathcal{O}_X$-linear, and trivial if and only if the connection preserves the filtration on $\mathcal{E}$.

We always assume that the following holds, because without it we can prove little:

Basic assumption. *The Hodge cohomology of $\mathcal{E}_X$ is torsion-free over V, and the Hodge spectral sequence for $H^*(X, \mathcal{E}_X \otimes \Omega^*_{X/V})$ degenerates.*

Obviously once the Hodge cohomology is torsion-free, it suffices to verify degeneration of the Hodge spectral sequence after tensoring with the fraction field K of V. For example this holds for constant coefficients.

By general facts the crystalline cohomology $H^*(X/R_V)$ can be represented by a finite complex $M^*(X/R_V, \mathcal{E})$ of filtered free R_V-modules. This holds by [B], Ch. VII, Th. 1.1.1, over each noetherian quotient $R_V/F^q(R_V)$, and one can paste these together (R_V itself is not noetherian). See the appendix for details. We may also assume that all differentials in $gr_F(M^*(X, R_V, \mathcal{E}))$ vanish modulo the maximal ideal of $gr_F(R_V)$.

This implies that in fact the differentials vanish modulo $F^1(R_V)$. We shall see that for Frobenius crystals it follows that the differentials vanish, so that the complex $M^*(X/R_V, \mathcal{E})$ is quasi-isomorphic to its cohomology.

Crystalline cohomology satisfies Poincaré-duality: If $\mathcal{E}^*$ denotes the dual crystal to $\mathcal{E}$ ($\mathcal{E}^*_{\mathcal{U}}$ = filtered dual of $\mathcal{E}_{\mathcal{U}}$), and if X/V is proper and smooth of pure relative dimension d, there exists a filtered quasi-isomorphism

$$M^*(X/R_V, \mathcal{E}) \cong Hom(M^*(X/R_V, \mathcal{E}^*), R_V\{-d\}[-2d]).$$

Here $R_V\{-d\}$ denotes R_V with filtration shifted by d (so $F^d(R_V\{-d\}) = R_V$), and $R_V\{-d\}[-2d]$ the complex whose only nonzero term is $R_V\{-d\}$ in degree $2d$. The existence of a map as above follows from that of the trace map in crystalline cohomology (see [B], Ch. VII, 1.4), and that it is a filtered quasi-isomorphism follows from Serre duality on X. The duality isomorphism is canonical up to homotopy.

R_V admits a unique semilinear (relative to φ on V_0) continuous Frobenius endomorphism ϕ defined by $\phi(T) = T^p$. As it extends modulo p to the Frobenius endomorphism ϕ_X of X, and as a crystal depends only on the reduction of X modulo p, we can define a Frobenius-crystal as a crystal $\mathcal{E}$ together with an isomorphism $\phi_X^*(\mathcal{E})[1/p] \cong \mathcal{E}[1/p]$. This means that for each local embedding $U \subset \mathcal{U}$ of an open subset $U \subset X$ into a smooth formal R_V-scheme $\mathcal{U}$, and any Frobenius lift $\phi_{\mathcal{U}}$ on $\mathcal{U}$, we obtain on the divided power-hull $D_U(\mathcal{U})$ an isomorphism $\Phi_{\mathcal{U}} : \phi_{\mathcal{U}}^*(\mathcal{E}_{\mathcal{U}})[1/p] \cong \mathcal{E}_{\mathcal{U}}[1/p]$, compatible with pullbacks f^* for maps $f : \mathcal{U}_1 \to \mathcal{U}_2$ which respect Frobenius lifts. As in [Fa2], IV, e), this also can be expressed by choosing embeddings $U \subset \mathcal{U}$, Frobenius-lifts, etc. as above for a set of U's covering X, and specifying the necessary compatibilities on the overlaps as in [Fa2], II, d), using the connection.

The dual of a Frobenius-crystal is a Frobenius-crystal again, Frobenius acts on the cohomology by similitudes for Poincaré duality (it respects the pairing up to some Tate-twist), and thus by quasi-isomorphisms after inverting p. It follows that for a Frobenius-crystal $\mathcal{E}$ there exists a quasi-isomorphism

$$\Phi : M^*(X/R_V, \mathcal{E}) \otimes_{\varphi} R_V[1/p] \cong M^*(X/R_V, \mathcal{E})[1/p].$$

Now assume in addition that the differentials of $M^*(X/R_V, \mathcal{E})$ vanish modulo $F^1(R_V)$. This means that the pushout (= tensor product) of this complex via $R_V \to V \subset K$ has trivial differentials, and its cohomology has the same total dimension as the complex itself. Since Φ^n defines a quasi-isomorphism between $\phi^{n*}M^*[1/p]$ and $M^*[1/p]$, we see that the pushout of $\phi^{n*}M^*[1/p]$ also has trivial differentials. This means that if h is a matrix coefficient of some differential, we may consider $h \in K_0[[T]]$ as a formal power series convergent in the open disk of elements of valuation $> 1/e \cdot (p-1)$, and then h vanishes not only at π but also at all p^n-powers of π. By well known principles of rigid analysis it follows that h vanishes, i.e. $M^*(X/R_V, \mathcal{E})$ itself has trivial differentials.

It now follows that the individual $M^i(X/R_V, \mathcal{E})$ are Frobenius-crystals on R_V: They have a connection $\bigtriangledown$, nilpotent modulo p, and there is a horizontal isomorphism

$$M^i(X/R_V, \mathcal{E}) \otimes_\varphi R_V[1/p] \cong M^i(X/R_V, \mathcal{E})[1/p].$$

If $M_0^i = M^i(X/R_V, \mathcal{E}) \otimes V_0$ denotes the fibre at the origin ($T = 0$), this implies that

$$M^i(X/R_V, \mathcal{E}) \otimes_\varphi R_V[1/p] \cong M_0^i \otimes_{V_0} R_V[1/p]$$

as Frobenius-isocrystal, i.e. the isomorphism respects Frobenius and connection. For example (see [Fa3], lemma 3.1 and remark after it) if one chooses an isomorphism α of $R_V[1/p]$-modules which respects Frobenius modulo T, then the limit (as $n \to \infty$) of $\Phi^n(\alpha)$ exists in $R_V[1/p]$, and this limit fulfills all requirements. This is nothing else than the main result (Th. 2.4) from [BO], which in turn derives from B. Dwork's classical observation that Frobenius increases the radius of convergence.

Theorem 1. *Suppose X/V is proper and smooth and $\mathcal{E}$ is a filtered Frobenius-crystal on X such that the Hodge cohomology of $\mathcal{E}$ is torsion-free over V, and such that the Hodge spectral sequence relative to V degenerates. Then the crystalline cohomology $H^*(X/R_V, \mathcal{E})$ is represented by a complex $M^*(X/R_V, \mathcal{E})$ of filtered free R_V-modules, with trivial differentials. Furthermore if we invert p, then all $H^i(X/R_V, \mathcal{E})[1/p]$ are induced from K_0-vector spaces with Frobenius automorphism.*

There are logarithmic variants. Let us start with the easier one, to which we shall refer as "having a divisor at infinity":

Assume $D \subset X$ is a divisor with simple normal crossings, relative to V. Then one can consider the logarithmic crystalline topos, as in [Fa2], IV, c). As sites one takes PD-immersions $U \subset \mathcal{U}$ such that the line bundles $\mathcal{O}(D_i)$ (D_i an irreducible component of D) with their global section 1 lift to line bundles $\mathcal{L}_i$ on $\mathcal{U}$, and sections $f_i \in \Gamma(\mathcal{U}, \mathcal{L}_i)$. Furthermore maps have to extend to these line bundles, and respect the canonical section. In this situation one defines filtered crystals as before, and the whole theory works except that one has to replace usual differentials by differentials with logarithmic poles, and that there are two cohomology theories corresponding to usual cohomology and to cohomology with compact support.

For Frobenius-crystals one considers local Frobenius-lifts $\phi_\mathcal{U}$ and isomorphisms $\phi_\mathcal{U}^*(\mathcal{L}_i) \cong \mathcal{L}_i^{\otimes p}$, sending f_i to f_i^p. This way one easily obtains a logarithmic analogue of Theorem 1.

A more ambitious logarithmic theory can be found in [K], see pg. 222 there for a discussion of how it is related to the formalism used here. In the terminology of [K] one obtains a fine log-structure which locally can be given as follows: Choose local equations f_i for the irreducible components D_i. Then the free monoid generated by the D_i maps to $(O_X, \cdot)$ and defines a prelog-structure. The associated (fine) log-structure is independent of choices. One advantage of the approach in [K] is its good behavior with respect to étale localisation. However the important direct construction of diagonal classes from [Fa2] is missing in this theory, and thus the additional generality does not carry over to the comparison with étale cohomology.

It is more challenging to introduce a logarithmic structure on the base. We shall refer to this as "the case of semistable X"; see for example [Fa4] or [Fa5], or [H2], [HK], [Mo]. For this we consider $Spec(V)$ as a logarithmic scheme over V_0 (which has no logarithmic structure), such that the divisor at infinity is $Spec(k)$. That is, we have one line bundle $\mathcal{L} \cong \mathcal{O}$, with global section π. $Spec(R_V)$ is still the versal PD-thickening; we lift $\mathcal{L}$ as the trivial line bundle on it, and π to its section T. Also φ is obviously an admissible Frobenius-lift.

Now assume that X is proper and semistable over V. The latter means that locally in X there are smooth V-maps from X to

$$Spec(V[T_1, \dots, T_r]/(T_1 \cdot \dots \cdot T_r - \pi)).$$

Assume furthermore that all irreducible components $D_1, \dots, D_r$ of the special fibre $X \otimes_V k$ are smooth over k. Then we obtain a logarithmic structure on X and a log-map from X to $Spec(V)$ as follows:

On X let $\mathcal{L}_i = \mathcal{O}(D_i)$, with global section $f_i = 1 \in \Gamma(X, \mathcal{L}_i)$. Furthermore identify the tensor product of all $\mathcal{L}_i$ with the pullback of the trivial bundle on $Spec(V)$, such that the product of the f_i corresponds to the pullback of its canonical section π.

Again the corresponding approach in [K] uses the prelog-structure (on X as well as on $Spec\,(R_V)$) associated to the free monoid generated by the irreducible components, mapping to local generators of the corresponding ideals. As before the additional generality of the theory in [K] does not help for the comparison with étale cohomology, because a good theory of diagonal classes does not exist.

Furthermore the log-crystalline topos uses logarithmic PD-thickenings over R_V, similar to before except that we need compatibility with the logarithmic structure on the base. Again almost everything goes through, using logarithmic relative differentials. The only point which is not quite straightforward is the existence of a trace map

$$H^{2d}(X/R_V, \mathcal{O}) \to R_V\{-d\},$$

which is needed to set up Poincaré duality. However Berthelot's proof can be made to work also in this case:

Using the Cousin complex one has to construct local trace maps on the crystalline cohomology with support in a closed point of $X_0 = X \otimes_V k$, and show a reciprocity-law for any curve $Z_0 \subset X_0$. The first is done as in [B], VII, Prop. 1.2.8, by lifting open subsets $U_0 \subset X_0$ to log-smooth formal R_V-schemes $\mathcal{U}$, using that the highest exterior power of the relative logarithmic differentials is equal to the relative dualising complex. For the reciprocity law one shows that for suitable choices of $\mathcal{U}$'s one can glue the *r-th* infinitesimal neighbourhoods of Z_0 in $\mathcal{U}$ to a global logarithmic scheme $Z^{(r)}$. The proof is the same as in [B], VII, Cor. 1.3.7, replacing "smooth" by "log-smooth" everywhere. The key fact is that the obstructions lie in H^2 of a coherent sheaf on Z_0.

After this the whole theory goes through, except that after inverting p the logarithmic connection on $M^i(X/R_V, \mathcal{E})[1/p] \cong M_0^i \otimes R_V[1/p]$ differs from the constant connection by $N \cdot dT/T$, where $N \in End(M_0^i)$ is its residue at the origin. Thus

Theorem 1*. *Suppose that X is proper and semistable over V, and that $\mathcal{E}$ is a filtered logarithmic Frobenius-crystal on X such that the Hodge cohomology of $\mathcal{E}$ is torsion-free over V, and such that the Hodge spectral sequence degenerates. Then the crystalline cohomology $H^*(X/R_V, \mathcal{E})$ is represented by a complex $M^*(X/R_V, \mathcal{E})$ of filtered free R_V-modules, with trivial differentials. Furthermore if we invert p, then all $H^i(X/R_V, \mathcal{E})[1/p]$ are induced from K_0-vector spaces H_0^i with Frobenius* *automorphism. Under this isomorphism the connection becomes*

$$\bigtriangledown = d + N \cdot dT/T,$$

where $N \in End(H_0^i)$ is nilpotent and satisfies $p \cdot \Phi_0 \cdot N = N \cdot \Phi_0$.

So far things seem to depend on the choice of the uniformizer π of V (I have to thank A. Ogus for pointing this out to me). However this is only up to canonical isomorphism: Without logarithmic structure on $Spec(V)$ this is clear, since $Spec(R_V)$ is just the versal PD-thickening. In the logarithmic case two different uniformisers differ by multiplication by a unit in V^*. We can lift it to a unit $u \in R_V^*$, unique up to $1 + F^1(R_V)$. Then multiplication by u defines an automorphism of logarithmic schemes, which is the identity on $Spec(V)$. We can extend it to any logarithmic PD-thickening of the semistable X, by multiplying the local equations f_i for D_i by units $u_i \in R_V^*$ whose product is u. The resulting isomorphism on logarithmic crystalline cohomology is independent of all choices. Quite similarly we also can consider Frobenius-lifts on R_V different from the standard one: One then has to use the connection $\bigtriangledown$ to change the Frobenius on $M^i(X/R_V, \mathcal{E})$, as in [Fa2], II, Th. 2.3.

Remark. There is also a relative theory, for log-smooth proper maps $X \to Y$ which are generically smooth at infinity. (The referee has pointed out that in more generality the arguments apply to maps which are log-smooth and "of Cartier type" as defined in [K], Definition 4.8.) One assumes that the relative Hodge cohomology is torsion-free, and that the relative Hodge-spectral sequence degenerates. Berthelot's argument implies the existence of a trace map with values in the formal completion of X at a closed point, or better the formal completion of a universal PD-thickening. This suffices to prove that Frobenius is nondegenerate (after inverting p), and that the derived direct image splits into its individual cohomology groups. The theory over $\mathbb{Q}_p$ has been developed already in [Fa2], VI), and the rest is quite analogous to what we have done before.

3. The category $\mathcal{MF}$

One can get a more precise version of the fact that Frobenius is nondegenerate, if one puts restrictions on the relative dimensions. We want to explain that the theory of the category $\mathcal{MF}_{[0,p-2]}$ of [Fa2], II, carries over, provided one restricts to objects without p-torsion. This makes it necessary to use a different method of proof since devissage is no longer available. Instead we shall reduce to the $\mathbb{Q}_p$-case. We start with the category $\mathcal{MF}_{[0,a]}(V)$. Note first that on R_V itself the restriction of ϕ to $F^q(R_V)$ is for $0 \le q \le p-1$ divisible by p^q, because $\phi(f) = f^\phi(T^p) = T^{p\cdot e} + \sum_{0 \le i < e} \phi(a_i) \cdot T^{p \cdot i}$ is divisible by p, and even becomes a unit after dividing by p. We thus can define $\phi_q = \phi/p^q$ on $F^q(R_V)$, for $0 \le q \le p-1$.

Definition 2. An object of $\mathcal{MF}_{[0,a]}(V)$ $(a \le p-2)$ is a filtered R_V-module M with a connection $\bigtriangledown$ (nilpotent modulo p) and a Frobenius Φ, such that:

i) M is filtered free as an R_V-module, with a basis m_i having filtration degrees q_i, $0 \le q_i \le a$.

ii) The connection $\bigtriangledown$ satisfies Griffiths transversality:

$$\bigtriangledown(\partial/\partial T)(F^q(M)) \subset F^{q-1}(M).$$

iii) Φ is a $\bigtriangledown$ -horizontal semilinear endomorphism $\Phi = \Phi_M$ of M whose restriction to $F^q(M)$ is divisible by p^q, for $0 \leq q \leq a$. Furthermore the elements $\Phi_{q_i}(m_i) = \Phi(m_i)/p^{q_i}$ form a new R_V-basis of M.

As $\phi_1(f)$ is a unit, the last condition can be reformulated as the fact that Φ_a induces an isomorphism

$$gr_F^a(M) \otimes_\phi (R_V/p \cdot R_V) \cong M/p \cdot M.$$

Maps are defined in the obvious manner.

It follows that M defines a Frobenius-crystal $\mathcal{E}$ on $Spec(V)$ relative to V_0, that is, on any PD-thickening $\mathcal{U}$ of $Spec(V)$ with a Frobenius-lift $\Phi_\mathcal{U}$ we have a vector bundle $\mathcal{E}_\mathcal{U}$ and maps $\Phi_i : \Phi_\mathcal{U}^*(F^i(\mathcal{E}_\mathcal{U})) \to \mathcal{E}_\mathcal{U}$, for $0 \leq i \leq a$, functorial with compatible maps $f : \mathcal{U}_1 \to \mathcal{U}_2$, etc. Of course $\mathcal{E}_\mathcal{U}, F^i(\mathcal{E}_\mathcal{U})$ and Φ_i are defined by choosing (locally) a PD-map $\mathcal{U} \to \mathrm{Spec}(R_V)$ and pulling back. The connection makes this independant of choices. Also if $M_0 = M \otimes V_0$ denotes the fibre of M at the origin, it follows as before that $M[1/p] \cong M_0 \otimes R_V[1/p]$ as a module with connection. For M's originating from the cohomology this has been discussed in section 2 (preceding Theorem 1), and the same arguments apply here. There is also a logarithmic version. In the logarithmic version the connection is logarithmic, i.e. only $\bigtriangledown(T\frac{\partial}{\partial T})$ and not $\bigtriangledown(\partial/\partial T)$ is defined. Otherwise the conditions i), ii), iii) remain the same. On $M[\frac{1}{p}] \cong M_0 \otimes R_V[1/p]$ the connection becomes $\bigtriangledown = d + N \cdot dT/T$.

Also for a smooth V-scheme X we can define the category $\mathcal{MF}_{[0,a]}(X/R_V)$ as the category of filtered crystals with Frobenius-maps $\Phi_i : \Phi^*(F^i(\mathcal{E})) \to \mathcal{E}$, for any PD-embedding $U \subset \mathcal{U}$ admitting a Frobenius-lift. These should be compatible for maps, and locally $\mathcal{E}$ should admit a filtered basis e_i (of degree q_i, $0 \leq q_i \leq a$) such that the $\Phi_{q_i}(e_i)$ form another basis. Again one can define logarithmic versions, if one replaces everywhere "connections" by "logarithmic connections" and "Frobenius-lift" by "logarithmic Frobenius-lift". Any such $\mathcal{E}$ is given by its evaluations on R_V-smooth local liftings $\mathcal{U}$ of X. These define V_0-smooth local liftings of the special fibre $X_0 = X \otimes_V k$, by dividing by the ideal generated by divided powers of T, and the $\mathcal{E}_\mathcal{U}$ modulo this ideal define a Frobenius-crystal $\mathcal{E}_0$ on X_0, relative V_0. As before one checks (using Frobenius) that the corresponding isocrystal $\mathcal{E}_0[1/p]$ induces $\mathcal{E}[1/p]$.

Now for such an $\mathcal{E}$ assume that the Hodge cohomology has no torsion, and that the Hodge spectral sequence degenerates. It then follows from a local calculation, using the Cartier isomorphism, that if $a + b \leq p - 2$, the cohomology $H^b(X/R_V, \mathcal{E})$ lies in $\mathcal{MF}_{[0,a+b]}(V)$. (See [Fa2], IV, Th. 4.1: Show that for a perfect filtered complex M^* representing the direct image, Φ_a induces a quasi-isomorphism

$$gr_F^{a+b}(M^*) \otimes_\phi (R_V/p \cdot R_V) \xrightarrow{\approx} M/p \cdot M.$$

As explained in [Fa2] it suffices to check this for the crystalline cohomologies of affines in X. There for constant coefficients this is the Cartier isomorphism, and devissage reduces to that.)

Remark. More general for a map $f : X \to Y$ which is log-smooth and generically smooth at infinity (or of "Cartier-type" as in [K]), this reasoning applies if the relative Hodge cohomology has no torsion, and the relative Hodge spectral sequence degenerates. Under these circumstances for any $\mathcal{E} \in \mathcal{MF}_{[0,a]}(X/R_V)$, $R^b f_*(\mathcal{E})$ lies in $\mathcal{MF}_{[0,a+b]}(Y/R_V)$, provided $a + b \leq p - 2$.

4. The Fontaine ring $B^+(V)$ and the functor $\mathbb{D}$

Let us first list for reference some easily derived properties of the ring $B^+(V)$. Let $\bar{V}$ denote the integral closure of V in an algebraic closure $\bar{K}$ of K, and $\bar{V}^\wedge$ its p-adic completion. Consider the ring $S = lim.proj._{n\in\mathbb{N}}(\bar{V}/p\cdot\bar{V})$, consisting of sequences $\{x_n | n \geq 0\}$ with $x_n \in \bar{V}/p\cdot\bar{V}$, $x_n = x_{n+1}^p$. We can also identify it with such sequences in $\bar{V}^\wedge$. S is a valuation ring of characteristic p which is perfect, i.e. Frobenius is an isomorphism on S, and admits a continuous action of $Gal(\bar{K}/K)$. There is a map $\alpha : \mathbb{Q}_p(1) \to S^*$, whose definition is obvious if we identify $\mathbb{Q}_p(1)$ with sequences $\{\zeta_n | n \geq 0\}$ of p-power roots of unity such that $\zeta_n = \zeta_{n+1}^p$. Also for any $x \in \bar{V}$ choose a compatible sequence x_n of p^n-th roots of x, to obtain an element $\underline{x} \in S$, well defined up to multiplication by $\alpha(\mathbb{Z}_p(1))$. We can form the Witt vectors $W(S)$, and $W(S)/p\cdot W(S) = S$. Thus elements of $W(S)$ are sequences $(x_0, x_1, \dots)$ with $x_i = (x_{i,0}, x_{i,1}, x_{i,2}, \dots) \in S$ as above. Furthermore there is a surjective homomorphism $\theta : W(S) \to \bar{V}^\wedge$ defined by

$$\theta(x_0, x_1, \dots) = \sum p^n \cdot x_{n,n}.$$

It is known that the kernel of θ is a principal ideal, for example generated by $p\cdot 1 - (\underline{p}, 0, 0, \dots)$. More generally let $f(t)$ denote the minimal polynomial (over V_0) of the uniformiser π of V. Then $\xi = f((\underline{\pi}, 0, 0, \dots))$ lies in a kernel (θ) and is a generator: It suffices to check modulo p where θ becomes the projection onto the first component $S \to \bar{V}/p\cdot\bar{V}$. Its kernel is generated by any element in S of valuation 1, for example by $\underline{\pi}^e \equiv f(\underline{\pi})\, mod\, p$.

We define $B^+(V)$ as the completed divided power hull of $\ker(\theta)$. It is an algebra over V_0, has a complete filtration $F^n(B^+(V))$ by the divided powers, and $gr_F^n(B^+(V))$ is free over $gr_F^0(B^+(V)) = \bar{V}^\wedge$ with basis the image of $\xi^n/n!$. Note that as the associated gradeds have no p-torsion, the various ways to define p-adic topologies on subquotients all lead to the same result. From the definition it follows that $B^+(V)$ admits a continuous Galois-action (respecting filtrations) and a Frobenius ϕ. Frobenius extends because we could have defined $B^+(V)$ as the completed PD-hull of $p\cdot W(S) + ker(\theta)$, using $p > 2$. Furthermore for $i < p$, φ is on $F^i(B^+(V))$ divisible by p^i. Call $\phi_i = \phi/p^i$ on $F^i(B^+(V))$ (for $i < p$). The referee has pointed out that in [Fa2] the filtration is slightly different in degrees $\geq p$. The definition here is clearer, works as well, and the historical reasons for the other convention do not seem so convincing anymore.

We also identify α with its composition with the Teichmüller map:

$$\alpha : \mathbb{Q}_p(1) \to B^+(V)^*.$$

There is a well-defined map

$$\beta = log(\alpha) : \mathbb{Z}_p(1) \to F^1(B^+(V)),$$

and $\Phi\cdot\beta = p\cdot\beta$. We sometimes denote by β_0 the image of a generator of $\mathbb{Z}_p(1)$. If $\tilde{\xi}$ (resp. $\tilde{\xi}_0$) denotes the image of ξ (resp. β_0) in $gr_F^1(B^+(V))$, one has $\tilde{\beta}_0 = \lambda\cdot\tilde{\xi}$, with $\lambda \in \bar{V}^\wedge$ satisfying $v(\lambda) = 1/(p-1)$. v is the normalised ($v(p) = 1$) valuation. If $B_{DR}^+(V)$ denotes the completion of $B^+(V)\otimes\mathbb{Q}_p$ in the filtration topology, it follows that

$$gr_F^n(B_{DR}^+(V)) \cong \bar{K}^\wedge(n).$$

For many purposes it suffices to consider the quotient

$$B^+(V)/(F^p(B^+(V)) + p \cdot B^+(V)) \cong W(S)/(\xi^p, p) \cong S/(\xi^p) \cong \bar{V}/p \cdot \bar{V}.$$

(See [Fa2], pg. 30. The discussion simplifies with our definition of F^p.) The induced filtration on it is defined by

$$F^i(\bar{V}/p \cdot \bar{V}) = p^{i/p} \cdot \bar{V}/p \cdot \bar{V},\ \varphi_i(x) = x^p/(-p)^i$$

(for $0 \leq i \leq p-2$). For example ([Fa2], Ch. II) one can derive that for $0 \leq i \leq p-2$ any $x \in F^i(B^+(V))$ with $\varphi_i(x) = x$ is in $\mathbb{Z}_p \cdot \beta_0^i$, by first showing the analogue for $B^+(V)/p \cdot B^+(V)$. Fontaine ([Fo2]) even proves such an assertion (which we do not need) without any restriction on i, but this requires stronger methods. He uses an injection of $B^+(V)$ into bivectors. (His ring B_{cris} is smaller but has the same Frobenius-eigenvectors.)

He also defines $B(V)$ as $B^+(V)[p^{-1}, \beta^{-1}]$, and $B_{DR}(V)$ as its F-completion. As K is separable over K_0, we can lift it to a subring of $B_{DR}(V)$. Another important property (also due, as everything here, to Fontaine) is the following:

Lemma 3. i) *The map $B^+(V) \otimes K \to B_{DR}(V)$ is injective.*
ii) *For each n, all $Gal(\bar{K}/K)$-invariants in $B^+(V)(-n)$ are K_0-multiples of β^n.*

Proof. One easily reduces to the case where k is algebraically closed. The algebraic closure of K_0 in $B^+(V) \otimes \mathbb{Q}_p$ is a field stable under Frobenius and $Gal(\bar{K}/K)$, and has trivial intersection with $F^1(B^+(V))$. If L is a finite extension of K, the invariants under $Gal(\bar{K}/L)$ thus form a finite extension of K_0 contained in L (under the projection to $gr_F^0(B^+(V))$, and stable under Frobenius. The action of Frobenius on this extension K' has only finitely many slopes, and it follows that elements of nonzero slope are nilpotent and vanish. So K' is generated over K_0 by Frobenius-invariants, which lie in $\mathbb{Q}_p$. Moreover any element in the algebraic closure is invariant under $Gal(\bar{K}/L)$, for some L, and thus lies in K_0. This implies injectivity for the map above. For the second assertion, use that for the Galois-invariants in $B_{DR}(V)(-n)$ one obtains $K \cdot \beta^n$, and the injectivity of the map in i). □

In [Fo3], Fontaine has also defined a subring $B_{st} \subset B_{DR}$. We give a slight variant adapted to our purposes:

Under $Gal(\bar{K}/K)$ the element $\underline{\pi} \in B^+(V)$ changes by elements in $\alpha(\mathbb{Z}_p(1))$. We thus can define a continuous 1-cocycle on $Gal(\bar{K}/K)$ with values in $F^1(B_{DR}(V))$ by sending $\sigma \in Gal(\bar{K}/K)$ to $log(\sigma(\underline{\pi})/\underline{\pi})$. As $F^1(B_{DR}(V))$ has trivial continuous Galois cohomology there exists a unique element $u \in F^1(B_{DR}(V))$ with $\sigma(u) - u = log(\sigma(\underline{\pi})/\underline{\pi})$. In fact one can choose $u = log(\underline{\pi}/\pi) \in F^1(B_{DR}(V))$.

Define $B_{st}^+(V) = B^+(V)[u]$, and $B_{st}(V) = B_{st}^+(V)[\beta_0^{-1}]$. Using the Galois action one shows easily that u is algebraically independent over the fraction-field of $B^+(V)$, and thus we can extend Frobenius to $B_{st}^+(V)$ by setting $\varphi(u) = p \cdot u$. Also it follows that the Galois-invariants in $B_{st}(V)$ are still equal to K_0.

Finally let as before $R = V_0[[T]]$ and write $V = R/(f)$, and form the completed divided power-hull R_V. There is a unique continuous homomorphism of V_0-algebras $V \to W(S)$ sending T to $[\underline{\pi}]$, and this map respects Frobenius but not Galois actions. As the filtrations on R_V (respectively $B^+(V)$) are defined by divided powers of $f(T)$ (respectively $f(\underline{\pi})$), the map respects filtrations and induces an isomorphism

$$gr_F(B^+(V)) \cong gr_F(R_V) \otimes_V \overline{V}^\wedge.$$

It follows that the map becomes faithfully flat on the level of associated graded, and thus for each n, $B^+(V)/F^n$ is faithfully flat over R_V/F^n.

Although the maps do not respect Galois-actions (trivial on R_V), if we extend to $B^+(V)$ we can use divided powers. Thus for a crystal M over R_V the tensor product with $B^+(V)$ has a canonical Galois-action. Equivalently we can consider the preimage of V in $B^+(V)$ as a projective limit of PD-thickenings, thus evaluate the crystal M on it and push forward to all of $B^+(V)$. In more detail for $\sigma \in Gal(\bar{K}/K)$ write $\sigma([\underline{\pi}]) = [\underline{\pi}] + z$, with $z \in F^1(B^+(V))$. Then σ acts on $M \otimes_{R_V} B^+(V)$ by the rule

$$\sigma(m \otimes b) = \sum_{n \geq 0} \bigtriangledown(\frac{\partial}{\partial T})^n(m) \otimes \sigma(b) \cdot \frac{z^n}{n!}.$$

This also works in the logarithmic context, using the log-structure on $B^+(V)$ defined by $[\underline{\pi}]$, $\varphi([\underline{\pi}]) = [\underline{\pi}]^p$.

Definition 4. Suppose $M \in \mathcal{MF}_{[0,p-2]}(V)$. Define

$$\mathbb{D}(M) = Hom_{R_V,F,\Phi_*}(M, B^+(V)) = Hom_{B^+(V),F,\Phi_*}(M \otimes_{R_V} B^+(V), B^+(V)).$$

Here the right-hand side means homomorphisms respecting Frobenius and filtrations. $\mathbb{D}(M)$ is a $\mathbb{Z}_p$-module which admits a continuous action of $Gal(\bar{K}/K)$. If $\mathbb{D}(M)^*$ denotes the dual of $\mathbb{D}(M)$, we obviously have a map $\rho : M \otimes B^+(V) \to \mathbb{D}(M)^* \otimes B^+(V)$.

Theorem 5. i) *$\mathbb{D}(M)$ is free over $\mathbb{Z}_p$, of rank equal to the R_V-rank of M.*

ii) *For $M \in \mathcal{MF}_{[0,a]}(V)$ we have*

$$\beta_0^a \cdot \mathbb{D}(M)^* \otimes_{\mathbb{Z}_p} B^+(V) \subset M \otimes_{R_V} B^+(V) \subset \mathbb{D}(M)^* \otimes_{\mathbb{Z}_p} B^+(V).$$

These inclusions are strict for the filtrations, i.e. they induce injections on gr_F.

iii) *The functor $\mathbb{D}$ from $\mathcal{MF}_{[0,p-2]}(V)$ to $\mathbb{Z}_p - Gal(\bar{K}/K)$-modules is fully faithful.*

Proof. Note that for the first two parts the Galois action on $\mathbb{D}(M)$, and thus the connection on M, do not matter. Also for the special case $M = R_V\{a\}$, the free R_V-module with one generator m in degree a, $\bigtriangledown$-parallel and fixed by Φ_a, $\mathbb{D}(M) = \mathbb{Z}_p(a)$ (Tate-twist) consists of maps sending m to a $\mathbb{Z}_p$-multiple of β_0^a.

i) As in [Fa2], II e) (and already noted above), there is a map $B^+(V)/p \cdot B^+(V) \to \bar{V}/p \cdot \bar{V}$, with kernel F^p. The induced morphism from R_V to $\bar{V}/p \cdot \bar{V}$ is the inverse of Frobenius on V_0, and sends T to the canonical root $\underline{\pi}^{1/p}$. If we filter $\bar{V}/p \cdot \bar{V}$ by $F^i(\bar{V}/p \cdot \bar{V}) = p^{i/p} \cdot \bar{V}/p \cdot \bar{V}$, and let $\varphi_i(x) = x^p/(-p)^i$ for $x \in F^i(\bar{V}/p \cdot \bar{V})$, we obtain an isomorphism ([Fa2], pg. 37)

$$Hom_{R_V,F,\Phi_*}(M, B^+(V)/p \cdot B^+(V)) \cong Hom_{R_V,F,\Phi_*}(M, \bar{V}/p \cdot \bar{V}).$$

As we have slightly changed definitions we repeat the argument here:

Given a map $f : M \to \bar{V}/p\bar{V}$ respecting filtrations and φ_μ, lift the $f(m_\mu)$ somehow to elements of $F^{q_\mu}(B^+(V))$. Then define a lifting $\tilde{f} : M \to B^+(V)/p \cdot B^+(V)$ by mapping $\Phi_{q_\mu}(m_\mu)$ to Φ_{q_μ} (this lift). As Φ_{q_μ} vanishes modulo p on F^{p-1} $(q_\mu \leq p-2)$ one checks that $\tilde{f}$ is the unique lift respecting filtration and Frobenius.

Now the right hand side can be described as follows:

Choose a filtered basis m_μ of M, with filtration-degrees q_μ, $1 \leq \mu \leq r$. Furthermore define an invertible $r \times r$-matrix $A = (a_{\mu\nu})$ with entries in R_V by

$$\Phi_{q_\mu}(m_\mu) = \sum_\nu a_{\mu\nu} \cdot m_\nu.$$

Then $Hom_{R_V,F,\Phi_*}(M, \bar{V}/p \cdot \bar{V})$ can be identified with r-tuples $x_\mu = p^{q_\mu/p} \cdot y_\mu \in \bar{V}/p \cdot \bar{V}$ satisfying

$$x_\mu^p \equiv (-p)^{q_\mu} \cdot \sum_\nu a_{\mu\nu} \cdot x_\nu \quad mod\, p^{q_\mu+1}.$$

By Newton's method any such tuple lifts uniquely to a solution in $\bar{V}$ of

$$x_\mu^p = (-p)^{q_\mu} \cdot \sum_\nu a_{\mu\nu} \cdot x_\nu,$$

if we lift the image of A in $\bar{V}/p \cdot \bar{V}$ in some way to an invertible matrix with entries in $\bar{V}$. The argument is in [Fa2], pg. 38. For $\sigma \geq \frac{p-2}{p-1}$ we define a solution mod p^σ as a tuple x_μ satisfying the congruence above modulo $p^{q_\mu+\sigma}$. Two such solutions are considered equal if the x_μ coincide modulo p^σ. Then any solution modulo p^σ lifts uniquely to a solution modulo $p^{\sigma'}$, for $\sigma \leq \sigma' \leq \sigma + \frac{2}{p}$. Thus if $\mathcal{B}$ denotes the $\bar{V}$-algebra generated by indeterminates Y_μ, $1 \leq \mu \leq r$, and relations

$$Y_\mu^p = \sum_\nu a_{\mu\nu} \cdot (-p)^{q_\nu/p} \cdot Y_\nu,$$

and $\mathcal{B}^{norm}$ its normalization, then

$$Mor_{R_V,F,\Phi_*}(M, \bar{V}/p \cdot \bar{V}) = Mor_{\bar{V}}(\mathcal{B}, \bar{V}) = Mor_{\bar{V}}(\mathcal{B}^{norm}, \bar{V}).$$

But $\mathcal{B}$ is obviously finite flat over $\bar{V}$, of rank p^r, and one checks that $\mathcal{B}[1/p]$ is étale over $\bar{K}$. Thus $\mathcal{B}^{norm}$ is isomorphic to the product of p^r-copies of $\bar{V}$, and

$$Mor_{R_V,F,\Phi_*}(M, B^+(V)/p \cdot B^+(V)) = Mor_{R_V,F,\Phi_*}(M, \bar{V}/p \cdot \bar{V})$$

has order p^r.

However one checks that any homomorphism into $B^+(V)/p \cdot B^+(V)$ lifts to $B^+(V)$: This comes down ([Fa2], pg. 38) to solving equations

$$Y_\mu^p - \sum_\nu a_{\mu\nu} \cdot (-p)^{p_\nu/p} \cdot Y_\nu = z_\mu \in \bar{V},$$

which is always possible in $\bar{V}$ as these define a finite flat $\bar{V}$-algebra. For the reader's convenience we again repeat the argument from [Fa2]:

Suppose we have lifted modulo p^r. To go one step further lift the images of the basis m_μ somehow to elements of $F^{q_\mu}(B^+(V))$. The resulting map may not respect Φ_{q_μ} but its commutator with Φ_{q_μ} takes values in $p^r \cdot B^+(V)/p^{r+1} \cdot B^+(V) \cong B^+(V)/pB^+(V)$. Map to $\bar{V}/p\bar{V}$, and call the corresponding obstructions z_μ. By solving the equations above we can change lifts to make these zero. Finally mapping $\Phi_{q_\mu}(m_\mu)$ to Φ_{q_μ} (new lift) gives the desired extension. Thus lifting an $\mathbb{F}_p$-basis of $Hom_{R_V,F,\Phi_*}(M, B^+(V)/p \cdot B^+(V))$ gives a $\mathbb{Z}_p$-basis of $\mathbb{D}(M)$.

ii) Let M^t denote the internal Hom of M into $R_e\{p-2\}$, that is, M^t has a filtered basis m^μ of degrees $p-2-q_\mu$, and

$$\Phi_{p-2-q_\mu}(m^\mu) = \sum_\nu b_{\mu\nu} \cdot m^\nu,$$

where $B = (b_{\mu\nu})$ denotes the inverse matrix to the transpose A^t. Then

$$m = \sum_{\mu} m_\mu \otimes m^\mu \in M \otimes_{R_V} M^t$$

defines a homomorphism from $R_V\{p-2\}$ to $M \otimes M^t$, and evaluation on m gives a pairing

$$\mathbb{D}(M) \times \mathbb{D}(M^t) \to \mathbb{D}(R_V\{p-2\}) = \mathbb{Z}_p(p-2).$$

We claim that this is a perfect duality. It suffices to check this modulo p, and to show the following:

Suppose $\epsilon \in Hom_{R_V,F,\Phi_*}(M^t, B^+(V)/p \cdot B^+(V))$ has the property that

$$(\delta \otimes \epsilon)(m) \in p \cdot B^+(V),$$

for all

$$\delta \in Hom_{R_V,F,\Phi_*}(M, B^+(V)/p \cdot B^+(V)).$$

Then ϵ vanishes. □

If we identify $Hom_{R_V,F,\Phi_*}(M, B^+(V)/p \cdot B^+(V))$ with $Hom_{\bar{V}}(\mathcal{B}^{norm}, \bar{V})$, and lift the elements $\epsilon(m^\mu) \in B^+(V)/p \cdot B^+(V)$ to elements $p^{(p-2-q_\mu)/p} \cdot z^\mu$, $z^\mu \in \bar{V}$, as in the proof of i), it follows that the element ($\mathcal{B}$ and Y_μ as previously defined)

$$z = \sum_{\mu} Y_\mu \cdot z^\mu \in \mathcal{B}$$

has the property that $p^{(p-2)/p} \cdot z$ lies in $p \cdot \mathcal{B}^{norm}$. Furthermore from Φ_*-linearity

$$p^{(p-2)/p} \cdot z \equiv z^p \,\text{modulo}\, p \cdot \mathcal{B}.$$

By Newton's method we can lift this congruence to a precise solution $\tilde{z}$ in the p-adic completion of $\mathcal{B}$, and this solution coincides with z modulo $p^{2/p} \cdot \mathcal{B}$. Then $w = \tilde{z}/p^{2/p}$ lies in the p-adic completion of $\mathcal{B}^{norm}$, which is a product of p^r copies of $\bar{V}^\wedge$, and satisfies $w^p = w/p$. Thus $w = 0$, and all z^μ are divisible by $p^{2/p}$. This in turn implies that $\Phi_{p-2-q_\mu}(\epsilon(m^\mu))$ is divisible by p, thus vanishes modulo p. Thus the pairing is nondegenerate.

Now the rest is easy:

We obtain a map from $\mathbb{D}(M)^*$ to $\mathbb{D}(M^t)(2-p)$. Since M is free over R_V, multiplication by β^{p-2} then defines

$$\mathbb{D}(M)^* \to Hom_{R_V,F,\Phi_*}(M^t, B^+(V)\{p-2\}) \to M \otimes B^+(V)$$

such that the composition

$$\mathbb{D}(M)^* \otimes_{\mathbb{Z}_p} B^+(V) \to M \otimes_{R_V} B^+(V) \to \mathbb{D}(M)^* \otimes_{\mathbb{Z}_p} B^+(V)$$

is multiplication by β^{p-2}. As β_0 is not a zero-divisor in $B^+(V)$, and both sides are free $B^+(V)$-modules of rank r, the same holds for the composition in the opposite order, and we easily derive assertion ii) for $a = p-2$. For other $a \leq p-2$ the proof is the same, replacing $p-2$ by a throughout. The assertion about filtrations follows because all our maps preserve them, except for multiplication by β_0^a which shifts them by a, and because β is injective on $gr_F(B^+(V))$.

iii) It follows from ii) and the fact that $gr_F(B^+(V))$ is faithfully flat over $gr_F(R_V)$ that $\mathbb{D}$ is faithful. Now assume given a Galois-homomorphism from $\mathbb{D}(M_2)$ to $\mathbb{D}(M_1)$. First we invert p and note that $M[\frac{1}{p}]$ is induced from the crystal $M_0[\frac{1}{p}]$ on k/V_0 (without reference to filtrations), or in the general logarithmic case from

$(M_0[\frac{1}{p}], N)$ (defined on $Spec(k)/V_0$ with the log-structure given by 0). If N is trivial, $M_0[1/p]$ thus coincides by Lemma 3 ii) with the Galois-invariants in $\mathbb{D}(M)^* \otimes B^+(V)[1/p]$ (the map being induced from the inclusion in ii), and by ii) our Galois-homomorphism induces a parallel and Frobenius-invariant map from $M_1[1/p]$ to $M_2[1/p]$, which respects filtrations. For general N, $exp(N \otimes u)$ intertwines our Galois-action on $M_0 \otimes B_{st}(V)$ (which uses twisting by N) and the action via the second factor, and we recover $M_0[1/p]$ as Galois-invariants in $\mathbb{D}(M)^* \otimes B_{st}(V)$, and reach the same conclusion. The explicit formula for intertwining works as in the discussion preceding Definition 4. An element of σ of the Galois-group changes the log-structure, as

$$\sigma[\underline{\pi}] = exp(\sigma(u) - u) \cdot [\underline{\pi}].$$

Since $\sigma(u) - u \in F^1(B_{st}(V))$, the induced isomorphism on crystals is described by the Taylor-series (N corresponds to $T\frac{\theta}{\theta T}$)

$$\sum_{n=0}^{\infty} \frac{(\sigma(u) - u)^n}{n!} \cdot N^n = exp((\sigma(u) - u)N).$$

It remains to show that this map (from M_1 to M_2) is integral. Multiplying by a suitable p-power we are reduced to showing the following.

Claim. Suppose $\delta : M_1 \to M_2$ is a map in $\mathcal{MF}_{[0,p-2]}(V)$ such that $\mathbb{D}(\delta)$ is divisible by p. Then δ itself is divisible by p.

Proof. In the following we will not use the Galois action any more. If m is a basis element of M_1, of degree q, then

$$m \otimes 1 \in \mathbb{D}(M_1)^* \otimes_{\mathbb{Z}_p} F^q(B^+(V)),$$

thus

$$\delta(m) \otimes 1 \in p \cdot \mathbb{D}(M_2)^* \otimes_{\mathbb{Z}_p} F^q(B^+(V)),$$

and

$$\alpha(m) \otimes \beta_0^{p-2} \in p \cdot F^q(B^+(V)) \cdot F^{p-2}(M_2 \otimes B^+(V)).$$

If m_μ denotes a filtered basis of M_2, of degrees q_μ, and if

$$\delta(m) = \sum c_\mu \cdot m_\mu,$$

it follows that

$$\beta_0^{p-2} \cdot c_\mu \in p \cdot F^q(B^+(V)) \cdot F^{p-2-q_\mu}(B^+(V)).$$

Let $\xi = f(\underline{\pi})$ denote the PD-generator of $F^1(B^+(V))$, and $\gamma = \beta_0/\xi$, which modulo (p, F^p) is equal to a unit multiple of $T^{e/(p-1)}$. Then using that $B^+(V)$ is graded faithfully flat over R_V we can deduce from that the following:

If $q_\mu \leq q$, then $T^{e\cdot(p-2)/(p-1)} \cdot c_\mu$ lies in the ideal of R_V generated by p, F^p, and all multiples

$$(p/(q+l)! \cdot (p-2-q_\mu+n)!) \cdot f(T)^{q-q_\mu+l+n},$$

l and n integers ≥ 0, $q - q_\mu + l + n < p$.

Modulo p this ideal lies in $F^s(R_V)$,

$$s = min(p - q_\mu, q+2) \geq q - q_\mu + 2.$$

It follows easily that each c_μ lies modulo p in $F^{q-q_\mu+1}(R_V)$, and thus $\delta(m)$ in $F^{q+1}(M_2)$ modulo p. Applying Φ_q we derive that

$$\delta(\Phi_q(m)) \in p \cdot M_2,$$

and thus δ is divisible by p. This finishes the proof of Theorem 5. □

A similar theory works for smooth V-schemes X. Any small enough affine $Spec(R) \subset X$ admits an étale map to a product of d copies of the multiplicative group, i.e. there is an étale homomorphism

$$V[T_1, T_1^{-1}, \ldots, T_d, T_d^{-1}] \to R.$$

If $\bar{R}$ denotes the normalisation of R in the maximal étale covering of $R[1/p]$, then $\bar{R}$ contains the extension R_∞ obtained by adjoining $\bar{V}$ as well as all p-power roots of the T_i. By [Fa1], Ch. I, Th. 3.1, $\bar{R}$ is almost étale over R_∞. This allows one to transfer many properties from R_∞ to $\bar{R}$. For example it follows that the Frobenius is surjective on $\bar{R}/p\cdot\bar{R}$, as this holds for R_∞. Now it is possible to repeat all steps in the construction of $B^+(V)$, replacing everywhere $\bar{V}$ by $\bar{R}$, and one obtains a ring $B^+(R)$. Also the proof of the main Theorem 5 works in this context, using the $\mathbb{Q}_p$-theory for isocrystals developed in [Fa2], V, f). We thus obtain:

Theorem 5*. *Define (for $M \in \mathcal{MF}_{[0,p-2]}(R)$)*

$$\mathbb{D}(M) = Hom_{R_V,F,\Phi}(M, B^+(V)) = Hom_{B^+(V),F,\Phi}(M \otimes_{R_V} B^+(R), B^+(R)).$$

i) *$\mathbb{D}(M)$ is free of rank equal to the rank of M.*

ii) *For $M \in \mathcal{MF}_{[0,a]}(V)$ we have*

$$\beta_0^a \cdot \mathbb{D}(M)^* \otimes_{\mathbb{Z}_p} \cdot B^+(V) \subset M \otimes_{R_e} B^+(V) \subset \mathbb{D}(M)^* \otimes_{\mathbb{Z}_p} B^+(V).$$

These inclusions are strict for the filtrations, i.e. they induce injections on gr_F.

iii) *The functor $\mathbb{D}$ from $\mathcal{MF}_{[0,p-2]}(V)$ to $\mathbb{Z}_p-Gal(\bar{R}/R)$-modules is fully faithful. Furthermore all this also works for divisors at infinity.*

Remark. One can extend the whole theory to "rings with toroidal singularities", which includes semistable singularities. This is due to recent progress in the theory of almost étale extensions (during the time the present paper was refereed).

5. Comparison

As in [Fa2], V, we can compare crystalline and étale cohomology. Suppose that X is proper and smooth over V, of relative dimension b, and $\mathcal{E} \in \mathcal{MF}_{[0,a]}(X/R_V)$ (defined in the beginning of section 3) such that $a+b \leq p-2$. We also assume that the Hodge-cohomology of $\mathcal{E}$ is torsion-free, and that the Hodge spectral sequence degenerates. Then $\mathbb{L} = \mathbb{D}(\mathcal{E})^*$ is a smooth $\mathbb{Z}_p$ -sheaf on $X \otimes_V K$. Using the Galois-cohomologies of $\bar{R}$'s we may form a cohomology theory $\mathcal{H}^*(X, \mathbb{L})$ as in [Fa2], III, c), e), which however turns out to be almost isomorphic to the étale cohomology $H^*(X \otimes \bar{K}, \mathbb{L}) \otimes \bar{V}^\wedge$ ([Fa2], II, Th. 3.3). Here "almost isomorphic" means that the associated graded is an almost isomorphism, that is, its kernel and cokernel are annihilated by the maximal ideal of $\bar{V}$. This is the definition from [Fa2], V, and there seems to be no easy description without refering to gr_F. I thank P. Berthelot for pointing this out to me. Also by an easy variant of [Fa2], V, a), there exists an almost defined map from $H^i(X/R_V, \mathcal{E}) \otimes B^+(V)$ to $H^i(X \otimes \bar{K}, \mathbb{L}) \otimes B^+(V)$, and Poincaré-duality provides an inverse up to β^{a+b}. It follows that $H^i(X \otimes \bar{K}, \mathbb{L})$ coincides with $\mathbb{D}(H^i(X/R_V, \mathcal{E}))^*$, and especially has no torsion. This uses that in the definition of $\mathbb{D}$ we may use "almost maps". However basically these are defined by solving certain equations (as in the proof of Theorem 5), and for these the assertion is obvious. The same reasoning works if we have divisors at infinity.

Theorem 6. *Suppose X is proper and smooth over $Spec(V)$, of relative dimension b, and $\mathcal{E} \in \mathcal{MF}_{[0,a]}(X/R_V)$. Let $\mathbb{L} = \mathbb{D}(\mathcal{E})^*$, a smooth étale sheaf on $X \otimes K$. Assume that $a + b \leq p - 2$, that the Hodge-cohomology of $\mathcal{E}$ has no torsion, and that the Hodge spectral sequence for $\mathcal{E}$ degenerates. Then for each i, $H^i(X \otimes_V \bar{K}, \mathbb{L})$ is isomorphic to $\mathbb{D}(H^i(X/R_V, \mathcal{E}))^*$. The same holds in the logarithmic case of divisors at infinity, for usual cohomology as well as cohomology with compact support. In particular the étale cohomology has no p-torsion as well.*

Obviously one can conjecture that everything also works for the semistable case in general. O. Hyodo and K. Kato have announced a comparison theorem for constant coefficients and small dimensions. Almost all the essential ingredients can be found in [H1], [H2] and [HK], but the result itself has not been published by them. Recently T. Tsuji has given a proof with all details.

More recently the almost étale theory has been extended to cover this case too (for curves see [Fa4], [Fa5]).

6. p-divisible groups

The whole theory works also for p-divisible groups. Suppose H is a p-divisible group over V. It gives rise to a filtered Frobenius-crystal $M = M(H)$ over R_V; see [BBM], [I], [MM] or [Me]. For example M can be chosen as the dual of the Lie-algebra of the universal vector-extension of H. Also $M(H)$ is an object of $\mathcal{MF}_{[0,1]}(V/R_V)$. In fact M (without filtration) depends up to isomorphism only on the reduction of H modulo p. As after composing with a sufficiently high power of Frobenius this reduction is induced from $k = V_0/p{\cdot}V_0 \subset V/p{\cdot}V$, and as Frobenius on M is an isogeny, it follows that $M[1/p] \cong M_0[1/p] \otimes R_V$ as Frobenius-isocrystal, where $M_0 = M/T{\cdot}M$ is the Dieudonné module of the special fibre $H \otimes k$. (Compare [BO], Th. 1.3.)

Also as $B^+(V)$ is a projective limit of PD-thickenings of $\bar{V}^\wedge$ we can form the crystalline cohomology relative $B^+(V)$. This is equal to the pushout

$$H^1(H \otimes_V \bar{V}^\wedge / B^+(V)) = M \otimes_{R_V} B^+(V),$$

and any $\bar{V}$- homomorphisms between H's induce a map between cohomologies. For example any element of the Tate-module $T_p(H)$ defines over $\bar{V}$ a homomorphism $\mathbb{Q}_p/\mathbb{Z}_p \to H$. Now $H^1(\mathbb{Q}_p/\mathbb{Z}_p)$ is canonically isomorphic to R_V, as the universal vector-extension of $\mathbb{Q}_p/\mathbb{Z}_p$ is obtained from

$$0 \to \mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

via pushout ($\mathbb{Z}_p \subset R_V$). We deduce a pairing from $T_p(H) \times M(H) \otimes B^+(V)$ into $B^+(V)$, respecting Frobenius filtrations, and Galois-operation. Equivalently, if we define the étale cohomology of H as

$$H^1_{et}(H) = T_p(H)^*,$$

we obtain a canonical map

$$\rho : M(H) \otimes B^+(V) \to H^1_{et}(H) \otimes B^+(V).$$

One can check that for p-divisible groups associated to abelian varieties ρ coincides with the previous comparsion map (although we do not need this assertion here).

Sketch. One shows that both maps define the same Hodge-Tate structure. (This suffices by Galois-invariance.) But this comes down to the computation in [Fa6], proof of Th. 4, where it is shown that Tate's and Fontaine's method give the same Hodge-Tate structure.

To study ρ we first consider the special case of $H = G_m[p^\infty]$, where $T_p(H) = \mathbb{Z}_p(1)$. The homology of $\mathbb{Q}_p/\mathbb{Z}_p$ can be defined via the universal vector-extension, which in this case is just the pushout $(\mathbb{Z}_p \to \mathbb{G}_a)$ of the extension

$$\mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p.$$

The pairing

$$\mathbb{Z}_p(1) \times \mathbb{Q}_p/\mathbb{Z}_p \to \bar{V}^{\wedge *}$$

lifts via α to

$$\mathbb{Z}_p(1) \times \mathbb{Q}_p \to B^+(V)^*.$$

The induced tangent-map on the universal vector-extension is obtained by taking logarithms, and we obtain (what else could it be)

$$\beta : \mathbb{Z}_p(1) \to F^1(B^+(V)) :$$

For $\lambda \in \mathbb{Z}_p(1) = \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \mu_{p^\infty})$, its unique lift $E(\mathbb{Q}_p/\mathbb{Z}_p) \to \hat{\mathbb{G}}_m$ over $B^+(V)$ is induced by $\alpha \circ \lambda : \mathbb{Q}_p \to B^+(V)^*$.

For general H we can apply this by composing with the maps $H \to G_m[p^\infty]$ induced by $T_p(H^*) = H^1_{et}(H)(1)$. We derive that ρ has a left-inverse up to β, i.e. the composition of these maps gives $\beta \cdot id$ on $H^1_{et}(H) \otimes B^+(V)$. However as all modules are free of rank h over $B^+(V)$, and β_0 is a non-zero-divisor, the composition the other way round is also $\beta \cdot id$. Thus we have shown:

Theorem 7. *There exists a functorial injection, respecting Frobenius, filtrations and Galois operations*

$$\rho : M(H) \otimes B^+(V) \to H^1_{et}(H) \otimes B^+(V).$$

The cokernel of ρ is annihilated by β_0. In particular if $p > 2$, then $T_p(H) = \mathbb{D}(M(H))$.

The last assertion follows because obviously $T_p(H)$ is contained in $\mathbb{D}(M(H))$, and the quotient $H^1_{et}(M)/(\mathbb{D}(M(H))^* \otimes_{\mathbb{Z}_p} B^+(V))$ is annihilated by β_0.

Note that after inverting p the left side can be identified with the tensor product $M_0 \otimes B^+(V)$, and we can recover $M_0[1/p]$ as invariants under Galois in $H^1_{et}(H) \otimes B^+(V)$. Similarly after extension to $B^+_{DR}(V)$ we recover the Hodge filtration on $H^1_{DR}(H/V)$. This is the content of Fontaine's comparison theorem. However there is more content to this result than assertions over $\mathbb{Q}_p$, as for example the following shows.

Definition 8. An étale Tate-cycle of degree r on H is a Galois invariant multilinear-form $\psi_{et} : T_p(H)^{\otimes 2r} \to \mathbb{Z}_p(r)$, or equivalently a Galois-invariant class in $\psi_{et} \in (H^1_{et}(H))^{\otimes 2r}(r)$. A crystalline Tate-cycle of degree r is a class $\psi \in F^r(M(H)^{\otimes 2r})$ which is $\bigtriangledown$ -parallel and fixed by $\Phi_r = \Phi/p^r$.

By Fontaine's theory the $\mathbb{Q}_p$-vector-spaces spanned by Tate-cycles correspond under ρ, i.e. $\rho(\psi \otimes 1) = \psi_{et} \otimes \beta^r$.

Corollary 9. *If $r \leq p-2$, then the comparison respects integrality, that is, ψ is integral if and only if ψ_{et} is.*

Proof. Consider ψ as a map from $M(H)^{\otimes r}$ to $M(H^*)^{\otimes r}$, and apply Theorem 5. □

Remark. The bound cannot be extended to $p-1$, as $\beta^{p-1}/p \in B^+(V)$ is integral. Take $H = (\mathbb{Q}_p/\mathbb{Z}_p \oplus \mu_{p\infty})^{p-1}$, $T_p(M) = \mathbb{Z}_p^{p-1} \oplus \mathbb{Z}_p(1)^{p-1}$, ψ_{et} = the unique Galois-linear alternating form $\Lambda^{2(p-1)} T_p(M) \to \mathbb{Z}_p(p-1)$, ψ its crystalline analogue. Then ψ/p is not integral in crystalline cohomology, but $\psi_{et} \otimes \beta^{p-1}/p$ is integral in $H^{2(p-1)}_{et}(H) \otimes B^+(V)$.

7. Deformation theory

Suppose H_0 is a p-divisible group over V_0, of dimension d and height $h = d + d^*$, where d^* is the dimension of the dual group H_0^*. Let M_0 denote its Dieudonné-module (free of rank h over V_0). Then $H_0 \otimes_{V_0} k$ has a versal deformation H over $A = V_0[[t_1, \ldots, t_n]]$, $n = d \cdot d^*$, that is, any other deformation can be induced from H via base change. Furthermore we can choose coordinates t_i such that $H_0 = H \otimes_A A/(t_i)$. Let $M(H) = H^1_{DR}(H/A)$ denote the associated contravariant Dieudonné module (see [BBM], [I], [MM], [Me]). It is an object in $\mathcal{MF}_{[0,1]}(A)$. This holds even for $p = 2$, if we interpret it as follows:

Let $\phi : A \to A$ denote the Frobenius-lift on A which extends the Frobenius on V_0 and sends t_i to t_i^p. Then there exists a canonical ϕ- linear endomorphism $\Phi : M \to M$ (by reduction mod p and functoriality). Furthermore the restriction of Φ to F is divisible by p, $\Phi|F = p \cdot \Phi_1$, and Φ induces an isomorphism

$$\Phi : (M + p^{-1} \cdot F) \otimes_{A\,\phi} A \cong M.$$

One checks that $\bigtriangledown$ induces an integrable connection on the left-hand side, and Φ becomes horizontal. Also it follows that modulo the ideal generated by the *p-th* powers t_i^p, M is canonically isomorphic to M_0. That is, there exists a horizontal (w.r.t. $\bigtriangledown$ and $1 \otimes d$) isomorphism

$$M \otimes_A A/(t_i^p) \cong M_0 \otimes_{V_0} A/(t_i^p).$$

Using this canonical isomorphism the versality-condition on A can be restated as the fact that A is formally étale over the Grassmannian of d-planes in M_0. Now consider a complete local V_0-algebra R, with residue field k, and a PD-ideal $I \subset R$ such that all divided powers of I are p-adically closed. We also assume that R has no torsion, which implies that the divided powers are compatible with the standard powers of p.

If $\bar{H}$ is a p-divisible group over $\bar{R} = R/I$ which deforms $H_0 \otimes_{V_0} k$, we can lift it to a p-divisible group over R, and it is induced from H via some map $\alpha : A \to R$. The induced module $M(\bar{H}/R) = M(H/A) \otimes_A R$ depends up to canonical isomorphism only on the reduction of α modulo I:

If α_1 and α_2 coincide modulo I, as usual an isomorphism between the two induced modules given by the Taylor-series ($J = (j_1, \ldots, j_d)$ multi-index)

$$\sum \bigtriangledown(\partial^J)(m) \cdot (\alpha_2(t) - \alpha_1(t))^J / J!.$$

The sum converges because $\bigtriangledown$ is topologically nilpotent. Also $M(\bar{H}/R)$ is canonically filtered compatible with the divided power filtration on R, and this filtration is induced from A. Furthermore if R admits a Frobenius-lift ϕ_R, there is an induced ϕ_R-linear Φ_R on $M(\bar{H}/R)$, whose restriction to F^1 is canonically divisible by p, that is, $\Phi_R|F^1 = p \cdot \Phi_{R,1}$. Again these are induced from the universal data over A, but because the Frobenius-lifts may not be compatible one has to use the connection for this:

Define elements $z_i \in R$ such that $p \cdot z_i = \phi_R(\alpha(t_i)) - \alpha(t_i^p)$. Then although the connection $\bigtriangledown$ on M (over A) does not induce a connection on $\tilde{M} = M + p^{-1} \cdot F$, the Taylor-series

$$\sum \bigtriangledown(\partial^J)(m) \cdot p^{|J|}/J! \cdot z^J$$

still converges and defines an isomorphism between the two pushforwards of $\tilde{M}$, via $\phi_R \cdot \alpha$, respectively $\alpha \cdot \phi_A$. (The inverse is a similar series with z_i replaced by $-z_i$.) Under this isomorphism the two Φ's correspond. This follows for example by composing with Verschiebung, i.e. the adjoint of Frobenius on the dual group H^*.

Next we want to demonstrate that any such crystal (as $M(\bar{H}/R)$) can be induced from A. However for technical reasons we assume that

$$R = V_0[[x_1, \dots , x_r]],\ I = (0),\ \phi_R(x_i) = x_i^p,\ \phi_R|V_0 = \phi.$$

Suppose on R we are given a free module M_R of rank $h = d + d^*$, a direct summand $F_R \subset M_R$ of rank d, and an isomorphism $\Phi : \tilde{M}_R \otimes_{R\,\phi} R \cong M$. Furthermore we assume that modulo the x_j these are isomorphic to M_A modulo the t_i, i.e. the canonical pushforwards to V_0 are isomorphic.

Theorem 10. *There exists a lifting $\alpha : A \to R$ such that (M_R, F_R, Φ_R) is the pushforward of (M, F, Φ), taking into account the difference in Frobenius-lifts as above. In particular (M_R, F_R, Φ_R) is induced by a deformation of H_0. In addition M_R admits a unique connection $\bigtriangledown_R$ such that Φ_R is $\bigtriangledown_R$-horizontal, and $\bigtriangledown_R$ is induced from $\bigtriangledown$ on M.*

Proof. For any choice of α there is a filtered isomorphism $M_R \cong M \otimes_A R$, unique up to transformation with an element $g \in Aut(M_R, F_R)$. The problem is to adjust α and g such that the Φ's correspond. For this assume that this holds modulo some power $\mathfrak{r}^m$ of the augmentation-ideal $\mathfrak{r} = (x_j)$. We want to modify α and g by elements in $\mathfrak{r}^m$ to achieve this also modulo the next power $\mathfrak{r}^{m+1}$. This is a little bit complicated because we have to consider the corrections due to the difference in Frobenius-lifts. These are given by the Taylor-series

$$(*) : \sum \bigtriangledown(\partial^J)(m) \cdot p^{|J|}/J! \cdot z^J$$

as above. If we change $\alpha(t_i)$ by elements $\delta\alpha(t_i) \in \mathfrak{r}^m$, the z_i change by elements $\delta z_i \in \mathfrak{r}^{m+1}$. We thus use the following strategy: First modify g (by something in $\mathfrak{r}^m$) to make the diagram

$$\begin{tikzcd}
M \otimes_{A,\alpha} R \arrow[r, "\sim^g"] & M_R \\
\tilde{M} \otimes_{A,\alpha\circ\varphi_A} R \arrow[u, "\wr\, \Phi_A \otimes 1"'] & \\
\tilde{M} \otimes_{A,\varphi_R\circ\alpha} R \arrow[u, "\wr\, (*)"'] \arrow[r, "g\otimes 1", "\sim"'] & \tilde{M}_R \otimes_{R\,\varphi_R} R \arrow[uu, "\wr\, \Phi_R"']
\end{tikzcd}$$

commute modulo $\mathfrak{r}^{m+1} \cdot M_R$. As for a typical element $\tilde{m} \in \tilde{M}$ the image of $\tilde{m} \otimes 1$ under $\Phi_R \circ (g \otimes 1)$ does not change modulo $\mathfrak{r}^{m+1}$ if we modify g by something in $\mathfrak{r}^m$;

there is a unique choice of $g(mod\, \mathfrak{r}^{m+1})$ which achieves this. After that we modify $\alpha(t_i)$ by elements in $\mathfrak{r}^m$ to make the new g respect filtrations, using versality. One now checks that this change does not destroy the commutativity (modulo $\mathfrak{r}^{m+1}$) of the diagram above. This proves the first part of the theorem.

For the second note that two connections differ by a form $\beta \in End(M_R) \otimes_R \Omega_R$, with $d\Phi_R(\beta) = \beta$. However as $d\Phi_R(\mathfrak{r}^m \cdot \Omega_R)$ is contained in $p \cdot \mathfrak{r}^{m+1} \cdot \Omega_R$ and as $p \cdot \Phi_R$ is integral on $End(M_R)$, β must vanish. According to the referee unicity is well known. $\square$

Remarks. i) As an application we can replace A by A_1 = formal completion of $Aut(M_0)/V_0$ at the origin (over Spec(k)). In suitable coordinates A_1 satisfies all conditions. Define a crystal $M_1 = M_0 \otimes A_1$ (respecting filtrations), $\Phi_1 = g \cdot (\Phi_0 \otimes \phi_{A_1})$, $g \in A_1$ the universal element. One checks that any crystal over R as above can also be induced from M_1, however less unique as from M, and of course M_1 can also be induced from M. The advantage of M_1 is that it has an easy explicit description.

ii) Suppose $G \subset Aut(M_0)$ is a smooth connected subgroup such that its Lie-algebra in $End(M_0)$ is stable under Frobenius (= conjugation with Φ_0). Then we can redo the construction above with the formal completion of G at the origin. One easily derives that the connection respects the G-action, that is, of the form $d + \beta$, with $\beta \in Lie(G) \otimes \Omega_G$. (The coefficients of the power-series β are determined by a recursion which only involves elements of Lie(G). It starts with $-g^{-1}dg$.)

iii) Now assume that in addition the Lie-algebra $Lie(G) \subset End(M_0)$ is a subobject in $\mathcal{MF}_{[-1,1]}(V_0)$, i.e. if F^i denotes the filtration on $End(M_0)$ (located in degrees $-1, 0, +1$), then $Lie(G)$ is isomorphic to the Frobenius-transform of

$$p^{-1}(F^1 \cap Lie(G)) + F^0 \cap Lie(G) + p \cdot Lie(G).$$

(See also [FL], 1.5.)

Now the affine algebra $A_{GL(M_0)}$ of $GL(M_0)$ is naturally an object (or better a filtering union of objects) in $\mathcal{MF}(V_0)$. Furthermore the affine algebra A_G of G is a quotient, and the kernel of $A_{GL(M_0)} \to A_G$ consists of $f \in A_{GL(M_0)}$ with the property that for any element $Z \in U(Lie(G))$ of the enveloping algebra of $Lie(G)$, $Z(f)$ vanishes at the origin. Thus this kernel is a subobject. It follows that there exists an $E_0 \in \mathcal{MF}(V_0)$, a representation of $GL(M_0)$ on E_0 given by a map $E_0 \to E_0 \otimes A_{GL(M_0)}$ in $\mathcal{MF}(V_0)$, and an element $l \in F_0(E_0)$ fixed by Φ_0 such that G is a normaliser of the line L_0 spanned by l. Furthermore E_0 can be chosen by applying certain tensor-operations (like duals, or exterior or symmetric powers and Tate-twists) to M_0.

Now suppose we have a deformation M_R of M_0 such that $L_0 \subset E_0$ extends to an inclusion of objects in $\mathcal{MF}(R)$. Then we claim that M_R is induced from the formal completion of G along the origin.

We can follow the previous arguments if we show that there exists an isomorphism of filtered spaces between M_R and $M_0 \otimes R$ which respects L_0. This can be done by infinitesimal lifting, and we come down to the following assertion, which holds because the evaluation at l, $End(M_0) \to E_0/L_0$, is a map in $\mathcal{MF}$, and thus is strict for filtrations ([FL], Lemma 1.9): If $X \in End(M_0)$ respects L_0, then $X \in F^0(End(M_0)) + Lie(G)$. By removing some unnecessary variables we obtain a "versal deformation respecting Tate-cycles", which has tangent space $Lie(G)/F^0(Lie(G))$.

We close with a negative result. Recall ([D]) that the Drinfeld upper half-plane is the complement of all $\mathbb{Q}_p$-rational hyperplanes in $\mathbb{P}^{d-1}$. It classifies certain types of p-divisible groups, as follows: Let B denote the division-algebra over $\mathbb{Q}_p$ with invariant $1/d$. Consider (over extensions of $\mathbb{Z}_p$) formal groups of dimension d and height $h = d^2$ which admit right-multiplication by the integers $\mathcal{O}_B$, and fulfill a certain condition on the tangent-space (which is automatic in characteristic zero). It is known that all of these are isogenous in characteristic p: For example if k is algebraically closed there exists a unique (up to isogeny) crystal X of dimension d and slope $1/d$, and its endomorphisms are isomorphic to B. Then the Dieudonné module of our formal group must be of the form $X \otimes Y$, where Y has dimension d, average slope zero, and all slopes between $-1/d$ and $(d-1)/d$. This can only happen if Y has slope zero.

One can define a formal model Ω for the upper half-plane which parametrises these formal $\mathcal{O}_B$-modules together with a quasi-isogeny (= isogeny up to inverting p) to a fixed one, modulo p. It admits an operation of $PGL(d, \mathbb{Q}_p)$, and is quasi-compact modulo this action. Locally in Ω the denominator of this quasi-isogeny is bounded, that is, a fixed multiple of it is an actual isogeny. As Ω is quasi-compact modulo the operation of $PGL(d)$ it follows that there exists a power p^N such that for any two such modules over a valuation-ring V, their reductions modulo p admit between them an isogeny of degree $\leq p^N$.

Now Voskuil has defined analogous spaces contained in the Grassmannian of a-planes in $(a+b)$ -space, for any two coprime integers a, b (see [V], Ch. IV). Special cases arise from formal groups with $\mathcal{O}_B$-multiplication, for B now the division-algebra with invariants $a/(a+b)$. We show that for these the above isogeny-property fails, that is, there exists no a priori bound p^N.

We do this in the simplest case $a = 2$, $b = 3$, and assume $p > 2$. The idea behind this construction is that now the Dieudonné-module is again of the form $X \otimes Y$, with X of slope $2/5$ and Y of dimension 5 and slopes between $-2/5$ and $3/5$. The symmetric space parametrises Y's of slope zero. However this is not automatic: Y might also have slopes $-1/3$ and $+1/2$. Our example is constructed using a deformation from a good Y (pure slope zero) to such a bad one.

Let V_0 denote the completed maximal unramified extension of $\mathbb{Z}_p$. By $L \subset V_0^5$ we denote the line generated by the integers in the unramified extension of $\mathbb{Q}_p$ of degree 5, embedded in five possible ways into K_0. Then V_0^5 is also the direct sum of the Frobenius-transforms $\phi^j(L)$, $0 \leq j \leq 4$. We define a Dieudonné-module M_0 over V_0 as follows: M_0 is $\mathbb{Z}/5 \cdot \mathbb{Z}$ -graded, with components $M_{0,i}$, $0 \leq i \leq 4$. In turn the $M_{0,i}$ are submodules in K_0^5, and of the form

$$M_{0,i} = \sum p^{n_{ij}} \cdot \phi^j(L),$$

where the exponents n_{ij} are given by the matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

One checks that

$$M_{0,0} \subset M_{0,3} \subset M_{0,1} \subset M_{0,4} \subset M_{0,2} \subset p^{-1} \cdot M_{0,0}.$$

Also if $F_{0,i} \subset M_{0,i}$ denotes the subspace spanned by the first two summands (involving L and $\phi(L)$), then $\phi(F_{0,i}) \subset p \cdot F_{0,i+1}$ for $i = 0, 2, 3, 4$, respectively $\phi(F_{0,i}) \subset M_{0,i+1}$ for $i = 2, 4$. Finally if $z \in End(K_0^5)$ denotes the nilpotent endomorphism which sends L isomorphically to $p^{-1} \cdot \phi^2(L)$ and vanishes on all other components $\phi^j(L)$ it follows that z respects all $M_{0,i}$. Also $p \cdot z \cdot \phi^2$ respects V_0^5, and has modulo p a nontrivial eigenvector with eigenvalue t.

After these definitions we can do business. The integer of the unramified extension of $\mathbb{Q}_p$ of degree 5 operate on $M_{0,i}$ via the *i-th* power of Frobenius. Furthermore there is an endomorphism Π of M_0, with $\Pi^5 = p$, which on components is given by the inclusions

$$M_{0,0} \subset M_{0,3} \subset M_{0,1} \subset M_{0,4} \subset M_{0,2} \subset p^{-1} \cdot M_{0,0}$$

above, except that we multiply the last one by p. These two generate a copy of $\mathcal{O}_B$ acting on M_0, respecting $F_0 =$ direct sum of the $F_{0,i}$. Furthermore we define Frobenius by $\Phi_0 = \Pi^2 \cdot \phi$; it follows that (M_0, F_0, Φ_0) is an object in $\mathcal{M}\mathcal{F}_{[0,1]}(V_0)$ with $\mathcal{O}_B$-multiplication, which integrates to a formal group H_0. Furthermore z defines an endomorphism of M_0 commuting with $\mathcal{O}_B$. We thus can define another formal group H over $R = V_0[[t]]$ by $(M_R, F_R) = (M_0, F_0) \otimes R$, $\Phi_R = (1 + t \cdot z) \cdot \Phi_0$. H admits $\mathcal{O}_B$-multiplication, by using either Theorem 5^* or Dieudonné-theory. If we choose a very ramified quotient V of R, then the two induced p-divisible groups $H_0 \otimes_V V/pV$ and $H \otimes_V V/pV$ over V/pV do not admit an $\mathcal{O}_B$-linear isogeny of degree $\leq p^N$:

Otherwise let R_V denote the completed PD-hull of such a V. R_V is the p-adic completion of the polynomials in $\{t, t^{ne}/n!\}$, e the ramification index of V. Any isogeny over V/pV would induce such an isogeny of lattices over R_V. However M_0 has the property that the powers (of the real Frobenius ϕ) $\{(\Pi^{-2} \cdot \Phi_0)^n,\ n \geq 0\}$ have uniformly bounded denominator, for example because these powers respect the lattice spanned by V_0^{25}. In contrast to that first over R, $p^n \cdot (\Pi^{-2} \cdot \Phi)^{3n}$ has on the lattice spanned by V_0^{25} modulo p an eigenvector with eigenvalue $t^{1+p^3+\dots p^{3(n-1)}}$, and so over R_V the denominators of $(\Pi^{-2} \cdot \Phi)^{3n}$ grow like p^n up to a point (depending on e), which can be moved very far out by making e bigger than the exponent of the t-power above. It then follows that even over R_V the two lattices are sufficiently different so that no isogeny of degree $\leq p^N$ can exist between them.

8. Appendix: Finiteness and variants

In this appendix we list some basic results about finiteness, which are easily proven but not all documented in the literature. We also explain how to treat the prime $p = 2$. First, let R denote a commutative ring with a decreasing filtration by ideals

$$R = F^0 R \supseteq F^1 R \supseteq F^2 R \supseteq \dots,\ F^a R \cdot F^b R \subseteq F^{a+b} R.$$

A filtered R-module M is an R-module with a filtration $F^q M$ such that

i) $F^q M = M$ for $q << 0$.

ii) $F^a R \cdot F^b M \subseteq F^{a+b} M$.

An example is $R\{a\}$, which is R with its filtration shifted by $a(F^q R\{a\} = F^{q-a}(R))$. M is called filtered free if it is the direct sum (possibly infinite) of $R\{a_i\}$'s with the a_i uniformly bounded below. M is called filtered projective if it is a filtered direct summand of a filtered free module. Consider filtered complexes K^*,

$$\ldots \to K^{n-2} \to K^{n-1} \to K^n \to K^{n+1} \to \ldots,$$

where each K^n is filtered, $K^n = 0$ for $n >> 0$, and the differentials preserve filtrations. A filtered map $\alpha : K^* \to L^*$ is called a filtered quasi-isomorphism if α induces quasi-isomorphisms $F^qK^* \xrightarrow{\approx} F^qL^*$ for each q. Equivalently this means that α as well as $gr_F(\alpha)$ are quasi-isomorphisms.

i) For any K^* there exists a quasi-isomorphism $L^* \to K^*$ with each L^n filtered free.

ii) If each L^n is filtered projective and if K^* is filtered acyclic (i.e. each F^qK^* is acyclic), then each map $L^* \to K^*$ is filtered homotopic to zero.

iii) If each L^n is filtered projective and if $\alpha : K_1^* \to K_2^*$ is a filtered quasi-isomorphism, then α induces an isomorphism on filtered homotopy-classes of filtered maps from L^* to K_1^*, respectively K_2^*. (Consider the mapping cone Cone (α).)

Now form the filtered derived category $D_{fil}(R)$ by inverting filtered quasi-isomorphisms. Then filtered homotopic maps become equal in $D_{fil}(R)$, and $D_{fil}(R)$ is equivalent to the category of filtered free (or filtered projective) complexes with filtered homotopy-classes of filtered maps.

Now suppose $I \subseteq R$ is an ideal with $I^2 = (0), \bar{R} = R/I$ (with induced filtration), K^* a filtered complex of R-modules (bounded above). Suppose furthermore there exists an exact sequence of filtered complexes (i.e. each F^q is exact)

$$0 \to K_1^* \to K^* \to K_2^* \to 0$$

with K_1 and K_2 annihilated by I. We then obtain a well-defined map of $\bar{R}$-complexes $\pi : I \otimes_{\bar{R}} K_2^* \to K_1^*$. Suppose furthermore we have given a filtered quasi-isomorphism $\bar{\alpha} : \bar{L}^* \to K_2^*$ with $\bar{L}^*$ filtered projective over $\bar{R}$.

Proposition 11. *The following are equivalent:*

i) $\bar{L}^*$ and $\bar{\alpha}$ can be lifted to a filtered quasi-isomorphism

$$\alpha : L^* \to K^*$$

with L^* filtered projective.

ii) The induced map

$$I \otimes_{\bar{R}} \bar{L}^* \to K_1^*$$

is a filtered quasi-isomorphism. Here

$$F^q(I \otimes_{\bar{R}} \bar{L}^n) = \sum_{a+b=q} \mathit{Image}\,((F^a \cap I) \otimes F^b\bar{L}^n),$$

and one checks that our map respects filtrations.

Proof. i) $\Rightarrow$ ii) One checks that $I \otimes_{\bar{R}} \bar{L}^* \xrightarrow{\approx} IL^*$ (with induced filtration) is a quasi-isomorphism, and so is $IL^* \xrightarrow{\approx} K_1^*$. ii) $\Rightarrow$ i) Each $\bar{L}^n$ lifts to a filtered projective R-module L^n. (This holds for filtered free objects, and one can lift projections over nilpotent ideals.) We construct by decreasing induction on n the maps $\alpha_n : L^n \to K^n$ and $d_n : L^n \to L^{n+1}$. First choose some filtered lifts α_n^* and d_n^* of $\bar{\alpha}_n$ and $\bar{d}_n$. Then $(d_n \circ \alpha_n^* - \alpha_{n+1} \circ d_n^*, d_{n+1} \circ d_n^*)$ defines a filtered map

$$L^n \to K_1^{n+1} \oplus I \otimes_{\bar{R}} \bar{L}^{n+2}$$

whose image is d-closed in the mapping cone of $I \otimes_{\bar{R}} \bar{L}^* \to K_1^*$.

As L^n is filtered projective and the mapping cone is filtered acyclic, it is the boundary of a filtered map $L^n \to K_1^n \oplus I \otimes_{\bar{R}} \bar{L}^{n+1}$ whose components are the corrections to α_n^* and d_n^*. □

Corollary 12. *Suppose $R \supseteq I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq I_{n+1}$ is a decreasing sequence of ideals in R, such that $I_1 \cdot I_n \subseteq I_{n+1}$, so $I = I_1$ is nilpotent modulo each I_n. Assume furthermore we have a projective system of filtered complexes K_n^* (bounded above) over $R_n = R/I_n$ (with induced filtration), such that the transition maps $\pi_n : K_n \to K_{n-1}$ are filtered surjective, and $Ker(\pi_n)$ is annihilated by I. Assume furthermore we have a filtered quasi-isomorphism $\alpha_1 : L_1^* \xrightarrow{\approx} K_1^*$ with L_1^* filtered projective, and that the obvious maps*

$$I_{n-1}/I_n \otimes_{R_1} L_1^* \to Ker\,(\pi_n)$$

are filtered quasi-isomorphisms. Then L_1^* lifts to a compatible system of filtered projective complexes L_n^* over R_n, and α_1^* to a compatible system of filtered quasi-isomorphisms $\alpha_n : L_n^* \xrightarrow{\approx} K_n^*$.

We want to apply this to crystalline cohomology. In [B] there are three different definitions for the crystalline topoi, each using certain PD-embeddings. All the theorems are formulated for the case that some power p^N vanishes on the base-scheme ([B], pg. 179). A variant is the nilpotent crystalline topos, where instead one requires that the embeddings are PD-nilpotent ([B], pg. 187). Finally one can use the extended topos where one requires that $n!I^{[n]} = (0)$ for $n >> 0$ ([B], pg. 587). We now assume that $F^0R = I_1$ is a PD-ideal, that all I_n are sub-PD-ideals, and that $R = \varprojlim R/I_n$ is complete for the I_n-topology. We also require that the PD-embedding

$$Spec(R/I_1) = S_1 \hookrightarrow Spec(R/I_n) = S_n$$

satisfies for all n the conditions to be in the relevant crystalline topos, that is, either

- some power $p^N \in I_n$
- some $I_1^{[N]} \subseteq I_n$
- some $N! \cdot I_1^{[N]} \subseteq I_n$,

where N may increase with n. In particular I_1/I_n is a nilideal, that is, each element in it is nilpotent. It follows that for any finitely generated ideal $I \subseteq I_1$, some power (depending on n) of I is contained in I_n. Finally we assume that R is a $\mathbb{Z}_p$-algebra, and that all PD-structures are compatible with the divided powers on $p \cdot \mathbb{Z}_p$.

Then for crystals on S_n-schemes we may define crystalline cohomology. For affine schemes $Spec(A)$ one computes it by writing $A = B/J$ with B a smooth R_n-algebra, forming the divided power hull $D_J(B)$ (divided powers compatible with those on I_1/I_n), completing it in either

- the discrete topology (if $p^N \in I_n$)
- the PD-topology defined by $D_J(J)^{[N]}$
- the topology defined by $N! \cdot D_J(J)^{[N]}$.

In each case one can evaluate the crystal on it, then form the de Rham complex by tensoring with Ω^*_{B/R_n}, and its cohomology represents crystalline cohomology. That it is independent of the choice of B follows from the Poincaré lemma. A variant uses filtered-free crystals $\mathcal{E}$. Then $\mathcal{E}(\widehat{D_J(B)})$ and $\mathcal{E}(\widehat{D_J(B)}) \otimes \Omega^*_{B/A}$ are also filtered, and the cohomology is represented by a filtered complex over R_n. In particular if

A itself is smooth over R_1, we can lift it to a smooth R_n-algebra B, and then the crystalline cohomology is represented by the filtered complex $(\mathcal{E}(B) \otimes \Omega^0_{B/R_n}, \bigtriangledown)$.

Now in general assume we are given a smooth scheme X over R_1, and a locally filtered free crystal $\mathcal{E}$ on X/R_n. Then if X is separated we can compute its crystalline cohomology by choosing an affine covering $X = \bigcup_i Spec(A_i)$, smooth embeddings $Spec(A_i) \hookrightarrow Spec(B_i)$, forming the de Rham complexes for the PD-hulls of $Spec(A_i) \cap \ldots \cap Spec(A_{i_1}) \hookrightarrow Spec(B_{i_1} \otimes_{R_n} \ldots \otimes B_{i_r})$, arranging them into a double complex and forming the associated simple complex K_n^*. If $\mathcal{E}$ is locally filtered free, K_n^* is naturally filtered. Also for two different choices of coverings or embeddings the resulting complexes K_n^* and $\tilde{K}_n^*$ are related by a diagram of filtered quasi-isomorphism:

$$\begin{array}{ccc} & & K_n^* \\ & \approx\nearrow & \\ \overset{\approx}{K}{}_n^* & & \\ & \searrow\approx & \\ & & \tilde{K}_n^*, \end{array}$$

where $\overset{\approx}{K}{}_n^*$ is obtained by using the "double" covering $X = \bigcup_i Spec(A_i) \bigcup_j Spec(\tilde{A}_j)$. We may represent K_n^* by a filtered free complex, and the construction commutes with base-change (devissage from the affine case). If X is not separated, one has to use affine hypercoverings. However we do not need this case.

Now assume that X_1 is proper over R_1.

Proposition 13. *The filtered complex K_1^* can be represented by a filtered projective L^* such that:*

i) $L^n = (0)$ unless $0 \leq n \leq 2 \cdot rel \cdot dim(X)$,

ii) all L^n are filtered direct summands in finite direct sums of copies of $R_1\{a\}$'s. (Such an L_1^* is called strictly perfect.)

Proof. There exists a finitely generated $\mathbb{Z}$-subalgebra of R_1 over which X and $\mathcal{E}$ are already defined (EGA IV, §8). We thus may assume that R_1 is noetherian. The assertion is equivalent to the fact that $gr_F(K_1^*)$ can be represented by a complex of finitely generated projective R_1-modules, concentrated in degrees $[0, 2{\cdot}rel.dim.(X)]$. However $gr^q_F(K_1^*)$ represents $\mathbb{R}\Gamma(X, gr^q_F(\mathcal{E} \otimes \Omega^*_{X/R_1}))$, where $gr^q_F(\mathcal{E} \otimes \Omega^*_{X/R_1})$ is a complex of vector bundles, and the assertion follows from well-known facts about coherent cohomology. □

Finally assume that we have a compatible system $\mathcal{E}_n$ of locally filtered free crystals on X/R_n. Then choosing a covering $X = \bigcup_i Spec(A_i)$ and compatible systems of liftings of A_i to smooth R_n-algebras $B_{i,n}$ (possible since I_1/I_n is nilpotent), we obtain a compatible system of filtered complexes K_n^* as in Corollary 12. We claim that modulo the ideal

$$I = p \cdot I_1/I_n \subseteq R_n = R/I_n$$

K_n is filtered quasi-isomorphic to a strictly perfect L_n^* as in Proposition 13. If this holds modulo ideals $\mathfrak{a}$ and $\mathfrak{b}$, then also modulo $\mathfrak{a} \cdot \mathfrak{b}$, by a simple devissage. We are thus reduced to

Proposition 13*. *Assume $I_n = p \cdot R = (0)$. Then K_n is filtered quasi-isomorphic to strictly perfect L_n^*.*

Proof. In characteristic p Frobenius annihilates any PD-ideal. Hence firstly it lifts canonically to $\mathrm{Frob} : R_1 \to R_n = R$. Secondly relative Frobenius defines a morphism of ringed topoi

$$\mathrm{Frob}_X : (X_1/R_n)_{crys} \to (X_1 \otimes_{R_1 \mathrm{Frob}} R_n)_{Zar},$$

and the derived direct image $\mathbb{R}\,\mathrm{Frob}_{X,*}(\mathcal{E}_n)$ is locally representable by a strictly perfect filtered complex on $(X_1 \otimes_{R_1 \mathrm{Frob}} R)_{Zar}$, namely the de Rham complex of any smooth lift of X_1. Conclude by applying Zariski-cohomology.

Now $p \cdot I_1/I_n \subseteq R_n$ is (globally) nilpotent, by the assumptions on I_n. Thus by Corollary 12 (applied to its powers) K_n^* is also filtered quasi-isomorphic to a strictly perfect L_n^* (over R_n). Furthermore one can choose those compatible, that is, $L_n^* \to K_n^*$ is induced from $L_{n+1}^* \to K_{n+1}^*$:

This follows because the first filtered quasi-isomorphism can be lifted modulo some finitely generated subideal $I \subseteq I_n/I_{n+1}$ (using that some L_{n+1}^* already exists) which is globally nilpotent, and one can use Corollary 12 again.

The projective limit $L^* = \varprojlim L_n^*$ is then a filtered complex of R-modules whose components L^n are all filtered direct summands in completions of direct sums of copies of $R\{a\}$'s. We claim that up to canonical isomorphism L^* as an object in the filtered derived category is independent of all choices. The key fact is the following:

Consider a projective system of filtered R_n-complexes R_n^* with filtered surjective transition maps $\mathbb{K}$ (i.e. $F^q(K_n^m) \twoheadrightarrow F^q(K_{n-1}^m)$). If all K_n^* are filtered acyclic, then any filtered map

$$\alpha : L^* \to K^* = \varprojlim_n K_n^*$$

is filtered homotopic to zero:

Construct by decreasing induction filtered homotopies

$$\zeta_m : L^m \to K^{m-1}$$

with $\alpha_m = d_{m-1} \circ s_m + s_{m+1} \circ d_m$. As

$$\begin{aligned}(\alpha_m - s_{m+1} \circ d_m)(F^q(L^m)) &\subseteq \varprojlim(F^q(K_n^m) \cap ker(d_m)) \\ = \varprojlim(d_{m-1}(F^q(K_n^{m-1}))) &= d_{m-1}(\varprojlim(F^q(K^{m-1})))\end{aligned}$$

(EGA III) one easily constructs s_m using projectivity. (If L^m is the completion of a filtered free module, one only has to lift generators.) It follows that any projective system of maps $K_n^* \xrightarrow{\approx} \tilde{K}_n^*$ which are filtered quasi-isomorphisms (for each n) induces an isomorphism on homotopy-classes of maps of L^* into K^*, respectively $\tilde{K}^*$. Now different choices of coverings, embeddings, etc. give rise to diagrams

$$\begin{tikzcd} & \tilde{K}^* \\ \tilde{\tilde{K}}^* \arrow[ur, "\approx"] \arrow[dr, "\approx"'] & \\ & K^*, \end{tikzcd}$$

with projective systems of filtered quasi-isomorphisms (and we leave it to the reader to check transitivity if one compares three choices). Applying the above observation we obtain that L^* is well-defined up to canonical filtered homotopy-equivalence. $\square$

Theorem 14. *Assume X is smooth over R_1, and $\mathcal{E} = (\mathcal{E}_n)$ is a compatible system of locally filtered free crystals on X/R_n. Then the construction above gives an object $L^* = \mathbb{R}\Gamma_{crys}(X/R, \mathcal{E})$ in the filtered derived category of R, unique up to canonical quasi-isomorphism. Moreover it can be represented by a complex L^* all of whose terms are completions of filtered projective modules, and L^* is unique up to canonical filtered homotopy.*

If X/R_1 is proper, we can in addition assume that L^* is concentrated in degrees $[0, 2 \cdot rel.dim.X]$, and that all L^m are filtered direct summands in finite direct sums of copies of $R\{a\}$'s. (L^* is strictly filtered perfect.)

Obviously these results extend to the case where X contains a normal crossing divisor D, and we consider logarithmic (locally filtered free) crystals. One just uses logarithmic differentials and de Rham complexes.

An interesting problem is the construction of Poincaré duality. If X is proper of pure relative dimension d over R_1, one needs a trace map

$$tr : H^{2 \cdot d}(X/R, \mathcal{O}_X) \to R\{d\}.$$

It suffices to exhibit a compatible system of traces

$$tr : H^{2d}(X/R_n, \mathcal{O}_X) \to R_n\{d\}.$$

The construction in [B], ch. VII, 1, can be used if R_q is artinian (one has to lift certain curves). In general I know a construction for projective X, but cannot quite treat the proper case. In any case if it exists it induces a perfect pairing

$$\mathbb{R}\Gamma_{crys}(X/R, \mathcal{E}) \times \mathbb{R}\Gamma_{crys}(X/\mathbb{R}, \mathcal{E}^\vee) \to R\{a\}[-2d]$$

(as the pairing is perfect modulo I_1).

Finally for the comparison to étale cohomology one has to use the appropriate versions of $B^+(V)$. Namely $B^+(V)$ is obtained by forming the PD-hull of $I = \xi \cdot W(S) \subseteq W(S)$ (see section 4) and completing it. The topology used for the completion should be either the PD-topology (for the nilpotent site), the p-adic topology (for Berthelot's crystalline site) or the topology defined by powers $n!I^{[n]}$ (for the extended site). As the comparison in [Fa2], V, works already with coefficients $W(S)/\xi^n \cdot W(S)$, it carries over to all completions. This allows one to treat the prime $p = 2$, which however is often excluded for other reasons.

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