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ON A CORRESPONDENCE BETWEEN CUSPIDAL REPRESENTATIONS OF GL_{2n} AND $\widetilde{\mathrm{Sp}}_{2n}$

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Introduction

Let K be a number field and $\mathbb{A}$ its ring of adeles. Let η be an irreducible, automorphic, cuspidal representation of $\mathrm{GL}_m(\mathbb{A})$. Assume that η is self-dual. By Langlands conjectures, we expect that η is the functorial lift of an irreducible, automorphic, cuspidal representation σ of $G(\mathbb{A})$, where G is an appropriate classical group defined over K. We know by [J.S.1] that the partial (as well as the complete) L-function $L^S(\eta \otimes \eta, s)$ has a (simple) pole at $s = 1$. Thus, since $L^S(\eta \otimes \eta, s) = L^S(\eta, \mathrm{Sym}^2, s)L^S(\eta, \Lambda^2, s)$, either $L^S(\eta, \Lambda^2, s)$ or $L^S(\eta, \mathrm{Sym}^2, s)$ has a pole at $s = 1$. If $m = 2n$, then one expects that in the first case $G = \mathrm{SO}_{2n+1}$, and in the second case $G = \mathrm{SO}_{2n}$. If $m = 2n + 1$, we know that $L^S(\eta, \Lambda^2, s)$ is entire (see [J.S.2]), and hence $L^S(\eta, \mathrm{Sym}^2, s)$ has a pole at $s = 1$. Here, one expects that $G = \mathrm{Sp}_{2n}$.

In [G.R.S.1], we started a program to prove the existence of σ as above. See also [G.R.S.2], [G.R.S.3]. We proposed to prove this existence by *explicit construction*. We constructed a space $V_{\sigma(\eta)}$ of automorphic forms on $G(\mathbb{A})$, invariant under right translations. The elements of $V_{\sigma(\eta)}$ are Bessel coefficients, or Fourier-Jacobi coefficients, restricted to $G(\mathbb{A})$, of residues at $s = 1$ of certain Eisenstein series on an appropriate ("larger") group $H(\mathbb{A})$, induced from η. The space $V_{\sigma(\eta)}$ has an extra property. It contains all irreducible, automorphic, cuspidal, *generic* (with respect to a given nondegenerate character) representations σ of $G(\mathbb{A})$ which lift weakly to η (i.e. the unramified parameters of σ, at almost all places, are determined by those of η through the L-embedding ${}^LG \hookrightarrow \mathrm{GL}_m(\mathbb{C})$). This is a first step towards the well-known conjecture that every tempered L-packet on $G(\mathbb{A})$ contains a generic representation. Since we believe in the existence of functorial lifting from G to GL_m, we expect that $V_{\sigma(\eta)}$ is nontrivial. This remained as a conjecture in [G.R.S.1]. From now on we restrict ourselves to the following special case. Let $m = 2n$, and assume that $L^S(\eta, \Lambda^2, s)$ has a pole at $s = 1$. Now, we add the assumption that $L^S(\eta, \frac{1}{2}) \neq 0$. (It can be checked that $L^S(\eta, \frac{1}{2}) \neq 0$, iff $L(\eta, \frac{1}{2}) \neq 0$, for a unitary η.) Fix a nontrivial character ψ of $K\backslash\mathbb{A}$. One expects that the "backward" lift of η to $\mathrm{SO}_{2n+1}(\mathbb{A})$ is lifted from the metaplectic group $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, via the theta correspondence associated to ψ (see [F]). Indeed, in this case we constructed in [G.R.S.1] a space $V_{\sigma(\eta)}$, as above, of automorphic forms on $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$. We proved in [G.R.S.1, Theorem 13] that the elements of $V_{\sigma(\eta)}$ are cuspidal in the sense that

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all their constant terms along radicals of parabolic subgroups are identically zero. The main (global) result of this paper is

Main Theorem (global). *Let η be an irreducible, automorphic, cuspidal, self-dual representation of $\mathrm{GL}_{2n}(\mathbb{A})$. Assume that $L^S(\eta, \Lambda^2, s)$ has a pole at $s = 1$ and that $L^S(\eta, \frac{1}{2}) \neq 0$. Then the space of automorphic cusp forms $V_{\sigma(\eta)}$ on $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ is nontrivial.*

The space $V_{\sigma(\eta)}$ affords an automorphic, cuspidal, genuine representation $\sigma(\eta)$ of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$. The construction of $\sigma(\eta)$ and the results of [G.R.S.1] show

Theorem 1. *The cuspidal representation $\sigma(\eta)$ contains all irreducible, automorphic, cuspidal, genuine representations σ of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, which have a nontrivial ψ^{-1}-Whittaker coefficients (i.e. ψ^{-1}-generic) and for which $L^S_\psi(\sigma \otimes \eta, s)$ has a pole at $s = 1$. Each irreducible summand of $\sigma(\eta)$ is ψ^{-1}-generic.*

As explained in [G.R.S.4, Sec. 3.1], there is no canonical definition of the (standard) L-function for $\sigma_\nu \otimes \eta_\nu$ on $\widetilde{\mathrm{Sp}}_{2n}(K_\nu) \times \mathrm{GL}_{2n}(K_\nu)$ for almost all places ν. It depends on a choice of a nontrivial character of K_ν. It turns out that

$$L_{\psi_\nu}(\sigma_\nu \otimes \eta_\nu, s) = L(\theta_{\psi_\nu}(\sigma_\nu) \otimes \eta_\nu, s) \tag{0.1}$$

whenever all data is unramified. Here $\theta_{\psi_\nu}(\sigma_\nu)$ is the unramified representation of $\mathrm{SO}_{2n+1}(K_\nu)$, obtained from σ_ν, under the local theta correspondence θ_{ψ_ν}, with respect to ψ_ν, from $\widetilde{\mathrm{Sp}}_{2n}(K_\nu)$ to $\mathrm{SO}_{2n+1}(K_\nu)$. The r.h.s. of (0.1) is the standard L-function for $\mathrm{SO}_{2n+1} \times \mathrm{GL}_{2n}$. This also suggests that the notion of an L-group can be extended to $\widetilde{\mathrm{Sp}}_{2n}(F)$. Its L-group should be the L-group of $\mathrm{SO}_{2n+1}(F)$, i.e. $\mathrm{Sp}_{2n}(\mathbb{C})$. (See [Sa] where this is justified in terms of Hecke algebras.) Note again that the choice of a nontrivial character of K_ν enables us to associate an unramified representation of $\widetilde{\mathrm{Sp}}_{2n}(K_\nu)$ to a semisimple conjugacy class in $\mathrm{Sp}_{2n}(\mathbb{C})$. Thus, the standard embedding $\mathrm{Sp}_{2n}(\mathbb{C}) \subset \mathrm{GL}_{2n}(\mathbb{C})$ should yield a (weak) lift from genuine automorphic representations σ of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ to automorphic representations π of $\mathrm{GL}_{2n}(\mathbb{A})$, only after we fixed ψ. In light of (0.1), π should be the weak lift of $\theta_\psi(\sigma)$, the representation of $\mathrm{SO}_{2n+1}(\mathbb{A})$, obtained from σ under the theta correspondence with respect to ψ. Note that if σ on $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ has a ψ-weak lift to η on $\mathrm{GL}_{2n}(\mathbb{A})$, then, of course, $L^S_\psi(\sigma \otimes \eta, s)$ has a pole at $s = 1$. (The converse is not trivial. In a sequel to this paper, we will show, using some of the new results of this paper, that $\sigma(\eta)$ is exactly the direct sum of all irreducible, automorphic, genuine, cuspidal and ψ^{-1}-generic representations σ of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$ which, under the ψ-weak lift to $\mathrm{GL}_{2n}(\mathbb{A})$, lift to η. This will show, using Theorem 1, that if σ is ψ^{-1}-generic on $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, and $L^S_\psi(\sigma \otimes \eta, s)$ has a pole at $s = 1$, then η is the ψ-weak lift of σ.)

The global theory sketched so far has a beautiful local counterpart. The constructions and theorems are motivated by the global results. However, the passage is not trivial. This local theory occupies the majority of the contents of this paper. Let F be a local nonarchimedean field of characteristic zero. Fix a nontrivial character of F still denoted by ψ. We construct an explicit map, which associates an irreducible, supercuspidal representation $\sigma(\tau)$ of the metaplectic group $\widetilde{\mathrm{Sp}}_{2n}(F)$ to an irreducible self-dual, supercuspidal representation τ of $\mathrm{GL}_{2n}(F)$, such that $L(\tau, \Lambda^2, s)$ has a pole at $s = 0$. (See [Sh1] for the definition of $L(\tau, \Lambda^2, s)$.) The representation $\sigma(\tau)$ has a Whittaker model, with respect to the standard nondegenerate character determined by ψ^{-1}. The defining property of $\sigma(\tau)$, and this is in essence the main local result of this work, is

Main Theorem (local). *The representation $\sigma(\tau)$ is the unique irreducible representation σ of $\widetilde{\mathrm{Sp}}_{2n}(F)$, which is supercuspidal, ψ^{-1}-generic and such that the gamma factor $\gamma(\sigma \times \tau, s, \psi)$ has a pole at $s = 1$.*

The gamma factor just mentioned is the local gamma factor which appears in the corresponding local theory of global integrals of Shimura type, which represent the standard (partial) L-function for generic cusp forms on $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A}) \times \mathrm{GL}_{2n}(\mathbb{A})$. (See [G.R.S.4]. In general, the local gamma factor $\gamma(\sigma \otimes \tau, s, \psi)$ may be defined similarly for any irreducible ψ^{-1}-generic representation of $\widetilde{\mathrm{Sp}}_{2k}(F) \times \mathrm{GL}_m(F)$. We expect that it is equal, at least up to a simple exponential, to the local gamma factor associated (say, by Shahidi, or as in [So]) to $\theta_\psi(\sigma) \otimes \tau$ on $\mathrm{SO}_{2k+1}(F) \times \mathrm{GL}_m(F)$, where $\theta_\psi(\sigma)$ is the local ψ-theta lift of σ to $\mathrm{SO}_{2k+1}(F)$.)

Here we get the added property that $\sigma(\tau)$ is irreducible, which we do not have yet in the global case, but, of course, conjecture to be true. Due to our lack of knowledge of the local representation theory of $\widetilde{\mathrm{Sp}}_{2n}(F)$, we do not yet have a good definition of the local L-factor $L_\psi(\sigma \otimes \eta, s)$ and the local ε-factor $\varepsilon(\sigma \otimes \tau, s, \psi)$, such that

$$\gamma(\sigma \otimes \tau, s, \psi) = \frac{\varepsilon(\sigma \otimes \tau, s, \psi) L_\psi(\widehat{\sigma} \otimes \tau, 1-s)}{L_\psi(\sigma \otimes \tau, s)}. \tag{0.2}$$

However, the gamma factor on the l.h.s. of (0.2) is well defined (details will be given in the paper) and the condition that $\gamma(\sigma \otimes \tau, s, \psi)$ has a pole at $s = 1$ should really reflect the condition that $L_\psi(\widehat{\sigma} \otimes \tau, s)$ has a pole at $s = 0$.

We will now recall some details from [G.R.S.1], [G.R.S.4] in order to complete the introductory global picture and thence draw the outline of the local theory, noting the beautiful analogy that runs throughout. We start with the global integrals mentioned above. These are introduced in [G.R.S.4]. Let us briefly explain how they are constructed. Let π be an irreducible, automorphic, cuspidal representation of $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$. We assume that π is genuine and has nontrivial Whittaker coefficients, with respect to a standard nondegenerate character, corresponding to ψ^{-1}. (In the text we denote this Whittaker character by ψ_k.) Let η be an irreducible, automorphic, cuspidal representation of $\mathrm{GL}_m(\mathbb{A})$. We assume that $k < m$. Consider the representation $\rho_{\eta,s}$ of $\mathrm{Sp}_{2m}(\mathbb{A})$, induced from the Siegel parabolic subgroup and $\eta \otimes |\det \cdot|^{s-1/2}$, and let $E_\eta(g, s)$ be the corresponding Eisenstein series. The global integrals mentioned above have the form

$$\int\limits_{\mathrm{Sp}_{2k}(K)\backslash \mathrm{Sp}_{2k}(\mathbb{A})} \varphi(g) J_{k,\psi}(\omega_\psi^{(k)}(g,1)\phi, E_\eta(g,s))dg \tag{0.3}$$

where φ is a cusp form in the space of π, $\omega_\psi^{(k)}$ denotes the Weil representation of $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$ attached to ψ (it acts on Schwartz-Bruhat functions $\phi \in S(\mathbb{A}^k)$) and $J_{k,\psi}(\quad,\quad)$ denotes a Fourier-Jacobi coefficient. The integral (0.3) is Eulerian, and for decomposable data, it represents $\frac{L_\psi^S(\pi \otimes \eta, s)}{L^S(\eta, \Lambda^2, 2s) L^S(\eta, s+\frac{1}{2})}$ where $L_\psi^S(\pi \otimes \eta, s)$ is the partial, standard L-function of $\pi \otimes \eta$, corresponding to ψ.

Let $m = 2n$, and let $L^S(\eta, \Lambda^2, s)$ have a pole at $s = 1$. Assume also that $L^S(\eta, \frac{1}{2}) \neq 0$. In this case, $E_\eta(g,s)$ has a (simple) pole at $s = 1$, and then, for $L_\psi^S(\pi \otimes \eta, s)$ to have a pole at $s = 1$, we must have (using that $L^S(\eta, \Lambda^2, 2s) \cdot L^S(\eta, s + \frac{1}{2})$ is holomorphic and nonzero at $s = 1$)

$$J_{k,\psi}(\omega_\psi^{(k)}(g,\varepsilon)\phi, \mathrm{Res}_{s=1} E_\eta(g,s)) \not\equiv 0$$

for $(g,\varepsilon) \in \widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$. We prove in [G.R.S.1],

Theorem 2. *We have, for $k < n$,*

$$J_{k,\psi}(\omega_\psi^{(k)}(g,\varepsilon)\phi, \mathrm{Res}_{s=1} E_\eta(g,s)) = 0,$$

and hence $L_\psi^S(\pi \otimes \eta, s)$ is holomorphic at $s = 1$, for $k < n$.

The holomorphy of $L_\psi^S(\pi \otimes \eta, s)$ is clear if we assume that π has a ψ-weak lift to $\mathrm{GL}_{2k}(\mathbb{A})$, and then we expect that η as above is the image of such a lift if $k = n$. Therefore, we introduced, in [G.R.S.1], the automorphic representation $\sigma(\eta) = \sigma_n(\eta)$ of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, which acts by right translations in the space (where $-$ denotes complex conjugation)

$$V_{\sigma(\eta)} = \{(g,\varepsilon) \mapsto \overline{J_{n,\psi}(\omega_\psi^{(n)}(g,\varepsilon)\phi, \mathrm{Res}_{s=1} E_\eta(g,s))}\} \ .$$

Once we have the main global theorem of this paper (i.e. $V_{\sigma(\eta)} \neq 0$), then Theorem 1, mentioned earlier, is proven in [G.R.S.1].

We now present in more details the local (supercuspidal) counterpart of the global theory sketched above. For an irreducible, self-dual supercuspidal representation τ of $\mathrm{GL}_{2n}(F)$, such that $L(\tau, \Lambda^2, s)$ has a pole at $s = 0$, we construct the following representation of $\widetilde{\mathrm{Sp}}_{2n}(F)$:

$$\widehat{\sigma}_n(\tau) = J_{\mathcal{H}_n}\Big(J_{N_{n+1},\chi_n^{-1}}(\pi_\tau) \otimes \omega_\psi^{(n)}\Big). \tag{0.4}$$

Here π_τ is the Langlands quotient of $\rho_{\tau,1}$, the representation of $\mathrm{Sp}_{4n}(F)$, induced from the Siegel parabolic subgroup, and $\tau \otimes |\det\cdot|^{1/2}$. π_τ is the analog of the residue at $s = 1$ of the Eisenstein series. J_U (resp. $J_{U,\chi}$) denotes the Jacquet functor with respect to a unipotent group U and the trivial character (resp. the character χ of U). N_{n+1} is the unipotent radical of the standard parabolic subgroup Q_{n+1}, which preserves a flag of isotropic subspaces of dimensions from 1 to $n+1$, and χ_n is the character of N_{n+1}, which equals ψ on each simple root group inside N_{n+1}, and is trivial on the other root groups. $\mathcal{H}_n$ is the Heisenberg group in $2n+1$ variables. It is embedded in the unipotent radical of the standard parabolic subgroup of $\mathrm{Sp}_{2n+2}(F)$, which preserves a line, and we take the natural embedding of $\mathrm{Sp}_{2n+2}(F)$ inside Levi (Q_{n+1}). Moreover, replacing n by k, $k < 2n$, at each place in (0.4), we obtain a sequence of representations $\widehat{\sigma}_k(\tau)$ of $\widetilde{\mathrm{Sp}}_{2k}(F)$. We prove

The tower property. *For the first k_0, such that $\widehat{\sigma}_{k_0}(\tau) \neq 0$, the representation $\widehat{\sigma}_{k_0}(\tau)$ is supercuspidal, i.e. its Jacquet functors, with respect to unipotent radicals of parabolic subgroups, vanish. (In particular $\widehat{\sigma}_{k_0}(\tau)$ is semisimple.)*

This is the exact analog of tower (2.3) in [G.R.S.1]. The tower property is valid even if we just require that τ is supercuspidal and replace π_τ by any constituent of $\rho_{\tau,s}$. The nonvanishing of $\widehat{\sigma}_k(\tau)$ is guaranteed for $k = n$.

Theorem A. *Let θ be a generic representation of* $\mathrm{GL}_{2n}(F)$*. Then, for a subrepresentation π of $\rho_{\theta,\frac{1}{2}}$, we have*

$$J_{\mathcal{H}_n}(J_{N_{n+1},\chi_n^{-1}}(\pi)\otimes\omega_\psi^{(n)})\neq 0.$$

In particular (for τ as above),

$$\hat{\sigma}_n(\tau)\neq 0\ .$$

The main property of the residue at $s=1$ of the Eisenstein series (on $\mathrm{Sp}_{4n}(\mathbb{A})$) $E_\eta(g,s)$ is that this residue has a nontrivial period along $\mathrm{Sp}_{2n}(\mathbb{A})\times\mathrm{Sp}_{2n}(\mathbb{A})$ [G.R.S.1, Theorem 2]. We prove the local analog.

Theorem B. *For τ as above, the Langlands quotient π_τ admits nontrivial* $\mathrm{Sp}_{2n}(F)\times\mathrm{Sp}_{2n}(F)$*-invariant functionals, and hence (by* [G.R.S.1, Theorem 17]*)*

$$\widehat{\sigma}_k(\tau)=0\ ,\qquad \textit{for}\quad k<n\ .$$

The tower property and Theorem A now prove that $\widehat{\sigma}_n(\tau)$ is supercuspidal, and hence semisimple. The irreducibility of $\widehat{\sigma}_n(\tau)$ follows from the following results:

(i) *Each summand of $\widehat{\sigma}_n(\tau)$ is ψ-generic* (*Proposition* 4.2).
(ii) *There is a certain unipotent subgroup $E_{2n}\subset\mathrm{Sp}_{4n}(F)$ and a character $\psi^{(2n)}$ of E_{2n}* (*Sec.* 4.1)*, such that the dimension of the space of ψ-Whittaker functionals on (the space of) $\widehat{\sigma}_n(\tau)$ is equal to* $\dim J_{E_{2n},\psi^{(2n)}}(\pi_\tau)$.
(iii) $\dim J_{E_{2n},\psi^{(2n)}}(\pi_\tau)=\dim J_{V_{2n},\widetilde{\psi}}(\pi_\tau)=1$, *where V_{2n} is the standard maximal unipotent subgroup of* $\mathrm{Sp}_{4n}(F)$*, and $\widetilde{\psi}$ is the character, which is trivial on the Siegel radical and is equal to the standard ψ-nondegenerate character on the* GL_{2n}*-Levi part of V_{2n}.*

It is interesting that these arguments are local variants, adapted from the proof of the main global theorem ($V_{\sigma(\eta)}\neq 0$).

The representation $\sigma(\tau)$ which appears in the main local theorem is the contragredient of $\widehat{\sigma}_n(\tau)$.

The paper is organized as follows. Section 1 is an extended introduction. We explain how the local theory of the global integrals (0.3) gives rise to the representations $\widehat{\sigma}_k(\tau)$. We define representations $\tilde{\sigma}_k(\tau)$ of $\widetilde{\mathrm{Sp}}_{2k}(F)$, such that there is a surjective morphism $\hat{\sigma}_k(\tau)\longrightarrow\tilde{\sigma}_k(\tau)$. The elements of the space $V_{\tilde{\sigma}_k(\tau)}$ of $\tilde{\sigma}_k(\tau)$ are ψ-Whittaker functions on $\widetilde{\mathrm{Sp}}_{2k}(F)$, which appear as inner integrals to the local integrals which emerge from (0.3). We explain in Section 1.2 that for a supercuspidal, ψ^{-1}-generic representation σ of $\widetilde{\mathrm{Sp}}_{2k}(F)$, $k<2n$, $\gamma(\sigma\times\tau,s,\psi)$ has a pole at $s=1$, if and only if $\widehat{\sigma}$ is a summand of $\tilde{\sigma}_k(\tau)$ (Corollary 1.2.3). The main theorem of Section 1 is Theorem 1.3, which is Theorem A above. We discovered this theorem only recently, and since it stands in such a generality (arbitrary θ...) we preferred it over our older proof outlined in [G.R.S.3, Theorem 9]. In Section 2, we prove the tower property of the representations $\{\widehat{\sigma}_k(\tau)\}_{k<2n}$. In Section 3, we prove Theorem B. For this, we need to relate two local exterior square gamma factors of τ: the one defined by Shahidi (as a local coefficient) and the one which emerges from the work of Jacquet-Shalika, by examining the local theory which corresponds to their Rankin-Selberg integrals which represent the (partial) exterior square L-function. In Section 4, we prove the irreducibility of $\widehat{\sigma}_n(\tau)$. We prove the results (ii) and (iii) above in a much larger generality. In Section 5 we prove the main global theorem (i.e. $V_{\sigma(\eta)}\neq 0$). The proof here does not depend on the material presented in the previous section, although the analogy is clear. We have chosen to place the global result in this section in order to maintain some sort of continuity in the paper. Still we made Section 5 self-contained for the convenience of the reader who is interested in the global result first. We marked the precise reference for each notation which appeared previously. However, we did not refrain from pointing out the analogy with the local theory. Section 6 is an appendix, where we prove results that we need on the local theory of L-functions for $\widetilde{\mathrm{Sp}}_{2k} \times \mathrm{GL}_m$ (Sections 6.1 and 6.2) and on exterior square gamma factors for τ (Section 6.3).

General notation. We write the elements of the symplectic group $\mathrm{Sp}_{2k}(F)$, with respect to the skew-symmetric matrix $\begin{pmatrix} & w_k \\ -w_k & \end{pmatrix}$ where $w_k = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}$.

If x is a $k \times k$ matrix, such that $w_k x$ is symmetric, we sometimes denote

$$\ell(x) = \begin{pmatrix} I_k & x \\ & I_k \end{pmatrix} \ , \quad \overline{\ell}(x) = \begin{pmatrix} I_k & \\ x & I_k \end{pmatrix} \ .$$

Of course, $\ell(x)$ and $\overline{\ell}(x)$ lie in $\mathrm{Sp}_{2k}(F)$. If $a \in \mathrm{GL}_k(F)$, we sometimes denote

$$m(a) = \begin{pmatrix} a & \\ & a^* \end{pmatrix}$$

where $a^* = w_k {}^t a^{-1} w_k$. (The dependence on k will always be made clear.)

For a representation π, we denote a space of its realization by V_π.

Let U be a unipotent group, χ a character of U, and π a smooth representation of U acting on V_π. We denote by $J_{U,\chi}(V_\pi)$ or $J_{U,\chi}(\pi)$ the space

$$V_\pi \Big/ \mathrm{Span}\left\{\pi(u)v - \chi(u)v \Big| u \in U, v \in V\right\}$$

which we also call the Jacquet module of π with respect to χ. When $\chi = 1$, we abbreviate to $J_U(V_\pi)$ or $J_U(\pi)$. We denote by $j_{U,\chi}$ the projection $j_{U,\chi} : V_\pi \longrightarrow J_{U,\chi}(V_\pi)$.

We denote by $\{e_{ij}\}_{i,j=1}^k$ the standard basis of the matrix space $M_k(F)$. Thus, e_{ij} is the $k \times k$ matrix, which has 1 in the (i,j)-th coordinate and zero elsewhere. Again, the dependence on k will be made clear.

We denote by Ind^c compact induction.

Finally, let us recall the notions of Whittaker coefficients, Whittaker model and genericity. Let G be a reductive split group over K. Fix a maximal split torus and a maximal unipotent subgroup U. This corresponds to a choice of a basis of simple roots for the corresponding root system. For each simple root α, let x_α denote the corresponding root coordinate inside U. Let χ be a character of $U(\mathbb{A})$, trivial on $U(K)$. We say that χ is generic if it is nontrivial on $\{x_\alpha(r) | r \in \mathbb{A}\}$, for each simple root α. Thus, there is a set $\{a_\alpha \in K^* \mid \alpha - \text{simple root}\}$, such that $\chi(x_\alpha(t)) = \psi(a_\alpha t)$, for each simple root α and $t \in \mathbb{A}$ (and $\chi(x_\alpha(t)) = 1$ for each positive nonsimple root α). Let χ be a generic character of $U(\mathbb{A})$. An automorphic representation σ of $G(\mathbb{A})$, acting on a space V_σ of automorphic forms, is χ-generic if

$$\int\limits_{U(K)\backslash U(\mathbb{A})} \varphi(u)\chi^{-1}(u)du \not\equiv 0 \ , \qquad \varphi \in V_\sigma \ .$$

This Fourier coefficient is called a χ-Whittaker coefficient. Similarly, over a local field, say $F = K_\nu$, a generic character of $U(F)$ is a character χ of $U(F)$, such that χ is nontrivial on $\{x_\alpha(r) \mid r \in F\}$, for each simple root α. A smooth representation σ of $G(F)$ acting in V_σ is χ-generic if $\mathrm{Hom}_{U(F)}(\sigma, \chi) \neq 0$. A nontrivial element ℓ of the last space is called a χ-Whittaker functional, and a χ-Whittaker model of σ (defined by ℓ) is the space of functions on $G(F)$, $\{g \mapsto \ell(\sigma(g)v) \mid v \in V_\sigma\}$. Note that σ is χ-generic iff $J_{U(F),\chi}(V_\sigma) \neq 0$. These notions adapt easily to $\widetilde{\mathrm{Sp}}_{2k}(F)$ and $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$. Here we choose U to be the standard maximal unipotent subgroup. There is a canonical splitting of the two-fold cover over $U(F)$ (resp. $U(\mathbb{A})$). See, for example, [M.V.W, p. 43]. In this case, ψ above defines a standard generic character by applying ψ to the sum of entries in the second diagonal of an element of U. We sometimes abbreviate to "ψ-generic", "ψ-Whittaker", etc.

1. Preliminaries, motivations and the main theorems

1.1. Gamma factors for $\widetilde{\mathrm{Sp}}_{2k}(F) \times \mathrm{GL}_m(F)$ $(k < m)$. In [G.R.S.4], we introduced global integrals of Shimura type, which represent the standard L-function for generic cusp forms on $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A}) \times \mathrm{GL}_m(\mathbb{A})$, where $\mathbb{A}$ is the adele ring of a number field K. We explained in [G.R.S.1], [G.R.S.3] how they lead to a map from irreducible, automorphic cuspidal representations η on $\mathrm{GL}_{2n}(\mathbb{A})$, such that $L^S(\eta, \Lambda^2, s)$ has a pole at $s = 1$ and $L(\eta, \frac{1}{2}) \neq 0$, to irreducible automorphic, cuspidal ψ-generic representations of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, which should be the inverse to the functorial lift (once we fix a nontrivial character ψ of $k\backslash\mathbb{A}$).

We motivate and explain the analogous local counterpart of the above map by means of the local theory of the Shimura type integrals studied in [G.R.S.4].

Let F be a local nonarchimedean field of characteristic zero. Fix a nontrivial character of F, denoted again by ψ. Let σ (resp. τ) be an irreducible representation of $\widetilde{\mathrm{Sp}}_{2k}(F)$ (resp. $\mathrm{GL}_m(F)$). We assume that τ is generic and that σ is genuine and generic with respect to the character ψ_k^{-1} of V_k, the standard maximal unipotent subgroup of $\widetilde{\mathrm{Sp}}_{2k}(F)$, where

$$\psi_k : v \mapsto \psi\Big(\sum_{i=1}^{k} v_{i,i+1}\Big). \tag{1.1}$$

Consider the representation

$$\rho_{\tau,s} = \mathrm{Ind}_{P_m}^{\mathrm{Sp}_{2m}(F)} \tau \otimes |\det\cdot|^{s-1/2} \qquad \text{(normalized induction)}$$

induced from P_m, the Siegel parabolic subgroup of $\mathrm{Sp}_{2m}(F)$. We think of an element $\varphi_{\tau,s}$ in the space of $\rho_{\tau,s}$ as a complex function on $\mathrm{Sp}_{2m}(F) \times \mathrm{GL}_m(F)$ such that for $a \in \mathrm{GL}_m(F)$

$$\varphi_{\tau,s}\left(\begin{pmatrix} a & * \\ 0 & a^* \end{pmatrix} g, I_m\right) = |\det a|^{s+\frac{m}{2}} \varphi_{\tau,s}(g, a)$$

and $a \mapsto \varphi_{\tau,s}(g, a)$ lies in the Whittaker model of τ with respect to the character

$$\psi'_m(z) = \psi\left(\sum_{i=1}^{k-1} z_{i,i+1} + 2z_{k,k+1} + \sum_{i=k+1}^{m-1} z_{i,i+1}\right). \tag{1.2}$$

Consider the parabolic subgroups $Q_{m,i} = D_{m,i} \ltimes N_{m,i}$ of $\mathrm{Sp}_{2m}(F)$, where

$$D_{m,i} = \left\{ \begin{pmatrix} a_1 &&&&&&\\ & \ddots &&&&&\\ && a_{m-i} &&&&\\ &&& g &&&\\ &&&& a_{m-i}^{-1} &&\\ &&&&& \ddots &\\ &&&&&& a_1^{-1} \end{pmatrix} \middle| \begin{matrix} a_j \in F^* \\ g \in \mathrm{Sp}_{2i}(F) \end{matrix} \right\},$$

$$N_{m,i} = \left\{ \begin{pmatrix} z & * & * \\ & I_{2i} & * \\ && z^* \end{pmatrix} \in \mathrm{Sp}_{2m}(F) \middle| z \in Z_{m-i} \right\}.$$

Here Z_{m-i} is the standard maximal unipotent subgroup of $\mathrm{GL}_{m-i}(F)$. Assume that $k < m$. Let χ_k be the following character of $N_{m,k+1}$:

$$\chi_k(v) = \psi\Big(\sum_{i=1}^{m-k-1} v_{i,i+1} \Big) , \qquad v \in N_{m,k+1} . \tag{1.3}$$

χ_k is the restriction to $N_{m,k+1}$ of the standard generic character defined by ψ. Consider the following subgroup of $N_{m,k}$:

$$H_k = \left\{ h = \begin{pmatrix} I_{m-k-1} &&&& \\ & 1 & x & z & \\ && I_{2k} & x' & \\ &&& 1 & \\ &&&& I_{m-k-1} \end{pmatrix} \in \mathrm{Sp}_{2m}(F) \right\}. \tag{1.4}$$

H_k is naturally identified with $N_{m,k}/N_{m,k+1}$ and is isomorphic to the Heisenberg group $\mathcal{H}_k$ on F^{2k} equipped with the symplectic form defined by $2 \begin{pmatrix} & w_k \\ -w_k & \end{pmatrix}$, where $w_k = \begin{pmatrix} && 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 && \end{pmatrix}$ $(k \times k)$. The isomorphism is given by $j_{m,k}(x;z) = h$, as in (1.4). Note that we use a slightly different isomorphism than the one in [G.R.S.4], the purpose being the use of the Whittaker model of σ with respect to ψ_k in (1.1) rather than the character $\widetilde{\psi}_k$ in [G.R.S.4, Sec. 1].

Let $\omega_\psi^{(k)}$ be the Weil representation of $\mathcal{H}_k \rtimes \widetilde{\mathrm{Sp}}_{2k}(F)$ which corresponds to the character $(0;z) \mapsto \psi(z)$ of the center of $\mathcal{H}_k$. $\omega_\psi^{(k)}$ acts on $S(F^k)$—the space of Schwartz Bruhat functions on F^k. For $k = 0$, we define $\widetilde{\mathrm{Sp}}_0(F) = \{I\}$, $\mathcal{H}_0 = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} | z \in F \right\}$ and $\omega_\psi^{(0)} = \psi$. The local integrals which emerge from [G.R.S.4] are

$$J(W,\phi,\varphi_{\tau,s}) = \int\limits_{g \in V_k \backslash \widetilde{\mathrm{Sp}}_{2k}(F)} \int\limits_{h \in \mathcal{Y}_k \backslash \mathcal{H}_k} \int\limits_{v \in N'_{m,k+1} \backslash N_{m,k+1}} W(g)\omega_\psi^{(k)}(h \cdot g)\phi(\xi_0) \cdot f_{\tau,s}(\gamma_{m,k} v j_{m,k}(hg))\chi_k(v)dvdhdg. \tag{1.5}$$

Here, W lies in the ψ_k^{-1} Whittaker model of σ, ϕ is in $S(F^k)$, $\xi_0 = (0\cdots 01)$,

$$\gamma_{m,k} = \begin{pmatrix} 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & -I_{m-k} \\ I_{m-k} & 0 & 0 & 0 \\ 0 & 0 & I_k & 0 \end{pmatrix}, \tag{1.6}$$

and $j_{m,k}(hg) = j_{m,k}(h)j_{m,k}(g)$, where, for $g \in \mathrm{Sp}_{2k}(F)$,

$$j_{m,k}(g) = \begin{pmatrix} I_{m-k} & & \\ & g & \\ & & I_{m-k} \end{pmatrix}. \tag{1.7}$$

$$N'_{m,k+1} = \left\{ \begin{pmatrix} z & 0 & u & 0 \\ & I_k & 0 & u' \\ & & I_k & 0 \\ & & & z^* \end{pmatrix} \in \mathrm{Sp}_{2m}(F) \,\middle|\, \begin{matrix} z \in Z_{m-k} \\ u \text{ with zero bottom row} \end{matrix} \right\},$$

$$\mathcal{Y}_k = \{(0, y; 0) \in \mathcal{H}_k \mid y \in F^k\},$$

$$f_{\tau,s}(h) = \varphi_{\tau,s}(h; I_m).$$

The integral (1.5) converges absolutely for s in a right half plane, and for a holomorphic section $\varphi_{\tau,s}$, it has a meromorphic continuation to the whole complex plane, being a rational function of q^{-s}, where q is the number of elements in the residue field of F. J satisfies the following properties:

(1.8)
$$J(W, \phi, \rho_{\tau,s}(v)\varphi_{\tau,s}) = \chi_k^{-1}(v) J(W, \phi, \varphi_{\tau,s}), \qquad \text{for} \quad v \in N_{m,k+1},$$

(1.9)
$$J(\sigma(g,\varepsilon)W, \omega_\psi^{(k)}(h \cdot (g,\varepsilon))\phi, \rho_{\tau,s}(j_{m,k}(hg))\varphi_{\tau,s}) = J(W, \phi, \varphi_{\tau,s})$$
$$\text{for} \quad (g,\varepsilon) \in \widetilde{\mathrm{Sp}}_{2k}(F), h \in \mathcal{H}_k;$$

J satisfies a functional equation. The second side of the functional equation can be deduced from the global integrals, by applying an intertwining operator M_s in the Eisenstein series. This translates to the local case as follows. Take M_s to be the local intertwining operator, acting on the space of $\rho_{\tau,s}$, which corresponds to the Weyl element $\begin{pmatrix} & I_k \\ -I_k & \end{pmatrix}$. M_s takes $\rho_{\tau,s}$ to $\rho_{\hat{\tau},1-s}$. Define $\widetilde{J}(W, \phi, M_s(\varphi_{\tau,s}))$ by substituting $M_s(\varphi_{\tau,s})(r, d_{m,k})$ in $J(W, \phi, \varphi_{\tau,s})$, instead of $\varphi_{\tau,s}(r, I_m)$, where $d_{m,k} = \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_m \end{pmatrix}$, such that $\frac{t_i}{t_{i+1}} = -1$, for all $i \neq m-k$, and $\frac{t_{m-k}}{t_{m-k+1}} = -2$. (This is done in order to preserve the transformation rule in (1.2).) $\widetilde{J}(W, \phi, M_s(\varphi_{\tau,s}))$ satisfies the equivariance properties (1.8), (1.9). An adaptation to the local case of the proof of [G.R.S.4, Theorem 5.1] shows that there exists a meromorphic function $\gamma(\sigma \times \tau, s, \psi)$, such that

$$\frac{\gamma(\sigma \times \tau, s, \psi)}{\gamma(\tau, \Lambda^2, 2s-1, \psi)} J(W, \phi, \varphi_{\tau,s}) = \widetilde{J}(W, \phi, M_s(\varphi_{\tau,s})). \tag{1.10}$$

This amounts to proving that except for a finite number of values of q^{-s}, there is, up to scalars, a unique trilinear form in the variables $(W, \phi, \varphi_{\tau,s})$ which satisfies

(1.8), (1.9). We give the details in Sections 6.1, 6.2. In (1.10), $\gamma(\tau,\Lambda^2,2s-1,\psi)$ is the Shahidi local coefficient of τ, which corresponds to Λ^2 [Sh1].

Consider the inner integral of $J(W,\phi,\varphi_{\tau,s})$ (1.5) (with $g=I$):

(1.11)

$$J_{m,k}(\phi,\varphi_{\tau,s})=\int\limits_{h\in\mathcal{Y}_k\backslash\mathcal{H}_k}\int\limits_{v\in N'_{m,k+1}\backslash N_{m,k+1}}\omega_\psi^{(k)}(h)\phi(\xi_0)f_{\tau,s}(\gamma_{m,k}vj_{m,k}(h))\chi_k(v)dvdh.$$

Note that

$$J_{m,k}(\omega_\psi^{(k)}(u\cdot h)\phi,\rho_{\tau,s}(vj_{m,k}(u\cdot h))\varphi_{\tau,s})=\psi_k(u)\chi_k^{-1}(v)J_{m,k}(\phi,\varphi_{\tau,s}) \tag{1.12}$$

for $v\in N_{m,k+1}, h\in\mathcal{H}_k, u\in V_k$. This shows that

$$W_{\phi,\varphi_{\tau,s}}(g,\varepsilon)=J_{m,k}(\omega_\psi^{(k)}(g,\varepsilon)\phi,\rho_{\tau,s}(j_{m,k}(g))\varphi_{\tau,s}) \tag{1.13}$$

is a ψ-Whittaker function on $\widetilde{\mathrm{Sp}}_{2k}(F)$, and more generally, $J_{m,k}$ is an element of the dual to the Jacquet module $J_{V_k,\psi_k}\Big[J_{\mathcal{H}_k}\Big(J_{N_{m,k+1},\chi_k^{-1}}(\rho_{\tau,s})\otimes\omega_\psi^{(k)}\Big)\Big]$. It is possible to adapt the proof of Theorem 5.1 in [G.R.S.4] and show that the embedding in $V_{\rho_{\tau,s}}$ of the subspace of functions $S(\mathcal{O}_{m,k};P_m,\tau_s)$, supported inside the (open) orbit $\mathcal{O}_{m,k}=P_m\cdot\gamma_{m,k}N_{m,k}j_{m,k}(\mathrm{Sp}_{2k}(F))$, induces an isomorphism of $\widetilde{\mathrm{Sp}}_{2k}(F)$-modules

(1.14)

$$J_{\mathcal{H}_k}\Big(J_{N_{m,k+1},\chi_k^{-1}}(V_{\rho_{\tau,s}})\otimes\omega_\psi^{(k)}\Big)\cong J_{\mathcal{H}_k}\Big(J_{N_{m,k+1},\chi_k^{-1}}(S(\mathcal{O}_{m,k};P_m,\tau_s))\otimes\omega_\psi^{(k)}\Big).$$

Details are given in Section 6.1. From this, it is easy to conclude

Proposition. *The integral* (1.11)*, which defines $J_{m,k}(\phi,\varphi_{\tau,s})$, stabilizes for large compact open subgroups of $\mathcal{Y}_k\backslash\mathcal{H}_k\times N'_{m,k+1}\backslash N_{m,k+1}$, and hence $J_{m,k}(\phi,\varphi_{\tau,s})$ is holomorphic for all s.*

The reason for the proposition is that, due to (1.14), we may replace $\varphi_{\tau,s}$ in $J_{m,k}(\phi,\varphi_{\tau,s})$ by an element $\varphi'_{\tau,s}\in S(\mathcal{O}_{m,k};P,\tau_s)$, for which $J_{m,k}(\phi,\varphi'_{\tau,s})$ converges absolutely, since the corresponding integrand is compactly supported.

Note that the proposition implies the meromorphic continuation of the integrals $J(W,\phi,\varphi_{\tau,s})$, since we can write

$$J(W,\phi,\varphi_{\tau,s})=\int\limits_{V_k\backslash\mathrm{Sp}_{2k}(F)}W(g)J_{m,k}\Big(\omega_\psi^{(k)}(g)\phi,\rho_{\tau,s}(j_{m,k}(g))\varphi_{\tau,s}\Big)dg \tag{1.15}$$

and now we use the Iwasawa decomposition of $\mathrm{Sp}_{2k}(F)$ and the asymptotic expansion of $W(g)$.

1.2. Existence of a pole at $s=1$, for $\gamma(\sigma\times\tau,s,\psi)$. We assume from now on that $m=2n$. Let τ be supercuspidal, and rewrite the functional equation (1.10) as

$$\gamma(\sigma\times\tau,s,\psi)J(W,\phi,\varphi_{\tau,s})=L(\widehat{\tau},\Lambda^2,2(1-s))\widetilde{J}(W,\phi,M_s^*(\varphi_{\tau,s})) \tag{1.16}$$

where

$$M_s^*=\frac{\varepsilon(\tau,\Lambda^2,2s-1,\psi)}{L(\tau,\Lambda^2,2s-1)}M_s\ .$$

We use the local factors defined by Shahidi and the fact that

$$\gamma(\tau,\Lambda^2,z,\psi)=\varepsilon(\tau,\Lambda^2,z,\psi)\frac{L(\hat{\tau},\Lambda^2,1-z)}{L(\tau,\Lambda^2,z)}\ .$$

Shahidi showed that $\frac{1}{L(\tau,\Lambda^2,2s-1)}M_s$ is holomorphic and nontrivial, and thence M_s^* is holomorphic and nontrivial, since $\varepsilon(\tau,\Lambda^2,2s-1,\psi)$ is monomial. See [Sh1], [Sh2].

Proposition. *Assume that σ and τ are supercuspidal. Then $J(W,\phi,\varphi_{\tau,s})$ and $\widetilde{J}(W,\phi,M_s^*(\varphi_{\tau,s}))$ are holomorphic.*

Proof. Since σ is supercuspidal, W has compact support, modulo V_k. Now use (1.15) and the proposition in 1.1 to obtain the holomorphicity of $J(W,\phi,\varphi_{\tau,s})$. Since $M_s^*(\varphi_{\tau,s})$ is holomorphic, and $\widetilde{J}(W,\phi,M_s^*(\varphi_{\tau,s}))$ has entirely the same structure, the same proof works in this case as well. ∎

Redenote $N_{2n,i}=N_i$, $N'_{2n,i}=N'_i$, $j_{2n,k}=j_k$, $\gamma_{2n,k}=\gamma_k$, $J_{2n,k}=J_k$, etc. Define

(1.17)
$$\widetilde{J}_k(\phi,M_s^*(\varphi_{\tau,s}))=\int\limits_{\mathcal{Y}_k\backslash\mathcal{H}_k}\int\limits_{N'_{k+1}\backslash N_{k+1}}\omega_\psi^{(k)}(h)\phi(\xi_0)M_s^*(\varphi_{\tau,s})(\gamma_k v j_k(h),d_k)\chi_k(v)dvdh$$

(d_k is $d_{2n,k}$ from Section 1.1). We write, for the record,

(1.18)
$$\widetilde{J}(W,\phi,M_s^*(\varphi_{\tau,s}))=\int\limits_{V_k\backslash\mathrm{Sp}_{2k}(F)}W(g)\widetilde{J}_k(\omega_\psi^{(k)}(g)\phi,\rho_{\widehat{\tau},1-s}(j_k(g))M_s^*(\varphi_{\tau,s}))dg.$$

Corollary 1. *Assume that σ and τ are supercuspidal. Then the only possible poles of $\gamma(\sigma\times\tau,s,\psi)$ occur among those of $L(\widehat{\tau},\Lambda^2,2(1-s))$. In particular, if τ is self-dual, the only possible poles of $\gamma(\sigma\times\tau,s,\psi)$ occur on the line $\mathrm{Re}(s)=1$.*

Proof. From the last proposition, the only poles of the r.h.s. of (1.16) occur among those of $L(\widehat{\tau},\Lambda^2,2(1-s))$. In [G.R.S.4, Prop. 6.6], we showed that data $(W,\phi,\varphi_{\tau,s})$ can be chosen, so that $J(W,\phi,\varphi_{\tau,s})=1$, for all s. The functional equation (1.16) now implies the corollary. In case τ is self-dual, the only possible poles of $L(\tau,\Lambda^2,2(1-s))$ occur on the line $\mathrm{Re}(s)=1$. See [Sh1]. ∎

Corollary 2. *Assume that σ and τ are supercuspidal and that τ is self-dual. Then $\gamma(\sigma\times\tau,s,\psi)$ has a pole at $s=1$, if and only if $L(\tau,\Lambda^2,s)$ has a pole at $s=0$ and*

$$\int\limits_{V_k\backslash\mathrm{Sp}_{2k}(F)}W(g)\widetilde{J}_k(\omega_\psi^{(k)}(g)\phi,\rho_{\tau,0}(j_k(g))M_1^*(\varphi_{\tau,1}))dg\not\equiv 0. \tag{1.19}$$

Assume that τ is self-dual and $L(\tau,\Lambda^2,s)$ has a pole at $s=0$. The condition (1.19) suggests that we consider the following space of functions on $\widetilde{\mathrm{Sp}}_{2k}(F)$:

$$V_{\tilde{\sigma}_k(\tau)}=\mathrm{Span}\left\{(g,\varepsilon)\mapsto\widetilde{J}_k(\omega_\psi^{(k)}(g,\varepsilon)\phi,\rho_{\tau,0}(j_k(g))M_1^*(\varphi_{\tau,1}))\Big|\begin{matrix}\phi\in s(F^k)\\ \varphi_{\tau,1}\in V_{\rho_{\tau,1}}\end{matrix}\right\}. \tag{1.20}$$

By (1.12) and (1.13), $V_{\tilde{\sigma}_k}(\tau)$ consists of Whittaker functions on $\widetilde{\mathrm{Sp}}_{2k}(F)$, with respect to the standard character ψ_k of V_k ((1.1)). $V_{\tilde{\sigma}_k(\tau)}$ is invariant to right translations by $\widetilde{\mathrm{Sp}}_{2k}(F)$, and hence affords a smooth representation $\tilde{\sigma}_k(\tau)$ of $\widetilde{\mathrm{Sp}}_{2k}(F)$. The condition (1.19) means that σ is paired into $\tilde{\sigma}_k(\tau)$, which is equivalent, due to the supercuspidality of σ, to

Corollary 3. *Under the above assumptions, $\gamma(\sigma\times\tau,s,\psi)$ has a pole at $s=1$, if and only if $\widehat{\sigma}$ is a summand of $\tilde{\sigma}_k(\tau)$.*

1.3. The question of nonvanishing of $V_{\tilde{\sigma}_k}(\tau)$. In (1.20), the elements $M_1^*(\varphi_{\tau,1})$ form an invariant subspace of $V_{\rho_{\tau,0}}$. Let us consider, for a given irreducible generic representation θ of $\mathrm{GL}_{2n}(F)$, for a given subrepresentation $(\pi(\theta), V_{\pi(\theta)})$ of $\rho_{\theta,\frac{1}{2}}$, and for $k < 2n$,

$$V_{k,\pi(\theta)} = \mathrm{Span}\left\{(g,\varepsilon) \mapsto J_k\left(\omega_\psi^{(k)}(g,\varepsilon)\phi, \rho_{\theta,\frac{1}{2}}(j_k(g))\varphi\right)\Big| \begin{matrix}\phi \in S(F^k)\\ \varphi \in V_{\pi(\theta)}\end{matrix}\right\}. \tag{1.21}$$

This space affords a representation $\tilde{\sigma}_{k,\pi(\theta)}$ (by right translations of $\widetilde{\mathrm{Sp}}_{2k}(F)$). As in (1.3), this is a space of Whittaker functions, with respect to ψ_k. From (1.12), the map

$$\varphi \otimes \phi \mapsto \Big((g,\varepsilon) \mapsto J_k(\omega_\psi^{(k)}(g,\varepsilon)\phi, \rho_{\theta,\frac{1}{2}}(j_k(g))\varphi)\Big) \tag{1.22}$$

defines a surjective morphism of $\widetilde{\mathrm{Sp}}_{2k}(F)$-modules

$$J_{\mathcal{H}_k}\left(J_{N_{k+1},\chi_k^{-1}}(V_{\pi(\theta)}) \otimes \omega_\psi^{(k)}\right) \longrightarrow V_{k,\pi(\theta)}. \tag{1.23}$$

Note the case $\theta = \tau \otimes |\det\cdot|^{-1/2}$, where τ is self-dual, supercuspidal and such that $L(\tau, \Lambda^2, s)$ has a pole at $s = 0$. Here $\rho_{\theta,\frac{1}{2}} = \rho_{\tau,0}$. Shahidi proved that $\rho_{\tau,0}$ has two constituents. One irreducible subrepresentation (nongeneric) π_τ, and one irreducible (generic) quotient ρ_τ. π_τ is the image of M_1 (or M_1^*) applied to $\rho_{\tau,1}$. It is the Langlands quotient of $\rho_{\tau,1}$. Thus, in this case, the only nontrivial $\pi(\theta)$ is π_τ. Here $\tilde{\sigma}_k(\tau) = \tilde{\sigma}_{k,\pi(\theta)}$.

Denote by $\widehat{\sigma}_{k,\pi(\theta)}$ the representation of $\widetilde{\mathrm{Sp}}_{2k}(F)$ on the l.h.s. of (1.23), and in case $\theta = \tau \otimes |\det\cdot|^{-1/2}$, τ-self-dual supercuspidal, and with $L(\tau, \Lambda^2, s)$ having a pole at $s = 0$, denote $\widehat{\sigma}_k(\tau) = \widehat{\sigma}_{k,\pi_\tau}$.

Theorem. *We have, in general,*

$$\tilde{\sigma}_{n,\pi(\theta)} \neq 0 \ , \tag{1.24}$$

and, in particular,

$$\widehat{\sigma}_{n,\pi(\theta)} \neq 0. \tag{1.25}$$

Proof. We will show that $J_n(\phi, \varphi) \not\equiv 0$, as ϕ and φ vary in $S(F^n)$ and $V_{\pi(\theta)}$, respectively. Explicating (1.11), we have

$$J_n(\phi,\varphi) = \int \phi(\xi_0 + u_n)\varphi\left(\begin{pmatrix} I_n & & & \\ & I_n & & \\ u & v & I_n & \\ & u' & & I_n\end{pmatrix}\gamma_n, 1\right)\psi(v_{n,1})d(u,v). \tag{1.26}$$

Here u_n is the n-th row of u. Since ϕ is arbitrary, the nonvanishing of (1.26) in (ϕ, φ) is equivalent to

$$\int\limits_{u_n=0} \varphi\left(\begin{pmatrix} I_n & & & \\ & I_n & & \\ u & v & I_n & \\ 0 & u' & & I_n\end{pmatrix}, 1\right)\psi(v_{n,1})d(u,v) \not\equiv 0 \ , \tag{1.27}$$

as φ varies in $V_{\pi(\theta)}$.

Denote, for $x=(x_1,\dots,x_{n-1})$, $y=\begin{pmatrix} y_1\\ \vdots \\ y_{n-1}\end{pmatrix}$ and $1\le i\le n-1$, $2\le j\le n$,

$$\overline{\ell}_i(x)=\overline{\ell}\Big(\sum_{s=1}^{n-1}x_s(e_{i,s}+e_{2n+1-s,2n+1-i})\Big),$$
$$\ell^j(y)=\ell\Big(\sum_{s=1}^{n-1}y_s(e_{s,j}+e_{2n+1-j,2n+1-s})\Big).$$

Let, for $0\le i\le n-1$,

$$(1.28)\quad \overline{X}(i,n)=\left\{\overline{x}=\begin{pmatrix} I_{n-1} & & & \\ & I_{n+1} & & \\ u & v & I_{n+1} & \\ 0 & u' & & I_{n-1}\end{pmatrix} \in \mathrm{Sp}_{4n}(F)\,\middle|\, \begin{matrix} u_{n+1}=u_n=\cdots=u_{i+1}=0\\ v_{n,1}=v_{n+1,1}=0\end{matrix}\right\}.$$

Here, u_j is the j-th row of u. The domain of integration in (1.27) is $\overline{X}(n-1,n)$. Define, for $\overline{x}$ in $\overline{X}(i,n)$, as in (1.28)

$$\psi_i(\overline{x})=\psi(v_{n,2})\ ,$$

and consider

$$I_i(\varphi)=\int\limits_{\overline{X}(i,n)}\varphi(\overline{x},1)\psi_i(\overline{x})d\overline{x}\ .$$

The integral (1.27) is $I_{n-1}(\varphi)$. We first show that, for $1\le i\le n-1$,

$$I_i(\varphi)\not\equiv 0 \Longleftrightarrow I_{i-1}(\varphi)\not\equiv 0,\quad \text{as}\quad \varphi\quad \text{varies in}\quad V_{\pi(\theta)}\ .$$

Indeed, we may assume that φ is a linear combination of vectors of the form

$$\xi*f=\int\limits_{F^n}\xi(y)\rho_{\theta,\frac{1}{2}}\Big(\ell^{i+1}(y)\Big)f dy$$

where $\xi\in S(F^{n-1})$ and $f\in V_{\pi(\theta)}$. (Here we allow an abuse of notation when we use "$*$".) We have

$$I_i(\xi*f)=\int\limits_{\overline{X}(i,n)}\int\limits_{F^n}\xi(y)f\Big(\overline{x}\cdot\ell^{i+1}(y),1\Big)\psi_i(\overline{x})dyd\overline{x}\ .$$

Write $\overline{x}$ in $\overline{X}(i,n)$ in the form

$$(1.29)\qquad \overline{x}=\overline{x}_0\overline{\ell}_1(u_1)\overline{\ell}_2(u_2)\cdot\ldots\cdot\overline{\ell}_i(u_i)$$

where $\overline{x}_0\in\overline{X}(0,n)$. For $i'\le i$, we have

$$\overline{\ell}_{i'}(u_{i'})\ell^{i+1}(y)\overline{\ell}_{i'}(u_{i'})^{-1}=z(i')\ell^{i+1}(y)$$

where

$$z(i')=m(I_{2n}-(u_{i'}\cdot y)e_{2n-i,2n-i'+1})\ .$$

It is easy to see that $z(i')$ normalizes $\overline{X}(i'-1,n)$, and preserves $\psi_{i'-1}$ and the measure $d\overline{x}$ on $\overline{X}(i'-1,n)$. Similarly, $\ell^{i+1}(y)$ normalizes $\overline{X}(0,n)$, and preserves ψ_0 and the measure $d\overline{x}$ on $\overline{X}(0,n)$. Finally, note that

$$f(z(i')h,1)=\psi(\delta_{i',i}(u_{i'}\cdot y))\ .$$

All this implies that (using obvious notation)

$$\begin{aligned}I_i(\xi * f) &= \int\limits_{\overline{X}(i,n)}\int\limits_{F^n}\xi(y)\psi^{-1}(u_i\cdot y)f(\overline{x},1)\psi_i(\overline{x})dyd\overline{x}\\ &= \int\limits_{\overline{X}(i,n)}\widehat{\xi}(u_i)f(\overline{x},1)\psi_i(\overline{x})d\overline{x}=I_{i-1}(\widehat{\xi}*f)\ .\end{aligned}$$

Here we used the notation (1.29) for $\overline{x}$. This proves our assertion. Now consider $I_0(\varphi)$. Let, for $r=\begin{pmatrix}r_1\\ \vdots\\ r_{n-1}\end{pmatrix}$ and $p=(p_1,\dots,p_{n-1})$,

$$\begin{aligned}\overline{e}(r)&=\overline{\ell}\Big(\sum_{s=1}^{n-1}r_s(e_{s,n}+e_{n+1,2n+1-s})\Big),\\ e(p)&=\ell\Big(\sum_{s=1}^{n-1}p_s(e_{n-1,s}+e_{2n+1-s,n+2})\Big).\end{aligned}$$

Again, take φ to be a linear combination of vectors of the form

$$\xi * f=\int\limits_{F^{n-1}}\xi(p)\rho_{\theta,\frac{1}{2}}(e(p))f\,dp$$

where $\xi\in S(F^{n-1})$ and $f\in V_{\pi(\theta)}$. Denote

$$\begin{aligned}\overline{S}&=\left\{\overline{v}=\begin{pmatrix}I_n&&&\\&I_n&&\\&v&I_n&\\&&&I_n\end{pmatrix}\in \mathrm{Sp}_{4n}(F)\right\},\\ \psi_{\overline{S}}(\overline{v})&=\psi(v_{n,1})\ .\end{aligned}$$

As before, we get

$$I_0(\xi * f)=\int\limits_{\overline{S}}\int\limits_{F^n}\widehat{\xi}(r)f(a\cdot\overline{e}(r),1)\psi_{\overline{S}}(a)drda\ . \tag{1.30}$$

Put

$$A(f)=\int\limits_{\overline{S}}f(a,1)\psi_{\overline{S}}(a)da\ .$$

Thus (in obvious notation) $I_0(\xi * f) = A(\widehat{\xi} * f)$. So far, we proved that $J_n(\phi, \varphi) \not\equiv 0$ if and only if $A(\varphi) \not\equiv 0$. Denote, for $n \le i \le 2n-1$,

$$K_{i+1} = \Big\{ k_{i+1}(t_1, \dots, t_{2n-i}) = \overline{\ell}\Big(\sum_{j=1}^{2n-i-1} t_j(e_{j,i+1} + e_{2n-i,2nr1-j}) + t_{2n-i}e_{2n-i,i+1} \Big) \Big\},$$

$$R_i = \Big\{ r_i(x_1, \dots, x_{2n-i}) = \ell\Big(\sum_{j=1}^{2n-i} x_j(e_{i,j} + e_{2n+1-j,2n+1-i}) \Big) \Big\} \ .$$

Let, for $n \le i \le 2n-1$,

$$K^i = \prod_{\ell=i+1}^{2n} K_\ell,$$

$$\psi^i = \psi_{\overline{S}}\Big|_{K^i},$$

$$A^{(i)}(\varphi) = \int\limits_{K^i} f(k,1)\psi^i(k)dk \ .$$

Note that $K^n = \overline{S}$, $A^{(n)} = A$, $\psi^n = \psi_{\overline{S}}$ and $\psi^i = 1$, for $i \ge n+1$. Again, we may assume that φ has the form

$$\xi * f = \int\limits_{R_i} \xi(r)\rho_{\theta,\frac{1}{2}}(r) f dr \ ; \quad \xi \in S(R_i), f \in V_{\pi(\theta)} \ .$$

It is easy to check that, for $r = r_i(x_1, \dots, x_{2n-i}) \in R_i$ and $k = k' \cdot k_{i+1}(t_1, \dots, t_{2n-i})$, $k' \in K^{i+1}$, we have that $krk^{-1} \in V_{2n}$, and

$$f\Big(krk^{-1} \cdot k, 1) = \psi^{-1}(\alpha \cdot \sum_{j=1}^{2n-i} t_j x_j\Big) f(k,1)$$

($\alpha = 2$ if $i = n$, and $\alpha = 1$ if $i \ge n+1$). This implies

(1.31)
$$A^{(i)}(\xi * f) = \int\limits_{K^{i+1}} \int\limits_{F^{2n-i}} \widetilde{\xi}(t_1, \dots, t_{2n-i})\rho_{\theta,\frac{1}{2}}(k_{i+1}(t_1, \dots, t_{2n-i})) f(k',1) dt dk' \ .$$

Here

$$\widetilde{\xi}(t_1, \dots, t_n) = \int\limits_{F^n} \xi(r_n(x_1, \dots, x_n))\psi^{-1}\Big(2\sum_{j=1}^{n-1} t_j x_j + x_n(2t_n - 1)\Big)dx$$

and for $i \ge n+1$

$$\widetilde{\xi}(t_1, \dots, t_{2n-i}) = \int\limits_{F^{2n-i}} \xi(r_i(x_1, \dots, x_{2n-i}))\psi^{-1}\Big(\sum_{j=1}^{2n-i} t_j x_j\Big)dx \ .$$

Clearly $\widetilde{\xi}$ varies over $S(F^{2n-i})$ as ξ varies over $S(R_i)$. Since the l.h.s of (1.31) has the form $A^{(i+1)}(\widetilde{\xi} * f)$, where this time

$$\widetilde{\xi} * f = \int\limits_{F^{2n-i}} \widetilde{\xi}(t_1, \dots, t_{2n-i})\rho_{\theta,\frac{1}{2}}(k_{i+1}(t_1, \dots, t_{2n-i})) f dt,$$

this shows that $A^{(i)}(\varphi) \not\equiv 0$, if and only if $A^{(i+1)}(\varphi) \not\equiv 0$, as φ varies in $V_{\pi(\theta)}$, for $n \leq i \leq 2n-1$, where $A^{(2n)}(\varphi) = \varphi(1,1)$. Since $A^{(2n)}(\varphi) \not\equiv 0$, on $V_{\pi(\theta)}$, we conclude that $J_n(\phi, \varphi) \not\equiv 0$, and the theorem is proved. $\square$

Here we record the following special case.

Corollary. *Let τ be an irreducible, self-dual, supercuspidal representation of $\mathrm{GL}_{2n}(F)$, such that $L(\tau, \Lambda^2, s)$ has a pole at $s = 0$. Then $\tilde{\sigma}_n(\tau) \neq 0$, and, in particular, $\widehat{\sigma}_n(\tau) \neq 0$.*

In Section 2 we will prove that $\widehat{\sigma}_k(\tau) = 0$, for $k < n$ (and hence $\tilde{\sigma}_k(\tau) = 0$, for $k < n$), and that $\widehat{\sigma}_n(\tau)$ (and hence $\tilde{\sigma}_n(\tau)$) is supercuspidal. (Recall that we have a surjection $\widehat{\sigma}_n(\tau) \to \tilde{\sigma}_n(\tau)$, and hence a surjection of Jacquet modules $J_R(\widehat{\sigma}_n(\tau)) \to J_R(\tilde{\sigma}_n(\tau))$ for each unipotent radical R of a parabolic subgroup of $\mathrm{Sp}_{2n}(F)$. Thus, $J_R(\tilde{\sigma}_n(\tau)) = 0$, for each such R, meaning that $\tilde{\sigma}_n(\tau)$ is supercuspidal.)

We end this section with the following definition. Let π be a smooth representation of $\mathrm{Sp}_{4n}(F)$. Define for $k < 2n$

$$\widehat{\sigma}_{k,\pi} = J_{\mathcal{H}_k}\left(J_{N_{k+1},\chi_k^{-1}}(\pi) \otimes \omega_\psi^{(k)}\right) .$$

This is a representation of $\widetilde{\mathrm{Sp}}_{2k}(F)$. We will consider such representations (in this generality) in the course of this work.

1.4. The representations $\sigma_k(\tau)$ and functoriality. Let τ be an irreducible, self-dual, supercuspidal representation of $\mathrm{GL}_{2n}(F)$, such that $L(\tau, \Lambda^2, s)$ has a pole at $s = 0$. By the principle of functoriality (adapted to metaplectic groups once we fix a nontrivial character, say ψ, of F), we expect that τ is the functorial lift of a supercuspidal representation σ of $\widetilde{\mathrm{Sp}}_{2n}(F)$. Although this notion is not well defined yet, we certainly expect that $\gamma(\sigma \times \tau, s, \psi)$ has a pole at $s = 1$, and if we impose that σ is generic (with respect to ψ_n^{-1}), then σ should be uniquely determined by the pole condition. By Corollary 3 in Section 1.2, such a σ exists, if and only if $\widehat{\sigma}$ is a summand of $\tilde{\sigma}_n(\tau)$, which by Corollary 1.2.3 is nontrivial. Thus, if we prove that $\tilde{\sigma}_n(\tau)$ is supercuspidal, then any summand of $\tilde{\sigma}_n(\tau)$ will provide an example of $\widehat{\sigma}$ as above.

1.5. The main local theorem. The main local theorem of this paper is

Theorem. *Let τ be an irreducible, self-dual, supercuspidal representation of $\mathrm{GL}_{2n}(F)$, such that $L(\tau, \Lambda^2, s)$ has a pole at $s = 0$. Then the representation $\widehat{\sigma}_n(\tau)$ is nontrivial, supercuspidal and irreducible.*

We have already proved the nontriviality of $\widehat{\sigma}_n(\tau)$ (Theorem 1.3). The surjection (1.23) $\widehat{\sigma}_n(\tau) \longrightarrow \tilde{\sigma}_n(\tau)$ implies that

$$\widehat{\sigma}_n(\tau) \cong \tilde{\sigma}_n(\tau)$$

and we conclude, denoting by $\sigma(\tau)$ the contragredient of $\widehat{\sigma}_n(\tau)$,

Corollary. *Let τ be as above. There is a unique genuine, irreducible, supercuspidal representation σ of $\widetilde{\mathrm{Sp}}_{2n}(F)$, which is generic with respect to ψ_n^{-1} and is such that $\gamma(\sigma \times \tau, s, \psi)$ has a pole at $s = 1$. This is the representation $\sigma(\tau)$.*

Remark. Although there is no precise theory of L-packets yet available for metaplectic groups like $\widetilde{\mathrm{Sp}}_{2n}(F)$, we consider that the last corollary establishes in principle the fact that $\sigma(\tau)$ is the unique ψ-generic representative of "the L-packet determined by τ and ψ".

1.6. Main steps of the proof. We will prove the following theorems, for τ as in Theorem 1.5.

Theorem 1. *(The tower property): Assume that $\widehat{\sigma}_j(\tau) = 0$ for all $j < k$. Then $\widehat{\sigma}_k(\tau)$ is either zero or supercuspidal.*

Theorem 2. *(Vanishing): We have*

$$\widehat{\sigma}_k(\tau) = 0 \ , \quad \textit{for all} \quad k < n \ .$$

These two theorems imply that $\widehat{\sigma}_n(\tau)$ is supercuspidal.

We will get the irreducibility of $\widehat{\sigma}_n(\tau)$ by showing that all summands of $\widehat{\sigma}_n(\tau)$ are ψ_n-generic, and then show that up to scalars $\widehat{\sigma}_n(\tau)$ admits exactly one ψ_n-Whittaker functional. Note the following Corollary to Theorem 2, Corollary 1.2.1 and Corollary 1.2.3.

Corollary. *Let σ be an irreducible, genuine, supercuspidal representation of $\widetilde{\mathrm{Sp}}_{2k}(F)$. Assume that $k < n$ and that σ is ψ_k^{-1}-generic. Then $\gamma(\sigma \times \tau, s, \psi)$ is holomorphic.*

1.7. The global case. Our original case of study in [G.R.S.1] was the global case. Let K be a number field, $\mathbb{A}$ its ring of adeles and η an irreducible, automorphic, cuspidal, self-dual representation of $\mathrm{GL}_{2n}(\mathbb{A})$, such that $L^S(\eta, \frac{1}{2}) \neq 0$ and $L^S(\eta, \Lambda^2, s)$ has a pole at $s = 1$. Then the Eisenstein series, defined for $\mathrm{Re}(s) \gg 0$, and for a holomorphic section $\varphi_{\eta,s}$ for (the global version of) $\rho_{\eta,s} = \mathrm{Ind}_{P_{2n}(\mathbb{A})}^{\mathrm{Sp}_{4n}(\mathbb{A})} \eta \otimes |\det|^{s-1/2}$ (P_{2n} = Siegel parabolic subgroup of Sp_{4n}) by

$$E(g, \varphi_{\eta,s}) = \sum_{\gamma \in P_{2n}(K) \backslash \mathrm{Sp}_{4n}(K)} \varphi_{\eta,s}(\gamma g; I_{2n}),$$

has a simple pole at $s = 1$. For $k < 2n$, we defined $\sigma_k(\eta)$ to be the representation by right translations of $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A})$ on the following space of automorphic functions:

$$V_{\sigma_k(\eta)} = \mathrm{Span}\{(g,\varepsilon) \mapsto \overline{J_k(\omega_\psi^{(k)}(g,\varepsilon)\phi, \mathrm{Res}_{s=1} E(j_k(g), \varphi_{\eta,s}))}\} \tag{1.32}$$

where

(1.33)

$$\begin{aligned} &J_k(\omega_\psi^{(k)}(g,\varepsilon)\phi, \mathrm{Res}_{s=1} E(j_k(g), \varphi_{\eta,s})) \\ &= \int\limits_{\mathcal{H}_k(F)\backslash \mathcal{H}_k(\mathbb{A})} \int\limits_{N_{k+1}(F)\backslash N_{k+1}(\mathbb{A})} \theta_{\psi,\phi}^{(k)}(h \cdot (g,\varepsilon)) \,\mathrm{Res}_{s=1} E(hvj_k(g), \varphi_{\eta,s}) \chi_k(v) dv dh \end{aligned}$$

($\omega_\psi^{(k)}$ is the corresponding global Weil representation and $\theta_{\psi,\phi}^{(k)}$ is the corresponding theta series). We proved in [G.R.S.1] that $\sigma_k(\eta) = 0$ for all $k < n$. We will prove, in Section 5,

Theorem. *In the global set-up just described, we have*

$$V_{\sigma_n(\eta)} \neq 0 \ .$$

Recall from [G.R.S.1] that $\sigma_n(\eta)$ contains all irreducible, automorphic, cuspidal, ψ^{-1}-generic representations σ of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{A})$, such that $L^S_\psi(\sigma\times\eta, s)$ has a pole at $s=1$. We denote $\sigma(\eta)=\sigma_n(\eta)$.

2. The tower property of the representations $\{\widehat{\sigma}_k(\tau)\}_{k<2n}$

2.1. Statement of the tower property. Let $1\le p\le k<2n$, and consider the unipotent radical

$$R_p=\left\{\begin{pmatrix} I_p & x & y \\ & I_{2(k-p)} & x' \\ & & I_p\end{pmatrix}\in \mathrm{Sp}_{2k}(F)\right\}$$

and the parabolic subgroup

$$Q_{k-p}=\left\{\begin{pmatrix} m & * & * & * & * \\ & z & * & * & * \\ & & I_{2(k-p)} & * & * \\ & & & z^* & * \\ & & & & m^*\end{pmatrix}\in \mathrm{Sp}_{4n}(F)\middle| \begin{matrix} m\in \mathrm{GL}_p(F) \\ z\in Z_{2n-k}\end{matrix}\right\}.$$

Regard $\widehat{\sigma}_j(\tau)$ as an $\widetilde{Sp}_{2j}(F)\cdot N_j$-module.

Theorem. *We have a vector space isomorphism*

$$J_{R_p}(V_{\hat{\sigma}_k(\tau)})\cong \mathrm{Ind}^{c^{Q_{k-p}}}_{N_{k-p}} V_{\widehat{\sigma}_{k-p}(\tau)}\ .$$

This isomorphism is the local analog of formula (2.44) in [G.R.S.1]. This also implies that if $\widehat{\sigma}_j(\tau)=0$ for all $j<k$, then $\widehat{\sigma}_k(\tau)$ is either zero or supercuspidal, which is Theorem 1.6.1.

2.2. A general lemma. We start with a general lemma, which will be used repeatedly in this paper.

Let $\mathcal{U}$ be a maximal nilpotent Lie subalgebra of $\mathrm{Lie}(\mathrm{Sp}_{4n}(F))$. Let $\mathcal{A},\mathcal{C},\mathcal{X}$ and $\mathcal{Y}$ be Lie subalgebras of $\mathcal{U}$ and let A,C,X,Y be the corresponding unipotent subgroups of $\mathrm{Sp}_{4n}(F)$. Let χ be a nontrivial character of C. We make the following assumptions:

(i) $C,X,Y\subset A$.
(ii) X and Y are abelian, normalize C and preserve χ.
(iii) The commutators $x^{-1}y^{-1}xy$ lie in C, for all $x\in X$, $y\in Y$. In particular, Y normalizes $D=CX$ and X normalizes $B=CY$.
(iv) $A=D\rtimes Y=B\rtimes X$.
(v) The set

$$\{x\mapsto \chi(x^{-1}y^{-1}xy)\mid y\in Y\}$$

is the group of all characters of X. Moreover, writing $x=\exp E, y=\exp S$, for $E\in\mathcal{X}$, $S\in\mathcal{Y}$, we have

$$\chi(xyx^{-1}y^{-1})=\psi((E,S))$$

where $(,)$ is a nondegenerate, bilinear pairing between $\mathcal{X}$ and $\mathcal{Y}$.

$$\begin{tikzcd} & BX = A = DY & \\ B = CY \arrow[ur, dash, "X"] & & D = CX \arrow[ul, dash, "Y"'] \\ & C \arrow[ul, dash, "Y"] \arrow[ur, dash, "X"'] & \end{tikzcd} \tag{2.1}$$

Lemma. *Assume (i)-(iv). Let π be a smooth representation of A. Extend χ trivially to characters χ_B of B and χ_D of D. Then we have an isomorphism of C-modules*

$$J_{B,\chi_B}(\pi) \cong J_{D,\chi_D}(\pi) \ . \tag{2.2}$$

Proof. Since X and Y normalize C and both preserve χ, X and Y act on $J_{C,\chi}(\pi)$. Consider $J_X(J_{C,\chi}(\pi)) = J_{D,\chi_D}(\pi)$. We have a natural surjection (over D)

$$T : J_{C,\chi}(\pi) \longrightarrow J_{D,\chi_D}(\pi) \tag{2.3}$$

which induces a map of A-modules

$$i : J_{C,\chi}(\pi) \longrightarrow \mathrm{Ind}_D^A \, J_{D,\chi_D}(\pi) \tag{2.4}$$

determined by

$$i(\xi)(y) = T(\overline{\pi}(y)\xi) \ , \qquad y \in Y, \tag{2.5}$$

where $\overline{\pi}$ is the representation of A in $J_{C,\chi}(\pi)$.

We will show that i is injective and $\mathrm{Im}(i) \subseteq \mathrm{Ind}_D^{cA} \, J_{D,\chi_D}(\pi)$, and then taking Jacquet modules in (2.4), with respect to Y, we obtain

$$J_{B,\chi_B}(\pi) = J_Y(J_{C,\chi}(\pi)) \hookrightarrow J_Y\Big(\mathrm{Ind}_D^{cA} \, J_{D,\chi_D}(\pi)\Big) \cong J_{D,\chi_D}(\pi).$$

The last embedding is clearly surjective, and this will prove the lemma. We show the injectivity of i. Assume that ξ, in the space of $J_{C,\chi}(\pi)$, is such that $i(\xi) = 0$. From (2.3) and (2.5), this means that for each $y \in Y$, there is a compact open subgroup $\Omega_y \subset X$, such that

$$\int_\Omega \overline{\pi}(x \cdot y)\xi dx = 0 \tag{2.6}$$

for all compact open subgroups $\Omega_y \subseteq \Omega \subset X$.

By assumption (iii), $y^{-1}xy \in C$, and hence we may rewrite (2.6) as

$$\overline{\pi}(y) \int_\Omega \overline{\pi}(x)\pi(x^{-1}y^{-1}xy)\xi dx = 0,$$

i.e.

$$\int_\Omega \chi(x^{-1}y^{-1}xy)\overline{\pi}(x)\xi dx = 0. \tag{2.7}$$

By assumption (v), $x \mapsto \chi(x^{-1}y^{-1}xy)$ is an arbitrary character of X, as y varies in Y, and hence (2.7) means that ξ has zero image in all possible Jacquet modules of $J_{C,\chi}(\pi)$ with respect to X (and an arbitrary character). This implies that $\xi = 0$.

We now show that

$$\mathrm{Ind}_D^A J_{D,\chi_D}(\pi) = \mathrm{Ind}_D^{cA} J_{D,\chi_D}(\pi). \tag{2.8}$$

Let f be a function in (the space of) the l.h.s. of (2.8). It is determined by its values on Y. Since f is A-smooth, there is in X a small compact open subgroup R, such that

$$f(y \cdot x) = f(y) \qquad \forall y \in Y,\ x \in R\ .$$

We have

$$f(yx) = f(x \cdot (x^{-1}yx)) = f(x^{-1}yx) = f(x^{-1}yxy^{-1}y) = \chi(x^{-1}yxy^{-1})f(y)\ .$$

We may take $R = \exp \mathcal{O}$ where $\mathcal{O}$ is a small neighborhood of zero. Then, for $y = \exp S$, $S \in \mathcal{Y}$, we get

$$f(\exp S) = \psi^{-1}((E,S))f(\exp S)\ , \quad \forall E \in \mathcal{O}\ .$$

This implies that the function $y \mapsto f(y)$ is compactly supported on Y. The lemma is proved. □

2.3. Proof of the tower property. It is convenient to first perform conjugation by

$$\beta_p = \begin{pmatrix} & I_p & & & \\ I_{2n-k} & & & & \\ & & I_{2(k-p)} & & \\ & & & & I_{2n-k} \\ & & & I_p & \end{pmatrix}.$$

Let

$$V = \beta_p \cdot j_k(R_p \cdot \mathcal{H}_k)N_{k+1}\beta_p^{-1}.$$

Extend χ_k trivially to R_p and $\mathcal{H}_k$, and put, for $v \in V$

$$\alpha_k(v) = \chi_k(\beta_p^{-1}v\beta_p)\ .$$

Denote by $\pi_{\tau,\psi}$ the representation of $\beta_p j_k(\widetilde{\mathrm{Sp}}_{2k}(F) \cdot \mathcal{H}_k)N_{k+1}\beta_p^{-1}$ in $V_{\pi_\tau} \otimes S(F^k)$ defined by

$$\begin{aligned} &\pi_{\tau,\psi}(\beta_p u \beta_p^{-1})(\xi \otimes \phi) = \pi_\tau(\beta_p u \beta_p^{-1})\xi \otimes \phi\ ,\ u \in N_{k+1}, \\ &\pi_{\tau,\psi}(\beta_p j_k(r \cdot h)\beta_p^{-1})(\xi \otimes \phi) = \pi_\tau(\beta_p j_k(r \cdot h)\beta_p^{-1})\xi \otimes \omega_\psi^{(k)}(r \cdot h)\phi\ , \\ &r \in \widetilde{\mathrm{Sp}}_{2k}(F), h \in \mathcal{H}_k. \end{aligned} \tag{2.9}$$

Then we have a vector space isomorphism (induced by $\xi \otimes \phi \mapsto \pi_\tau(\beta_p)\xi \otimes \phi$)

$$J_{R_p}(V_{\widehat{\sigma}_k(\tau)}) \cong J_{V,\alpha_k^{-1}}(V_{\pi_{\tau,\psi}}).$$

The elements of V have the form

$$v = \begin{pmatrix} I_p & 0 & x & b & y \\ e & z & a & c & b' \\ & & I_{2(k-p)} & a' & x' \\ & & & z^* & 0 \\ & & & e' & I_p \end{pmatrix} \in \mathrm{Sp}_{4n}(F) \tag{2.10}$$

where $z \in Z_{2n-k}$. We have in the notation (2.10)

$$\pi_{\tau,\psi}(v)(\xi \otimes \phi) = \pi_\tau(v) \otimes \omega_\psi^{(k)}\left(\begin{pmatrix} I_p & x & y \\ & I_{2(k-p)} & x' \\ & & I_p \end{pmatrix} \cdot (e_{2n-k}, a_{2n-k}, (b')_{2n-k}; c_{2n-k,1})\right)\phi. \tag{2.11}$$

Here e_{2n-k} is the last row of e, etc. Note that e_{2n-k} and $(b')_{2n-k}$ are in F^p and a_{2n-k} is in $F^{2(k-p)}$. Finally

$$\alpha_k(v) = \psi(z_{12} + z_{23} + \cdots + z_{2n-k-1,2n-k}).$$

Let $\tilde{V}^{(1)}$ be the subgroup of V which consists of elements of the form (2.10), with $e_{2n-k} = 0$, and let $\mathcal{L}^{(1)}$ be its "complement" in V,

$$\mathcal{L}^{(1)} = \left\{\hat{\ell} = \begin{pmatrix} I_p &&&&&& \\ 0 & I_{2n-k-1} &&&&& \\ \ell & 0 & 1 &&&& \\ &&& I_{2(k-p)} &&& \\ &&&& 1 && \\ &&&& 0 & I_{2n-k-1} & \\ &&&& \ell' & 0 & I_p \end{pmatrix} \in \mathrm{Sp}_{4n}(F)\right\}. \tag{2.12}$$

Note that the complement $\mathcal{L}^{(1)}$ normalizes $\tilde{V}^{(1)}$ and preserves $\alpha_k\Big|_{\tilde{V}^{(1)}}$. Thus, $\mathcal{L}^{(1)}$ acts on $J_{\tilde{V}^{(1)},\alpha_k^{-1}}(V_{\pi_{\tau,\psi}})$. Write F^k as $F^p \oplus F^{k-p}$ and denote for $\phi \in S(F^k)$

$$\phi_{(2)}(t) = \phi(0,t)\ , \qquad t \in F^{k-p}.$$

Define

$$T : V_{\pi_\tau} \otimes S(F^k) \longrightarrow V_{\pi_\tau} \otimes S(F^{k-p})$$

by

$$T(\xi \otimes \phi) = \xi \otimes \phi_{(2)}\ .$$

Then, for $v \in \tilde{V}^{(1)}$,

$$T(\pi_{\tau,\psi}(v)(\xi \otimes \phi)) = \pi_\tau(v)\xi \otimes \omega_\psi^{(k-p)}(i_{k-p}(v))\phi_{(2)} \tag{2.13}$$

where $i_{k-p} : \tilde{V}^{(1)} \longrightarrow \mathcal{H}_{k-p}$ is the homomorphism (in the notation (2.10))

$$i_{k-p}(v) = (a_{2n-k}; c_{2n-k,1}).$$

For (2.13) we use the formulae for $\omega_\psi^{(k)}$ (and $\omega_\psi^{(k-p)}$). Denote by $\pi'_{\tau,\psi}$ the representation of $\tilde{V}^{(1)}$ on $V_{\pi_\tau} \otimes S(F^{k-p})$, defined by the r.h.s. of (2.13). Then T intertwines $\pi_{\tau,\psi}\Big|_{\tilde{V}^{(1)}}$ and $\pi'_{\tau,\psi}$.

Proposition 1. *We have an isomorphism of $\tilde{V}^{(1)}$-modules*

$$J_{V,\alpha_k^{-1}}(V_{\pi_{\tau,\psi}}) \cong J_{\tilde{V}^{(1)},\alpha_k^{-1}}(V_{\pi'_{\tau,\psi}}). \tag{2.14}$$

Proof. By passage to the quotients, T defines a surjection

$$T' : J_{\tilde{V}^{(1)},\alpha_k^{-1}}(V_{\pi_{\tau,\psi},\alpha_k^{-1}}) \longrightarrow J_{\tilde{V}^{(1)},\alpha_k^{-1}}(V_{\pi'_{\tau,\psi}})$$

which, in turn, induces a V-map

$$i : J_{\tilde{V}^{(1)},\alpha_k^{-1}}(V_{\pi_{\tau,\psi}}) \longrightarrow \mathrm{Ind}_{\tilde{V}^{(1)}}^{V}(J_{\tilde{V}^{(1)},\alpha_k^{-1}}(V_{\pi'_{\tau,\psi}}))$$

determined by (using notation 2.12))

$$i\Big[j_{\tilde V^{(1)},\alpha_k^{-1}}(\xi\otimes\phi)\Big](\hat\ell)=j_{\tilde V^{(1)},\alpha_k^{-1}}(\pi_\tau(\hat\ell)\otimes[\omega_\psi^{(k)}(\ell,0,0;0)\phi]_{(2)}) \tag{2.15}$$

for $\hat\ell\in\mathcal{L}^{(1)}$. Since

$$\omega_\psi^{(k)}(\ell,0,0;0)\phi(0,t)=\phi(\ell,t)\ ,\quad \text{for}\quad \ell\in F^p,t\in F^{k-p},$$

(2.15) implies that $Im(i)\subseteq \mathrm{Ind}_{\tilde V^{(1)}}^{c\,V}(J_{\tilde V^{(1)},\alpha_k^{-1}}(V_{\pi'_{\tau,\psi}}))$. Let us show that i is injective. For this, it is convenient to identify $V_{\pi_{\tau,\psi}}$ with $S(F^k;V_{\pi_\tau})$—the space of V_{π_τ}-valued Schwartz Bruhat functions on F^k. Similarly, we identify $V_{\pi'_{\tau,\psi}}$ with $S(F^{k-p};V_{\pi_\tau})$. Let

$$\widetilde{i}_k:V\longrightarrow R_p\mathcal{H}_k$$

be the homomorphism (in the notation (2.10))

$$\widetilde{i}_k(v)=\begin{pmatrix}I_p & x & y\\ & I_{2(k-p)} & x'\\ & & I_p\end{pmatrix}\cdot(e_{2n-k},a_{2n-k},(b')_{2n-k};c_{2n-k,1})\ .$$

Then the action of V on $S(F^k;V_{\pi_\tau})$ is given by

$$\pi_{\tau,\psi}(v)(\phi)=\pi_\tau(v)\circ(\omega_\psi^{(k)}(\widetilde{i}_k(v))\phi)=\omega_\psi^{(k)}(\widetilde{i}_k(v))(\pi_\tau(v)\circ\phi). \tag{2.16}$$

(The Weil representation $\omega_\psi^{(k)}$ acts on vector-valued functions by the same formulae as for scalar-valued functions.) Similarly $\tilde V^{(1)}$ acts on $S(F^{k-p};V_{\pi_\tau})$ by

$$\pi'_{\tau,\psi}(v)(\phi)=\pi_\tau(v)\circ(\omega_\psi^{(k-p)}(i_{k-p}(v)\phi))=\omega_\psi^{(k-p)}(i_{k-p}(v))(\pi_\tau(v)\circ\phi). \tag{2.17}$$

Let $\phi\in S(F^k;V_{\pi_\tau})$ be such that its class in $J_{\tilde V^{(1)},\alpha_k^{-1}}(V_{\pi_{\tau,\psi}})$ belongs to $\mathrm{Ker}(i)$. This means that the function (on $\mathcal{L}^{(1)}$) $\hat\ell\mapsto\pi_\tau(\hat\ell)\circ\phi^{(\ell)}$ has zero image in $J_{\tilde V^{(1)},\alpha_k^{-1}}(V_{\pi'_{\tau,\psi}})$, where $\phi^{(\ell)}=\Big[\omega_\psi^{(k)}(\ell,0,0;0)\phi\Big]_{(2)}$. Thus, for each $\ell\in F^p$, there is a compact open subgroup $C_\ell\subset\tilde V^{(1)}$, such that

$$\int\limits_C\alpha_k(v)\pi_\tau(v)\circ\omega_\psi^{(k-p)}(i_{k-p}(v))(\pi_\tau(\hat\ell)\circ\phi^{(\ell)})dv=0 \tag{2.18}$$

for all compact open subgroups $C_\ell\subseteq C\subset\tilde V^{(1)}$. By (2.16)-(2.18) we get

$$\int\limits_C\alpha_k(v)\pi_\tau(v\cdot\hat\ell)\Big[\omega_\psi^{(k)}(\widetilde{i}_k(v\cdot\hat\ell))\phi(0,t)\Big]dv=0\ ,\quad\forall t\in F^{k-p}. \tag{2.19}$$

Recall that $\hat\ell$ normalizes $\tilde V^{(1)}$ and preserves α_k. Hence, changing variables $v\mapsto\hat\ell v\cdot\hat\ell^{-1}$ in (2.19) gives

$$\int\limits_{C'}\alpha_k(v)\pi_\tau(v)\Big[\omega_\psi^{(k)}(\widetilde{i}_k(v))\phi(\ell,t)\Big]dv=0\ ,\quad\forall t\in F^{k-p}, \tag{2.20}$$

for all compact open subgroups $\hat\ell^{-1}C_\ell\hat\ell\subseteq C'\subset\widetilde{V}^{(1)}$. Using the formulae for $\omega_\psi^{(k)}(\widetilde{i}_k(v))$, it is easy to see that the function $\ell\mapsto\omega_\psi^{(k)}(\widetilde{i}_k(v))\phi(\ell,t)$ is uniformly smooth and has support contained in a compact set, independently of either $v\in$

$\tilde{V}^{(1)}$ or $t \in F^{k-p}$. It then follows from (2.20) that there is a compact open subgroup $C_0' \subset \tilde{V}^{(1)}$, such that

$$\int\limits_{C'} \chi_k(v)\pi_\tau(v) \circ \omega_\psi^{(k)}(\widetilde{i}_k(v))\phi dv = 0$$

for all compact open subgroups $C_0' \subseteq C' \subset \tilde{V}^{(1)}$. This means that ϕ has zero image in $J_{\tilde{V}^{(1)},\alpha_k^{-1}}(V_{\pi_{\tau,\psi}})$, and we proved that i is injective. Thus we have an injection of V-modules

$$i : J_{\tilde{V}^{(1)},\alpha_k^{-1}}(V_{\pi_{\tau,\psi}}) \hookrightarrow \mathrm{Ind}^{c\,V}_{\tilde{V}^{(1)}}(J_{\tilde{V}^{(1)},\alpha_k^{-1}}(V_{\pi'_{\tau,\psi}})),$$

and hence, taking Jacquet modules with respect to $\mathcal{L}^{(1)}$, we get an embedding of $\tilde{V}^{(1)}$-modules

$$\begin{aligned} J_{V,\alpha_k^{-1}}(V_{\pi_{\tau,\psi}}) &\hookrightarrow J_{\mathcal{L}^{(1)}}\, \mathrm{Ind}^{c\,V}_{\tilde{V}^{(1)}}(J_{\tilde{V}^{(1)},\alpha_k^{-1}}(V_{\pi'_{\tau,\psi}})) \\ &\cong J_{\tilde{V}^{(1)},\alpha_k^{-1}}(V_{\pi'_{\tau,\psi}}) \ . \end{aligned}$$

This last embedding is easily seen to be surjective, and this proves the proposition. $\square$

Lemma 2.2 will now serve as our apparatus which translates the Fourier expansion arguments of the proof of Theorem 8 in [G.R.S.1] to the local case. Redenote $B^{(2)} = \tilde{V}^{(1)}$. Let $C^{(2)}$ be the subgroup of $B^{(2)}$, which consists of elements of the form (2.10), such that $e_{2n-k-1} = e_{2n-k} = 0$. Let

$$Y^{(2)} = \left\{ \widehat{y} = \begin{pmatrix} I_p &&&&&& \\ 0 & I_{2n-k-2} &&&&& \\ y & 0 & 1 &&&& \\ &&& I_{2(k-p+1)} &&& \\ &&&& 1 && \\ &&&& 0 & I_{2n-k-2} & \\ &&&& y' & 0 & I_p \end{pmatrix} \in \mathrm{Sp}_{4n}(F) \right\},$$

$$X^{(2)} = \left\{ \check{x} = \begin{pmatrix} I_p & 0 & x &&&& \\ & I_{2n-k-1} & 0 &&&& \\ && 1 &&&& \\ &&& I_{2(k-p)} &&& \\ &&&& 1 & 0 & x' \\ &&&&& I_{2n-k-1} & 0 \\ &&&&&& I_p \end{pmatrix} \in \mathrm{Sp}_{4n}(F) \right\},$$

$$D^{(2)} = C^{(2)} \cdot X^{(2)} \ , \quad A^{(2)} = D^{(2)} \cdot Y^{(2)},$$

$$\chi^{(2)} = \alpha_k^{-1}\Big|_{C^{(2)}} .$$

We consider $V_{\pi'_{\tau,\psi}}$ as an $A^{(2)}$-module by letting $X^{(2)}$ act through π_τ only, i.e. $\check{x}$ in $X^{(2)}$ takes the class of $\xi \otimes \phi \in V_{\pi_\tau} \otimes S(F^{k-p})$ to the class of $\pi_\tau(\check{x})\xi \otimes \phi$. It is easy to check that the diamond

$$
\begin{tikzcd}[column sep=small]
& A^{(2)} \arrow[dl, dash, "X^{(2)}"'] \arrow[dr, dash, "Y^{(2)}"] & \\
B^{(2)} \arrow[dr, dash, "Y^{(2)}"'] & & D^{(2)} \arrow[dl, dash, "X^{(2)}"] \\
& C^{(2)} &
\end{tikzcd}
$$

satisfies assumptions (i)-(v) of Section 2.2, and hence by Lemma 2.2, we have

$$J_{V,\alpha_k^{-1}}(V_{\pi_{\tau,\psi}}) \cong J_{\tilde{V}^{(1)},\alpha_k^{-1}}(V_{\pi'_{\tau\psi}}) = J_{B^{(2)},\chi^{(2)}_{B^{(2)}}}(V_{\pi'_{\tau,\psi}}) \cong J_{D^{(2)},\chi^{(2)}_{D^{(2)}}}(V_{\pi'_{\tau,\psi}})\ .$$

We repeat this process by taking $B^{(3)} = D^{(2)}$, "deleting one row and adding one column". In general, for $2 \le j \le 2n-k$, let $C^{(j)}$ be the group of elements of $\mathrm{Sp}_{4n}(F)$ which have the form

$$v = \begin{pmatrix} I_p & 0 & u & * & * & * & * \\ e & z_1 & b & * & * & * & * \\ 0 & 0 & z_2 & * & * & * & * \\ & & & I_{2(k-p)} & * & * & * \\ & & & & z_2^* & v'* & u' \\ & & & & & z_1^* & 0 \\ & & & & & e' & I_p \end{pmatrix} \tag{2.21}$$

where $z_1 \in Z_{2n-k-j}$, $z_2 \in Z_j$ and the first two columns of u are zero. Let

$$Y^{(j)} = \left\{ \widehat{y} = \begin{pmatrix} I_p & & & & & & \\ 0 & I_{2n-k-j} & & & & & \\ y & 0 & 1 & & & & \\ & & & I_{2(k-p+j-1)} & & & \\ & & & & 1 & & \\ & & & & 0 & I_{2n-k-j} & \\ & & & & y' & 0 & I_p \end{pmatrix} \in \mathrm{Sp}_{4n}(F) \right\},$$

$$X^{(j)} = \left\{ \check{x} = \begin{pmatrix} I_p & 0 & x & & & & \\ & I_{2n-k-j+1} & 0 & & & & \\ & & 1 & & & & \\ & & & I_{2(k-p+j-2)} & & & \\ & & & & 1 & 0 & x' \\ & & & & & I_{2n-k-j+1} & 0 \\ & & & & & & I_p \end{pmatrix} \in \mathrm{Sp}_{4n}(F) \right\},$$

$$D^{(j)} = C^{(j)}X^{(j)}\ , \quad B^{(j)} = C^{(j)}Y^{(j)}\ , \quad A^{(j)} = D^{(j)}Y^{(j)}.$$

Finally let $\chi^{(j)}$ be the character of $C^{(j)}$ defined by (using the notation (2.21))

$$\chi^{(j)}(v) = \psi^{-1}(z_{12} + z_{23} + \cdots + z_{2n-k-1,2n-k})$$

where $(z_{12},\dots,z_{2n-k-1,2n-k})$ is the second diagonal of $\begin{pmatrix} z_1 & b \\ 0 & z_2 \end{pmatrix}$. It is easy to check that $\chi^{(j)}$ and the diamond

$$\begin{tikzcd} & A^{(j)} \arrow[dl, dash, "X^{(j)}"'] \arrow[dr, dash, "Y^{(j)}"] & \\ B^{(j)} \arrow[dr, dash, "Y^{(j)}"'] & & D^{(j)} \arrow[dl, dash, "X^{(j)}"] \\ & C^{(j)} & \end{tikzcd}$$

satisfy (i)-(v) in Section 2.2. Note that

$$B^{(j)} = D^{(j-1)}$$

and

$$\chi^{(j-1)}_{D^{(j-1)}}\Big|_{C^{(j)}} = \chi^{(j)}. \tag{2.22}$$

We let, for each j, $X^{(j)}$ act on $V_{\pi'_{\tau,\psi}}$, through π_τ only, and in this way $V_{\pi'_{\tau,\psi}}$ becomes an $A^{(j)}$-module for each j. We conclude from Lemma 2.2 that

$$J_{B^{(j)},\chi^{(j)}_{B^{(j)}}}(V_{\pi'_{\tau,\psi}}) \cong J_{D^{(j)},\chi^{(j)}_{D^{(j)}}}(V_{\pi'_{\tau,\psi}})$$

for all $2 \le j \le 2n-k$, and hence

$$J_{V,\alpha_k^{-1}}(V_{\pi_{\tau,\psi}}) \cong J_{D^{(2n-k)},\chi^{(2n-k)}_{D^{(2n-k)}}}(V_{\pi'_{\tau,\psi}}) \tag{2.23}$$

(this isomorphism is over the subgroup C of V which consists of elements of the form (2.2), such that $e=0$). Note that

$$D^{(2n-k)} = \left\{ \begin{pmatrix} I_p & u & * & * & * \\ & z & * & * & * \\ & & I_{2(k-p)} & * & * \\ & & & z^* & u' \\ & & & & I_p \end{pmatrix} \in \mathrm{Sp}_{4n}(F) \;\middle|\; \begin{matrix} z \in Z_{2n-k} \\ \text{and the first} \\ \text{column of } u \text{ is zero} \end{matrix} \right\} \subset N_{k-p}$$

and

$$\chi^{(2n-k)}_{D^{(2n-k)}} = \chi_{k-p}^{-1}\Big|_{D^{(2n-k)}}.$$

Consider the r.h.s. of (2.23) as an E-module, where

$$E = \left\{ \begin{pmatrix} m & x & & & \\ & 1 & & & \\ & & I_{2(2n-p-1)} & & \\ & & & 1 & x' \\ & & & & m^* \end{pmatrix} \in \mathrm{Sp}_{4n}(F) \right\}.$$

E is isomorphic to the parabolic subgroup of $\mathrm{GL}_{p+1}(F)$ of type $(p,1)$ (the so-called mirabolic subgroup). By [B.Z.], the Jordan Hölder decomposition over E of $J_{D^{(2n-k)},\chi^{-1}_{\tau,\psi}}(\pi'_{\tau,\psi})$ is expressed through the various derivatives, which all, except one, clearly involve the Jacquet modules of π_τ with respect to unipotent radicals of the form $\begin{pmatrix} I_{p-\ell} & * & * \\ & I_{2(2n-p+\ell)} & * \\ & & I_{p-\ell} \end{pmatrix}$, for $0 \leq \ell < p$. (See [B.Z.] for the notion of a derivative of a smooth representation of the mirabolic subgroup of $\mathrm{GL}_{p+1}(F)$.)

Remark 1. The analysis done so far in this section is valid if we replace π_τ by any smooth representation of $\mathrm{Sp}_{4n}(F)$. In particular (2.23) is true if we replace π_τ by any smooth representation of $\mathrm{Sp}_{4n}(F)$.

Now we use the supercuspidality of τ and the fact that π_τ is the Langlands quotient of $\rho_{\tau,1}$. It follows that the above derivatives vanish, since π_τ is concentrated on the Siegel parabolic subgroup. (Note that for $0 \leq \ell < p \leq k < 2n$, we have $p-\ell < 2n$.) This implies that as E-modules,

$$J_{D^{(2n-k)},\chi_{k-p}^{-1}}(\pi'_{\tau,\psi}) \cong \mathrm{Ind}_{Z_{p+1}}^{c^E}\left[J_{Z_{p+1},\psi^{-1}}(J_{D^{(2n-k)},\chi_{k-p}^{-1}}(\pi'_{\tau,\psi}))\right]. \tag{2.24}$$

Here ψ denotes (by a slight abuse of notation) the standard nondegenerate character of Z_{p+1}, which corresponds to ψ. It is clear from the definitions that

$$J_{Z_{p+1},\psi^{-1}}\left(J_{D^{(2n-k)},\chi_{k-p}^{-1}}(\pi'_{\tau,\psi})\right) = \widehat{\sigma}_{k-p}(\tau).$$

This together with (2.23) completes the proof of Theorem 2.1. $\square$

Remark 2. The argument above is valid if we replace π_τ by $\rho_{\tau,s}$ or any constituent of $\rho_{\tau,s}$, as long as τ is supercuspidal.

To end this section, we would like to write the last two remarks (Remarks 1, 2) as two explicit propositions.

Let π be a smooth representation of $\mathrm{Sp}_{4n}(F)$. Consider, for $1 \leq p \leq k < 2n$, the representation π_ψ of $\beta_p j_k(\widetilde{\mathrm{Sp}}_{2k}(F)\mathcal{H}_k)N_{k+1}\beta_p^{-1}$ in $V_\pi \otimes S(F^k)$, defined by formulae (2.1), where we replace π_τ by π. Let π_ψ^* be the representation of $D^{(2n-k)}$ in $V_\pi \otimes S(F^{k-p})$ defined by

$$\pi_\psi^*(u)(\xi \otimes \phi) = \pi(u)\xi \otimes \omega_\psi^{(k-p)}(u')\phi$$

where $u \mapsto u'$ is the projection of $D^{(2n-k)}$ on $\mathcal{H}_{k-p}$, defined by

$$u' = (u_{2n-k+p,2n-k+p+1}, \dots, u_{2n-k+p,2n+k-p+1}) \ .$$

Remark 1 says

Proposition 2. *Let π be a smooth representation of $\mathrm{Sp}_{4n}(F)$. For $1 \leq p \leq k < 2n$, define V, $D^{(2n-k)}$ and α_k as before. Then we have a vector space isomorphism*

$$J_{V,\alpha_k^{-1}}(V_{\pi_\psi}) \cong J_{D^{(2n-k)},\chi_{k-p}^{-1}}(V_{\pi_\psi^*}) \ .$$

The precise content of Remark 2 is

Proposition 3. *Let τ be a supercuspidal representation of $\mathrm{GL}_{2n}(F)$. Let π^τ be a subquotient of $\rho_{\tau,\frac{1}{2}}$. Then in the above notation, we have a vector space isomorphism*

$$J_{R_p}\left(V_{\widehat{\sigma}_{k,\pi^\tau}}\right) \cong \mathrm{Ind}_{N_{k-p}}^{c^{Q_{k-p}}} V_{\widehat{\sigma}_{k-p,\pi^\tau}}.$$

3. Vanishing of the representations $\{\widehat{\sigma}_k(\tau)\}_{k<n}$

In this section, τ is an irreducible, self-dual, supercuspidal representation of $\mathrm{GL}_{2n}(F)$, such that $L(\tau, \Lambda^2, s)$ has a pole at $s=0$.

3.1. The case $k = 0$. In this case N_0 is the standard maximal unipotent subgroup of $\mathrm{Sp}_{4n}(F)$ and χ_0 is its standard nondegenerate character, which corresponds to ψ. Thus $\widehat{\sigma}_0(\tau) = J_{N_0,\chi_0^{-1}}(\pi_\tau)$. Since π_τ is not generic (with respect to any nondegenerate character of N_0), we conclude that

$$\widehat{\sigma}_0(\tau) = 0 \ .$$

3.2. A reduction. Let C denote the center of the Heisenberg group $\mathcal{H}_k$, and let

$$N^{(k)} = N_{k+1} \cdot j_k(C).$$

This is a subgroup of $N_k = N_{k+1} \cdot j_k(\mathcal{H}_k)$. Let $\chi_{(k)}$ be the character of $N^{(k)}$ defined by

$$\chi_{(k)}(u \cdot j_k(0, 0; t)) = \chi_k(u)\psi(t) \ . \tag{3.1}$$

We have the following analog of Lemma 15 in [G.R.S.1].

Lemma. *For a smooth representation π of $\mathrm{Sp}_{4n}(F)$, we have $\widehat{\sigma}_{k,\pi} = 0$, if and only if*

$$J_{N^{(k)},\chi_{(k)}^{-1}}(\pi) = 0. \tag{3.2}$$

Proof. Assume that $\widehat{\sigma}_{k,\pi} \neq 0$. Thus, $J_{\mathcal{H}_k}\left(J_{N_{k+1},\chi_k^{-1}}(\pi) \otimes \omega_\psi^{(k)}\right) \neq 0$. Let b be a nontrivial element of the dual of the last space. (We sometimes confuse a representation and its space.) We regard b as an $\mathcal{H}_k$-invariant bilinear form $b(v, \phi)$ on $J_{N_{k+1},\chi_k^{-1}}(V_\pi) \otimes S(F^k)$. For fixed v, $b_v(\phi) = b(v, \phi)$ is a smooth distribution, and hence there is a unique smooth function $f_v(z)$ on F^k, such that

$$b(v, \phi) = \int\limits_{F^k} \phi(z) f_v(z) dz \ .$$

The $\mathcal{H}_k$-equivariance of b implies that

$$f_{\overline{\pi}(j_k(u,0;0)j_k(0,y;t))v}(z) = \psi^{-1}(2zw_k \cdot {}^t y + t) f_v(z + u) \ .$$

Here $\overline{\pi}$ denotes the action on the Jacquet module. Thus, f_v, which is nontrivial, is fully determined by one value, as v varies. Choose $f_v(0)$. The linear form $\ell(v) = f_v(0)$ is nontrivial and satisfies

$$\ell\Big(\overline{\pi}(j_k(0, y; t))v\Big) = \psi^{-1}(t)\ell(v).$$

We showed that b is uniquely determined by ℓ, which is an element of the dual of $J_{\tilde{N}^{(k)},\tilde{\chi}_{(k)}^{-1}}(\pi)$, where

$$\tilde{N}^{(k)} = N_{k+1} j_k(\mathcal{Y}_k \cdot C)$$

and $\widetilde{\chi}_{(k)}$ is the character of $\tilde{N}^{(k)}$ obtained by extending $\chi_{(k)}$ trivially to $j_k(\mathcal{Y}_k)$. We actually showed a vector space isomorphism $V^*_{\hat{\sigma}_{k,\pi}} \cong J_{\tilde{N}^{(k)},\tilde{\chi}_{(k)}^{-1}}(V_\pi)^*$. In particular, $\widehat{\sigma}_{k,\pi} \neq 0$, if and only if $J_{\tilde{N}^{(k)},\chi_{(k)}^{-1}}(\pi) \neq 0$. Since $N^{(k)}$ is a subgroup of $\tilde{N}^{(k)}$, it is now clear that if $\widehat{\sigma}_{k,\pi} \neq 0$, then $J_{N^{(k)},\chi_{(k)}^{-1}}(\pi) \neq 0$. Assume now that $J_{\tilde{N}^{(k)},\tilde{\chi}_k^{-1}}(\pi) = 0$. If $J_{N^{(k)},\chi_{(k)}^{-1}}(\pi) \neq 0$, then, since $\mathcal{Y}_k \cdot C$ is abelian, the (abelian) group $j_k(\mathcal{Y}_k)$ acts on $J_{N^{(k)},\chi_k^{-1}}(V_\pi)$, and hence there is a character χ' of $j_k(\mathcal{Y}_k)$, such that $J_{j_k(\mathcal{Y}_k),\chi'}(J_{N^{(k)},\chi_{(k)}^{-1}}(V_\pi)) \neq 0$. χ' has the form

$$\chi'(j_k(0,y;0)) = \psi^{-1}(\sum_{i=1}^{k} x_i y_i) \ ,$$

for $x_1, \dots, x_k \in F$. Let φ be a nontrivial linear functional on V_π, such that

$$\varphi(\pi(v'v)\xi) = \chi'(v')\chi_{(k)}^{-1}(v)\varphi(\xi) \ , \quad \text{for} \quad v' \in j_k(\mathcal{Y}_k), v \in N^{(k)}.$$

Define $u = -\frac{1}{2}(x_k, \dots, x_2, x_1)$ and

$$\varphi'(\xi) = \varphi(\pi(j_k(u,0;0))\xi) \ .$$

Then φ' is a nontrivial linear functional on V_π, such that

$$\varphi'(\pi(v)\xi) = \tilde{\chi}_{(k)}^{-1}(v)\varphi'(\xi) \ , \quad \text{for} \quad v \in \tilde{N}^{(k)} \ ,$$

which implies that $J_{\tilde{N}^{(k)},\tilde{\chi}_{(k)}^{-1}}(\pi) \neq 0$, a contradiction. □

3.3. $\mathrm{Sp}_{2n}(F) \times \mathrm{Sp}_{2n}(F)$**-invariant functionals.** We recall (in Theorem 1) Theorem 17 in [G.R.S.1]. Let H be the image of the direct sum embedding of $\mathrm{Sp}_{2n}(F) \times \mathrm{Sp}_{2n}(F)$ inside $\mathrm{Sp}_{4n}(F)$.

Theorem 1. *Let π be an irreducible, representation of $\mathrm{Sp}_{4n}(F)$. Assume that the space of π admits nontrivial H-invariant functionals. Then*

$$J_{N^{(k)},\chi_{(k)}^{-1}}(\pi) = 0 \ , \qquad \textit{for} \quad 0 \le k < n \ .$$

In particular,

$$\tilde{\sigma}_{k,\pi} = 0 \ , \qquad \textit{for} \quad 0 \le k < n \ .$$

Remark. This theorem is valid if we change $\chi_{(k)}$ to another character of $N^{(k)}$ as long as it remains nontrivial on each root subgroup of $N^{(k)}$ on which χ_k is nontrivial. The proof in [G.R.S.1] works for such characters as well. Thus, in order to prove that $\widehat{\sigma}_k(\tau) = 0$, for $k < n$, it will suffice to show

Theorem 2. *The representation π_τ admits nontrivial H-invariant functionals.*

Proof. Recall again Shahidi's local coefficient

$$\gamma(\tau, \Lambda^2, z, \psi) = \varepsilon(\tau, \Lambda^2, z, \psi) \frac{L(\widehat{\tau}, \Lambda^2, 1-z)}{L(\tau, \Lambda^2, z)} \ .$$

Our assumption is that τ is self-dual, supercuspidal and that $L(\tau, \Lambda^2, z)$ has a pole at $z = 0$. Jacquet and Shalika wrote a global integral which represents the (partial) exterior square L-function for automorphic cuspidal representations on $\mathrm{GL}_{2n}(\mathbb{A})$ ([J.S.2]). Their global theory yields a corresponding local theory, which centers around a functional equation, the details of which we now explain. It has the form

$$\widetilde{\gamma}(\tau, \Lambda^2, s, \psi)\mathcal{L}(W, \phi, s) = \widetilde{\mathcal{L}}(W, \widehat{\phi}, 1-s) \tag{3.3}$$

where $\widetilde{\gamma}(\tau, \Lambda^2, s, \psi)$ is a rational function in q^{-s}. $\mathcal{L}(W, \phi, s)$ is the "local integral" which depends on W, an element in the Whittaker model of τ (which we take, for simplicity, with respect to the standard character, defined by ψ^{-1}), and on $\phi \in S(F^n)$, which defines the following section for $\mathrm{Ind}_{P_{n-1,1}}^{\mathrm{GL}_n(F)} \alpha_s$:

$$f_{\phi,s}(g) = |\det g|^s \int_{F^*} \phi(t(0\cdots 01)g)|t|^{ns}\omega_\tau(t)d^*t. \tag{3.4}$$

Here ω_τ is the central character of $\tau(\omega_\tau^2 = 1)$, $P_{n-1,1}$ is the parabolic subgroup of $\mathrm{GL}_n(F)$ of type $(n-1,1)$ and

$$\alpha_s\begin{pmatrix} m & * \\ 0 & a\end{pmatrix} = \Big(\frac{|\det m|}{|a|^{n-1}}\Big)^{s-\frac{1}{2}}\omega_\tau^{-1}(a)\ , \quad m \in \mathrm{GL}_{n-1}(F), a \in F^*\ .$$

Finally,

(3.5)

$$\mathcal{L}(W,\phi,s) = \int_{C_nZ_n\backslash \mathrm{GL}_n(F)} \int_{X\in\mathfrak{b}_\mathfrak{n}\backslash M_n(F)} W\left(\mu\cdot\begin{pmatrix} I_n & X \\ & I_n\end{pmatrix}\begin{pmatrix} g & \\ & g\end{pmatrix}\right)\psi(trX)f_{\phi,s}(g)dX dg$$

where $\mathfrak{b}_\mathfrak{n}$ is the space of $n\times n$ upper triangular matrices, C_n is the center of $\mathrm{GL}_n(F)$ and μ is the Weyl element defined by

$$\mu_{2i-1,i} = \mu_{2i,n+i} = 1\ ,\ i = 1,\dots,n,$$
$$\mu_{\ell,j} = 0 \quad \text{for} \quad (\ell,j)\notin\{(2i-1,i),(2i,n+i)\mid i=1,\dots,n\}\ .$$

The integral (3.5) converges absolutely for $\mathrm{Re}(s)\gg 0$ and is rational in q^{-s}. The r.h.s. of (3.3) has a similar structure

$$\begin{aligned}\widetilde{\mathcal{L}}(W,\hat{\phi},1-s) &= \int_{C_nZ_n\backslash \mathrm{GL}_n(F)} \int_{X\in\mathfrak{b}_\mathfrak{n}\backslash M_n(F)} W\left(\mu\begin{pmatrix} I_n & X \\ & I_n\end{pmatrix}\begin{pmatrix} w_n{}^tg^{-1} & \\ & w_n{}^t\, g^{-1}\end{pmatrix}\right) \\ &\qquad\cdot\psi(trX)f_{\hat{\phi},1-s}(g)dXdg\ .\end{aligned} \tag{3.6}$$

Hence $\widehat{\phi}$ is the Fourier transform of ϕ.

The proof of (3.3) can be inferred from the corresponding Euler product expansion of the global integral of [J.S.2], as we do in Sections 6.1, 6.2. Exact details will appear in a forthcoming paper by J. Cogdell and I. Piatetski-Shapiro.

Both $\widetilde{\gamma}(\tau,\Lambda^2,s,\psi)$ and $\gamma(\tau,\Lambda^2,s,\psi)$ should be equal (at least up to a monomial). We show this in Section 6.3. In particular, $\gamma(\tau,\Lambda^2,s,\psi)$ has a zero at $s=0$ if and only if $\widetilde{\gamma}(\tau,\Lambda^2,s,\psi)$ does. Consider then the functional equation (3.3) at $s=0$. Now, exactly the same proof as that of Proposition 1 in Section 7.1 of [J.S.2] shows that $\widetilde{\mathcal{L}}(W,\widehat{\phi},1-s)$ is holomorphic at $s=0$. This implies that $\mathcal{L}(W,\phi,s)$ has a pole at $s=0$. Again, a close look at the proof of Proposition 1 in Section 7.1 of [J.S.2] shows that since (by our assumption of supercuspidality of τ) W has compact support modulo $C_{2n}Z_{2n}$, the function $(x,g)\mapsto W\left(\mu\cdot\begin{pmatrix} I_n & x \\ & I_n\end{pmatrix}\begin{pmatrix} g & \\ & g\end{pmatrix}\right)$ has compact support in $\mathfrak{b}_\mathfrak{n}\backslash M_n(F)\times C\cdot Z_n\backslash \mathrm{GL}_n(F)$. This shows that the integral (3.5) is absolutely convergent and holomorphic, in this case, as long as $f_{\phi,s}$ is holomorphic. Thus, if $\mathcal{L}(W,\phi,s)$ has a pole at $s=0$, $f_{\phi,s}$ must have a pole at $s=0$. Note that (3.4) is just a Tate integral. We conclude that $\omega_\tau = 1$ (!) and

(3.7)

$$\begin{aligned}
&0 \not\equiv \operatorname{Res}_{s=0} \mathcal{L}(W,\phi,s)\\
&= \int\limits_{C_nZ_n\backslash \mathrm{GL}_n(F)} \int\limits_{b_n\backslash M_n(F)} W\left(\mu \begin{pmatrix} I_n & X \\ & I_n \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix}\right)\psi(trX)\operatorname{Res}_{s=0} f_{\phi,s}(g)dXdg\\
&= c\cdot\phi(0) \int\limits_{C_nZ_n\backslash \mathrm{GL}_n(F)} \int\limits_{b_n\backslash M_n(F)} W\left(\mu \begin{pmatrix} I_n & X \\ & I_n \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix}\right)\psi(trX)dXdg
\end{aligned}$$

for some constant c. The last integral in (3.7) defines a Shalika functional on τ, i.e. a nontrivial linear functional λ on V_τ such that

$$\lambda\left(\tau\left(\begin{pmatrix} I_n & x \\ & I_n \end{pmatrix} \quad \begin{pmatrix} g & \\ & g \end{pmatrix}\right)v\right) = \psi^{-1}(trX)\lambda(v)\ .$$

For all $x \in M_n(F)$, $g \in \mathrm{GL}_n(F)$ and $v \in V_\tau$. By Section 6.1 in [J.R.], it follows that V_τ admits a nontrivial $\mathrm{GL}_n(F)\times\mathrm{GL}_n(F)$-invariant functional, where $\mathrm{GL}_n(F)\times\mathrm{GL}_n(F)$ is embedded by $(g_1,g_2)\mapsto \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}$. Choose such a functional ℓ (for example, $I_0(v,0)$ on p. 117 in [J.R.]). It defines an H-map

(3.8) $\qquad T_\ell : \rho_{\tau,1} \longrightarrow \operatorname{Ind}_{P_n\times P_n}^{\mathrm{Sp}_{2n}(F)\times\mathrm{Sp}_{2n}(F)} \delta_{P_n\times P_n}^{1/2} \qquad$ (normalized induction)

as follows. Think of the elements of $\rho_{\tau,1} = \operatorname{Ind}_{P_{2n}}^{\mathrm{Sp}_{4n}(F)} \tau\otimes|\det\cdot|^{1/2}$ as V_τ-valued functions on $\mathrm{Sp}_{4n}(F)$. For a function f in the space of $\rho_{\tau,1}$, define, for $(g_1,g_2)\in \mathrm{Sp}_{2n}(F)\times\mathrm{Sp}_{2n}(F)$,

$$T_\ell(f)(g_1,g_2) = \ell[f(g_1,g_2)]\ .$$

We identify $\mathrm{Sp}_{2n}(F)\times\mathrm{Sp}_{2n}(F)$ and its image H in $\mathrm{Sp}_{4n}(F)$. We have

$$T_\ell(f)\left(\begin{pmatrix} a & x \\ 0 & a^* \end{pmatrix} g_1\ ,\quad \begin{pmatrix} b & y \\ 0 & b^* \end{pmatrix} g_2\right) = |\det ab|^{n+1} T_\ell(f)(g_1,g_2)\ .$$

The representation on the r.h.s. of (3.8) has the identity representation (of H) as its quotient. The composition of this quotient map and T_ℓ provides a nontrivial H-invariant form b on $\rho_{\tau,1}$. Recall that $\rho_{\tau,1}$ has two constituents: one irreducible subrepresentation W_τ, which is generic, and one irreducible quotient, which is π_τ. The case $k=0$ of Theorem 1 (in this section) shows that $b(W_\tau)=0$, and hence b defines a nontrivial H-invariant form on the quotient $W_\tau\backslash\rho_{\tau,1} = \pi_\tau$.

We conclude from Theorems 3.3.1, 3.3.2 and Lemma 3.2 that

$$\widehat{\sigma}_k(\tau) = 0 \qquad \text{for} \qquad 0 \le k < n,$$

which is Theorem 1.6.2.

This completes the proof of Theorem 3.3.2. ∎

Let us also record the following corollary of the proof of Theorem 3.3.2.

Corollary. *Let τ be an irreducible, self-dual, supercuspidal representation of $\mathrm{GL}_{2n}(F)$, such that $L(\tau,\Lambda^2,s)$ has a pole at $s=0$. Then the central character (ω_τ) of τ is trivial.*

4. Irreducibility of $\widehat{\sigma}_n(\tau)$

In this section, we complete the proof of Theorem 1.5 (our main local theorem) and show the irreducibility of $\widehat{\sigma}_n(\tau)$. We assume that τ is irreducible, self-dual, and supercuspidal, such that $L(\tau, \Lambda^2, s)$ has a pole at $s = 0$.

4.1. The group E_{2n}. Let E_{2n} denote the subgroup of $\mathrm{Sp}_{4n}(F)$, which consists of elements of the form:

$$\left(\begin{array}{ccccc|ccccc}
I_2 & z_1 & & & * & * & & & & * \\
 & I_2 & z_2 & & & & & & & \\
 & & \ddots & & & \vdots & & & & \vdots \\
 & & & I_2 & z_{n-1} & & & & & \\
 & & & & I_2 & y & \cdots & & & * \\
\hline
 & & & & & I_2 & z'_{n-1} & & & \\
 & & & & & & I_2 & & & * \\
 & & & & & & & \ddots & & \\
 & & & & & & & & z'_2 & \\
 & & & & & & & & I_2 & z'_1 \\
 & & & & & & & & & I_2
\end{array}\right) \tag{4.1}$$

Define

$$\psi^{(2n)}(e) = \psi(tr(z_1 + z_2 + \cdots + z_{n-1}) + y_{12} - y_{21}) \ .$$

This is a character of E_{2n}.

4.2. Main steps of the proof. Let U_{2n} be the unipotent radical of (the Siegel parabolic subgroup) P_{2n}. We have

$$J_{U_{2n}}(\pi_\tau) \simeq \tau \otimes |\det \cdot|^n \tag{4.2}$$

as $\mathrm{GL}_{2n}(F)$-modules. In particular $J_{V_{2n},\widetilde{\psi}}(\pi_\tau)$ is one-dimensional, where V_{2n} is the standard maximal unipotent subgroup of $\mathrm{Sp}_{4n}(F)$ and $\widetilde{\psi}$ is trivial on U_{2n} and is any nondegenerate character on $m(Z_{2n})$. We will prove

Theorem 1. *We have a vector space isomorphism*

$$J_{E_{2n},\psi^{(2n)}}(V_{\pi_\tau}) \cong J_{V_{2n},\widetilde{\psi}}(V_{\pi_\tau}) \tag{4.3}$$

and in particular

$$\dim J_{E_{2n},\psi^{(2n)}}(V_{\pi_\tau}) = 1 \ . \tag{4.4}$$

Theorem 2. *We have a vector space isomorphism*

$$J_{E_{2n},\psi^{(2n)}}(V_{\pi_\tau}) \cong J_{V_n,\psi_n}(J_{\tilde{N}^{(n)},\tilde{\chi}^{-1}_{(n)}}(V_{\pi_\tau})). \tag{4.5}$$

Recall that V_n is the standard maximal unipotent subgroup of $\mathrm{Sp}_{2n}(F)$ and ψ_n is its standard nondegenerate character (1.1).

Let us show how the last two theorems imply the irreducibility of $\widehat{\sigma}_n(\tau)$. We already know that $\widehat{\sigma}_n(\tau)$ is nontrivial and supercuspidal. Thus, $\widehat{\sigma}_n(\tau)$ is a direct sum of irreducible (supercuspidal) representations of $\widetilde{\mathrm{Sp}}_{2n}(F)$. (A supercuspidal representation is semisimple.)

Proposition. *Each summand σ of $\widehat{\sigma}_n(\tau)$ is ψ_n-generic, i.e. $J_{V_n,\psi_n}(\sigma) \neq 0$.*

Proof. We have an embedding

$$Bil_{\widetilde{\mathrm{Sp}}_{2n}(F)}(\widehat{\sigma}, \widehat{\sigma}_n(\tau)) \longrightarrow Bil_{\widetilde{\mathrm{Sp}}_{2n}(F)}\left(\widehat{\sigma}, J_{\mathcal{H}_n}\left(J_{N_{n+1},\chi_n^{-1}}(\rho_{\tau,1}) \otimes \omega_\psi^{(n)}\right)\right) \ .$$

Thus, the last space is nontrivial. Theorem 6.2(c) in the Appendix implies that $\widehat{\sigma}$ must be ψ_n^{-1}-generic, and hence σ is ψ_n-generic. □

To conclude the irreducibility, we have

Theorem 3. *The representation $\widehat{\sigma}_n(\tau)$ has a unique ψ_n-Whittaker model, i.e.*

$$\dim J_{V_n,\psi_n}(\widehat{\sigma}_n(\tau)) = 1 \ .$$

Proof. The proof of Lemma 3.2 can be used to show that for a smooth representation π of $\mathrm{Sp}_{2n}(F)$

$$\left[J_{V_k,\psi_k}(V_{\widehat{\sigma}_{k,\pi}})\right]^* \cong \left[J_{V_k,\psi_k}(J_{\tilde{N}^{(k)},\tilde{\chi}_{(k)}^{-1}}(V_\pi))\right]^*. \tag{4.6}$$

Thus, for $\pi = \pi_\tau$ and $k = n$,

$$\dim J_{V_n,\psi_n}(V_{\widehat{\sigma}_n(\tau)}) = \dim J_{V_n,\psi_n}(J_{\tilde{N}^{(n)},\tilde{\chi}_{(n)}^{-1}}(V_{\pi_\tau})) = 1 \ ,$$

by (4.4) and (4.5). □

4.3. Proof of Theorem 4.2.1. We will prove the following more general theorem. Let H be the image of the direct sum embedding of $\mathrm{Sp}_{2n}(F) \times \mathrm{Sp}_{2n}(F)$ inside $\mathrm{Sp}_{4n}(F)$.

Theorem. *Let π be an irreducible representation of $\mathrm{Sp}_{4n}(F)$. Assume that V_π admits nontrivial H-invariant functionals. Then we have a vector space isomorphism*

$$J_{E_{2n},\psi^{(2n)}}(V_\pi) \cong J_{V_{2n},\tilde{\psi}}(V_\pi) \ .$$

Note that by Theorem 3.2 the last isomorphism is valid for $\pi = \pi_\tau$, and this will prove Theorem 4.2.1. We now start with the proof.

As we have done in Section 2.3, the map $\xi \mapsto \pi(a)\xi$ defines an isomorphism (of vector spaces)

$$J_{E_{2n},\psi^{(2n)}}(V_\pi) \cong J_{(E_{2n})^a,\psi_a^{(2n)}}(V_\pi) \tag{4.7}$$

where $(E_{2n})^a = aE_{2n}a^{-1}$ and $\psi_a^{(2n)}(x) = \psi^{(2n)}(a^{-1}xa)$, $x \in (E_{2n})^a$. We apply (4.7) twice. First, for

$$a = \begin{pmatrix} b &&&&& \\ & \ddots &&&& \\ && b &&& \\ &&& b^* && \\ &&&& \ddots & \\ &&&&& b^* \end{pmatrix} , \quad \text{where} \quad b = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \tag{4.8}$$

we have $(E_{2n})^a = E_{2n}$, and for $e \in E_{2n}$ (of the form (4.1)),

$$\psi_a^{(2n)}(e) = \psi(tr(z_1 + \cdots + z_{n-1}) + y_{1,1}).$$

Second, conjugate by the Weyl element ν defined by

$$\begin{aligned} &\nu_{i,2i-1} = 1, && \text{for} \quad i = 1, \ldots, 2n, \\ &\nu_{2n+i,2i} = -1, && \text{for} \quad i = 1, \ldots, n, \\ &\nu_{2n+i,2i} = 1, && \text{for} \quad i = n+1, \ldots, 2n, \\ &\text{otherwise}, \quad \nu_{i,j} = 0. \end{aligned} \tag{4.9}$$

Up to signs, ν is μ^{-1} of Section 3.3 (with $4n$ replacing $2n$). Denote

$$B = \nu E_{2n} \nu^{-1},$$
$$\chi(\nu e \nu^{-1}) = \psi_a^{(2n)}(e).$$

Then

$$J_{E_{2n},\psi^{(2n)}}(V_\pi) \cong J_{B,\chi}(V_\pi)\ .$$

The subgroup B consists of those elements of $\mathrm{Sp}_{4n}(F)$ which have the form

$$v = \begin{pmatrix} z & x \\ y & z' \end{pmatrix} \tag{4.10}$$

where $z, z' \in Z_{2n}$ and x and y are upper triangular and nilpotent. For v of the form (4.10),

$$\chi(v) = \psi(z_{12} + z_{23} + \cdots + z_{n,n+1} - z_{n+1,n+2} - \cdots - z_{2n-1,2n}). \tag{4.11}$$

Our goal is to "fatten" x in (4.10), "using" y, by successive applications of Lemma 2.2 until we get from $J_{B,\chi}$ to $J_{V_{2n},\widetilde{\psi}}$. Let us introduce some notation. Let

$$\mathcal{X} = \{x \in M_{2n}(F) \Big| \begin{pmatrix} I_{2n} & x \\ & I_{2n} \end{pmatrix} \in \mathrm{Sp}_{4n}(F)\}.$$

For a subspace $S \subset \mathcal{X}$, write $\ell(S) = \{\ell(x) \mid x \in S\}$ and $\overline{\ell}(S) = \{\overline{\ell}(x) \mid x \in S\}$. Put

$$\mathcal{X}_0 = \{x \in \mathcal{X} \ \mid \ x \quad \text{is nilpotent and upper triangular}\}. \tag{4.12}$$

An element in B can be written in the form

$$v = \ell(x) m(z) \overline{\ell}(y) \tag{4.13}$$

where $x, y \in \mathcal{X}_0$ and $z \in Z_{2n}$. Let

$$\mathcal{Y}_{1,2} = \{x \in \mathcal{X}_0 \mid x_{12} (= x_{2n-1,2n}) = 0\} = \begin{pmatrix} 0 & 0 & * & \cdots & * \\ & 0 & * & \cdots & * \\ & & \ddots & & \\ & & & * & * \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}.$$

Let $C^{(1,2)}$ be the subgroup of elements of the form (4.13) such that $y \in \mathcal{Y}_{1,2}$. Thus

$$C^{(1,2)} = \ell(\mathcal{X}_0) m(Z_{2n}) \overline{\ell}(\mathcal{Y}_{1,2})\ .$$

Let

$$Y^{(1,2)} = \overline{\ell}(\mathcal{Y}^{1,2})$$

where

$$\mathcal{Y}^{1,2} = F \cdot (e_{12} + e_{2n-1,2n}) = \left\{ \begin{pmatrix} 0 & t & \cdots & & 0 \\ & 0 & & & \\ \vdots & & \ddots & & \vdots \\ & & & 0 & t \\ 0 & & & & 0 \end{pmatrix} | t \in F \right\}.$$

Denote

$$X^{(1,1)} = \ell(\mathcal{X}^{1,1})$$

where

$$\mathcal{X}^{1,1} = F \cdot (e_{11} + e_{2n,2n}) = \left\{ \begin{pmatrix} t & 0 & \cdots & & 0 \\ & 0 & & & \\ \vdots & & \ddots & & \vdots \\ & & & 0 & \\ 0 & & & & t \end{pmatrix} | t \in F \right\}.$$

Let

$$\chi^{(1,2)} = \chi\Big|_{C^{(1,2)}} \ , \ B^{(1,2)} = B \ , \ D^{(1,2)} = C^{(1,2)} X^{(1,1)} \ , \ A^{(1,2)} = D^{(1,2)} Y^{(1,2)} \ .$$

It is easy to check that $\chi^{(1,2)}$ and the diamond

$$\begin{tikzcd}[column sep=small] & A^{(1,2)} \arrow[dl, dash, "X^{(1,1)}"'] \arrow[dr, dash, "Y^{(1,2)}"] & \\ B^{(1,2)} \arrow[dr, dash, "Y^{(1,2)}"'] & & D^{(1,2)} \arrow[dl, dash, "X^{(1,1)}"] \\ & C^{(1,2)} & \end{tikzcd}$$

satisfy assumptions (i)-(v) of Section 2.2. We conclude that

$$J_{B^{(1,2)},\chi^{(1,2)}_{B^{(1,2)}}}(V_\pi) \cong J_{D^{(1,2)},\chi^{(1,2)}_{D^{(1,2)}}}(V_\pi) \ . \tag{4.14}$$

Put

$$\mathcal{X}_{1,1} = \mathcal{X}_0 \oplus \mathcal{X}^{1,1} \ .$$

Then

$$D^{(1,2)} = \ell(\mathcal{X}_{1,1}) m(Z_{2n}) \overline{\ell}(\mathcal{Y}_{1,2}).$$

$\chi^{(1,2)}_{D^{(1,2)}}$ is the character of $D^{(1,2)}$ which is trivial on $\ell(\mathcal{X}_{1,1}) \cdot \overline{\ell}(\mathcal{Y}_{1,2})$ and is χ on $m(Z_{2n})$. Let $1 \leq i < j$. Define

(4.15)
$$\mathcal{Y}_{i,j} = \left\{ x \in \mathcal{X}_0 \mid x_{r,\ell} = 0 \quad \text{for} \quad r, \ell \leq j-1, \quad \text{and} \quad x_{r,j} = 0 \quad \text{for} \quad r \geq i \right\},$$

$$\mathcal{Y}^{i,j} = F(e_{i,j} + e_{2n-j+1,2n-i+1}) \ . \tag{4.16}$$

(Note that if $i+j=2n+1$, then $\mathcal{Y}^{i,j}=Fe_{i,j}$.) Define for $1\le s\le r\le 2n$

$$\mathcal{X}^{r,s}=F(e_{r,s}+e_{2n+1-s,2n+1-r}) \tag{4.17}$$

and for $1\le s\le r\le n$

$$\mathcal{X}_{r,s}=\mathcal{X}_0\oplus(\bigoplus_{q\le\ell\le r-1}\mathcal{X}^{\ell,q})\oplus(\bigoplus_{q=s}^{r}\mathcal{X}^{r,q}). \tag{4.18}$$

For example, for $n=4$

$$\mathcal{X}_{3,2}=\begin{pmatrix} * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \end{pmatrix}.$$

Let, for $1\le i<j\le n+1$,

$$C^{(i,j)}=\ell(\mathcal{X}_{j-1,i+1})m(Z_{2n})\overline{\ell}(\mathcal{Y}_{i,j})\ , \qquad \text{if} \qquad i+1\le j-1, \tag{4.19}$$

and if $i=j-1$,

$$C^{(j-1,j)}=\ell(\mathcal{X}_{j-2,1})m(Z_{2n})\overline{\ell}(\mathcal{Y}_{j-1,j})\ , \tag{4.20}$$

where we put $\mathcal{X}_{0,1}=\mathcal{X}_0$.

Let, for $1\le i<j\le n+1$,

$$\begin{gathered} Y^{(i,j)}=\overline{\ell}(\mathcal{Y}^{i,j}),\quad X^{(r,s)}=\ell(\mathcal{X}^{r,s}),\quad B^{(i,j)}=C^{(i,j)}Y^{(i,j)}, \\ D^{(i,j)}=C^{(i,j)}X^{(j-1,i)},\quad A^{(i,j)}=D^{(i,j)}Y^{(i,j)}\ . \end{gathered} \tag{4.21}$$

Let $\chi^{(i,j)}$ be the character of $C^{(i,j)}$, which is trivial on $\ell(\mathcal{X}_{j-1,i+1})\cdot\overline{\ell}(\mathcal{Y}_{i,j})$ (resp. on $\ell(\mathcal{X}_{j-2,1})\cdot\overline{\ell}(\mathcal{Y}_{j-1,j})$) and is χ on $m(Z_{2n})$. One can check that $\chi^{(i,j)}$ and the diamond

$$\begin{tikzcd} & A^{(i,j)} \arrow[dl, dash, "X^{(i,j)}"'] \arrow[dr, dash, "Y^{(i,j)}"] & \\ B^{(i,j)} \arrow[dr, dash, "Y^{(i,j)}"'] & & D^{(i,j)} \arrow[dl, dash, "X^{(j-1,i)}"] \\ & C^{(i,j)} & \end{tikzcd}$$

satisfy assumptions (i)-(v) of Section 2.2, and hence

$$J_{B^{(i,j)},\chi^{(i,j)}_{B^{(i,j)}}}(V_\pi)\cong J_{D^{(i,j)},\chi^{(i,j)}_{D^{(i,j)}}}(V_\pi) \tag{4.22}$$

for all $1\le i<j\le n+1$. Note that

$$D^{(i,j)}=B^{(i-1,j)} \quad \text{and} \quad \chi^{(i,j)}_{D^{(i,j)}}=\chi^{(i-1,j)}_{B^{(i-1,j)}}\ , \quad 2\le i<j\le n+1\ ,$$

and

$$D^{(1,j)}=B^{(j,j+1)} \quad \text{and} \quad \chi^{(1,j)}_{D^{(1,j)}}=\chi^{(j,j+1)}_{B^{(j,j+1)}}\ , \quad 1\le j\le n\ .$$

We conclude that

$$J_{B^{(1,2)},\chi}(V_\pi) \cong J_{D^{(1,n+1)},\chi^{(1,n+1)}_{D^{(1,n+1)}}}(V_\pi). \tag{4.23}$$

Note that

$$D^{(1,n+1)} = \ell(\mathcal{X}_{1,n})m(Z_{2n})\overline{\ell}(\mathcal{Y}_{1,n+1}).$$

In case $n = 4$,

$$\mathcal{X}_{1,n} = \begin{pmatrix} * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \end{pmatrix} \quad , \quad \mathcal{Y}_{1,n+1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that, so far in this proof, we did not use any particular property of V_π. Define for $n+1 \leq r \leq 2n$ and $1 \leq s \leq 2n+1-r$

$$\mathcal{X}_{r,s} = \mathcal{X}_{1,n} \oplus \Big(\bigoplus_{\substack{n+1\leq \ell \leq r-1 \\ 1 \leq q \leq 2n+1-\ell}} \chi^{\ell,q}\Big) \oplus \Big(\bigoplus_{q=s}^{2n+1-r} \mathcal{X}^{r,q} \Big). \tag{4.24}$$

In case $n = 4$,

$$\mathcal{X}_{6,2} = \begin{pmatrix} * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * \end{pmatrix}.$$

Consider the action of $X^{(n+1,n)}$ on the r.h.s of (4.23). Note that $X^{(n+1,n)}$ is the center of $\mathcal{H}_{n-1}$ (embedded in $\mathrm{Sp}_{4n}(F)$). We have, for any nontrivial character ξ of $X^{(n+1,n)}$,

$$J_{X^{(n+1,n)},\xi}(J_{D^{(1,n+1)},\chi^{(1,n+1)}_{D^{(1,n+1)}}}(V_\pi)) = 0. \tag{4.25}$$

Indeed, $D^{(1,n+1)}X^{(n+1,n)} \supset N^{(n-1)}$ (see Section 3.2) and $\widetilde{\chi}_{n-1} = \chi^{(1,n+1)}_{D^{(1,n+1)}} \cdot \xi \Big|_{N^{(n-1)}}$ is a character of $N^{(n-1)}$ of the same type of $\chi_{(n-1)}$ (in the sense of the remark following Theorem 3.3.1). Hence there is a surjection

$$J_{N^{(n-1)},\widetilde{\chi}_{n-1}}(V_\pi) \longrightarrow J_{X^{(n+1,n)}D^{(1,n+1)},\xi\cdot\chi^{(1,n+1)}_{D^{(1,n+1)}}}(V_\pi). \tag{4.26}$$

But the Jacquet module on the l.h.s. of (4.26) is zero by Section 3, and hence (4.25) follows. We conclude that $X^{(n+1,n)}$ acts trivially on $J_{D^{(1,n+1)},\chi^{(1,n+1)}_{D^{(1,n+1)}}}(V_\pi)$. Define

$B^{(n-1,n+2)} = D^{(1,n+1)}X^{(n+1,n)}$ and extend $\chi^{(1,n+1)}_{D^{(1,n+1)}}$ to $B^{(n-1,n+2)}$ by making it trivial on $X^{(n+1,n)}$. Denote the resulting character by $\chi^{(n-1,n+2)}_{B^{(n-1,n+2)}}$. Thus, we have

$$J_{B^{(n-1,n+2)},\chi^{(n-1,n+2)}_{B^{(n-1,n+2)}}}(V_\pi) \cong J_{D^{(1,n+1)},\chi^{(1,n+1)}_{D^{(1,n+1)}}}(V_\pi). \tag{4.27}$$

Now, we can continue as before, "replacing the $n-1$ coordinates of $\bigoplus_{i=1}^{n-1}\mathcal{Y}^{i,n+2}$ into $\mathcal{X}_{n+1,1}$". Define as before, for $1 \le i \le n-1$, $j \ge n+2$,

$$\begin{aligned} &C^{(i,j)} = \ell(\mathcal{X}_{j-1,i+1})m(Z_{2n})\overline{\ell}(\mathcal{Y}_{i,j}), B^{(i,j)} = C^{(i,j)}Y^{(i,j)}, \\ &D^{(i,j)} = C^{(i,j)}X^{(j-1,i)}, A^{(i,j)} = D^{(i,j)}Y^{(i,j)} . \end{aligned} \tag{4.28}$$

Let $\chi^{(i,n+2)}$ be the character of $C^{(i,n+2)}$ which is trivial on $\ell(\mathcal{X}_{n+1,i+1})\overline{\ell}(\mathcal{Y}_{i,n+2})$ and is χ on $m(Z_{2n})$. We are again at the situation of Lemma 2.2, and we conclude from (4.27), after $n-1$ successive applications, that

$$J_{D^{(1,n+1)},\chi^{(1,n+1)}_{D^{(1,n+1)}}}(V_\pi) \cong J_{D^{(1,n+2)},\chi^{(1,n+2)}_{D^{(1,n+2)}}}(V_\pi) . \tag{4.29}$$

Next, we repeat the previous argument, using the assumption on π, Theorem 3.3.1 ($k = n-2$) and the remark which follows, and Lemma 3.2 to show that $X^{(n+2,n-1)}$ acts trivially on the r.h.s. of (4.29), and then proceed as before, using $C^{(i,n+3)}, B^{(i,n+3)}, D^{(i,n+3)}, A^{(i,n+3)}$ for $i \le n-2$, and so on. Note that in these stages of the proof, we use as above the assumption on π, Theorem 3.3.1 and Lemma 3.2, for $0 \le k \le n-1$. We get

$$J_{D^{(1,n+2)},\chi^{(1,n+2)}_{D^{(1,n+2)}}}(V_\pi) \cong \cdots \cong J_{D^{(1,2n)},\chi^{(1,2n)}_{D^{(1,2n)}}}(V_\pi). \tag{4.30}$$

Note that $D^{(1,2n)} = \ell(\mathcal{X}_{2n-1,1})m(Z_{2n})$ and $\chi^{(1,2n)}_{D^{(1,2n)}}$ is the character which is trivial on $\ell(\mathcal{X}_{2n-1,1})$ and χ on $m(Z_{2n})$. Note that $X^{(2n,2n)}D^{(1,2n)} = V_{2n}$, the standard maximal unipotent subgroup of $\mathrm{Sp}_{4n}(F)$. As before, $X^{(2n,2n)}$ acts trivially on $J_{D^{(1,2n)},\chi^{(1,2n)}_{D^{(1,2n)}}}(V_\pi)$, since π is nongeneric with respect to any nondegenerate character of V_{2n}. Again, this follows from Theorem 3.3.1 and the remark which follows. We conclude that

$$J_{D^{(1,2n)},\chi^{(1,2n)}_{D^{(1,2n)}}}(V_\pi) \cong J_{V_{2n},\widetilde{\psi}}(V_\pi) .$$

This completes the proof of the theorem. □

4.4. Proof of Theorem 4.2.2. This theorem is general. We prove

Theorem. *Let π be a smooth representation of $\mathrm{Sp}_{4n}(F)$. Then we have a vector space isomorphism*

$$J_{E_{2n},\psi^{(2n)}}(V_\pi) \cong J_{V_n,\psi_n}\Big(J_{\tilde{N}^{(n)},\tilde{\chi}^{-1}_{(n)}}(V_\pi)\Big).$$

We start with $J_{V_n,\psi_n}\Big(J_{\tilde{N}^{(n)},\tilde{\chi}^{-1}_{(n)}}(V_\pi)\Big)$. Again, we will first use a conjugation by $\omega = m(\widetilde{\omega})$, where $\widetilde{\omega}$ is the following Weyl element of $\mathrm{GL}_{2n}(F)$:

$$\begin{aligned} &\widetilde{\omega}_{2i,i} = 1 , && i = 1, \dots, n, \\ &\widetilde{\omega}_{2i-1,i+n} = 1 , && i = 1, \dots, n, \\ &\widetilde{\omega}_{i,j} = 0 , && \text{otherwise.} \end{aligned} \tag{4.31}$$

Note that $\widetilde{\omega} = \mu \cdot \begin{pmatrix} & I_n \\ I_n & \end{pmatrix}$, where μ is the Weyl element used in Section 3.3. Denote

$$\omega \cdot j_n(V_n)\tilde{N}^{(n)} \cdot \omega^{-1} = B. \tag{4.32}$$

The elements of B have the form

$$b = \begin{pmatrix} T & X \\ 0 & T^* \end{pmatrix} \tag{4.33}$$

where T has the following description:

$$T = \begin{pmatrix} 1 & t_{12} & & \cdots & & t_{1,2n} \\ t_{21} & 1 & & & \cdots & t_{2,2n} \\ t_{31} & t_{32} & 1 & & & t_{3,2n} \\ & \vdots & & & & \vdots \\ t_{2n-1,1} & t_{2n-1,2} & & & 1 & t_{2n-1,2n} \\ t_{2n,1} & t_{2n,2} & & & t_{2n,2n-1} & 1 \end{pmatrix}. \tag{4.34}$$

Put, for $i, j \le 2n-1$, $\overline{t}_j = \begin{pmatrix} t_{j+1,j} \\ \vdots \\ t_{2n,j} \end{pmatrix}$, $t_i = (t_{i,i+1}, \dots, t_{i,2n})$. Then, for $j = 1, \dots, n$,

$$\overline{t}_{2j-1} = \begin{pmatrix} * \\ 0 \\ * \\ 0 \\ \vdots \\ * \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \overline{t}_{2j} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \tag{4.35}$$

(in case $j = n$, $\overline{t}_{2n-1} = (0)$), and for $i \le n$

$$t_{2i-1} = (0 * 0 * \cdots * 0)$$

and t_{2i} is arbitrary in F^{2n-2i}. Denote by $T(n)$ the subgroup of such matrices T. For example, in case $n = 4$

$$T = \begin{pmatrix} 1 & 0 & * & 0 & * & 0 & * & 0 \\ * & 1 & * & * & * & * & * & * \\ 0 & 0 & 1 & 0 & * & 0 & * & 0 \\ * & 0 & * & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & 0 \\ * & 0 & * & 0 & * & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let χ be the following character of B (in the notation (4.33)):

$$\begin{aligned} \chi(b) = \psi((t_{13} + t_{24}) + (t_{35} + t_{47}) \\ + \cdots + (t_{2n-3,2n-1} + t_{2n-2n,2n}) + (x_{2n-1,2} - x_{2n,1})). \end{aligned} \tag{4.36}$$

Note that

$$\chi(b) = (\psi_n \cdot \tilde{\chi}_{(n)}^{-1})(\omega^{-1} b \omega) \ .$$

Thus,

$$J_{V_n,\psi_n}\Big(J_{\tilde{N}^{(n)},\tilde{\chi}_{(n)}^{-1}}(V_\pi)\Big) \cong J_{B,\chi}(V_\pi). \tag{4.37}$$

We will use Lemma 2.2, as we did before, to "fill in the zeroes of t_{2i-1}, from right to left, using $\bar{t}_{2i-1}$" and thus "obtain E_{2n} from B, and $\psi^{(2n)}$ from χ". Let

$$\begin{aligned}
&Y^{(i,1)} = \{m(I_{2n} + y e_{2i,1})\} \ , \quad i = 1, \dots, n-1, \\
&X^{(1,j)} = \{m(I_{2n} + x e_{1,2j})\} \ , \quad j = 2, 3, \dots, n, \\
&B^{(n-1,1)} = B, \\
&B^{(i,1)} = \{b \in B^{(n-1,1)} \mid b_{j,1} = 0 \ , \ \forall_{j>2i}\} \cdot \prod_{j=i+2}^{n} X^{(1,j)}, \quad i \le n-2, \\
&C^{(i,1)} = \{b \in B^{(i,1)} \mid b_{2i,1} = 0\}, \\
&D^{(1,i+1)} = C^{(i,1)} X^{(1,i+1)}, \quad A^{(1,i+1)} = D^{(1,i+1)} Y^{(i,1)} \ .
\end{aligned} \tag{4.38}$$

Define a character $\chi^{(i,1)}$ by the same formula (4.36), except replace t_{rs} by c_{rs} and $x_{2n-1,2} - x_{2n,1}$ by $c_{2n-1,2n+2} - c_{2n,2n+1}$, for $c \in C^{(i,1)}$. Extend $\chi^{(i,1)}$ trivially to $D^{(1,i+1)}$ and to $B^{(i,1)}$, denoting the extensions by $\chi^{(i,1)}_{D^{(1,i+1)}}$ and $\chi^{(i,1)}_{B^{(i,1)}}$. Note that $D^{(1,i+1)} = B^{(i-1,1)}$ and $\chi^{(i,1)}_{D^{(1,i+1)}}\Big|_{C^{(i-1,1)}} = \chi^{(i-1,1)}$. The character $\chi^{(i,1)}$ and the diamond

$$\begin{tikzcd}
 & A^{(1,i+1)} \arrow[dl, dash, "X^{(1,i+1)}"'] \arrow[dr, dash, "Y^{(i,1)}"] & \\
B^{(i,1)} \arrow[dr, dash, "Y^{(i,1)}"'] & & D^{(1,i+1)} \arrow[dl, dash, "X^{(1,i+1)}"] \\
 & C^{(i,1)} &
\end{tikzcd}$$

satisfy assumptions (i)-(v) of Section 2.2, and we conclude that

$$J_{B^{(i,1)},\chi^{(i,1)}_{B^{(i,1)}}}(V_\pi) \cong J_{D^{(1,i+1)},\chi^{(i,1)}_{D^{(i,1)}}}(V_\pi) \cong J_{B^{(i-1,1)},\chi^{(i-1,1)}_{B^{(i-1,1)}}}(V_\pi)$$

for $i = n-1, n-2, \dots, 2$. Thus

$$J_{B,\chi}(V_\pi) \cong J_{D^{(1,2)},\chi^{(1,1)}_{D^{(1,2)}}}(V_\pi) \ .$$

To visualize the above subgroups, it is enough to visualize their GL_{2n}-Levi part (since they all lie in P_{2n}, and contain the full unipotent radical of P_{2n}).

$$Y^{(i,1)}: \begin{array}{c} \\ \\ \\ 2i \longrightarrow \\ \\ \\ \end{array} \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ \vdots & & & \\ * & & \ddots & \\ \vdots & & & \\ 0 & & & 1 \end{pmatrix} \quad , \quad X^{(1,j)}: \begin{array}{c} \\ \begin{pmatrix} 1 & 0 & \cdots & \overset{\overset{2j}{\downarrow}}{*} & \cdots & 0 \\ & 1 & & & & \\ & & & \ddots & & \\ & & & & & 1 \end{pmatrix} \end{array}.$$

$B^{(i,1)}$: the first column in its GL_{2n} part has the form

$$\begin{pmatrix} 1 \\ * \\ 0 \\ * \\ 0 \\ * \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{c} \\ \\ \\ \\ \\ \longleftarrow 2i \\ \\ \\ \\ \end{array}$$

and the first row has the form $(1\ 0 * 0 * \cdots 0 \overset{\overset{2i+3}{\downarrow}}{*} * \cdots *)$, the rest of the GL_{2n} part is like that of T in (4.35).

$C^{(i,1)}$: like $B^{(i,1)}$, only that $b_{2i,1} = 0$.

$D^{(1,i+1)}$: like $C^{(i,1)}$, only that the first row of its GL_{2n}-part has the form:

$$\left(1\ 0 * 0 * \cdots 0 \underset{\underset{2i+1}{\uparrow}}{*} * \cdots *\right)$$

Note that the GL_{2n}-part of $D^{(1,2)}$ looks like $\begin{pmatrix} I_2 & * \\ 0 & T' \end{pmatrix}$, where $T' \in T(n-1)$. In general, for $1 \leq r, s \leq n$ let

$$Y^{(r,s)} = \{m(I_{2n} + ye_{2r,2s-1})\} \ , \quad X^{(r,s)} = \{m(I_{2n} + xe_{2r-1,2s})\} \ . \tag{4.39}$$

Let, for $1 \leq j \leq i \leq n-1$,

$$B^{(i,j)} = \tilde{B}^{(i,j)} \prod_{s=i+2}^{n} X^{(j,s)} \ , \tag{4.40}$$

where

$$\tilde{B}^{(i,j)} = \left\{ \begin{pmatrix} T & X \\ 0 & T^* \end{pmatrix} \in \mathrm{Sp}_{4n}(F) \middle| T = \begin{pmatrix} I_2 & & * & * \\ & \ddots & & \\ & & I_2 & * \\ & & & T' \end{pmatrix} \ , \ \begin{matrix} T' \in T(n-j+1) \\ T_{\ell,2j-1} = 0, \forall \ell > 2i \end{matrix} \right\},$$

$$C^{(i,j)} = \{b \in B^{(i,j)} \mid b_{2i,2j-1} = 0\} \ , \ D^{(j,i+1)} = C^{(i,j)} X^{(j,i+1)} \ , \tag{4.41}$$

$$A^{(j,i+1)} = D^{(j,i+1)} Y^{(i,j)} \ .$$

(We let $X^{(j,n+1)} = \{I_{4n}\}$, $T(2) = \{I_{2n}\}$.) Let $\chi^{(i,j)}$ be the character of $C^{(i,j)}$ defined by (4.36), where, for $c \in C^{(i,j)}$, we replace $t_{r,s}$ by $c_{r,s}$ and $x_{2n-1,2} - x_{2n,1}$ by $c_{2n-1,2n+1} - c_{2n,2n+1}$. Note that $D^{(j,i+1)} = B^{(i-1,j)}$, for $i \geq j+1$, and $D^{(j,j+1)} = B^{(n-1,j+1)}$. As before, we have the usual compatibility relations among the $\chi^{(i,j)}$ and their (trivial) extensions to $D^{(j,i+1)}, B^{(i,j)}$. One checks that $\chi^{(i,j)}, A^{(j,i+1)}, B^{(i,j)}, C^{(i,j)}, D^{(j,i+1)}, \ Y^{(i,j)}, X^{(j,i+1)}$ satisfy assumptions (i)-(v) of Section 2.2. All in all, we conclude that

$$\begin{aligned} J_{B,\chi}(V_\pi) &\cong J_{D^{(1,2)},\chi^{(1,1)}_{D^{(1,2)}}}(V_\pi) \cong \cdots \cong J_{D^{(j,j+1)},\chi^{(j,j)}_{D^{(j,j+1)}}}(V_\pi) \\ &\cong \cdots \cong J_{D^{(n-1,n)},\chi^{(n-1,n-1)}_{D^{(n-1,n)}}}(V_\pi) \ . \end{aligned}$$

Note that $D^{(n-1,n)} = E_{2n}$ and $\chi^{(n-1,n-1)}_{D^{(n-1,n)}} = \psi^{(2n)}$. This completes the proof of the theorem. $\square$

We record the following corollary to (4.6) and the theorems in Sections 4.3 and 4.4. Recall that H denotes the image of the direct sum embedding of $\mathrm{Sp}_{2n}(F) \times \mathrm{Sp}_{2n}(F)$ inside $\mathrm{Sp}_{4n}(F)$.

Corollary. *Let π be an irreducible representation of $\mathrm{Sp}_{4n}(F)$. Assume that V_π admits a nontrivial H-invariant functional. Then we have a vector space isomorphism*

$$\Big[J_{V_n,\psi_n}(V_{\widehat{\sigma}_{n,\pi}})\Big]^* \cong \Big[J_{V_{2n},\widetilde{\psi}}(V_\pi)\Big]^*.$$

5. The global case

In this section we place ourselves in the global set-up of Section 1.7, and we prove Theorem 1.7, which states that $\sigma_n(\eta) \neq 0$, the space of $\sigma_n(\eta)$ being given by (1.33), for $k = n$. (The reader will note that the proof is analogous to that of Theorems 4.2.1 and 4.2.2, the repeated use of Lemma 2.2 being replaced by corresponding Fourier expansions. Of course, nothing of these is used here. We keep the notation of Section 4, except we use it for the global set-up; however we give a precise reference for each notation.) Denote (in the notation of (1.28))

$$\mathcal{E}(g,\varphi) = \mathrm{Res}_{s=1} E(g, \varphi_{\eta,s})$$

where we put, for short, $\varphi = \varphi_{\eta,1}$. As is clear from the proof of [G.R.S.1, Lemma 15], $\sigma_n(\tau) \neq 0$, if and only if

$$I(\varphi) = \int\limits_{\tilde{N}^{(n)}_K \backslash \tilde{N}^{(n)}_{\mathbb{A}}} \mathcal{E}(v,\varphi)\widetilde{\chi}_{(n)}(v)dv \not\equiv 0. \tag{5.1}$$

Recall that

$$\tilde{N}^{(n)} = \left\{ v = \begin{pmatrix} z & x & * & * & * & * \\ & 1 & 0 & y & t & * \\ & & I_n & 0 & y' & * \\ & & & I_n & 0 & * \\ & & & & 1 & x' \\ & & & & & z^* \end{pmatrix} \in \mathrm{Sp}_{4n} \middle| z \in Z_{n-1} \right\}$$

and, for $v \in \tilde{N}^{(n)}(\mathbb{A})$,

$$\tilde{\chi}_{(n)}(v) = \psi\Big(\sum_{i=1}^{n-2} z_{i,i+1} + x_{n-1} + t\Big)$$

(Z_k denotes the standard maximal unipotent subgroup of GL_k). We also proved in [G.R.S.1, Chapter 3] that for $k < n$ and $\alpha \in K^*$

$$\int_{N_K^{(k)} \backslash N_{\mathbb{A}}^{(k)}} \mathcal{E}(v, \varphi)\chi^{-1}_{(k),\alpha}(v)dv \equiv 0 \tag{5.2}$$

where

$$N^{(k)} = \left\{ v = \begin{pmatrix} z & x & * & * & * \\ & 1 & 0 & t & * \\ & & I_{2k-2} & 0 & * \\ & & & 1 & x' \\ & & & & z^* \end{pmatrix} \in \mathrm{Sp}_{4n} \,\Big|\, z \in Z_{n-k} \right\}$$

and, for $v \in N^{(k)}(\mathbb{A})$,

$$\chi_{(k),\alpha}(v) = \psi\Big(\sum_{i=1}^{n-k-1} z_{i,i+1} + x_{n-k} + \alpha t\Big) .$$

Note that it is (5.2) that is responsible for $\sigma_k(\eta) = 0$ for $k < n$. We remark that the proof in [G.R.S.1, Chapter 3] of (5.2) was for $\alpha = 1$, but the same proof works word for word for $\alpha \in K^*$. Denote (and note the analogy with Theorem 4.2.1)

$$R(\varphi) = \int_{E_{2n}(K)\backslash E_{2n}(\mathbb{A})} \mathcal{E}(v, \varphi)\overline{\psi}^{(2n)}(v)dv$$

(E_{2n} and $\psi^{(2n)}$ are defined in (4.1)). Consider the elements a and ν of $\mathrm{Sp}_{4n}(K)$ defined in (4.8) and (4.9). Put $\nu_0 = \nu a$. We have

$$R(\varphi) = \int_{E_{2n}(K)\backslash E_{2n}(\mathbb{A})} \mathcal{E}(\nu_0 v, \varphi)\overline{\psi}^{(2n)}(v)dv = \int_{B_K\backslash B_{\mathbb{A}}} \mathcal{E}(v \cdot \nu_0, \varphi)\chi^{-1}(v)dv \tag{5.3}$$

where B and χ (global versions) are defined in (4.10) and (4.11). Put, for $1 \le i < j$,

$$R_{i,j}(\varphi) = \int_{Y^*_{i,j}(\mathbb{A})} \int_{B^{(i,j)}(K)\backslash B^{(i,j)}(\mathbb{A})} \mathcal{E}(by^*\nu_0, \varphi)\chi^{-1}(b)dbdy^* \tag{5.4}$$

where $B^{(i,j)}$ is defined in (4.21) and (4.28) and

(5.5)

$$Y^*_{i,j} = \overline{\ell}\left[\mathrm{Span}\left\{ e_{r,s} + e_{2n+1-s,2n+1-r} \mid \begin{array}{c} 1 \le r < s \le j-1, \quad \text{and if } s = j, \\ \text{then } r = i+1, i+2, \dots, j-1 \end{array} \right\}\right].$$

An element of $B^{(i,j)}$ has the form $\ell(x_{j-1,i+1})m(z)\overline{\ell}(y)$ or $\ell(x_{j-2,1})m(z)\overline{\ell}(y)$ according to whether $i < j-1$ or $i = j-1$, where $x_{r,s} \in \mathcal{X}_{r,s}$, defined in (4.18), $z \in Z_{2n}$ and $y \in \mathcal{Y}_{i+1,j}$ (defined in (4.15)). Here $\chi(b) = \chi(m(z))$. Note that $R_{1,2}(\varphi) = R(\varphi)$.

Now, let us write the Fourier expansion along $\mathcal{X}^{(j-1,i)}(K)\backslash\mathcal{X}^{(j-1,i)}(\mathbb{A})$, defined in (4.17), of $x \mapsto \mathcal{E}(\ell(x)h,\varphi)$:

$$\mathcal{E}(h,\varphi) = \sum_{\alpha\in K}\int\limits_{K\backslash\mathbb{A}} \mathcal{E}(\ell(t(e_{j-1,i}+e_{2n+1-i,2n+2-j}))h,\varphi)\psi(\alpha t)dt \ . \tag{5.6}$$

Substitute (5.6) in (5.4) and decompose

$$B^{(i,j)}(K)\backslash B^{(i,j)}(\mathbb{A}) = C^{(i,j)}(K)\backslash C^{(i,j)}(\mathbb{A})\cdot Y^{(i,j)}(K)\backslash Y^{(i,j)}(\mathbb{A})$$

with $db = dcdy$. These groups are defined in (4.21). (Recall that $Y^{(i,j)}$ normalizes $C^{(i,j)}$ and preserves χ on $C^{(i,j)}$.) We get

(5.7)

$$\begin{aligned} R_{ij}(\varphi) = &\int\limits_{Y^*_{i,j}(\mathbb{A})}\ \int\limits_{Y^{(i,j)}(K)\backslash Y^{(i,j)}(\mathbb{A})}\ \int\limits_{C^{(i,j)}(K)\backslash C^{(i,j)}(\mathbb{A})} \\ &\cdot\sum_{\alpha\in K}\mathcal{E}(\ell(t(e_{j-1,i}+e_{2n+1-i,2n+2-j}))cy\cdot y^*\nu_0,\varphi)\psi(\alpha t)\chi^{-1}(c)dcdydy^* \ . \end{aligned}$$

Now, use that $\mathcal{E}(\overline{\ell}(\alpha(e_{i,j}+e_{2n+1-j,2n+1-i}))h) = \mathcal{E}(h)$, for $\alpha\in K$, and conjugate $\overline{\ell}_{i,j}(\alpha) = \overline{\ell}(\alpha(e_{i,j}+e_{2n+1-j,2n+1-i}))$ all the way to the right in (5.7). (Here, we use in full the global analog of assumptions (i)-(v) of Section 2.2.)

We get

$$\begin{aligned} R_{i,j}(\varphi) &= \int\limits_{Y^*_{i,j}(\mathbb{A})}\ \int\limits_{Y^{(i,j)}(K)\backslash Y^{(i,j)}(\mathbb{A})} \\ &\quad\cdot\sum_{\alpha\in K}\ \int\limits_{D^{(i,j)}(K)\backslash D^{(i,j)}\mathbb{A})} \mathcal{E}(v\cdot\overline{\ell}_{i,j}(\alpha)yy^*\nu_0,\psi)\chi^{-1}(v)dvdydy^* \\ &= \int\limits_{Y^*_{i-1,j}(\mathbb{A})}\ \int\limits_{B^{(i-1,j)}(K)\backslash B^{(i-1,j)}(\mathbb{A})} \mathcal{E}(by^*\nu_0,\varphi)\chi^{-1}(b)dbdy^* \end{aligned} \tag{5.8}$$

where, as in Section 4.3, $B^{(0,j)} = B^{(j,j+1)}$. $D^{(i,j)}$ is defined in (4.21). Here to shorten the notation, we keep using $\chi(v)$ as the character of $B^{(i,j)}$ which is $\chi(m(z))$ on $m(z)$ and trivial on the other factors. (Note the clear analogy with Section 4.3.) We conclude from (5.8) that

$$R_{i,j}(\varphi) = R_{i-1,j}(\varphi) \quad \text{if } 2\le i<j\le n+1$$

and

$$R_{1,j}(\varphi) = R_{j,j+1}(\varphi) \quad \text{if } 1\le j\le n,$$

and hence

$$R(\varphi) = R_{1,n+1}(\varphi)\ .$$

Now perform ("in $R_{1,n+1}(\varphi)$") a Fourier expansion, as before, along $\mathcal{X}^{(n,1)}(K)\backslash \mathcal{X}^{(n,1)}(\mathbb{A})$ to get (as in (5.7), (5.8))

$$R(\varphi) = \int\limits_{Y^*_{n-1,n+2}(\mathbb{A})}\ \int\limits_{D^{(1,n+1)}(K)\backslash D^{(1,n+1)}(\mathbb{A})} \mathcal{E}(by^*\nu_0,\varphi)\chi^{-1}(b)dbdy^*. \tag{5.9}$$

Recall that an element of $D^{(1,n+1)}$ has the form $b = \ell(x_{1,n})m(z)\overline{\ell}(y)$, where $x_{1,n} \in \chi_{1,n}$, $z \in Z_{2n}$ and $y \in \mathcal{Y}_{1,n+1}$. ($\chi(b) = \chi(m(z))$, defined in (4.11).) Note that, so far, we did not use any property of $\mathcal{E}(h,\varphi)$, except for being an automorphic function. Now write the Fourier expansion of $\mathcal{E}(h,\varphi)$ along $X^{(n+1,n)}(K)\backslash X^{(n+1,n)}(\mathbb{A})$ (defined in (4.21)):

$$\mathcal{E}(h,\varphi) = \sum_{\alpha\in K} \int\limits_{K\backslash\mathbb{A}} \mathcal{E}\Big(\ell(te_{n+1,n})h,\varphi\Big)\psi(\alpha t)dt. \tag{5.10}$$

The contribution of all nontrivial characters in (5.10) (i.e. $\alpha \neq 0$) to (5.9) is zero, since, for $\alpha \in K^*$,

$$\int\limits_{K\backslash\mathbb{A}} \int\limits_{D^{(1,n+1)}(K)\backslash D^{(1,n+1)}(\mathbb{A})} \mathcal{E}\Big(\ell(te_{n+1,n})by^*\nu_0,\varphi\Big)\chi^{-1}(b)\psi(\alpha t)dtdb$$

contains an inner integral of the form

$$\int\limits_{N^{(n-1)}(K)\backslash N^{(n-1)}(\mathbb{A})} \mathcal{E}(bg,\varphi)\chi^{-1}_{(n-1),\alpha}(b)db$$

for $g \in D^{(1,n+1)}(\mathbb{A})Y^*_{0,n+1}(\mathbb{A})\nu_0$, which is identically zero, by (5.2). Thus, in (5.9), we get

$$\begin{aligned} R(\varphi) &= R_{n-1,n+2}(\varphi) \\ &= \int\limits_{Y^*_{n-1,n+2}(\mathbb{A})} \int\limits_{B^{(n-1,n+2)}(K)\backslash B^{(n-1,n+2)}(\mathbb{A})} \mathcal{E}(by^*\nu_0,\varphi)\chi^{-1}(b)dbdy^* \end{aligned} \tag{5.11}$$

where as in (4.28) $B^{(n-1,n+2)} = D^{(1,n+1)}X^{(n+1,n)}$, and χ, as usual, extends trivially to $X^{(n+1,n)}(\mathbb{A})$. Now we continue as in (5.6) ("replacing the $n-1$ coordinates of $\bigoplus_{i=1}^{n-1}\mathcal{Y}^{i,n+2}$ into $\chi_{n+1,1}$"). Assume (by induction) that $R(\varphi) = R_{i,n+2}(\varphi)$, for $i \leq n-1$. Write the Fourier expansion of $\mathcal{E}(h,\varphi)$ along $X^{(n+1,i)}(K)\backslash X^{(n+1,i)}(\mathbb{A})$ for $i = n-1, n-2, \ldots$:

$$\mathcal{E}(h,\varphi) = \sum_{\alpha\in K} \int\limits_{K\backslash\mathbb{A}} \mathcal{E}\Big(\ell(t(e_{n+1,i}+e_{2n+1-i,n}))h,\varphi\Big)\psi(\alpha t)dt \tag{5.12}$$

and substitute in (5.4) in $R_{i,n+2}(\varphi)$. The steps (5.7), (5.8) are valid here as well and we get that $R_{i,n+2}(\varphi) = R_{i-1,n+2}(\varphi)$, for $2 \leq i \leq n-1$. Thus

$$R(\varphi) = R_{1,n+2}(\varphi)\ .$$

As in (5.9), using the Fourier expansion (of $\mathcal{E}(h,\varphi)$) along $X^{(n+1,1)}(K)\backslash X^{(n+1,1)}(\mathbb{A})$, we get

$$R(\varphi) = \int\limits_{Y^*_{n-2,n+3}(\mathbb{A})} \int\limits_{D^{(1,n+2)}(K)\backslash D^{(1,n+2)}(\mathbb{A})} \mathcal{E}(by^*\nu_0,\varphi)\chi^{-1}(b)dbdy^* \tag{5.13}$$

where, as usual, for $b = \ell(x_{1,n+1})m(z)\overline{\ell}(y) \in D^{(1,n+2)}(\mathbb{A})$ ($x_{1,n+1} \in \chi_{1,n+1}(\mathbb{A})$, $z \in Z_{2n}(\mathbb{A})$, $y \in \mathcal{Y}_{1,n+2}(\mathbb{A})$),

$$\chi(b) = \chi(m(z)).$$

Now repeat the argument, which started at (5.10). Write the Fourier expansion of $\mathcal{E}(h,\varphi)$ along $X^{(n+2,n-1)}(K)\backslash X^{(n+2,n-1)}(\mathbb{A})$. The contribution of each nontrivial character in this expansion to (5.13) is zero, since it contains an inner integral of the form

$$\int_{N^{(n-2)}(K)\backslash N^{(n-2)}(\mathbb{A})} \mathcal{E}(bg,\varphi)\chi^{-1}_{(n-2),\alpha}(b)db$$

which is identically zero, for $\alpha \in K^*$, by (5.2). Thus, in (5.13) we get

$$\begin{aligned} R(\varphi) &= R_{n-2,n+3}(\varphi) \\ &= \int_{\mathcal{Y}^*_{n-2,n+3}(\mathbb{A})} \int_{B^{(n-2,n+3)}(K)\backslash B^{(n-2,n+3)}(\mathbb{A})} \mathcal{E}(by^*\nu_0,\varphi)\chi^{-1}(b)dbdy^*. \end{aligned} \tag{5.14}$$

Now we use Fourier expansions of $\mathcal{E}(h,\varphi)$ along $X^{(n+2,i)}(K)\backslash X^{(n+2,i)}(\mathbb{A})$ for $i = n-2, n-3, \dots$, repeat steps (5.7), (5.8) which are valid here and get $R(\varphi) = R_{n-2,n+3}(\varphi) = R_{n-3,n+3}(\varphi) = R_{n-4,n+3}(\varphi) = \cdots = R_{1,n+3}(\varphi)$. We continue in this manner until we get, for $j \geq n+2$,

$$R(\varphi) = R_{1,j}(\varphi).$$

As in (5.9), using the Fourier expansion along $X^{(j-1,1)}(K)\backslash X^{(j-1,1)}(\mathbb{A})$ we get

$$R(\varphi) = \int_{Y^*_{2n-j,j+1}(\mathbb{A})} \int_{D^{(1,j)}(K)\backslash D^{(1,j)}(\mathbb{A})} \mathcal{E}(by^*\nu_0,\varphi)\chi^{-1}(b)dbdy^*. \tag{5.15}$$

As before, using (5.2), the contribution to (5.15) of nontrivial characters in the Fourier expansion of $\mathcal{E}(h,\varphi)$ along $X^{(j,2n+1-j)}(K)\backslash X^{(j,2n+1-j)}(\mathbb{A})$ is zero. Thus, as in (5.11) (and (5.13)) $R(\varphi) = R_{2n-j,j+1}(\varphi)$. Now, we use Fourier expansions of $\mathcal{E}(h,\varphi)$ along $X^{(j,i)}(K)\backslash X^{(j,i)}(\mathbb{A})$, for $i = 2n-j, 2n-j-1, \dots$, repeat steps (5.7), (5.8), which are still valid, and get $R(\varphi) = R_{2n-j,j+1}(\varphi) = R_{2n-j-1,j+1}(\varphi) = \cdots = R_{1,j+1}(\varphi)$ and as in (5.15), we get

$$R(\varphi) = \int_{Y^*_{2n-j-1,j+2}(\mathbb{A})} \int_{D^{(1,j+1)}(K)\backslash D^{(1,j+1)}(\mathbb{A})} \mathcal{E}(by^*\nu_0,\varphi)\chi^{-1}(b)dbdy^* .$$

At the final step, we get

$$R(\varphi) = \int_{Y^*_{0,2n+1}(\mathbb{A})} \int_{D^{(1,2n)}(K)\backslash D^{(1,2n)}(\mathbb{A})} \mathcal{E}(by^*\nu_0,\varphi)\chi^{-1}(b)dbdy^* \tag{5.16}$$

where $Y^*_{0,2n+1} = \overline{\ell}(\mathcal{X}_0)$ ($\mathcal{X}_0$ is defined in (4.12)). Note that $D^{(1,2n)}$ is the standard maximal unipotent subgroup of Sp_{4n} and that

$$\chi\begin{pmatrix} z & * \\ 0 & z^* \end{pmatrix} = \chi(m(z)) \text{ , for } \quad z \in Z_{2n}(\mathbb{A}).$$

Thus, the inner integral of (5.16) reads as

$$\int_{Z_{2n}(K)\backslash Z_{2n}(\mathbb{A})} M(\varphi)(m(z)\overline{\ell}(x)\nu_0)\chi^{-1}(m)(z))dz \tag{5.17}$$

where

$$M(\varphi)(h) = \mathrm{Res}_{s=1}\, M(s)\varphi_{\eta,s}(h),$$

the residue at $s=1$ of the intertwining operator on $\rho_{\eta,s}$. Note that (5.17) is Eulerian, by uniqueness of the Whittaker model for η. Let us denote the r.h.s. of (5.17) by $M(\varphi)_{W,\chi}(\overline{\ell}(x)\nu_0)$. We record this as

Theorem 1. *Let η be an irreducible, self-dual, automorphic, cuspidal representation of $\mathrm{GL}_{2n}(\mathbb{A})$, such that $L^S(\eta,\Lambda^2,s)$ has a pole at $s=1$, and $L^S(\eta,\frac{1}{2})\neq 0$. Then*

$$\int\limits_{E_{2n}(K)\backslash E_{2n}(\mathbb{A})} \mathrm{Res}_{s=1}\, E(v,\varphi_{\eta,s})\overline{\psi}^{(2n)}(v)dv = \int\limits_{\mathcal{X}_0(\mathbb{A})} M(\varphi)_{W,\chi}(\overline{\ell}(x)\nu_0)dx\ . \tag{5.18}$$

Here

$$\chi(z)=\psi(z_{12}+z_{23}+\cdots+z_{n,n+1}-z_{n+1,n+2}-\cdots-z_{2n-1,2n})\ ,\quad z\in Z_{2n}(\mathbb{A})\ .$$

□

The next step is to relate the l.h.s. of (5.18) with $I(\varphi)$ in (5.1) (as we did in Theorem 4.2.2). Consider the ψ_n-Whittaker coefficient of $I(\varphi)$ along V_n,

$$\begin{aligned} S(\varphi) &= \int\limits_{V_n(K)\backslash V_n(\mathbb{A})} I(j_n(u)\cdot\varphi)\psi_n^{-1}(u)du \\ &= \int\limits_{V_n(K)\backslash V_n(\mathbb{A})}\ \int\limits_{\tilde{N}_K^{(n)}\backslash\tilde{N}_{\mathbb{A}}^{(n)}} \mathcal{E}(v\cdot j_n(u),\varphi)\widetilde{\chi}_{(n)}(v)\cdot\psi_n^{-1}(u)dvdu\ . \end{aligned}$$

As in (5.3), this time using conjugation by ω defined in (4.31), we get

$$S(\varphi)=\int\limits_{B_K\backslash B_{\mathbb{A}}} \mathcal{E}(v\cdot\omega,\varphi)\chi^{-1}(v)dv \tag{5.19}$$

where now B is defined by (4.32)-(4.35) and χ by (4.36). Introduce for $1\leq j\leq i\leq n-1$

$$S_{i,j}(\varphi)=\int\limits_{Y_{i,j}^*(\mathbb{A})}\ \int\limits_{B^{(i,j)}(K)\backslash B^{(i,j)}(\mathbb{A})} \mathcal{E}(vy^*\omega,\varphi)\chi_{i,j}^{-1}(v)dvdy^* \tag{5.20}$$

where $B^{(i,j)}$ is defined in (4.40) and $\chi_{i,j}$ is the character of $B^{(i,j)}(\mathbb{A})$ defined as in (4.36), i.e.

$$\begin{aligned} \chi_{i,j}(b)=\psi((b_{13}+b_{24})+(b_{35}+b_{47}) \\ +\cdots+(b_{2n-3,2n-1}+b_{2n-2,2n})+(b_{2n-1,2n+2}-b_{2n,2n+1}))\ . \end{aligned} \tag{5.21}$$

Here

$$Y_{ij}^*=\left\{m(T)\middle|\ \begin{array}{l} T\in T(n) \\ T_{rs}=0,\quad \text{if}\quad r<s,\ \text{or}\ s>2j-1,\ \text{or}\ s=2j-1\ \text{and}\ r>2i \end{array}\right\}.$$

$T(n)$ is defined in (4.34), (4.35). Note that $S_{n-1,1}(\varphi)=S(\varphi)$. Now write the Fourier expansion of $x\mapsto\mathcal{E}(xh,\varphi)$ along $X^{(j,i+1)}(K)\backslash X^{(j,i+1)}(\mathbb{A})$ (defined in (4.39)):

$$\mathcal{E}(h,\varphi)=\sum_{\alpha\in K}\int\limits_{K\backslash\mathbb{A}} \mathcal{E}(m(I_{2n}+te_{2j-1,2i+2})h,\varphi)\psi(\alpha t)dt. \tag{5.22}$$

We are exactly at the same situation as in (5.6), and we continue in the same way as in (5.7), (5.8), using $C^{(i,j)}$, $Y^{(i,j)}$ defined in (4.39), (4.41), to get

$$S(\varphi) = S_{n-1,n}(\varphi). \tag{5.23}$$

Since $B^{(n-1,n)} = E_{2n}$ and $\chi_{n-1,n} = \psi^{(2n)}$, we get from (5.23)

$$S(\varphi) = \int\limits_{Y^*_{n-1,n}(\mathbb{A})} \int\limits_{E_{2n}(K)\backslash E_{2n}(\mathbb{A})} \mathcal{E}(vy^*\omega, \varphi)\overline{\psi}^{(2n)}(v)dvdy^*. \tag{5.24}$$

Note that

$$Y^*_{n-1,n} = \left\{ m(T) \middle| \begin{array}{l} T \in T(n) \\ T \quad \text{is lower triangular} \end{array} \right\}. \tag{5.25}$$

Note that (5.24) is valid for any automorphic form on $\mathrm{Sp}_{4n}(\mathbb{A})$ (we did not use any property of $\mathcal{E}$). We record this in

Theorem 2. *For any automorphic form ξ on $\mathrm{Sp}_{4n}(\mathbb{A})$, we have*

$$\begin{aligned} &\int\limits_{V_n(K)\backslash V_n(\mathbb{A})} \int\limits_{\tilde{N}^{(n)}_K\backslash \tilde{N}^{(n)}_{\mathbb{A}}} \xi(v \cdot j_n(u))\widetilde{\chi}_{(n)}(v)\psi_n^{-1}(u)dvdu \\ &= \int\limits_{Y^*_{n-1,n}(\mathbb{A})} \int\limits_{E_{2n}(K)\backslash E_{2n}(\mathbb{A})} \xi(vy^*\omega)\overline{\psi}^{(2n)}(v)dvdy^* \ . \end{aligned} \tag{5.26}$$ $\square$

We conclude from (5.18) and (5.26)

Corollary. *Under the assumption of Theorem 5.1, we have*

$$\begin{aligned} &\int\limits_{V_n(K)\backslash V_n(\mathbb{A})} \int\limits_{\tilde{N}^{(n)}_K\backslash \tilde{N}^{(n)}_{\mathbb{A}}} \mathrm{Res}_{s=1}\, E(vj_n(u), \varphi_{\eta,s})\widetilde{\chi}_{(n)}(v)\psi_n^{-1}(u)dvdu \\ &= \int\limits_{Y^*_{n-1,n}(\mathbb{A})} \int\limits_{\mathcal{X}_0(\mathbb{A})} M(\varphi)_{W,\chi}(\overline{\ell}(x)\nu_0 y^*\omega)dxdy^* \end{aligned} \tag{5.27}$$

where χ is defined in Theorem 5.1, $\mathcal{X}_0$ in (4.12) and $Y^*_{n-1,n}$ in (5.18).

To conclude that $\sigma_n(\eta) \neq 0$, which is equivalent to $I(\varphi) \not\equiv 0$ (in (5.1)), it remains to prove the following two lemmas.

Lemma 1. *For any automorphic representation π of $\mathrm{Sp}_{4n}(\mathbb{A})$*

(5.28)

$$\int\limits_{Y^*_{n-1,n}(\mathbb{A})} \int\limits_{E_{2n}(K)\backslash E_{2n}(\mathbb{A})} \xi(vy^*)\overline{\psi}^{(2n)}(v)dvdy^* \not\equiv 0, \quad \textit{as} \quad \xi \quad \textit{varies in} \quad V_\pi,$$

if and only if

$$\int\limits_{E_{2n}(F)\backslash E_{2n}(\mathbb{A})} \xi(v)\overline{\psi}^{(2n)}(v)dv \not\equiv 0, \quad \textit{as} \quad \xi \quad \textit{varies in} \quad V_\pi \ .$$

Lemma 2. *Let L be an $\mathrm{Sp}_{4n}(\mathbb{A})$-invariant space of smooth functions on $Z_K X_{\mathbb{A}} \backslash \mathrm{Sp}_{4n}(\mathbb{A})$ where X is the unipotent radical of the Siegel parabolic subgroup and $Z = m(Z_{2n})$. Assume that the representation of $\mathrm{Sp}_{4n}(\mathbb{A})$ on L is of moderate growth, and the elements of L satisfy*

$$f(m(z)g) = \chi(m(z))f(g) \ ,$$

for $z \in Z_{2n}(\mathbb{A})$. Then

$$\int\limits_{\mathcal{X}_0(\mathbb{A})} f(\overline{\ell}(x))dx \not\equiv 0 \ , \quad \text{as} \quad f \quad \text{varies in} \quad L \tag{5.29}$$

(notation of (5.27)).

Proof of Lemma 1. Note that $Y^*_{n-1,n}$ is abelian. Write it as the product $Y^*_{n-1,n} = \prod\limits_{i=1}^{n-1} K_i$, where

$$K_i = \{k_i(t_1, \dots, t_i) = m(I_{2n} + \sum_{j=1}^{i} t_j e_{2i,2j-1})\} \ .$$

Define

$$R_s = \{r_s(t_1, \dots, t_{s-1}) = m(I_{2n} + \sum_{i=1}^{s-1} t_i e_{2i-1,2s})\} \ .$$

Denote the inner dv-integral of (5.28) by $\varphi_\xi(y^*)$. For $1 \le i \le n-1$, $K^i = \prod\limits_{\ell=1}^{i} K_\ell$ put

$$a_i(\xi) = \int\limits_{K^i(\mathbb{A})} \varphi_\xi(y)dy.$$

Note that (5.28) is $a_{n-1}(\xi)$. By [D.M.] (applied at the infinite places) we can write

$$\xi = \sum_\alpha \int\limits_{\mathbb{A}^i} \phi_\alpha(x_1, \dots, x_i)\pi(r_{i+1}(x_1, \dots, x_i)) \cdot (\xi_\alpha)d(x_1, \dots, x_i)$$

where $\phi_\alpha \in S(\mathbb{A}^i)$ and $\xi_\alpha \in V_\pi$. We get

$$a_i(\xi) = \sum_\alpha \int\limits_{\mathbb{A}^i} \int\limits_{K^i(\mathbb{A})} \phi_\alpha(x_1, \dots, x_i)\varphi_{\xi_\alpha}(y \cdot r_{i+1}(x_1, \dots, x_i))dyd(x_1, \dots, x_i) \ .$$

Write $K^i = K^{i-1} \cdot K_i$ and $y = y' k_i(t_1, \dots, t_i)$ accordingly. Note that

$$y r_{i+1}(x_1, \dots, x_i)y^{-1} \in E_{2n}$$

and

$$\psi^{(2n)}(y r_{i+1}(x_1 \cdots x_i)y^{-1}) = \psi(\sum_{i=1}^{n} x_i t_i) \ .$$

We get

$$a_i(\xi) = \sum_\alpha \int\limits_{K^{i-1}(\mathbb{A})} \int\limits_{\mathbb{A}^i} \widehat{\phi}_\alpha(t_1, \dots, t_i)\varphi_{\xi_\alpha}(y' k_i(t_1, \dots, t_i)) dy' d(t_1, \dots, t_i)$$
$$= a_{i-1}(\xi')$$

where $\xi' = \sum\limits_\alpha \int\limits_{\mathbb{A}^i} \widehat{\phi}_\alpha(t_1, \dots, t_i)\pi(k_i(t_1, \dots, t_i)) \cdot (\xi_\alpha) d(t_1, \dots, t_i) \cdot \xi'$ varies over all of V_π as we vary ϕ_α and ξ_α. Thus $a_i(\xi) \not\equiv 0$ if and only if $a_{i-1}(\xi) \not\equiv 0$, where $a_0(\xi) = \varphi_\xi(1)$. Thus, $a_{n-1}(\xi) \not\equiv 0$ if and only if $\varphi_\xi(1) \not\equiv 0$. $\square$

Proof of Lemma 2. (The proof here is similar to part of the proof of Theorem 1.3.) Denote

(5.30) $K_{i+1} =$
$$\begin{cases} \Big\{k_{i+1}(t_1, \dots, t_i) = \overline{\ell}\Big(\sum\limits_{j=1}^{i} t_j(e_{j,i+1} + e_{2n-i,2n+1-j})\Big)\Big\}, & \text{for } i \le n, \\ \Big\{k_{i+1}(t_1, \dots, t_{2n-i}) = \overline{\ell}\Big(\sum\limits_{j=1}^{2n-i-1} t_j(e_{j,i+1} + e_{2n-i,2n+1-j}) + t_{2n-i}e_{2n-i,i+1}\Big)\Big\}, & \\ & \text{for } n+1 \le i \le 2n-1. \end{cases}$$

Then

$$\overline{\ell}(\mathcal{X}_0) = \prod_{\ell=2}^{2n} K_\ell \ .$$

Put

$$K^i = \prod_{\ell=i+1}^{2n} K_\ell \ .$$

Define

(5.31)
$$R_s = \begin{cases} \Big\{r_s(x_1, \dots, x_s) = \ell\Big(\sum\limits_{j=1}^{s} x_j(e_{s,j} + e_{2n+1-j,2n+1-s})\Big)\Big\}, & \text{for } s \le n, \\ \Big\{r_s(x_1, \dots, x_{2n-s}) & \\ \qquad = \ell\Big(\sum\limits_{j=1}^{2n-s} x_j(e_{s,j} + e_{2n+1-j,2n+1-s})\Big)\Big\}, & \text{for } n+1 \le s \le 2n. \end{cases}$$

Put, for $1 \le i \le 2n-1$ and $f \in L$,

$$a_i(f) = \int\limits_{K^i(\mathbb{A})} f(y) dy \ .$$

Note that the integral (5.29) is $a_1(f)$. Write, as in the previous lemma,

$$f = \sum_\alpha \int\limits_{R_i(\mathbb{A})} \phi_\alpha(r) r \cdot f_\alpha dk$$

for $\phi_\alpha \in S(R_i(\mathbb{A}))$, $f_\alpha \in L$. Write $y \in K^i(\mathbb{A})$ in the form $y = y' \cdot k$, where $y' \in K^{i+1}(\mathbb{A})$ and $k \in K_{i+1}(\mathbb{A})$, of the form (5.30). Write $r \in R_i(\mathbb{A})$ in the form

(5.31). Then $yry^{-1} \in V_{2n}(\mathbb{A})$ and

$$f_\alpha((yry^{-1}) \cdot y) = \psi(\pm \sum x_j t_j) f_\alpha(y)$$

($\pm$ according to whether $i > n$ or $i \leq n$). We are now at exactly the same situation of the last lemma, and we conclude that $a_i(f) \not\equiv 0$ if and only if $a_{i+1}(f) \not\equiv 0$, for $i = 1, \ldots, 2n-1$, where $a_{2n}(f) = f(1)$. This proves the lemma, and completes the proof of the (global) nonvanishing of $\sigma_n(\tau)$. $\square$

6. Appendix

6.1. Proof of the isomorphism (1.14). We use the notation of Section 1.1.

Let τ be an irreducible, generic representation of $\mathrm{GL}_m(F)$. Let σ be an irreducible, generic, genuine representation of $\widetilde{\mathrm{Sp}}_{2k}(F)$. We assume that $k < m$. Each side of the functional equation (1.10) defines an $\widetilde{\mathrm{Sp}}_{2k}(F)$-invariant pairing between σ and $J_{\mathcal{H}_k}\Big(J_{N_{m,k+1},\chi_k^{-1}}(\rho_{\tau,s}) \otimes \omega_\psi^{(k)}\Big)$. Then consider

$$P_m \backslash \mathrm{Sp}_{2m}(F)/j_{m,k}\Big(\mathrm{Sp}_{2k}(F) \cdot \mathcal{H}_k\Big) N_{m,k+1}. \tag{6.1}$$

Note that $j_{m,k}(\mathcal{H}_k) N_{m,k+1} = N_{m,k}$. By Lemma 4.1 in [G.R.S.4], representatives of (6.1) have the form

$$\gamma_{m,i,w} = \gamma_{m,i} \cdot m(w)\ , \quad k \leq i \leq m,$$

where

$$\gamma_{m,i} = \begin{pmatrix} & & -I_{m-i} \\ & I_{2i} & \\ I_{m-i} & & \end{pmatrix}$$

and

$$w \in W_{\mathrm{GL}_{m-i}} \times W_{\mathrm{GL}_{i-k}} \backslash W_{\mathrm{GL}_{m-k}};$$

here W_{GL_r} denotes the Weyl group of GL_r. Denote

$$\mathcal{O}_{i,w}^{(m,k)} = P_m \gamma_{m,i,w} j_{m,k}(\mathrm{Sp}_{2k}(F)) N_{m,k}.$$

Note that $\mathcal{O}_{k,1}^{(m,k)}$ is the unique open orbit. For E a union of orbits $\mathcal{O}_{i,w}^{(m,k)}$, let $S(E, P_m, \tau_s)$ denote the space of smooth functions φ on E, with values in the Whittaker model $W(\tau, \psi'_m)$ (see (1.2)), such that, for $g \in \mathrm{Sp}_{2m}(F)$, $a, r \in \mathrm{GL}_m(F)$,

$$\varphi\left(\begin{pmatrix} a & * \\ 0 & a^* \end{pmatrix} g, r\right) = |\det a|^{s+\frac{m}{2}} \varphi(g, ra)\ ,$$

and such that the support of φ is compact modulo P_m.

We may arrange the orbits in a sequence

$$\mathcal{O}_{m,1}^{(m,k)} = P_m j_{m,k}(\mathrm{Sp}_{2k}(F)) = \Omega_1, \Omega_2, \ldots, \Omega_L = \mathcal{O}_{k,1}^{(m,k)}\ ,$$

such that $F_i = \bigcup_{j=1}^{i} \Omega_j$ is closed in $\mathrm{Sp}_{2m}(F)$. We then have an exact sequence

$$0 \longrightarrow S(\Omega_{i+1}, P_m, \tau_s) \xrightarrow{e} S(F_{i+1}, P_m, \tau_s) \xrightarrow{r} S(F_i, P_m, \tau_s) \longrightarrow 0. \tag{6.2}$$

In (6.2), the map e is the natural embedding, and the map r is the restriction to F_i. Applying the Jacquet functor $J_{N_{m,k+1},\chi_k^{-1}}$ to (6.2), we get the exact sequence

$$\begin{aligned} 0 \to J_{N_{m,k+1},\chi_k^{-1}}(S(\Omega_{i+1},P_m,\tau_s)) &\to J_{N_{m,k+1},\chi_k^{-1}}(S(F_{i+1},P_m,\tau_s)) \\ &\to J_{N_{m,k+1},\chi_k^{-1}}(S(F_i,P_m,\tau_s)) \to 0\ . \end{aligned} \tag{6.3}$$

Note that tensoring with $\omega_\psi^{(k)}$ (acting on $S(F^k)$) preserves exactness. Applying $\otimes S(F^k)$ to (6.3) and then $J_{\mathcal{H}_k}$ gives the exact sequence

$$\begin{aligned} 0 &\to J_{\mathcal{H}_k}\Big(J_{N_{m,k+1},\chi_k^{-1}}(S(\Omega_{i+1},P_m,\tau_s)) \otimes S(F^k)\Big) \\ &\to J_{\mathcal{H}_k}\Big(J_{N_{m,k+1},\chi_k^{-1}}(S(F_{i+1},P_m\tau_s)) \otimes S(F^k)\Big) \\ &\to J_{\mathcal{H}_k}\Big(J_{N_{m,k+1},\chi_k^{-1}}(S(F_i,P_m,\tau_s)) \otimes S(F^k)\Big) \to 0\ . \end{aligned}$$

This reduces the study of $J_{\mathcal{H}_k}\Big(J_{N_{m,k+1},\chi_k^{-1}}(V_{\rho_{\tau,s}} \otimes S(F^k))\Big)$ to the study of the various $J_{\mathcal{H}_k}\Big(J_{N_{m,k+1},\chi_k^{-1}}(S(\mathcal{O}_{i,w}^{(m,k)},P_m,\tau_s) \otimes S(F^k))\Big)$, $k \le i \le m$. We have

$$S(\mathcal{O}_{i,w}^{(m,k)},P_m,\tau_s) \simeq \mathrm{Ind}_{R_{i,w}}^{c^{\mathrm{Sp}_{2k}(F)\cdot N_{m,k}}}(\delta_{P_m}^{1/2}\tau_s)^{\gamma_{m,i,w}}$$

as $\mathrm{Sp}_{2k}(F)\cdot N_{m,k}$-modules. The induction is not normalized. Here

$$\begin{aligned} &R_{i,w} = \gamma_{m,i,w}^{-1}P_m\gamma_{m,i,w} \cap \mathrm{Sp}_{2k}(F)\cdot N_{m,k}, \\ &(\delta_{P_m}^{1/2}\tau_s)^{\gamma_{m,i,w}}(r) = \delta_{P_m}^{1/2}\tau_s(\gamma_{m,i,w} r \gamma_{m,i,w}^{-1})\ , \quad r \in R_{i,w}, \\ &(\delta_{P_m}^{1/2}\tau_s)\begin{pmatrix} a & x \\ 0 & a^* \end{pmatrix} = |\det a|^{s+\frac{m}{2}}\,\tau(a)\ . \end{aligned}$$

We claim that $J_{N_{m,k+1},\chi_k^{-1}}\left(\mathrm{Ind}_{R_{i,w}}^{c^{\mathrm{Sp}_{2k}(F)\cdot N_{m,k}}}(\delta_{P_m}^{1/2}\tau_s)^{\gamma_{m,i,w}}\right) = 0$, for $i \ge k+1$ and $w \ne I_m$. Indeed, by Lemma 4.3 in [G.R.S.4] there is a simple root subgroup $x(t)$ inside $N_{m,k+1}$, such that $\gamma_{m,i,w}^{-1}x(t)\gamma_{m,i,w}$ lies in the unipotent radical of P_m. This shows that $x(t) \in R_{i,w} \cap N_{m,k+1}$, $(\delta_{P_m}^{1/2}\tau_s)^{\gamma_{m,i,w}}(x(t)) = id$, while $\chi_k(x(t)) = \psi(t)$.

Put $\gamma'_{m,i} = \gamma_{m,i,1}$. We now show that, for $i \ge k+1$,

$$J_{\mathcal{H}_k}\left(J_{N_{m,k+1},\chi_k^{-1}}\left(\mathrm{Ind}_{R_{i,1}}^{c^{\mathrm{Sp}_{2k}(F)\cdot N_{m,k}}}(\delta_{P_m}^{1/2}\tau_s)^{\gamma'_{m,i}}\right) \otimes \omega_\psi^{(k)}\right) = 0\ . \tag{6.4}$$

We may realize the inner space in the last Jacquet module as

$$S\Big(F^k; J_{N_{m,k+1},\chi_k^{-1}}\Big(\mathrm{Ind}_{R_{i,1}}^{c^{\mathrm{Sp}_{2k}(F)\cdot N_{m,k}}}(\delta_{P_m}^{1/2}\tau_s)^{\gamma'_{m,i}}\Big)\Big)$$

– the space of $J_{N_{m,k+1},\chi_k^{-1}}\Big(\mathrm{Ind}_{R_{i,1}}^{c^{\mathrm{Sp}_{2k}(F)\cdot N_{m,k}}}(\delta_{P_m}^{1/2}\tau_s)^{\gamma'_{m,i}}\Big)$-vector valued Schwartz-Bruhat functions on F^k. We have

$$R_{i,1} = \left\{ \begin{pmatrix} z_1 & 0 & 0 & c & y & 0 \\ & z_2 & a & b & x & y' \\ & & r & d & b' & c' \\ & & & r^* & a' & 0 \\ & & & & z_2^* & 0 \\ & & & & & z_1 \end{pmatrix} \in \mathrm{Sp}_{2m}(F) \middle| \begin{matrix} z_1 \in Z_{m-i} \\ z_2 \in Z_{i-k} \end{matrix} \right\}.$$

Thus the functions f in $\mathrm{Ind}_{R_{i,1}}^{c^{\mathrm{Sp}_{2k}(F)\cdot N_{m,k}}}(\delta_{P_m}^{1/2}\tau_s)^{\gamma'_{m,i}}$ are determined by the values

$$f\left(\begin{pmatrix} I_{m-i} & e & t & 0 & 0 & r \\ & I_{i-k} & 0 & 0 & 0 & 0 \\ & & I_k & 0 & 0 & 0 \\ & & & I_k & 0 & t' \\ & & & & I_{i-k} & e' \\ & & & & & I_{n-i} \end{pmatrix} j_{m,k}(g)\right), \quad g \in \mathrm{Sp}_{2k}(F),$$

and

$$f\left(\begin{pmatrix} z_1 & 0 & 0 & c & b & 0 \\ & z_2 & a & b & x & b' \\ & & r & d & b' & c' \\ & & & r^* & a' & 0 \\ & & & & z_2^* & 0 \\ & & & & & z_1^* \end{pmatrix} h\right) = |\det r|^{s+\frac{m}{2}}\tau\begin{pmatrix} z_1^* & & \\ -y' & z_2 & a \\ -c' & 0 & r \end{pmatrix} f(h)\ .$$

Let

$$B_{i,k} = \left\{ \begin{pmatrix} z_1 & 0 & 0 \\ u & z_2 & a \\ v & 0 & I_k \end{pmatrix} \middle| \begin{matrix} z_1 \in Z_{m-i}, z_2 \in Z_{i-k} \\ \text{last row of } a \text{ is zero} \end{matrix} \right\}$$

and consider the character $\psi_{i,k}$ of $B_{i,k}$ defined by $\psi^{-1}(z_1)\psi(z_2)$. Then it is easy to see that there is an embedding

$$J_{N_{m,k+1},\chi_k^{-1}}\left(\mathrm{Ind}_{R_{i,1}}^{c^{\mathrm{Sp}_{2k}(F)\cdot N_{m,k}}}(\delta_{P_m}^{1/2}\tau_s)^{\gamma'_{m,i}}\right) \hookrightarrow \mathrm{Ind}_{P_k}^{\mathrm{Sp}_{2k}(F)} J_{B_{i,k},\psi_{i,k}^{-1}}(\tau)\cdot|\det\cdot|^{s-\frac{m}{2}+i}\ .$$

We regard $J_{B_{ik},\psi_{i,k}^{-1}}(\tau)$ as a $\mathrm{GL}_k(F)$-module, where $\mathrm{GL}_k(F)$ is embedded in $\mathrm{GL}_m(F)$ by $g \mapsto \begin{pmatrix} I_{m-k} & \\ & g \end{pmatrix}$. (The inductions are not normalized.) The embedding $\overline{p}$ is given through its pull-back to $\mathrm{Ind}_{R_{i,1}}^{c^{\mathrm{Sp}_{2k}(F)\cdot N_{m,k}}}(\delta_{P_m}^{1/2}\tau_s)^{\gamma'_{m,i,i}}$ by

$$p(f)(g) = \int \psi(e_{m-i,1}) j_{B_{i,k}} \cdot \left(f\left(\begin{pmatrix} I_{n-i} & e & t & 0 & 0 & r \\ & I_{i-k} & 0 & 0 & 0 & 0 \\ & & I_k & 0 & 0 & 0 \\ & & & I_k & 0 & t' \\ & & & & I_{i-k} & e' \\ & & & & & I_{m-i} \end{pmatrix} j_{m,k}(g)\right)\right) d(e,t,r)\ .$$

The action of the center of $\mathcal{H}_k$ on $S\left(F^k; J_{N_{m,k+1},\chi_k^{-1}}\left(\mathrm{Ind}_{R_{i,1}}^{c^{\mathrm{Sp}_{2k}(F)\cdot N_{m,k}}}(\delta_{P_m}^{1/2}\tau_s)^{\gamma'_{m,i}}\right)\right)$ is by

$$((0,0;t)\cdot\varphi)(x) = \psi(t) j_{m,k}(0,0;t)\cdot(\varphi(x)).$$

It is easy to see that

$$\overline{p}(j_{m,k}(0,0;t)\cdot\xi) = \overline{p}(\xi)$$

for $t \in F,\ \xi \in J_{N_{m,k+1},\chi_k^{-1}}\left(\mathrm{Ind}_{R_{i,1}}^{\mathrm{Sp}_{2k}(F)\cdot N_{m,k}}(\delta_{P_m}^{1/2}\tau_s)^{\gamma'_{m,i}}\right)$. Since $\overline{p}$ is injective, we find that

$$j_{m,k}(0,0;t)\cdot\xi = \xi,$$

and so

$$(0,0;t)\cdot\varphi=\psi(t)\varphi$$

for

$$\begin{aligned} t\in F\ ,\quad \varphi\in S\Big(F^k; J_{N_{m,k+1},\chi_k^{-1}}\Big(\mathrm{Ind}_{R_{i,1}}^{c^{\mathrm{Sp}_{2k}(F)\cdot N_{m,k}}}(\delta_{P_m}^{1/2}\tau_s)^{\gamma'_{m,i}}\Big)\Big)\\ \simeq J_{N_{m,k+1},\chi_k^{-1}}\Big(\mathrm{Ind}_{R_{i,1}}^{c^{\mathrm{Sp}_{2k}(F)\cdot N_{m,k}}}(\delta_{P_m}^{1/2}\tau_s)^{\gamma'_{m,i}}\Big)\otimes\omega_\psi^{(k)};\end{aligned}$$

this implies (6.4). Denote $\mathcal{O}_{m,k}=\mathcal{O}_{k,1}^{(m,k)}$. We showed that the natural embedding of $J_{\mathcal{H}_k}\Big(J_{N_{m,k+1},\chi_k^{-1}}(S(\mathcal{O}_{m,k},P_m\tau_s))\otimes\omega_\psi^{(k)}\Big)$ in $J_{\mathcal{H}_k}\Big(J_{N_{m,k+1},\chi_k^{-1}}(V_{\rho_{\tau,s}})\otimes\omega_\psi\Big)$ is surjective. This is the isomorphism (1.14). □

6.2. The local functional equation for $\widetilde{\mathrm{Sp}}_{2k}(F)\times\mathrm{GL}_m(F)$ $(k<m)$. We keep the notation of Section 1.1 and of the previous section. Here we show that, except for a finite number of values of q^{-s}, there is, up to scalar multiples, a unique $\widetilde{\mathrm{Sp}}_{2k}(F)$-invariant pairing between σ and

$$J_{\mathcal{H}_k}\Big(J_{N_{m,k+1},\chi_k^{-1}}(V_{\rho_{\tau,s}})\otimes\omega_\psi^{(k)}\Big)\cong J_{\mathcal{H}_k}\Big(J_{N_{m,k+1},\chi_k^{-1}}(S(\mathcal{O}_{m,k},P_m,\tau_s))\otimes\omega_\psi^{(k)}\Big)\ .$$

This implies the functional equation (1.10). It is convenient to choose the representative $\gamma_{m,k}$ (see (1.6)) for the orbit $\mathcal{O}_{m,k}$ (rather than $\gamma_{m,k,1}$). Denote the stabilizer by R_k. We have

$$S(\mathcal{O}_{m,k},P_m,\tau_s)\simeq\mathrm{Ind}_{R_k}^{c^{\mathrm{Sp}_{2k}(F)N_{m,k}}}(\delta_{P_m}^{1/2}\tau_s)^{\gamma_{m,k}}$$

(the induction is not normalized). Note that

$$R_k=\left\{\begin{pmatrix} z&0&c&0\\ &I_k&0&c'\\ &&I_k&0\\ &&&z^*\end{pmatrix}\in\mathrm{Sp}_{2m}(F)\Big| z\in Z_{m-k}\right\}\cdot j_{m,k}(P_k),$$

$$(\delta_{P_m}^{1/2}\tau_s)^{\gamma_{m,k}}\begin{pmatrix} I_{m-k}&&&\\ &a&*&\\ &&a^*&\\ &&&I_{m-k}\end{pmatrix}=|\det a|^{s+\frac{m}{2}}\tau\begin{pmatrix} a&\\ &I_{m-k}\end{pmatrix},$$

$$(\delta_{P_m}^{1/2}\tau_s)^{\gamma_{m,k}}\begin{pmatrix} z&0&c&0\\ &I_k&0&c'\\ &&I_k&0\\ &&&z^*\end{pmatrix}=\tau\begin{pmatrix} I_k&-c'\\ &z^*\end{pmatrix}\ ,\quad z\in Z_{m-k}\ .$$

Let

$$B_k=\left\{\begin{pmatrix} I_k&y\\ &z\end{pmatrix}\Big|\begin{matrix} z\in Z_{m-k},\\ \text{the first column of } y \text{ is zero}\end{matrix}\right\}$$

and consider the character η_k of B_k defined by $\eta_k\begin{pmatrix} I_k&y\\ &z\end{pmatrix}=\psi^{-1}(z)$. As before, we have (with unnormalized induction notation)

(6.5)

$$J_{N_{m,k+1},\chi_k^{-1}}\Big(\mathrm{Ind}_{R_k}^{c^{\mathrm{Sp}_{2k}(F)\cdot N_{m,k}}}(\delta_{P_m}^{1/2}\tau_s)^{\gamma_{m,k}}\Big)\cong\mathrm{Ind}_{P_k\cdot\mathcal{Y}_k}^{c^{\mathrm{Sp}_{2k}(F)\cdot\mathcal{H}_k}}j_{B_k,\eta_k}(\tau)\cdot|\det\cdot|^{s-\frac{m}{2}+k}$$

where $\begin{pmatrix} a & * \\ & a^* \end{pmatrix} \cdot (0,y;0)$ in $P_k \cdot \mathcal{Y}_k$ acts on $j_{B_k,\eta_k}(\tau)$ through

$$\tau\left(\begin{pmatrix} a & \\ & I_{m-k} \end{pmatrix}\begin{pmatrix} I_k & -y' & 0 \\ & 1 & 0 \\ & & I_{m-k-1} \end{pmatrix}\right).$$

The isomorphism $\overline{p}$ of (6.5) is given through its pull back p to

$$\mathrm{Ind}_{R_k}^{c^{\mathrm{Sp}_{2k}(F)\cdot N_{m,k}}}(\delta_{P_m}^{1/2}\tau_s)^{\gamma_{m,k}}$$

by

$$\begin{aligned} &p(f)\Big(j_{m,k}((u,0;t)\cdot g)\Big) \\ &= \int j_{B_k,\eta_k}\left(f\left(\begin{pmatrix} I_{m-k-1} & 0 & x & 0 & e_1 & e_2 \\ & 1 & u & 0 & t & e_1' \\ & & I_k & 0 & 0 & 0 \\ & & & I_k & u' & x' \\ & & & & 1 & 0 \\ & & & & & I_{m-k-1} \end{pmatrix} j_{m,k}(g)\right)\right) d(x,e). \end{aligned}$$

So far, we showed that

$$J_{N_{m,k+1},\chi_k^{-1}}(\rho_{\tau,s}) \cong \mathrm{Ind}_{P_k\cdot\mathcal{Y}_k}^{c^{\mathrm{Sp}_{2k}(F)\cdot\mathcal{H}_k}}\left(j_{B_k,\eta_k}(\tau)\cdot|\det\cdot|^{s-\frac{m}{2}+k}\right)$$

(as $\mathrm{Sp}_{2k}(F)\mathcal{H}_k$-modules).

Extend σ trivially to $\mathcal{H}_k$. Denote the extension by σ_1. We have

$$\begin{aligned} (6.6) \quad & Bil_{\widetilde{\mathrm{Sp}}_{2k}(F)}\Big(\sigma, J_{\mathcal{H}_k}(\mathrm{Ind}_{P_k\mathcal{Y}_k}^{c^{\mathrm{Sp}_{2k}(F)\cdot\mathcal{H}_k}}(j_{B_k,\eta_k}(\tau)\cdot|\det\cdot|^{s-\frac{m}{2}+k}) \otimes \omega_\psi^{(k)})\Big) \\ &\simeq Bil_{\widetilde{\mathrm{Sp}}_{2k}(F)\mathcal{H}_k}\Big(\sigma_1, \mathrm{Ind}_{P_k\mathcal{Y}_k}^{c^{\widetilde{\mathrm{Sp}}_{2k}(F)\cdot\mathcal{H}_k}}\Big(j_{B_k,\eta_k}(\tau)\cdot|\det\cdot|^{s-\frac{m}{2}+k}\otimes \omega_\psi^{(k)}\Big|_{\widetilde{P}_k\cdot\mathcal{Y}_k}\Big)\Big) \\ &\simeq Bil_{\widetilde{P}_k\cdot\mathcal{Y}_k}\Big(\sigma_1, j_{B_k,\eta_k}(\tau)\cdot|\det\cdot|^{s-\frac{m}{2}+k}\otimes\omega_\psi^{(k)}\Big|_{\widetilde{P}_k\cdot\mathcal{Y}_k}\Big). \end{aligned}$$

Our objective is then to show that the space (6.6) is at most one-dimensional, outside a finite set of values of q^{-s}. Regard an element A of the space (6.6) as a trilinear form on $V_\sigma \times V_\tau \times S(F^k)$, satisfying

$$\begin{aligned} (6.7) \quad & A\left(\sigma\langle\begin{pmatrix} a & x \\ & a^* \end{pmatrix},\varepsilon\rangle\xi, \quad \tau\left(\begin{pmatrix} a & \\ & z \end{pmatrix}\begin{pmatrix} I_k & -y' & e \\ & 1 & 0 \\ & & I_{m-k-1} \end{pmatrix}\right)v,\right. \\ &\qquad\qquad \left.\omega_\psi\left(\langle\begin{pmatrix} a & x \\ 0 & a^* \end{pmatrix},\varepsilon\rangle(0,y;0)\right)\phi\right) \\ &\qquad\qquad\qquad = \psi(z)|\det a|^{-s+\frac{m}{2}-k}A(\xi,v,\phi)\ , \end{aligned}$$

for $z \in Z_{m-k}$. Consider the exact sequence

$$0 \longrightarrow S(F^k\backslash\{0\}) \xrightarrow{i} S(F^k) \xrightarrow{j} \mathbb{C} \longrightarrow 0$$

where i is the natural embedding and $j(\phi) = \phi(0)$. If $A(\xi,v,\phi)$ vanishes on the image of i, for all ξ and v, then, since

$$\omega_\psi\left(\langle\begin{pmatrix} a & x \\ 0 & a^* \end{pmatrix},\varepsilon\rangle(0,y,0)\right)\phi(0) = \varepsilon\gamma_{\psi,\det a}|\det a|^{1/2}\phi(0)\ ,$$

A defines a pairing $\overline{A}$ on $V_\sigma \times V_\tau$ such that

$$(6.8)\quad \overline{A}\left(\sigma\left(\langle\begin{pmatrix} a & x \\ & a^* \end{pmatrix}, \varepsilon\rangle\right)\xi, \tau\begin{pmatrix} a & b \\ & z \end{pmatrix} v\right) = \varepsilon\gamma_{\psi,\det a}^{-1}|\det a|^{s+\frac{m-1}{2}-k}\psi(z)\overline{A}(\xi, v)$$

for $z \in Z_{m-k}$.

Note that if σ is supercuspidal, then, by (6.8), $\overline{A}$ must be zero. In general, (6.8) shows that, if $\overline{A}$ is nonzero, then σ pairs into the representation of $\widetilde{\mathrm{Sp}}_{2k}(F)$ induced from $\widetilde{P}_k$ and $\gamma_{\psi,\det}\cdot|\det\cdot|^{s-\frac{m-1}{2}+k}$ times the derivative (in the sense of [B.Z.]) of τ obtained by the Jacquet module of τ, with respect to $\{\begin{pmatrix} I_k & b \\ & z \end{pmatrix} | z \in Z_{m-k}\}$ and the character $\begin{pmatrix} I_k & b \\ & z \end{pmatrix} \mapsto \psi^{-1}(z)$. This implies that q^s lies in a certain finite set $S^0_{\sigma,\tau}$ (which depends on σ and τ). Outside this finite set, $\overline{A}$ must be zero, and then A is fully determined by its restriction to $V_\sigma \times V_\tau \times S(F^k\backslash\{0\})$. Note that the representation of $\widetilde{P}_k \cdot \mathcal{Y}_k$, acting through $\omega_\psi^{(k)}$ on $S(F^k\backslash\{0\})$, is isomorphic to $\mathrm{Ind}_{\widetilde{P}^0_{k-1}\cdot\mathcal{Y}_k}^{c\widetilde{P}_k\mathcal{Y}_k} |\det\cdot|^{1/2}\gamma_{\psi,\det}\cdot c_\psi$ (nonnormalized induction), where

$$P^0_{k-1} = \left\{\begin{pmatrix} a & x \\ 0 & a^* \end{pmatrix} \in P_k | a = \begin{pmatrix} b & * \\ 0 & 1 \end{pmatrix}, b \in \mathrm{GL}_{k-1}(F)\right\},$$

$$c_\psi\left(\begin{pmatrix} I_k & x \\ & I_k \end{pmatrix}\cdot(0, y; 0)\right) = \psi(x_{k1} + 2y_1)\ .$$

Thus (6.7) and Frobenius reciprocity imply that for $q^{-s} \notin S^0_{\sigma,\tau}$, A defines a bilinear form $\widetilde{A}$ on $V_\sigma \times V_\tau$, satisfying

$$\begin{aligned} &\widetilde{A}\left(\sigma\langle\begin{pmatrix} a & x \\ 0 & a^* \end{pmatrix}, \varepsilon\rangle\xi, \tau\left(\begin{pmatrix} a & \\ & I_{n-k} \end{pmatrix}\begin{pmatrix} I_k & e \\ & z \end{pmatrix}\right)v\right) \\ (6.9)\qquad &= \varepsilon\gamma_{\psi,\det a}^{-1}|\det a|^{-s+\frac{m-1}{2}+k}\psi^{-1}(x_{k1})\psi'_m\begin{pmatrix} I_k & e \\ & z \end{pmatrix}\widetilde{A}(\xi, v) \end{aligned}$$

for $\begin{pmatrix} a & x \\ 0 & a^* \end{pmatrix} \in P^0_{k-1}$. (See (1.2).) Note that $\widetilde{A}$ factors through the product $J_\sigma \times J_\tau$ of Jacquet modules of σ, with respect to $\left\{\begin{pmatrix} I_k & x \\ & I_k \end{pmatrix}\right\}$ and $\psi(x_{k1})$, and of τ, with respect to $\left\{\begin{pmatrix} I_k & e \\ & z \end{pmatrix} |z \in Z_{m-k}\right\}$ and ψ'^{-1}_m. We regard J_σ and J_τ as $\tilde{M}^0_{k-1}$-modules, where

$$M^0_{k-1} = \left\{\begin{pmatrix} b & * \\ 0 & 1 \end{pmatrix}\middle| b \in \mathrm{GL}_{k-1}(F)\right\}.$$

Now we continue exactly as in [So], p. 52 (using the analog of Proposition 8.2 in [G.PS.], for J_σ, as explained in Proposition 11.2 of [G.PS.]). We conclude that outside a finite set $S_{\sigma,\tau}$ of values of q^{-s}, all the various derivatives of J_σ and J_τ which correspond to nonminimal standard parabolic subgroups of M^0_{k-1} contribute zero to the space of bilinear forms A' on $J_\sigma \times J_\tau$, satisfying

$$A'(\overline{\sigma}\langle a,\varepsilon\rangle\xi, \overline{\tau}(a)v) = \varepsilon\gamma_{\gamma,\det a}^{-1}|\det a|^{-s+\frac{m-1}{2}+k}A'(\xi, v) \tag{6.10}$$

for $a \in M_{k-1}^0$, $\xi \in J_\sigma$, $v \in J_\tau$.

Thus, outside $S_{\sigma,\tau}$, the only possible contribution to (6.10) comes from the derivatives of J_σ and J_τ which correspond to the Whittaker models with respect to ψ_k^{-1} and ψ'_m, respectively. This contribution is of dimension one. The proof of the functional equation (1.10) is now complete.

Note again the special case where σ is supercuspidal. We have already remarked that $S_{\sigma,\tau}^0 = \phi$. Moreover, $S_{\sigma,\tau} = \phi$, as well, since clearly any derivative of J_σ with respect to a nonminimal parabolic subgroup in M_{k-1}^0 involves a Jacquet module with respect to a unipotent radical of $\mathrm{Sp}_{2k}(F)$, and hence equals zero. Thus the only possible contribution to (6.10) results from the derivatives of J_σ and J_τ which correspond to the Whittaker models with respect to ψ_k^{-1} and ψ'_m. Summing up,

Theorem. *Let σ and τ be irreducible representations of $\widetilde{\mathrm{Sp}}_{2k}(F)$ and $\mathrm{GL}_m(F)$, respectively ($k < m$). Denote*

$$d_\psi(\sigma,\tau,s) = \dim Bil_{\widetilde{\mathrm{Sp}}_{2k}(F)}\Big(\sigma, J_{\mathcal{H}_k}\Big(J_{N_{m,k+1},\chi_k^{-1}}(\rho_{\tau,s}) \otimes \omega_\psi^{(k)}\Big)\Big).$$

Then $d_\psi(\sigma,\tau)$ is a finite number. Moreover

(a) *There is a finite set $S_{\sigma,\tau} \subset \mathbb{C}$, such that for $q^{-s} \notin S_{\sigma,\tau}$*

$$d_\psi(\sigma,\tau,s) \le 1\ .$$

(b) *If σ is ψ_k^{-1}-generic and τ is generic, then, for $q^{-s} \notin S_{\sigma,\tau}$,*

$$d_\psi(\sigma,\tau,s) = 1\ .$$

(c) *If σ is supercuspidal and τ is generic, then $d_\psi(\sigma,\tau,s) \le 1$, for all $s \in \mathbb{C}$, and $d_\psi(\sigma,\tau,s) = 1$ if and only if σ is ψ_k^{-1}-generic.* □

6.3. A result on exterior square gamma factors. In this section, we prove a result needed in the proof of Theorem 3.3.2. Here we let τ be an irreducible, supercuspidal representation of $\mathrm{GL}_{2n}(F)$. Recall that $\gamma(\tau,\Lambda^2,s,\psi)$ and $\widetilde{\gamma}(\tau,\Lambda^2,s,\psi)$ denote the corresponding exterior square gamma factors following Shahidi [Sh1] and Jacquet-Shalika [J.S.2], respectively.

Proposition. *There is an exponential $c(s) = a^{\alpha s+\beta}$, such that*

$$\gamma(\tau,\Lambda^2,s,\psi) = c(s)\widetilde{\gamma}(\tau,\Lambda^2,s,\psi).$$

In particular, $\gamma(\tau,\Lambda^2,s,\psi)\Big|_{s=0} = 0$ if and only if $\widetilde{\gamma}(\tau,\Lambda^2,s,\psi)\Big|_{s=0} = 0$.

Proof. Embed τ inside an irreducible, automorphic, cuspidal representation π of $\mathrm{GL}_{2n}(\mathbb{A})$, where $\mathbb{A}$ is the adele ring of a number field k, such that at the place ν_0, $k_{\nu_0} = F$, and if $\pi \cong \bigotimes \pi_\nu$, then $\pi_{\nu_0} \cong \tau$. We may take π_ν to be unramified, at all finite places $\nu \neq \nu_0$. Also, there is a nontrivial character $\widetilde{\psi} = \bigotimes \psi_\nu$ of $k\backslash\mathbb{A}$, such that $\psi_{\nu_0} = \psi$. (See [Sh2], Section 4.) The global functional equation for the corresponding theory of the exterior square L-function amounts to

$$\prod_{\nu\in S_\infty}\gamma(\pi_\nu,\Lambda^2,s,\psi_\nu)\gamma(\tau,\Lambda^2,s,\psi) = \frac{L^S(\widehat{\pi},\Lambda^2,1-s)}{L^S(\pi,\Lambda^2,s)}. \tag{6.11}$$

Here S_∞ is the set of archimedean places of k, and $S = S_\infty \cup \{\nu_0\}$. A similar equation holds for $\widetilde{\gamma}$ as well. The global integral of [J.S.2] is

$$I(\varphi,\phi,s) = \int\limits_{C_n(\mathbb{A})\,\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A})} \int\limits_{X\in M_n(k)\backslash M_n(\mathbb{A})} \varphi\left(\begin{pmatrix} I_n & X \\ & I_n \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} \right) \cdot \widetilde{\psi}(trX) E(g, f_{\phi,s}) dx dg$$

where φ is a cusp form in the space of π, $\phi \in S(\mathbb{A}^n)$ and $E(g, f_{\phi,s})$ is the Eisenstein series which corresponds to the section $f_{\phi,s}$ (defined by (3.4), except F^* is replaced by $\mathbb{A}^*$ and ω_τ by ω_π). The global functional equation for $I(\varphi,\phi,s)$ is obtained from the corresponding functional equation of the Eisenstein series (see [J.S.1], p. 546)

$$E(g, f_{\phi,s}) = E({}^tg^{-1}, \widetilde{f}_{\hat{\phi},1-s})$$

where $\widehat{\phi}$ is the Fourier transform of ϕ (with respect to $\psi(x\cdot {}^ty)$) and $\widetilde{f}_{\hat{\phi},1-s}$ is obtained from (3.4) by replacing $s \mapsto 1-s$, $\omega_\tau \mapsto \omega_\pi^{-1}, F^* \mapsto \mathbb{A}^*$. Thus,

$$I(\varphi,\phi,s) = \widetilde{I}(\varphi,\widehat{\phi},1-s) \tag{6.12}$$

where

$$\begin{aligned} &\widetilde{I}(\varphi,\widehat{\phi},1-s) \\ &= \int\limits_{C_n(\mathbb{A})\,\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A})} \int\limits_{M_n(F)\backslash M_n(\mathbb{A})} \varphi\left(\begin{pmatrix} I_n & X \\ & I_n \end{pmatrix} \begin{pmatrix} w_n{}^tg^{-1} & \\ & w_n{}^tg^{-1} \end{pmatrix} \right) \\ &\qquad \cdot \widetilde{\psi}(trX) E(g, \widetilde{f}_{\hat{\phi},1-s}) dx dg. \end{aligned}$$

Writing the Euler product expansion for decomposable data at each side of (6.12), we get

$$\begin{aligned} &\mathcal{L}(W_\infty,\phi_\infty,s)\mathcal{L}(W_{\nu_0},\phi_{\nu_0},s)L^S(\pi,\Lambda^2,s) \\ &\quad = \tilde{\mathcal{L}}(W_\infty,\widehat{\phi}_\infty,1-s)\tilde{\mathcal{L}}\Big(W_{\nu_0},\widehat{\phi}_{\nu_0},1-s\Big)L^S(\widehat{\phi},\Lambda^2,1-s) \end{aligned} \tag{6.13}$$

where we write the $\tilde{\psi}^{-1}$-Whittaker function of φ with respect to $\widetilde{\psi}$ as $\prod W_\nu$, the product of local ψ_ν^{-1}-Whittaker functions, and we let $W_\infty = \prod\limits_{\nu\in S_\infty} W_\nu$. (Similar notation for ϕ.) The factors $\mathcal{L}$ and $\tilde{\mathcal{L}}$ in (6.13) are defined as in (3.5) and (3.6), and can be seen to extend to meromorphic functions in $\mathbb{C}$. From (6.13) and (3.3), we conclude that there is a meromorphic function $\widetilde{\gamma}(\pi_\infty,\Lambda^2,s,\psi_\infty)$ such that

$$\widetilde{\gamma}(\pi_\infty,\Lambda^2,s,\psi_\infty)\mathcal{L}(W_\infty,\phi_\infty,s) = \tilde{\mathcal{L}}(W_\infty,\widehat{\phi}_\infty,1-s) \tag{6.14}$$

and

$$\widetilde{\gamma}(\pi_\infty,\Lambda^s,s,\psi_\infty)\widetilde{\gamma}(\tau,\Lambda^2,s,\psi) = \frac{L^S(\widehat{\pi},\Lambda^2,1-s)}{L^S(\pi,\Lambda^2,s)} \tag{6.15}$$

where we denote $\pi_\infty = \bigotimes_{\nu\in S_\infty}\pi_\nu$. Of course, we replace W_∞ in (6.14) by any linear combination of elements of the form $\prod\limits_{\nu\in S_\infty} W_\infty$. Here we remark that the local functional equation can be obtained at each archimedean place separately. The proof is an appropriate adaptation of the global Euler product expansion. Details will appear in a forthcoming paper of J. Cogdell and I. Piatetski-Shapiro.

Thus, we can define local gamma factors $\widetilde{\gamma}(\pi_\nu, \Lambda^2, s, \psi_\nu)$ at each archimedean place as well, but we won't need this here.

From (6.11) and (6.15),

$$\gamma(\pi_\infty, \Lambda^2, s, \psi_\infty)\gamma(\tau, \Lambda^2, s, \psi) = \widetilde{\gamma}(\pi_\infty, \Lambda^2, s, \psi_\infty)\widetilde{\gamma}(\tau, \Lambda^2, s, \psi) \tag{6.16}$$

where we denote $\gamma(\pi_\infty, \Lambda^2, s, \psi_\infty) = \prod\limits_{\nu \in S_\infty} \gamma(\pi_\nu, \Lambda^2, s, \psi_\nu)$. We know that the local Shahidi gamma factor has the form $\frac{\varepsilon(s)L(1-s)}{L(s)}$, where $\varepsilon(s)$ is an exponential and $L(s)$ is a product of one-dimensional local L-functions (see [Sh2]). Thus, at the place ν_0, $L(s) = \prod\limits_{i=1}^{r}(1 - a_i q^{-s})^{-1}$, and at an archimedean place $L(s)$ is, up to an exponential, of the form $\prod\limits_{i=1}^{m} \Gamma(\frac{1}{2}(s + s_i))$ or $\prod\limits_{i=1}^{m} \Gamma(s + s_i)$.

The function $\widetilde{\gamma}(\tau, \Lambda^2, s, \psi)$ is rational in q^{-s}. This is clear from the functional equation (3.3). The following two properties can be proved:

(A) Given $s_0 \in \mathbb{C}$, there exist W_∞ in $W(\pi_\infty, \psi_\infty^{-1})$, the ψ_∞^{-1}-Whittaker model of π_∞, and $\phi_\infty \in S\Big(\prod\limits_{\nu \in S_\infty} k_\nu^n\Big)$, such that $\mathcal{L}(W_\infty, \phi_\infty, s)$ is nonzero at $s = s_0$. (We do not exclude a pole.)

(B) There exist "Euler factors" $G(\pi_\infty, s)$ and $G(\widehat{\pi}_{\infty,s})$ of the form

$$P_0(s) \prod_{i=1}^{r} \Gamma(\frac{1}{2}(s + s_i))\Gamma(\frac{1}{2}(ns + s_0))$$

or

$$P_0(s) \prod_{i=1}^{r} \Gamma(s + s_i)\Gamma(ns + s_0),$$

where $P_0(s)$ is a polynomial, such that $\frac{\mathcal{L}(W_\infty, \phi_\infty, s)}{G(\pi_\infty, s)} = g_{W_\infty, \phi_\infty}(s)$ and $\frac{\tilde{\mathcal{L}}(W_\infty, \phi_\infty, s)}{G(\hat{\pi}_\infty, s)} = \widetilde{g}_{W_\infty, \phi_\infty}(s)$ are holomorphic for all $W_\infty \in W(\pi_\infty, \psi_\infty^{-1})$ and $\phi_\infty \in S(\prod\limits_{\nu \in S_\infty} k_\nu^n)$.

Rewrite (6.14), using (B), as

$$\widetilde{\gamma}(\pi_\infty, \Lambda^2, s, \psi_\infty) = \frac{G(\widehat{\pi}_\infty, 1 - s)\widetilde{g}_{W_\infty, \widehat{\phi}_\infty}(1 - s)}{G(\pi_\infty, s)g_{W_\infty, \phi_\infty}(s)}. \tag{6.17}$$

Let

$$R(q^{-s}) = \frac{\gamma(\tau, \Lambda^2, s, \psi)}{\widetilde{\gamma}(\tau, \Lambda^2, s, \psi)}.$$

This is a rational function of q^{-s}. By (6.16)

$$\widetilde{\gamma}(\pi_\infty, \Lambda^2, s, \psi_\infty) = R(q^{-s})\gamma(\pi_\infty, \Lambda^2, s, \psi_\infty). \tag{6.18}$$

Thus, from (6.17)

$$R(q^{-s})\gamma(\pi_\infty, \Lambda^2, s, \psi_\infty) = \frac{G(\widehat{\pi}_\infty, 1 - s)}{G(\pi_\infty, s)} \cdot \frac{\widetilde{g}_{W_\infty, \widehat{\phi}_\infty}(1 - s)}{g_{W_\infty, \phi_\infty}(s)}. \tag{6.19}$$

Write $R(q^{-s}) = \frac{P(q^{-s})}{Q(q^{-s})}$, a quotient of disjoint polynomials $P(x)$ and $Q(x)$ in $\mathbb{C}[x]$. Let $x - a$, $a \neq 0$, be a factor of either $P(x)$ or $Q(x)$. Since $q^{-s} = a$ has infinitely many solutions, which all lie on a line parallel to the imaginary axis, we can pick a solution s_0, far enough from the real axis so that it is not a pole or a zero of

$\gamma(\pi_\infty, \Lambda^2, s, \psi_\infty)$ (recall that it has the form $\frac{\varepsilon(s)L(1-s)}{L(s)}$), and also s_0 is not a pole or a zero of the quotient of Euler factors $\frac{G(\hat{\pi}_\infty,1-s)}{G(\pi_\infty,s)}$. From (6.19), s_0 is a zero, or a pole of $\frac{\tilde{g}_{W_\infty,\hat{\phi}_\infty}(1-s)}{g_{W_\infty,\phi_\infty}(s)}$ for all W_∞ and ϕ_∞ (according to whether $x-a$ is a factor of $P(x)$ or of $Q(x)$). Since $\widetilde{g}_{W_\infty,\widehat{\phi}_\infty}(1-s)$ and $g_{W_\infty\phi_\infty}(s)$ are holomorphic, we conclude that s_0 is a zero of $\widetilde{g}_{W_\infty,\widehat{\phi}_\infty}(1-s)$ (resp. of $g_{W_\infty,\phi_\infty}(s)$), for all W_∞, ϕ_∞. This contradicts (A). Thus $R(x) = \alpha X^{m_0}$, for $m_0 \in \mathbb{Z}$, and then (6.18) implies that $\widetilde{\gamma}(\pi_\infty, \Lambda^2, s, \psi_\infty) = \alpha q^{-m_0 s}\gamma(\pi_\infty, \Lambda^2, s, \psi_\infty)$. This completes the proof of the proposition. □

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