

A CONVERSE TO A THEOREM OF ADAMYAN, AROV AND KREIN

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INTRODUCTION

A classical theorem of Pick gives a criterion for interpolation by analytic functions in the open unit disc \mathbb{D} subject to an H^∞ -norm bound. This result has substantial generalizations in two different directions. On the one hand, one can replace H^∞ by the multiplier algebras of certain Hilbert function spaces, some of them having no connection with analyticity [Ag1, Ag2]. On the other hand, one can obtain a criterion for interpolation by meromorphic functions with a prescribed number of poles in \mathbb{D} and with an L^∞ -norm bound on the unit circle \mathbb{T} ; this is a classical result of Akhiezer [Ak], now better known in the form of the far-reaching generalizations due to Adamyan, Arov and Krein [AAK]. In each case the criterion is in terms of the signature of a “Pick matrix” constructed from the interpolation data and the reproducing kernel of the appropriate Hilbert function space (i.e. H^2 in the case of the AAK theorem). It is therefore conceivable that there might be a common generalization which would hold for a significant class of function spaces. After all, the analogue of Pick’s theorem is true for the Dirichlet space D of analytic functions in \mathbb{D} with finite Dirichlet integral [Ag1] and for the space $W^{1,2}[a, b]$ of L^2 functions f on $[a, b]$ for which $f' \in L^2(a, b)$ [Ag2]. Might not an analogue of the Akhiezer-Adamyan-Arov-Krein theorem hold for these spaces? This natural question was posed in [Q2].

Pick’s theorem has long held the attention of analysts as one of the most elegant of all interpolation results. However, there are grounds beyond the aesthetic ones for its continued prominence. Knowledge that the analogue of Pick’s theorem holds for a particular Hilbert function space gives a powerful tool for the conversion of L^∞ -type problems to Hilbert space problems; this principle is brought out in [MS], where among other things it is used to give a relatively simple proof of Carleson’s theorem on interpolation sequences for H^∞ and also to characterize geometrically the interpolation sequences for multipliers of the Dirichlet space. Since about 1980 the Pick property has played an important role in linear control theory. The feedback controllers which internally stabilize a given linear system can be described by analytic functions satisfying a finite set of interpolation conditions, so

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that constructing a stabilizing controller with gain bounded by 1 is a Pick interpolation problem [DFT]. Applications of the Pick property to circuit theory can be found in [DGK, H]. Akhiezer's generalization of Pick's theorem (usually in the Adamyan-Arov-Krein version) has led to an effective approach to model reduction [G].

Pick's and Akhiezer's theorems have also served as test problems for techniques in function theory and operator theory. The applications call for far-reaching refinements of the classical theorems, and numerous approaches have been developed to proving generalizations. Closely related approaches of Sarason [S] and Adamyan, Arov and Krein [AAK] prove them using compressions of the shift operator and Hankel operations on H^2 respectively. Other powerful techniques are the commutant lifting theory [FF], the invariant subspace theory for indefinite Hardy spaces [BH] and at least four other radically different methods. A recent development along novel lines is a series of generalizations to functions of severable variables by means of a general lifting theorem by Cotlar and Sadosky [CS].

The approaches we have listed are all closely tied to the Hardy space H^2 and to the Szegő kernel. Investigation of the Pick property for other Hilbert spaces of functions in [Ag1, Ag2] had to proceed by a more "bare hands" method, involving a point-by-point construction of interpolating functions. So it is with the present paper, where we study the extension of Akhiezer's theorem to other function spaces. The original theorem of Akhiezer is couched in terms of meromorphic functions. Here we study functions on an arbitrary set, possibly without differentiable structure, and so we need a reformulation of Akhiezer's theorem which has an analogue in the general setting. Let H be a Hilbert space of functions on a set Ω and let $k : \Omega \times \Omega \rightarrow \mathbb{C}$ be the reproducing kernel of H . Consider distinct points $\lambda_1, \dots, \lambda_n \in \Omega$ and complex numbers z_1, \dots, z_n ; we regard these as data for an interpolation problem. The *Pick matrix* corresponding to these data is the $n \times n$ matrix $[(1 - z_i \bar{z}_j)k(\lambda_i, \lambda_j)]_{i,j=1}^n$. One says [Q1] that *Pick's theorem holds for (H, Ω)* if the following two statements are equivalent for any set of interpolation data:

- (1) there exists a function $\varphi : \Omega \rightarrow \mathbb{C}$ which is a contractive multiplier of H and satisfies $\varphi(\lambda_i) = z_i$, $1 \leq i \leq n$;
- (2) the corresponding Pick matrix is positive semidefinite.

The classical result [P] is usually stated as a criterion for the existence of an analytic interpolating function bounded by one in modulus, but it can also be formulated as the statement that Pick's theorem holds for H^2 . The results of [Ag1, Ag2] are that Pick's theorem holds for D and for $W^{1,2}[a, b]$. Similarly we can formulate the Akhiezer-Adamyan-Arov-Krein generalization of Pick's theorem by saying that when H is H^2 and Ω is \mathbb{D} , for any finite set of interpolation data and any non-negative integer m , the following two statements are equivalent:

- (1) there exists a function φ defined at all but at most m points of Ω which contractively multiplies an m -codimensional closed subspace of H into H and satisfies $\varphi(\lambda_i) = z_i$, $1 \leq i \leq n$;
- (2) the corresponding Pick matrix has at most m negative eigenvalues.

For $H = H^2$, condition (1) amounts to saying there is an interpolating meromorphic function with at most m poles bounded by 1 in modulus on \mathbb{T} (actually a precise statement is slightly more cumbersome since account must be taken of certain singular cases), which is the formulation given by Akhiezer. Accordingly we say that (H, Ω) has the *m -AAK property* if (1) is equivalent to (2) for any finite set of interpolation data (a more precise definition is given below). The question

we address is:

Which Hilbert function spaces have the m -AAK property for all non-negative integers m ?

Our main result, Theorem 3.12, answers this question. It constitutes a converse to Akhiezer’s result that H^2 has the m -AAK property for all m : it asserts that, subject to modest hypotheses on H , if H has the 0-AAK property and the 1-AAK property, then the reproducing kernel of H is a Szegő kernel – roughly speaking, (H, Ω) is a subspace of H^2 up to rescaling and a change of variable. It follows that D and $W^{1,2}[a, b]$ are examples of spaces having the 0-AAK property but not the 1-AAK property. The restriction of the Bergman space of the disc, $L^2_\alpha(\mathbb{D})$, to any 3 points is an example of a space having the 1-AAK property but not the 0-AAK property (actually this is a slight over-simplification: see Corollary 3.4 below). In the course of establishing Theorem 3.12 we prove our other main result, Theorem 1.8, which characterizes the m -AAK property for spaces of functions on infinite sets in terms of the restrictions of the kernel to finite sets. This result contains the Akhiezer-Adamyman-Arov-Krein theorem. It enables us to concentrate on the restrictions of the kernel to small sets; for example, it turns out that the signs of the determinants of the (entrywise) reciprocals of the 3×3 principal submatrices of the kernel play a key role (Theorem 3.1).

DEFINITIONS AND NOTATION

By a *Hilbert function space* we mean a pair (H, Ω) where Ω is a set and H is a Hilbert space of functions on Ω such that evaluation at any point of Ω is a continuous linear functional on H . When there is no risk of confusion we may sometimes write H in place of (H, Ω) ; however we shall often need to be precise about the domain under consideration. Thus, if $\Omega' \not\subseteq \Omega$, $(H|\Omega', \Omega')$ is a different Hilbert function space from (H, Ω) , where the vertical bar denotes restriction and $H|\Omega' \stackrel{\text{def}}{=} \{f|\Omega' : f \in H\}$. Nevertheless the natural mapping $f \mapsto f|\Omega'$ may be a unitary mapping between the Hilbert spaces H and $H|\Omega'$. Indeed, this will be so if and only if Ω' is a *set of uniqueness for H* , meaning that if $f, g \in H$ and $f|\Omega' = g|\Omega'$, then $f = g$. A Hilbert function space (H, Ω) has a *reproducing kernel*, that is, a function $k : \Omega \times \Omega \rightarrow \mathbb{C}$ with defining property

$$\langle f, k_\lambda \rangle = f(\lambda) \quad \text{for all } f \in H \text{ and } \lambda \in \Omega$$

where $k_\lambda = k(\cdot, \lambda) \in H$. We say that k is a *non-singular kernel* if the vectors $k_\lambda, \lambda \in \Omega$, are linearly independent.

Now consider a fixed Hilbert space H of functions on a set Ω with non-singular kernel k . For $\psi : D_\psi \subset \Omega \rightarrow \mathbb{C}$ and $h \in H$ we denote by ψh the pointwise product of ψ and $h|D_\psi$, so that ψh is a function on D_ψ . For $E \subset H$ we denote by ψE the set $\{\psi h : h \in E\}$, so that ψE is a set of functions on D_ψ . If E is a closed linear subspace of H , if D_ψ is a set of uniqueness and $\psi E \subset H|D_\psi$ we define the linear operator $M_{\psi, E} : E \rightarrow H$ by $M_{\psi, E}h = g$ where g is the unique element of H which extends ψh . By the closed graph theorem $M_{\psi, E}$ is continuous. If D_ψ is a set of uniqueness of H we shall say that $\alpha \in \Omega$ is a *pseudopole* of ψ (relative to H) if there exists $h \in H$ such that $h(\alpha) = 0$ and ψh has an extension to a function $g \in H$ (necessarily unique) such that $g(\alpha) \neq 0$. Of course this can only happen if $\alpha \notin D_\psi$.

Definition 0.1. Let ψ be a complex-valued function on a subset D_ψ of Ω and let $m \in \mathbb{Z}^+$, the set of non-negative integers. We say that ψ is an *m -pseudomultiplier*

of H on Ω if

- (1) D_ψ is a set of uniqueness for H ;
- (2) $\Omega \setminus D_\psi$ consists of pseudopoles of ψ and contains at most m points;
- (3) there is a closed subspace E of codimension m in H such that $\psi E \subset H|_{D_\psi}$.

For an m -pseudomultiplier ψ of H we define $s_m(\psi)$ to be the infimum of $\|M_{\psi,E}\|$ over all closed m -codimensional subspaces E of H such that $\psi E \subset H|_{D_\psi}$. We say that ψ is a *pseudomultiplier* of H if it is an m -pseudomultiplier of H for some non-negative integer m .

There are two standard examples of 1-pseudomultipliers to keep in mind, both on (H^2, \mathbb{D}) : $\varphi_1(z) = 1/z$ and $\varphi_2 = \chi_{\{0\}}$, the characteristic or indicator function of the singleton set $\{0\}$. Note that $s_1(\varphi_1) = 1$ and $s_1(\varphi_2) = 0$. It might appear that the assumptions about D_φ in Definition 0.1 are unduly strong, but the main result of [AY], Theorem 2.1, shows that weaker assumptions lead to essentially the same objects, subject only to the mild hypotheses that D_φ be a set of uniqueness for the class $HH \stackrel{\text{def}}{=} \{fg : f, g \in H\}$ and that k be non-singular.

Definition 0.2. Let (H, Ω) be a Hilbert function space with reproducing kernel k and let m be a non-negative integer. We say that (H, Ω) has the *m -AAK property* if the following two conditions are equivalent for any choice of $n \in \mathbb{N}$, distinct points $\lambda_1, \dots, \lambda_n \in \Omega$ and arbitrary complex numbers z_1, \dots, z_n :

- (1) there exists an m -pseudomultiplier φ of (H, Ω) such that $s_m(\varphi) \leq 1$ and $\varphi(\lambda_j) = z_j$, $1 \leq j \leq n$;
- (2) the matrix

$$(0.1) \quad [(1 - z_i \bar{z}_j)k(\lambda_i, \lambda_j)]_{i,j=1}^n$$

has at most m negative eigenvalues, counting multiplicities.

Our goal is to characterize Hilbert function spaces with the 0-AAK property and the 1-AAK property. The conclusion is in terms of *Szegő kernels*. We say that a kernel $k : \Omega \times \Omega \rightarrow \mathbb{C}$ is a Szegő kernel if there exist functions $a : \Omega \rightarrow \mathbb{C} \setminus \{0\}$ and $b : \Omega \rightarrow \mathbb{D}$ such that b is injective and

$$k(\lambda, \mu) = \frac{a(\lambda)\bar{a}(\mu)}{1 - b(\lambda)\bar{b}(\mu)}.$$

1. ONE POINT EXTENSIONS

In this section we characterize the m -AAK property in terms of operators on finite-dimensional spaces. Our approach is based on constructing pseudomultipliers one point at a time. It is a little surprising that, say, a meromorphic function can successfully be constructed in this finitary, point-by-point manner, but study of the case $m = 0$ (the Pick property) shows that it is indeed possible [Ag1, Ag2]. One thinks in terms of associated multiplication operators and keeps control of the operator norm. When $m > 0$ this technique has to be modified since multiplication by a pseudomultiplier is no longer a bounded operator on H . Instead of the operator norm we use the m th s -number or singular value of an operator, and we are faced with the problem of extending operators with preservation of the m th s -number. This is a more subtle problem than the corresponding one for the operator norm [CG], and it cannot always be achieved exactly.

Throughout this section we consider a fixed Hilbert function space (H, Ω) with non-singular reproducing kernel k . We first wish to derive a necessary condition for

H to have the m -AAK property. The idea is simplest in the case $m = 0$. Suppose that H has the 0-AAK (or Pick) property. That is, for any choice of distinct points $\lambda_1, \dots, \lambda_n \in \Omega$ and any $z_1, \dots, z_n \in \mathbb{C}$ such that

$$(1.1) \quad [(1 - z_i \bar{z}_j)k(\lambda_i, \lambda_j)]_{i,j=1}^n \geq 0$$

there exists a contractive multiplier φ of H such that $\varphi(\lambda_i) = z_i, 1 \leq i \leq n$. There is a standard operator-theoretic way of viewing the Pick condition (1.1). Introduce the space

$$(1.2) \quad \mathcal{M} \stackrel{\text{def}}{=} \text{span}\{k_{\lambda_1}, \dots, k_{\lambda_n}\} \subset H$$

and the linear operator T on \mathcal{M} given by

$$(1.3) \quad Tk_{\lambda_j} = \bar{z}_j k_{\lambda_j}, \quad 1 \leq j \leq n.$$

The matrix of $1 - T^*T$ with respect to the basis $k_{\lambda_1}, \dots, k_{\lambda_n}$ is precisely the Pick matrix, i.e. the left hand side of (1.1), and so the condition (1.1) is equivalent to $\|T\| \leq 1$. The Pick property of H thus tells us that if the data λ_j, z_j are such that $\|T\| \leq 1$, then there is an interpolating multiplier φ such that $\|M_\varphi\| \leq 1$, where $M_\varphi \in \mathcal{L}(H)$ is the operation of multiplication by φ . It is readily seen, in this case, that T is the restriction of M_φ^* to its invariant subspace \mathcal{M} , so that we can regard the Pick property as an extension property for certain operators on finite-dimensional subspaces of H . This viewpoint leads to a necessary condition for (H, Ω) to have the Pick property. Let T, \mathcal{M} and φ be as above. Choose any $\alpha \in \Omega \setminus \{\lambda_1, \dots, \lambda_n\}$ and let

$$\tilde{\mathcal{M}} = \text{span}\{k_{\lambda_1}, \dots, k_{\lambda_n}, k_\alpha\} \cap k_\alpha^\perp.$$

Then $\tilde{\mathcal{M}}$ is an n -dimensional subspace of H and a basis of $\tilde{\mathcal{M}}$ is $\tilde{k}_{\lambda_1}, \dots, \tilde{k}_{\lambda_n}$ where

$$(1.4) \quad \tilde{k}_{\lambda_j} = P_{\tilde{\mathcal{M}}} k_{\lambda_j} = k_{\lambda_j} - \frac{k(\alpha, \lambda_j)}{k(\alpha, \alpha)} k_\alpha.$$

Let \tilde{T} be the compression of M_φ^* to $\tilde{\mathcal{M}}$. Then

$$(1.5) \quad \begin{aligned} \tilde{T}\tilde{k}_{\lambda_j} &= P_{\tilde{\mathcal{M}}} M_\varphi^* \tilde{k}_{\lambda_j} = P_{\tilde{\mathcal{M}}} (\bar{\varphi}(\lambda_j) k_{\lambda_j} - \frac{k(\alpha, \lambda_j)}{k(\alpha, \alpha)} \bar{\varphi}(\alpha) k_\alpha) \\ &= \bar{z}_j \tilde{k}_{\lambda_j} \end{aligned}$$

since $P_{\tilde{\mathcal{M}}} k_\alpha = 0$. Now the n -dimensional operator \tilde{T} is determined by (1.5), which expresses \tilde{T} entirely in terms of the data and the chosen point α . On the other hand, since \tilde{T} is a compression of the contraction M_φ^* , we must have $\|\tilde{T}\| \leq 1$. We have shown that the following is a necessary condition that (H, Ω) (or equivalently, the kernel k) have the Pick property:

For any choice of distinct points $\lambda_1, \dots, \lambda_n, \alpha \in \Omega$ and of $z_1, \dots, z_n \in \mathbb{C}$, the operators T, \tilde{T} given by (1.3) and (1.5) respectively satisfy $\|\tilde{T}\| \leq \|T\|$.

We shall now show that, subject to a mild hypothesis, the expected variant is valid with ‘‘Pick’’ replaced by ‘‘ m -AAK’’ and $\|\cdot\|$ replaced by s_m .

Lemma 1.1. *Suppose that $D_\varphi \subset \Omega$ is a set of uniqueness for H and for HH . If $\varphi : D_\varphi \subset \Omega \rightarrow \mathbb{C}$ is a pseudomultiplier of H and $\alpha \in \Omega \setminus D_\varphi$, then any $h \in H$ such that $\varphi h \in H|_{D_\varphi}$ must vanish at α .*

Proof. By the definition of pseudomultiplier, α is a pseudopole of φ . That is, there exist $u, g \in H$ such that $u(\alpha) = 0$, $\varphi u = g|_{D_\varphi}$ and $g(\alpha) \neq 0$. Suppose $\varphi h = f|_{D_\varphi}$ where $f \in H$. On D_φ we have

$$gh - uf = \varphi uh - u\varphi h = 0.$$

Thus gh and uf agree on D_φ , which by hypothesis is a set of uniqueness for HH . Thus $gh = uf$. Since $u(\alpha) = 0$ and $g(\alpha) \neq 0$, we have $h(\alpha) = 0$. \square

Remark. The conclusion would not follow if the hypothesis on HH were removed. [AY, Example 2.2] exhibits a 1-pseudomultiplier φ and $\alpha \in \Omega \setminus D_\varphi$ such that $\varphi h \in H|_{D_\varphi}$ for all $h \in H$, while not all h vanish at α .

Lemma 1.2. *Suppose every cofinite set of uniqueness for H in Ω is also a set of uniqueness for HH . If m is a non-negative integer, $\lambda_1, \dots, \lambda_n, \alpha$ are distinct points of Ω , $z_1, \dots, z_n \in \mathbb{C}$ and φ is an m -pseudomultiplier of H such that $\varphi(\lambda_j) = z_j$, $1 \leq j \leq n$, then the operator \tilde{T} defined by (1.5) satisfies*

$$s_m(\tilde{T}) \leq s_m(\varphi).$$

Proof. Let $\varepsilon > 0$. By hypothesis there exists an m -codimensional subspace E of H such that $\varphi E \subset H|_{D_\varphi}$ and the operator $X_\varphi : E \rightarrow H$ given by

$$(1.6) \quad (X_\varphi f)|_{D_\varphi} = \varphi f, \quad f \in E,$$

satisfies

$$\|X_\varphi\| < s(\varphi) + \varepsilon.$$

Observe that $X_\varphi^* : H \rightarrow E$ satisfies, for any $\lambda \in D_\varphi$,

$$X_\varphi^* k_\lambda = \overline{\varphi(\lambda)} P_E k_\lambda.$$

Hence, for $1 \leq j \leq n$ there exists $u_j \in E^\perp$ such that

$$\begin{aligned} X_\varphi^* k_{\lambda_j} &= \overline{\varphi(\lambda_j)} k_{\lambda_j} + u_j \\ &= \overline{z_j} k_{\lambda_j} + u_j. \end{aligned}$$

Let $T_1 = P_{\tilde{\mathcal{M}}} X_\varphi^* |_{\tilde{\mathcal{M}}} : \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}}$. For $1 \leq j \leq n$ and some $c_j \in \mathbb{C}$,

$$\begin{aligned} T_1 \tilde{k}_{\lambda_j} &= P_{\tilde{\mathcal{M}}} X_\varphi^* (k_{\lambda_j} - c_j k_\alpha) \\ &= P_{\tilde{\mathcal{M}}} (\overline{z_j} k_{\lambda_j} + u_j - c_j X_\varphi^* k_\alpha) \\ &= \overline{z_j} \tilde{k}_{\lambda_j} + P_{\tilde{\mathcal{M}}} u_j - c_j P_{\tilde{\mathcal{M}}} X_\varphi^* k_\alpha \\ &= \tilde{T} \tilde{k}_{\lambda_j} + P_{\tilde{\mathcal{M}}} u_j - c_j P_{\tilde{\mathcal{M}}} X_\varphi^* k_\alpha. \end{aligned}$$

Thus

$$\text{Range}(T_1 - \tilde{T}) \subset \text{span}\{P_{\tilde{\mathcal{M}}} E^\perp, P_{\tilde{\mathcal{M}}} X_\varphi^* k_\alpha\}.$$

We claim that the right hand side has dimension at most m . For suppose $\alpha \in D_\varphi$. Then for some $u \in E^\perp$

$$X_\varphi^* k_\alpha = \overline{\varphi(\alpha)} k_\alpha + u$$

and so, since $P_{\tilde{\mathcal{M}}} k_\alpha = 0$,

$$P_{\tilde{\mathcal{M}}} X_\varphi^* k_\alpha = P_{\tilde{\mathcal{M}}} u \in P_{\tilde{\mathcal{M}}} E^\perp.$$

Hence

$$\text{Range}(T_1 - \tilde{T}) \subset P_{\tilde{\mathcal{M}}} E^\perp$$

which has dimension $\leq m$. Alternatively $\alpha \notin D_\varphi$. In that case, by Lemma 1.1, every element of E vanishes at α , i.e. $E \subset k_\alpha^\perp$. Hence $k_\alpha \in E^\perp$, and since $P_{\tilde{\mathcal{M}}}k_\alpha = 0$ it follows that

$$\dim P_{\tilde{\mathcal{M}}}E^\perp \leq \dim E^\perp - 1 \leq m - 1,$$

and so

$$\dim \text{span}\{P_{\tilde{\mathcal{M}}}E^\perp, P_{\tilde{\mathcal{M}}}X_\varphi^*k_\alpha\} \leq m.$$

In either case $T_1 - \tilde{T}$ has rank at most m . Since $s_m(\tilde{T})$ is the distance of \tilde{T} from the set of operators of rank at most m we have

$$\begin{aligned} s_m(\tilde{T}) &\leq \|\tilde{T} - (\tilde{T} - T_1)\| = \|T_1\| \\ &\leq \|X_\varphi^*\| < s_m(\varphi) + \varepsilon. \end{aligned}$$

Since ε was arbitrary we have

$$s_m(\tilde{T}) \leq s_m(\varphi).$$

□

Corollary 1.3. *Suppose that (H, Ω) has the m -AAK property and that every cofinite set of uniqueness for H in Ω is also a set of uniqueness for HH . For any choice of distinct points $\lambda_1, \dots, \lambda_n, \alpha \in \Omega$ and of $z_1, \dots, z_n \in \mathbb{C}$, the operators T and \tilde{T} given by (1.3) and (1.5) respectively satisfy*

$$s_m(\tilde{T}) \leq s_m(T).$$

Proof. Suppose λ_j, z_j are such that $s_m(T) \leq 1$. Then $1 - T^*T$ has at most m negative eigenvalues. Expressing $1 - T^*T$ as a matrix with respect to $k_{\lambda_1}, \dots, k_{\lambda_n}$ we find that the Pick matrix $[(1 - z_i \bar{z}_j)k(\lambda_i, \lambda_j)]_{i,j=1}^n$ has at most m negative eigenvalues. By the m -AAK property, there is an m -pseudomultiplier φ of H such that $\varphi(\lambda_j) = z_j$ and $s_m(\varphi) \leq 1$. By Lemma 1.2 it follows that $s_m(\tilde{T}) \leq 1$. Hence

$$s_m(\tilde{T}) \leq s_m(T).$$

□

For any operator A we denote $(A^*A)^{1/2}$ by $|A|$.

Corollary 1.4. *Suppose that (H, Ω) has the m -AAK property for every m and that every cofinite set of uniqueness for H in Ω is also a set of uniqueness for HH . Then, for any choice of data $\lambda_1, \dots, \lambda_n \in \Omega, z_1, \dots, z_n \in \mathbb{C}$, the operators T, \tilde{T} given by (1.3), (1.5) respectively satisfy*

$$|T| \cong |\tilde{T}|.$$

Here \cong denotes unitary equivalence.

Proof. By Corollary 1.3 we have $s_m(\tilde{T}) \leq s_m(T)$ for all m . Furthermore

$$s_0(T)s_1(T) \dots s_{n-1}(T) = |\det T| = |z_1 z_2 \dots z_n|$$

and likewise $s_0(\tilde{T})s_1(\tilde{T}) \dots s_{n-1}(\tilde{T}) = |z_1 z_2 \dots z_n|$. Provided that all z_j are non-zero it follows that $s_j(T) = s_j(\tilde{T}), 1 \leq j \leq n$, and hence that $|T| \cong |\tilde{T}|$. To see that the conclusion remains true even if some z_j are zero we appeal to a theorem of Specht (see for example [K, Theorem 63]), according to which two operators A, B on n -dimensional Hilbert spaces are unitarily equivalent if and only if

$$\text{trf}(A, A^*) = \text{trf}(B, B^*)$$

for every monomial f in a pair of non-commuting variables. It is clear that $\operatorname{tr}f(T, T^*)$ and $\operatorname{tr}f(\tilde{T}, \tilde{T}^*)$ vary continuously as functions of z_1, \dots, z_n . Since they agree when the z_j are all non-zero they must agree for all z_j . Thus $|T| \cong |\tilde{T}|$. \square

Let us introduce some terminology for the conditions arising in Corollary 1.3.

Definition 1.5. Let (H, Ω) be a Hilbert function space with kernel k and let m be a non-negative integer. We say that H has the *compression property for s_m* if, for any $n \in \mathbb{N}$ and any choice of distinct points $\lambda_1, \dots, \lambda_n, \alpha \in \Omega$ and of $z_1, \dots, z_n \in \mathbb{C}$, the operators T, \tilde{T} on

$$\mathcal{M} \stackrel{\text{def}}{=} \operatorname{span}\{k_{\lambda_1}, \dots, k_{\lambda_n}\}, \quad \tilde{\mathcal{M}} \stackrel{\text{def}}{=} \operatorname{span}\{\mathcal{M}, k_\alpha\} \cap k_\alpha^\perp$$

given by

$$Tk_{\lambda_j} = \bar{z}_j k_{\lambda_j}, \quad \tilde{T}P_{\tilde{\mathcal{M}}}k_{\lambda_j} = \bar{z}_j P_{\tilde{\mathcal{M}}}k_{\lambda_j}, \quad 1 \leq j \leq n,$$

respectively, satisfy

$$s_m(\tilde{T}) \leq s_m(T).$$

It will often be enough to assume that the relation $s_m(\tilde{T}) \leq s_m(T)$ holds for the particular values $n = 2$ and $n = 3$; in these cases we shall say that H has the compression property for s_m for pairs of points or for triples of points respectively.

Thus Corollary 1.3 can be expressed as follows. Suppose every cofinite set of uniqueness for H in Ω is also a set of uniqueness for HH . Then the m -AAK property for H implies the compression property for s_m . In the case $m = 0$ the converse is also true [Ag1], but when $m > 0$ it is not so.

Example 1.6. Let $0 < \alpha < 1$, let $\Omega = \{-\alpha, 0, \alpha\}$ and let H be the restriction of the Hardy space H^2 to Ω , so that the reproducing kernel of the 3-dimensional space H is

$$k(\lambda, \mu) = 1/(1 - \lambda\mu), \quad \lambda, \mu \in \Omega.$$

Define $\varphi : \{-\alpha, \alpha\} \rightarrow \mathbb{C}$ by $\varphi(\lambda) = 1/\lambda$. The Pick matrix corresponding to these data is

$$\frac{1}{\alpha^2} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix},$$

which has one negative and one zero eigenvalue. However φ cannot be extended to a 1-pseudomultiplier of H with preservation of s_1 . For suppose we take $\varphi(0) = \zeta$. A simple calculation shows that the 3×3 Pick matrix corresponding to the augmented data has determinant $4/\alpha^2$, independently of ζ . Since the Pick matrix is non-singular we have $s_1(\varphi) \neq 1$. It cannot be that $s_1(\varphi) < 1$ since a restriction of φ has s_1 equal to 1, and hence we have $s_1(\varphi) > 1$. Thus H does not have the 1-AAK property. However, H has the compression property for s_1 (for example, Theorem 1.9 below).

We shall need the following special case of Theorem 3.1 of [GRSW].

Lemma 1.7. *Let G be an n -dimensional Hilbert space, let E be a subspace of codimension 1 in G , let A be a linear operator on E and let m be a non-negative integer. Let $g \in G \setminus E$ and define $A(c)$ on G by*

$$A(c)(x + \lambda g) = Ax + c\lambda g$$

for $x \in E, \lambda \in \mathbb{C}$. Then

$$\inf_{c \in \mathbb{C}} s_m(A(c)) = \max\{s_m(A), s_m(\tilde{A})\}$$

where \tilde{A} is the compression of $A(c)$ to g^\perp .

Note that \tilde{A} does not depend on c . It is well known that in the case $m = 0$ the infimum is attained at some $c \in \mathbb{C}$: this is a version of ‘‘Parrott’s Lemma’’ (see for example [Ag1]). When $m > 1$ the infimum need not be attained. The issue is discussed in much greater generality in [CG].

One of the two main results of this paper is a converse to Corollary 1.3, in other words, a statement that the compression property for s_m implies the m -AAK property. We have just seen in Example 1.6 that this implication does not hold in complete generality. It transpires that there is a big difference here between finite and infinite Ω . In the example, to preserve the property $s_1(\varphi) = 1$, we really need to take $\varphi(0)$ to be ∞ , or in other words, to define $\varphi(\lambda)$ to be $1/\lambda$. In the case of finite Ω this results in the domain of φ being the complement of a single point, hence not being a set of uniqueness. For most function spaces of interest on infinite domains the complement of a finite set will be a set of uniqueness, and so a pseudomultiplier can perfectly well have some ‘‘infinities’’.

Theorem 1.8. *Let (H, Ω) be a Hilbert function space with non-singular kernel, let m be a non-negative integer and suppose that every cofinite set in Ω is a set of uniqueness for H and for HH . Then H has the m -AAK property if and only if H has the compression property for s_m .*

Proof. The forward implication is a consequence of Corollary 1.3. Suppose that H has the compression property for s_m . Consider distinct points $\lambda_1, \dots, \lambda_n \in \Omega$ and any $z_1, \dots, z_n \in \mathbb{C}$ such that the Pick matrix (0.1) has m or fewer negative eigenvalues. We must construct an interpolating m -pseudomultiplier φ such that $s_m(\varphi) \leq 1$. As we have seen, the operator T on

$$\mathcal{M} \stackrel{\text{def}}{=} \text{span}\{k_{\lambda_1}, \dots, k_{\lambda_n}\}, \quad Tk_{\lambda_j} = \bar{z}_j k_{\lambda_j},$$

satisfies $s_m(T) \leq 1$.

Let \mathcal{F} denote the collection of finite sets F satisfying $\{\lambda_1, \dots, \lambda_n\} \subset F \subset \Omega$. Pick $F \in \mathcal{F}$ and $\varepsilon > 0$. We shall construct

$$\varphi_{F\varepsilon} : F \rightarrow \mathbb{C}$$

such that $\varphi_{F\varepsilon}(\lambda_j) = z_j, 1 \leq j \leq n$, and $s_m(\varphi_{F\varepsilon}) < 1 + \varepsilon$. Write

$$F = \{\lambda_1, \dots, \lambda_n, \lambda_{n+1}, \dots, \lambda_N\}$$

and let $\mathcal{M}_F = \text{span}\{k_\lambda : \lambda \in F\}$. Further, let $\tilde{T}, \tilde{\mathcal{M}}$ be as in Definition 1.5 with $\alpha = \lambda_{n+1}$. By the hypothesis that H have the compression property for s_m ,

$$s_m(\tilde{T}) \leq s_m(T) \leq 1.$$

By Lemma 1.7, if we define $T(c)$ on

$$\mathcal{M}_{n+1} \stackrel{\text{def}}{=} \mathcal{M}_{\{\lambda_1, \dots, \lambda_{n+1}\}}$$

by

$$T(c)k_{\lambda_j} = \begin{cases} \bar{z}_j k_{\lambda_j}, & 1 \leq j \leq n, \\ \bar{c} k_{\lambda_{n+1}}, & j = n + 1, \end{cases}$$

then

$$\inf_c s_m(T(c)) = \max\{s_m(T), s_m(\tilde{T})\} \leq 1.$$

Choose $c \in \mathbb{C}$ such that $s_m(T(c)) \leq 1 + \varepsilon/2$ and let $z_{n+1}(\varepsilon) = c$. Write $T(c) = T_{n+1}$ on \mathcal{M}_{n+1} , so that $s_m(T_{n+1}) \leq 1 + \varepsilon/2$. By repeating the argument we can find $z_{n+2}(\varepsilon) \in \mathbb{C}$ such that $s_m(T_{n+2}) \leq 1 + \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2}$. Continuing inductively we get a sequence $z_j(\varepsilon)$, $n+1 \leq j \leq N$, such that the corresponding operators T_j , $n+1 \leq j \leq N$, all satisfy

$$(1.7) \quad s_m(T_j) \leq 1 + \varepsilon.$$

Define $\varphi_{F\varepsilon}(\lambda_j)$ to be z_j for $1 \leq j \leq n$ and $z_j(\varepsilon)$ for $n+1 \leq j \leq N$, and define $T_{F\varepsilon}$ to be T_N . Then for each $F \in \mathcal{F}$ and $\varepsilon > 0$ we have $\varphi_{F\varepsilon} : F \rightarrow \mathbb{C}$ and an operator $T_{F\varepsilon}$ on \mathcal{M}_F such that $s_m(\varphi_{F\varepsilon}) = s_m(T_{F\varepsilon}) \leq 1 + \varepsilon$ and $\varphi_{F\varepsilon}(\lambda_j) = z_j$, $1 \leq j \leq n$. The idea, roughly speaking, is to construct an interpolating m -pseudomultiplier φ by taking the pointwise limit of $\varphi_{F\varepsilon}$ as $F \rightarrow \Omega$ and $\varepsilon \rightarrow 0$.

We wish to define an ultrafilter on $\mathcal{F} \times (0, 1)$. For each $F \in \mathcal{F}$ and $\varepsilon > 0$ let

$$Q(F, \varepsilon) = \{(G, x) : G \in \mathcal{F}, F \subset G, 0 < x < \varepsilon\}.$$

$Q(F, \varepsilon)$ is a subset of $\mathcal{F} \times (0, 1)$, and

$$Q(F, \varepsilon) \cap Q(F', \varepsilon') = Q(F \cup F', \min\{\varepsilon, \varepsilon'\}).$$

Thus the sets $Q(F, \varepsilon)$ constitute a filter base on $\mathcal{F} \times (0, 1)$ as F ranges over \mathcal{F} and ε ranges over $(0, 1)$. Pick an ultrafilter \mathcal{U} on $\mathcal{F} \times (0, 1)$ which refines this filter base.

Define $\hat{\varphi} : \Omega \rightarrow \mathbb{C} \cup \{\infty\}$ by

$$\hat{\varphi}(\lambda) = \lim_{\mathcal{U}} \varphi_{F\varepsilon}(\lambda), \quad \text{all } \lambda \in \Omega.$$

That is, for fixed λ , we define $\hat{\varphi}(\lambda)$ to be the unique limit point in the compact Hausdorff space $\mathbb{C} \cup \{\infty\}$ of the ultrafilter on $\mathbb{C} \cup \{\infty\}$ consisting of those subsets A for which

$$\{(F, \varepsilon) : F \in \mathcal{F}, \varepsilon > 0, \varphi_{F\varepsilon}(\lambda) \in A\}$$

belongs to \mathcal{U} . Now let

$$\mathcal{P} = \{\lambda \in \Omega : \hat{\varphi}(\lambda) = \infty\},$$

the set of singularities of $\hat{\varphi}$. We shall see shortly that \mathcal{P} contains at most m points. Let φ be the restriction of $\hat{\varphi}$ to $\Omega \setminus \mathcal{P}$. We wish to show that φ is the desired interpolating pseudomultiplier with $s_m(\varphi) \leq 1$.

Certainly $\varphi(\lambda_j) = z_j$ for $1 \leq j \leq n$, since $\varphi_{F\varepsilon}(\lambda_j) = z_j$ for all $F \in \mathcal{F}$ and $\varepsilon \in (0, 1)$. By the spectral theorem, since $s_m(T_{F\varepsilon}) \leq 1 + \varepsilon$, there exists a positive operator $B_{F\varepsilon} \leq (1 + \varepsilon)^2 1$ and vectors $u_{F\varepsilon}^0, \dots, u_{F\varepsilon}^{m-1} \in \mathcal{M}_F$ such that

$$T_{F\varepsilon}^* T_{F\varepsilon} = B_{F\varepsilon} + \sum_{j=0}^{m-1} u_{F\varepsilon}^j \otimes u_{F\varepsilon}^j,$$

where $B_{F\varepsilon} u_{F\varepsilon}^j = 0$, $0 \leq j \leq m-1$. For $\lambda \in F$ we have

$$T_{F\varepsilon} k_\lambda = \bar{\varphi}_{F\varepsilon}(\lambda) k_\lambda,$$

and hence, for $\lambda, \mu \in F$,

$$(1.8) \quad \bar{\varphi}_{F\varepsilon}(\lambda) \varphi_{F\varepsilon}(\mu) \langle k_\lambda, k_\mu \rangle = \langle B_{F\varepsilon} k_\lambda, k_\mu \rangle + \sum_{j=0}^{m-1} \bar{u}_{F\varepsilon}^j(\lambda) u_{F\varepsilon}^j(\mu).$$

In particular, for $\lambda \in F$ we have

$$\begin{aligned} \sum_{j=0}^{m-1} |u_{F^\varepsilon}^j(\lambda)|^2 &= |\varphi_{F^\varepsilon}(\lambda)|^2 \|k_\lambda\|^2 - \langle B_{F^\varepsilon} k_\lambda, k_\lambda \rangle \\ &\leq |\varphi_{F^\varepsilon}(\lambda)|^2 \|k_\lambda\|^2. \end{aligned}$$

Thus, if we define $g_{F^\varepsilon}^j(\lambda)$ for $\lambda \in F$, $0 \leq j \leq m - 1$, by

$$g_{F^\varepsilon}^j(\lambda) = \begin{cases} u_{F^\varepsilon}^j(\lambda)/\varphi_{F^\varepsilon}(\lambda) & \text{if } \varphi_{F^\varepsilon}(\lambda) \neq 0, \\ 0 & \text{if } \varphi_{F^\varepsilon}(\lambda) = 0, \end{cases}$$

then we obtain

$$|g_{F^\varepsilon}^j(\lambda)| \leq \|k_\lambda\|, \quad 0 \leq j \leq m - 1,$$

and

$$u_{F^\varepsilon}^j = \varphi_{F^\varepsilon} g_{F^\varepsilon}^j \text{ on } F.$$

On dividing through by $\bar{\varphi}_{F^\varepsilon}(\lambda)\varphi_{F^\varepsilon}(\mu)$ in (1.8) we have, for $\lambda, \mu \in \Omega$ such that $\bar{\varphi}_{F^\varepsilon}(\lambda)\varphi_{F^\varepsilon}(\mu) \neq 0$,

$$(1.9) \quad \langle k_\lambda, k_\mu \rangle = \frac{1}{\bar{\varphi}_{F^\varepsilon}(\lambda)\varphi_{F^\varepsilon}(\mu)} \langle B_{F^\varepsilon} k_\lambda, k_\mu \rangle + \sum_{j=0}^{m-1} \bar{g}_{F^\varepsilon}^j(\lambda) g_{F^\varepsilon}^j(\mu).$$

For $\lambda \in \Omega$ and $0 \leq j \leq m - 1$ let

$$a_j(\lambda) = \lim_{\mathcal{U}} g_{F^\varepsilon}^j(\lambda).$$

Then $|a_j(\lambda)| \leq \|k_\lambda\|$ for all $\lambda \in \Omega$. By weak compactness $\lim_{\mathcal{U}} B_{F^\varepsilon}$ exists in the weak operator topology. Call this limit B and note that $\|B\| \leq 1 + \varepsilon$ for all $\varepsilon > 0$, so that $\|B\| \leq 1$. Take limits along \mathcal{U} in (1.9) to obtain, for all $\lambda, \mu \in \Omega$ such that $\varphi(\lambda) \neq 0$ and $\varphi(\mu) \neq 0$,

$$(1.10) \quad \langle k_\lambda, k_\mu \rangle = \frac{1}{\bar{\varphi}(\lambda)\varphi(\mu)} \langle B k_\lambda, k_\mu \rangle + \sum_{j=0}^{m-1} \bar{a}_j(\lambda) a_j(\mu),$$

and note that this relation remains valid (with the natural interpretation) if $\hat{\varphi}(\lambda)$ or $\hat{\varphi}(\mu) = \infty$. In particular, for $\lambda, \mu \in \mathcal{P}$,

$$\langle k_\lambda, k_\mu \rangle = \sum_{j=0}^{m-1} \bar{a}_j(\lambda) a_j(\mu).$$

This implies that the matrix $[k(\mu, \lambda)]_{\lambda, \mu \in \mathcal{P}}$ has rank at most m . In view of the non-singularity of k , there can be at most m points in \mathcal{P} .

Let \mathcal{R} be the closed linear span of $\{k_\lambda : \lambda \in \Omega, \varphi(\lambda) \neq 0\}$. The relation (1.10) implies that we can define an isometry $V : \mathcal{R} \rightarrow H \oplus \mathbb{C}^m$ by

$$(1.11) \quad V k_\lambda = \begin{cases} \bar{\varphi}(\lambda)^{-1} B^{\frac{1}{2}} k_\lambda \oplus (\bar{a}_0(\lambda), \dots, \bar{a}_{m-1}(\lambda)) & \text{if } \hat{\varphi}(\lambda) \neq 0 \text{ or } \infty, \\ 0 \oplus (\bar{a}_0(\lambda), \dots, \bar{a}_{m-1}(\lambda)) & \text{if } \hat{\varphi}(\lambda) = \infty. \end{cases}$$

Write V in the form

$$V = \begin{bmatrix} C \\ 1 \otimes f_0 \\ \vdots \\ 1 \otimes f_{m-1} \end{bmatrix}$$

for some $f_0, \dots, f_{m-1} \in \mathcal{R}$. Since $V^*V = 1$,

$$C^*C + f_0 \otimes f_0 + \dots + f_{m-1} \otimes f_{m-1} = 1.$$

By (1.11), for all λ such that $\varphi(\lambda) \neq 0$,

$$(1.12) \quad Ck_\lambda = \begin{cases} \bar{\varphi}(\lambda)^{-1}B^{\frac{1}{2}}k_\lambda & \text{if } \hat{\varphi}(\lambda) \neq \infty, \\ 0 & \text{if } \hat{\varphi}(\lambda) = \infty, \end{cases}$$

$$(1.13) \quad f_j(\lambda) = a_j(\lambda).$$

Define $C^\sharp : H = \mathcal{R} \oplus \mathcal{R}^\perp \rightarrow H$ to be C on \mathcal{R} and 0 on \mathcal{R}^\perp . Let $L = B^{\frac{1}{2}}C^\sharp \in \mathcal{L}(H)$.

By (1.12), if $\hat{\varphi}(\lambda) \neq 0$ or ∞ ,

$$\begin{aligned} L^*k_\lambda &= (C^\sharp)^*B^{\frac{1}{2}}k_\lambda = (C^\sharp)^*\bar{\varphi}(\lambda)Ck_\lambda \\ &= \bar{\varphi}(\lambda)C^*Ck_\lambda = \bar{\varphi}(\lambda)\left(1 - \sum_{j=0}^{m-1} f_j \otimes f_j\right)k_\lambda. \end{aligned}$$

This relation remains valid when $\varphi(\lambda) = 0$. Indeed, it follows from taking limits along \mathcal{U} in (1.8) that $\varphi(\lambda) = 0$ implies $B^{\frac{1}{2}}k_\lambda = 0$ and hence that $L^*k_\lambda = (C^\sharp)^*B^{\frac{1}{2}}k_\lambda = 0$. Thus, for all $\lambda \in \Omega \setminus \mathcal{P}$,

$$L^*k_\lambda = \bar{\varphi}(\lambda)(1 - A)k_\lambda$$

where A is the Hermitian contraction

$$A = f_0 \otimes f_0 + \dots + f_{m-1} \otimes f_{m-1}$$

of rank at most m . Dualizing this relation we obtain, for all $h \in H$ and $\lambda \in \Omega \setminus \mathcal{P}$,

$$(Lh)(\lambda) = \varphi(\lambda)(h(\lambda) - Ah(\lambda)).$$

$\text{Ker } A$ is a closed subspace of H of codimension at most m , and for $h \in \text{Ker } A$ we have

$$\varphi h|_{\Omega \setminus \mathcal{P}} = Lh|_{\Omega \setminus \mathcal{P}} \in H|_{\Omega \setminus \mathcal{P}}$$

and

$$\|Lh\| = \|B^{\frac{1}{2}}C^*h\| \leq \|h\|.$$

Thus φ multiplies a closed m -codimensional subspace of H contractively into $H|_{\Omega \setminus \mathcal{P}}$, and so $s_m(\varphi) \leq 1$.

The function φ is almost the m -pseudomultiplier we are trying to construct. Its domain is the set $D \stackrel{\text{def}}{=} \Omega \setminus \mathcal{P}$, which contains all but at most m points of Ω and so by assumption is a set of uniqueness for H . It is defined at each λ_i and satisfies $\varphi(\lambda_i) = z_i$, $1 \leq i \leq n$. All that remains is to ensure that the points in the complement of the domain of φ are pseudopoles of φ .

Consider $\mu \in \mathcal{P}$. We claim that either μ is a pseudopole of φ or we can define $\varphi(\mu)$ to be a (finite) number so as to preserve the relation

$$(1.14) \quad Lh(\lambda) = \varphi(\lambda)h(\lambda)$$

for all λ in the domain of φ and all $h \in \text{Ker } A$. There are three cases.

Case 1. Suppose $k_\mu \notin AH$. Then there exists $h \in \text{Ker } A$ such that $h(\mu) \neq 0$. Define $\varphi(\mu)$ to be $Lh(\mu)/h(\mu)$. This formula does define $\varphi(\mu)$ uniquely, for if h_1, h_2 are two elements of $\text{Ker } A$ which do not vanish at μ , then we have, from (1.14),

$$h_1(\lambda) \cdot Lh_2(\lambda) = h_1(\lambda)\varphi(\lambda)h_2(\lambda) = Lh_1(\lambda) \cdot h_2(\lambda)$$

for all $\lambda \in \Omega \setminus \mathcal{P}$. Since $\Omega \setminus \mathcal{P}$ is a set of uniqueness for HH it follows that $Lh_1 \cdot h_2 = h_1 \cdot Lh_2$, and so Lh_1/h_1 and Lh_2/h_2 agree at μ . Clearly (1.14) remains true for $\lambda = \mu$.

Case 2. Suppose $k_\mu \in AH$ and $L \text{Ker } A \subset k_\mu^\perp$. Then for any $h \in \text{Ker } A$ we have $h(\mu) = 0$ and $Lh(\mu) = 0$, so that (1.14) remains true for any choice of $\varphi(\mu)$.

Case 3. Suppose $k_\mu \in AH$ and $L \text{Ker } A \not\subset k_\mu^\perp$. Then there exists $h \in \text{Ker } A \subset k_\mu^\perp$ such that $h(\mu) = 0$ and φh extends to $Lh \in H$ which does not vanish at μ . That is, μ is a pseudopole of φ . This establishes the claim.

For each $\mu \in \mathcal{P}$ which is not a pseudopole of φ assign $\varphi(\mu)$ an appropriate finite value to preserve (1.14). Then φ is the desired interpolating m -pseudomultiplier such that $s_m(\varphi) \leq 1$. It follows that H has the m -AAK property. \square

The last result contains the theorem of Akhiezer, Adamyan, Arov and Krein, which is obtained on taking $H = H^2$, $\Omega = \mathbb{D}$ in the following statement.

Theorem 1.9. *Let k be a Szegő kernel on a set Ω and let H be the corresponding Hilbert space of functions. Then (H, Ω) has the compression property for s_m for every non-negative integer m . Moreover, if every cofinite subset of Ω is a set of uniqueness for H and HH , then (H, Ω) has the m -AAK property for every non-negative integer m .*

Proof. Consider distinct points $\lambda_1, \dots, \lambda_n \in \Omega$ and $z_1, \dots, z_n \in \mathbb{C}$. Let $\mathcal{M}, \tilde{\mathcal{M}}, T$ and \tilde{T} be as in Definition 1.5. We must show that $s_m(\tilde{T}) \leq s_m(T)$. By hypothesis there exist functions $a : \Omega \rightarrow \mathbb{C}$, $b : \Omega \rightarrow \mathbb{D}$ such that b is injective and

$$k(\lambda, \mu) = \frac{a(\lambda)\bar{a}(\mu)}{1 - b(\lambda)\bar{b}(\mu)}.$$

We have, for $\lambda = \lambda_i, 1 \leq i \leq n$,

$$\tilde{k}_\lambda = P_{\tilde{\mathcal{M}}} k_\lambda = k_\lambda - \frac{\langle k_\lambda, k_\alpha \rangle}{k(\alpha, \alpha)} k_\alpha.$$

A straightforward calculation now gives

$$(1.15) \quad \tilde{k}_\lambda(\mu) = c_\alpha(\mu) k_\lambda(\mu) \bar{c}_\alpha(\lambda),$$

where

$$c_\alpha(\mu) = \frac{b(\alpha) - b(\mu)}{1 - \bar{b}(\alpha)b(\mu)}.$$

Note that $c_\alpha(\cdot)$ is non-zero on $\{\lambda_1, \dots, \lambda_n\}$ since α is taken to be distinct from each λ_i . Define $U : \mathcal{M} \rightarrow \tilde{\mathcal{M}}$ by

$$Uk_i = \bar{c}_\alpha(\lambda_i)^{-1} \tilde{k}_i$$

where k_i denotes k_{λ_i} . The relation (1.15) says precisely that

$$\langle Uk_i, Uk_j \rangle = \langle k_i, k_j \rangle,$$

that is, U is unitary. Moreover

$$\begin{aligned} \tilde{T}Uk_j &= \bar{c}_\alpha(\lambda_j)^{-1} \tilde{T}\tilde{k}_j = c_\alpha(\lambda_j)^{-1} \bar{z}_j \tilde{k}_j \\ &= \bar{z}_j Uk_j, \end{aligned}$$

so that $\tilde{T}U = UT$. Hence \tilde{T} is unitarily equivalent to T , and so $s_m(\tilde{T}) = s_m(T)$. That is, H has the compression property for s_m . The second assertion is immediate from Theorem 1.8. \square

2. SZEGŐ KERNELS

One of the main questions we address is: what converses of the Adamyan-Arov-Krein theorem hold? That is, if H is a Hilbert function space, does the validity of AAK-type extension properties force the reproducing kernel of H to be a Szegő kernel? In this section we establish some characterizations of Szegő kernels. A kernel k on a set Ω is a complex-valued function on $\Omega \times \Omega$ which is Hermitian symmetric, i.e. satisfies $k(\mu, \lambda)^\top = k(\lambda, \mu)$ for all $\lambda, \mu \in \Omega$.

Recall that in this paper we are adopting a narrow definition of Szegő kernel: a kernel k on a set Ω is Szegő if there exist functions

$$a : \Omega \rightarrow \mathbb{C} \setminus \{0\}, \quad b : \Omega \rightarrow \mathbb{D}$$

such that b is injective and

$$k(\lambda, \mu) = \frac{a(\lambda)\bar{a}(\mu)}{1 - b(\lambda)\bar{b}(\mu)}, \quad \lambda, \mu \in \Omega.$$

Note that a Szegő kernel does not vanish at any point of $\Omega \times \Omega$. Such a kernel k is indeed positive and non-singular; this is a simple consequence of the fact that the standard Szegő kernel $(1 - \lambda\bar{\mu})^{-1}$ on \mathbb{D} has these properties. Let us define the rank of a Hermitian symmetric function $F : \Omega \times \Omega \rightarrow \mathbb{C}$ to be the dimension of the ‘‘column space’’, that is, the dimension of the subspace $\text{span}\{F(\cdot, \mu) : \mu \in \Omega\}$ of \mathbb{C}^Ω . If k is the reproducing kernel of a Hilbert function space H on a set Ω and $\alpha \in \Omega$, we denote by k^α the reproducing kernel of the subspace k_α^\perp of H :

$$k^\alpha(\lambda, \mu) = k(\lambda, \mu) - \frac{k(\lambda, \alpha)k(\alpha, \mu)}{k(\alpha, \alpha)}.$$

Lemma 2.1. *Let k be a non-singular positive kernel on a set Ω containing at least 2 points and suppose k does not vanish at any point of $\Omega \times \Omega$. The following are equivalent:*

- (1) k is a Szegő kernel;
- (2) for some $\alpha \in \Omega$, k^α/k has rank one;
- (3) for every $\alpha \in \Omega$, k^α/k has rank one;
- (4) $1/k$ has rank two.

Proof. (1) \Rightarrow (3) It is a straightforward calculation that if k is given by (2), then

$$k^\alpha(\lambda, \mu) = \varphi(\lambda) k(\lambda, \mu) \bar{\varphi}(\mu)$$

where

$$\varphi(\lambda) = \frac{b(\lambda) - b(\alpha)}{1 - \bar{b}(\alpha)b(\lambda)},$$

whence k^α/k has rank one.

(3) \Rightarrow (2) is immediate.

(2) \Rightarrow (4) It follows from (2) that there exists a function $\varphi : \Omega \rightarrow \mathbb{C}$ such that

$$1 - \frac{k(\lambda, \alpha)k(\alpha, \mu)}{k(\lambda, \mu)k(\alpha, \alpha)} = \varphi(\lambda)\bar{\varphi}(\mu)$$

for all $\lambda, \mu \in \Omega$. Hence

$$\frac{1}{k(\lambda, \mu)} = \frac{k(\alpha, \alpha)}{k(\lambda, \alpha)k(\alpha, \mu)} (1 - \varphi(\lambda)\bar{\varphi}(\mu))$$

and so the column space of $1/k$ is spanned by the functions $1/k(\cdot, \alpha)$ and $\varphi/k(\cdot, \alpha)$. The rank of $1/k$ is thus at most 2. If the rank were 1, then k would also have rank 1, and since Ω contains more than one point the non-singularity of k would be contradicted. Thus the rank of $1/k$ is 2.

(4) \Rightarrow (1) Suppose that the column space of $1/k$ is spanned by linearly independent functions φ, ψ . Note that since k is positive and non-singular, any 2×2 principal submatrix of $1/k$ has negative determinant, and hence $1/k$ cannot be a positive kernel; nor can it be a negative kernel, since it takes positive values on the diagonal of $\Omega \times \Omega$. By hypothesis, there exist functions u, v on Ω such that

$$\frac{1}{k(\lambda, \mu)} = \varphi(\lambda)\bar{u}(\mu) + \psi(\lambda)\bar{v}(\mu), \quad \lambda, \mu \in \Omega.$$

By Hermitian symmetry we have also

$$\frac{1}{k(\lambda, \mu)} = u(\lambda)\bar{\varphi}(\mu) + v(\lambda)\bar{\psi}(\mu).$$

Since $\bar{\varphi}$ and $\bar{\psi}$ are linearly independent, u and v lie in the column space of $1/k$, say

$$u = \bar{c}_1\varphi + \bar{c}_2\psi, \quad v = \bar{d}_1\varphi + \bar{d}_2\psi.$$

We therefore have

$$\frac{1}{k(\lambda, \mu)} = \begin{bmatrix} \bar{\varphi}(\mu) & \bar{\psi}(\mu) \end{bmatrix} \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} \varphi(\lambda) \\ \psi(\lambda) \end{bmatrix}.$$

The 2×2 matrix on the right hand side is Hermitian and therefore can be factorized as $P^* \text{diag}\{\pm 1, \pm 1\}P$ for some non-singular matrix $P = [p_{ij}]$. Replacing φ, ψ by $p_{11}\varphi + p_{12}\psi$ and $p_{21}\varphi + p_{22}\psi$ respectively, we have

$$\frac{1}{k(\lambda, \mu)} = \pm\varphi(\lambda)\bar{\varphi}(\mu) \pm \psi(\lambda)\bar{\psi}(\mu).$$

Since $1/k$ can be neither positive nor negative as a kernel, there must be one plus and one minus and we have

$$\frac{1}{k(\lambda, \mu)} = \varphi(\lambda)\bar{\varphi}(\mu) - \psi(\lambda)\bar{\psi}(\mu).$$

In particular, for any $\lambda \in \Omega$,

$$|\varphi(\lambda)|^2 - |\psi(\lambda)|^2 = \frac{1}{k(\lambda, \lambda)} > 0,$$

so that φ does not vanish on Ω and $|\psi/\varphi| < 1$. Let $a = 1/\varphi$, $b = \psi/\varphi$. Then a, b map Ω into $\mathbb{C} \setminus \{0\}, \mathbb{D}$ respectively, and

$$k(\lambda, \mu) = \frac{1}{\varphi(\lambda)\bar{\varphi}(\mu) - \psi(\lambda)\bar{\psi}(\mu)} = \frac{a(\lambda)\bar{a}(\mu)}{1 - b(\lambda)\bar{b}(\mu)}$$

for any $\lambda, \mu \in \Omega$. Since k is non-singular, b is necessarily injective, and so k is a Szegő kernel. □

It is clear that any restriction of a Szegő kernel is again Szegő. In order to determine whether a given kernel is Szegő it is enough to check restrictions to small subsets.

Lemma 2.2. *Let k be a non-singular positive kernel on a set Ω containing at least 4 points. If every restriction of k to a 4-point subset of Ω is Szegő, then k is Szegő.*

Proof. Fix $\alpha \in \Omega$. The hypothesis implies that k does not vanish on $\Omega \times \Omega$, and hence we can define a kernel F on $\Omega \setminus \{\alpha\}$ by $F = k^\alpha/k$. By Lemma 2.1, the restriction of F to any 3-point subset of $\Omega \setminus \{\alpha\}$ has rank 1. Suppose k is not Szegő: then there exist $\mu_1, \mu_2 \in \Omega \setminus \{\alpha\}$ such that $F(\cdot, \mu_1), (F\cdot, \mu_2)$ are linearly independent. Observe that if f_1, f_2 are linearly independent functions and x_1 is a point such that $f_1(x_1) \neq 0$, then there is some point x_2 such that $[f_i(x_j)]_{i,j=1}^2$ has rank 2. Apply this observation to $F(\cdot, \mu_1), F(\cdot, \mu_2)$ with $x_1 = \mu_1$: we can do this since $F(\mu_1, \mu_1) \neq 0$, else we should have $k(\mu_1, \mu_1)k(\alpha, \alpha) = |k(\mu_1, \alpha)|^2$, contradicting the non-singularity of k . Hence there exists $\mu_3 \in \Omega \setminus \{\alpha\}$ such that

$$\begin{bmatrix} F(\mu_1, \mu_1) & F(\mu_1, \mu_2) \\ F(\mu_3, \mu_1) & F(\mu_3, \mu_2) \end{bmatrix}$$

has rank 2. This is a submatrix of $[F(\mu_i, \mu_j)]_{i,j=1}^3$, which accordingly has rank at least 2, a contradiction. \square

It is plausible to guess that, in order to show a kernel is Szegő, it suffices to show that restrictions to 3-point subsets are Szegő. The following example shows it does not.

Example 2.3. Let $\alpha > 2 + \sqrt{5}$ and let

$$K = \begin{bmatrix} \alpha^2 & \alpha & \alpha & 1 \\ \alpha & \frac{\alpha+1}{2} & \frac{\alpha+1}{2} & 1 \\ \alpha & \frac{\alpha+1}{2\alpha} & \frac{2\alpha}{\alpha+1} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

It may be verified that $K > 0$ and that every principal 3×3 submatrix of $1/K$ is singular, but

$$\det(1/K) = \frac{4(\alpha - 1)^3}{\alpha^2(\alpha + 1)} \neq 0.$$

That is, every 3-point restriction of K is Szegő, but K is not itself Szegő.

3. CHARACTERISATION OF COMPRESSION PROPERTIES

Theorem 1.8 reduces the study of the AAK property to concrete questions about kernels on finite sets, and so enables us to find the kernels with the desired properties by elementary algebra. Let us investigate the compression property for s_1 in the case of a Hilbert space H of functions on a set Ω containing a small number of points. If Ω has only one or two points the property is trivial.

Theorem 3.1. *Let H be a Hilbert space of functions on a three-point set Ω with non-singular kernel k . Then H has the compression property for s_1 if and only if*

$$(3.1) \quad \left(\prod_{i,j=1}^3 k_{ij} \right) \det \left[\frac{1}{k_{ij}} \right]_{i,j=1}^3 \leq 0,$$

while H has the compression property for s_0 if and only if

$$\left(\prod_{i,j=1}^3 k_{ij} \right) \det \left[\frac{1}{k_{ij}} \right]_{i,j=1}^3 \geq 0.$$

Note. The left hand sides in these two conditions are to be understood as polynomials in the k_{ij} , so that the statements are meaningful and valid even if some k_{ij} are zero. Indeed, one can write this polynomial as

$$\begin{vmatrix} k_{12}k_{13} & k_{11}k_{13} & k_{11}k_{12} \\ k_{22}k_{23} & k_{21}k_{23} & k_{21}k_{22} \\ k_{32}k_{33} & k_{31}k_{33} & k_{31}k_{32} \end{vmatrix}.$$

If all k_{ij} are non-zero, then $\prod k_{ij} > 0$ and so (3.1) is equivalent to $\det [1/k_{ij}] \leq 0$.

We shall establish a number of equivalent conditions for the compression property. As is implicit in the above statement, we are assuming that $\Omega = \{1, 2, 3\}$ and are writing k_{ij} for $k(i, j)$. Let K denote the matrix $[k_{ij}]_{i,j=1}^3$ and $1/K$ the matrix $[1/k_{ij}]_{i,j=1}^3$.

Lemma 3.2. *The following conditions are equivalent.*

- (1) H has the compression property for s_1 ;
- (2) if

$$A = [k_{ij}]_{i,j=1}^2, \quad \tilde{A} = \left[k_{ij} - \frac{k_{i3}k_{3j}}{k_{33}} \right]_{i,j=1}^2,$$

then

$$A * A^{-T} \leq \tilde{A} * \tilde{A}^{-T};$$

$$(3) k_{11}k_{22}k_{33} \det K \leq (k_{11}k_{33} - |k_{13}|^2)(k_{11}k_{22} - |k_{12}|^2)(k_{22}k_{33} - |k_{23}|^2);$$

$$(4) \left(\prod_{i,j=1}^3 k_{ij} \right) \det(1/K) \leq 0.$$

Moreover, the following conditions are also equivalent.

- (1') H has the compression property for s_0 ;
- (2') for A, \tilde{A} as in (2) above,

$$A * A^{-T} \geq \tilde{A} * \tilde{A}^{-T};$$

$$(3') k_{11}k_{22}k_{33} \det K \geq (k_{22}k_{33} - |k_{23}|^2)(k_{11}k_{33} - |k_{13}|^2)(k_{11}k_{22} - |k_{12}|^2);$$

$$(4') \left(\prod_{i,j=1}^3 k_{ij} \right) \det(1/K) \geq 0.$$

The following conditions are also equivalent.

- (1'') H has the compression property for both s_0 and s_1 ;
- (2'') for A, \tilde{A} as in (2) above,

$$A * A^{-T} = \tilde{A} * \tilde{A}^{-T};$$

$$(3'') k_{11}k_{22}k_{33} \det K = (k_{22}k_{33} - |k_{23}|^2)(k_{11}k_{33} - |k_{13}|^2)(k_{11}k_{22} - |k_{12}|^2);$$

$$(4'') \left(\prod_{i,j=1}^3 k_{ij} \right) \det(1/K) = 0;$$

- (5'') there exists a non-singular diagonal matrix Λ such that $K^{-T} = \Lambda K \Lambda^*$.

For this and subsequent theorems we need to calculate the characteristic polynomials of T^*T and $\tilde{T}^*\tilde{T}$, where T and \tilde{T} are the operators occurring in the definition of the compression property.

Lemma 3.3. *Let $\lambda_1, \dots, \lambda_n, \alpha, z_1, \dots, z_n, \mathcal{M}, \tilde{\mathcal{M}}, T$ and \tilde{T} be as in Definition 1.5. The characteristic polynomials of T^*T , $\tilde{T}^*\tilde{T}$ are p, \tilde{p} respectively, where*

$$p(t) = \Delta^{-1} \det [(t - z_i \bar{z}_j) k(\lambda_i, \lambda_j)]_{i,j=1}^n,$$

$$\tilde{p}(t) = \tilde{\Delta}^{-1} \det [(t - z_i \bar{z}_j) \tilde{k}(\lambda_i, \lambda_j)]_{i,j=1}^n,$$

$$\Delta = \det [k(\lambda_i, \lambda_j)]_{i,j=1}^n, \quad \tilde{\Delta} = \det [\tilde{k}(\lambda_i, \lambda_j)]_{i,j=1}^n$$

and

$$\tilde{k}(\lambda_i, \lambda_j) = k(\lambda_i, \lambda_j) - \frac{k(\lambda_i, \alpha)k(\alpha, \lambda_j)}{k(\alpha, \alpha)}.$$

Proof. Routine linear algebra shows that the matrix of $A \in \mathcal{L}(\mathcal{M})$ with respect to the basis $k_{\lambda_1}, \dots, k_{\lambda_n}$ of \mathcal{M} is

$$(3.2) \quad K^{-1} [\langle Ak_{\lambda_j}, k_{\lambda_i} \rangle]_{i,j=1}^n$$

where

$$K = [k(\lambda_i, \lambda_j)]_{i,j=1}^n.$$

Indeed, the (i, j) -entry of the matrix of A is $\langle Ak_{\lambda_j}, w_i \rangle$ where w_1, \dots, w_n is the dual basis to $k_{\lambda_1}, \dots, k_{\lambda_n}$. If we write $w_i = \sum_r \gamma_{ir} k_{\lambda_r}$, then we have

$$\delta_{ij} = \langle k_{\lambda_j}, w_i \rangle = \sum_r \bar{\gamma}_{ir} \langle k_{\lambda_j}, k_{\lambda_r} \rangle = \sum_r \bar{\gamma}_{ir} k(\lambda_r, \lambda_j),$$

that is, $[\bar{\gamma}_{ij}]_{i,j=1}^n = K^{-1}$, while

$$\langle Ak_{\lambda_j}, w_i \rangle = \sum_r \bar{\gamma}_{ir} \langle Ak_{\lambda_j}, k_{\lambda_r} \rangle,$$

which is the (i, j) -entry of the matrix (3.2).

On applying this observation to T^*T we find that

$$T^*T \sim K^{-1} [\langle Tk_{\lambda_j}, Tk_{\lambda_i} \rangle] = K^{-1} [z_i \bar{z}_j k(\lambda_i, \lambda_j)],$$

and hence

$$\begin{aligned} p(t) &= \det(t - T^*T) \\ &= \Delta^{-1} \det[(t - z_i \bar{z}_j) k(\lambda_i, \lambda_j)] \end{aligned}$$

as required. Similar reasoning gives the formula for \tilde{p} (recall equations (1.4), (1.5)). \square

Proof of Lemma 3.2. We prove (1) \Leftrightarrow (2). We have to consider subsets $\lambda_1, \dots, \lambda_n, \alpha$ of distinct points of $\Omega = \{1, 2, 3\}$. If $n = 1$, then both T and \tilde{T} in Definition 1.5 have rank one, and so $s_1(T) = s_1(\tilde{T}) = 0$. Take $n = 2$, $\lambda_1 = 1, \lambda_2 = 2, \alpha = 3$, and corresponding data values $z_1, z_2 \in \mathbb{C}$. By the foregoing lemma the characteristic polynomial of T^*T is

$$\begin{aligned} p(t) &= \Delta^{-1} \det[(t - z_i \bar{z}_j) k(i, j)]_{i,j=1}^2 \\ &= t^2 - \beta t + |z_1 z_2|^2, \end{aligned}$$

where we find, after a simple calculation, that

$$\beta = \langle A * A^{-T} z, z \rangle_{\mathbb{C}^2} \text{ where } z = [z_1 \quad z_2]^T.$$

Hence

$$s_1(T)^2 = \frac{1}{2}(\beta - \sqrt{\beta^2 - 4|z_1 z_2|^2}),$$

and likewise

$$s_1(\tilde{T})^2 = \frac{1}{2}(\tilde{\beta} - \sqrt{\tilde{\beta}^2 - 4|z_1 z_2|^2})$$

where

$$\tilde{\beta} = \langle \tilde{A} * \tilde{A}^{-T} z, z \rangle_{\mathbb{C}^2}.$$

On differentiating with respect to β we find that $s_1(T)$ is a strictly decreasing function of β on $(2|z_1 z_2|, \infty)$, and so $s_1(\tilde{T}) \leq s_1(T)$ if and only if

$$\langle A * A^{-T} z, z \rangle \leq \langle \tilde{A} * \tilde{A}^{-T} z, z \rangle.$$

This holds for all $z_1, z_2 \in \mathbb{C}$ if and only if

$$A * A^{-T} \leq \tilde{A} * \tilde{A}^{-T}.$$

It follows that (1) \Rightarrow (2); conversely, (2) implies that $s_1(\tilde{T}) \leq s_1(T)$ whenever we take $\lambda_1 = 1, \lambda_2 = 2, \alpha = 3$. We shall see later that the inequality remains valid when $\alpha = 1$ or 2 .

A similar calculation, where we take a $+$ sign in the formula for the solution of a quadratic equation, shows that $s_0(T)$ is a strictly *increasing* function of β , and so $s_0(\tilde{T}) \leq s_0(T)$ if and only if $A * A^{-T} \geq \tilde{A} * \tilde{A}^{-T}$. Thus (1') \Rightarrow (2').

(2) \Leftrightarrow (3). For any invertible matrix M , the row sums of $M * M^{-T}$ are all 1. Hence $A * A^{-T} - \tilde{A} * \tilde{A}^{-T}$ has row sums zero, and is therefore of rank 1. Now

$$A * A^{-T} - \tilde{A} * \tilde{A}^{-T} = \frac{1}{\Delta} \begin{bmatrix} k_{11}k_{22} & -k_{12}k_{21} \\ -k_{21}k_{12} & k_{11}k_{22} \end{bmatrix} - \frac{1}{\tilde{\Delta}} \begin{bmatrix} \tilde{k}_{11}\tilde{k}_{22} & -\tilde{k}_{12}\tilde{k}_{21} \\ -\tilde{k}_{21}\tilde{k}_{12} & \tilde{k}_{11}\tilde{k}_{22} \end{bmatrix},$$

a rank one Hermitian matrix with equal diagonal entries. Such a matrix is negative if and only if its (1,1) entry (which is half its trace) is non-positive. Thus condition (2) is equivalent to

$$\frac{k_{11}k_{22}}{\Delta} \leq \frac{\tilde{k}_{11}\tilde{k}_{22}}{\tilde{\Delta}}.$$

Since

$$K = \begin{bmatrix} A & u \\ u^* & k_{33} \end{bmatrix}, \quad u = \begin{bmatrix} k_{13} \\ k_{23} \end{bmatrix},$$

we have

$$\det K = k_{33} \det(A - k_{33}^{-1} u u^*) = k_{33} \det \tilde{A} = k_{33} \tilde{\Delta}.$$

Hence (3) is equivalent to

$$k_{11}k_{22}k_{33}^{-1} \det K \leq \tilde{k}_{11}\tilde{k}_{22} \det A,$$

that is, to

$$k_{11}k_{22}k_{33} \det K \leq k_{33}^2 \left(k_{11} - \frac{k_{13}k_{31}}{k_{33}} \right) \left(k_{22} - \frac{k_{23}k_{32}}{k_{33}} \right) (k_{11}k_{22} - k_{12}k_{21}).$$

This simplifies to

$$k_{11}k_{22}k_{33} \det K \leq (k_{22}k_{33} - |k_{23}|^2)(k_{11}k_{33} - |k_{13}|^2)(k_{11}k_{22} - |k_{12}|^2).$$

Thus (2) \Leftrightarrow (3). An easy modification of the argument shows that (2') \Leftrightarrow (3').

Note that condition (3) is symmetric with respect to permutations of $\Omega = \{1, 2, 3\}$. From (2) \Leftrightarrow (3) it follows that (2) is equivalent to the corresponding inequality obtained by choosing α to be 1 or 2 instead of 3. This completes the proofs that (2) \Rightarrow (1) and (2') \Rightarrow (1').

(3) \Leftrightarrow (4) and (3') \Leftrightarrow (4') are consequences of the identity

$$\begin{aligned} & k_{11}k_{22}k_{33} \det K - (k_{22}k_{33} - |k_{23}|^2)(k_{11}k_{33} - |k_{13}|^2)(k_{11}k_{22} - |k_{12}|^2) \\ &= \left(\prod_{i,j=1}^3 k_{ij} \right) \det(1/K) \end{aligned}$$

which is a matter of straightforward verification.

The equivalence of (1'') to (4'') is immediate. The proof that (2'') \Leftrightarrow (5'') will be given below; see the Remark following the proof of Theorem 3.9 \square

Corollary 3.4. *Let H be the restriction of the Bergman space $L_a^2(\mathbb{D})$ to a three-point subset of \mathbb{D} . Then H has the compression property for s_1 but not for s_0 .*

Proof. Let $\Omega = \{\lambda_1, \lambda_2, \lambda_3\} \subset \mathbb{D}$ consist of three distinct points and $H = L_a^2(\mathbb{D})|_{\Omega}$. Then H can be identified with the 3-dimensional subspace $\text{span}\{k_{\lambda_1}, k_{\lambda_2}, k_{\lambda_3}\}$ of $L_a^2(\mathbb{D})$, where k is the Bergman kernel of \mathbb{D} ,

$$k(\lambda, \mu) = (1 - \lambda\bar{\mu})^{-2}.$$

The reproducing kernel k_H of H is the restriction of k to $\Omega \times \Omega$. Thus, for $1 \leq i, j \leq 3$,

$$1/k_H(\lambda_i, \lambda_j) = 1 - 2\lambda_i\bar{\lambda}_j + \lambda_i^2\bar{\lambda}_j^2.$$

That is,

$$1/K_H = [1/k_H(\lambda_i, \lambda_j)]_{i,j=1}^3 = V^* \text{diag}\{1, -2, 1\}V$$

where V is the non-singular Vandermonde matrix

$$V = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}.$$

Thus

$$\det(1/K_H) = -2|\det V|^2 < 0.$$

It follows from Theorem 3.1 that H has the compression property for s_1 but not for s_0 . \square

Corollary 3.5. *Let H be a Hilbert space of functions on a three-point set Ω with non-singular kernel k , and suppose that k takes the value zero at some point of $\Omega \times \Omega$. Then H has the compression property for s_1 . Moreover, H has the compression property for s_0 if and only if k vanishes at at least four points of $\Omega \times \Omega$.*

Proof. Suppose $\Omega = \{1, 2, 3\}$ and $k_{12} = 0$. By Theorem 3.1 and the note following it H has the compression property for s_1 if and only if

$$\begin{vmatrix} 0 & k_{11}k_{13} & 0 \\ k_{22}k_{23} & 0 & 0 \\ k_{32}k_{33} & k_{31}k_{33} & k_{31}k_{32} \end{vmatrix} \leq 0,$$

that is, if and only if

$$-k_{11}k_{22}|k_{23}k_{13}|^2 \leq 0,$$

which is true. On the other hand, the compression property for s_0 holds in H if and only if the last inequality holds with equality, which is so if and only if $k_{23} = 0$ or $k_{13} = 0$. \square

On combining Theorem 3.1 with Lemma 2.1 and Corollary 3.5 we obtain the following.

Theorem 3.6. *Let H be a Hilbert function space with non-singular kernel k on a set Ω containing at least 3 points. Suppose that H has the compression property for both s_0 and s_1 for pairs of points of Ω . Then for any 3-point subset Ω' of Ω , the restriction of k to $\Omega' \times \Omega'$ is either a Szegő kernel or has at least 4 zeros on $\Omega' \times \Omega'$.*

Example 2.3 shows that it does not follow, for non-vanishing k , that k is itself a Szegő kernel. We therefore need to study compression properties on 4-point sets. Let $\Omega = \{1, 2, 3, 4\}$, let k be a positive non-singular kernel on Ω and let H be the corresponding Hilbert space of functions on Ω . We shall analyze the extension of functions from $\{1, 2, 3\}$ to Ω with preservation of s_1 . For data $z_1, z_2, z_3 \in \mathbb{C}$ let T, \tilde{T} be the operators of Definition 1.5; we need to find the conditions on k that ensure $s_1(\tilde{T}) \leq s_1(T)$ whatever the z_j . By Lemma 3.3 the characteristic polynomial of T^*T acting on $\text{span}\{k_1, k_2, k_3\}$ is

$$p(t) \stackrel{\text{def}}{=} \Delta^{-1} \det[(t - z_i \bar{z}_j)k_{ij}]_{i,j=1}^3$$

where $\Delta = \det[k_{ij}]_{i,j=1}^3$. Regard the j th column of the matrix $[(t - z_i \bar{z}_j)k_{ij}]$ as the sum of the two columns $[tk_{ij}]_{i=1}^3$ and $[z_i \bar{z}_j k_{ij}]_{i=1}^3$, and expand the right hand side accordingly as the sum of eight determinants, of which one, for example, is

$$\begin{vmatrix} z_1 \bar{z}_1 k_{11} & tk_{12} & z_1 \bar{z}_3 k_{13} \\ z_2 \bar{z}_1 k_{21} & tk_{22} & z_2 \bar{z}_3 k_{23} \\ z_3 \bar{z}_1 k_{31} & tk_{32} & z_3 \bar{z}_3 k_{33} \end{vmatrix}.$$

On expanding further we find that the coefficient of t in $p(t)$ is

$$\Delta^{-1} \sum_{i,j=1}^3 k_{ij} C_{ij} Z_i \bar{Z}_j$$

where C_{ij} is the (i, j) cofactor of $C = [k_{ij}]_{i,j=1}^3$ and $Z^T = [z_2 z_3 \quad z_3 z_1 \quad z_1 z_2]$. Since $\Delta^{-1} C_{ij}$ is the (j, i) entry of C^{-1} we can write this in the form

$$\langle C * C^{-T} Z, Z \rangle_{\mathbb{C}^3}.$$

Likewise the coefficient of t^2 is

$$-\Delta^{-1} \sum_{ij} k_{i,j} C_{ij} z_i \bar{z}_j = \langle C * C^{-T} z, z \rangle_{\mathbb{C}^3}$$

where $z^T = [z_1 \quad z_2 \quad z_3]$. Let us introduce the following notation for a 3×3 matrix $A > 0$:

$$p_A(t) \stackrel{\text{def}}{=} t^3 - \langle A * A^{-T} z, z \rangle t^2 + \langle A * A^{-T} Z, Z \rangle t - |z_1 z_2 z_3|^2$$

where $z^T = [z_1 \quad z_2 \quad z_3]$, $Z^T = [z_2 z_3 \quad z_3 z_1 \quad z_1 z_2]$. Since the characteristic polynomials of T^*T and $\tilde{T}^* \tilde{T}$ are monic and have constant terms equal to $-|\det T|^2 = -|z_1 z_2 z_3|^2$, we obtain the following.

Lemma 3.7. *Let T, \tilde{T} be as in Definition 1.5 for the points 1, 2, 3 and $\alpha = 4$. The characteristic polynomials of T^*T and $\tilde{T}^*\tilde{T}$ are p_C and $p_{\tilde{C}}$ respectively, where*

$$C = [k_{ij}]_{i,j=1}^3, \quad \tilde{C} = \left[k_{ij} - \frac{k_{i4}k_{4j}}{k_{44}} \right]_{i,j=1}^3.$$

To say that $s_1(\tilde{T}) \leq s_1(T)$ is thus to say that the second zero of $p_{\tilde{C}}$ is less than or equal to the second zero of p_C .

Lemma 3.8. *Let A, B be positive definite 3×3 matrices such that, for any choice of $z_1, z_2, z_3 \in \mathbb{C}$, the second zero of p_A is no greater than the second zero of p_B . Then*

$$A * A^{-T} = B * B^{-T}.$$

Proof. Write $A * A^{-T} = [a_{ij}]$, $B * B^{-T} = [b_{ij}]$. Take $z_3 = 0$. Then

$$(3.3) \quad p_A(t) = t(t^2 - \langle A * A^{-T} z, z \rangle t + a_{33}|z_1 z_2|^2).$$

Since all zeros of p_A are non-negative,

$$\langle A * A^{-T} z, z \rangle^2 \geq 4a_{33}|z_1 z_2|^2,$$

and moreover the second zero of p_A is the lesser zero of the quadratic factor on the right hand side of (3.3). We shall use the following elementary observation. Suppose β_j, γ_j are positive real numbers for $j = 1, 2$ such that $\beta_j^2 \geq 4\gamma_j$, and let $q_j(t) = t^2 - \beta_j t + \gamma_j$. Then the lesser zero of q_1 is no greater than that of q_2 if and only if

$$\gamma_1 \left(\beta_2 + \sqrt{\beta_2^2 - 4\gamma_2} \right) \leq \gamma_2 \left(\beta_1 + \sqrt{\beta_1^2 - 4\gamma_1} \right).$$

Apply this observation with $p_A = tq_1, p_B = tq_2$ (so that $\beta_1 = \langle A * A^{-T} z, z \rangle$, $\gamma_1 = a_{33}|z_1 z_2|^2$, etc.). We deduce that, for all $z_1, z_2 \in \mathbb{C}$,

$$(3.4) \quad \begin{aligned} & a_{33}|z_1 z_2|^2 \left(\langle B * B^{-T} z, z \rangle + \sqrt{\langle B * B^{-T} z, z \rangle^2 - 4b_{33}|z_1 z_2|^2} \right) \\ & \leq b_{33}|z_1 z_2|^2 \left(\langle A * A^{-T} z, z \rangle + \sqrt{\langle A * A^{-T} z, z \rangle^2 - 4a_{33}|z_1 z_2|^2} \right). \end{aligned}$$

Divide through by $|z_1 z_2|^2$ and then put $z_1 = 1, z_2 = 0$ to obtain

$$a_{33}b_{11} \leq b_{33}a_{11}.$$

By symmetry the opposite inequality also holds, and thus

$$a_{ii}b_{jj} = b_{ii}a_{jj}, \quad 1 \leq i, j \leq 3.$$

Hence there exists $\tau > 0$ such that $a_{ii} = \tau b_{ii}$, $1 \leq i \leq 3$. Cancel $|z_1 z_2|^2$ from (3.4) and put $z_2 = \xi \in \mathbb{R}$ to get

$$\begin{aligned} & a_{33} \left(\langle B * B^{-T} z, z \rangle + \sqrt{\langle B * B^{-T} z, z \rangle^2 - 4b_{33}|z_1|^2 \xi^2} \right) \\ & \leq b_{33} \left(\langle A * A^{-T} z, z \rangle + \sqrt{\langle A * A^{-T} z, z \rangle^2 - 4a_{33}|z_1|^2 \xi^2} \right) \end{aligned}$$

for all $z_1 \in \mathbb{C}$, $\xi \in \mathbb{R}$, where $z^T = [z_1 \ \xi \ 0]$. The two sides of this inequality are equal when $\xi = 0$, and so the inequality remains true if we differentiate and put $\xi = 0$. Now

$$\begin{aligned} \frac{d}{d\xi} \sqrt{\langle B * B^{-T} z, z \rangle^2 - 4\xi^2 |z_1|^2 b_{33}} \Big|_{\xi=0} &= \frac{d}{d\xi} \langle B * B^{-T} z, z \rangle \Big|_{\xi=0} \\ &= 2\operatorname{Re} (b_{21} z_1). \end{aligned}$$

Hence

$$a_{33} 2\operatorname{Re} (b_{21} z_1) \leq b_{33} 2\operatorname{Re} (a_{21} z_1).$$

Thus, for all $z_1 \in \mathbb{C}$,

$$\operatorname{Re} \{z_1(a_{33}b_{21} - b_{33}a_{21})\} \leq 0,$$

and hence

$$a_{33}b_{21} - b_{33}a_{21} = 0.$$

Since $a_{33} = \tau b_{33}$, we have $a_{21} = \tau b_{21}$. By symmetry $a_{ij} = \tau b_{ij}$ for each i, j , and so

$$A * A^{-T} = \tau B * B^{-T}.$$

Since the row sums of $A * A^{-T}$ and $B * B^{-T}$ are all 1, we must have $\tau = 1$, and so

$$A * A^{-T} = B * B^{-T}.$$

□

We shall say that two $n \times n$ matrices A, B are *diagonally congruent* if there is a non-singular diagonal matrix D such that $A = D^* B D$. In the event that the entries of A and B are all non-zero it is equivalent to saying that the (entrywise) ratio A/B has rank 1.

Theorem 3.9. *Let H be a Hilbert space of functions with a non-vanishing and non-singular kernel k on the 4-point set $\Omega = \{1, 2, 3, 4\}$. The following conditions are equivalent.*

- (1) H has the compression property for s_1 for triples of points;
- (2) if

$$A = [k_{ij}]_{i,j=1}^3, \quad \tilde{A} = \left[k_{ij} - \frac{k_{i4}k_{4j}}{k_{44}} \right]_{i,j=1}^3,$$

then $A * A^{-T} = \tilde{A} * \tilde{A}^{-T}$;

- (3) K^{-T} is diagonally congruent to K , where $K = [k_{ij}]_{i,j=1}^4$.

Proof. (1) \Rightarrow (2) In the definition of compression property (Definition 1.5) take $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$ and $\alpha = 4$. For any choice of data $z_1, z_2, z_3 \in \mathbb{C}$ we have $s_1(\tilde{T}) \leq s_1(T)$. By Lemma 3.7 it follows that the second zero of $p_{\tilde{A}}$ does not exceed that of p_A , and so, by Lemma 3.8, (2) holds.

(2) \Leftrightarrow (3) We shall show that both are equivalent to a fourth condition involving

$$c \stackrel{\text{def}}{=} k_{44}, \quad u \stackrel{\text{def}}{=} [k_{14} \ k_{24} \ k_{34}]^T,$$

in fact to

- (4) there exists a non-singular diagonal matrix P such that $A^{-T} = P \tilde{A} P^*$ and

$$AP^* \bar{u} = \left(\frac{c}{c - u^* A^{-1} u} \right)^{\frac{1}{2}} u.$$

Note that $\tilde{A} = A - c^{-1}uu^*$ and that $c - u^*A^{-1}u$, being the Schur complement of A in K , is positive. Moreover one may verify that $\tilde{A}^{-1} = A^{-1} + ww^*$ where

$$(3.5) \quad w = \frac{A^{-1}u}{(c - u^*A^{-1}u)^{\frac{1}{2}}}.$$

Hence

$$\begin{aligned} \tilde{A} * \tilde{A}^{-T} &= \tilde{A} * (A^{-T} + \bar{w}w^T) \\ &= (A - c^{-1}uu^*) * A^{-T} + \tilde{A} * \bar{w}w^T, \end{aligned}$$

and so (2) is equivalent to

$$(3.6) \quad c^{-1}uu^* * A^{-T} = \bar{w}w^T * \tilde{A}$$

Suppose (2) holds. The components of u are non-zero by hypothesis, and hence those of w are too. Let $P = \text{diag}\{c^{\frac{1}{2}}\bar{w}_j/u_j\}$; then (1.5) gives $A^{-T} = P\tilde{A}P^*$, while $P^*\bar{u} = c^{\frac{1}{2}}w$, and so

$$AP^*\bar{u} = c^{\frac{1}{2}}Aw = \left(\frac{c}{c - u^*A^{-1}u} \right)^{\frac{1}{2}} u.$$

Thus (2) \Rightarrow (4). Conversely, if (4) holds, then $P^*\bar{u} = c^{\frac{1}{2}}w$ and the relation $A^{-T} = P\tilde{A}P^*$ yields (3.6). Thus (4) \Rightarrow (2).

(3) \Leftrightarrow (4) Note that

$$K = \begin{bmatrix} A & u \\ u^* & c \end{bmatrix} = \begin{bmatrix} 1 & c^{-1}u \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{A} & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ c^{-1}u^* & 1 \end{bmatrix}$$

so that

$$\begin{aligned} K^{-T} &= \begin{bmatrix} 1 & 0 \\ -c^{-1}u^T & 1 \end{bmatrix} \begin{bmatrix} \tilde{A}^{-T} & 0 \\ 0 & c^{-1} \end{bmatrix} \begin{bmatrix} 1 & -c^{-1}\bar{u} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \tilde{A}^{-T} & -c^{-1}\tilde{A}^{-T}\bar{u} \\ -c^{-1}u^T\tilde{A}^{-T} & c^{-1} + c^{-2}u^T\tilde{A}^{-T}\bar{u} \end{bmatrix}. \end{aligned}$$

Suppose (4) holds. Then $\tilde{A}^{-T} = PAP^*$ and so

$$\begin{aligned} K^{-T} &= \begin{bmatrix} PAP^* & -c^{-1}PAP^*\bar{u} \\ -c^{-1}u^T PAP^* & c^{-1} + c^{-2}u^T PAP^*\bar{u} \end{bmatrix} \\ &= \begin{bmatrix} PAP^* & -c^{-1}\kappa PAu \\ -c^{-1}\kappa U^* AP^* & c^{-1} + c^{-2}u^T PAP^*\bar{u} \end{bmatrix} \end{aligned}$$

where $\kappa = c^{\frac{1}{2}}(c - u^*A^{-1}u)^{-\frac{1}{2}}$. Now

$$\begin{aligned} c^{-1} + c^{-2}u^T P A P^* \bar{u} &= c^{-1} + c^{-2}\kappa^2 u^* A^{-1}u \\ &= c^{-1} + c^{-1}(c - u^*A^{-1}u)^{-1}u^* A^{-1}u \\ &= (c - u^*A^{-1}u)^{-1} = c^{-1}\kappa^2. \end{aligned}$$

It follows that

$$\begin{aligned} K^{-T} &= \begin{bmatrix} P A P^* & -c^{-1}\kappa P u \\ -c^{-1}\kappa u^* P^* & c^{-1}\kappa^2 \end{bmatrix} \\ &= \begin{bmatrix} P & 0 \\ 0 & -c^{-1}\kappa \end{bmatrix} \begin{bmatrix} A & u \\ u^* & c \end{bmatrix} \begin{bmatrix} P^* & 0 \\ 0 & -c^{-1}\kappa \end{bmatrix} \\ &= \Lambda K \Lambda^*, \end{aligned}$$

where Λ is the non-singular diagonal matrix $\text{diag}\{P, -c^{-1}\kappa\}$. Thus (4) \Rightarrow (3).

Suppose (3) holds: say $K^{-T} = \Lambda K \Lambda^*$ where Λ is a non-singular diagonal matrix which we write as $\text{diag}\{Q, \gamma\}$ for some non-singular diagonal 3×3 matrix Q and $\gamma \neq 0$. By virtue of (3) we have

$$\begin{aligned} (3.7) \quad \tilde{A}^{-T} &= Q A Q^*, \\ -c^{-1}\tilde{A}^{-T}\bar{u} &= \bar{\gamma} Q u, \\ c^{-1} + c^{-2}u^T \tilde{A}^{-T}\bar{u} &= c|\gamma|^2. \end{aligned}$$

From the first two equations we have

$$Q A Q^* \bar{u} = -c\bar{\gamma} Q u$$

and hence

$$(3.8) \quad A Q^* \bar{u} = -c\bar{\gamma} u,$$

and moreover

$$\begin{aligned} c^{-2}u^T \tilde{A}^{-T}\bar{u} &= c^{-2}u^T \tilde{A}^{-T} \tilde{A}^T \tilde{A}^{-T}\bar{u} \\ &= \gamma u^* Q^* \tilde{A}^T \bar{\gamma} Q u \\ &= |\gamma|^2 u^* Q^* (Q^{-*} A^{-1} Q^{-1}) Q u \\ &= |\gamma|^2 u^* A^{-1} u, \end{aligned}$$

which combines with the third equation to yield

$$c^{-1} + |\gamma|^2 u^* A^{-1} u = c|\gamma|^2$$

and hence

$$|\gamma|^2 = c^{-1}(c - u^*A^{-1}u)^{-1}.$$

It follows that there is a complex number ξ of unit modulus such that

$$\xi\gamma = -|\gamma| = -c^{-\frac{1}{2}}(c - u^*A^{-1}u)^{-\frac{1}{2}}.$$

Let $P = \xi Q$. Then equations (3.7), (3.8) become

$$\begin{aligned} \tilde{A}^{-T} &= P A P^*, \\ A P^* \bar{u} &= c|\gamma|u \\ &= c^{\frac{1}{2}}(c - u^*A^{-1}u)^{-\frac{1}{2}}u. \end{aligned}$$

Thus (3) \Rightarrow (4).

It remains to prove that (2) \Rightarrow (1). Suppose, then, that $A * A^{-T} = \tilde{A} * \tilde{A}^{-T}$. Then $p_A = p_{\tilde{A}}$, so that T, \tilde{T} (as defined at the beginning of the proof) have the same singular values, and in particular $s_1(\tilde{T}) \leq s_1(T)$. However, to prove (1) one must show that the corresponding relation holds for any triple of points from Ω , not just $\{1, 2, 3\}$. This conclusion follows from the equivalence of (2) and (3) and the symmetry of condition (3). \square

Remarks. 1. The proof of (2) \Leftrightarrow (3) remains valid for $n \times n$ positive matrices K with trivial modifications. The case $n = 3$ is (2') \Leftrightarrow (5') in Lemma 3.2.

2. One way that (2) can hold is if \tilde{A} is diagonally congruent to A , and this is in fact another characterization of Szegő kernels. One might suspect that the conditions of the theorem are *equivalent* to k being a Szegő kernel, but this is false.

Example 3.10. There are non-vanishing positive non-singular kernels on $\{1, 2, 3, 4\}$ which have the compression property for s_1 for triples of points but not for pairs of points. Such kernels of course cannot be Szegő kernels. To see this consider positive definite 2×2 matrices M, N with non-zero entries. Up to diagonal congruence they have the form

$$M = \begin{bmatrix} 1 & 1 \\ 1 & m \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 1 \\ 1 & n \end{bmatrix}$$

where $m, n > 1$. It is easy to see that M, N are diagonally congruent to their inverse transposes. This property is clearly preserved under tensor products, and so the 4×4 matrix $M \otimes N$ has the properties (1) to (3) of Theorem 3.9. However, the leading principal 3×3 submatrix of $M \otimes N$ is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & m & 1 \\ 1 & 1 & n \end{bmatrix}$$

and the determinant of the reciprocal of this matrix is $(m^{-1} - 1)(n^{-1} - 1)$, which is positive. By Theorem 3.1, $M \otimes N$ does not have the compression property for s_1 for pairs of points.

With the aid of Theorem 3.9 and a matrix algebra programme it is easy to show that the Bergman space $L_a^2(\mathbb{D})$ does not have the compression property for s_1 , despite the fact that its 3-point restrictions do. Choose 4 points in \mathbb{D} ($0, \pm\frac{1}{2}, \frac{3}{4}$ will do), and form the corresponding matrix $K = [(1 - \lambda_i \bar{\lambda}_j)^{-2}]_{i,j=1}^4$. A couple of commands will produce the singular values of the matrix K^{-T}/K . They turn out to be all non-zero, so K^{-T}/K is not rank 1, or in other words, K^{-T} is not diagonally congruent to K . Thus $L_a^2(\mathbb{D})$ does not have the compression property for s_1 for triples of points.

We now show that the assumption of quite modest compression properties forces a kernel to be Szegő.

Theorem 3.11. *Let H be a Hilbert space of functions with non-singular kernel k on a set Ω containing at least 4 points. Suppose that k does not vanish on $\Omega \times \Omega$ and that*

- (1) H has the compression property for s_0 and s_1 for pairs of points, and
- (2) H has the compression property for s_1 for triples of points.

Then k is a Szegő kernel.

Proof. In view of Lemma 2.2 we can assume that $\Omega = \{1, 2, 3, 4\}$. Write

$$K = [k_{ij}]_{i,j=1}^4 = \begin{bmatrix} A & u \\ u^* & c \end{bmatrix}.$$

By assumption (1) and Lemma 3.2 there is a non-singular diagonal matrix P such that $A^{-T} = PAP^*$. Another way of saying this is: there is a column vector $p \in \mathbb{C}^3$ with non-zero components such that

$$A^{-T} = pp^* * A.$$

By assumption (2) and Theorem 3.9 we have also

$$A * A^{-T} = \tilde{A} * \tilde{A}^{-T}$$

where $\tilde{A} = A - c^{-1}uu^*$. Since $\tilde{A}^{-1} = A^{-1} + ww^*$ for a suitable column matrix w we have

$$A * A^{-T} = (A - c^{-1}uu^*) * (A^{-T} + \bar{w}w^T)$$

and hence

$$\begin{aligned} 0 &= A * \bar{w}w^T - c^{-1}uu^* * A^{-T} - c^{-1}uu^* * \bar{w}w^T \\ &= A * (\bar{w}w^T - c^{-1}uu^* * pp^*) - c^{-1}uu^* * \bar{w}w^T. \end{aligned}$$

Thus

$$A = \frac{c^{-1}uu^* * \bar{w}w^T}{\bar{w}w^T - c^{-1}uu^* * pp^*}$$

(note that our hypotheses force all components of u and w to be non-zero, and hence those of the denominator are non-zero too). Consequently

$$\begin{aligned} 1/A &= \frac{c\bar{w}w^T - uu^* * pp^*}{uu^* * \bar{w}w^T} \\ &= \frac{c}{uu^*} - \frac{pp^*}{\bar{w}w^T}. \end{aligned}$$

Hence

$$\begin{aligned} 1/K &= \begin{bmatrix} 1/A & 1/u \\ 1/u^* & 1/c \end{bmatrix} \\ &= \begin{bmatrix} \frac{c}{uu^*} - \frac{pp^*}{\bar{w}w^T} & \frac{1}{u} \\ \frac{1}{u^*} & \frac{1}{c} \end{bmatrix}. \end{aligned}$$

On multiplying the second block row by $-c/u$ and adding to the first block row we infer that $1/K$ is row-equivalent to

$$\begin{bmatrix} -\frac{pp^*}{\bar{w}w^T} & 0 \\ \frac{1}{u^*} & \frac{1}{c} \end{bmatrix}$$

which has rank 2. Thus $1/K$ has rank 2, which is to say that k is Szegő. □

Theorem 3.12. *Let H be a Hilbert space of functions with non-singular kernel k on a set Ω containing at least 4 points. Suppose that k does not vanish on $\Omega \times \Omega$ and that every cofinite set of uniqueness for H in Ω is also a set of uniqueness for HH . If H has the 0-AAK property and the 1-AAK property, then k is a Szegő kernel.*

Proof. It follows from Theorem 1.8 that H has the compression property for s_0 and s_1 . By Theorem 3.11, k is a Szegő kernel. \square

We leave open the following question: Is there a significant class of Hilbert function spaces which have the 1-AAK property but not the 0-AAK property?

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