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L_1 STABILITY FOR 2×2 SYSTEMS OF HYPERBOLIC CONSERVATION LAWS

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1. Introduction

Consider the Cauchy problem for the 2×2 system of conservation laws,

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad t \geq 0, -\infty < x < \infty, \tag{1.1}$$

$$u(x, 0) = u_0(x), \qquad -\infty < x < \infty. \tag{1.2}$$

We assume that the system is strictly hyperbolic, i.e. the matrix $\frac{\partial f(u)}{\partial u}$ has real and distinct eigenvalues $\lambda_1(u) < \lambda_2(u)$ for all u under consideration, with the corresponding right eigenvectors $r_i(u)$, $i = 1, 2$. Each characteristic field is assumed to be either linearly degenerate or genuinely nonlinear [11], i.e. $r_i(u) \cdot \bigtriangledown\lambda_i(u) \equiv 0$ or $r_i(u) \cdot \bigtriangledown\lambda_i(u) \neq 0, \ i = 1, 2$.

The purpose of this paper is to construct a nonlinear functional $H(t) = H(u_1(., t), u_2(., t))$, which is equivalent to the L_1 norm of the difference $u_1 - u_2$ between two weak solutions of (1.1) and (1.2) and is time-decreasing. It depends explicitly on the wave patterns of these two solutions. The construction of the functional will allow us to identify the nonlinear effects of the wave behaviour on the L_1 topology of the solution space.

The Hugoniot curves do not coincide with, but are a second order bifurcation from, the rarefaction curves. This and other nonlinear coupling cause new waves to be created when waves interact. The classical work of James Glimm [9] on the global existence of weak solutions introduces a nonlinear functional $F(u)$ to control the increase of the total variation of a solution u due to interactions. The work of Glimm and that of Glimm-Lax [10] yield the boundedness of total wave interactions. The subsequent wave tracing [14] shows that, so far as the L_1 norm is concerned and subject to the aforementioned, in the construction of the nonlinear functional $H(t)$, we need only deal with solutions consisting of nonlinear waves linearly superimposed.

For a scalar conservation law, it is well known that the solution operator is an L_1 contraction semigroup [20]. Thus we may simply take $H(t) = ||u_1 - u_2||_{L_1}$. In this paper we consider the simplified situation of 2×2 conservation laws, so that there exists a set of Riemann invariants $r(u),\ s(u)$. We need to assess the deviation of $r(u)$ and $s(u)$ from the solutions of certain scalar equations. For this we introduce

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several nonlinear functionals in Section 3. In considering the scalar function $r(u_1)$, for instance, we realize that the other Riemann invariant $s(u_1)$ is not constant. The same holds for $r(u_2)$ and can cause the increase of the L_1 norm of $r(u_1) - r(u_2)$. We introduce a nonlinear functional $Q_d(t)$ to register the effects of this coupling (Definition 3).

In the simplified situation when one of the Riemann invariants is constant, say $s(u_1) = s(u_2) =$ constant, there is still the issue that in general no scalar conservation law governs $r(u_1)$ and $r(u_2)$. However, an approximate scalar conservation law exists and the error can be controlled by another functional. The new functional is called the entropy functional because it generalizes the traditional concept of entropy in the second law of thermodynamics. Its introduction is a key idea for the present paper. The traditional entropy inequality says that the L_2 norm of the solution of a scalar convex conservation law decreases in time. This basic property cannot be generalized to the difference of two solutions; the L_2 norm of the difference of two solutions of a scalar conservation law may increase at an arbitrarily large rate. The entropy functional is defined, instead, in terms of the L_1 distance between two solutions as well as the variation of each solution (Definition 4). The entropy functional is constructed by directly making use of the nonlinearity of the flux function $f(u)$.

The L_2 functional yields the estimate on the third order of shocks. Our entropy functional yields estimates for the difference of the third orders of shocks pertaining to different solutions (Claim 1 and Lemma 6.3). This allows us to control part of the effect (on the L_1 distance) of the variation of the j-Riemann invariant across the i-shocks, $i \neq j$, in the solutions. We need to introduce another functional $L_h(t)$ (Definition 2) for the complete control of the third order variation just mentioned. The description of this functional involves the decomposition of the 'domain of influence' of a given shock (Definition 1). The main point is to guarantee that some of the effect of the minor variation of a shock on the L_1 distance is conservative. This accounts for naming the functional $L_h(t)$ the 'hamiltonian' when taken together with the linear functional $L(t)$, which is the main linear functional measuring the L_1 distance of the two solutions.

There are two simplified situations, for which the functionals are easier to define. In [16], we consider the case when one of the solutions is a constant state. The functional, besides the basic linear $L(t)$ measuring the L_1 distance, consists of $Q_d(t)$ and the traditional entropy integral.

Another simplified situation occurs when the Hugoniot curves are identical to the rarefaction curves. In this case, we only need $L(t)$ and $Q_d(t)$. The situation is simple because there is a canonical way to identify scalar conservation laws to govern waves pertaining to each characteristic family [18].

In Section 2 we describe a simplified version of the wave tracing scheme and the analysis of wave interaction and cancellation; cf. [16]. In each small time strip in the wave tracing scheme, the approximate solutions are reduced to a linear superposition of nonlinear waves propagating with constant strengths and speeds. In Section 3, we define a functional $\bar{H}(t)$ for the approximate solutions constructed by Glimm's scheme. The corresponding functional $H(t)$ for the simplified approximate solutions in the wave tracing scheme is also defined. The difference between these two functionals is mainly due to the randomness of the scheme and the wave interactions and cancellations. These have been studied in [9] and [14] and have been shown to vanish when the grids tend to zero and the random sequence defining the scheme is chosen to be equidistributed. Our main effort is to show that the functional $H(t)$ decreases in time. This is done through a series of lemmas, which are stated in Section 3 and proved in later sections.

In preparation for the main estimates on the functionals, we classify the third-order errors in Section 4. The primary goal is to identify those errors which will be controlled by the generalized entropy functional. Through this classification, it becomes clear why we need the hamiltonian functional $L_h(t)$.

In Sections 5 and 6, we carry out our main estimates on the derivative of the functional $H(t)$ at the time when it is differentiable. In the last section, Section 7, we finish our analysis by considering the jumps of the functional at the time of wave interactions using the Glimm functional. Our result yields the L_1 stability of the weak solutions constructed by Glimm's scheme.

In recent years, there has been much progress on this basic well-posedness problem. In [2], Bressan studied the problem when the two solutions are infinitesimally close. The analysis makes use of the fact that the topologies of the shock waves are close in this case. This is used to study the continuous dependence of the solutions on its initial data for 2×2 systems in [1] and for $n \times n$ systems in [3]. The idea is to construct a Riemann semigroup by homotopically deforming one solution to the other. This line of approach requires the monitoring of the changes of the topology of shocks in the approximate solutions. Thus a clever construction procedure of approximation is needed.

We have obtained a robust way of measuring the L_1 distance between the solutions and therefore we do not require any particular approximation scheme. In fact, our analysis would apply to any approximate scheme based on the characteristic method; cf. [5] and [8]. The L_1 stability yields trivially the uniqueness of solutions for the initial value problem for such schemes. Somewhat more general uniqueness formulations have been given by various authors ([3], [4] and references therein). For attempts on the uniqueness based on the L_2 norm, see [19], [15], [6], [7], and [13]. For comments on noncontractiveness in the L_1 norm, see [21].

We have succeeded in our program for 2×2 systems of conservation laws when each of the characteristics is either genuinely nonlinear or linear degenerate. The general situation is being pursued by the authors.

2. Wave tracing

In this section we briefly recall Glimm's scheme and the wave tracing method and reformulate the basic theorems for 2×2 systems for later uses.

Choose any mesh lengths $\Delta x = r$ and $\Delta t = s$, $\frac{r}{s}$ bounded, and satisfying the Courant-Friedrich-Lewy condition:

$$\frac{r}{s} \geq 2 \sup_{i=1,2} |\lambda_i(u)|,$$

for all u under consideration. The approximate solution $u_r(x,t;a_m)$ is constructed inductively according to a prechosen random sequence $\{a_m\}_{m=1}^{\infty}$, $0 < a_m < 1$, as follows: Set

$$u_r(x,0;a_m) = u_0(ir), \qquad \text{for} \quad ir < x < (i+1)r.$$

Suppose that $u_r(x,t;a_m)$ has been defined for $t < js$. Then we set

$$u_r(x,js;a_m) = u_r((i+a_j)r - 0, js - 0; a_m), \quad ir < x < (i+1)r,$$

for any $i = 0, \pm 1, \pm 2, \cdots$. Thus $u_r(x, js; a_m)$ is a step function of x with possible discontinuities at (ir, js). We then define $u_r(x, t; a_m)$, $js < t < (j+1)s$, by resolving these discontinuities (Lax [11]), so that in the zone $js < t < (j+1)s$ the approximate solution is an exact solution and consists of elementary waves issued from (ir, js), $i = 0, \pm 1, \pm 2, \cdots$.

For simplicity, we assume that the system (1.1) is genuinely nonlinear. The rarefaction curve $R_i(u_0)$ is the integral curve of $r_i(u)$ through a given state u_0, and the shock curve $S_i(u_0)$ is the Rankine-Hugoniot curve which is tangent to $R_i(u_0)$ at u_0 and satisfies, for any $u \in S_i(u_0)$,

$$(u - u_0)\sigma(u, u_0) = f(u) - f(u_0),$$

for some $\sigma(u, u_0)$, the shock speed, satisfying

$$\lim_{u \to u_0} \sigma(u, u_0) = \lambda_i(u_0).$$

These curves are divided into

$$R_i^+(u_0)(\text{resp. } R_i^-(u_0)) = \{u \in R_i(u_0) \mid \lambda_i > (\text{resp. } <)\, \lambda_i(u_0)\},$$
$$S_i^+(u_0)(\text{resp. } S_i^-(u_0)) = \{u \in S_i(u_0) \mid \sigma(u, u_0) > (\text{resp. } <)\, \lambda_i(u_0)\}.$$

A state u can be connected to u_0 on the left by an i-rarefaction (or i-shock) wave if $u \in R_i^+(u_0)$ (or $u \in S_i^-(u_0)$). The shock wave (u_0, u), $u \in S_i^-(u_0)$, is stable in the sense of Lax: $\lambda_i(u_0) > \sigma(u, u_0) > \lambda_i(u)$. The Riemann problem (u_l, u_r) for (1.1) with two constant initial states:

$$u(x, 0) = \begin{cases} u_l, & \text{for} \quad x < 0, \\ u_r, & \text{for} \quad x > 0, \end{cases}$$

is solved by finding vectors $u_i, i = 0, 1, 2$, $u_0 = u_l$, $u_2 = u_r$, $u_1 \in S_1^-(u_0) \cup R_1^+(u_0)$, so that the solution consists of elementary i-waves (u_{i-1}, u_i), $i = 1, 2$. For any nonsingular parameter μ_i along $S_i^- \cup R_i^+$, for example arclength, we define the strength of the i-wave in (u_l, u_r) as

$$(u_l, u_r)_i = \mu_i(u_i) - \mu_i(u_{i-1}), \quad i = 1, 2.$$

We choose μ_i so that shock waves have negative strengths and rarefaction waves have positive strengths. The following theorem on wave interaction is due to Glimm [9].

Theorem 2.1. *For any nearby states u_l, u_m, u_r, there exist bounds $0(1)$ depending only on the system (1.1), such that*

$$(u_l, u_r)_i = (u_l, u_m)_i + (u_m, u_r)_i + 0(1)Q(u_l, u_m, u_r), \quad i = 1, 2, \tag{2.1}$$

where Q measures the potential amount of interaction and is defined as follows:

A j-wave on the left interacts with a k-wave on the right if either $j > k$, or $j = k$ and at least one of the waves is a shock wave. We set

$$Q(u_l, u_m, u_r) = \sum_{j,k} (u_l, u_m)_j (u_m, u_r)_k,$$

the summation being over all interacting waves.

In the Glimm scheme the waves heading toward a grid point (ir, js) are solutions of two Riemann problems, say, (u_l, u_m) and (u_m, u_r). We denote by $Q(\Delta) = Q(\Delta_{i,j}) \equiv Q(u_l, u_m, u_r)$ the amount of interaction and by $C_i(\Delta) = |(u_l, u_m)_i| +$

$|(u_m, u_r)_i| - |(u_l, u_m)_i + (u_m, u_r)_i|$ the amount of cancellation that occurs. (2.1) can be rewritten as:

$$|(u_l, u_r)_i| = |(u_l, u_m)_i| + |(u_m, u_r)_i| + C_i(\Delta) + 0(1)Q(\Delta).$$

The following theorem on wave interaction and cancellation is due to Glimm [9] and Glimm-Lax [10].

Theorem 2.2. *Suppose that the initial data* (1.2) *has sufficiently small total variation denoted by T.V. Then*

(a) *total variation* $\{u_r(\cdot, t) \mid -\infty < x < \infty\} \leq 2T.V.$, *for all t;*
(b) $Q = \sum_\Delta Q(\Delta) \leq 2(T.V.)^2$*;*
(c) $C = \sum_\Delta C(\Delta) \leq T.V. + Q$*;*
(d) $F(J_2) - F(J_1) \leq -\sum_\Delta (C(\Delta) + Q(\Delta))$,

where J_2 is an immediate successor of J_1, and in (d) *Δ is the diamond between J_1 and J_2, and $F(J) = L(J) + KQ(J)$, $Q(J) = \sum\{|ab| \mid a$ and b are strengths of interacting waves crossing $J\}$, $L(J) = \sum\{|a| \mid a$ is the strength of wave crossing $J\}$, and K is a large constant. Moreover, for almost all choices of random sequence* **a**, *the approximate solutions $\{u_r(x, t)\}$ tend to an exact solution locally in L_1 for a sequence of mesh sizes tending to zero.*

For the consistency of the scheme, the random sequence $\mathbf{a} = \{a_m\}_{m=1}^\infty$ needs to be equidistributed in $(0, 1)$, that is,

$$\lim_{k\to\infty} \frac{B(\mathbf{a}, k, I)}{k} = \mu(I),$$

for any subinterval I of $(0, 1)$. Here $B(\mathbf{a}, k, I)$ denotes the number of m, $1 \leq m \leq k$, with $a_m \in I$, and $\mu(I)$ is the length of I.

For any fixed time T, we choose large integers N, M such that

$$(N - 1)Ms < T \leq NMs.$$

Let $\epsilon = \frac{1}{N}$, and divide the interval $(0, 1)$ into N subintervals of equal length ϵ. Let $\{I_i\}_{i=1}^N$ be the collection of unions of any such subintervals. We set

$$\delta = \sup_{1\leq p\leq N, 1\leq i\leq N} \left(\frac{B(a_{m+(p-1)M}, M, I_i)}{M} - \mu(I_i)\right), \tag{2.2}$$

which tends to zero as $M \to \infty$ for any fixed N.

For a given time zone $\Lambda_p \equiv \{t : (p - 1)Ms \leq t < pMs\}$, $1 \leq p \leq N$, the elementary waves $\{v_k^h(i, j), \lambda_k^h(i, j)\}$ in the approximate solution are partitioned into subwaves, and each rarefaction wave is divided into rarefaction shocks with each strength being less than ϵ. The subwaves are a disjoint union of surviving waves $\{\tilde{v}_k^h(i, j), \tilde{\lambda}_k^h(i, j)\}$ and those either created or cancelled due to interactions, denoted by $\{\tilde{\tilde{v}}_k^h(i, j), \tilde{\tilde{\lambda}}_k^h(i, j)\}$. In the following theorem we generalize slightly that of Liu [14] (cf. [16]), and replace the surviving subwaves with a simplified approximate wave pattern $\{\bar{v}_k^h(i, j), \lambda^*(\bar{v}_k^h(i, j))\}$ of linear superposition of nonlinear waves with the following properties:

Theorem 2.3. *There exists a simplified wave pattern*

$$\bar{u}_r \equiv \{\bar{v}_k^h(i, (p - 1)M), \lambda^*(\bar{v}_k^h(i, j))\}$$

in Λ_p such that each wave propagates along the straight line $l(\bar{v}_k^h)$ with slope

$\lambda^*(\bar{v}_k^h(i,j))$. *Moreover, it satisfies the following properties:*

(i) $\sum_{i,h,k} |\tilde{\tilde{v}}_k^h(i,j)| \le [Q(\Lambda_p) + C(\Lambda_p)]$.

(ii) *There is a one to one correspondence between* $\{\tilde{v}_k^h(i,j), \tilde{\lambda}_k^h(i,j)\}$ *and* $\{\bar{v}_k^h(i,(p-1)M), \lambda^*(\bar{v}_k^h(i,(p-1)M))\}$,

$$(\tilde{v}_k^h(i,j), \tilde{\lambda}_k^h(i,j)) \to (\bar{v}_k^h(i_j,(p-1)M), \lambda^*(\bar{v}_k^h(i_j,(p-1)M))),$$

such that $l(\bar{v}_k^h)$ *connects the grid points of* $\{\tilde{v}_k^h(i,j)\}$ *at* $t = (p-1)Ms$ *and* $t = pMs$. *Furthermore, if* $\tilde{v}_j = (u_j^-, u_j^+)$ *and* $\tilde{v}_{j+1} = (u_{j+1}^-, u_{j+1}^+)$ *are two adjacent surviving waves and* $\tilde{v}_j$ *is to the left of* $\tilde{v}_{j+1}$, *then the wave in the simplified wave pattern corresponding to* $\tilde{v}_{j+1}$ *is* $\bar{v}_{j+1} = (u_j^+, u_{j+1}^+)$.

(iii) $\sum_{i,h,k} |\bar{v}_k^h(i_j,(p-1)M) - \tilde{v}_k^h(i,j)| = 0(1)(Q(\Lambda_p) + C(\Lambda_p))$.

(iv)

$$\sum_{i,h,k} |\bar{v}_k^h(i,(p-1)M)| \max_{(p-1)M \le j \le pM} |\lambda_k^h(i_j,j) - \lambda^*(v_k^h(i,j))| \le 0(1)(Q(\Lambda_p) + C(\Lambda_p) + T.V.(\delta + \epsilon)).$$

(v) $\int_{-\infty}^{\infty} |u_r(x,t) - \bar{u}_r(x,t)| dx = 0(1)(Q(\Lambda_p) + C(\Lambda_p) + T.V.(\delta + \epsilon))Ms$, $(p-1)Ms \le t < pMs$, *where the bounds* $0(1)$ *are independent of* i, j, *and* r. *Here* δ *is defined in* (2.2). *The term* $0(1)T.V.\delta$ *is due to the equidistribution of the random sequence* $\{a_m\}_{m=1}^{\infty}$ *for* $(p-1)M \le m \le pM$ *and the wave speed* $\lambda_k^h(i,j)$ *being replaced by the slope of* $l(v_k^h(i,j))$.

3. Functionals and the main theorem

Let r, s be the first and second Riemann invariants, respectively. The strength of an i-wave is measured by the i-Riemann invariant, $i = 1, 2$. Let α and β denote a 1-wave and 2-wave, respectively, and, without any ambiguity, also their strengths.

For a simplified wave pattern, the Riemann invariants are step functions of the spatial coordinate x. Since the rarefaction curve $R^+(u_0)$ and the shock curve $S^-(u_0)$ have a second-order tangency at $u = u_0$, the secondary strength, the *minor waves*, of the i-waves measured by the j-th Riemann invariant, $j \ne i$, are

$$|\hat{\alpha}| = \begin{cases} 0, & \alpha \ge 0, \\ 0(1)|\alpha|^3, & \alpha < 0, \end{cases} \qquad |\hat{\beta}| = \begin{cases} 0, & \beta \ge 0, \\ 0(1)|\beta|^3, & \beta < 0. \end{cases} \tag{3.1}$$

For definiteness, we assume that they jump down from left to right.

Consider two approximate solutions $u^1(x,t)$ and $u^2(x,t)$ constructed by Glimm's scheme with $u^1(x,t) - u^2(x,t) \in L_1$. We use $\alpha_i^p(t)$ to denote 1-waves and $\beta_i^p(t)$ for 2-waves in $u^p(x,t)$, $p = 1, 2$, located at $x(\alpha_i^p(t))$ and $x(\beta_i^p(t))$ at time t. The Riemann invariants corresponding to $u^p(x,t)$ are denoted by $r^p(x,t)$ and $s^p(x,t)$, $p = 1, 2$. For any wave γ, we write $\gamma = ((r_-(\gamma), s_-(\gamma)), (r_+(\gamma), s_+(\gamma)))$ to indicate the left and right states.

The linear functional $L(t)$ is equivalent to the L_1 distance of $u^1(x,t)$ and $u^2(x,t)$ and is defined as follows:

$$\begin{aligned} L(t) &\equiv \int_{\mathbf{R}} (\gamma^-(x,t) + \gamma^+(x,t)) dx, \\ \gamma^-(x,t) &\equiv |r^1(x,t) - r^2(x,t)|, \\ \gamma^+(x,t) &\equiv |s^1(x,t) - s^2(x,t)|. \end{aligned} \tag{3.2}$$

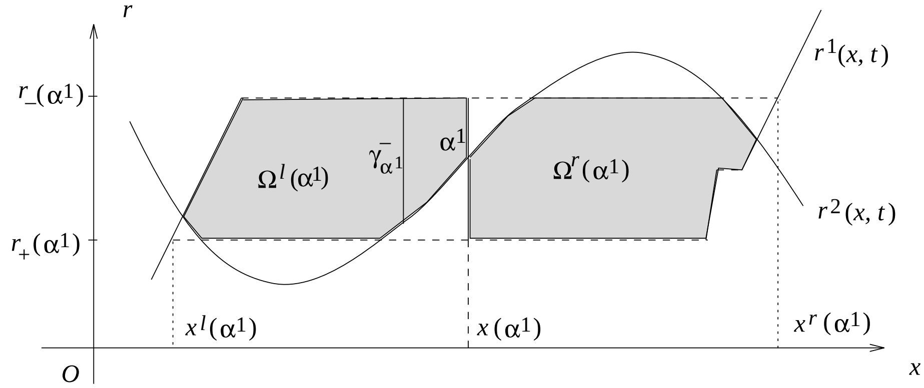

FIGURE 3.1

For the definition of the other parts of the functional $H(t)$, we need the notion of *domain of influence* $\Omega(\gamma)$ for each shock wave or minor wave γ:

Definition 1 $(\Omega(\gamma))$**.** For any fixed time t consider the graphs of $r^1(x,t)$ and $r^2(x,t)$ on the r-x plane. For a 1-shock or minor wave of the second family γ in $u^1(x,t)$, we set $x = x^l(\gamma)$ (or $x^r(\gamma)$) to be the largest (or smallest) x with $r^1(x,t) = r_+(\gamma)$ (or $r^1(x,t) = r_-(\gamma)$). The subset $\Omega(\alpha^1) = \Omega^l(\alpha^1) \cup \Omega^r(\alpha^1)$ of the region between the two graphs and $x^l(\gamma) < x < x^r(\gamma)$ is defined as follows:

Let $\gamma = \alpha^1$ be a 1-shock wave in $u^1(x,t)$. Then $\Omega^l(\alpha^1)$ is the maximal region to the left of α^1 with the property that $x^l(\alpha^1) < x < x(\alpha^1)$, $r_+(\alpha^1) < r < r_-(\alpha^1)$ for $(x,r) \in \Omega^l(\alpha^1)$. The region $\Omega^r(\alpha^1)$ is defined similarly. The part of $\gamma^-(x,t)$ which lies in $\Omega(\alpha^1)$ is denoted by $\gamma^-_{\alpha^1}(x,t)$; cf. Figure 3.1.

The definition of $\Omega(\theta)$ for another shock or minor wave θ is similar.

Since the strength of a minor wave $\hat{\gamma}$ is of cubic order of the strength of the shock wave γ, the function $C_\gamma = \frac{|\hat{\gamma}|}{|\gamma|^3}$ is positive and depends continuously on the states of γ. In order to study the nonlinear effect of the minor waves on the L_1 distance of two solutions, we introduce the following functional $L_h(t)$.

Definition 2 $(L_h(t))$**.** For a 2-shock wave β^1 in $u^1(x,t)$, set:

$$L_h(\beta^1) = C_{\beta^1} \int_{x^l(\beta^1)}^{x^r(\beta^1)} \xi_{\beta^1}(x,t)(\gamma^+_{\beta^1}(x,t))^3 dx,$$

where $\xi_{\beta^1}(x,t)$ is defined as follows:

(1) $x(\beta^1) < x < x^r(\beta^1)$. Let $\{\theta^1_i\}$, $i = 1, 2, \cdots, n$, be a sequence of consecutive 2-shock waves in $u^1(x,t)$ with $(x(\theta^1_i), x^r(\theta^1_i)) \subset (x(\beta^1), x^r(\beta^1))$ (cf. Figure 3.2). Then:

(i) when

$$r^1(x,t) \geq r^2(x,t) - \sum_{i=1}^{n} C_{\theta^1_i}(\gamma^+_{\theta^1_i}(x,t))^3,$$

set $\xi_{\beta^1}(x,t) = 1$;

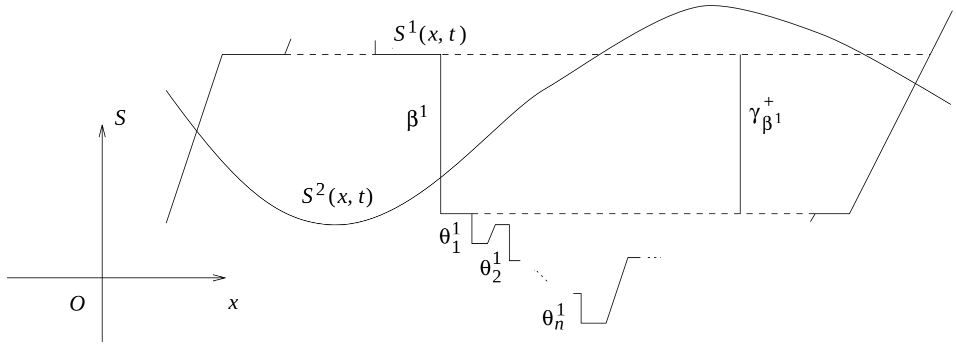

FIGURE 3.2

(ii) when

$$\sum_{i=1}^{n} C_{\theta_i^1}(\gamma^+_{\theta_i^1}(x,t))^3 \le r^2(x,t) - r^1(x,t) \le \sum_{i=1}^{n} C_{\theta_i^1}(\gamma^+_{\theta_i^1}(x,t))^3 + C_{\beta^1}(\gamma^+_{\beta^1}(x,t))^3,$$

set

$$\xi_{\beta^1}(x,t) = \frac{\sum_{i=1}^{n} C_{\theta_i^1}(\gamma^+_{\theta_i^1}(x,t))^3 + C_{\beta^1}(\gamma^+_{\beta^1}(x,t))^3 - r^2(x,t) + r^1(x,t)}{C_{\beta^1}(\gamma^+_{\beta^1}(x,t))^3};$$

(iii) when

$$r^2(x,t) - r^1(x,t) \ge \sum_{i=1}^{n} C_{\theta_i^1}(\gamma^+_{\theta_i^1}(x,t))^3 + C_{\beta^1}(\gamma^+_{\beta^1}(x,t))^3,$$

set $\xi_{\beta^1}(x,t) = 0$.

(2) $x^l(\beta^1) < x < x(\beta^1)$. We also have a sequence of consecutive 2-shock waves $\{\theta_i^1\}$, $i = 1, 2, \cdots, n$, in $u^1(x,t)$ with $(x^l(\theta_i^1), x(\theta_i^1)) \subset (x^l(\beta^1), x(\beta^1))$, and:

(i) when

$$r^2(x,t) \ge r^1(x,t) - \sum_{i=1}^{n} C_{\theta_i^1}(\gamma^+_{\theta_i^1}(x,t))^3,$$

set $\xi_{\beta^1}(x,t) = 1$;

(ii) when

$$\sum_{i=1}^{n} C_{\theta_i^1}(\gamma^+_{\theta_i^1}(x,t))^3 \le r^1(x,t) - r^2(x,t) \le \sum_{i=1}^{n} C_{\theta_i^1}(\gamma^+_{\theta_i^1}(x,t))^3 + C_{\beta^1}(\gamma^+_{\beta^1}(x,t))^3,$$

set

$$\xi_{\beta^1}(x,t) = \frac{\sum_{i=1}^{n} C_{\theta_i^1}(\gamma^+_{\theta_i^1}(x,t))^3 + C_{\beta^1}(\gamma^+_{\beta^1}(x,t))^3 - r^1(x,t) + r^2(x,t)}{C_{\beta^1}(\gamma^+_{\beta^1}(x,t))^3};$$

(iii) when

$$r^1(x,t) - r^2(x,t) \ge \sum_{i=1}^{n} C_{\theta_i^1}(\gamma^+_{\theta_i^1}(x,t))^3 + C_{\beta^1}(\gamma^+_{\beta^1}(x,t))^3,$$

set $\xi_{\beta^1}(x,t) = 0$.

These quantities can be defined in a similar way for $\theta = \beta^2, \alpha^1, \alpha^2$.

Then we set

$$\eta^+(x,t) = \sum_{\alpha(t)} \eta_{\alpha(t)}(x,t), \qquad \eta^-(x,t) = \sum_{\beta(t)} \eta_{\beta(t)}(x,t),$$

where

$$\eta_\theta(x,t) = \begin{cases} 0, & x \le x^l(\theta), \quad \text{or} \quad x > x^r(\theta), \\ C_\theta \xi_\theta(x,t)(\gamma_\theta^+(x,t))^3, & x^l(\theta) < x \le x^r(\theta), \end{cases}$$

and the summations are over all waves θ in the two solutions. Finally our functional is defined as:

$$L_h(t) = \int_{\mathbf{R}} (\eta^-(x,t) + \eta^+(x,t))dx.$$

For later use, we introduce the following notation for any wave θ:

$$L_h^r(\theta) = \int_{x(\theta)}^{x^r(\theta)} \eta_\theta(x,t)dx, \quad L_h^l(\theta) = \int_{x^l(\theta)}^{x(\theta)} \eta_\theta(x,t)dx.$$

Clearly,

$$L_h(t) = \sum_\theta (L_h^r(\theta) + L_h^l(\theta)).$$

The linear terms $L(t)$ and $L_h(t)$ will be considered together from now on:

$$\tilde{L}(t) = L(t) + L_h(t) = \int_{\mathbf{R}} (\Theta^-(x,t) + \Theta^+(x,t))dx,$$

$$\Theta^\pm = \gamma^\pm(x,t) + \eta^\pm(x,t).$$

Our next functional $Q_d(t)$ measures the coupling of waves and is the same as in [16] save for the addition of $L_h(t)$ to the linear functional $L(t)$:

Definition 3 ($Q_d(t)$)**.** For each 1-wave $\alpha^p(t)$ in $u^p(x,t)$, $p = 1, 2$, set

$$Q_d^-(\alpha^p(t)) = |\alpha^p(t)| \int_{-\infty}^{x(\alpha^p(t))} \Theta^+(x,t)dx.$$

For each 2-wave $\beta^p(t)$ in $u^p(x,t)$, $p = 1, 2$, set

$$Q_d^+(\beta^p(t)) = |\beta^p(t)| \int_{x(\beta^p(t))}^{\infty} \Theta^-(x,t)dx.$$

Set

$$Q_d(t) = Q_d^-(t) + Q_d^+(t),$$

$$Q_d^-(t) = \sum_{\alpha(t)} Q_d^-(\alpha(t)), \; Q_d^+(t) = \sum_{\beta(t)} Q_d^+(\beta(t)),$$

with the summations over all waves in the two solutions.

Similar to the above, the addition of $L_h(t)$ to the linear functional induces a slight modification to the generalized entropy functional in [17] as mentioned in the Introduction:

Definition 4 ($\Delta(t)$)**.** Suppose the function $\Delta r(x,t) = r^1(x,t) - r^2(x,t)$ changes signs at z_i^r: $\Delta r(x,t) \ge 0$ for $x \in (z_i^r, z_{i+1}^r)$, i odd; and $\Delta r(x,t) \le 0$ for $x \in (z_i^r, z_{i+1}^r)$, i even.

Consider a 1-wave $\alpha(t)$ in $u^p(x,t)$. We have

(I) $p = 1$.

(i) For $x(\alpha(t)) \in (z^r_j, z^r_{j+1}]$, j odd, set

$$\begin{aligned}\Delta(\alpha(t)) &= |\alpha(t)|\{\int_{x(\alpha(t))}^{z^r_{j+1}} \gamma^-(x,t)dx + \sum_{l>j, l=\text{odd}} \int_{z^r_l}^{z^r_{l+1}} \gamma^-(x,t)dx \\ &\quad + \sum_{l<j, l=\text{even}} \int_{z^r_l}^{z^r_{l+1}} \gamma^-(x,t)dx + U^1(\alpha)\},\end{aligned}$$

where $U^1(\alpha) = \sum_{\beta(t)} U^1_\alpha(\beta(t))$ and

$$U^1_\alpha(\beta(t)) = \begin{cases} L^r_h(\beta(t)) + L^l_h(\beta(t))\ |_{x<x(\alpha)}, & x(\alpha) < x(\beta(t)), \\ L^l_h(\beta(t)) + L^r_h(\beta(t))\ |_{x\geq x(\alpha)}, & x(\alpha) \geq x(\beta(t)), \end{cases}$$

where the summation is over all 2-shocks $\beta(t)$ in the two solutions at time t.

(ii) For $x(\alpha(t)) \in (z^r_j, z^r_{j+1}]$, j even, set

$$\begin{aligned}\Delta(\alpha(t)) &= |\alpha(t)|\{\int_{z^r_j}^{x(\alpha(t))} \gamma^-(x,t)dx + \sum_{l>j, l=\text{odd}} \int_{z^r_l}^{z^r_{l+1}} \gamma^-(x,t)dx \\ &\quad + \sum_{l<j, l=\text{even}} \int_{z^r_l}^{z^r_{l+1}} \gamma^-(x,t)dx + U^1(\alpha)\}.\end{aligned}$$

Similarly, we can define $\Delta(\alpha(t))$ when α is in $u^2(x,t)$, and $\Delta(\beta(t))$ for 2-waves $\beta(t)$. Thus our generalized entropy functional is defined as follows:

$$\Delta(t) = \Delta^-(t) + \Delta^+(t),$$

$$\Delta^-(t) = \sum_{\alpha(t)} \Delta(\alpha(t)),\ \ \Delta^+(t) = \sum_{\beta(t)} \Delta(\beta(t)),$$

with the summations over all waves in the solutions at time t.

The definition of $L_h(t)$ depends on the formation of the region $\Omega(\gamma)$ for a shock wave or minor wave γ. This induces jumps of $L_h(t)$ at $t = n\Delta t,\ 1 \leq n \leq N-1$, and also across the interaction points in $\Lambda_p = \{(x,t):\ x \in R^1,\ (p-1)Ms < t < pMs\}$. These jumps are controlled by the following functional:

Definition 5 ($D(t)$)**.** For any 1-shock waves α^p_i and α^p_j, $j < i$, let $x^r_i(\alpha^p_j) = \min\{x^r(\alpha^p_j), x(\alpha^p_i)\}$; and for any 2-shock wave β^p_k, let $x^r_i(\hat\beta^p_k) = \min\{x^r(\hat\beta^p_k), x(\alpha^p_i)\}$ if $x(\alpha^p_i) > x(\beta^p_k)$, and $x^r_i(\hat\beta^p_k) = x^l(\hat\beta^p_k)$ otherwise. Similarly, for any shock waves β^p_i and β^p_j, $i < j$, let $x^l_i(\beta^p_j) = \max\{x^l(\beta^p_j), x(\beta^p_i)\}$; and for any shock wave α^p_k, let $x^l_i(\hat\alpha^p_k) = \max\{x^l(\hat\alpha^p_k), x(\beta^p_i)\}$ if $x(\beta^p_i) < x(\alpha^p_k)$, and $x^l_i(\hat\alpha^p_k) = x^r(\hat\alpha^p_k)$ otherwise. Set

$$D(t) = D^-(t) + D^+(t),$$

$$\begin{aligned}D^-(t) &= \sum_{\alpha^p_i, \alpha^p_j, i>j} \int_{x^l_i(\alpha^p_j)}^{x^r_i(\alpha^p_j)} \gamma^-_{\alpha^p_j}(x,t)(\gamma^-_{\alpha^p_i}(x,t))^2 dx \\ &\quad + \sum_{\alpha^p_i, \beta^p_k, x(\alpha^p_i)>x(\beta^p_k)} \int_{x^l(\hat\beta^p_k)}^{x^r_i(\hat\beta^p_k)} \gamma^-_{\hat\beta^p_k}(x,t)(\gamma^-_{\alpha^p_i}(x,t))^2 dx \\ &\quad + \sum_{\alpha^p_i} \int_{x^l(\alpha^p_i)}^{x(\alpha^p_i)} (\gamma^-(x,t) - \sum_{\alpha^p_j} \gamma^-_{\alpha^p_j}(x,t) - \sum_{\beta^p_k} \gamma^-_{\hat\beta^p_k}(x,t))(\gamma^-_{\alpha^p_i}(x,t))^2 dx,\end{aligned}$$

$$
\begin{aligned}
D^+(t) &= \sum_{\beta_i^p,\beta_j^P,i<j}\int_{x_i^l(\beta_j^p)}^{x^r(\beta_j^p)}\gamma^+_{\beta_j^p}(x,t)(\gamma^+_{\beta_i^p}(x,t))^2dx \\
&+ \sum_{\beta_i^p,\alpha_k^p,x(\beta_i^p)<x(\alpha_k^p)}\int_{x_i^l(\hat\alpha_k^p)}^{x^r(\hat\alpha_k^p)}\gamma^+_{\hat\alpha_k^p}(x,t)(\gamma^+_{\beta_i^p}(x,t))^2dx \\
&+ \sum_{\beta_i^p}\int_{x(\beta_i^p)}^{x^r(\beta_i^p)}(\gamma^+(x,t)-\sum_{\beta_j^p}\gamma^+_{\beta_j^p}(x,t)-\sum_{\alpha_k^p}\gamma^+_{\hat\alpha_k^p}(x,t))(\gamma^+_{\beta_i^p}(x,t))^2dx,
\end{aligned}
$$

where the summations are over all waves in the two solutions.

With the above functionals, we can define the desired nonlinear functionals $\bar H(t)$ for the approximate solutions in Glimm's scheme, and $H(t)$ for the simplified wave patterns in the wave tracing scheme: For $u_r^i(x,t)$, $0\le t\le T$, $i=1,2$, we define

$$\bar H(t)=(1+k_1\bar F(t))\bar{\tilde L}(t)+k_2(\bar Q_d(t)+\bar\Delta(t))+k_3\bar D(t),$$

where $F(t)$ is the sum of the two Glimm's functionals for $u^1(x,t)$ and $u^2(x,t)$ defined in Theorem 2.2, and k_1, k_2 and k_3 are positive constants to be chosen later. Here the " $^-$ " denotes that all the wave patterns used in the definition of the functionals are from $u_r^i(x,t)$, $i=1,2$. For the simplified wave patterns of Section 2, we define

$$H(t)=(1+k_1F((p-1)Ms))\tilde L(t)+k_2(Q_d(t)+\Delta(t))+k_3D(t),$$

for $(p-1)Ms\le t\le pMs$, $p=1,2,\cdots,N$. Notice that besides the differences of the wave patterns for their definitions, the functional $H(t)$ differs from $\bar H(t)$ in that the Glimm functional $F(t)$ in $H(t)$ is replaced by its value $F((p-1)Ms)$ at the beginning of the time $t=(p-1)Ms$ in Λ_p.

Now we can state the main lemmas in this paper; their proofs will be given in the following sections. Our main theorem, Theorem 3.1, will be an easy consequence of these lemmas.

We denote the open time interval $((p-1)Ms,pMs)$ by I_p. According to the construction of the approximate solutions for the simplified wave patterns, I_p is a union of two disjoint sets, denoted by I_p^c and I_p^d, where I_p^d are the countable interaction times. $H(t)$ is differentiable for $t\in I_p^c$. For simplicity in presentation, we may set the approximate solutions to be constants when $x\gg 1$ and $x\ll 1$ so that I_p^d are the finite interaction times.

Main Lemma 1. *Suppose that the total variation of the initial data T.V. is sufficiently small, and $u_0^1(x)-u_0^2(x)\in L_1(R)$. Then, for $t\in I_p^c$,*

$$\frac{d}{dt}H(t)\le ce(\Lambda_p),\quad p=1,2,\cdots,N, \tag{3.3}$$

for some choices of the constants k_1, k_2 and k_3 and $e(\Lambda_p)=Q(\Lambda_p)+C(\Lambda_p)+T.V.(\delta+\epsilon)$.

Hereafter c denotes the generic positive constant which is independent of T and s. The function δ is given in (2.2) and ε is the upper bound for the strength of each rarefaction shock in the simplified wave pattern. Both vanish as the mesh sizes tend to zero. For the jump of the functional $H(t)$ crossing each interaction time $t\in I_p^d$, we have the following lemma.

Main Lemma 2. *Under the hypotheses of Main Lemma 1, there exist constants k_1, k_2 and k_3 independent of T and s such that, for $t \in I_p^d$,*

$$H(t+) - H(t-) \le c(F(t-) - F(t+))L(t). \tag{3.4}$$

The following lemma shows that the difference between $H(t)$ and $\bar{H}(t)$ is a sum of a time-decreasing functional and a term which vanishes as the mesh size s tends to zero.

Main Lemma 3. *Under the hypotheses of Main Lemma 1, we have*

(3.5)
$$\begin{aligned}&\bar{H}(pMs+) - H(pMs-) + H((p-1)Ms+) - \bar{H}((p-1)Ms+)\\ &\qquad \le ce(\Lambda_p)Ms - \frac{k_1}{2}(Q(\Lambda_p) + C(\Lambda_p))L((p-1)Ms+), \quad p = 1, 2, \cdots, N.\end{aligned}$$

Theorem 3.1. *Under the hypotheses of Main Lemma 1, for the exact weak solutions $u^1(x,t)$ and $u^2(x,t)$ of (1.1) constructed by Glimm's scheme, there exists a constant G independent of time such that*

$$||u^1(x,t) - u^2(x,t)||_{L_1} \le G||u^1(x,s) - u^2(x,s)||_{L_1},$$

for any s, t, $0 \le s \le t < \infty$.

Proof. Without loss of generality, we will show that $||u^1(x,T) - u^2(x,T)||_{L_1} \le G||u^1(x,0) - u^2(x,0)||_{L_1}$ for any time T. By Main Lemmas 2 and 3, we have

(3.6)
$$H(pMs-) - H((p-1)Ms+) \le ce(\Lambda_p)Ms + c(Q(\Lambda_p) + C(\Lambda_p))L((p-1)Ms+);$$

(3.7)
$$\begin{aligned}&\bar{H}(pMs+) - H(pMs-) + H((p-1)Ms+) - \bar{H}((p-1)Ms+)\\ &\qquad \le -\frac{k_1}{2}(Q(\Lambda_p) + C(\Lambda_p))L((p-1)Ms+) + ce(\Lambda_p)Ms, \quad p = 1, 2, \cdots, N.\end{aligned}$$

Summing up (3.6)–(3.7) and choosing $k_1 \ge 2c$, we have

$$\bar{H}(pMs+) - \bar{H}((p-1)Ms+) \le ce(\Lambda_p)Ms, \quad p = 1, 2, \cdots, N. \tag{3.8}$$

Similarly, summing up (3.8) with respect to p from 1 to N yields

$$\bar{H}(T) \le \bar{H}(0) + c(Q(T) + C(T))Ms + cT.V.(\delta + \epsilon)T.$$

For fixed T, $(N-1)Ms < T < NMs$, we have $M, N \to \infty$ as the mesh size s tends to zero. Thus, by the definition of δ and ϵ, we have $\delta \to 0$ and $\epsilon \to 0$. Hence we have,

$$c(Q(T) + C(T))Ms + cT.V.(\delta + \epsilon)T = c(Q(T) + C(T))T/N + cT.V.(\delta + \epsilon)T \to 0,$$

as $s \to 0$.

Notice that for any fixed M and N, the functional $\bar{H}(t)$ is equivalent to the L_1 distance of the approximate solutions $\{u_r^1(x,t)\}$ and $\{u_r^2(x,t)\}$. Moreover, by Theorem 2.2, the approximate solutions $\{u_r^1(x,t)\}$ and $\{u_r^2(x,t)\}$ converge to the exact solutions locally in the L_1 norm. Consequently, there exists a constant G independent of T and s such that

$$||u^1(x,T) - u^2(x,T)||_{L_1} \le G||u^1(x,0) - u^2(x,0)||_{L_1}.$$

This completes the proof of the theorem. □

As an immediate consequence of the above theorem, we have the following uniqueness theorem.

Theorem 3.2. *Under the hypotheses of Main Lemma 1, the initial value problem for* (1.1) *has a unique weak solution constructed by Glimm's scheme.*

4. Third-order errors

The goal of this section is to study the error due to the third-order minor waves on the L_1 distance between the two solutions. The following Claims show that a certain difference of third-order terms between two solutions is dominated by a quantity which will be shown in Section 6 to be controllable by the derivative of the functionals $Q_d(t)$ and $\Delta(t)$. We will also show in the next section that this difference is the error produced by the derivative of $L(t) + L_h(t)$.

For any shock wave β^1 in $u^1(x,t)$, $\theta(\beta^1)$ denotes the part of a shock wave θ in $u^2(x,t)$ which lies in $\Omega(\beta^1)$. Without any ambiguity, $\theta(\beta^1)$ represents both the strength of the wave as well as the s-interval of its end states. For later use, $\bar{\theta}(\beta^1)$ denotes the maximum subinterval of $\theta(\beta^1)$ such that there does not exist x between $x(\beta^1)$ and $x(\theta)$ with $s^2(x) \in \bar{\theta}(\beta^1)$ (Figure 4.1). We denote by $s^b(\bar{\theta}(\beta^1))$ and $s^t(\bar{\theta}(\beta^1))$ the minimum and maximum s values of $\bar{\theta}(\beta^1)$, respectively. If θ is a minor wave we use the notation $\theta^f(\beta^1)$ and $\bar{\theta}^f(\beta^1)$ instead. Similar notation holds for the shock waves β^2, α^1 and α^2. $\bar{\theta}(\beta)$ can be understood as the portion of θ for which there is a corresponding part of β.

Remark 1. Notice that when a 1-shock wave α in one solution is crossed by the r-curve of the other solution, the L_1 contraction of the scalar equation for r yields a good quadratic term of the order $-|\alpha^t||\alpha^b|$ in $\frac{d}{dt}L^-(t)$. Here $|\alpha^t|$ and $|\alpha^b|$ denote the strengths of the upper and lower parts of α above and below the r-curve, respectively. The same is true for other shock waves.

We also denote the same dividing of a minor wave θ by θ^t and θ^b. Without any ambiguity, the above notation holds also when the shock wave or minor wave θ is not crossed by the Riemann invariant curve of the other solution, and in this case either θ^t or θ^b is zero. Below we introduce the quantities $\gamma_x^{\pm}$, $\gamma_\theta^{\pm}(x,t)$, in order to classify the relative positions of minor waves through $\gamma_{x(\theta)*}^{\pm}$. The classification is needed to estimate $d/dt(L(t) + L_h(t))$ and $d/dt(Q_d(t))$ in the subsequent sections.

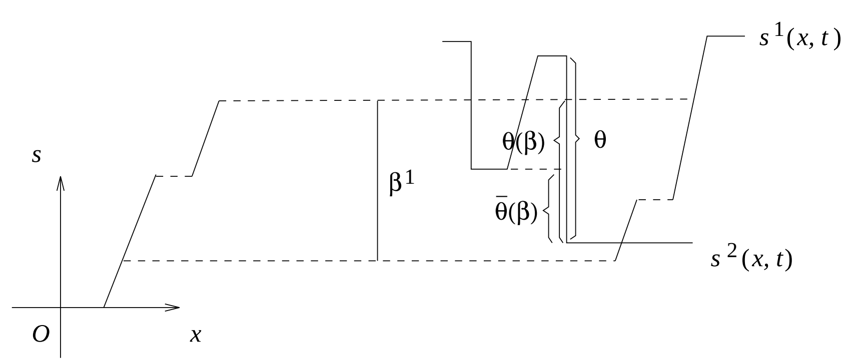

FIGURE 4.1

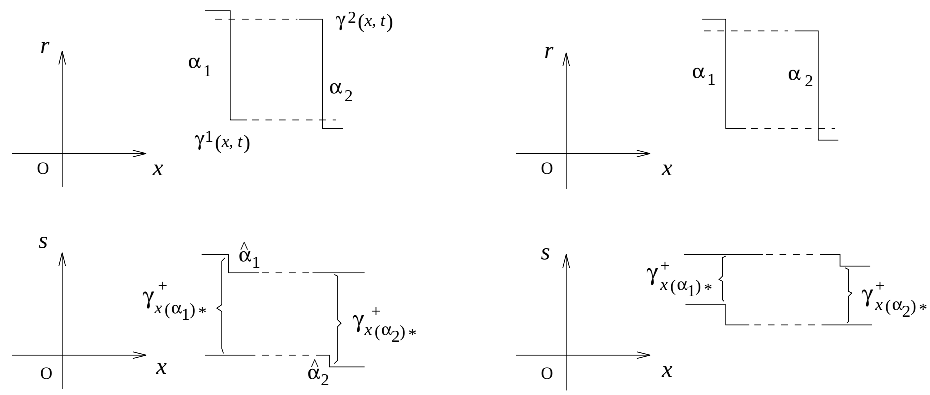

FIGURE 4.2

In the following Claims we express the third-order errors in terms of these quantities (Figure 4.2).

For each location x we have $\gamma_x^\pm$ and for each wave θ we have $\gamma_\theta^\pm$ defined as follows:

$$\gamma_x^+ (\text{or } \gamma_x^-) = 0, \qquad \text{if } (s^1_{x+} - s^2_{x+})(s^1_{x-} - s^2_{x-})(\text{or } (r^1_{x+} - r^2_{x+})(r^1_{x-} - r^2_{x-})) \le 0;$$
$$\gamma_x^+ (\text{or } \gamma_x^-) = \min\{\gamma^+_{x+}, \gamma^+_{x-}\}(\text{or } \min\{\gamma^-_{x+}, \gamma^-_{x-}\}),$$
$$\text{if } (s^1_{x+} - s^2_{x+})(s^1_{x-} - s^2_{x-})(\text{or } (r^1_{x+} - r^2_{x+})(r^1_{x-} - r^2_{x-})) > 0.$$

For any wave θ, the part of $\gamma_x^\pm$ which lies in $\Omega(\theta)$ is denoted by $\gamma_\theta^\pm(x,t)$.

For any waves α, we define $\gamma^+_{x(\alpha)*}$ as follows: When α is a rarefaction wave, or a shock with $\max_\theta\{\bar\theta(\alpha)\} < \mu|\alpha|$, set $\gamma^+_{x(\alpha)*} = \gamma^+_{x(\alpha)}$, where $\frac{1}{2} < \mu < 1$ is a fixed constant.

If there exists a $\theta(\alpha)$ such that $|\bar\theta(\alpha)| \ge \mu|\alpha|$, then we have the following two cases. If $|\theta| > 2|\alpha|$, we define $\gamma^+_{x(\alpha)*} = \gamma^+_{x(\alpha)}$. Otherwise, we let $\gamma^+_{x(\alpha)*} = \gamma^+_{x(\alpha)} + |\hat\alpha^b|$ if $x(\theta) < x(\alpha)$; and $\gamma^+_{x(\alpha)*} = \gamma^+_{x(\alpha)} + |\hat\alpha^t|$ if $x(\theta) > x(\alpha)$.

Similarly we can define $\gamma^-_{x(\beta)*}$ for each 2-wave β. For any shock wave $\bar\alpha$, the part of $\gamma^-_{x(\beta)*}$ which lies in $\Omega(\bar\alpha)$ is denoted by $\gamma^-_{\bar\alpha}(x(\beta)*,t)$; and for any shock wave $\bar\beta$, the part of $\gamma^+_{x(\alpha)*}$ which lies in $\Omega(\bar\beta)$ is denoted by $\gamma^+_{\bar\beta}(x(\alpha)*,t)$.

The first claim estimates the difference of the third-order error $|\beta|^3$ of a given shock β and that of the other waves $\bar\theta(\beta)$ it corresponds to.

Claim 1. *For a 2-shock wave β in $u^1(x,t)$,*

$$\begin{aligned}
&|\beta|^3 - \sum_\theta |\bar\theta(\beta)|^3 - \sum_\theta |\bar\theta^f(\beta)|^3 \\
&\quad \le c(|\beta||\beta^t||\beta^b| + \sum_{\beta^2} |\beta^2(\beta)|\gamma^+_\beta(x(\beta^2),t)(|\beta^2(\beta)| + \gamma^+_\beta(x(\beta^2),t)) \\
&\qquad + \sum_{\beta^1} |\beta^1|\gamma^+_\beta(x(\beta^1),t)(|\beta^1| + \gamma^+_\beta(x(\beta^1),t)) \\
&\qquad + T.V. \sum_{\alpha^2} |\alpha^2|\gamma^+_\beta(x(\alpha^2),t) + T.V. \sum_{\beta^1} |\beta^1|\epsilon),
\end{aligned}$$

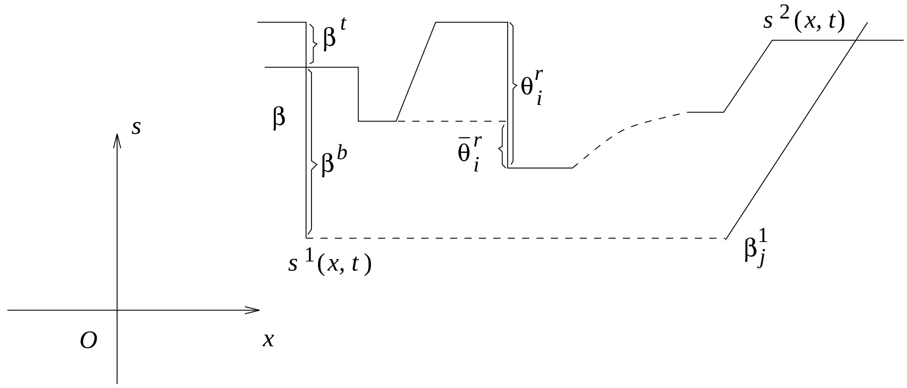

FIGURE 4.3

where the summations are over β^2, α^2 *with* $x(\beta^2), x(\alpha^2) \in (x^l(\beta), x^r(\beta))$*, and* β^1 *represents the 2-rarefaction waves in* $u^1(x,t)$ *which are on the boundary of* $\Omega(\beta)$*. A similar estimate holds for other types of shock waves.*

Proof of Claim 1. Consider $\{\bar\theta_i^r(\beta)\}$, $i = 1, 2, \cdots, n$, and $\{\bar\theta_j^{f,r}(\beta)\}$, $j = 1, 2, \cdots, n_f$, the set of all waves to the right of $x(\beta)$. Without loss of generality, we let $x(\theta_{n_f}^{f,r}) < x(\theta_n^r)$ and denote by $\{\beta_j^1\}$ the rarefaction waves in $u^1(x,t)$ forming the boundary of $\Omega(\beta)$ between $x(\beta)$ and $x(\theta_n^r)$. These waves, $\{\bar\theta_i^r(\beta)\} \cup \{\bar\theta_i^{f,r}(\beta)\} \cup \{\beta_j^1\}$, are renamed, from left to right, by $\{\zeta_i\}$, $i = 1, 2, \cdots, m$; cf. Figure. 4.3. We have

$$\begin{aligned}
& |\beta^b|^3 - \sum_{i=1}^{n} |\bar\theta_i^r|^3 - \sum_{i=1}^{n_f} |\bar\theta_i^{f,r}|^3 \\
= \quad & \sum_{j=1}^{m} \{(|\beta^b| - \sum_{i=1}^{j-1} |\zeta_i|)^3 - (|\beta^b| - \sum_{i=1}^{j} |\zeta_i|)^3 - |\zeta_j|^3\} \\
+ \quad & \sum_j |\beta_j^1|^3 + (\gamma_\beta^+(x(\theta_n)+, t))^3 \\
\le \quad & c \sum_{j=1}^{m} |\zeta_j| \gamma_\beta^+(x(\zeta_j), t)(|\zeta_j| + \gamma_\beta^+(x(\zeta_j), t)) \\
+ \quad & (\gamma_\beta^+(x(\theta_n)+, t))^3 + cT.V. \sum_j |\beta_j^1| \epsilon.
\end{aligned}$$

If ζ_j is a minor wave with the corresponding shock wave α_j, we have

$$|\zeta_j| \gamma_\beta^+(x(\zeta_j), t)(|\zeta_j| + \gamma_\beta^+(x(\zeta_j), t)) \le cT.V. |\alpha_j| \gamma_\beta^+(x(\alpha_j), t).$$

It is easy to see that

$$\gamma_\beta^+(x(\theta_n)+, t))^3 \le c \sum_{\beta^1} |\beta^1| \gamma_\beta^+(x(\beta^1), t)(|\beta^1| + \gamma_\beta^+(x(\beta^1), t)),$$

where the summation is over part of the rarefaction waves β_j^1 in $u^1(x,t)$ on the boundary of $\Omega(\beta)$. Thus

$$\begin{aligned}
& |\beta^b|^3 - \sum_{i=1}^{n} |\bar{\theta}_i^r|^3 - \sum_{i=1} |\bar{\theta}_i^{f,r}|^3 \\
\leq\ & c\{\sum_{\beta^2} |\beta^2(\beta)|\gamma_\beta^+(x(\beta^2),t)(|\beta^2(\beta)| + \gamma_\beta^+(x(\beta^2),t)) \\
+\ & \sum_{\beta^1} |\beta^1|\gamma_\beta^+(x(\beta^1),t)(|\beta^1| + \gamma_\beta^+(x(\beta^1),t)) \\
+\ & T.V. \sum_{\alpha^2} |\alpha^2|\gamma_\beta^+(x(\alpha^2),t) + T.V. \sum_{\beta^1} |\beta^1|\epsilon\},
\end{aligned}$$

with the summations over waves as stated in the claim. A similar estimate holds for $|\beta^t|^3 - \sum_i |\bar{\theta}_i^l|^3 - \sum_j |\bar{\theta}_j^{f,l}|^3$. Combining these estimates gives the proof of the claim. □

As an immediate consequence of Claim 1 we have the following Claim 2.

Claim 2. *Under the set-up of Claim 1, if* $\max_{i,j}\{|\bar{\theta}_i(\beta)|, |\bar{\theta}_j^f(\beta)|\} < \mu|\beta|$*, then, for any fixed constant* $\mu \in (0,1)$*,*

$$\begin{aligned}
|\beta|^3 \leq\ & c_1(\mu)\{|\beta||\beta^t||\beta^b| + \sum_{\beta^2} |\beta^2(\beta)|\gamma_\beta^+(x(\beta^2),t)(|\beta^2(\beta)| + \gamma_\beta^+(x(\beta^2),t)) \\
+\ & \sum_{\beta^1} |\beta^1|\gamma_\beta^+(x(\beta^1),t)(|\beta^1| + \gamma_\beta^+(x(\beta^1),t)) \\
+\ & T.V. \sum_{\alpha^2} |\alpha^2|\gamma_\beta^+(x(\alpha^2),t) + T.V. \sum_{\beta^1} |\beta^1|\epsilon\},
\end{aligned}$$

where $c_1(\mu)$ *is bounded and depends on* μ *with* $\lim_{\mu\to 1-} c_1(\mu) = \infty$*, and the summations are over the waves as stated in Claim 1. A similar estimate also holds for* α*.*

The following claim specifies the condition that there exists a wave θ with $\bar{\theta}(\beta)$ almost of the same magnitude as that of β, so that the difference of the two third-order terms is controllable by the generalized entropy functional and $Q_d(t)$.

Claim 3. *Under the set-up of Claim 1, if there exists a* $\bar{\theta}_{i_0}(\beta)$*, such that* $|\bar{\theta}_{i_0}(\beta)| \geq \mu|\beta|$*, with* $\mu \in (0,1)$*, then*

$$\begin{aligned}
|\beta|^3 - |\bar{\theta}_{i_0}(\beta)|^3 \leq\ & c_2(\mu)\{|\beta||\beta^t||\beta^b| \\
+\ & \sum_{\beta^2} |\beta^2(\beta)|\gamma_\beta^+(x(\beta^2),t)(|\beta^2(\beta)| + \gamma_\beta^+(x(\beta^2),t)) \\
+\ & \sum_{\beta^1} |\beta^1|\gamma_\beta^+(x(\beta^1),t)(|\beta^1| + \gamma_\beta^+(x(\beta^1),t)) \\
+\ & T.V. \sum_{\alpha^2} |\alpha^2|\gamma_\beta^+(x(\alpha^2),t) + T.V. \sum_{\beta^1} |\beta^1|\epsilon\},
\end{aligned}$$

where $c_2(\mu)$ *is a constant depending on* μ *and* $\lim_{\mu\to 0+} c_2(\mu) = \infty$*, and the summations are over the waves as stated in Claim 1. A similar estimate holds for* α*.*

The proof of Claim 3 is based on the discussion in Claim 1 and using the fact that $\gamma^+_\beta(x,t) \geq \mu|\beta|$ for x between $x(\beta)$ and $x(\theta_{i_0})$. We omit it here.

From now on, we use $c(\mu)$ to denote a generic positive constant depending on μ which is uniformly bounded away from zero and infinity when μ is in any fixed compact subset of $(0,1)$.

For a shock wave β in $u^1(x,t)$, define

$$\begin{aligned} E(\beta) &= |\beta||\beta^t||\beta^b| + |\beta|\gamma^-_{x(\beta)*} + |\beta|\gamma^+_{x(\beta)}(|\beta| + \gamma^+_{x(\beta)}) \\ &+ \sum_{\beta^1} |\beta^1|(\gamma^-_{x(\beta^1)*} + \gamma^+_{x(\beta^1)}(|\beta^1| + \gamma^+_{x(\beta^1)})) \\ &+ \sum_{\beta^2} |\beta^2(\beta)|(\gamma^-_{x(\beta^2)*} + \gamma^+_{x(\beta^2)}(|\beta^2| + \gamma^+_{x(\beta^2)})) \\ &+ \sum_{\alpha} |\alpha|\gamma^+_\beta(x(\alpha)*,t) + \sum_{\bar\beta} |\bar\beta|\gamma^+_\beta(x(\bar\beta),t)(|\bar\beta| + \gamma^+_\beta(x(\bar\beta),t)) + e(\Omega(\beta)), \end{aligned}$$

where the summations over β^1 and β^2 are the same as those stated in Claim 1, the summations of α and $\bar\beta$ are over all the waves in $(x^l(\beta), x^r(\beta))$, and $e(\Omega(\beta))$ is the part of $e(\Lambda_p)$ at time t which is related to $\Omega(\beta)$. Similar definitions hold for β in $u^2(x,t)$ and also for 1-shock waves. From our Claims, $E(\beta)$ can be used later to denote the terms which can be used to control the shock wave strength to the cubic power or the difference of shock wave strengths to the cubic power.

Claim 4. *In the region $\Omega(\beta)$, if there exists a $\theta^f(\beta)$ with $|\bar\theta^f(\beta)| \geq \mu|\beta|$, then there exists $\tilde\alpha$ or $\tilde\beta$, not necessarily located in $\Omega(\beta)$, such that*

$$|\beta|^3 \leq c|\beta|E(\tilde\alpha) \quad \textit{or} \quad |\beta|^3 \leq c|\beta|E(\tilde\beta). \tag{4.1}$$

Proof. In the following, we will prove a slightly stronger conclusion. In the region where there exists $\theta^f(\beta)$ with $|\bar\theta^f(\beta)| \geq \mu|\beta|$, we denote the shock wave corresponding to $\theta^f(\beta)$ by α_1, i.e. $\theta^f = \hat\alpha_1$. Now we consider the region $\Omega(\alpha_1)$. By using Claim 2, if $\max_{i,j}\{|\bar\theta_i(\alpha_1)|, |\bar\theta^f_j(\alpha_1)|\} < \mu|\alpha_1|$, then $|\alpha_1|^3 \leq c(\mu)E(\alpha_1)$. In this case, we have

$$|\beta|^3 \leq \mu^{-1}|\beta|^2|\hat\alpha_1| \leq c\mu^{-1}|\beta|E(\alpha_1),$$

and (4.1) holds. If there exists $\theta^f(\alpha_1)$ such that $|\bar\theta^f(\alpha_1)| \geq \mu|\alpha_1|$, then we denote the shock wave corresponding to $\theta^f(\alpha_1)$ by β_1, and consider the region $\Omega(\beta_1)$. Similarly, if $\max_{i,j}\{|\bar\theta_i(\beta_1)|, |\bar\theta^f_j(\beta_1)|\} < \mu|\beta_1|$, then we have $|\beta_1|^3 \leq c(\mu)E(\beta_1)$. In this case, we have

$$|\beta|^3 + |\alpha_1|^3 \leq c(\mu)|\alpha_1|E(\beta_1).$$

Otherwise, we consider the region $\Omega(\beta_1)$. This procedure produces a sequence of shock waves, denoted by $\{\alpha_1, \beta_1, \alpha_2, \cdots,\}$. By our construction, there are only finitely many shocks in the approximate solutions at any given time, and so this sequence is finite. Without loss of generality, we assume that the last shock wave in the sequence is of the first family, denoted by α_n. For simplicity, we include the shock wave β as β_0. Now in the region $\Omega(\alpha_n)$, we have one of the following cases.

When $\max_{i,j}\{|\bar\theta_i(\alpha_n)|, |\bar\theta^f_j(\alpha_n)|\} < \mu|\alpha_n|$, then $|\alpha_n|^3 \leq c(\mu)E(\alpha_n)$ and we have

$$\sum_{i=0}^{n-1} |\beta_i|^3 + \sum_{i=1}^{n-1} |\alpha_i|^3 \leq c(\mu)|\beta_{n-1}|E(\alpha_n).$$

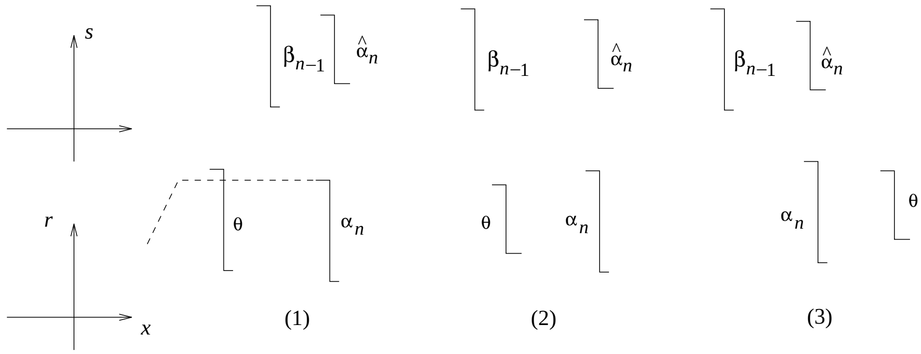

FIGURE 4.4

When there exists $\theta(\alpha_n)$ such that $|\bar{\theta}(\alpha_n)| \geq \mu|\alpha_n|$, then we have two subcases. If $|\theta| > 2|\alpha_n|$, then, according to Claim 2, $|\alpha_n|^3$ can be controlled by $c(\mu)E(\gamma)$ for some γ and (4.1) is true. If $|\theta| \leq 2|\alpha_n|$, then, according to the relative position of θ to α_n, we have three cases; cf. Figure. 4.4. Without loss of generality, we assume $x(\beta_{n-1}) < x(\alpha_n)$.

If $x(\tilde{\alpha}_n) < x(\beta_{n-1})$, then $|\beta_{n-1}|\gamma^-_{x(\beta_{n-1})*} \geq c(\mu)|\beta_{n-1}|^{\frac{4}{3}}$. Hence

$$\sum_{i=0}^{n-1} |\beta_i|^3 + \sum_{i=1}^{n-1} |\alpha_i|^3 \leq c(\mu)||\beta_{n-1}|E(\beta_{n-1}).$$

If $x(\beta_{n-1}) < x(\theta) < x(\alpha_n)$, then $|\theta|\gamma^+_{x(\theta)*} \geq c(\mu)|\beta_{n-1}|^{\frac{4}{3}}$. In either case we have (4.1).

Finally if $x(\alpha_n) < x(\theta)$, by the definition of $\gamma^{\pm}_{x*}$, we have

$$\gamma^+_{x(\alpha_n)*} = \gamma^+_{x(\alpha_n)} + |\hat{\alpha}^t_n|.$$

Since $|\hat{\alpha}^t_n| \geq \mu^{-1}|\beta_{n-1}|$, we have $|\alpha_n|\gamma^+_{x(\alpha_n)*} \geq c(\mu)|\alpha_n||\beta_{n-1}| \geq c(\mu)|\beta_{n-1}|^{\frac{4}{3}}$. Therefore, we also have the conclusion of the claim. This completes the proof of the claim. □

Remark 2. Combining Claims 1–4, for any shock wave γ, the only case that $|\gamma|^3$ cannot be uniformly controlled by the terms $E(\gamma_i)$ with respect to μ is when there exists θ with $|\bar{\theta}(\gamma)| \geq \mu|\gamma|$ and $|\theta| \leq 2|\gamma|$.

5. LINEAR FUNCTIONALS

We will show in the following sections that when N is sufficiently large, there exists an upper bound for $\frac{d}{dt}H(t)$, $t \in I^c_p$, with some appropriately chosen constants k_i, $i = 1, 2, 3$. By (iv) of Theorem 2.3, there exists an error term $e(\Lambda_p)$ when we consider $\frac{d}{dt}H(t)$, for $t \in I^c_p$. For brevity, we only put this error term back at the end of each estimate. This section studies the linear functionals $L(t)$ and $L_h(t)$.

For simplicity we assume that at each point (x,t), $t \in I^c_p$, there exists at most one wave belonging either to $u^1(x,t)$ or $u^2(x,t)$.

5.1. **Estimation of $\frac{dL(t)}{dt}$.** We consider only $\frac{d}{dt}L^-(t)$; the same method applies to $\frac{d}{dt}L^+(t)$. We need to consider the functions $r^p(x,t)$ and $s^p(x,t)$, $p=1,2$, together because of the coupling of waves of different families. To study this complex coupling, we will carry out the analysis in two steps. The first step, Section 5.1.1, concentrates on the simplified situation of no minor waves and deals with the change of wave speeds due to different bases. The second, harder, step, Section 5.1.2, deals with the effects of minor waves of one family on the derivative of the integral of the Riemann invariant of the other family.

Our basic idea is to compare the r values of $u^1(x,t)$ and $u^2(x,t)$ with solutions of scalar conservation laws, for which we have L_1 contraction. This induces an error due to the fact that the s value of a wave in $u^2(x,t)$ may be different from the s value of $u^1(x,t)$. The scalar conservation laws are:

Definition 6. For a 1-shock wave α, we define four speeds related to α: $\lambda_2(\hat{\alpha})$ is the speed of the wave with states $(s_-(\alpha), s_+(\alpha))$ of the scalar equation

$$s_t + \lambda_2(r_+(\alpha), s)s_x = 0;$$

$\check{\lambda}_2(\hat{\alpha})$ is the speed of the wave with states $(s_-(\alpha), s_+(\alpha))$ of the scalar equation

$$s_t + \lambda_2(r_-(\alpha), s)s_x = 0;$$

$\hat{\lambda}_1(\alpha)$ is the speed of the wave with states $(r_-(\alpha), r_+(\alpha))$ of the scalar equation

$$r_t + \lambda_1(r, s_-(\alpha))r_x = 0;$$

and $\check{\lambda}_1(\alpha)$ is the speed of the wave with states $(r_-(\alpha), r_+(\alpha))$ of the scalar equation

$$r_t + \lambda_1(r, s_+(\alpha))r_x = 0.$$

A similar definition holds for a 2-shock wave β. The above scalar equations are made conservative by simple integration of the characteristics to yield the flux functions.

5.1.1. $\hat{\alpha} \equiv 0$ *and* $\hat{\beta} \equiv 0$ *for all* α *and* β. With Definition 6, we have

$$\frac{d}{dt}L^-(t) = \sum_{\alpha} \lambda_1(\alpha)(\gamma^-(x(\alpha)-,t) - \gamma^-(x(\alpha)+,t)),$$

where the summation is over 1-waves α in $u^1(x,t)$ and $u^2(x,t)$. For each 1-wave α in $u^2(x,t)$, we consider the corresponding wave $\tilde{\alpha}$ with the same r values as α, but with s value equal to $s^1(x(\alpha),t)$. This yields the wave speed $\hat{\lambda}_1(\tilde{\alpha})$ for $\tilde{\alpha}$ as the solution of the scalar conservation law

$$r_t + \lambda_1(r, s^1(x(\alpha),t))r_x = 0. \tag{5.1}$$

Rewrite $dL^-(t)/dt$ above as:

$$\begin{aligned}\frac{d}{dt}L^-(t) &= \sum_{\alpha^1} \hat{\lambda}_1(\alpha^1)(\gamma^-(x(\alpha^1)-,t) - \gamma^-(x(\alpha^1)+,t)) \\ &+ \sum_{\alpha^2} \hat{\lambda}_1(\tilde{\alpha}^2)(\gamma^-(x(\alpha^2)-,t) - \gamma^-(x(\alpha^2)+,t)) \\ &+ \sum_{\alpha^2} (\hat{\lambda}_1(\alpha^2) - \hat{\lambda}_1(\tilde{\alpha}^2))(\gamma^-(x(\alpha^2)-,t) - \gamma^-(x(\alpha^2)+,t)) + I_L^{2,1},\end{aligned}$$

where

$$I_L^{2,1} = \sum_{\alpha} (\lambda_1(\alpha) - \hat{\lambda}_1(\alpha))(\gamma^-(x(\alpha)-,t) - \gamma^-(x(\alpha)+,t)),$$

and the summation is over all 1-waves in $u^1(x,t)$ and $u^2(x,t)$. By continuity, $|\hat{\lambda}_1(\alpha^2)-\hat{\lambda}_1(\tilde{\alpha}^2)| = 0(1)\gamma^+(x(\alpha^2),t)$ and $|\gamma^-(x(\alpha)-,t)-\gamma^-(x(\alpha)+,t)| = |\alpha|$, and so

$$|\sum_{\alpha^2}(\hat{\lambda}_1(\alpha)-\hat{\lambda}_1(\tilde{\alpha}))(\gamma^-(x(\alpha^2)-,t)-\gamma^-(x(\alpha^2)+,t))| = 0(1)\sum_{\alpha^2}|\alpha^2|\gamma^+((\alpha^2),t).$$

(Notice that here $\gamma^+(x(\alpha),t)$ and $\gamma^-(x(\beta),t)$ are well defined. But they need to be redefined carefully when we consider minor waves in Section 5.1.2.) Thus

$$\begin{aligned} \frac{d}{dt}L^-(t) \le & \sum_{\alpha^1}\hat{\lambda}_1(\alpha^1)(\gamma^-(x(\alpha^1)-,t)-\gamma^-(x(\alpha^1)+,t)) + I_L^{2,1} \\ & + \sum_{\alpha^2}\hat{\lambda}_1(\tilde{\alpha}^2)(\gamma^-(x(\alpha^2)-,t)-\gamma^-(x(\alpha^2)+,t)) \\ & + c\sum_{\alpha^2}|\alpha^2|\gamma^+(x(\alpha^2),t). \end{aligned} \tag{5.2}$$

If $s^1(x,t)$ is constant, then by the L_1 contraction of scalar conservation laws, the sum of the first two terms in the above inequality is nonpositive. In general, we need to consider the error due to the variation of the function $s^1(x,t)$. Recall that $s^1(x,t)$ is a step function. Let β_i^1 and β_{i+1}^2 be two adjacent 2-waves in $u^1(x,t)$. By using the fact that $s^1(x(\beta_i^1)+,t) = s^1(x(\beta_{i+1}^1)-,t)$ and $|\lambda_1(\cdot,s^1(x(\beta_i^1)+,t)) - \lambda_1(\cdot,s^1(x(\beta_i^1)-,t))| = 0(1)|\beta_i^1|$, the L_1 contraction for scalar conservation laws yields:

$$\begin{aligned} & \sum_{\alpha^1}\hat{\lambda}_1(\alpha^1)(\gamma^-(x(\alpha^1)-,t)-\gamma^-(x(\alpha^1)+,t)) \\ & \quad + \sum_{\alpha^2}\hat{\lambda}_1(\tilde{\alpha}^2)(\gamma^-(x(\alpha^2)-,t)-\gamma^-(x(\alpha^2)+,t))) \\ & \le c\sum_{\beta^1}|\beta^1|\gamma^-(x(\beta^1),t). \end{aligned} \tag{5.3}$$

Combining (5.2) and (5.3) gives

$$\frac{d}{dt}L^-(t) \le c(\sum_{\alpha^2}|\alpha^2|\gamma^+(x(\alpha^2),t) + \sum_{\beta^1}|\beta^1|\gamma^-(x(\beta^1),t)) + I_L^{2,1}. \tag{5.4}$$

Similarly, we have the following estimate for $\frac{d}{dt}L^+(t)$:

$$\frac{d}{dt}L^+(t) \le c(\sum_{\alpha^1}|\alpha^1|\gamma^+(x(\alpha^1),t) + \sum_{\beta^2}|\beta^2|\gamma^-(x(\beta^2),t)) + I_L^{2,2}, \tag{5.5}$$

where

$$I_L^{2,2} = \sum_{\beta}(\lambda_2(\beta)-\hat{\lambda}_2(\beta))(\gamma^+(x(\beta)-,t)-\gamma^+(x(\beta)+,t)),$$

and the summation is over 2-waves in $u^1(x,t)$ and $u^2(x,t)$. Combining (5.4) and (5.5), we have

$$\frac{d}{dt}L(t) \le c(\sum_{\alpha}|\alpha|\gamma^+(x(\alpha),t) + \sum_{\beta}|\beta|\gamma^-(x(\beta),t)) + I_L^2, \tag{5.6}$$

where the summations are over all 1-waves α and 2-waves β in $u^i(x,t)$, $i=1,2$, and $I_L^2 = I_l^{2,1} + I_L^{2,2}$.

Remark 3. The estimate (5.6) shows that the derivative of $L(t)$ is simple if there are no minor waves. In fact, for an $n \times n$ system of conservation laws whose shock wave curve coincides with the rarefaction wave curve, there exists a canonical measure for wave strengths such that the desired nonlinear functional is simply $H(t) = (1 + k_1 F(t))L(t) + k_2 Q_d(t)$ for some positive constants k_1 and k_2; cf. [18].

5.1.2. *General case with minor waves.* Based on the discussion in Section 5.1.1, it remains to study the new error terms due to the minor waves, i.e., the effect of 2-waves on the L_1 norm of the 1-Riemann invariant. This is the main part of the error and will be studied later again together with L_h. The study of this extra error can be divided into the following consideration:

(A1) $\hat{\alpha}_i^p \neq 0$, $i \in \mathbf{Z}$ and $p = 1, 2$;
(A2) $\hat{\beta}_i^p \neq 0$, $i \in \mathbf{Z}$ and $p = 1, 2$.

We change the s value of the waves in $u^2(x,t)$ to that of $u^1(x,t)$. If there is a shock wave β_i^1 in $u^1(x,t)$, then the error term related to the minor wave $\hat{\beta}_i^1$ can be written as:

$$(\lambda_2(\beta_i^1) - \lambda_1(\hat{\beta}_i^1))(\gamma^-(x(\beta_i^1)-,t) - \gamma^-(x(\beta_i^1)+,t)) + 0(1)|\beta_i^1|\gamma^-_{x(\beta_i^1)} + A_2^-(\beta_i^1),$$

where $A_2^-(\beta_i^1)$ represents the error due to the definition of $\lambda_1(\hat{\beta}_i^1)$ when it is evaluated at $s = s_+(\beta_i^1)$. When $|\hat{\beta}_i^{1,t}||\hat{\beta}_i^{1,b}| \neq 0$, the L_1 contraction for scalar conservation laws yields a good term of order $-|\hat{\beta}_i^{1,t}||\hat{\beta}_i^{1,b}|$ and we have

$$|A_2^-(\beta_i^1)| \leq |\lambda_1(\hat{\beta}_i^1) - \check{\lambda}_1(\hat{\beta}_i^1)||\hat{\beta}_i^{1,t}| - c|\hat{\beta}_i^{1,t}||\hat{\beta}_i^{1,b}|,$$

for a positive constant c.

If there is a shock wave β_i^2 in $u^2(x,t)$, then the error term related to the minor wave $\hat{\beta}_i^2$ can be written as:

$$(\lambda_2(\beta_i^2) - \lambda_1(\hat{\beta}_i^2))(\gamma^-(x(\beta_i^2)-,t) - \gamma^-(x(\beta_i^2)+,t)) + A_2^-(\beta_i^2),$$
$$A_2^-(\beta_i^2) \equiv (\lambda_1(\hat{\beta}_i^2) - \lambda_1(\tilde{\hat{\beta}}^2{}_i))(\gamma^-(x(\beta_i^2)-,t) - \gamma^-(x(\beta_i^2)+,t)),$$

where $\tilde{\hat{\beta}}^2{}_i$ is a wave with states $(r_-(\beta_i^2), r_+(\beta_i^2))$ of the scalar equation

$$r_t + \lambda_1(r, s^1(x(\beta_i^2),t))r_x = 0.$$

Here we have noted that $s^1(x(\beta_i^2)-,t) = s^1(x(\beta_i^2)+,t)$.

Notice that $|A_2^-(\beta_i^2)|$ is bounded by

$$c|\hat{\beta}_i^2||s^1(x(\beta_i^2),t) - s_+(\beta_i^2)| = c|\beta_i^2|^3(\gamma^+_{x(\beta_i^2)} + |\beta_i^{2,b}|).$$

Therefore both $A_2^-(\beta_i^1)$ and $A_2^-(\beta_i^2)$ can be bounded by the terms of order $|\beta_i^1|^4$ and $(|\beta_i^2|^4 + |\beta_i^2|^2\gamma^+_{x(\beta_i^2)}(|\beta_i^2| + \gamma^+_{x(\beta_i^2)}))$, respectively. Thus if $|\beta_i^1|^3$ (or $|\beta_i^2|^3$) can be bounded by $E(\gamma)$ for some γ, uniformly with respect to μ, then $A_2^-(\beta_i^1)$ (or $A_2^-(\beta_i^2)$) can be uniformly bounded also. Hence we only need to consider the case when $|\beta_i^1|^3$ (or $|\beta_i^2|^3$) cannot be controlled by terms like $E(\gamma)$ uniformly with respect to μ. For this, we need to modify the estimation procedure in Section 5.1.1 as follows:

Assume that there exists a shock wave β_i^1 such that in the region $\Omega(\beta_i^1)$, there is a $\theta(\beta_i^1)$ satisfying $|\bar{\theta}(\beta_i^1)| \geq \mu|\beta_i^1|$. If we denote the shock wave corresponding to $\theta(\beta_i^1)$ by β_i^2, then by Claims 1–4 we can also assume that $|\beta_i^2| \leq 2|\beta_i^1|$. Then instead of changing the s value of waves in $u^2(x,t)$ to the s value of $u^1(x,t)$ for all cases as in Section 5.1.1, we do as follows: If $x(\beta_i^1) < x(\beta_i^2)$, then we change

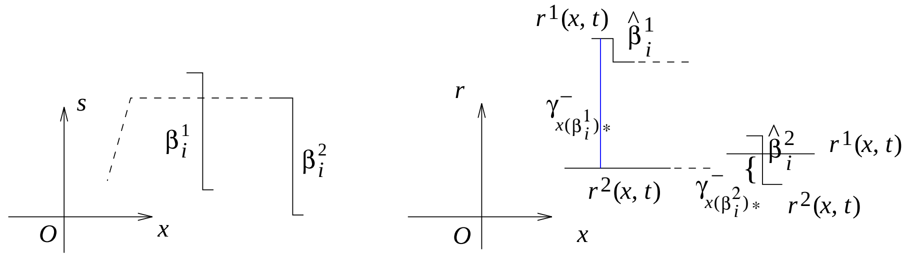

FIGURE 5.1

the s value of waves in $u^2(x,t)$ which lies in $(x(\beta_i^1), x(\beta_i^2)]$ to the corresponding s value in $u^1(x,t)$. Otherwise we change the s value from $u^1(x,t)$ to $u^2(x,t)$. By the construction of the regions for shock waves, such a procedure is well defined with an error term bounded by terms of order $(|\beta|\gamma^-_{x(\beta)*} + |\alpha|\gamma^-_{x(\alpha)}(|\alpha| + \gamma^-_{x(\alpha)}))$ for some β and α. Now, for definiteness, we assume $x(\beta_i^1) < x(\beta_i^2)$; cf. Figure 5.1.

From the definition of $\gamma^\pm_{x*}$, we have $\gamma^-_{x(\beta_i^1)*} = \gamma^-_{x(\beta_i^1)} + |\hat{\beta}_i^{1,t}|$ and

$$A_2^-(\beta_i^1) \le c|\beta_i^1|\gamma^-_{x(\beta_i^1)*}.$$

As for $A_2^-(\beta_i^2)$, we know that $|\beta_i^2|^2|\beta_i^{2,b}| \le c(\mu)(E(\beta_i^1) + E(\beta_i^2))$ and so $A_2^-(\beta_i^2)$ can be controlled by the terms of order

$$\{|\alpha|(\gamma^+_{x(\alpha)*} + \gamma^-_{x(\alpha)}(|\alpha| + \gamma^-_{x(\alpha)})) + |\beta|(\gamma^-_{x(\beta)*} + \gamma^+_{x(\beta)}(|\beta| + \gamma^+_{x(\beta)}))\}$$

uniformly with respect to μ.

We now study (A_1). According to the above discussions, the error terms due to the 1-shock wave α_i^1 in $\frac{d}{dt}L^-(t)$ can be written as:

$$(\lambda_1(\alpha_i^1) - \hat{\lambda}_1(\alpha_i^1))(\gamma^-(x(\alpha_i^1)-,t) - \gamma^-(x(\alpha_i^1)+,t)) + A_1^-(\alpha_i^1) + 0(1)|\hat{\alpha}_i^1|\gamma^-_{x(\alpha_i^1)},$$

where $A_1^-(\alpha_i^1) = -(\hat{\lambda}_1(\alpha_i^1) - \check{\lambda}_1(\alpha_i^1))|\alpha_i^{1,b}|$. Here, again, we have used the fact that if $|\alpha_i^{1,t}||\alpha_i^{1,b}| \ne 0$, then the L_1 contraction of scalar conservation law gives a good term of order $-|\alpha_i^{1,t}||\alpha_i^{1,b}|$. Notice that $|A_1^-(\alpha_i^1)| \le c|\alpha_i^1|^4$.

For a shock wave α_i^2, the error term related to α_i^2 in $\frac{d}{dt}L^-(t)$ can be written as (cf. (5.2)):

$$(\lambda_1(\alpha_i^2) - \hat{\lambda}_1(\alpha_i^2))(\gamma^-(x(\alpha_i^2)-,t) - \gamma^-(x(\alpha_i^2)+,t)) + A_1^-(\alpha_i^2),$$

where

$$A_1^-(\alpha_i^2) = (\hat{\lambda}_1(\alpha_i^2) - \hat{\lambda}_1(\tilde{\alpha}_i^2))(\gamma^-(x(\alpha_i^2)-,t) - \gamma^-(x(\alpha_i^2)+,t)).$$

Notice that $|A_1^-(\alpha_i^2)| \le c|s^1(x(\alpha_i^2),t) - s_-(\alpha_i^2)||\alpha_i^2| \le c(|\alpha_i^2|\gamma^+_{x(\alpha_i^2)} + |\alpha_i^2|^4)$. Hence for any shock wave α_i^1, $|A_1^-(\alpha_i^1)|$ can also be uniformly controlled with respect to the same μ, when $|\alpha_i^1|^3$ can be uniformly controlled by $E(\gamma)$ for some γ. A similar argument holds for $|A_1^-(\alpha_i^2)|$. Therefore, we only need to consider the case when $|A_1^-(\alpha_i^1)|$ or $|A_1^-(\alpha_i^2)|$ cannot be uniformly controlled by some $E(\gamma)$ with respect to μ.

For a shock wave α_i^1, we assume that there exists a $\theta(\alpha_i^1)$ with $|\bar{\theta}(\alpha_i^1)| \ge \mu|\alpha_i^1|$. We denote the shock wave corresponding to $\theta(\alpha_i^1)$ by α_i^2. According to Remark 2, we need only consider the case when $|\alpha_i^2| \le 2|\alpha_i^1|$ and thereby study the sum of

$A_1^-(\alpha_i^1)$ and $A_1^-(\alpha_i^2)$. For definiteness, we assume $x(\alpha_i^1) < x(\alpha_i^2)$. With these, we have

$$|\alpha_i^{1,t}||\alpha_i^1| + |\alpha_i^{2,b}||\alpha_i^2| \le c(\mu)(|\alpha_i^{1,t}||\alpha_i^{1,b}| + |\alpha_i^{2,t}||\alpha_i^{2,b}|). \tag{5.7}$$

From Claims 1–4, we have

$$\begin{aligned}(|r_-(\alpha_i^1) - r_-(\alpha_i^2)| + |r_+(\alpha_i^1) - r_+(\alpha_i^2)|)(|\alpha_i^1|^2 + |\alpha_i^2|^2) \\ \le c(\mu)(E(\alpha_i^1) + E(\alpha_i^2)),\end{aligned} \tag{5.8}$$

$$|s_-(\alpha_i^1) - s_-(\alpha_i^2)| \le \gamma^+_{x(\alpha_i^1)*} + \sum_{x(\beta_j^1)\in(x(\alpha_i^1),x(\alpha_i^2))} |\beta_j^1| + \sum_{x(\alpha_j^1)\in(x(\alpha_i^1),x(\alpha_i^2))} |\hat{\alpha}_j^1|, \tag{5.9}$$

$$|s_+(\alpha_i^1) - s_+(\alpha_i^2)| \le \gamma^+_{x(\alpha_i^2)*} + \sum_{x(\beta_j^2)\in(x(\alpha_i^1),x(\alpha_i^2))} |\beta_j^2| + \sum_{x(\alpha_j^2)\in(x(\alpha_i^1),x(\alpha_i^2))} |\hat{\alpha}_j^2|. \tag{5.10}$$

Based on (5.7)–(5.10), we can estimate $|A_1^-(\alpha_i^1) + A_1^-(\alpha_i^2)|$ as follows:

$$\begin{aligned}
& |A_1^-(\alpha_i^1) + A_1^-(\alpha_i^2)| \\
\le\ & |-(\hat{\lambda}_1(\alpha_i^1) - \check{\lambda}_1(\alpha_i^1))|\alpha_i^{1,b}| + (\hat{\lambda}_1(\alpha_i^2) - \check{\lambda}_1(\alpha_i^2))(|\alpha_i^{2,t}| + \gamma^-_{x(\alpha_i^2)}) \\
+\ & |\check{\lambda}_1(\alpha_i^2) - \hat{\lambda}_1(\tilde{\alpha}_i^2)|(\gamma^-(x(\alpha_i^2)-,t) - \gamma^-(x(\alpha_i^2)+,t)) \\
\le\ & c\{(|r_-(\alpha_i^1) - r_-(\alpha_i^2)| + |r_+(\alpha_i^1) - r_+(\alpha_i^2)| + |s_-(\alpha_i^1) - s_-(\alpha_i^2)| \\
+\ & |s_+(\alpha_i^1) - s_+(\alpha_i^2)|)(|\alpha_i^1|^3 + |\alpha_i^1|^3) \\
+\ & (|\alpha_i^{1,t}||\alpha_i^1| + |\alpha_i^{2,b}||\alpha_i^2|)(|\alpha_i^1| + |\alpha_i^1|) + |\alpha_i^2|\gamma^+_{x(\alpha_i^2)*}\} \\
\le\ & c\{|\alpha_i^1|\gamma^+_{x(\alpha_i^1)*} + |\alpha_i^2|\gamma^+_{x(\alpha_i^2)*} + |\alpha_i^1||\alpha_i^{1,t}||\alpha_i^{1,b}| + |\alpha_i^2||\alpha_i^{2,t}||\alpha_i^{2,b}| \\
+\ & \sum_{x(\beta_j^p)\in(x(\alpha_i^1),x(\alpha_i^2)),p=1,2} |\alpha_i^1||\beta_j^p|\gamma^-_{\alpha_i^1}(x(\beta_j^p),t) \\
+\ & \sum_{x(\alpha_j^p)\in(x(\alpha_i^1),x(\alpha_i^2)),p=1,2} |\alpha_i^1||\alpha_j^p|^2\gamma^-_{\alpha_i^1}(x(\alpha_j^p),t) + |\alpha_i^1|E(\alpha_i^1) + |\alpha_i^2|E(\alpha_i^2)\}.
\end{aligned}$$

Combining Section 5.1.1 and the above estimates, we have

$$\begin{aligned}
\frac{d}{dt}L^-(t) \quad \le \quad & c\{\sum_\alpha |\alpha|\gamma^+_{x(\alpha)*} + \sum_\beta |\beta|\gamma^-_{x(\beta^1)*} \\
+ \quad & \sum_\alpha |\alpha|\gamma^-_{x(\alpha)}(|\alpha| + \gamma^-_{x(\alpha)}) + \sum_\beta |\beta|\gamma^+_{x(\beta)}(|\beta| + \gamma^+_{x(\beta)})\}
\end{aligned}$$

where the summations are over waves in $u^1(x,t)$ and $u^2(x,t)$. Since

$$|\lambda(\alpha) - \hat{\lambda}(\alpha)| = 0(1)|\alpha|^2, \qquad |\gamma^-(x(\beta)-,t) - \gamma^-(x(\beta)+,t)| \le |\hat{\beta}|,$$
$$|\gamma^-(x(\alpha)-,t) - \gamma^-(x(\alpha)+,t)| \le |\alpha|, \qquad |\gamma^\pm_\cdot| \le T.V.,$$

we have

$$\begin{aligned}
\frac{d}{dt}L^-(t) \quad \le \quad & c\{\sum_\alpha |\alpha|\gamma^+_{x(\alpha)*} + \sum_\beta |\beta|\gamma^-_{x(\beta)*} + \sum_\alpha |\alpha|^3 + \sum_\beta |\beta|^3 \\
+ \quad & \sum_\alpha |\alpha|\gamma^-_{x(\alpha)}(|\alpha| + \gamma^-_{x(\alpha)}) + \sum_\beta |\beta|\gamma^+_{x(\beta)}(|\beta| + \gamma^+_{x(\beta)})\}.
\end{aligned}$$

A similar estimate holds for $\frac{d}{dt}L^+(t)$. Hence we have the following two lemmas for $\frac{d}{dt}L(t)$, with the error term $ce(\Lambda_p)$ put back.

Lemma 5.1. *Under the hypotheses of Main Lemma 1, for each $t \in I_p^c$, we have*

(5.11)
$$\begin{aligned}\frac{d}{dt}L(t) \le c\{&\sum_\alpha |\alpha|^3 + \sum_\beta |\beta|^3 + \sum_\alpha |\alpha|\gamma^+_{x(\alpha)*} + \sum_\beta |\beta|\gamma^-_{x(\beta)*} + e(\Lambda_p)\\ &+ \sum_\alpha |\alpha|\gamma^-_{x(\alpha)}(|\alpha| + \gamma^-_{x(\alpha)}) + \sum_\beta |\beta|\gamma^+_{x(\beta)}(|\beta| + \gamma^+_{x(\beta)})\}\\ &- c_1\{\sum_\alpha |\alpha^t||\alpha^b| + \sum_\beta |\beta^t||\beta^b|\},\end{aligned}$$

where c_1 is a positive constant and the last two summations are over all shock waves.

Since some of the terms in $\frac{d}{dt}L(t)$ should be considered together with $\frac{d}{dt}L_h(t)$, we write another form of the estimation for $\frac{d}{dt}L(t)$ for later use as follows:

Lemma 5.2. *Under the hypotheses of Main Lemma 1, for each $t \in I_p^c$, we have*

(5.12)
$$\begin{aligned}\frac{d}{dt}L(t) \le c\{&\sum_\alpha |\alpha|\gamma^+_{x(\alpha)*} + \sum_\beta |\beta|\gamma^-_{x(\beta)*} + \sum_\alpha |\alpha|\gamma^-_{x(\alpha)}(|\alpha| + \gamma^-_{x(\alpha)})\\ &+ \sum_\beta |\beta|\gamma^-_{x(\beta)}(|\beta| + \gamma^-_{x(\beta)}) + e(\Lambda_p)\} + I_L^1 + I_L^2\\ &- c_1\{\sum_\alpha |\alpha^t||\alpha^b| + \sum_\beta |\beta^t||\beta^b|\},\end{aligned}$$

where

$$\begin{aligned}I_L^1 = &\sum_\alpha (\lambda_1(\alpha) - \lambda_2(\hat{\alpha}))(\gamma^+(x(\alpha)-, t) - \gamma^+(x(\alpha)+, t))\\ &+ \sum_\beta (\lambda_2(\beta) - \lambda_1(\hat{\beta}))(\gamma^-(x(\beta)-, t) - \gamma^-(x(\beta)+, t)).\end{aligned}$$

5.2. **Estimation of $\frac{d}{dt}L_h(t)$.** In this subsection, we are going to study $\frac{d}{dt}L_h(t)$, which, together with $\frac{d}{dt}L(t)$, will largely eliminate the error term I_L^1 in $\frac{d}{dt}L(t)$.

Lemma 5.3. *Under the hypotheses of Main Lemma 1, for each $t \in I_p^c$,*

$$\begin{aligned}\frac{d}{dt}L_h(t) \;\le\; & c\{\sum_\alpha |\alpha|\gamma^-_{x(\alpha)}(\gamma^-_{x(\alpha)} + |\alpha|) + \sum_\beta |\beta|\gamma^+_{x(\beta)}(\gamma^+_{x(\beta)} + |\beta|) + \sum_\alpha |\alpha|\gamma^+_{x(\alpha)*}\\ &+ \sum_\beta |\beta|\gamma^-_{x(\beta)*} + T.V.(\sum_\alpha |\alpha^t||\alpha^b| + \sum_\beta |\beta^t||\beta^b|) + e(\Lambda_p)\}\\ &+ \sum_\alpha B(\alpha) + \sum_\beta B(\beta),\end{aligned}$$

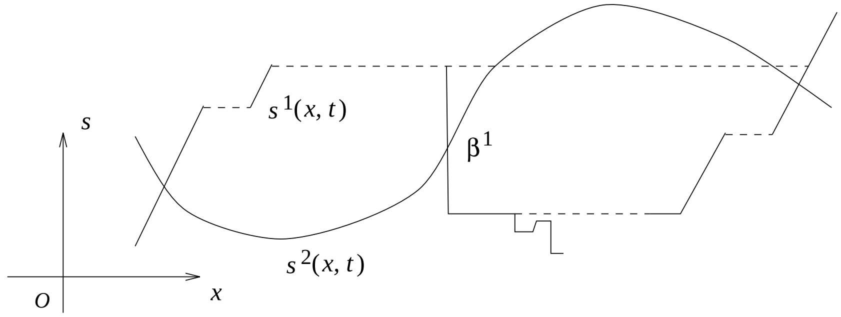

FIGURE 5.2

where $B(\alpha)$ and $B(\beta)$ are defined as follows: For a 2-shock wave γ, consider the waves inside $\Omega(\gamma)$ (cf. Figure 5.2), and set

$$\begin{aligned} B(\gamma) &= C_\gamma\{\xi_\gamma(x(\gamma)-,t)(\gamma_\gamma^+(x(\gamma)-,t))^3 \\ &- \xi_\gamma(x(\gamma)+,t)(\gamma_\gamma^+(x(\gamma)+,t))^3)\}(\lambda_2(\gamma)-\lambda_1(\hat\gamma)) \\ &+ C_\gamma\sum_i\{\xi_\gamma(x(\theta_i(\gamma))-,t)(\gamma_\gamma^+(x(\theta_i(\gamma))-,t))^3 \\ &- \xi_\gamma(x(\theta_i(\gamma))+,t)(\gamma_\gamma^+(x(\theta_i(\gamma))+,t))^3\}(\lambda_2(\theta_i(\gamma))-\lambda_1(\hat\theta_i(\gamma))), \end{aligned}$$

where $\lambda_2(\theta_i(\gamma)) = \lambda_2(\theta_i)$, $\lambda_1(\hat\theta_i(\gamma)) = \lambda_1(\hat\theta_i)$. A similar definition holds for a 1-shock wave γ by replacing $\lambda_2(\theta)-\lambda_1(\hat\theta)$ by $\lambda_1(\theta)-\lambda_2(\hat\theta)$.

Remark 4. The summations $B(\alpha)$ and $B(\beta)$ will be related to I_L^1 in $\frac{d}{dt}L(t)$. A rough estimate is that they are bounded by the sum of shock wave strengths to the cubic power plus some error terms:

$$\begin{aligned} &\sum_\alpha B(\alpha)+\sum_\beta B(\beta)\le c\{\sum_\alpha|\hat\alpha|+\sum_\beta|\hat\beta| \\ +\quad &\sum_\alpha|\alpha|\gamma^-_{x(\alpha)}(\gamma^-_{x(\alpha)}+|\alpha|)+\sum_\beta|\beta|\gamma^+_{x(\beta)}(\gamma^+_{x(\beta)}+|\beta|)\}. \end{aligned}$$

Proof of Lemma 5.3. Consider $\frac{d}{dt}L_h(\beta)$, β in $u^1(x,t)$. In the following, we first replace the speeds of the waves in $u^1(x,t)$ appearing in $\frac{d}{dt}L_h(\beta)$ by the corresponding wave speeds related to the one of $u^2(x,t)$.

For each x between $x^l(\beta)$ and $x^r(\beta)$, there may be some shock wave or rarefaction wave of $u^1(x,t)$ or $u^2(x,t)$ which affects the value of $\frac{d}{dt}L_h(\beta)$. We study such a wave in the following four cases; each case containing two subcases:

(I) A shock wave of $u^1(x,t)$; $(\mathbf{I})_\mathbf{1}$: 1-shock; $(\mathbf{I})_\mathbf{2}$: 2-shock;
(II) A rarefaction wave of $u^1(x,t)$: $(\mathbf{II})_\mathbf{1}$: 1-rarefaction; $(\mathbf{II})_\mathbf{2}$: 2-rarefaction;
(III) A shock wave of $u^2(x,t)$: $(\mathbf{III})_\mathbf{1}$: 1-shock; $(\mathbf{III})_\mathbf{2}$: 2-shock;
(IV) A rarefaction wave of $u^2(x,t)$: $(\mathbf{IV})_\mathbf{1}$: 1-rarefaction; $(\mathbf{IV})_\mathbf{2}$: 2-rarefaction.

$(\mathbf{I})_\mathbf{1}$: For definiteness we assume that the shock α is located to the left of β, $x^r(\beta) > x(\alpha) > x(\beta)$ (Figure 5.3). By definition, the contribution of α at $x = x(\alpha)$

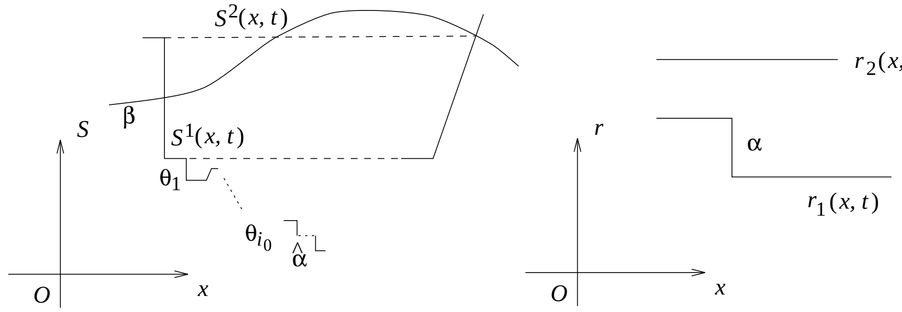

FIGURE 5.3

is:

$$\begin{aligned}
&C_\beta^{-1}\frac{d}{dt}L_h(\beta)\mid_{x=x(\alpha)}\\
&\quad=\lambda_1(\alpha)(\xi_\beta(x-,t)(\gamma_\beta^+(x-,t))^3-\xi_\beta(x+,t)(\gamma_\beta^+(x+,t))^3)\\
&\quad=\lambda_1(r^2(x,t),s^2(x,t))(\xi_\beta(x-,t)(\gamma_\beta^+(x-,t))^3-\xi_\beta(x+,t)(\gamma_\beta^+(x+,t))^3)\\
&\qquad+(\lambda_1(\alpha)-\lambda_1(r^2(x,t),s^2(x,t)))(\xi_\beta(x-,t)(\gamma_\beta^+(x-,t))^3\\
&\qquad-\xi_\beta(x+,t)(\gamma_\beta^+(x+,t))^3),
\end{aligned}$$

where the first term in the last equality will be estimated along with other similar terms later, and the second term $I(\alpha,\beta)$ is estimated as follows: At $x=x(\alpha)$, $|\lambda_1(\alpha)-\lambda_1(r^2(x,t),s^2(x,t))|\le 0(1)(\gamma_{x(\alpha)}^+ +\gamma_{x(\alpha)}^- +|\alpha|)$ and so:

$$\begin{aligned}
I(\alpha,\beta)\equiv\;&|(\lambda_1(\alpha)-\lambda_1(r^2(x,t),s^2(x,t)))(\xi_\beta(x-,t)(\gamma_\beta^+(x-,t))^3\\
&\qquad\qquad-\xi_\beta(x+,t)(\gamma_\beta^+(x+,t))^3)|\\
\le\;& c(\gamma^+(x,t)+\gamma_{x(\alpha)}^- +|\alpha|)|\xi_\beta(x-,t)(\gamma_\beta^+(x-,t))^3-\xi_\beta(x+,t)(\gamma_\beta^+(x+,t))^3|.
\end{aligned}$$

Now we consider the total sum of this kind of errors. Since $\gamma^+(x-,t)=\gamma^+(x+,t)$ at $x=x(\alpha)$, $I(\alpha,\beta)$ vanishes unless $\xi_\beta(x-,t)\ne\xi_\beta(x+,t)$ there. We denote all those 2-shock waves in $u^1(x,t)$ by $\{\theta_i\}$, $i=1,2,\cdots,n$, which satisfy $\xi_{\theta_i}(x-,t)\ne\xi_{\theta_i}(x+,t)$ at $x=x(\alpha)$. According to the definition of $\xi(x,t)$, $\xi_{\theta_i}(x-,t)\ne\xi_{\theta_i}(x+,t)$ at $x=x(\alpha)$ if and only if

$$\begin{aligned}
&\sum_{l=i}^{i_0}C_{\theta_l}(\gamma_{\theta_l}^+(x,t))^3\ge r^2(x-,t)-r^1(x-,t)\ge 0,\\
&\sum_{l=i+1}^{i_0}C_{\theta_l}(\gamma_{\theta_l}^+(x,t))^3\le r^2(x-,t)-r^1(x-,t)+|\alpha|,
\end{aligned}$$

where θ_{i_0}, $i_0\ge n$, is a 2-shock wave in $u^1(x,t)$ immediately to the left of $x(\alpha)$. In what follows, we make essential use of the notation $\gamma_\beta^\pm(x,t)$ and $\gamma_{x(\beta)*}^\pm$ defined in Section 4. At $x = x(\alpha)$,

$$\sum_{l=1}^{n} I(\alpha,\theta_l) \leq (\gamma^+_{x(\alpha)} + \gamma^-_{x(\alpha)} + |\alpha|) \times |\sum_{l=1}^{n} |\xi_{\theta_l}(x-,t)(\gamma^+_{\theta_l}(x-,t))^3 - \xi_{\theta_l}(x+,t)(\gamma^+_{\theta_l}(x+,t))^3|.$$

Since $\gamma^-_{x(\alpha)} \leq \sum_{l=1}^{i_0} C_{\theta_l}(\gamma^+_{\theta_l}(x,t))^3$ and $\sum_{l=1}^{n}(\gamma^+_{\theta_l}(x,t))^3 \leq 0(1)|\alpha|$,

$$\begin{aligned}\sum_{l=1}^{n} I(\alpha,\theta_l) &\leq c\left(|\alpha|\sum_{l=1}^{n}(\gamma^+_{\theta_l}(x,t))^3 + \gamma^+_{x(\alpha)}\sum_{l=1}^{n}(\gamma^+_{\theta_l}(x,t))^3 + |\alpha|\gamma^+_{x(\alpha)}\right)\\ &\leq c|\alpha|\gamma^+_{x(\alpha)*}.\end{aligned}$$

Thus, the second error term for $(\mathbf{I})_\mathbf{1}$ can be bounded by $c|\alpha|\gamma^+_{x(\alpha)*}$.

$(\mathbf{I})_\mathbf{2}$: Let θ, $\theta \neq \beta$, be the 2-shock wave in $u^1(x,t)$. For definiteness, we assume that $x(\beta) < x(\theta) < x^r(\beta)$. We have at $x = x(\theta)$,

$$\begin{aligned}&C_\beta^{-1}\frac{d}{dt}L_h(\beta)\mid_{x=x(\theta)}\\ &\quad= \lambda_2(\theta)(\xi_\beta(x-,t)(\gamma^+_\beta(x-,t))^3 - \xi_\beta(x+,t)(\gamma^+_\beta(x+,t))^3)\\ &\quad= \lambda_1(r^2(x,t),s^2(x,t))(\xi_\beta(x-,t)(\gamma^+_\beta(x-,t))^3 - \xi_\beta(x+,t)(\gamma^+_\beta(x+,t))^3)\\ &\qquad+ (\lambda_2(\theta)-\lambda_1(r^2(x,t),s^2(x,t))(\xi_\beta(x-,t)(\gamma^+_\beta(x-,t))^3-\xi_\beta(x+,t)(\gamma^+_\beta(x+,t))^3).\end{aligned}$$

The first term of the last equality will be estimated later; the second term can be estimated as follows: As in $(\mathbf{I})_\mathbf{1}$, we need to estimate the sum of the terms of the form:

$$I(\theta,\beta) = |(\lambda_2(\theta) - \lambda_1(r^2(x,t),s^2(x,t))(\xi_\beta(x-,t)(\gamma^+_\beta(x-,t))^3 - \xi_\beta(x+,t)(\gamma^+_\beta(x+,t))^3)|.$$

Notice that $\gamma^+_\beta(x-,t) = \gamma^+_\beta(x+,t)$ at $x = x(\theta)$. $I(\theta,\beta)$ vanishes unless $\xi_\beta(x-,t) \neq \xi_\beta(x+,t)$ there. We denote by $\{\theta_i\}$, $i = 1,2,\cdots,n$, those 2-shock waves in $u^1(x,t)$ consecutively from left to right, which satisfy $\xi_{\theta_i}(x-,t) \neq \xi_{\theta_i}(x+,t)$ at $x = x(\theta)$. Furthermore, those shock waves of $u^1(x,t)$ between $x(\theta_n)$ and $x(\theta)$ are denoted by $\{\theta_j\}$ consecutively, $j = n+1, n+2, \cdots, m-1$, and $\theta_m = \theta$. According to the definition of ξ, $\xi_{\theta_l}(x-,t) \neq \xi_{\theta_l}(x+,t)$ if and only if

$$\begin{aligned}&\sum_{j=l}^{m} C_{\theta_j}(\gamma^+_{\theta_j}(x,t))^3 \geq r^2(x-,t) - r^1(x-,t) \geq 0,\\ &\sum_{j=l+1}^{m} C_{\theta_j}(\gamma^+_{\theta_j}(x,t))^3 \leq r^2(x-,t) - r^1(x-,t) + |\hat{\theta}|, \quad 1 \leq l \leq n.\end{aligned}$$

Since

$$\sum_{l=1}^{n} |\xi_{\theta_l}(x-,t)(\gamma^+_{\theta_l}(x-,t))^3 - \xi_{\theta_l}(x+,t)(\gamma^+_{\theta_l}(x+,t))^3| \leq |\hat{\theta}|,$$

we have

$$\sum_{l=1}^{n} I(\theta,\theta_l) \le c\sum_{l=1}^{n} |\xi_{\theta_l}(x-,t)(\gamma^+_{\theta_l}(x-,t))^3 - \xi_{\theta_l}(x+,t)(\gamma^+_{\theta_l}(x+,t))^3|$$
$$\le c|\hat\theta|^{\frac{1}{3}}(\sum_{l=1}^{n}\gamma^+_{\theta_l}(x,t))^{\frac{2}{3}}$$
$$\le c|\theta|(\gamma^+_{x(\theta)})^2.$$

Therefore, the error terms $I(\theta,\theta_l)$ of $(\mathbf{I})_\mathbf{2}$ can be controlled by $0(1)|\theta|(\gamma^+_{x(\theta)})^2$.

$(\mathbf{II})_\mathbf{1}$: Let α be a 1-rarefaction wave in $u^1(x,t)$. Similar to $(\mathbf{I})_\mathbf{1}$, we have

$$C_\beta^{-1}\frac{d}{dt}L_h(\beta)\mid_{x=x(\alpha)} = \lambda_1(\alpha)(\xi_\beta(x-,t)(\gamma^+_\beta(x-,t))^3 - \xi_\beta(x+,t)(\gamma^+_\beta(x+,t))^3)$$
$$= \lambda_1(r^2(x,t),s^2(x,t))(\xi_\beta(x-,t)(\gamma^+_\beta(x-,t))^3 - \xi_\beta(x+,t)(\gamma^+_\beta(x+,t))^3)$$
$$+ (\lambda_1(\alpha)-\lambda_1(r^2(x,t),s^2(x,t))(\xi_\beta(x-,t)(\gamma^+_\beta(x-,t))^3-\xi_\beta(x+,t)(\gamma^+_\beta(x+,t))^3)$$
$$\le \lambda_1(r^2(x,t),s^2(x,t))(\xi_\beta(x-,t)(\gamma^+_\beta(x-,t))^3 - \xi_\beta(x+,t)(\gamma^+_\beta(x+,t))^3)$$
$$+ c|\alpha|\gamma^+_{x(\alpha)*}.$$

$(\mathbf{II})_\mathbf{2}$: Denote by θ the 2-rarefaction wave in $u^1(x,t)$. There are two cases depending on the location of θ in the region $\Omega(\beta)$:

Case 1. When θ is on the boundary of $\Omega(\beta)$, similar to $(\mathbf{I})_\mathbf{2}$, we have, at $x = x(\theta)$,

$$C_\beta^{-1}\frac{d}{dt}L_h(\beta)\mid_{x=x(\theta)} = \lambda_2(\theta)(\xi_\beta(x-,t)(\gamma^+_\beta(x-,t))^3 - \xi_\beta(x+,t)(\gamma^+_\beta(x+,t))^3)$$
$$= \lambda_1(r^2(x,t),s^2(x,t))(\xi_\beta(x-,t)(\gamma^+_\beta(x-,t))^3 - \xi_\beta(x+,t)(\gamma^+_\beta(x+,t))^3)$$
$$+ (\lambda_2(\theta) - \lambda_1(r^2(x,t),s^2(x,t))(\xi_\beta(x-,t)(\gamma^+_\beta(x-,t))^3 - \xi_\beta(x+,t)(\gamma^+_\beta(x+,t))^3)$$
$$\le \lambda_1(r^2(x,t),s^2(x,t))(\xi_\beta(x-,t)(\gamma^+_\beta(x-,t))^3 - \xi_\beta(x+,t)(\gamma^+_\beta(x+,t))^3)$$
$$+ c(|\theta|\epsilon + |\theta|\gamma^+_{x(\theta)}(|\theta| + \gamma^+_{x(\theta)})).$$

Case 2. When θ is not on the boundary of $\Omega(\beta)$, $\gamma^+_\beta(x-,t) = \gamma^+_\beta(x+,t)$ at $x = x(\theta)$. Hence $\xi_\beta(x-,t)(\gamma^+_\beta(x-,t))^3 - \xi_\beta(x+,t)(\gamma^+_\beta(x+,t))^3 = 0$ unless $\xi_\beta(x-,t) \ne \xi_\beta(x+,t)$. As in $(\mathbf{I})$, we consider the summation of all these nonzero terms. By the definition of ξ, we have

$$\sum_\beta |\xi_\beta(x-,t)(\gamma^+_\beta(x-,t))^3 - \xi_\beta(x+,t)(\gamma^+_\beta(x+,t))^3| \le \delta(\theta),$$

where the summation is over all 2-shock waves β in $u^1(x,t)$ satisfying $\xi_\beta(x-,t) \ne \xi_\beta(x+,t)$ at $x = x(\theta)$, and

$$\delta(\theta) = (|\theta| + \gamma^+_{x(\theta)})^3 - (\gamma^+_{x(\theta)})^3$$
$$\le 0(1)(|\theta|\epsilon + |\theta|\gamma^+_{x(\theta)}(|\theta| + \gamma^+_{x(\theta)})).$$

Notice that the terms $|\alpha|\epsilon$ can be summed and the sum is bounded by $T.V.\epsilon$, which tends to zero as the grid size approaches zero. The cases **(III)** and **(IV)** can be discussed similarly. Combining all these estimates and putting back the error term $e(\Lambda_p)$, which has been omitted, we have

(5.13)
$$\frac{d}{dt}L_h(\beta) \le c\{\sum_\alpha |\alpha|\gamma^+_{x(\alpha)*} + \sum_\beta |\beta|\gamma^-_{x(\beta)*} + \sum_\alpha |\alpha|\gamma^-_{x(\alpha)}(|\alpha| + \gamma^-_{x(\alpha)}) + \sum_\beta |\beta|\gamma^+_{x(\beta)}(|\beta| + \gamma^+_{x(\beta)}) + e(\Lambda_p)\} + J_1 + B(\beta),$$

where J_1 and $B(\beta)$ are terms which we have so far not estimated:

$$J_1 = \sum_i \tilde{\lambda}_1^i(\xi_\beta(x_i-,t)\gamma^+_\beta(x_i-,t))^3 - \xi_\beta(x_i+,t)(\gamma^+_\beta(x_i+,t))^3) + \lambda_1(\hat{\beta})(\xi_\beta(x(\beta)-,t)\gamma^+_\beta(x(\beta)-,t))^3 - \xi_\beta(x(\beta)+,t)(\gamma^+_\beta(x(\beta)+,t))^3).$$

Here $\{x_i\}$ are the locations of waves in $u^1(x,t)$ and $u^2(x,t)$ between $x^l(\beta)$ and $x^r(\beta)$. When the wave at x_i is in $u^1(x,t)$, $\tilde{\lambda}_1^i$ denotes the wave speed of the first family evaluated at $(r^2(x_i,t), s^2(x_i,t))$. When the wave at x_i is in $u^2(x,t)$, $\tilde{\lambda}_1^i$ equals $\lambda_1(\alpha^2)$ or $\lambda_1(\hat{\beta}^2)$.

Now we estimate J_1. Notice that $|\gamma^+_\beta(x,t)| \le \epsilon$ when $x = x^l(\beta)$ or $x^r(\beta)$. We denote $\lambda_1(r^2(x(\beta),t), s^2(x(\beta),t))$ by $\tilde{\lambda}_1^{i_0}$, and $x(\beta) = x_{i_0}$ for some i_0. According to the above discussions, for each wave between $x^l(\beta)$ and $x^r(\beta)$, there is an x_i corresponding to it. Hence, $\xi_\beta(x_i+,t)(\gamma^+_\beta(x_i+,t))^3 = \xi_\beta(x_{i+1}-,t)(\gamma^+_\beta(x_{i+1}-,t))^3$ for all i. Summation by parts of J_1 gives

(5.14)
$$J_1 = J_2 + (\lambda_1(\hat{\beta}) - \lambda_1(r^2(x(\beta),t), s^2(x(\beta),t)))(\xi_\beta(x(\beta)-,t)(\gamma^+_\beta(x(\beta)-,t))^3 - \xi_\beta(x(\beta)+,t)(\gamma^+_\beta(x(\beta)+,t))^3) + c|\beta|\epsilon,$$

where $J_2 = \sum_i \xi_\beta(x_i+,t)(\gamma^+_\beta(x_i+,t))^3(\tilde{\lambda}_1^{i+1} - \tilde{\lambda}_1^i)$. Now we estimate J_1 in two cases.

<u>Case 1.</u> If $|\beta|^3$ can be uniformly controlled by terms like $E(\gamma)$ for some γ, then

$$\begin{aligned}|J_1| &\le c|\beta|\{|\beta|\gamma^-_{x(\beta)*} + |\beta|\gamma^+_{x(\beta)}(|\beta| + \gamma^+_{x(\beta)}) + E(\gamma)\} + |J_2| \\ &\le c|\beta|\{|\beta|\gamma^-_{x(\beta)*} + |\beta|\gamma^+_{x(\beta)}(|\beta| + \gamma^+_{x(\beta)}) + E(\gamma) + \sum_{\alpha^2}|\alpha^2|\gamma^+_\beta(x(\alpha),t) \\ &\quad + \sum_{\beta^2}|\beta^2|\gamma^+_\beta(x(\beta^2),t)(|\beta^2| + \gamma^+_\beta(x(\beta^2),t)) + T.V.\epsilon\},\end{aligned}$$

where the summations are over α^2 and β^2 in $(x^l(\beta), x^r(\beta))$ and the $T.V.$ is also over $(x^l(\beta), x^r(\beta))$.

<u>Case 2.</u> In this case, we assume that there exists a $\theta(\beta)$ such that $|\bar{\theta}(\beta)| \ge \mu|\beta|$. If we denote the shock wave corresponding to $\theta(\beta)$ by $\tilde{\beta}$, according to Remark 2, we only need to consider the case when $|\tilde{\beta}| \le 2|\beta|$. For definiteness, we assume that $x(\beta) < x(\tilde{\beta})$. Now we estimate each term in $|J_1|$ as follows:

By the definition of ξ_β, we have

$$C_\beta\xi_\beta(x(\beta)+,t)(\gamma^+_\beta(x(\beta)+,t))^3) = |\hat{\beta}^t|,$$

and, according to Claims 1–4, $|\xi_\beta(x(\beta)-,t)(\gamma_\beta^+(x(\beta)-,t))^3|$ can be uniformly controlled by $cE(\beta)$. Hence

$$\begin{aligned}
&|(\lambda_1(\hat\beta)-\lambda_1(r^2(x(\beta),t),s^2(x(\beta),t)))(\xi_\beta(x(\beta)-,t)(\gamma_\beta^+(x(\beta)-,t))^3\\
&\qquad\qquad\qquad\qquad -\xi_\beta(x(\beta)+,t)(\gamma_\beta^+(x(\beta)+,t))^3)|\\
&\le c\{(|\beta^b|+\gamma^+_{x(\beta)}+\gamma^-_{x(\beta)})|\hat\beta^t|+E(\beta)\}\\
&\le c\{E(\beta)+|\beta|\gamma^-_{x(\beta)*}+|\beta|\gamma^+_{x(\beta)}(|\beta|+\gamma^+_{x(\beta)})\}\le cE(\beta).
\end{aligned}$$

Next we consider each term in J_2. Since the difference between two adjacent $\tilde\lambda_i$ is dominated by the sum of wave strengths of the corresponding 1-waves or minor waves of the second family in $u^2(x,t)$, it is easy to check that, except for the term corresponding to $\tilde\beta$, the other terms can be bounded by $E(\beta)+T.V.\epsilon$.

It remains to consider the term $\xi_\beta(x(\tilde\beta)-,t)(\gamma_\beta^+(x(\tilde\beta)-,t))^3(\lambda_1(\hat{\tilde\beta})-\tilde\lambda_1^i)$ for some $\tilde\lambda_1^i$. Notice that in the above discussions we may replace $\tilde\lambda_1^i$ by $\lambda_1(r_-(\tilde\beta),s_-(\tilde\beta))$. Assume that there exists a sequence of 2-shock waves in $u^1(x,t)$ between $x(\beta)$ and $x(\tilde\beta)$, denoted by $\{\theta_i\}_{i=1}^n$. Then

$$\begin{aligned}
&|\xi_\beta(x(\tilde\beta)-,t)(\gamma_\beta^+(x(\tilde\beta)-,t))^3(\lambda_1(\hat{\tilde\beta})-\lambda_1(r_-(\tilde\beta),s_-(\tilde\beta)))|\\
&\quad\le c|\beta|\big|(C_\beta\gamma_\beta^+(x(\tilde\beta)-,t))^3+C_\beta\sum_{i=1}^n(\gamma^+_{\theta_i}(x(\tilde\beta)-,t))^3\\
&\qquad\qquad -(r^2(x(\tilde\beta)-,t)-r^1(x(\tilde\beta)-,t)+|\hat{\tilde\beta}^t|)\big|\\
&\quad\le c\{|\tilde\beta|\gamma^-_{x(\tilde\beta)*}+|\beta|\sum_{i=1}^n(\gamma^+_{\theta_i}(x(\tilde\beta)-,t))^3+|\beta||C_\beta\gamma_\beta^+(x(\tilde\beta)-,t))^3-|\hat{\tilde\beta}||\}\\
&\quad\le c\{|\tilde\beta|\gamma^-_{x(\tilde\beta)*}+\sum_{i=1}^n|\theta_i|^2\gamma_\beta^+(x(\theta_i),t)(|\theta_i|+\gamma_\beta^+(x(\theta_i),t))+E(\beta)+E(\tilde\beta)\}.
\end{aligned}$$

Notice that

$$\sum_{\gamma^1}|\alpha|\gamma^+_{\gamma^1}(x(\alpha)-,t)\le 2|\alpha|\gamma^+_{x(\alpha)-},\qquad \sum_{\gamma^2}|\beta|\gamma^-_{\gamma^2}(x(\beta)+,t)\le 2|\beta|\gamma^-_{x(\beta)+},$$

and

$$\begin{aligned}
&\sum_{\gamma^1}|\alpha|\gamma^-_{\gamma^1}(x(\alpha),t)(|\alpha|+\gamma^-_{\gamma^1}(x(\alpha),t))\le 2|\alpha|\gamma^-_{x(\alpha)}(|\alpha|+\gamma^-_{x(\alpha)}),\\
&\sum_{\gamma^2}|\beta|\gamma^+_{\gamma^2}(x(\beta),t)(|\beta|+\gamma^+_{\gamma^2}(x(\beta),t))\le 2|\beta|\gamma^+_{x(\beta)}(|\beta|+\gamma^+_{x(\beta)}),
\end{aligned}$$

where γ^i denotes the i-wave and the summations are over the corresponding waves in the region of consideration.

Therefore, combining Case 1 and Case 2, we have

$$\begin{aligned}
\sum_\beta\frac{d}{dt}L_h(\beta)\le c\{&\sum_\alpha|\alpha|\gamma^+_{x(\alpha)*}+\sum_{\bar\beta}|\bar\beta|\gamma^-_{x(\bar\beta)*}+\sum_\alpha|\alpha|\gamma^-_{x(\alpha)}(|\alpha|+\gamma^-_{x(\alpha)})\\
&+\sum_{\bar\beta}|\bar\beta|\gamma^+_{x(\bar\beta)}(|\bar\beta|+\gamma^+_{x(\bar\beta)})+e(\Lambda_p)\}+B(\beta),
\end{aligned}$$

where the summations are over all waves in $u^1(x,t)$ and $u^2(x,t)$. A similar estimate holds when β is in $u^2(x,t)$.

We also have similar estimates for $\frac{d}{dt}L_h(\alpha)$, when α is a shock wave in either $u^1(x,t)$ or $u^2(x,t)$. This completes the proof of the lemma. □

With the estimates for $\frac{d}{dt}L(t)$ and $\frac{d}{dt}L_h(t)$, we are ready to state and prove the following main lemma of this section.

Lemma 5.4. *Under the hypotheses of Main Lemma 1, for each $t \in I_p^c$, we have*

(5.15)

$$\frac{d}{dt}\tilde{L}(t) \leq c\{T.V.\sum_{\alpha}|\alpha^t||\alpha^b| + T.V.\sum_{\beta}|\beta^t||\beta^b| + \sum_{\alpha}|\alpha|\gamma^+_{x(\alpha)*} + \sum_{\beta}|\beta|\gamma^-_{x(\beta)*} + \sum_{\alpha}|\alpha|\gamma^-_{x(\alpha)}(|\alpha| + \gamma^-_{x(\alpha)}) + \sum_{\beta}|\beta|\gamma^+_{x(\beta)}(|\beta| + \gamma^+_{x(\beta)}) + e(\Lambda_p)\},$$

where the summations are over all waves in $u^1(x,t)$ and $u^2(x,t)$.

Proof. From now on, if we do not explicitly state it, all the summations and total variations are over the corresponding interval of discussion. This will not cause ambiguity. The proof is based on Claims 1–4 of the previous section. From here on, we fix a μ between $\frac{1}{2}$ and 1. Notice that the only error terms we need to consider here are the shock wave strengths to a cubic order. Following the notation used in Lemma 5.3, we consider a 2-shock wave β in $u^1(x,t)$. We have the following two cases.

Case 1. For any i, $|\bar{\theta}_i(\beta)| < \mu|\beta|$.

Case 2. There exists i_0 such that $|\bar{\theta}_{i_0}(\beta)| \geq \mu|\beta|$.

For Case 1, according to Claim 2, $|\beta|^3$ can be controlled by the terms on the right hand side of (5.15). Therefore we only need to consider Case 2. Since there are terms $0(1)T.V.|\beta^t||\beta^b|$ on the right hand side of (5.15), we may assume, without loss of generality, that either β^t or β^b is zero. A similar assumption applies to θ_{i_0}. We also assume that the minor waves of β and θ_{i_0} are not crossed by the r curve of the other solution in the r-x plane. The cases when it crosses can be discussed similarly. By using the same notation and techniques of proving Claims 1–4, we have from Lemma 5.3 that

(5.16)

$$\begin{aligned}\frac{d}{dt}L_h(\beta)(t) \leq C_\beta(&\xi_\beta(x(\beta)-,t)(\gamma^+_\beta(x(\beta)-,t))^3 \\ &- \xi_\beta(x(\beta)+,t)(\gamma^+_\beta(x(\beta)+,t))^3)(\lambda_2(\beta) - \lambda_1(\hat{\beta})) \\ &+ C_\beta(\xi_\beta(x(\theta_{i_0}(\beta))-,t)(\gamma^+_\beta(x(\theta_{i_0}(\beta))-,t))^3 - \xi_\beta(x(\theta_{i_0}(\beta))+,t) \\ &\times(\gamma^+_\beta(x(\theta_{i_0}(\beta))+,t))^3)(\lambda_2(\theta_{i_0}(\beta)) - \lambda_1(\hat{\theta}_{i_0}(\beta))) \\ &+ cE(\beta).\end{aligned}$$

A similar estimate holds for $\frac{d}{dt}L_h(\alpha)(t)$. According to the positions of the minor waves $\hat{\beta}$ and $\hat{\theta}_{i_0}(\beta)$, we have the following four subcases. For definiteness, we assume that $x(\theta_{i_0}) < x(\beta)$.

Subcase 1. $\xi_\beta(x(\theta_{i_0}),t) = 1$, $\xi_\beta(x(\beta)-,t) = 0$, and $\xi_{\theta_{i_0}}(x(\theta_{i_0})+,t) = 1$; cf. Figure 5.4.

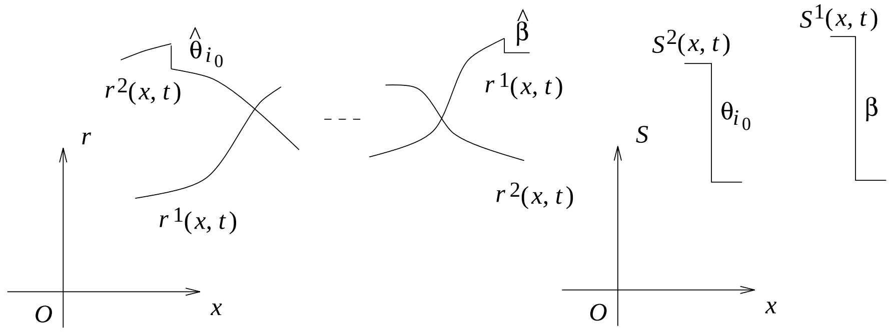

FIGURE 5.4

Using the definition of ξ, we also know that $\xi_{\theta_{i_0}}(x(\beta)-,t) = 0$ unless there are some shock waves in $u^2(x,t)$ between $x(\theta_{i_0})$ and $x(\beta)$. In that case, the nonzero part of

$$\xi_{\theta_{i_0}}(x(\beta(\theta_{i_0}))-,t)(\gamma^+_{\theta_{i_0}}(x(\beta(\theta_{i_0}))-,t))^3 - \xi_{\theta_{i_0}}(x(\beta(\theta_{i_0}))+,t)(\gamma^+_{\theta_{i_0}}(x(\beta(\theta_{i_0}))+,t))^3$$

can be controlled by $E(\theta_{i_0})$, where $\beta(\theta_{i_0})$ denotes the part of β in $\Omega(\theta_{i_0})$. Thus by (5.16) we have

$$\frac{d}{dt}L_h(\beta) \leq -C_\beta|\bar{\theta}_{i_0}(\beta)|^3(\lambda_2(\theta_{i_0}) - \lambda_1(\hat{\theta}_{i_0})) + cE(\beta)$$

and

$$\frac{d}{dt}L_h(\theta_{i_0}) \leq -|\hat{\theta}_{i_0}|(\lambda_2(\theta_{i_0}) - \lambda_1(\hat{\theta}_{i_0})) + cE(\theta_{i_0}).$$

Since

$$|C_\beta|\bar{\theta}_{i_0}(\beta)|^3(\lambda_2(\theta_{i_0}) - \lambda_1(\hat{\theta}_{i_0})) - |\hat{\beta}|(\lambda_2(\beta) - \lambda_1(\hat{\beta}))| \leq cE(\beta),$$

we have

$$\begin{aligned} &\frac{d}{dt}(L_h(\beta) + L_h(\theta_{i_0})) \leq -|\hat{\beta}|(\lambda_2(\beta) - \lambda_1(\hat{\beta})) \\ - \quad &|\hat{\theta}_{i_0}|(\lambda_2(\theta_{i_0}) + \lambda_1(\hat{\theta}_{i_0})) + c(E(\beta) + E(\theta_{i_0})). \end{aligned}$$

By (5.12) of Lemma 5.2, the terms in I^1_L related to β and θ_{i_0} are cancelled by $\frac{d}{dt}(L_h(\beta) + L_h(\theta_{i_0}))$ up to an error term, which is bounded by $c(E(\beta) + E(\theta_{i_0}))$. Now we are going to show that the terms in I^2_L related to β and θ_{i_0}, denoted by $I^2_L(\beta, \theta_{i_0})$, can also be controlled by $c(E(\beta) + E(\theta_{i_0}))$. These terms can be written as (cf. Figure 5.4)

$$\begin{aligned} I^2_L(\beta,\theta_{i_0}) &= (\lambda_2(\beta) - \hat{\lambda}_2(\beta))(\gamma^+_{x(\beta)-} - \gamma^+_{x(\beta)+}) \\ &+ (\lambda_2(\theta_{i_0}) - \hat{\lambda}_2(\theta_{i_0}))(\gamma^+_{x(\theta_{i_0})-} - \gamma^+_{x(\theta_{i_0})+}) \\ &= g(\beta)|\beta|^3 - g(\theta_{i_0})|\theta_{i_0}|^3, \end{aligned}$$

where $g(\beta)$ and $g(\theta_{i_0})$ are Lipschitz continuous functions. Based on the above discussion, we have

$$\begin{aligned} I^2_L(\beta,\theta_{i_0}) &\leq c(||\beta|^3 - |\theta_{i_0}|^3| + |g(\beta) - g(\theta_{i_0})||\theta_{i_0}|^3) \\ &\leq c(E(\beta) + E(\theta_{i_0})). \end{aligned}$$

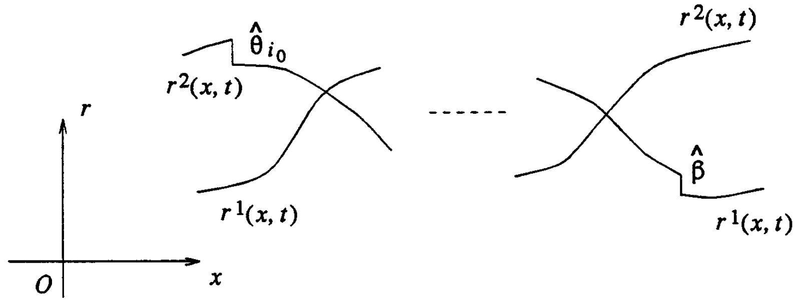

FIGURE 5.5

Therefore the error terms in both I_L^1 and I_L^2 related to β and θ_{i_0}, together with $\frac{d}{dt}(L_h(\beta)+L_h(\theta_{i_0}))$, can be controlled by $c(E(\beta)+E(\theta_{i_0}))$, uniformly with respect to μ.

Subcase 2. $\xi_\beta(x(\beta)-,t)=\xi_\beta(x(\theta_{i_0})+,t)=1$ and $\xi_{\theta_{i_0}}(x(\theta_{i_0})+,t)=\xi_{\theta_{i_0}}(x(\beta)-,t)=1$ (Figure 5.5). From (5.16)

$$\begin{aligned}\frac{d}{dt}L_h(\beta) &\le |\hat{\beta}|(\lambda_2(\beta)-\lambda_1(\hat{\beta}))\\ &- C_\beta|\bar{\theta}_{i_0}(\beta)|^3(\lambda_2(\theta_{i_0})-\lambda_1(\hat{\theta}_{i_0}))+cE(\beta)\end{aligned}\tag{5.17}$$

and

$$\begin{aligned}\frac{d}{dt}L_h(\theta_{i_0}) &\le -|\hat{\theta}_{i_0}|(\lambda_2(\theta_{i_0})-\lambda_1(\hat{\theta}_{i_0}))\\ &+ C_{\theta_{i_0}}|\bar{\beta}(\theta_{i_0})|^3(\lambda_2(\beta)-\lambda_1(\hat{\beta}))+cE(\theta_{i_0}).\end{aligned}\tag{5.18}$$

Combining (5.17) and (5.18) gives

$$\frac{d}{dt}(L_h(\beta)+L_h(\theta_{i_0}))\le c(E(\beta)+E(\theta_{i_0})),$$

where we have used Claims 1–4. By using Claims 1–4 again and the discussion similar to Subcase 1, we can show that the terms related to β and θ_{i_0} in I_L^1 and I_L^2 can be controlled by $c(E(\beta)+E(\theta_{i_0}))$.

The other two subcases, $\xi_{\theta_{i_0}}(x(\theta_{i_0})+,t)=\xi_\beta(x(\beta)-,t)=0$ and $\xi_\beta(x(\beta)-,t)=1$, $\xi_{\theta_{i_0}}(x(\theta_{i_0})+,t)=0$, can be discussed similarly. This completes the proof of the lemma. □

Remark 5. The following estimates, obtained in Lemma 5.4, are needed when we estimate $\frac{d}{dt}Q_d(t)$ and $\frac{d}{dt}\Delta(t)$ because, in $Q_d(t)$ and $\Delta(t)$, the L_1 norms of different regions are multiplied by different factors.

(1) In Subcases 1 and 4 when either $I_L^1(\beta)>0, I_L^1(\theta_{i_0})>0$, or $I_L^1(\beta)<0, I_L^1(\theta_{i_0})<0$, we have

$$\frac{d}{dt}L_h(\beta)\le -I_L^1(\beta)+cE(\beta),\qquad \frac{d}{dt}L_h(\theta_{i_0})\le -I_L^1(\theta_{i_0})+cE(\theta_{i_0}).$$

(2) In Subcases 2 and 3 when either $I_L^1(\beta)>0$, $I_L^1(\theta_{i_0})<0$ or $I_L^1(\beta)<0$, $I_L^1(\theta_{i_0})>0$, we have

$$\frac{d}{dt}L_h(\beta)\le cE(\beta),\qquad \frac{d}{dt}L_h(\theta_{i_0})\le cE(\theta_{i_0}).$$

6. Nonlinear functionals

6.1. **Estimation of $\frac{d}{dt}Q_d(t)$.** As stated in Lemma 5.4, there are two main types of error terms for $\frac{d}{dt}\tilde{L}(t)$: the first is $\sum_\alpha |\alpha|\gamma^+_{x(\alpha)*} + \sum_\beta |\beta|\gamma^-_{x(\beta)*}$ and the second is $\sum_\alpha |\alpha|\gamma^-_{x(\alpha)}(|\alpha| + \gamma^-_{x(\alpha)}) + \sum_\beta |\beta|\gamma^+_{x(\beta)}(|\beta| + \gamma^+_{x(\beta)})$. The first will be controlled by the good terms of $\frac{d}{dt}Q_d(t)$ and the second by those of $\frac{d}{dt}\Delta(t)$.

The functional $Q_d(t)$ measures the strong decoupling of the effect of one family of waves on the L_1 norm of other characteristic distances. This is due mainly to the strict hyperbolicity of the system, $\lambda_1 < \lambda_2$.

Lemma 6.1. *Under the hypotheses of Main Lemma 1, for each $t \in I_p^c$, we have*

$$\begin{aligned}
\frac{d}{dt}Q_d(t) &\le -c_1(\sum_\alpha |\alpha|\gamma^+_{x(\alpha)*} + \sum_\beta |\beta|\gamma^-_{x(\beta)*}) \\
&+ cT.V.\{\sum_\alpha |\alpha|\gamma^-_{x(\alpha)}(|\alpha| + \gamma^-_{x(\alpha)}) + \sum_\beta |\beta|(\gamma^+_{x(\beta)}(|\beta| + \gamma^+_{x(\beta)}) \\
&+ \sum_\alpha |\alpha^t||\alpha^b| + \sum_\beta |\beta^t||\beta^b|) + e(\Lambda_p)\},
\end{aligned}$$

where c_1 is a positive constant independent of T and s.

Proof. First, we consider $\frac{d}{dt}Q_d^-(\alpha^p)(t)$ when α^p is a rarefaction wave in $u^p(x,t)$. Notice that $s^i(x(\alpha^p)-,t) = s^i(x(\alpha^p)+,t)$, $i = 1,2$. By Definition 3 of $Q_d^-(\alpha^p)(t)$ in Section 3 and the discussion on $\frac{d}{dt}\tilde{L}(t)$ in Section 5, and by the crucial fact that $\lambda_1 < \lambda_2$, we have

$$\begin{aligned}
\frac{d}{dt}Q_d^-(\alpha^p)(t) &= \frac{d}{dt}(|\alpha^p|\int_{-\infty}^{x(\alpha^p)} \Theta^+(x,t)dx) \\
&\le c|\alpha^p|\{\sum_\alpha |\alpha|(\gamma^+_{x(\alpha)*} + \gamma^-_{x(\alpha)}(|\alpha| + \gamma^-_{x(\alpha)}) + |\alpha^t||\alpha^b|)\ |_{x(\alpha)<x(\alpha^p)} \\
&+ \sum_\beta |\beta|(\gamma^-_{x(\beta)*} + \gamma^+_{x(\beta)}(|\beta| + \gamma^+_{x(\beta)}) + |\beta^t||\beta^b|)\ |_{x(\beta)<x(\alpha^p)}\} \\
&- \bar{c}|\alpha^p|\gamma^+_{x(\alpha^p)} + |\alpha^p|\{\sum_{\alpha, x(\alpha)<x(\alpha^p)} B(\alpha) + I_L^{1,+} + I_L^{2,+}\}\ |_{x<x(\alpha^p)},
\end{aligned} \tag{6.1}$$

where $\bar{c}$ is a positive constant, and the summations $B(\alpha)$, I_L^1 and I_L^2 are only for the parts corresponding to the region to the left of $x(\alpha^p)$; $I_L^{i,\pm}$, $i = 1,2$, denotes the part of I_L^i corresponding to $\Theta^\pm(x,t)$ respectively.

Next we discuss the case when α^p is a shock wave. In this case, part of $\hat{\alpha}^p$ may appear in the integral $\int_{-\infty}^{x(\alpha^p)} \Theta^+(x,t)dx$. By definition, this part is $|\hat{\alpha}^{p,t}|$, which is of cubic order of the strength of α^p. Hence we have the following cases. If $|\alpha|^3$ is uniformly controlled with respect to μ, then we still have (6.1). Next we consider the case when there exists $\theta(\alpha^p)$ such that $|\bar{\theta}(\alpha^p)| \ge \mu|\alpha^p|$ in the region $\Omega(\alpha^p)$. And, as before, we can assume, by Remark 2, that $|\theta| \le 2|\alpha^p|$. Depending on the relative positions of α^p and θ, there are two cases: $x(\alpha^p) < x(\theta)$ and $x(\theta) < x(\alpha^p)$. According to the discussion on $\frac{d}{dt}L(t)$, we also have (6.1).

Similarly, we can prove

$$\begin{aligned}\frac{d}{dt}Q_d^+(\beta^p)(t) &\leq |\beta^p|\{c(\sum_\alpha |\alpha|(\gamma^+_{x(\alpha)*}+\gamma^-_{x(\alpha)}(|\alpha|+\gamma^-_{x(\alpha)})+|\alpha^t||\alpha^b|)\mid_{x(\alpha)>x(\beta^p)}\\ &+ \sum_\beta |\beta|(\gamma^-_{x(\beta)*}+\gamma^+_{x(\beta)}(|\beta|+\gamma^+_{x(\beta)})+|\beta^t||\beta^b|)\mid_{x(\beta)>x(\beta^p)}\}\\ &- \bar{c}|\beta^p|\gamma^-_{x(\beta^p)*}+|\beta^p|\{\sum_{\beta,x(\beta)>x(\beta^p)} B(\beta)+I_L^{1,-}+I_L^{2,-}\}\mid_{x>x(\beta^p)},\end{aligned}$$

for any β^p, where the summations I_L^1, I_L^2 and $B(\beta)$ are for the corresponding regions to the right of $x(\beta^p)$.

Combining the estimates for $\frac{d}{dt}Q_d^-(\alpha^p)(t)$ and $\frac{d}{dt}Q_d^+(\beta^p)(t)$ yields

$$\begin{aligned}\frac{d}{dt}Q_d(t) &\leq -\bar{c}(\sum_\alpha |\alpha|\gamma^+_{x(\alpha)*}+\sum_\beta|\beta|\gamma^-_{x(\beta)*})\\ &+ cT.V.(\sum_\alpha|\alpha|\gamma^-_{x(\alpha)}(|\alpha|+\gamma^-_{x(\alpha)})+\sum_\beta|\beta|\gamma^+_{x(\beta)}(|\beta|+\gamma^+_{x(\beta)})\\ &+ \sum_\alpha|\alpha^t||\alpha^b|+\sum_\beta|\beta^t||\beta^b|)+R,\end{aligned}\tag{6.2}$$

where

$$\begin{aligned}R = &\sum_\alpha|\alpha|\{\sum_{\alpha',x(\alpha')<x(\alpha)} B(\alpha')+I_L^+\}\mid_{x<x(\alpha)}\\ &+\sum_\beta|\beta|\{\sum_{\beta',x(\beta')>x(\beta)} B(\beta')+I_L^-\}\mid_{x>x(\beta)},\end{aligned}$$

and $I_L^\pm = I_L^{1,\pm}+I_L^{2,\pm}$. It remains to estimate the term R. Notice that

$$|R|\leq cT.V.(\sum_\alpha|\hat{\alpha}|+\sum_\beta|\hat{\beta}|+\sum_\alpha|\alpha|\gamma^-_{x(\alpha)}(|\alpha|+\gamma^-_{x(\alpha)})+\sum_\beta|\beta|\gamma^+_{x(\beta)}(|\beta|+\gamma^+_{x(\beta)})).$$

Thus for any shock wave α or β, if either $E(\gamma)\geq c(\mu)|\alpha|^3$ or $E(\gamma)\geq c(\mu)|\beta|^3$ for some γ, uniformly with respect to μ, then the error terms in R corresponding to α or β can be controlled by $E(\gamma)$ for some γ. Without loss of generality, we consider the case for a shock wave β_1 in $u^1(x,t)$. As before, we only need to consider the case when there exists a shock wave $\theta_i=\beta_2$ in $u^2(x,t)$ with $\bar{\theta}_i(\beta_1)\geq\mu|\beta_1|$. By Remark 2, we can assume that $|\beta_2|\leq 2|\beta_1|$. Depending on the x coordinate of β in the summation in R, we have the following three cases for

$$\sum_\beta|\beta|\{\sum_{\beta',x(\beta')>x(\beta)} B(\beta')+I_L^{1,-}\}\mid_{x>x(\beta)}.$$

<u>Case 1.</u> $x(\beta)>\max\{x(\beta_1),x(\beta_2)\}$. In this case, the major error terms in $|\beta|(B(\beta_i)+I_L^{1,-}(\beta_i))$, $i=1,2$, are not in R and so all the error terms related to β_i, $i=1,2$, in $\sum_\beta|\beta|\{\sum_{\beta',x(\beta')>x(\beta)}B(\beta')+I_L^{1,-}\}\mid_{x>x(\beta)}$ can be controlled by $c|\beta|(E(\beta_1)+E(\beta_2))$.

<u>Case 2.</u> $x(\beta)<\min\{x(\beta_1),x(\beta_2)\}$. In this case, based on the proof of Lemma 5.4, we can show that the error terms

$$|\beta||B(\beta_1)+B(\beta_2)+I_L^{1,-}(\beta_1)+I_L^{1,-}(\beta_2)|\mid_{x>x(\beta)}$$

in $\sum_\beta |\beta|\{\sum_{\beta',x(\beta')>x(\beta)} B(\beta') + I_L^{1,-}\} \,|_{x>x(\beta)}$ related to β_i, $i = 1, 2$, can be controlled by $c|\beta|(E(\beta_1) + E(\beta_2))$.

Case 3. $\min\{x(\beta_1), x(\beta_2)\} < x(\beta) < \max\{x(\beta_1), x(\beta_2)\}$. Without loss of generality, we assume that $x(\beta_1) < x(\beta_2)$. In this case we have an extra error term $|\beta|(B(\beta_2) + I_L^{1,-}(\beta_2)) \,|_{x>x(\beta)}$. Due to the truncation at $x = x(\beta)$ and based on the discussion for $\frac{d}{dt}\tilde{L}(t)$, we have

$$|B(\beta_2) + I_L^{1,-}(\beta_2)| \,|_{x>x(\beta)} \leq c(|\beta_2|^3 + (\gamma^+_{\beta_2}(x(\beta), t))^3 + E(\beta_2)).$$

Notice that $\gamma^+_{\beta_2}(x(\beta), t) \geq 0(1)\mu|\beta_1|$ and so

$$\begin{aligned} & |\beta||B(\beta_2) + I_L^{1,-}(\beta_2)| \,|_{x>x(\beta)} \\ \leq\ & c|\beta|(|\beta_2|^3 + (\gamma^+_{\beta_2}(x(\beta), t))^3 + E(\beta_2)) \\ \leq\ & c\{(|\beta_1| + |\beta_2|)|\beta|\gamma^+_{x(\beta)}(|\beta| + \gamma^+_{x(\beta)}) + |\beta|(E(\beta_1) + E(\beta_2)\}. \end{aligned}$$

A similar estimate holds for $\sum_\alpha |\alpha|\{\sum_{\alpha',x(\alpha')<x(\alpha)} B(\alpha') + I_L^{1,+}\} \,|_{x<x(\alpha)}$.

$\sum_\beta |\beta| I_L^{2,-} \,|_{x>x(\beta)}$ can be estimated similarly by considering the cubic strengths (or difference of cubic strengths) of 1-shock waves in $u^1(x,t)$ and $u^2(x,t)$.

Combining the above estimates for the above three cases, we have

$$\begin{aligned} |R| \ \leq\ & cT.V.\{\sum_\alpha |\alpha|(\gamma^+_{x(\alpha)*} + \gamma^-_{x(\alpha)}(|\alpha| + \gamma^-_{x(\alpha)})) + \sum_\alpha |\alpha^t||\alpha^b| \\ & + \sum_\beta |\beta|(\gamma^-_{x(\beta)*} + \gamma^+_{x(\beta)}(|\beta| + \gamma^+_{x(\beta)})) + \sum_\beta |\beta^t||\beta^b|\}. \end{aligned} \tag{6.3}$$

The lemma follows from (6.2) and (6.3). □

6.2. **Estimation of $\frac{d}{dt}\Delta(t)$.** In this subsection, we are going to estimate $\frac{d}{dt}\Delta(t)$ and show that it gives a good term of order

$$-(\sum_\alpha |\alpha|\gamma^-_{x(\alpha)}(|\alpha| + \gamma^-_{x(\alpha)}) + \sum_\beta |\beta|\gamma^+_{x(\beta)}(|\beta| + \gamma^+_{x(\beta)})).$$

Before doing this, we will present the simplified version for the scalar conservation law. This has been studied in [17] and is included here without proof.

Let $u^1(x,t)$ and $u^2(x,t)$ be two solutions of a scalar convex conservation law

$$u_t + f(u)_x = 0, \qquad f''(u) > 0. \tag{6.4}$$

Set

$$(u-v)_+ = \begin{cases} u - v, & u \geq v, \\ 0, & u < v, \end{cases} \qquad (u-v)_- = \begin{cases} 0, & u \geq v, \\ v - u, & u < v. \end{cases}$$

The generalized entropy functional $E(t) = E[u_1(\cdot,t), u_2(\cdot,t)]$ is defined as follows:

$$\begin{aligned} E(t) \ =\ & \int_{-\infty}^{\infty} |u_{1y}|(y,t)(\int_y^\infty (u_1 - u_2)_+(x,t)dx + \int_{-\infty}^y (u_1 - u_2)_-(x,t)dx)dy \\ + & \int_{-\infty}^{\infty} |u_{2y}|(y,t)(\int_y^\infty (u_2 - u_1)_+(x,t)dx + \int_{-\infty}^y (u_2 - u_1)_-(x,t)dx)dy. \end{aligned}$$

We will only consider piecewise constant functions in the simplified wave patterns in the next section. Thus for the solutions of the scalar conservation law here, the rarefaction waves are approximated by small rarefaction shocks, each with strength less than ϵ. This will induce errors of the order $T.V.\epsilon$ when we apply the L_1

contraction in the analysis below. The speed of a rarefaction shock $\alpha = (u^-, u^+)$ is denoted by

$$\lambda(\alpha) = \frac{f(u^+) - f(u^-)}{u^+ - u^-}.$$

The location of a wave α at time t is denoted by $x(\alpha) = x(\alpha(t))$. Let J_1 and J_2 be the sets of waves in u^1 and u^2, respectively. Then the above functional is rewritten as:

(6.5)

$$\begin{aligned} E(t) = & \sum_{\alpha \in J_1} |\alpha| \Big(\int_{x(\alpha)}^{\infty} (u_1 - u_2)_+(x,t)dx + \int_{-\infty}^{x(\alpha)} (u_1 - u_2)_-(x,t)dx \Big) \\ & + \sum_{\alpha \in J_2} |\alpha| \Big(\int_{x(\alpha)}^{\infty} (u_2 - u_1)_+(x,t)dx + \int_{-\infty}^{x(\alpha)} (u_2 - u_1)_-(x,t)dx \Big). \end{aligned}$$

Theorem 6.1. *For the convex scalar conservation law, the generalized functional* (6.5) *for the solutions* $u_1(x,t)$ *and* $u_2(x,t)$ *with total variations bounded by T.V. satisfies*

$$\frac{d}{dt}E(t) \le -C_1 \sum |\alpha| \gamma_{x(\alpha)} (\gamma_{x(\alpha)} + |\alpha|) + 0(1) T.V. \epsilon, \tag{6.6}$$

where the summation is over all waves α *at time* t *in both solutions.*

We are now ready to study the following estimate for the functional $\Delta(t)$:

Lemma 6.2. *Under the hypotheses of Main Lemma 1, for each* $t \in I_p^c$, *we have*

$$\begin{aligned} \frac{d}{dt}\Delta(t) \quad \le & \quad -c_1 \Big(\sum_{\alpha} |\alpha| \gamma^-_{x(\alpha)} (|\alpha| + \gamma^-_{x(\alpha)}) + \sum_{\beta} |\beta| \gamma^+_{x(\beta)} (|\beta| + \gamma^+_{x(\beta)}) \Big) \\ & + \quad cT.V. \Big(\sum_{\alpha} (|\alpha| \gamma^+_{x(\alpha)*} + |\alpha^t||\alpha^b|) + \sum_{\beta} (|\beta| \gamma^-_{x(\beta)*} + |\beta^t||\beta^b|) + e(\Lambda_p) \Big). \end{aligned}$$

Proof. The proof is based on the idea of obtaining the good terms in $E(t)$ for scalar conservation laws and the study in the last section for $\frac{d}{dt}\tilde{L}(t)$. Let $\alpha_i^1(t)$ be a 1-wave in $u^1(x,t)$ and $x(\alpha_i^1(t)) \in (z_j^r, z_{j+1}^r]$, j odd. By the definition of $\Delta(\alpha_i^1(t))$, we have

$$\begin{aligned} \frac{d}{dt}\Delta(\alpha_i^1(t)) \quad = & \quad |\alpha_i^1(t)| \Big\{ \frac{d}{dt} \int_{x(\alpha_i^1(t))}^{z_{j+1}^r} \gamma^-(x,t)dx + \sum_{l>j, l=\text{odd}} \frac{d}{dt} \int_{z_l^r}^{z_{l+1}^r} \gamma^-(x,t)dx \\ & + \quad \sum_{l<j, l=\text{even}} \frac{d}{dt} \int_{z_l^r}^{z_{l+1}^r} \gamma^-(x,t)dx + U^1(\alpha_i^1(t)) \Big\} \\ \le & \quad c|\alpha_i^1(t)| \Big\{ \sum_{\alpha} |\alpha| \gamma^-_{x(\alpha)} (|\alpha| + \gamma^-_{x(\alpha)}) + \sum_{\beta} |\beta| \gamma^+_{x(\beta)} (|\beta| + \gamma^+_{x(\beta)}) \\ & + \quad \sum_{\alpha} (|\alpha| \gamma^+_{x(\alpha)*} + |\alpha^t||\alpha^b|) + \sum_{\beta} (|\beta| \gamma^-_{x(\beta)*} + |\beta^t||\beta^b|) \Big\} \\ & - \quad \bar{c}|\alpha_i^1| \gamma^-_{x(\alpha_i^1)} (|\alpha_i^1| + \gamma^-_{x(\alpha_i^1)}) + |\alpha_i^1| \Big(\frac{d}{dt} U^1(\alpha_i^1) + J(\alpha_i^1) \Big), \end{aligned}$$

for a positive constant $\bar{c}$. Here $J(\alpha_i^1)$ is defined as follows:

$$J(\alpha_i^1) \quad = \quad (I_L^{1,-} + I_L^{2,-}) \mid_{(x(\alpha_i^1)), z_{j+1}^r) \cap (\bigcap_{l>j, l=\text{odd}} (z_l^r, z_{l+1}^r)) \cap (\bigcap_{l<j, l=\text{even}} (z_l^r, z_{l+1}^r))} .$$

Similar estimates hold for $\frac{d}{dt}\Delta(\alpha_i^2(t))$ and $\frac{d}{dt}\Delta(\beta_i^p(t))$, $p = 1, 2$, with $J(\beta_i^p(t))$ similarly defined. Summing all these estimates and using the assumption that $T.V.$ is sufficiently small, we have

$$\begin{aligned}
\frac{d}{dt}\Delta(t) \quad &\leq \quad -\frac{1}{2}\bar{c}(\sum_\alpha |\alpha|\gamma^-_{x(\alpha)}(|\alpha| + \gamma^-_{x(\alpha)}) + \sum_\beta |\beta|\gamma^+_{x(\beta)}(|\beta| + \gamma^+_{x(\beta)})) \\
&+ \quad cT.V.\{\sum_\alpha(|\alpha|\gamma^+_{x(\alpha)*} + |\alpha^t||\alpha^b|) + \sum_\beta(|\beta|\gamma^-_{x(\beta)*} + |\beta^t||\beta^b|)\} \\
&+ \quad \sum_\alpha |\alpha|(\frac{d}{dt}U^1(\alpha) + J(\alpha)) + \sum_\beta |\beta|(\frac{d}{dt}U^2(\beta) + J(\beta)),
\end{aligned} \tag{6.7}$$

where the last two sums are over shock waves.

It remains to estimate the last two summations in (6.7). First, notice that

$$\begin{aligned}
|J| \quad &= \quad |\sum_\alpha |\alpha|(\frac{d}{dt}U^1(\alpha) + J(\alpha)) + \sum_\beta |\beta|(\frac{d}{dt}U^2(\beta) + J(\beta))| \\
&\leq \quad cT.V.(\sum_\alpha |\hat{\alpha}| + \sum_\beta |\hat{\beta}| + \sum_\alpha |\alpha|\gamma^-_{x(\alpha)}(|\alpha| + \gamma^-_{x(\alpha)}) \\
&+ \quad \sum_\beta |\beta|\gamma^+_{x(\beta)}(|\beta| + \gamma^+_{x(\beta)})).
\end{aligned}$$

Therefore, for any given shock wave, say, β_1 of the second family in $u^1(x, t)$, we only need to consider the case when there exists a 2-shock wave θ in $u^2(x, t)$ with $|\bar{\theta}(\beta)| \geq \mu|\beta^1|$. Write $\beta_2 = \theta$. As before, we can assume $|\beta_2| \leq 2|\beta_1|$. In the following we consider the case when neither $\hat{\beta}_i$, $i = 1, 2$, is crossed by the r curve of the other solution in the r-x plane. The case when they cross can be discussed similarly. Let $x(\beta_1) \in (z^r_{j_1}, z^r_{j_1+1}]$ and $x(\beta_2) \in (z^r_{j_2}, z^r_{j_2+1}]$, and, without loss of generality, we assume that $j_1 \leq j_2$. This can be discussed by considering two cases: j_1 and j_2 are both even or both odd; one of j_1 and j_2 is odd and the other is even. For each case, there are three subcases depending on the positions of β_i, $i = 1, 2$, and α: $x(\alpha) < x(\beta_1)$, $x(\alpha) > x(\beta_2)$ and $x(\beta_1) < x(\alpha) < x(\beta_2)$. Since the estimation follows from the one for $\frac{d}{dt}\tilde{L}(t)$, we omit the details. Noticing that for any α and β

$$\begin{aligned}
\sum_{\gamma^1} |\alpha|\gamma^-_{\gamma^1}(x(\alpha), t)(|\alpha| + \gamma^-_{\gamma^1}(x(\alpha), t)) \quad &\leq \quad 2|\alpha|\gamma^-_{x(\alpha)}(|\alpha| + \gamma^-_{x(\alpha)}), \\
\sum_{\gamma^2} |\beta|\gamma^+_{\gamma^2}(x(\beta), t)(|\beta| + \gamma^+_{\gamma^2}(x(\beta), t)) \quad &\leq \quad 2|\beta|\gamma^+_{x(\beta)}(|\beta| + \gamma^+_{x(\beta)}),
\end{aligned}$$

where γ^i denotes the wave of the i-th family in $u^1(x, t)$ or $u^2(x, t)$, we have

$$\begin{aligned}
|J| \quad &\leq \quad cT.V.\{\sum_\alpha |\alpha|\gamma^-_{x(\alpha)}(|\alpha| + \gamma^-_{x(\alpha)}) + \sum_\beta |\beta|\gamma^+_{x(\beta)}(|\beta| + \gamma^+_{x(\beta)}) \\
&+ \quad \sum_\alpha(|\alpha|\gamma^+_{x(\alpha)*} + |\alpha^t||\alpha^b|) + \sum_\beta(|\beta|\gamma^-_{x(\beta)*} + |\beta^t||\beta^b|)\}.
\end{aligned} \tag{6.8}$$

Combining (6.7) and (6.8) completes the proof of the lemma. □

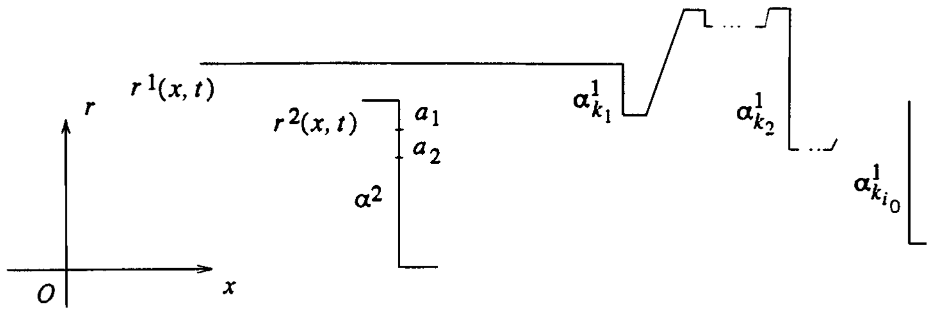

FIGURE 6.1

6.3. **Estimation of $\frac{d}{dt}D(t)$.** For $\frac{d}{dt}D(t)$, we have the following lemma.

Lemma 6.3. *Under the hypotheses of Main Lemma 1, we have, for each $t \in I_p^c$,*

$$\begin{aligned}\frac{d}{dt}D(t) &\leq c\{\sum_{\alpha}|\alpha|(\gamma^+_{x(\alpha)*} + \gamma^-_{x(\alpha)}(|\alpha| + \gamma^-_{x(\alpha)})) + \sum_{\beta}|\beta|(\gamma^-_{x(\beta)*} + \gamma^+_{x(\beta)}(|\beta| + \gamma^+_{x(\beta)}))\} \\ &+ cT.V.\{\sum_{\alpha}|\alpha^t||\alpha^b| + \sum_{\beta}|\beta^t||\beta^b| + e(\Lambda_p)\}.\end{aligned}$$

Proof. We estimate $\frac{d}{dt}D^-(t)$ by considering the following two cases.

Case 1. No "split" wave in $u^1(x,t)$ and $u^2(x,t)$ when forming the region $\Omega(\gamma)$ for any 1-shock wave γ and any minor wave corresponding to the second family. Since $D(t)$ is of cubic order, it is easy to show that

$$\begin{aligned}\frac{d}{dt}D^-(t) \leq c\{&\sum_{\alpha}|\alpha|(\gamma^+_{x(\alpha)*} + \gamma^-_{x(\alpha)}(|\alpha| + \gamma^-_{x(\alpha)})) \\ &+ \sum_{\beta}|\beta|(\gamma^-_{x(\beta)*} + \gamma^+_{x(\beta)}(|\beta| + \gamma^+_{x(\beta)}))\} \\ &+ cT.V.\{\sum_{\alpha}|\alpha^t||\alpha^b| + \sum_{\beta}|\beta^t||\beta^b| + e(\Lambda_p)\}.\end{aligned} \tag{6.9}$$

Case 2. Some waves "split" when forming the region $\Omega(\gamma)$ for a 1-shock wave or minor wave corresponding to the second family. We only consider the part $\frac{d}{dt}D^{-,p}(t)$ of $\frac{d}{dt}D^-(t)$ when $p = 1$. There are two subcases:

Subcase 1. The split wave α^1 in $u^1(x,t)$. If it is a rarefaction wave, then the error terms in $\frac{d}{dt}D^{-,1}(t)$ related to α^1 due to splitting can be bounded by $0(1)|\alpha^1|\epsilon$. If the split wave is a shock wave, then we do not have an extra error term.

Subcase 2. The split wave α^2 in $u^2(x,t)$. If α^2 is a rarefaction wave, then the error terms related to α^2 due to splitting are bounded by $0(1)|\alpha^2|\epsilon$. When α^2 is a shock wave, we assume that α^2 splits into n parts due to the forming of the regions $\Omega(\alpha^1_{k_i})$, $i = 1, 2, \cdots, n$, and we denote the length of these n parts from top to bottom by a_i, $i = 1, 2, \cdots, n$. We denote the region containing a_i by $\Omega(\alpha^1_{k_i})$, where $k_i < k_j$ if $i < j$. If the shock wave α^2 is crossed by the r curve of $u^1(x,t)$, then the $\frac{d}{dt}L(t)$ gives a good term of order $-|\alpha^{2,t}||\alpha^{2,b}|$. Thus we may assume that α^2 is not crossed by the r curve of $u^1(x,t)$; cf. Figure 6.1.

According to the discussion of Case 1, the extra error terms due to the splitting of α^2 in $\frac{d}{dt}D^{-,1}(t)$ can be bounded by

$$c\sum_{i=1}^{n-1} a_i(\sum_{j=i+1}^{n} a_j^2). \tag{6.10}$$

By Claims 1–4, we have either

$$|\alpha^2|^3 \le c(\mu)(E(\alpha^2)+E(\gamma)),$$

for some γ, or there exists a i_0 with

$$|\alpha^2|^3 - a_{i_0}^3 \le c(\mu)E(\alpha^2).$$

Notice that $|\alpha^2| = \sum_{i=1}^n a_i$ and we have

$$\sum_{i=1}^{n-1} a_i(\sum_{j=i+1}^{n} a_j^2) \le c(\mu)(E(\alpha^2)+E(\gamma)),$$

for some wave γ.

The case when the "split" wave is a minor wave in $u^2(x,t)$ can be discussed similarly. Combining all the above estimates and the discussion for Case 1, we have (6.9) in general for $\frac{d}{dt}D^{-,1}(t)$.

Similar estimates hold for $\frac{d}{dt}D^{-,2}(t)$ and $\frac{d}{dt}D^{+}(t)$. This completes the proof of the lemma. □

7. Proof of the main lemmas

In this section, we are first going to study the changes of the functional $H(t)$ when crossing the interaction points in Λ_p and $t = pMs$, $1 \le p \le N$. For the simplified wave pattern introduced in Section 2, even though the solutions are linear superpositions of a family of step functions in each small time strip Λ_p, the functionals $L_h(t)$ and $D(t)$ are not continuous, because they depend on the formation of the regions $\Omega(\gamma)$ when γ is a shock or minor wave. We will show that the jumps of the functional $H(t)$ when crossing these interaction points can be bounded by the change of Glimm's functional times the L_1 norm of the difference between two weak solutions plus an error term. Secondly, we compare the functionals $\bar{H}(t)$ and $H(t)$ at $t = pMs$ for $1 \le p \le N$, and show that the change of $\bar{H}(t)$ is bounded by an error term which approaches zero for any fixed time T as the approximate solutions tend to the weak solutions.

The paper will be completed after the proof of the main lemmas.

Proof of Main Lemma 1. Based on the discussion, for each $t \in I_p^c$, we have

$$\begin{aligned}
\frac{d}{dt}H(t) \quad &\le \quad (c+ck_3-c_1k_2)\{\sum_{\alpha}|\alpha|(\gamma^+_{x(\alpha)*}+\gamma^-_{x(\alpha)}(|\alpha|+\gamma^-_{x(\alpha)}))\\
&+ \quad \sum_{\beta}|\beta|(\gamma^-_{x(\beta)*}+\gamma^+_{x(\beta)}(|\beta|+\gamma^+_{x(\beta)}))\}\\
&+ \quad (cT.V.(k_2+k_3)-c_1)(\sum_{\alpha}|\alpha^t||\alpha^b|+\sum_{\beta}|\beta^t||\beta^b|)+ce(\Lambda_p),
\end{aligned}$$

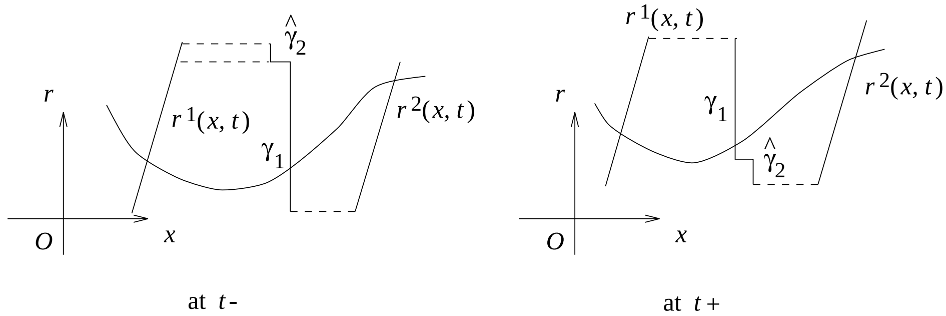

FIGURE 7.1

where we have used $k_1F(t) = 0(1)$ and $T.V. \ll 1$. Thus when

$$k_2 \geq c(1+k_3), \qquad T.V.(k_2+k_3) \leq c_1, \tag{7.1}$$

we have $\frac{d}{dt}H(t) \leq ce(\Lambda_p)$. □

Proof of Main Lemma 2. In each small time strip I_p, the functional $H(t)$ is continuous except on the set I_p^d. The discontinuity of $H(t)$ is due to the formation of the regions for shock waves and minor waves in the solutions, and it reflects on the discontinuity of the functionals $L_h(t)$ and $D(t)$ for $t \in I_p^d$. We have the following three cases.

Case 1. Rarefaction wave interacting with rarefaction wave. In this case, the regions for shock waves and minor waves are not changed at the interaction and the functionals $L_h(t)$ and $D(t)$ are continuous at the interaction point. Therefore the functional $H(t)$ is also continuous at this point.

Case 2. Shock wave interacting with shock wave. Denote the shock wave of the i-th family in the interaction by γ_i, $i = 1, 2$, and denote the interaction time by t. Without loss of generality, we let γ_i, $i = 1, 2$, be in $u^1(x,t)$.

From the definition of $D(t)$, we know that the functional $D(t)$ does not increase when crossing time t. So it remains to consider the change of $L_h(\gamma_i)$, $i = 1, 2$. Without loss of generality, we consider the change of $L_h(\gamma_1)$ as follows (cf. Figure 7.1).

For any $x \in (x^l(\gamma_1(t+)), x^r(\gamma_1(t+)))$, by the definition of $\xi_{\gamma_1}(x,t)$, we have

$$\begin{aligned} & \xi_{\gamma_1(t+)}(x,t+)(\gamma^-_{\gamma_1(t+)}(x,t+))^3 - \xi_{\gamma_1(t-)}(x,t-)(\gamma^-_{\gamma_1(t-)}(x,t-))^3 \\ \leq \;& \max\{0, (\gamma^-_{\gamma_1(t+)}(x,t+))^3 - (\gamma^-_{\gamma_1(t-)}(x,t-))^3\} \\ \leq \;& c\min\{|\gamma_1|^2, |\gamma_2|^2\}(\gamma^-_{\gamma_1(t-)}(x,t-) + \gamma^-_{\hat{\gamma}_2(t-)}(x,t-)) \\ \leq \;& c|\gamma_1||\gamma_2|(\gamma^-_{\gamma_1(t-)}(x,t-) + \gamma^-_{\hat{\gamma}_2(t-)}(x,t-)), \end{aligned}$$

where we note that $\gamma^-_{\gamma_1(t-)}(x,t-) + \gamma^-_{\hat{\gamma}_2(t-)}(x,t-) = \gamma^-_{\gamma_1(t+)}(x,t+) + \gamma^-_{\hat{\gamma}_2(t+)}(x,t+)$, $\gamma^-_{\gamma_1(t\pm)}(x,t\pm) \leq |\gamma_1|$ and $\gamma^-_{\hat{\gamma}_2}(t\pm)(x,t\pm) \leq |\hat{\gamma}_2|$. Thus,

$$L_h(\gamma_1)(t+) - L_h(\gamma_1)(t-) \leq c(F(t-) - F(t+))L(t).$$

The above estimate also holds for $L_h(\gamma_2)$.

Case 3. Shock wave interacting with rarefaction wave. As in Case 2, $D(t)$ does not increase and the change of $L_h(\gamma_1)$ or $L_h(\gamma_2)$ can be bounded by the change of

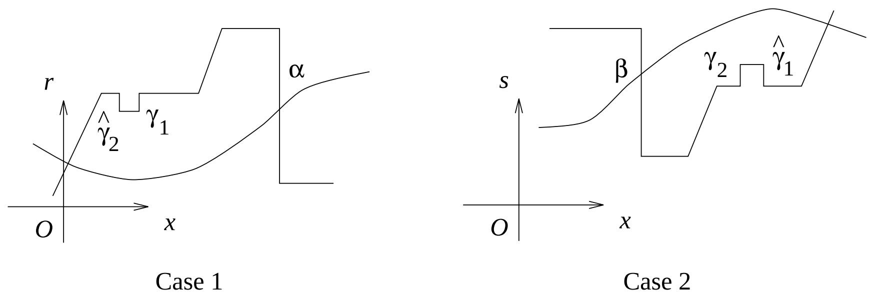

FIGURE 7.2

Glimm's functional times the L_1 distance of the two weak solutions except for the following case, denoted by Case E.

Case E. Part of the region of the minor wave will be included in the region of another shock wave after interaction. For definiteness, we let γ_1 be a rarefaction wave and γ_2 be a shock wave (cf. Figure 7.2).

For illustration, we assume that there exists a shock wave α which is in the same solution as γ_1 and that $\Omega(\alpha(t+)) = \Omega(\alpha(t-)) \cup \Omega(\hat{\gamma}_2(t-))$. Other cases can be discussed similarly. In this case, for any $x \in (x^l(\hat{\gamma}_2(t-)), x^r(\hat{\gamma}_2(t-)))$, we have

$$\begin{aligned} & \xi_\alpha(x,t+)(\gamma_\alpha^-(x,t+))^3 - \xi_\alpha(x,t-)(\gamma_\alpha^-(x,t-))^3 \\ \leq\ & c\{\gamma_{\hat{\gamma}_2}^-(x,t-)(\gamma_\alpha^-(x,t-))^2 + (\gamma_{\hat{\gamma}_2}^-(x,t-))^2(\gamma_{\hat{\gamma}_2}^-(x,t-) + \gamma_\alpha^-(x,t-))\} \\ \leq\ & c\{\gamma_{\hat{\gamma}_2}^-(x,t-)(\gamma_\alpha^-(x,t-))^2 + (F(t-) - F(t+))(\gamma_{\hat{\gamma}_2}^-(x,t-) + \gamma_\alpha^-(x,t-))\}. \end{aligned}$$

Furthermore,

$$D(t+) - D(t-)\ |_{\text{at } x} \leq -\gamma_{\hat{\gamma}_2}^-(x,t-)(\gamma_\alpha^-(x,t-))^2.$$

Thus

$$H(t+) - H(t-) \leq c(F(t-) - F(t+))L(t) + ((1 + cT.V.(k_1 + k_2)) - k_3)D_\alpha(\hat{\gamma}_2)(t-),$$

where

$$D_\alpha(\hat{\gamma}_2)(t-) = \sum_{\alpha, x(\alpha) > x(\gamma_2)} \int_{x^l(\hat{\gamma}_2)}^{\min\{x^r(\hat{\gamma}_2), x(\alpha)\}} \gamma_{\hat{\gamma}_2}^-(x,t-)(\gamma_\alpha^-(x,t-))^2 dx.$$

By choosing

$$k_3 \geq c(1 + T.V.(k_1 + k_2)), \tag{7.2}$$

we have

$$H(t+) - H(t-) \leq c(F(t-) - F(t+))L(t). \tag{7.3}$$

This also holds for other subcases in Case E by a similar argument.

Combining Cases 1–3 completes the proof of Main Lemma 2. □

Finally we complete the paper by proving Main Lemma 3.

Proof of Main Lemma 3. We estimate $\bar{H}(pMs+) - H(pMs-)$ first. By the definitions of $H(t)$ and $\bar{H}(t)$, we have

$$\bar{H}(pMs+) - H(pMs-) = \sum_{i=1}^{5} I_i,$$

where

$$\begin{aligned}
I_1 &= k_1(F(pMs) - F((p-1)Ms))(L + L_h)(pMs-),\\
I_2 &= (1 + k_1F(pMs))(\bar{L}(pMs+) - L(pMs-)),\\
I_3 &= (1 + k_1F(pMs))(\bar{L}_h(pMs+) - L_h(pMs-)) + K_3(\bar{D}(pMs+) - D(pMs-)),\\
I_4 &= k_2(\bar{Q}_d(pMs+) - Q_d(pMs-)),\\
I_5 &= k_2(\bar{\Delta}(pMs+) - \Delta(pMs-)).
\end{aligned}$$

We estimate I_i as follows: By (d) of Theorem 2.2 and (v) of Theorem 2.3, we have

$$I_1 + I_2 \le ce(\Lambda_p)Ms - k_1(Q(\Lambda_p) + C(\Lambda_p))L(pMs-). \tag{7.4}$$

Before estimating I_i, $i = 3, 4, 5$, we consider $\bar{D}(pMs+) - D(pMs-)$. The difference between the approximate solutions $u_r^i(x,t)$, $i = 1, 2$, at $t = pMs$ consists of three parts. The first part is due to the created waves in Λ_p; the second part is due to the interactions of waves of the same family at $t = pMs$; and the last part is due to the interactions of waves from different families at $t = pMs$. From the definition of $D(t)$, we know that the portion of $|\bar{D}(pMs+) - D(pMs-)|$ due to the first part is bounded by $c(T.V.e(\Lambda_p)Ms + (Q(\Lambda_p) + C(\Lambda_p))L(pMs))$. Hence it remains to estimate the effects of wave interactions on $\bar{D}(pMs+) - D(pMs-)$. We discuss this as follows:

Since the newly created waves from interactions have been considered, and the wave interactions in the case of waves from different families were considered in the discussion of the jumps of $H(t)$ inside Λ_p, we only need to consider the cases when waves of the same family either combine or cancel. For simplicity, we consider the change of $D^-(t)$ from $D^-(pMs-)$ to $\bar{D}^-(pMs+)$ due to the interaction of two shock waves α_i^p and α_{i+1}^p of the first family. Other cases can be discussed similarly. In this case, if we denote the shock wave of the first family after interaction by α, then the extra error terms not included before are

$$\begin{aligned}
&\sum_{\alpha_j^p, x(\alpha_j^p) < x(\alpha_i^p)} \int_{x^l(\alpha_j^p)}^{x_i^r(\alpha_j^p)} \gamma^-_{\alpha_j^p}(x, pMs)((\gamma^-_\alpha(x, pMs+))^2 - (\gamma^-_{\alpha_i^p}(x, pMs-))^2\\
&\qquad\qquad - (\gamma^-_{\alpha_{i+1}^p}(x, pMs-))^2)dx\\
&\quad + \sum_{\beta_k^p, x(\beta_k^p) < x(\alpha_i^p)} \int_{x^l(\hat{\beta}_k^p)}^{x_i^r(\hat{\beta}_k^p)} \gamma^-_{\hat{\beta}_k^p}(x, pMs)((\gamma^-_\alpha(x, pMs+))^2 - (\gamma^-_{\alpha_i^p}(x, pMs-))^2\\
&\qquad\qquad - (\gamma^-_{\alpha_{i+1}^p}(x, pMs-))^2)dx\\
&\quad + \int_{x^l(\alpha)}^{x(\alpha)} (\gamma^-(x, pMs) - \sum_{\alpha_j^p} \gamma^-_{\alpha_j^p}(x, pMs) - \sum_{\beta_k^p} \gamma^-_{\hat{\beta}_k^p}(x, pMs))((\gamma^-_\alpha(x, pMs+))^2\\
&\qquad\qquad - (\gamma^-_{\alpha_i^p}(x, pMs-))^2 - (\gamma^-_{\alpha_{i+1}^p}(x, pMs-))^2)dx\\
&\le c|\alpha_i^p||\alpha_{i+1}^p|L^-(pMs-)\\
&\le cQ(\alpha_i^p, \alpha_{i+1}^p)L^-(pMs-).
\end{aligned}$$

Therefore we have

$$\bar{D}(pMs+) - D(pMs-) \le c(e(\Lambda_p)Ms + (Q(\Lambda_p) + C(\Lambda_p))L(pMs)). \tag{7.5}$$

Now we turn to $\bar{L}_h(pMs+) - L_h(pMs-)$. As in the discussion for $\bar{D}(pMs+) - D(pMs-)$, the part of $|\bar{L}_h(pMs+) - L_h(pMs-)|$ due to the newly created waves inside Λ_p from wave interactions is bounded by $cT.V.e(\Lambda_p)Ms$. Thus all the wave interactions can be viewed as linear superpositions. We only need to consider the interactions of waves from the same family because the interactions of waves from different families were considered in the discussion of jumps of $H(t)$ inside Λ_p. We consider only wave combination; wave cancellation can be discussed similarly. Let α_i, $i = 1, 2$, be two shock waves of the same solution which interact at $t = pMs$ in Glimm's scheme, and let α be the main wave after the interaction. We have

$$\begin{aligned} & C_\alpha \xi_\alpha (\gamma_\alpha^-)^3 - C_{\alpha_1} \xi_{\alpha_1} (\gamma_{\alpha_1}^-)^3 - C_{\alpha_2} \xi_{\alpha_2} (\gamma_{\alpha_2}^-)^3 \\ \leq \quad & C_\alpha ((\gamma_\alpha^-)^3 - (\gamma_{\alpha_1}^-)^3 - (\gamma_{\alpha_2}^-)^3) + |C_\alpha - C_{\alpha_1}|(\gamma_{\alpha_1}^-)^3 + |C_\alpha - C_{\alpha_2}|(\gamma_{\alpha_2}^-)^3 \\ \leq \quad & c|\alpha_1||\alpha_2|\gamma_\alpha^-, \end{aligned}$$

where we have used $|C_\alpha - C_{\alpha_1}| = 0(1)|\alpha_2|$ and $|C_\alpha - C_{\alpha_2}| = 0(1)|\alpha_1|$. Hence we have that the extra error term for $|\bar{L}_h(pMs+) - L_h(pMs-)|$ is bounded by $cQ(\alpha_1, \alpha_2)L(pMs-)$. Recall that $\bar{L}_h(pMs+) - L_h(pMs-)$ may have a positive jump in Case E (cf. Figure 7.2), but in this case $\bar{D}(pMs+) - D(pMs-)$ has corresponding negative jumps. Thus, except for Case E,

(7.6)
$$\bar{L}_h(pMs+) - L_h(pMs-) \leq c(T.V.e(\Lambda_p)Ms + (Q(\Lambda_p) + C(\Lambda_p))L(pMs-)).$$

For Case E, there exists a positive constant c_2 such that

$$\begin{aligned} & \bar{L}_h(pMs+) - L_h(pMs-) + c_2(\bar{D}(pMs+) - D(pMs-)) \\ \leq \quad & c(T.V.e(\Lambda_p)Ms + (Q(\Lambda_p) + C(\Lambda_p))L(pMs-)). \end{aligned} \tag{7.7}$$

Combining (7.5)–(7.7), we have

$$I_3 \leq c(T.V.e(\Lambda_p)Ms + (Q(\Lambda_p) + C(\Lambda_p))L(pMs-)), \tag{7.8}$$

under condition (7.2).

Finally, we can discuss I_4 and I_5. The part of L_h in $Q_d(t)$ and $\Delta(t)$ has a factor bounded by $T.V.$ By choosing

$$k_3 \geq c(1 + T.V.(k_1 + k_2)),$$

we know that the effect of $L_h(t)$ on I_4 and I_5 can be bounded by $D(t)$ and $c(T.V.e(\Lambda_p)Ms + (Q(\Lambda_p) + C(\Lambda_p))L(pMs-))$. Therefore we may ignore the effect of $L_h(t)$ when considering I_4 and I_5.

By the definition of $Q_d(t)$, I_4 can be estimated by considering the following two terms: one is to measure the changes of the wave strengths times the corresponding L_1 norm, and the other is to measure the change of the L_1 norm times the wave strengths. According to the above discussion we have

$$I_4 \leq c(T.V.e(\Lambda_p)Ms + (Q(\Lambda_p) + C(\Lambda_p))L(pMs-)). \tag{7.9}$$

For I_5, note that the Riemann invariant curves for the approximate solutions in Glimm's scheme at $t = pMs$ and the simplified wave patterns at $t = pMs-$ are the same except for the newly created waves inside Λ_p at $t = pMs$. As in the discussion for I_4, the value of I_5 due to this kind of difference is bounded by $c(T.V.e(\Lambda_p)Ms + (Q(\Lambda_p) + C(\Lambda_p))L(pMs-))$.

In summary, under conditions (7.1) and (7.2), we have

$$\begin{aligned} &\bar{H}(pMs+) - H(pMs-) \leq ce(\Lambda_p)Ms \\ &+ \quad (c(k_2+k_3)-k_1)(Q(\Lambda_p)+C(\Lambda_p))L((p-1)Ms+). \end{aligned} \tag{7.10}$$

Similarly we can show that

$$\begin{aligned} &H((p-1)Ms+) - \bar{H}((p-1)Ms+) \leq ce(\Lambda_p)Ms \\ &+ \quad (1+c(k_2+k_3))(Q(\Lambda_p)+C(\Lambda_p))L((p-1)Ms+). \end{aligned} \tag{7.11}$$

Combining (7.10) and (7.11), we have

$$\begin{aligned} &\bar{H}(pMs+) - H(pMs-) + H((p-1)Ms+) - \bar{H}((p-1)Ms+) \\ \leq \quad &ce(\Lambda_p)Ms + (c(1+k_2+k_3)-k_1)(Q(\Lambda_p)+C(\Lambda_p))L((p-1)Ms+). \end{aligned}$$

Thus, if we choose

$$k_1 \geq 2c(1+k_2+k_3), \tag{7.12}$$

then (3.5) hold.

Notice that as long as $T.V. \ll 1$, there are constants k_i, $i = 1, 2, 3$, such that the conditions (7.1), (7.2) and (7.12) are satisfied. This completes the proof of Main Lemma 3. □

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