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ON THE IMAGE OF THE l-ADIC ABEL-JACOBI MAP FOR A VARIETY OVER THE ALGEBRAIC CLOSURE OF A FINITE FIELD

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0. Introduction

Let k_0 be a finite field of characteristic p whose algebraic closure we denote by $\bar{k}$. Let Y be a smooth projective k_0-variety. For each prime $l \neq p$ there is an l-adic Abel-Jacobi map

$$\mathbf{a}^r_{Y,l} : CH^r_{hom}(Y_{\bar{k}}) \to H^{2r-1}(Y_{\bar{k}}, \mathbb{Z}_l(r)) \otimes \mathbb{Q}_l/\mathbb{Z}_l,$$

which is a potentially useful tool for studying the Chow group of nullhomologous cycles on $Y_{\bar{k}}$. In this paper we investigate when $\mathbf{a}^r_{Y,l}$ is surjective. The first two results are conditional, because they depend on assuming the truth of the Tate conjecture (0.7).

(0.1) Theorem. *Suppose that* (0.7) *holds. If $p > 2$ and the dimension of Y is at most* 4*, then the set of prime numbers*

$$\mathfrak{L}_Y := \{l \text{ prime} : l \neq p \text{ and } \mathbf{a}^r_{Y,l} \text{ is not surjective for some } r\}$$

is finite.

In order to extend (0.1) to varieties of arbitrary dimension we find it necessary to make an assumption which goes beyond the Tate conjecture. Since such an assumption is easily formulated, we include it in §9.6 as Hypothesis **H**. With this hypothesis the same arguments which prove (0.1) yield immediately a result which is independent of the dimension of Y:

(0.2) Theorem. *Suppose that* (0.7) *and Hypothesis* **H** *hold. If $p > 2$, then $\mathfrak{L}_Y$ is a finite set.*

To obtain results which don't depend on conjectures, we restrict attention to varieties of a special form. Let $\pi : Y \to X$ be a non-isotrivial, semi-stable, elliptic surface with a section defined over a finite field k_0 of characteristic $p > 2$. Write m_π for the least common multiple of all m such that π has a singular fiber of Kodaira type I_m. Let W be the non-singular variety obtained by blowing up $Y \times_X Y$ along the singular locus.

(0.3) Theorem. *If $l \nmid 2 \cdot 5 \cdot p \cdot m_\pi$, then $\mathbf{a}^2_{W,l}$ is surjective.*

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An application of (0.3) and a theorem of Soulé [So] is the following extension of [Sch, 14.2]:

(0.4) Theorem. *Let $E \subset \mathbb{P}^2$ be the Fermat cubic curve. If $l \nmid 2 \cdot 3 \cdot p$, then $\mathbf{a}^2_{E^3,l}$ gives rise to an isomorphism*

$$CH^2_{hom}(E^3_{\bar{\mathbb{F}}_p}) \otimes \mathbb{Z}_l \to H^3(E^3_{\bar{\mathbb{F}}_p}, \mathbb{Q}_l/\mathbb{Z}_l(2)).$$

This last result may be reformulated in terms of the coniveau filtration on the third cohomology. The interesting piece of this filtration is

$$NH^3(Y_{\bar{k}}, \mathbb{Q}_l/\mathbb{Z}_l(2)) := \text{Ker } [H^3(Y_{\bar{k}}, \mathbb{Q}_l/\mathbb{Z}_l(2)) \to H^3(\bar{k}(Y), \mathbb{Q}_l/\mathbb{Z}_l(2))].$$

From work of Bloch and Merkuriev and Suslin one knows that

$$CH^2_{hom}(Y_{\bar{k}})_{tors} \otimes \mathbb{Z}_l \simeq NH^3(Y_{\bar{k}}, \mathbb{Q}_l/\mathbb{Z}_l(2)),$$

when $H^4(Y_{\bar{k}}, \mathbb{Z}_l(2))$ is torsion free (cf. [Ras, 3.6]). Thus (0.4) is equivalent to

$$NH^3(E^3_{\bar{\mathbb{F}}_p}, \mathbb{Q}_l/\mathbb{Z}_l(2)) = H^3(E^3_{\bar{\mathbb{F}}_p}, \mathbb{Q}_l/\mathbb{Z}_l(2)) \tag{0.5}$$

for $l \nmid 2 \cdot 3 \cdot p$.

It is interesting to speculate if (0.5) still holds when E^3 is replaced by any smooth, projective Y and l is allowed to be any prime different from p. A necessary condition for this to happen would be the surjectivity of $\mathbf{a}^2_{Y,l}$ $\forall\ l \neq p$. However, this would not be sufficient. In addition one would have to show that the torsion subgroup of $CH^2_{hom}(Y_{\bar{k}})$ is large. Soulé showed that $CH^2_{hom}(Y_{\bar{k}})$ is a torsion group when $Y_{\bar{k}}$ is a three-dimensional Abelian variety [So, 3.3]. One might hope that this result will eventually be extended to all smooth, projective varieties $Y_{\bar{k}}$, although substantial progress in this direction is not known to the author. The group $CH^2_{hom}(Y_{\bar{k}})_{tors}$, for certain special varieties $Y_{\bar{k}}$, is the subject of [Sch-T].

All the main results of this paper depend strongly on the assumption that the base field, $\bar{k}$, is the algebraic closure of a finite field. Analogous assertions about smooth, projective varieties over other algebraically closed fields are often false. For example, Bloch and Esnault considered the case of a variety, Y, defined over a number field with good, ordinary reduction at a place above l and satisfying $H^0(Y, \Omega^3_Y) \neq 0$. They proved that

$$NH^3(Y_{\bar{\mathbb{Q}}}, \mathbb{Q}_l/\mathbb{Z}_l(2)) \neq H^3(Y_{\bar{\mathbb{Q}}}, \mathbb{Q}_l/\mathbb{Z}_l(2)) \tag{0.6}$$

whenever certain technical hypotheses hold [Bl-Es]. In particular, (0.6) holds if $l \equiv 1 \mod 3$ and $Y_{\bar{\mathbb{Q}}} = E^3_{\bar{\mathbb{Q}}}$, where E is the Fermat cubic curve. This provides an interesting contrast with (0.5).

Further contrasts become evident when one compares the results described above with theorems and conjectures concerning the Hodge theoretic Abel-Jacobi map, $\mathfrak{a}^r_Y : CH^r_{hom}(Y_{\mathbb{C}}) \to J^r(Y_{\mathbb{C}})$, for varieties defined over $\mathbb{C}$. This map is known *not* to be surjective when $F^{r+1}H^{2r-1}(Y(\mathbb{C}), \mathbb{C}) \neq 0$ [Gri, 13.2]. In fact, Green [Gre] and Voisin (unpublished) show that the image of $\mathfrak{a}^2_V$ is torsion for a sufficiently general hypersurface, $V_{\mathbb{C}} \subset \mathbb{P}^4_{\mathbb{C}}$, of high degree. Combining this with the work of Bloch and Esnault [Bl-Es, 4.1] leads one to speculate that $\mathfrak{a}^2_V$ might be the zero map. A conjecture of Nori may be viewed as going even further to suggest that $CH^2_{hom}(V_{\mathbb{C}}) = 0$ [No, 7.2.5].

Before outlining the organization of the paper we state the Tate conjecture in the form required for (0.1) and (0.2):

(0.7) Conjecture. Let $k_0 \subset k$ be an arbitrary finite extension and let V/k be an arbitrary smooth, projective variety. Then

(0.7.1) the Frobenius element $\phi \in G_k$ acts semi-simply on $\bigoplus_{j \geq 0} H^j(V_{\bar{k}}, \mathbb{Q}_l)$ and

(0.7.2) the cycle class map $CH^s(V) \otimes \mathbb{Q}_l \to H^{2s}(V_{\bar{k}}, \mathbb{Q}_l(s))^{G_k}$ is surjective for all s.

We now describe the organization of the paper and the contents of the individual sections. The first section is devoted to definitions and basic properties of the l-adic intermediate Jacobian and the l-adic Abel-Jacobi map, $\mathbf{a}^r_{Y,l}$. We also consider a mod l version of $\mathbf{a}^r_{Y,l}$ which is easier to compute, but may still be used to show that $\mathbf{a}^r_{Y,l}$ is surjective.

The proofs of (0.1) and (0.2) make use of the theory of Lefschetz pencils. Central to the argument is the study of cycles supported in the fibers of such a pencil. Thus in §2 we consider a modified l-adic Abel-Jacobi map which is suitable for studying cycles supported in the fibers of a morphism from a variety to a curve. The point here is to reduce the problem of evaluating $\mathbf{a}^r_{Y,l}$ at a nullhomologous cycle to considerations involving only constructible sheaves on the curve. In fact this variant of the l-adic Abel-Jacobi map may be evaluated on Tate classes in the cohomology of the fibers, even if these are not known to correspond to algebraic cycles. We define a mod l version of this map and observe that evaluating it on a Tate class corresponds to computing a coboundary map associated with the relative cohomology sequence for a certain constructible sheaf, $\mathcal{M}$, of $\mathbb{F}_l$-vector spaces on the curve.

Sections 3 through 7 are devoted to the study of this coboundary map. The challenge here is to identify a finite field extension, $k_0 \subset k$, and a k rational point, $x \in X_k$, such that the coboundary map

$$H^2_x(X_{\bar{k}}, \mathcal{M})^{G_k} \to H^1(G_k, H^1(X_{\bar{k}}, \mathcal{M}))$$

is injective. Furthermore the domain of this map should contain a non-zero element which comes from a Tate class. Requiring that $H^2_x(X_{\bar{k}}, \mathcal{M})^{G_k}$ be one-dimensional turns out to be helpful here. The main technical result is Theorem (3.4).

The proof of Theorem (3.4) is begun in §4. We first pass to a finite Galois cover of the base curve in order to trivialize the sheaf $\mathcal{M}$. Write Γ for the Galois group of this cover. The original coboundary map may now be replaced with a $\mathbb{F}_l[\Gamma]$-linear coboundary map involving the cohomology of a constant sheaf on the covering curve.

Sections 5 and 6 are devoted to the detailed study of the cohomology of the covering curve as a module over $\mathbb{F}_l[\Gamma]$. The proof of Theorem (3.4) is completed in §7.

The next section contains a criterion for the existence of many Tate classes in the cohomology of the fibers of a morphism from a variety to a curve. In (8.2.2) the connection between $H^2_x(X_{\bar{k}}, \mathcal{M})^{G_k}$ being one-dimensional and Tate classes is discussed.

Section 9 is devoted to showing that the machinery developed in previous sections to evaluate the mod l Abel-Jacobi map at Tate classes in fibers applies to certain Lefschetz pencils. In particular we check that there are plenty of Tate classes in the fibers and that the hypotheses of Theorem (3.4) are fullfilled when $\mathcal{M}$ is constructed from the vanishing cohomology.

The proofs of (0.1) and (0.2) are given in §10. We start with the case of curves, where the results are well known, and proceed by induction on the dimension. A given variety Y will be blown up along the base locus of a Lefschetz pencil to obtain a variety W which maps to a curve. If the dimension of Y is odd, the techniques developed in the first part of the paper enable us to study the l-adic Abel-Jacobi map for the middle dimensional cohomology. By induction and the Tate conjecture this is the crucial case.

Theorem (0.3) is a generalization of [Sch, 0.4]. It is proved in §11 by recalling facts about elliptic surfaces and complex multiplication cycles and then applying Theorem (3.4). Now Theorem (0.4) follows from (0.3) much as [Sch, 14.2] followed from [Sch, 10.2]. The reader interested only in the proofs of (0.3) and (0.4) may skip §8, §9 and §10.

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Notational conventions. For a field K, $\bar{K}$ denotes a separable closure and $G_K := Gal(\bar{K}/K)$.

Variety means a geometrically integral, separated scheme of finite type over a field. A curve is a variety of dimension 1.

$k_0 \subset k_1 \subset k$ are finite fields.

l is an odd prime number, which is distinct from the characteristic of the base field. $\mathbb{F}_l$ is a field with l elements.

If H is an Abelian group, $H_{/\mathbf{t}}$ denotes the quotient of H by its torsion subgroup and $H[n]$ denotes the kernel of multiplication by n.

$Z^r(W)$ denotes the group of codimension r algebraic cycles on a variety W.

If W is smooth over a field K of characteristic $p \geq 0$, define

$$Z^r_{hom}(W) := \mathrm{Ker}[Z^r(W) \to \prod_{l \neq p} H^{2r}(W_{\bar{K}}, \mathbb{Z}_l(r))].$$

For V a closed subscheme of a variety W and $\mathcal{F}$ on W,

$$H^i_V(W, \mathcal{F})_0 := \mathrm{Ker}[H^i_V(W, \mathcal{F}) \to H^i(W, \mathcal{F})].$$

If $*$ is a short exact sequence of G_k-modules, then the first coboundary map in the associated long exact G_k-cohomology sequence will be denoted δ_*.

If M is a group representation, the contragredient representation will be denoted $M^\vee$.

1. The l-adic intermediate Jacobian and the l-adic Abel-Jacobi map

1.1. The l-adic intermediate Jacobian. Let W be a variety which is proper and smooth over a field K_0 which is finitely generated over the prime field. Fix a separable closure $\bar{K}$ of K_0 and let l be a prime distinct from the characteristic of K_0. For an Abelian group H, $H_{/\mathbf{t}}$ denotes the quotient by the torsion subgroup. We define the l-adic intermediate Jacobian

$$J^r_l(W) := \varinjlim_K H^1(G_K, H^{2r-1}(W_{\bar{K}}, \mathbb{Z}_l(r))_{/\mathbf{t}}), \tag{1.1.1}$$

where the direct limit is taken over subfields $K \subset \bar{K}$ which are finite extensions of K_0. In this paper we shall be exclusively concerned with the case in which $K_0 = k_0$ is a finite field. In this situation the structure of $J^r_l(W)$ turns out to be especially simple.

It is useful to introduce the more general notion of the l-adic intermediate Jacobian of a Galois module. Let H be a finitely generated $\mathbb{Z}_l$-module on which the absolute Galois group G_{k_0} acts continuously. Suppose further that no eigenvalue for the operation of the Frobenius element $\phi \in G_{k_0}$ on $H \otimes \mathbb{Q}_l$ is a root of unity. Such G_{k_0}-modules form a category and we may consider the functor to the category of Abelian groups defined by

$$J(H) := \varinjlim{}_k H^1(G_k, H_{/\mathbf{t}}), \tag{1.1.2}$$

where the limit is over intermediate fields $k_0 \subset k \subset \bar{k}$ of finite degree over k_0.

(1.1.3) Lemma. (i) *$J(H^{2r-1}(W_{\bar{k}}, \mathbb{Z}_l(r)))$ is defined and is isomorphic to $J_l^r(W)$.*
(ii) *$J(H) \simeq H \otimes \mathbb{Q}_l/\mathbb{Z}_l$ as G_{k_0}-modules.*
(iii) *For each finite extension $k_0 \subset k$, $J(H)^{G_k}$ is a finite group.*

Proof. (i) By Deligne's theorem [De1] the eigenvalues of the arithmetic Frobenius, ϕ, acting on $H^{2r-1}(W_{\bar{k}}, \mathbb{Z}_l(r)) \otimes \mathbb{Q}_l$ are algebraic numbers with complex absolute value different from 1. The first step in the proof of (ii) is to apply G_k-cohomology to the short exact sequence

$$0 \to H_{/\mathbf{t}} \to H \otimes \mathbb{Q}_l \to H \otimes \mathbb{Q}_l/\mathbb{Z}_l \to 0. \tag{1.1.4}$$

A topological generator of G_k is a power, ϕ^m, of ϕ. Since no root of unity is an eigenvalue of ϕ, both

$$H^0(G_k, H \otimes \mathbb{Q}_l) = \operatorname{Ker}(Id_{H\otimes\mathbb{Q}_l} - \phi^m)$$

and

$$H^1(G_k, H \otimes \mathbb{Q}_l) = \operatorname{Coker}(Id_{H\otimes\mathbb{Q}_l} - \phi^m)$$

are zero. Now assertion (ii) follows by taking direct limits. For (iii) observe that $(H \otimes \mathbb{Q}_l/\mathbb{Z}_l)^{G_k}$ is finite, since 1 is not an eigenvalue for the action of ϕ^m on $H \otimes \mathbb{Q}_l$. □

For future reference we record four additional facts related to the functor J.

(1.1.5) Lemma. (i) *Let $h : H \to H'$ be a homomorphism of finitely generated $\mathbb{Z}_l$-modules. Then $h \otimes Id : H \otimes \mathbb{Q}_l \to H' \otimes \mathbb{Q}_l$ is surjective iff $h \otimes Id : H \otimes \mathbb{Q}_l/\mathbb{Z}_l \to H' \otimes \mathbb{Q}_l/\mathbb{Z}_l$ is.*
(ii) *For any finite extension k/k_0, the natural map $H^1(G_k, H_{/\mathbf{t}}) \to J(H)^{G_k}$ is an isomorphism.*
(iii) *If H is torsion free, then there is a canonical isomorphism, $J(H)^{G_k}/l \to H^1(G_k, H/l)$.*
(iv) *Let $h : H \to H'$ be a homomorphism of finitely generated $\mathbb{Z}_l$-modules and let $A \subset H \otimes \mathbb{Q}_l/\mathbb{Z}_l$ be a subgroup. If the Tate module of A tensored with $\mathbb{Q}_l$ maps surjectively to $H' \otimes \mathbb{Q}_l$, then A maps surjectively to $H' \otimes \mathbb{Q}_l/\mathbb{Z}_l$.*

Proof. (i) follows from (1.1.4). (ii) follows from (1.1.3)(ii) and (1.1.4). For (iii) apply G_k-cohomology to

$$0 \to H \otimes \tfrac{1}{l}\mathbb{Z}_l/\mathbb{Z}_l \to H \otimes \mathbb{Q}_l/\mathbb{Z}_l \xrightarrow{\ l\ } H \otimes \mathbb{Q}_l/\mathbb{Z}_l \to 0.$$

For (iv) note that A is the direct sum of its maximal divisible subgroup and a finite group. The former may be written $H'' \otimes \mathbb{Q}_l/\mathbb{Z}_l$ where H'' is a $\mathbb{Z}_l$-submodule of H. Now the tautological map $H'' \otimes \mathbb{Q}_l/\mathbb{Z}_l \to H' \otimes \mathbb{Q}_l/\mathbb{Z}_l$ is surjective if and only if $H'' \otimes \mathbb{Q}_l \to H' \otimes \mathbb{Q}_l$ is surjective. □

1.2. The definition of the l-adic Abel-Jacobi map. For each finite extension k/k_0 there is an l-adic Abel-Jacobi map

$$\alpha^r_{l,k} : Z^r_{hom}(W_k) \to J^r_l(W)^{G_k}. \tag{1.2.1}$$

To evaluate $\alpha^r_{l,k}$ on a cycle z write

$$\lfloor z \rfloor \in H^{2r}_{|z|}(W_{\bar{k}}, \mathbb{Z}_l(r))_0 := \mathrm{Ker}[H^{2r}_{|z|}(W_{\bar{k}}, \mathbb{Z}_l(r)) \to H^{2r}(W_{\bar{k}}, \mathbb{Z}_l(r))] \tag{1.2.2}$$

for the fundamental class of z. By purity [Mi, VI.5.1], $H^{2r-1}_{|z|}(W_{\bar{k}}, \mathbb{Z}_l(r)) = 0$. Thus there is a short exact sequence of G_k-modules,

(1.2.3)
$$0 \to H^{2r-1}(W_{\bar{k}}, \mathbb{Z}_l(r)) \to H^{2r-1}((W - |z|)_{\bar{k}}, \mathbb{Z}_l(r)) \to H^{2r}_{|z|}(W_{\bar{k}}, \mathbb{Z}_l(r))_0 \to 0.$$

Applying the first coboundary map

$$\delta_{1.2.3} : H^{2r}_{|z|}(W_{\bar{k}}, \mathbb{Z}_l(r))^{G_k}_0 \to H^1(G_k, H^{2r-1}(W_{\bar{k}}, \mathbb{Z}_l(r)))$$

to $\lfloor z \rfloor$ and then taking the image under the tautological map

$$H^1(G_k, H^{2r-1}(W_{\bar{k}}, \mathbb{Z}_l(r))) \to H^1(G_k, H^{2r-1}(W_{\bar{k}}, \mathbb{Z}_l(r))_{/\mathbf{t}}) \simeq J^r_l(W)^{G_k}$$

gives $\alpha^r_{l,k}(z)$. Define

$$\mathbf{a}^r_{W,l} : Z^r_{hom}(W_{\bar{k}}) \to J^r_l(W), \qquad \mathbf{a}^r_{W,l} := \varinjlim_k \ \alpha^r_{l,k}. \tag{1.2.4}$$

(1.2.5) Proposition. *The map $\alpha^r_{l,k}$ factors through $CH^r_{hom}(W)$. Both $\alpha^r_{l,k}$ and $\mathbf{a}^r_{W,l}$ factor through $CH^r_{hom}(W_{\bar{k}})$ and are functorial with respect to correspondences.*

Proof. [Ja, 9.8] or [Sch, 1.10]. □

For a more detailed discussion of l-adic Abel-Jacobi maps, the reader is referred to [Ja] and [Sch, §1].

(1.2.6) Definition. The mod l Abel-Jacobi map, denoted $\bar{\alpha}^r_{l,k}$, is the composition of $\alpha^r_{l,k}$ with the tautological map $J^r_l(W)^{G_k} \to J^r_l(W)^{G_k}/l$.

1.3. The mod l Abel-Jacobi map for curves. Let C/k be a smooth complete curve and let c_0 be a degree one point. Consider the following diagram:

$$\begin{array}{ccc} \breve{C} & \xrightarrow{\breve{i}_0} & Pic^0(C) \\ \Big\downarrow{\scriptstyle \breve{m}_l} & & \Big\downarrow{\scriptstyle m_l} \\ C & \xrightarrow{i_0} & Pic^0(C), \end{array} \tag{1.3.1}$$

where $i_0(c) = \mathcal{O}_C(c - deg(c)c_0)$, m_l is multiplication by l, $\breve{C}$ is the fiber product and $\breve{i}_0$ and $\breve{m}_l$ are the canonical projections. For a fixed degree one point c distinct from c_0 define

$$\delta_{1.3.2} : H^2_{\{c,c_0\}}(C_{\bar{k}}, \mu_l)^{G_k}_0 \to H^1(G_k, H^1(C_{\bar{k}}, \mu_l))$$

to be the first coboundary map associated to the short exact sequence of G_k-modules,

$$0 \to H^1(C_{\bar{k}}, \mu_l) \to H^1((C - \{c, c_0\})_{\bar{k}}, \mu_l) \to H^2_{\{c,c_0\}}(C_{\bar{k}}, \mu_l)_0 \to 0. \tag{1.3.2}$$

(1.3.3) Lemma. *Suppose that G_k acts trivially on $H^1(C_{\bar{k}}, \mu_l)$. Then*
(i) $\mathrm{Ker}(m_l)$ *consists of degree one points.*
(ii) $k(\check{C})/k(C)$ *is Galois, in fact Abelian.*
(iii) *There are canonical isomorphisms*

$$\begin{aligned} H^1(G_k, H^1(C_{\bar{k}}, \mu_l)) &\simeq \mathrm{Hom}(G_k, H^1(C_{\bar{k}}, \mu_l)) \\ &\simeq H^1(C_{\bar{k}}, \mu_l) \simeq \mathrm{Ker}(m_l)(k) \simeq Gal(k(\check{C})/k(C)). \end{aligned}$$

Proof. (i) follows from the canonical identification $\mathrm{Ker}(m_l) \simeq H^1(C_{\bar{k}}, \mu_l)$. (ii) follows from (i) as does (iii) once one notes that the second isomorphism in (iii) is obtained by evaluating a homomorphism at the Frobenius element $\phi \in G_k$. ☐

(1.3.4) Proposition. (i) *The isomorphism* (1.1.5)(iii) *identifies* $\delta_{1.3.2}(c-c_0)$ *with the mod l Abel-Jacobi map* $\bar{\alpha}^r_{l,k}$ *for C evaluated at* $c - c_0$.

(ii) *The isomorphism in* (1.3.3)(iii) *identifies* $\delta_{1.3.2}(c-c_0)$ *with the Frobenius element* $Frob_c \in Gal(k(\check{C})/k(C))$.

Proof. (i) is straightforward. (ii) is a special case of [Sch, 1.14]. (The sign in *loc. cit.* is different since $Frob_w$ in *loc. cit.* was inadvertently defined to be the inverse of the usual Frobenius. The sign is irrelevant in the sequel.) ☐

1.4. A strategy for proving surjectivity of the l-adic Abel-Jacobi map. The following lemma gives a strategy for proving surjectivity of l-adic Abel-Jacobi maps which involves only computations mod l.

(1.4.1) Lemma. *Let $k_1 \subset k_2 \subset k_3 \subset \ldots$ be a sequence of finite extensions of k_0 such that $\bigcup_{n\in\mathbb{N}} k_n = \bar{k}$ and the composition*

$$\bar{\alpha}^r_{l,k_n} \;:\; Z^r_{hom}(W_{k_n}) \xrightarrow{\alpha^r_{l,k_n}} J^r_l(W)^{G_{k_n}} \longrightarrow J^r_l(W)^{G_{k_n}}/l$$

is surjective for each n. Then $\mathbf{a}^r_{W,l} : Z^r_{hom}(W_{\bar{k}}) \to J^r_l(W)$ is surjective.

Proof. Since $J^r_l(W)^{G_{k_n}}$ is a finite Abelian l-group, α^r_{l,k_n} is surjective if and only if $\bar{\alpha}^r_{l,k_n}$ is. The lemma follows, since $\mathbf{a}^r_{W,l}$ is the direct limit of the α^r_{l,k_n}'s. ☐

In practice it is useful to break the l-adic intermediate Jacobian up into pieces and to prove surjectivity for each individual piece separately using an argument similar to (1.4.1). In order to carry out this program it is necessary to discuss some variants of the l-adic Abel-Jacobi map.

2. Variants of the l-adic Abel-Jacobi map

2.1. Cycles supported in fibers and the Leray spectral sequence. Let $k_0 \subset k$ be an extension of finite fields. Assume the following

(2.1.1) Geometric situation. $f : W \to X$ is a flat, generically smooth morphism of smooth proper varieties over k_0. X is a curve, W has dimension $2m+1 \geq 3$, and the geometric fibers of f are connected. The inclusion of the largest open subset over which f is smooth is denoted $j : \dot{X} \to X$. The inclusion of the generic point is denoted $g : \eta \to X$. After base changing to k we have a diagram in which all squares are Cartesian:

$$\begin{array}{ccccccc} W' & \longrightarrow & W_k & \longleftarrow & \dot{W}_k & \xleftarrow{i_V} & V \\ {\scriptstyle f'}\big\downarrow & & {\scriptstyle f}\big\downarrow & & {\scriptstyle \dot{f}}\big\downarrow & & \big\downarrow \\ X' & \longrightarrow & X_k & \xleftarrow{j} & \dot{X}_k & \xleftarrow{i_x} & x \end{array}$$

Here i_x is a k-rational point and X' is the complement of the image of x in X_k.

Given the geometric situation (2.1.1) and a cycle $z \in Z^{m+1}_{hom}(W_k)$ supported on V there is a restriction map from (1.2.3) (with $r = m+1$) to

$$(2.1.2)\quad 0 \to \frac{H^{2m+1}(W_{\bar{k}}, \mathbb{Z}_l(m+1))}{i_{V*}(H^{2m+1}_V(W_{\bar{k}}, \mathbb{Z}_l(m+1)))} \to H^{2m+1}(W'_{\bar{k}}, \mathbb{Z}_l(m+1)) \to H^{2m+2}_V(W_{\bar{k}}, \mathbb{Z}_l(m+1))_0 \to 0.$$

Let $L^\bullet$ denote the filtration on $H^{2m+1}(W_{\bar{k}}, \mathbb{Z}_l(m+1))$ and $(L')^\bullet$ the filtration on $H^{2m+1}(W'_{\bar{k}}, \mathbb{Z}_l(m+1)))$ resulting from the Leray spectral sequence for f (respectively f'). From the commutative diagram

$$(2.1.3)\quad \begin{array}{ccc} H^{2m+1}_V(W_{\bar{k}}, \mathbb{Z}_l(m+1)) & \xrightarrow{i_{V*}} & H^{2m+1}(W_{\bar{k}}, \mathbb{Z}_l(m+1)) \\ \big\uparrow & & \big\uparrow \\ H^2_x(X_{\bar{k}}, R^{2m-1}f_*\mathbb{Z}_l(m+1)) & \longrightarrow & H^2(X_{\bar{k}}, R^{2m-1}f_*\mathbb{Z}_l(m+1)), \end{array}$$

in which the left-hand arrow is an isomorphism by purity and the bottom arrow is surjective, it follows that

$$(2.1.4)\qquad i_{V*}(H^{2m+1}_V(W_{\bar{k}}, \mathbb{Z}_l(m+1))) = L^2.$$

Thus

$$(2.1.5)\qquad 0 \to L^1/L^2 \to (L')^1 \to H^{2m+2}_V(W_{\bar{k}}, \mathbb{Z}_l(m+1))_0 \to 0$$

may be identified with an exact subsequence of (2.1.2). We may rewrite (2.1.5) in terms of the cohomology of $\mathcal{H} := R^{2m}f_*\mathbb{Z}_l(m+1)$ on the curve X. In the spectral sequences for f and f', $E_2^{1,1} = E_\infty^{1,1}$. Also $(L')^2 = 0$. One deduces that (2.1.5) is isomorphic to

$$(2.1.6)\qquad 0 \to H^1(X_{\bar{k}}, \mathcal{H}) \to H^1(X'_{\bar{k}}, \mathcal{H}) \to H^2_x(X_{\bar{k}}, \mathcal{H})_0 \to 0.$$

Write $\lfloor z \rfloor_V$ for the image of $\lfloor z \rfloor$ in $H^{2m+2}_V(W_{\bar{k}}, \mathbb{Z}_l(m+1))_0^{G_k}$ and $\lfloor z \rfloor_x$ for the image of $\lfloor z \rfloor_V$ in $H^2_x(X_{\bar{k}}, \mathcal{H})_0^{G_k}$. Throughout this paper δ_* will denote the first coboundary map on G_k-cohomology associated to a short exact sequence (*). With this notation $\delta_{2.1.5}(\lfloor z \rfloor_V)$ maps to $\delta_{2.1.6}(\lfloor z \rfloor_x)$. Since $\delta_{1.2.3}(\lfloor z \rfloor)$ (with $r = m+1$) maps to $\delta_{2.1.5}(\lfloor z \rfloor_V)$, we conclude that the coboundary map $\delta_{2.1.6}$ gives information concerning the value of the l-adic Abel-Jacobi map at a cycle supported in a smooth fiber of f.

2.2. Cycles supported in fibers and a criterion for surjectivity of the l-adic Abel-Jacobi map. Assume the geometric situation (2.1.1). Denote by

$Z^m(f^{-1}(x))_0$ the inverse image of $Z^{m+1}_{hom}(W_k)$ under the canonical injection $Z^m(f^{-1}(x)) \to Z^{m+1}(W_k)$. Define

$$Z^{m+1}_f(W_k) := im[\bigoplus_{x\in \dot{X}_k(k)} Z^m(f^{-1}(x))_0 \to Z^{m+1}_{hom}(W_k)]$$

and $Z^{m+1}_f(W_{\bar{k}}) = \varinjlim Z^{m+1}_f(W_k)$, where the limit is over finite intermediate fields $k_0 \subset k \subset \bar{k}$. Clearly $Z^{m+1}_f(W_{\bar{k}})$ may be regarded as the subgroup of $Z^{m+1}_{hom}(W_{\bar{k}})$ generated by cycles supported on the smooth fibers. There is a canonical map

$$\mathbf{a}_f : Z^{m+1}_f(W_{\bar{k}}) \to J(L^1/L^2) \tag{2.2.1}$$

whose value at a cycle z is computed as follows: First note that z is defined over a finite extension $k \supset k_0$ and is supported in a fiber V. Thus $z \in Z^m(V_k)_0$. Then $\mathbf{a}_f(z)$ is given by the image of $\delta_{2.1.5}(\lfloor z \rfloor_V) \in H^1(G_k, L^1/L^2)$ in $J(L^1/L^2)^{G_k}$. The image of $\mathbf{a}_f(z)$ under the natural map $J(L^1/L^2) \xrightarrow{\zeta_1} J(L^0/L^2)$ coincides with the image of $\mathbf{a}^{m+1}_{W,l}(z)$ under the natural map $J^{m+1}_l(W) \xrightarrow{\zeta_2} J(L^0/L^2)$.

Write $L^iJ^{m+1}_l(W)$ for the image of $J(L^i)$ in $J^{m+1}_l(W)$.

(2.2.2) Lemma. $\mathbf{a}^{m+1}_{W,l}(Z^{m+1}_f(W_{\bar{k}})) \subset L^1J^{m+1}_l(W)$.

Proof. By (1.1.3)(ii) J is a right exact functor. Thus the rows in the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L^1J^{m+1}_l(W) & \longrightarrow & J^{m+1}_l(W) & \longrightarrow & J(L^0/L^1) & \longrightarrow & 0 \\
 & & & & {\scriptstyle\zeta_2}\big\downarrow & & \big\| & & \\
 & & J(L^1/L^2) & \xrightarrow{\zeta_1} & J(L^0/L^2) & \longrightarrow & J(L^0/L^1) & \longrightarrow & 0
\end{array}$$

are exact. One deduces that $L^1J^{m+1}_l(W) = \zeta_2^{-1}(\zeta_1(J(L^1/L^2)))$ and the assertion follows. $\square$

(2.2.3) Proposition. *In order to show that* $\mathbf{a}^{m+1}_{W,l} : Z^{m+1}_{hom}(W_{\bar{k}}) \to J^{m+1}_l(W)$ *is surjective it suffices to verify that the following three conditions hold:*

(i) $\mathbf{a}^m_{V,l} : Z^m_{hom}(V_{\bar{k}}) \to J^m_l(V)$ *is surjective.*

(ii) $\mathbf{a}_f$ *is surjective.*

(iii) *The composition of* $\mathbf{a}^{m+1}_{W,l}$ *with the tautological map* $J^{m+1}_l(W) \xrightarrow{t} J(L^0/L^1)$ *is surjective.*

Proof. Assume (i). The canonical isomorphism

$$H^{2m-1}(V_{\bar{k}}, \mathbb{Z}_l(m)) \simeq H^{2m+1}_V(W_{\bar{k}}, \mathbb{Z}_l(m+1))$$

and (2.1.4) imply that $L^2J^{m+1}_l(W)$ is in the image of $\mathbf{a}^{m+1}_{W,l}$. Since $i_{V*}(Z^m_{hom}(V_{\bar{k}})) \subset Z^{m+1}_f(W_{\bar{k}})$, this shows that $L^2J^{m+1}_l(W) \subset \mathbf{a}^{m+1}_{W,l}(Z^{m+1}_f(W_{\bar{k}}))$. Assume (ii). Then

$$\zeta_2(\mathbf{a}^{m+1}_{W,l}(Z^{m+1}_f(W_{\bar{k}}))) = \zeta_1(J(L^1/L^2)).$$

Since $L^2J^{m+1}_l(W) = \mathrm{Ker}(\zeta_2)$, (i) and the proof of (2.2.2) imply $\mathbf{a}^{m+1}_{W,l}(Z^{m+1}_f(W_{\bar{k}})) = L^1J^{m+1}_l(W)$. Finally assume (iii). Then the surjectivity of $\mathbf{a}^{m+1}_{W,l}$ follows from $L^1J^{m+1}_l(W) = \mathrm{Ker}(t)$. $\square$

2.3. A variant of the l-adic Abel-Jacobi map associated to a subsheaf. Keep the notations of 2.2. In practice it often happens that the inverse system $\mathcal{H}(-1) = \{R^{2m} f_* \mathbb{Z}/l^n(m)\}_{n\in\mathbb{N}}$ contains a direct factor $\mathcal{E} := \{\mathcal{E}_n\}_{n\in\mathbb{N}}$ which satisfies:

$$\text{(i) } \forall\ \ x \in \dot{X}(\bar{k}),\ \mathcal{E}_x := \varprojlim(\mathcal{E}_n)_x \text{ is torsion free, and}$$

$$\text{(ii) } H^2_x(X_{\bar{k}}, \mathcal{E}(1)) = H^2_x(X_{\bar{k}}, \mathcal{E}(1))_0. \tag{2.3.1}$$

It is in this context that we wish to define the l-adic Abel-Jacobi map on Tate cycles. One application will be to the situation where f comes from a Lefschetz pencil of hyperplane sections and $\mathcal{E}$ comes from the vanishing cohomology (cf. §9).

For each $x \in \dot{X}(\bar{k})$ an open subgroup of G_k acts on $\mathcal{E}_x \simeq H^2_x(X_{\bar{k}}, \mathcal{E}(1))$. Write $\mathfrak{g}$ for the l-adic Lie algebra of the image. The subgroup annihilated by $\mathfrak{g}$, $\mathcal{E}^{\mathfrak{g}}_x$, is the subgroup of Tate cycles. Define $Z(\mathcal{E}) = \bigoplus_{x\in\dot{X}(\bar{k})} \mathcal{E}^{\mathfrak{g}}_x$. There is an abstract l-adic Abel-Jacobi map

$$\mathbf{a}_{\mathcal{E}} : Z(\mathcal{E}) \to J(H^1(X_{\bar{k}}, \mathcal{E}(1))), \tag{2.3.2}$$

which may be evaluated on a Tate class $z \in \mathcal{E}^{\mathfrak{g}}_x$ as follows: Choose a finite extension field $k \supset k_0$ such that $z \in \mathcal{E}^{G_k}_x$ and let $\delta_{2.3.3}$ be the first coboundary map associated to the short exact sequence of G_k-modules

$$0 \to H^1(X_{\bar{k}}, \mathcal{E}(1)) \to H^1(X'_{\bar{k}}, \mathcal{E}(1)) \to H^2_x(X_{\bar{k}}, \mathcal{E}(1))_0 \to 0. \tag{2.3.3}$$

Now $\mathbf{a}_{\mathcal{E}}(z)$ is defined to be the image of $\delta_{2.3.3}(z)$ under the natural map

$$H^1(G_k, H^1(X_{\bar{k}}, \mathcal{E}(1))) \to J(H^1(X_{\bar{k}}, \mathcal{E}(1))).$$

It is clear that $\mathbf{a}_{\mathcal{E}}(z)$ does not depend on the choice of k.

A choice of projection $q : \mathcal{H}(-1) \to \mathcal{E}$ gives rise to a map

$$q_{\#x}\ \ :\ \ Z^m(f^{-1}(x))_0 \to H^{2m+2}_{f^{-1}(x)}(W_{\bar{k}}, \mathbb{Z}_l(m+1))^{\mathfrak{g}} \simeq \mathcal{H}_x(-1)^{\mathfrak{g}} \xrightarrow{q_x} \mathcal{E}^{\mathfrak{g}}_x,$$

and hence to a map

$$q_\# := \bigoplus_{x\in\dot{X}(\bar{k})} q_{\#x} : Z^{m+1}_f(W_{\bar{k}}) \to Z(\mathcal{E}). \tag{2.3.4}$$

There is also a map

$$q_* : J(L^1/L^2) \simeq J(H^1(X_{\bar{k}}, \mathcal{H})) \xrightarrow{J(H^1(q))} J(H^1(X_{\bar{k}}, \mathcal{E}(1))).$$

(2.3.5) Lemma. $q_* \circ \mathbf{a}_f = \mathbf{a}_{\mathcal{E}} \circ q_\#$.

Proof. Straightforward. □

2.4. A criterion for the surjectivity of $\mathbf{a}_{\mathcal{E}}$ using only computations mod l. We first introduce a mod l version of the map $\mathbf{a}_{\mathcal{E}}$, which will be easier to compute than $\mathbf{a}_{\mathcal{E}}$ itself. Suppose that the inverse system $\{\mathcal{E}_n\}_n$ is a constructible l-adic sheaf in the sense of [Mi, p. 163] and that $\mathcal{E}_n$ is flat over $\mathbb{Z}/l^n$ for all n.

(2.4.1) Lemma. (i) *Suppose that* $H^0(X_{\bar{k}}, \mathcal{E}_1(1)) = 0$. *Then* $H^1(X_{\bar{k}}, \mathcal{E}(1))$ *is torsion free.*

(ii) *Suppose that* $j^*\mathcal{E}_n$ *is locally constant for all* n *and* $H^0(\dot{X}_{\bar{k}}, \mathcal{E}^\vee_1) = 0$. *Then* $H^1(X_{\bar{k}}, \mathcal{E}(1))/l \simeq H^1(X_{\bar{k}}, \mathcal{E}_1(1))$.

(iii) *If (i) and (ii) hold, then*

$$J(H^1(X_{\bar{k}}, \mathcal{E}(1)))^{G_k}/l \simeq H^1(G_k, H^1(X_{\bar{k}}, \mathcal{E}_1(1))).$$

Proof. (i) This follows from [Mi, V.1.11]. (ii) [Mi, V.1.11] also implies that

$$0 \to H^1(X_{\bar{k}}, \mathcal{E}(1))/l \to H^1(X_{\bar{k}}, \mathcal{E}_1(1)) \to H^2(X_{\bar{k}}, \mathcal{E}(1))[l] \to 0$$

is exact. It suffices to show that $H^2(X_{\bar{k}}, \mathcal{E}_n(1)) = 0$ for all n. This follows from Poincaré duality,

$$H^2(X_{\bar{k}}, \mathcal{E}_n(1)) \simeq H^2(X_{\bar{k}}, j_*j^*\mathcal{E}_n(1)) \simeq H^0(X_{\bar{k}}, j_*j^*\mathcal{E}_n^\vee)^\vee \simeq H^0(\dot{X}_{\bar{k}}, \mathcal{E}_n^\vee)^\vee,$$

and induction on n using the exactness of

$$0 \to H^0(\dot{X}_{\bar{k}}, \mathcal{E}_n^\vee) \to H^0(\dot{X}_{\bar{k}}, \mathcal{E}_{n+1}^\vee) \to H^0(\dot{X}_{\bar{k}}, \mathcal{E}_1^\vee).$$

(iii) This follows from the above and (1.1.5)(iii). □

Reduction mod l gives a map from (2.3.3) to

$$0 \to H^1(X_{\bar{k}}, \mathcal{E}_1(1)) \to H^1(X'_{\bar{k}}, \mathcal{E}_1(1)) \to H^2_x(X_{\bar{k}}, \mathcal{E}_1(1))_0 \to 0. \tag{2.4.2}$$

Given $z \in \mathcal{E}_x^{G_k}$, write $\bar{z} \in H^2(X_{\bar{k}}, \mathcal{E}_1(1))_0^{G_k}$ for the class of z mod l. The isomorphism in (2.4.1)(iii) sends the reduction of $\mathbf{a}_{\mathcal{E}}(z)$ mod l to $\delta_{2.4.2}(\bar{z})$.

Let $k_0 \subset k_1$ be a finite extension so that G_{k_1} acts trivially on $H^1(X_{\bar{k}}, \mathcal{E}_1(1))$. When $k_1 \subset k$, we may regard $\delta_{2.4.2}(\bar{z})$ as an element of $H^1(X_{\bar{k}}, \mathcal{E}_1(1))$ via

$$H^1(G_k, H^1(X_{\bar{k}}, \mathcal{E}_1(1))) \simeq \mathrm{Hom}(G_k, H^1(X_{\bar{k}}, \mathcal{E}_1(1))) \simeq H^1(X_{\bar{k}}, \mathcal{E}_1(1)),$$

where the last isomorphism sends a homomorphism to its value on the Frobenius element.

(2.4.3) Proposition. *Assume*

(i) *the hypotheses of* (2.4.1) *hold and*

(ii) *for each non-zero* $\epsilon \in H^1(X_{\bar{k}}, \mathcal{E}_1(1))$ *and each finite extension* $k_1 \subset k$*, there exist* $x \in \dot{X}(k)$ *and* $z \in \mathcal{E}_x^{G_k}$ *such that* $\delta_{2.4.2}(\bar{z})$ *is a non-zero multiple of* ϵ*.*

Then the map $\mathbf{a}_{\mathcal{E}} : Z(\mathcal{E}) \to J(H^1(X_{\bar{k}}, \mathcal{E}(1)))$ *is surjective.*

Proof. As in the proof of (1.4.1) it suffices to show that the composition

$$\bigoplus_{x \in \dot{X}(k)} \mathcal{E}_x^{G_k} \xrightarrow{\mathbf{a}_{\mathcal{E}}} J(H^1(X_{\bar{k}}, \mathcal{E}(1)))^{G_k} \longrightarrow J(H^1(X_{\bar{k}}, \mathcal{E}(1)))^{G_k}/l$$

is surjective for each k. Since the identification of $J(H^1(X_{\bar{k}}, \mathcal{E}(1)))^{G_k}/l$ with $H^1(X_{\bar{k}}, \mathcal{E}_1(1))$ takes $\mathbf{a}_{\mathcal{E}}(z)$ mod l to $\delta_{2.4.2}(\bar{z})$, this is an immediate consequence of the hypotheses. □

In order to apply Proposition (2.4.3) to prove surjectivity of Abel-Jacobi maps in situations where the Tate conjecture holds for the fibers of f, it is necessary to understand more about the coboundary map $\delta_{2.4.2}$. In the next section we take up this problem under the assumption that the étale sheaf $\mathcal{E}_1$ is the direct image of a sheaf on the generic point of X.

3. The coboundary map associated to a constructible sheaf on a curve

3.1. The set up. Let $\ddot{X} \subset X$ be a non-empty open affine subset of a smooth projective curve over a finite field k_0 of characteristic $p > 2$. Let $k_0 \subset k$ be a finite extension. Fix $x \in X_k(k)$ and let $g : \eta \to X$ denote the inclusion of the generic point. Let M be a finite-dimensional $\mathbb{F}_l$-vector space and let

$$\kappa : Gal(\overline{k(X)}/k(X)) \to \mathrm{Aut}_{\mathbb{F}_l}(M)$$

be a continuous representation, whose image will be denoted by Γ. The resulting étale sheaf on η will be denoted $\underline{M}$. Set $\mathcal{M} = g_*\underline{M}$. There is an exact sequence of G_k-modules

$$0 \to H^1(X_{\bar{k}}, \mathcal{M}) \to H^1(X'_{\bar{k}}, \mathcal{M}) \to H^2_x(X_{\bar{k}}, \mathcal{M})_0 \to 0, \tag{3.1.1}$$

where $X' := X - x$. We may drop the subscript $_0$ from the right-hand term precisely when the isomorphic groups

$$H^2(X_{\bar{k}}, \mathcal{M}) \simeq H^0(X_{\bar{k}}, \mathcal{M}^\vee(-1)) \simeq (M^\vee(-1))^\Gamma \tag{3.1.2}$$

are 0 [Mi, V.2.2(b)]. The coboundary map associated to (3.1.1) will be denoted

$$\delta_{3.1.1} : H^2_x(X_{\bar{k}}, \mathcal{M})_0^{G_k} \to H^1(G_k, H^1(X_{\bar{k}}, \mathcal{M})).$$

We now begin the task of analyzing the map $\delta_{3.1.1}$ under suitable hypotheses. Eventually the results will be applied when $\mathcal{M} = \mathcal{E}_1(1)$, in which case (3.1.1) is just (2.4.2).

3.2. Hypotheses on the base field. We make the following assumptions on the finite base field k:

(3.2.1) k is algebraically closed in the fixed field of $\operatorname{Ker}\kappa$.

(3.2.2) k contains l distinct l-th roots of unity.

By (3.2.1) there is a smooth, projective curve C defined over k such that $k(C)$ is the fixed field of $\operatorname{Ker}\kappa$. We write $\rho : C \to X_k$ for the corresponding morphism. Assume

(3.2.3) There is a degree one point x_0 of X such that each point in the fiber, $\rho^{-1}(x_0)$, has degree one.

(3.2.4) For any two points $c_0, b_0 \in \rho^{-1}(x_0)$ the class $c_0 - b_0 \in Pic^0(C)(k)$ is divisible by l.

(3.2.5) G_k acts trivially on $H^1(C_{\bar{k}}, \mu_l)$.

In order to state our final hypothesis on the base field k, we fix a degree one point $c_0 \in \rho^{-1}(x_0)$ and recall the curve $\breve{C}$ and the map $\breve{i}_0$ from (1.3).

Lemma. *$k(\breve{C})/k(X)$ is Galois.*

Proof. Since $k(\breve{C})/k(C)$ is Galois, by (3.2.5) it suffices to show that each $\gamma \in \Gamma = Gal(k(C)/k(X))$ is the image of some $\breve{\gamma} \in \operatorname{Aut}(k(\breve{C})/k(X))$. Write $\gamma_* \in \operatorname{Aut}(Pic^0(C))$ for the group automorphism induced by γ. Define $\gamma' \in \operatorname{Aut}(Pic^0(C))$ by $\gamma'(\mathfrak{d}) = \gamma_*(\mathfrak{d}) + \gamma(c_0) - c_0$. This is an automorphism of $Pic^0(C)$ as a principal homogeneous space. Now γ' induces the automorphism γ on the curve $i_0(C)$. By (3.2.4) there is a degree one point $\mathfrak{d}_\gamma \in Pic^0(C)$ such that $l\mathfrak{d}_\gamma \sim_{rat} \gamma(c_0) - c_0$. Define $\breve{\gamma} \in \operatorname{Aut}(Pic^0(C))$ by

$$\breve{\gamma}(\mathfrak{d}) = \gamma_*(\mathfrak{d}) + \mathfrak{d}_\gamma, \qquad \forall \mathfrak{d} \in Pic^0(C).$$

Since $l\breve{\gamma}(\mathfrak{d}) = \gamma'(l\mathfrak{d})$, $\breve{\gamma}$ stabilizes $\breve{i}_0(\breve{C})$ and thus gives an element of $\operatorname{Aut}(k(\breve{C})/k(X))$ which lifts γ. ☐

Let $\Sigma \subset X$ denote the branch locus of $\rho : C \to X$. The final assumption on k is:

(3.2.6) For every conjugacy class $\mathbf{F} \subset Gal(k(\breve{C})/k(X))$ there is a k-rational point $x \in (\ddot{X} \cap X - \Sigma)(k)$ with the property that $Frob_x = \mathbf{F}$.

No matter how M was chosen, one can always arrange that (3.2.1–6) hold by taking k to be a suitable finite extension of k_0. In fact there is a finite extension $k_0 \subset k_1$ such that (3.2.1–6) hold for every finite extension k of k_1. In the case of

(3.2.6) this assertion is a variant of the Tchebotarev density theorem (cf. [Lan] or [Sch, 9.9]).

3.3. Hypotheses on the monodromy representation. We need some assumptions on the structure of M as an $\mathbb{F}_l[\Gamma]$-module. Before stating these we note the following conventions: All $\mathbb{F}_l[\Gamma]$-modules in this paper are finitely generated left $\mathbb{F}_l[\Gamma]$-modules. If L is such a module, we denote by $L^\vee := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{F}_l)$ the dual module, and by P_L the projective cover [Al, §20]. There is, up to isomorphism, a unique way to write L as a product of indecompossible $\mathbb{F}_l[\Gamma]$-modules [Al, p. 23]. Thus one can speak of the multiplicity with which an indecomposible factor occurs in L.

We assume

(3.3.1) $\kappa : Gal(\overline{k(X)}/k(X)) \to \mathrm{Aut}_{\mathbb{F}_l}(M)$ is tamely ramified.

(3.3.2) $(M^\vee)^\Gamma = 0$.

(3.3.3) M is an absolutely irreducible $\mathbb{F}_l[\Gamma]$-module.

(3.3.4) $H^1(\Gamma, M) = 0$.

(3.3.5) $H^1(\Gamma, M^\vee) = 0$.

(3.3.6) There exists $\xi \in \Gamma$ of order prime to l such that $M^{\langle\xi\rangle} \simeq \mathbb{F}_l$.

(3.3.7) *Remark.* Very often M comes equipped with a Γ-invariant, non-degenerate bilinear form. In this case (3.3.4) and (3.3.5) are equivalent.

(3.4) Theorem. *Suppose that* (3.2.1–6) *and* (3.3.1–6) *hold. Then* $H^1(X_{\bar{k}}, \mathcal{M})$ *is a trivial* G_k*-module and for each*

$$\varepsilon \in H^1(G_k, H^1(X_{\bar{k}}, \mathcal{M})) \simeq \mathrm{Hom}(G_k, H^1(X_{\bar{k}}, \mathcal{M})) \simeq H^1(X_{\bar{k}}, \mathcal{M}),$$

there is a degree one point x *of* $\ddot{X} \cap X - \Sigma$ *such that*

$$H^2_x(X_{\bar{k}}, \mathcal{M})_0^{G_k} \simeq \mathbb{F}_l \tag{3.4.1}$$

and

$$\varepsilon \in \delta_{3.1.1}(H^2_x(X_{\bar{k}}, \mathcal{M})_0^{G_k}). \tag{3.4.2}$$

The proof of the theorem occupies the next four sections. A rough overview of the main steps follows.

Step 1. Define a functor

(3.4.3)

$$\Lambda : \text{ Finite } \mathbb{F}_l[\Gamma]\text{-modules} \longrightarrow \mathbb{F}_l\text{-vector spaces}; \qquad \Lambda(N) = (N \otimes_{\mathbb{Z}} M)^\Gamma$$

and interpret (3.1.1) as Λ applied to

$$0 \to H^1(C_{\bar{k}}, \mu_l) \to H^1((C - \rho^{-1}(x))_{\bar{k}}, \mu_l) \to H^2_{\rho^{-1}(x)}(C_{\bar{k}}, \mu_l)_0 \to 0. \tag{3.4.4}$$

Show $\delta_{3.1.1} = \Lambda(\delta_{3.4.4})$.

Step 2. Determine the multiplicity of $P_{M^\vee}$ as a direct factor in the $\mathbb{F}_l[\Gamma]$-modules $H^2_{\rho^{-1}(x)}(C_{\bar{k}}, \mu_l)_0^{G_k}$ and $H^1(C_{\bar{k}}, \mu_l)$. Show that for a suitable choice of x there exists an $\mathbb{F}_l[\Gamma]$-linear map, $\delta : H^2_{\rho^{-1}(x)}(C_{\bar{k}}, \mu_l)_0^{G_k} \to H^1(C_{\bar{k}}, \mu_l)$, with the property that the image of $\Lambda(\delta)$ may be identified with $\mathbb{F}_l\varepsilon$.

Step 3. Recall from (1.3.4) that the map $\delta_{3.4.4}$ is controlled by Frobenius conjugacy classes in $Gal(k(\check{C})/k(C))$ associated to certain degree zero divisors supported on $\rho^{-1}(x)$. Describe explicitly a conjugacy class $\mathbf{F} \subset Gal(k(\check{C})/k(X))$ with the property that $Frob_x = \mathbf{F}$ implies $dim_{\mathbb{F}_l}(H^2_x(X_{\bar{k}}, \mathcal{M})_0^{G_k}) = 1$ and that $\mathbb{F}_l\varepsilon$ may be identified with the image of $\Lambda(\delta_{3.4.4})$.

4. Cohomology on a finite cover

We keep the notation of the previous section. Assume that (3.2.1–2) and (3.3.1–2) hold. Let $g_C : \eta_C \to C$ denote the inclusion of the generic point. Set $\mathcal{M}_C := g_{C*}(\underline{M}|_{\eta_C})$. This is a constant sheaf. Let $\Xi \subset X$ be a finite set of closed points, which are disjoint from the branch locus Σ of ρ. Denote the stabilizer of a closed point $c \in C_{\bar{k}}$ by $\Gamma_c \subset \Gamma$ and the stalk of a sheaf $\mathcal{F}$ at c by $\mathcal{F}_c$.

(4.1) Proposition. (i) *There is a commutative diagram with exact rows:*

$$\begin{tikzcd}
0 \to H^1(X_{\bar{k}}, \mathcal{M}) \arrow[r] \arrow[d, "\beta"'] & H^1((X-\Xi)_{\bar{k}}, \mathcal{M}) \arrow[r] \arrow[d, "\beta'"'] & H^2_{\Xi}(X_{\bar{k}}, \mathcal{M}) \to 0 \arrow[d, "\beta_\bullet"', "\simeq"] \\
0 \to H^1(C_{\bar{k}}, \mathcal{M}_C)^\Gamma \arrow[r] & H^1((C-\rho^{-1}(\Xi))_{\bar{k}}, \mathcal{M}_C)^\Gamma \arrow[r] & H^2_{\rho^{-1}(\Xi)}(C_{\bar{k}}, \mathcal{M}_C)^\Gamma \to 0.
\end{tikzcd}$$

(ii) *If $H^1(\Gamma, M) \simeq 0$, then β and β' are injective.*

(iii) *If $H^2(\Gamma, M) \simeq 0$ and $H^1(\Gamma_c, (\mathcal{M}_C)_c) \simeq 0$ for all closed points $c \in C_{\bar{k}}$, then β and β' are surjective.*

Proof. By (3.3.2) $H^2(X_{\bar{k}}, \mathcal{M}) \simeq 0$. Thus the top row in the diagram is exact. The group Γ acts on $\underline{M}_{|\eta_C}$ and hence on $\mathcal{M}_C$. It follows that Γ acts on $\rho_*(\mathcal{M}_C)$. The subsheaf of Γ-invariant sections will be denoted $\rho_*^\Gamma(\mathcal{M}_C)$. One verifies directly that $\mathcal{M} \simeq \rho_*^\Gamma(\mathcal{M}_C)$. The vertical arrows in the diagram may be constructed by applying cohomology to the map of sheaves on $X_{\bar{k}}$,

$$\rho_*^\Gamma(\mathcal{M}_C) \to \rho_*(\mathcal{M}_C),$$

and then taking Γ-invariants. The injectivity and surjectivity of β may be analyzed with the help of the left exact functor,

$$T : \{\text{étale sheaves on } C_{\bar{k}} \text{ with } \Gamma\text{-action}\} \to \text{Abelian Groups},$$
$$T(\mathcal{F}) = H^0(C_{\bar{k}}, \mathcal{F})^\Gamma.$$

Writing T as a composition of two left exact functors in two different ways gives rise to two spectral sequences [Gro, 5.2.5],

$$E_2^{pq} = H^p(X_{\bar{k}}, R^q\rho_*^\Gamma\mathcal{F}) \Rightarrow R^{p+q}T(\mathcal{F}),$$
$$E_2^{pq} = H^p(\Gamma, H^q(C_{\bar{k}}, \mathcal{F})) \Rightarrow R^{p+q}T(\mathcal{F}).$$

When $\mathcal{F} = \mathcal{M}_C$, the corresponding 5-term exact sequences begin with

$$0 \to H^1(X_{\bar{k}}, \mathcal{M}) \to R^1T(\mathcal{M}_C) \to H^0(X_{\bar{k}}, R^1\rho_*^\Gamma(\mathcal{M}_C)) \qquad \text{and}$$
$$0 \to H^1(\Gamma, H^0(C_{\bar{k}}, \mathcal{M}_C)) \to R^1T(\mathcal{M}_C) \to H^1(C_{\bar{k}}, \mathcal{M}_C)^\Gamma \to H^2(\Gamma, H^0(C_{\bar{k}}, \mathcal{M}_C)).$$

Furthermore one can check that for each closed point $x \in X_{\bar{k}}$

$$R^1\rho_*^\Gamma(\mathcal{M}_C)_x \simeq H^1(\Gamma_c, (\mathcal{M}_C)_c),$$

where c is any point in the fiber $\rho^{-1}(x)$ (cf. [Gro, 5.3.1]). Since $M = H^0(C_{\bar{k}}, \mathcal{M}_C)$, assertions (ii) and (iii) follow for the first vertical arrow. An analogous argument with $X - \Xi$ replacing X gives the assertions for β'.

To deal with $\beta_\bullet$ we introduce a functor T_Ξ analogous to T defined by

$$T_\Xi(\mathcal{F}) = H^0_{\rho^{-1}(\Xi)}(C_{\bar{k}}, \mathcal{F})^\Gamma \simeq H^0_\Xi(X_{\bar{k}}, \rho_*^\Gamma\mathcal{F}).$$

There are spectral sequences

$$E_2^{pq} = H^p_\Xi(X_{\bar k}, R^q\rho_*^\Gamma\mathcal{F}) \Rightarrow R^{p+q}T_\Xi(\mathcal{F}),$$
$$E_2^{pq} = H^p(\Gamma, H^q_{\rho^{-1}(\Xi)}(C_{\bar k}, \mathcal{F})) \Rightarrow R^{p+q}T_\Xi(\mathcal{F}).$$

The sheaves $R^q\rho_*^\Gamma\mathcal{F}$ are supported on the branch locus of ρ for $q > 0$ [Gro, 5.3.1]. Since this closed subset does not meet Ξ, the first spectral sequence yields

$$H^n_\Xi(X_{\bar k}, \rho_*^\Gamma\mathcal{F}) \simeq R^nT_\Xi(\mathcal{F}).$$

By purity [Mi, VI.5.1], $H^n_{\rho^{-1}(\Xi)}(C_{\bar k}, \mathcal{M}_C) \simeq 0$ for $n \neq 2$. Consequently the natural maps

$$H^2_\Xi(X_{\bar k}, \rho_*^\Gamma\mathcal{M}_C) \to R^2T_\Xi(\mathcal{M}_C) \to H^2_{\rho^{-1}(\Xi)}(C_{\bar k}, \mathcal{M}_C)^\Gamma$$

are isomorphisms.

Finally the exactness of the bottom row in the diagram follows from the fact that $\beta_\bullet$ is an isomorphism. □

4.2. Recall the definition of the functor Λ from (3.4.3). Since $\mathcal{M}_C$ is the constant sheaf associated with M,

$$H^i(C_{\bar k}, \mathcal{M}_C) \simeq H^i(C_{\bar k}, \mathbb{Z}/l) \otimes_\mathbb{Z} M,$$

and similar isomorphisms hold when C is replaced by $C - \rho^{-1}(\Xi)$ or cohomology is replaced by cohomology with support. Fix a degree one point $x \in X_k - \Sigma$.

(4.2.1) Lemma. *Assume that* (3.2.1–2) *and* (3.3.1–4) *hold. Then* (4.1) *gives rise to a commutative diagram of G_k-modules with exact rows:*

(4.2.2)

$$\begin{array}{ccccc}
0 \to H^1(X_{\bar k}, \mathcal{M}) & \longrightarrow & H^1((X-x)_{\bar k}, \mathcal{M}) & \longrightarrow & H^2_x(X_{\bar k}, \mathcal{M}) \to 0 \\
{\scriptstyle\beta}\big\downarrow & & {\scriptstyle\beta'}\big\downarrow & & {\scriptstyle\beta_\bullet}\big\downarrow{\scriptstyle\simeq} \\
0 \to \Lambda(H^1(C_{\bar k}, \mathbb{Z}/l)) & \longrightarrow & \Lambda(H^1((C - \rho^{-1}(x))_{\bar k}, \mathbb{Z}/l)) & \longrightarrow & \Lambda(H^2_{\rho^{-1}(x)}(C_{\bar k}, \mathbb{Z}/l)_0) \to 0.
\end{array}$$

The maps β and β' are injective. Write $\delta_{4.2.2}$ for the first coboundary map in the long exact G_k-cohomology sequence associated to the bottom row of (4.2.2)*. Then there is a natural identification of $\delta_{3.1.1}$ with $\delta_{4.2.2}$.*

Proof. Observe that Λ applied to

$$i : H^2_{\rho^{-1}(x)}(C_{\bar k}, \mathbb{Z}/l)_0 \to H^2_{\rho^{-1}(x)}(C_{\bar k}, \mathbb{Z}/l)$$

is an isomorphism since i is injective, Λ is left exact, the cokernel of i is $H^2(C_{\bar k}, \mathbb{Z}/l)$, and $\Lambda(H^2(C_{\bar k}, \mathbb{Z}/l)) \simeq M^\Gamma = 0$ by (3.3.2–3). The first assertion follows. The injectivity of β and β' follows from (3.3.4) and (4.1)(ii). The final assertion is a consequence of $\delta_{4.2.2} \circ \beta_\bullet = \beta \circ \delta_{3.1.1}$. □

4.3. To complete the first step in the proof of (3.4) we need to compare $\delta_{4.2.2}$ and $\Lambda(\delta_{3.4.4})$. The following commutative diagram, in which the columns are short exact sequences, shows that

$$Gal(\overline{k(X)}/k(X))/Gal(\overline{k(X)}/\bar k(C)) \simeq \Gamma \times G_k$$

and in particular that the action of $Gal(\overline{k(X)}/k(X))$ on $H^1(C_{\bar{k}}, \mathbb{Z}/l)$ factors through $\Gamma \times G_k$:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & Gal(\overline{k(X)}/\bar{k}(C)) & \longrightarrow & Gal(\overline{k(X)}/k(C)) & & & & \\
& & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & Gal(\overline{k(X)}/\bar{k}(X)) & \longrightarrow & Gal(\overline{k(X)}/k(X)) & \longrightarrow & G_k & \longrightarrow & 1 \\
& & \downarrow & & \downarrow{\scriptstyle \kappa} & & & & \\
& & \Gamma & = & \Gamma. & & & &
\end{array}$$

Now $\delta_{4.2.2}$ may be identified with $\Lambda(\delta_{3.4.4})$ as the following abstract argument shows:

(4.3.1) Lemma. *Let*

(4.3.2) $$0 \to A_0 \to A_1 \xrightarrow{\varsigma} A_2 \to 0$$

be a short exact sequence of continuous $\mathbb{F}_l[\Gamma \times G_k]$-modules which are finite dimensional $\mathbb{F}_l$-vector spaces. Then

(i) *$H^j(G_k, A_i)$ is canonically an $\mathbb{F}_l[\Gamma]$-module for each i.*

(ii) *The coboundary map for G_k-cohomology*

$$\delta_{4.3.2} : A_2^{G_k} \to H^1(G_k, A_0)$$

is $\mathbb{F}_l[\Gamma]$-linear.

(iii) *If the map $\Lambda(\varsigma)$ in*

(4.3.3) $$0 \to \Lambda(A_0) \to \Lambda(A_1) \xrightarrow{\Lambda(\varsigma)} \Lambda(A_2)$$

is surjective, then the coboundary $\delta_{4.3.3}$ is defined and may be identified with $\Lambda(\delta_{4.3.2})$.

Proof. The assertions (i) and (ii) are clear since the actions of Γ and G_k commute. Since M is a trivial G_k-module, the coboundary $\delta_{4.3.4}$ associated to

(4.3.4) $$0 \to A_0 \otimes_{\mathbb{Z}} M \to A_1 \otimes_{\mathbb{Z}} M \to A_2 \otimes_{\mathbb{Z}} M \to 0$$

may be identified with the composition

$$(A_2 \otimes_{\mathbb{Z}} M)^{G_k} \simeq A_2^{G_k} \otimes_{\mathbb{Z}} M \xrightarrow{\delta_{4.3.2} \otimes Id_M} H^1(G_k, A_0) \otimes_{\mathbb{Z}} M \simeq H^1(G_k, A_0 \otimes_{\mathbb{Z}} M).$$

When $\Lambda(\varsigma)$ is surjective, $\delta_{4.3.3}$ may be identified with the restriction of $\delta_{4.3.4}$ to the Γ-invariants. But this is $(\delta_{4.3.2} \otimes Id_M)$ restricted to the Γ-invariants, which is $\Lambda(\delta_{4.3.2})$. □

This completes the first step in the proof of Theorem (3.4).

5. The structure of $H^1(C_{\bar{k}}, \mathbb{Z}/l)$ as an $\mathbb{F}_l[\Gamma]$-module

5.1. The conventions concerning $\mathbb{F}_l[\Gamma]$-modules introduced in §3.3 remain in force. To these we add the notations $soc(L)$ for the socle of L and P_0 for the projective cover of the trivial $\mathbb{F}_l[\Gamma]$-module, $\mathbb{F}_l$. The following facts will be used without further comment: P is projective as an $\mathbb{F}_l[\Gamma]$-module if and only if it is injective [Al, p. 41]. If it is projective, then so is $P^\vee$. If L is absolutely simple, then P_L is indecomposible [Al, p. 31], $soc(P_L) \simeq L$, and $P_{L^\vee} \simeq P_L^\vee$ [Al, Theorem 6 p. 43]. We shall also use

(5.1.1) Lemma. *For each $\mathbb{F}_l[\Gamma]$-module L, $H^i(\Gamma, L) \simeq \mathrm{Ext}^i_{\mathbb{F}_l[\Gamma]}(\mathbb{F}_l, L)$.*

Proof. Let $Q^\bullet \to \mathbb{Z}$ be a resolution by finitely generated, free $\mathbb{Z}[\Gamma]$-modules [Se3, VII.3]. By the snake lemma applied to

$$0 \to Q^\bullet \xrightarrow{l} Q^\bullet \to Q^\bullet/l \to 0, \tag{5.1.2}$$

$Q^\bullet/l \to \mathbb{F}_l$ is a free resolution in the category of $\mathbb{F}_l[\Gamma]$-modules. Apply $\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(\ \ ,L)$ to (5.1.2). Since $\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(Q^i/l, L) \simeq \mathrm{Hom}_{\mathbb{F}_l[\Gamma]}(Q^i/l, L)$ and multiplication by l annihilates $\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(Q^i, L)$, the complexes $\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(Q^\bullet, L)$ and $\mathrm{Hom}_{\mathbb{F}_l[\Gamma]}(Q^\bullet/l, L)$ are isomorphic. The lemma now follows from the definitions. □

Assume (3.3.3). Recall the functor Λ defined in (3.4.3):

$$\Lambda(L) = (L \otimes_{\mathbb{Z}} M)^\Gamma \simeq \mathrm{Hom}_{\mathbb{F}_l[\Gamma]}(L^\vee, M) \simeq \mathrm{Hom}_{\mathbb{F}_l[\Gamma]}(M^\vee, L) \simeq \mathbb{F}_l^s, \tag{5.1.3}$$

where s is the multiplicity with which M occurs in the maximal semi-simple quotient of $L^\vee$ or equivalently the multiplicity of $M^\vee$ in $soc(L)$ [Al, p. 47].

5.2. The information which we need concerning the structure of $H^1(C_{\bar{k}}, \mathbb{Z}/l)$ as an $\mathbb{F}_l[\Gamma]$-module is stated in the next proposition. Let $\varphi^\vee : P^\vee \to H^1(C_{\bar{k}}, \mathbb{Z}/l)$ denote the inclusion of a maximal projective submodule.

(5.2.1) Proposition. *Suppose that* (3.2.1–2) *and* (3.3.1–5) *hold. Then*

$$\beta : H^1(X_{\bar{k}}, \mathcal{M}) \to \Lambda(H^1(C_{\bar{k}}, \mathbb{Z}/l)) \qquad \textit{and} \qquad \Lambda(\varphi^\vee) : \Lambda(P^\vee) \to \Lambda(H^1(C_{\bar{k}}, \mathbb{Z}/l))$$

are injective maps with the same image.

Proof. Let $j : \dot{X} \to X$ denote the inclusion of an open affine subset of $X - \Sigma$. Set $\dot{C} = \rho^{-1}(\dot{X}_k)$. Consider the commutative diagram with exact rows and columns:

(5.2.2)

$$\begin{tikzcd}
 & & H^2(\Gamma, M) & \\
0 \arrow[r] & \Lambda(H^1(C_{\bar{k}}, \mathbb{Z}/l)) \arrow[r, "r"] & \Lambda(H^1(\dot{C}_{\bar{k}}, \mathbb{Z}/l)) \arrow[u, "e"] \arrow[r, "\chi"] & \Lambda(H^2_{C-\dot{C}}(C_{\bar{k}}, \mathbb{Z}/l)) \\
H^1_{X-\dot{X}}(X_{\bar{k}}, \mathcal{M}) \arrow[r] & H^1(X_{\bar{k}}, \mathcal{M}) \arrow[u, "\beta"] \arrow[r, "j^*"] & H^1(\dot{X}_{\bar{k}}, \mathcal{M}) \arrow[u, "\gamma"] \arrow[r, "r'"] & H^2_{X-\dot{X}}(X_{\bar{k}}, \mathcal{M}). \arrow[u, "\gamma'"]
\end{tikzcd}$$

The maps β and γ are injective since $H^1(\Gamma, M) = 0$ (3.3.4). Let ϖ be a left inverse of γ.

(5.2.3) Lemma. $r \circ \beta : H^1(X_{\bar{k}}, \mathcal{M}) \to \mathrm{Ker}(e) \cap \mathrm{Ker}(r' \circ \varpi)$ *is an isomorphism.*

Proof. ϖ maps $\mathrm{Ker}(e) \cap \mathrm{Ker}(r' \circ \varpi)$ isomorphically to $\mathrm{Ker}(r')$ and $\varpi \circ r \circ \beta = j^*$. Now j^* is injective because $r \circ \beta$ is. The lemma follows from the equality $im(j^*) = \mathrm{Ker}(r')$. □

Now (5.2.1) is a consequence of (5.2.3) and the following two lemmas:

(5.2.4) Lemma. *The image of* $r \circ \Lambda(\varphi^\vee) : \Lambda(P^\vee) \to \Lambda(H^1(\dot{C}_{\bar{k}}, \mathbb{Z}/l))$ *lies in* $\mathrm{Ker}(e) \cap \mathrm{Ker}(r' \circ \varpi)$.

(5.2.5) Lemma. $dim(\Lambda(P^\vee)) \geq h^1(X_{\bar{k}}, \mathcal{M})$.

Since $r \circ \Lambda(\varphi^\vee)$ is clearly injective, the previous lemmas imply that $dim(\Lambda(P^\vee)) = h^1(X_{\bar{k}}, \mathcal{M})$.

The remainder of this section is devoted to the proofs of (5.2.4) and (5.2.5).

5.3. This subsection is devoted to the proof of (5.2.4). Define

$$N = H^1(C_{\bar{k}}, \mathbb{Z}/l)^\vee = \pi_1(C_{\bar{k}})_{ab}/l\pi_1(C_{\bar{k}})_{ab},$$
$$\dot{N} = H^1(\dot{C}_{\bar{k}}, \mathbb{Z}/l)^\vee = \pi_1(\dot{C}_{\bar{k}})_{ab}/l\pi_1(\dot{C}_{\bar{k}})_{ab}.$$

As in (1.3.1) we write $\breve{C}_{\bar{k}} \to C_{\bar{k}}$ for the maximal abelian, unramified, exponent l cover. The maximal abelian, exponent l-cover which is unramified over $\dot{C}_{\bar{k}}$ will be denoted $\grave{C}_{\bar{k}} \to C_{\bar{k}}$. Define $\theta = Gal(\breve{C}_{\bar{k}}/X_{\bar{k}})$ and $\dot{\theta} = Gal(\grave{C}_{\bar{k}}/X_{\bar{k}})$. There is a commutative diagram with exact rows and surjective vertical maps:

$$\begin{tikzcd}
0 \arrow[r] & \dot{N} \arrow[r, "\dot{\zeta}"] \arrow[d, "\lambda_N"'] & \dot{\theta} \arrow[r] \arrow[d, "\lambda"] & \Gamma \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & N \arrow[r, "\zeta"] \arrow[d, "\varphi"'] & \theta \arrow[r] \arrow[d, "\psi"] & \Gamma \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & P \arrow[r, "\bar{\zeta}"] & \bar{\theta} \arrow[r] & \Gamma \arrow[r] & 1,
\end{tikzcd} \tag{5.3.1}$$

where $\bar{\theta} := \theta/\zeta(Ker(\varphi))$.

Since $H^1(\Gamma, M) = 0$, the Hochschild-Serre spectral sequence applied to (5.3.1) yields

$$\begin{tikzcd}
0 \arrow[r] & H^1(\dot{\theta}, M) \arrow[r, "\dot{\zeta}^*"] & \mathrm{Hom}_{\mathbb{F}_l[\Gamma]}(\dot{N}, M) \arrow[r, "e"] & H^2(\Gamma, M) \\
0 \arrow[r] & H^1(\theta, M) \arrow[r, "\zeta^*"] \arrow[u, "\lambda^*"] & \mathrm{Hom}_{\mathbb{F}_l[\Gamma]}(N, M) \arrow[r] \arrow[u, "\lambda_N^*"] & H^2(\Gamma, M) \arrow[u, equal] \\
0 \arrow[r] & H^1(\bar{\theta}, M) \arrow[r, "\bar{\zeta}^*"] \arrow[u, "\psi^*"] & \mathrm{Hom}_{\mathbb{F}_l[\Gamma]}(P, M) \arrow[r, "e'"] \arrow[u, "\varphi^*"] & H^2(\Gamma, M). \arrow[u, equal]
\end{tikzcd}$$

Since P is an injective $\mathbb{F}_l[\Gamma]$-module, $H^2(\Gamma, P) \simeq \mathrm{Ext}^2_{\mathbb{F}_l[\Gamma]}(\mathbb{F}_l, P) = 0$. Thus the bottom row in (5.3.1) admits a splitting $s : \Gamma \to \bar{\theta}$ [Br, IV.3] and e' is the zero map. Clearly the image of $\lambda_N^* \circ \varphi^*$ is contained in Ker(e). Since $\lambda_N^* \circ \varphi^*$ is precisely $r \circ \Lambda(\varphi^\vee)$ this proves one assertion in (5.2.4).

To prove that the image of $r \circ \Lambda(\varphi^\vee)$ is contained in Ker$(r' \circ \varpi)$, we consider the inertia group $\kappa : \dot{I} \to \dot{\theta}$ of a point $\grave{c} \in \grave{C}_{\bar{k}}$ above $x \in (X - \dot{X})_{\bar{k}}$. Write

$$r_x : H^2_{X-\dot{X}}(X_{\bar{k}}, \mathcal{M}) \to H^2_x((\dot{X} \cup \{x\})_{\bar{k}}, \mathcal{M}) \simeq H^2_x(X_{\bar{k}}, \mathcal{M})$$

for the restriction map.

(5.3.2) Lemma. *There is a left inverse $\dot{\upsilon}$ of $\dot{\zeta}^*$ and a commutative diagram:*

$$\begin{tikzcd}
\Lambda(H^1(C_{\bar{k}}, \mathbb{Z}/l)) \arrow[r, "r_x \circ r' \circ \varpi \circ r"] & H^2_x(X_{\bar{k}}, \mathcal{M}) \\
\mathrm{Hom}_{\mathbb{F}_l[\Gamma]}(N, M) \arrow[r, "\kappa^* \circ \dot{\upsilon} \circ \lambda_N^*"] \arrow[u, "\simeq"] & H^1(\dot{I}, M) \arrow[u, "\simeq"']
\end{tikzcd}$$

Proof. That the left-hand arrow is an isomorphism is a direct consequence of the definitions. To describe the map on the right fix a place of the separable closure,

$\overline{k(X)}$, of $\bar{k}(\grave{C})$ above $\grave{c}$. Let $\mathbf{I} \subset Gal(\overline{k(X)}/\bar{k}(X))$ denote the decomposition (=inertia) group at this place. This determines a field $K = \overline{k(X)}^{\mathbf{I}}$ and a discrete valuation ring $A \subset K$, which is a henselization of $\mathcal{O}_{X_{\bar{k}},x}$. Let a denote the closed point of $Spec(A)$. There are isomorphisms

$$H^2_x(X_{\bar{k}}, \mathcal{M}) \simeq H^2_a(Spec(A), \mathcal{M}) \simeq H^1(Spec(K), \mathcal{M}) \simeq H^1(\mathbf{I}, M). \tag{5.3.3}$$

(5.3.4) Sublemma. *The restriction map $H^1(\dot{I}, M) \to H^1(\mathbf{I}, M)$ is an isomorphism.*

Proof. The tame inertia group $\mathbf{I}^t$ is a product of an l-procyclic group $\mathbf{I}_l$ and a procyclic group $\mathbf{I}_{l'}$ whose quotients have no l-torsion. There is a corresponding decomposition $\dot{I} = \dot{I}_l \times \dot{I}_{l'}$. The Hochschild-Serre spectral sequence gives isomorphisms

$$H^1(\mathbf{I}, M) \simeq H^1(\mathbf{I}^t, M) \simeq H^1(\mathbf{I}_l, M') \qquad \text{and} \qquad H^1(\dot{I}, M) \simeq H^1(\dot{I}_l, M'), \tag{5.3.5}$$

where $M' := M^{\dot{I}_{l'}} = M^{\mathbf{I}_{l'}}$. Write I for the image of $\dot{I}$ in Γ and I_l for its l-primary subgroup. Define d by $l^d = |I_l|$ and write $I_l^{(j)}$ for the quotient of $\mathbf{I}_l$ with l^{d+j} elements. In particular $\dot{I}_l = I_l^{(1)}$. Write σ for a topological generator of $\mathbf{I}_l$. Now

$$H^1(\mathbf{I}_l, M') \simeq \varinjlim_j \, H^1(I_l^{(j)}, M')$$

and the restriction map

$$H^1(\dot{I}_l, M') \to H^1(I_l^{(j)}, M')$$

corresponds to the obvious inclusion

$$\frac{\mathrm{Ker}(1 + \sigma + ... + \sigma^{l^{d+1}-1})}{im(\sigma - 1)} \to \frac{\mathrm{Ker}(1 + \sigma + ... + \sigma^{l^{d+j}-1})}{im(\sigma - 1)}. \tag{5.3.6}$$

Decompose the $\mathbb{F}_l[\sigma]$-module M' as a direct sum of indecomposable factors isomorphic to

$$(\sigma - 1)^i \mathbb{F}_l[\sigma]/(\sigma - 1)^{l^d} \mathbb{F}_l[\sigma]$$

with $0 \le i < l^d$. Use the identity $1 + \sigma + ... + \sigma^{l^{d+j}-1} = (\sigma - 1)^{l^{d+j}-1}$ to check that both numerators in (5.3.6) are isomorphic to M', so (5.3.6) is an isomorphism for all j and (5.3.4) follows. $\square$

The commutative diagram of Galois groups,

$$\begin{array}{ccc}
\mathbf{I} & \longrightarrow & \pi_1(\dot{X}_{\bar{k}}, \bar{\eta}) \\
\downarrow & & \downarrow \\
\dot{I} & \longrightarrow & \dot{\theta},
\end{array}$$

gives a commutative diagram on cohomology,

$$\begin{tikzcd}
H^1(\dot{X}_{\bar{k}}, \mathcal{M}) \arrow[r, "r_x \circ r'"] \arrow[d, "\simeq"'] & H^2_x(X_{\bar{k}}, \mathcal{M}) \arrow[d, "\simeq"] \\
H^1(\pi_1(\dot{X}_{\bar{k}}, \bar{\eta}), M) \arrow[r] & H^1(\mathbf{I}, M) \\
H^1(\dot{\theta}, M) \arrow[r, "\kappa^*"] \arrow[u] & H^1(\dot{I}, M). \arrow[u]
\end{tikzcd}$$

The vertical arrows are isomorphisms by [Sch, 7.3] and (5.3.4). There is a commutative diagram:

$$\begin{tikzcd}
\Lambda(H^1(C_{\bar{k}}, \mathbb{Z}/l)) \arrow[r, "r"] & \Lambda(H^1(\dot{C}_{\bar{k}}, \mathbb{Z}/l)) & H^1(\dot{X}_{\bar{k}}, \mathcal{M}) \arrow[l, "\gamma"'] \arrow[r, "r_x \circ r'"] & H^2_x(X_{\bar{k}}, \mathcal{M}) \\
\mathrm{Hom}_{\mathbb{F}_l[\Gamma]}(N, M) \arrow[r, "\lambda_N^*"] \arrow[u, "\simeq"] & \mathrm{Hom}_{\mathbb{F}_l[\Gamma]}(\dot{N}, M) \arrow[u, "\simeq"] & H^1(\dot{\theta}, M) \arrow[l, "\dot{\zeta}^*"'] \arrow[r, "\kappa^*"] \arrow[u, "\simeq"] & H^1(\dot{I}, M) \arrow[u, "\simeq"]
\end{tikzcd}$$

We may reverse the arrows in the middle by choosing a left inverse ϖ of γ and the corresponding left inverse $\dot{\upsilon}$ of $\dot{\zeta}^*$. This proves (5.3.2). □

(5.3.7) Lemma. *The map* $\kappa^* \circ \dot{\upsilon} \circ \lambda_N^* \circ \varphi^* : \mathrm{Hom}_{\mathbb{F}_l[\Gamma]}(P, M) \to H^1(\dot{I}, M)$ *is zero.*

Proof. Since $\breve{C} \to C$ is unramified, the image $\lambda \circ \kappa(\dot{I}) \subset \theta$ is a lifting of the inertia group $I \subset \Gamma$. Since P is a projective $\mathbb{F}_l[I]$-module, $H^1(I, P) = 0$ and any two liftings of I to $\bar{\theta}$ are related by conjugation by an element of P [Br, IV.2]. Thus after modifying the section $s : \Gamma \to \bar{\theta}$ by conjugation by an appropriate element of P we have $\psi \circ \lambda \circ \kappa(\dot{I}) \subset s(\Gamma)$. Given any $f \in \mathrm{Hom}_{\mathbb{F}_l[\Gamma]}(P, M)$ define a map

$$F : \bar{\theta} \to M, \qquad F(p \cdot s(\gamma)) = f(p).$$

It is straightforward to verify that F is a crossed homomorphism and that the class of F in $H^1(\bar{\theta}, M)$ maps via the isomorphism $\bar{\zeta}^*$ to f. Since the map $F \circ \psi \circ \lambda \circ \kappa : \dot{I} \to M$ is identically zero and represents the class

$$\kappa^* \circ \lambda^* \circ \psi^* \circ (\bar{\zeta}^*)^{-1}(f) = \kappa^* \circ \dot{\upsilon} \circ \lambda_N^* \circ \varphi^*(f) \in H^1(\dot{I}, M),$$

the lemma follows. □

Now (5.3.2) and (5.3.7) show that $\lambda_N^* \circ \varphi^*(\mathrm{Hom}_{\mathbb{F}_l[\Gamma]}(P, M)) = r \circ \Lambda(\varphi^\vee)(\Lambda(P^\vee))$ lies in the kernel of $r_x \circ r' \circ \varpi$. Since this is true for every $x \in (X - \dot{X})_{\bar{k}}$, (5.2.4) follows.

5.4. This subsection is devoted to the proof of (5.2.5). Let $\dot{\varphi}^\vee : \dot{P}^\vee \to H^1(\dot{C}_{\bar{k}}, \mathbb{Z}/l)$ denote the inclusion of a maximal projective submodule.

(5.4.1) Lemma. $dim(\Lambda(\dot{P}^\vee)) = h^1(\dot{X}_{\bar{k}}, \mathcal{M})$.

Proof. There is a short exact sequence [Sch2, §8], [Se5, §8]

(5.4.2) $$0 \to P_0/soc(P_0) \to P_0 \oplus \mathbb{F}_l[\Gamma]^{h^1(\dot{X}_{\bar{k}})-1} \to H^1(\dot{C}_{\bar{k}}, \mathbb{Z}/l) \to 0.$$

Since M is absolutely irreducible, the multiplicity of $P_{M^\vee}$ in the middle term is

(5.4.3) $$dim_{\mathbb{F}_l}(M)(h^1(\dot{X}_{\bar{k}}) - 1) = -dim_{\mathbb{F}_l}(M)e(\dot{X}_{\bar{k}}) = -e(\mathcal{M}|_{\dot{X}_{\bar{k}}}) = h^1(\dot{X}_{\bar{k}}, \mathcal{M}),$$

where e denotes the euler characteristic and the last equality follows from $M^\Gamma = 0$ (3.3.2–3). The multiplicity with which $P_{M^\vee}$ appears as a direct summand of $H^1(\dot{C}_{\bar{k}}, \mathbb{Z}/l)$ is $dim(\Lambda(\dot{P}^\vee))$. By (5.4.3) this multiplicity may also be expressed as $h^1(\dot{X}_{\bar{k}}, \mathcal{M}) - dim(\Lambda(P_0/soc(P_0)))$. The isomorphism $rad(P_0)^\vee \simeq P_0/soc(P_0)$ and (5.1.3) allow us to write

$$\Lambda(P_0/soc(P_0)) = \mathrm{Hom}_{\mathbb{F}_l[\Gamma]}(rad(P_0), M) \simeq \mathrm{Ext}^1_{\mathbb{F}_l[\Gamma]}(\mathbb{F}_l, M) \simeq H^1(\Gamma, M).$$

Since the right-hand term is zero by (3.3.4), (5.4.1) follows. $\square$

For $x \in (X - \dot{X})_{\bar{k}}$ we denote by $I_x \subset \Gamma$ the inertia group at some point $c \in \rho^{-1}(x)$. The fact that changing the choice of $c \in \rho^{-1}(x)$ changes I_x by conjugation will be of no consequence in what follows.

(5.4.4) Lemma. $h^1(X_{\bar{k}}, \mathcal{M}) = h^1(\dot{X}_{\bar{k}}, \mathcal{M}) - \sum_{x \in (X-\dot{X})_{\bar{k}}} dim(M^{I_x})$.

Proof. In the exact sequence

$$H^1(X_{\bar{k}}, \mathcal{M}) \xrightarrow{j^*} H^1(\dot{X}_{\bar{k}}, \mathcal{M}) \to H^2_{X-\dot{X}}(X_{\bar{k}}, \mathcal{M}) \to H^2(X_{\bar{k}}, \mathcal{M}) \tag{5.4.5}$$

the map j^* is injective by the proof of (5.2.3). Recall that $(M^\vee)^\Gamma = 0$ (3.3.2), which implies $H^2(X_{\bar{k}}, \mathcal{M}) = 0$ (Poincaré duality [Mi, V.2.2(b)]). Thus we need only verify

$$h^2_x(X_{\bar{k}}, \mathcal{M}) = dim(M^{I_x})$$

for each $x \in (X - \dot{X})_{\bar{k}}$. This follows from

(5.4.6)
$$dim(M^{I_x}) = dim((M')^{I_l}) = dim(M'/im(\sigma - 1)) = h^1(\dot{I}_l, M') = h^2_x(X_{\bar{k}}, \mathcal{M}),$$

where the last two equalities follow from (5.3.3) and the proof of (5.3.4). $\square$

(5.4.7) Lemma. $dim(\Lambda(P^\vee)) \geq dim(\Lambda(\dot{P}^\vee)) - \sum_{x \in (X-\dot{X})_{\bar{k}}} dim(M^{I_x})$.

Proof. Consider the exact sequence of $\mathbb{F}_l[\Gamma]$-modules

$$0 \to H^1(C_{\bar{k}}, \mathbb{Z}/l) \to H^1(\dot{C}_{\bar{k}}, \mathbb{Z}/l) \to H^2_{C-\dot{C}}(C_{\bar{k}}, \mathbb{Z}/l) \xrightarrow{\wp} H^2(C_{\bar{k}}, \mathbb{Z}/l) \to 0. \tag{5.4.8}$$

By (5.4.2) $H^1(\dot{C}_{\bar{k}}, \mathbb{Z}/l) \simeq \dot{P}^\vee \oplus \Omega_0^{-2}$, where Ω_0^{-2} denotes the cokernel of the inclusion of $P_0/soc(P_0)$ in its injective hull [Al, §20]. Write

$$H^2_{C-\dot{C}}(C_{\bar{k}}, \mathbb{Z}/l)_0 := \mathrm{Ker}(\wp) \simeq Q \oplus U, \tag{5.4.9}$$

where Q is a maximal projective submodule. The surjection

$$H^1(\dot{C}_{\bar{k}}, \mathbb{Z}/l) \to H^2_{C-\dot{C}}(C_{\bar{k}}, \mathbb{Z}/l)_0 \xrightarrow{pr_Q} Q$$

induces a split surjection $\dot{P}^\vee \to Q$. Choose a submodule $P' \subset \dot{P}^\vee$ such that $\dot{P}^\vee \simeq Q \oplus P'$. Modding out Q from the middle terms in (5.4.8) gives a short exact sequence

$$0 \to H^1(C_{\bar{k}}, \mathbb{Z}/l) \to P' \oplus \Omega_0^{-2} \xrightarrow{h+g} U \to 0,$$

which defines the maps $h : P' \to U$ and $g : \Omega_0^{-2} \to U$. Let N_0 denote the image of $g(\Omega_0^{-2})$ in $U/rad(U) := \bar{U}$. Choose a submodule $N_1 \subset \bar{U}$ for which $\bar{U} = N_0 \oplus N_1$. Choose a submodule $P_1 \subset P'$ such that the composition $P_1 \xrightarrow{h} U \to \bar{U}$ identifies P_1 with a projective cover of N_1. Since $h(P_1) + g(\Omega_0^{-2})$ maps surjectively to $\bar{U}$, $h(P_1) + g(\Omega_0^{-2}) = U$. Choose a submodule $P_2 \subset P'$ such that $P' = P_2 \oplus P_1$ and set $h_i := h|_{P_i}$. Since P_2 is projective, there is a lifting $\tilde{h}_2$ of h_2 which makes the diagram

$$\begin{array}{ccc} P_1 \oplus \Omega_0^{-2} & \xrightarrow{h_1+g} & U \\ \uparrow{\scriptstyle \tilde{h}_2} & & \uparrow{\scriptstyle h_2} \\ P_2 & = & P_2 \end{array}$$

commute. Denote the image of

$$P_2 \to P_2 \oplus P_1 \oplus \Omega_0^{-2}, \qquad p \mapsto (p, -\tilde{h}_2(p))$$

by P_2'. Then $P' \oplus \Omega_0^{-2} \simeq P_2' \oplus P_1 \oplus \Omega_0^{-2}$ and

$$P_2' \subset \mathrm{Ker}(h+g) \simeq H^1(C_{\bar{k}}, \mathbb{Z}/l).$$

Thus P_2' is isomorphic to a submodule of $P^\vee$ and we have

$$dim(\Lambda(P^\vee)) \geq dim(\Lambda(P_2')) = dim(\Lambda(\dot{P}^\vee)) - dim(\Lambda(Q \oplus P_1)). \tag{5.4.10}$$

(5.4.11) Sublemma. (i) *$dim(\Lambda(Q \oplus P_U)) \geq dim(\Lambda(Q \oplus P_1))$.*

(ii) *$dim(\Lambda(Q \oplus P_U)) = dim(\mathrm{Hom}_{\mathbb{F}_l[\Gamma]}(H^2_{C-\dot{C}}(C_{\bar{k}}, \mathbb{Z}/l), M^\vee))$.*

(iii) *As $\mathbb{F}_l[\Gamma]$-modules, $H^2_{C-\dot{C}}(C_{\bar{k}}, \mathbb{Z}/l) \simeq \bigoplus_{x \in (X-\dot{X})_{\bar{k}}} \mathrm{Ind}_{I_x}^{\Gamma} \mathbb{F}_l$.*

(iv) *$\mathrm{Hom}_{\mathbb{F}_l[\Gamma]}(\mathrm{Ind}_{I_x}^{\Gamma} \mathbb{F}_l, M^\vee) \simeq (M^\vee)^{I_x}$.*

(v) *$dim(M^{I_x}) = dim((M^\vee)^{I_x})$.*

Proof. (i) $Q \oplus P_1$ is isomorphic to a direct factor of $Q \oplus P_U$.

(ii) $\Lambda(Q \oplus P_U) \simeq \mathrm{Hom}_{\mathbb{F}_l[\Gamma]}(M^\vee, Q \oplus P_U)$. Since $M^\vee$ is absolutely irreducible, this is isomorphic to

$$\mathrm{Hom}_{\mathbb{F}_l[\Gamma]}(Q \oplus P_U, M^\vee) \simeq \mathrm{Hom}_{\mathbb{F}_l[\Gamma]}(Q \oplus U, M^\vee) \simeq \mathrm{Hom}_{\mathbb{F}_l[\Gamma]}(H^2_{C-\dot{C}}(C_{\bar{k}}, \mathbb{Z}/l)_0, M^\vee)$$

by [Al, p. 43 Theorem 6]. Now apply $\mathrm{Hom}_{\mathbb{F}_l[\Gamma]}(\quad, M^\vee)$ to

$$0 \to H^2_{C-\dot{C}}(C_{\bar{k}}, \mathbb{Z}/l)_0 \to H^2_{C-\dot{C}}(C_{\bar{k}}, \mathbb{Z}/l) \to \mathbb{F}_l \to 0$$

and observe that $\mathrm{Hom}_{\mathbb{F}_l[\Gamma]}(\mathbb{F}_l, M^\vee) = 0$ and $\mathrm{Ext}^1_{\mathbb{F}_l[\Gamma]}(\mathbb{F}_l, M^\vee) \simeq H^1(\Gamma, M^\vee) = 0$ (3.3.5).

(iii) $H^2_{C-\dot{C}}(C_{\bar{k}}, \mathbb{Z}/l) \simeq H^2_{C-\dot{C}}(C_{\bar{k}}, \mu_l)$ may be interpreted as divisors supported on $(C - \dot{C})_{\bar{k}}$ with $\mathbb{Z}/l$-coefficients. The isomorphism is then a standard fact about permutation modules [Se, 3.3, Ex. 2].

(iv) [Al, p. 58, Lemma 6 (1)].

(v) Let σ be a generator of the cyclic group I_x. The kernel and cokernel of $\sigma - 1 : M \to M$ have the same dimension. $(M^\vee)^{I_x}$ is the dual of the cokernel. □

Combining (5.4.10) and (5.4.11)(i) gives

$$dim(\Lambda(P^\vee)) \geq dim(\Lambda(\dot{P}^\vee)) - dim(\Lambda(Q \oplus P_U)).$$

By (5.4.11)(ii)–(v) the right-hand side is equal to

$$dim(\Lambda(\dot{P}^\vee)) - \sum_{x \in (X-\dot{X})_{\bar{k}}} dim(M^{I_x}).$$

This completes the proof of (5.4.7). □

Now (5.2.5) follows by combining (5.4.7), (5.4.1), and (5.4.4).

6. The structure of $H^2_{\rho^{-1}(x)}(C_{\bar{k}}, \mu_l)^{G_k}_0$ as an $\mathbb{F}_l[\Gamma]$-module

(6.1) Proposition. *Assume* (3.2.1–4) *and* (3.3.1–3). *Let $x \in X(k)$ be disjoint from the branch locus of ρ. Let $\xi \in \Gamma$ be the Frobenius element at a point $c \in \rho^{-1}(x)$. Let $\mathbf{c} \in \rho^{-1}(x)_{\bar{k}}$ lie above c. Write e for the order of ξ and define $\varsigma = 1 + \xi + ... + \xi^{e-1} \in \mathbb{F}_l[\Gamma]$.*

(i) *There are isomorphisms of $\mathbb{F}_l[\Gamma]$-modules*

$$H^2_{\rho^{-1}(x)}(C_{\bar{k}}, \mu_l)^{G_k} \simeq \mathbb{F}_l[\Gamma]\varsigma\mathbf{c} \simeq \operatorname{Ind}^{\Gamma}_{\langle\xi\rangle} \mathbb{F}_l.$$

(ii) *The natural map*

$$H^2_{\rho^{-1}(\{x,x_0\})}(C_{\bar{k}}, \mu_l)^{G_k}_0 / H^2_{\rho^{-1}(x_0)}(C_{\bar{k}}, \mu_l)^{G_k}_0 \to H^2_{\rho^{-1}(x)}(C_{\bar{k}}, \mu_l)^{G_k} \tag{6.1.1}$$

is an isomorphism for x_0 as in (3.2.3).

(iii) *The coboundary map $\delta_{3.4.4}$ has a natural extension to an $\mathbb{F}_l[\Gamma]$-linear map*

$$\delta_{6.1.2} : H^2_{\rho^{-1}(x)}(C_{\bar{k}}, \mu_l)^{G_k} \to H^1(G_k, H^1(C_{\bar{k}}, \mu_l)). \tag{6.1.2}$$

Assume now that $gcd(e,l) = 1$. Define $d_\xi = dim_{\mathbb{F}_l} M^{\langle\xi\rangle}$. Then

(iv) $H^2_{\rho^{-1}(x)}(C_{\bar{k}}, \mu_l)^{G_k} \simeq (P^\vee_M)^{d_\xi} \oplus P_0 \oplus Q'$ *where Q' is a projective $\mathbb{F}_l[\Gamma]$-module such that* $\operatorname{Hom}_{\mathbb{F}_l[\Gamma]}(Q', M^\vee \oplus \mathbb{F}_l) = 0$.

(v) $H^2_{\rho^{-1}(x)}(C_{\bar{k}}, \mu_l)^{G_k}_0 \simeq (P^\vee_M)^{d_\xi} \oplus rad(P_0) \oplus Q'$.

(vi) *The natural inclusion*

$$\Lambda(H^2_{\rho^{-1}(x)}(C_{\bar{k}}, \mu_l)^{G_k}_0) \to \Lambda(H^2_{\rho^{-1}(x)}(C_{\bar{k}}, \mu_l)^{G_k})$$

is an isomorphism.

Proof. (i) Let $\mathfrak{d}$ denote the group of divisors on $C_{\bar{k}}$ with support in $\rho^{-1}(x)_{\bar{k}}$. The cycle class map gives a $\Gamma \times G_k$-equivariant isomorphism $\mathfrak{d}/l \simeq H^2_{\rho^{-1}(x)}(C_{\bar{k}}, \mu_l)$. Now $(\mathfrak{d}/l)^{G_k}$ is canonically identified with the $\mathbb{F}_l$-vector space with basis $\rho^{-1}(x)$. (The elements of $\rho^{-1}(x)$ are the Galois orbits in $\rho^{-1}(x)_{\bar{k}}$.) For $\mathbf{c} \in C_{\bar{k}}$ a point above c, $\varsigma\mathbf{c} \in (\mathfrak{d}/l)^{G_k}$. Since $\rho^{-1}(x) \simeq \Gamma/\langle\xi\rangle$, (i) is a standard fact about permutation modules [Se, 3.3, Ex. 2].

(ii) It suffices to show that the natural map

$$H^1(G_k, H^2_{\rho^{-1}(x_0)}(C_{\bar{k}}, \mu_l)_0) \to H^1(G_k, H^2_{\rho^{-1}(\{x,x_0\})}(C_{\bar{k}}, \mu_l)_0)$$

is injective. From the direct sum decomposition

$$H^2_{\rho^{-1}(\{x,x_0\})}(C_{\bar{k}}, \mu_l) \simeq H^2_{\rho^{-1}(x)}(C_{\bar{k}}, \mu_l) \oplus H^2_{\rho^{-1}(x_0)}(C_{\bar{k}}, \mu_l)$$

and the fact that $\rho^{-1}(x_0)$ supports a k-rational divisor of degree prime to l (3.2.3) it is easy to show that $H^2_{\rho^{-1}(x_0)}(C_{\bar{k}}, \mu_l)_0$ is a direct summand of the G_k-module $H^2_{\rho^{-1}(\{x,x_0\})}(C_{\bar{k}}, \mu_l)_0$.

(iii) By (1.3.4) and (3.2.4) the coboundary map $\delta_{6.1.3}$ associated to

$$0 \to H^1(C_{\bar{k}}, \mu_l) \to H^1((C - \rho^{-1}(\{x,x_0\}))_{\bar{k}}, \mu_l) \to H^2_{\rho^{-1}(\{x,x_0\})}(C_{\bar{k}}, \mu_l)_0 \to 0 \tag{6.1.3}$$

annihilates the submodule

$$H^2_{\rho^{-1}(x_0)}(C_{\bar{k}}, \mu_l)^{G_k}_0 \subset H^2_{\rho^{-1}(\{x,x_0\})}(C_{\bar{k}}, \mu_l)^{G_k}_0.$$

By (ii) $\delta_{6.1.3}$ induces a map having all the properties desired of $\delta_{6.1.2}$. To check that this map extends $\delta_{3.4.4}$ one applies the natural restriction map from the exact sequence (3.4.4) to (6.1.3).

(iv) Since $gcd(l,e)=1$, $\mathbb{F}_l$ is a projective $\mathbb{F}_l[\langle\xi\rangle]$-module. Thus $\mathrm{Ind}^{\Gamma}_{\langle\xi\rangle}\mathbb{F}_l$ is a projective $\mathbb{F}_l[\Gamma]$-module. The multiplicity with which $P_M^{\vee}$ occurs in this module is given by the dimension of

$$\mathrm{Hom}_{\mathbb{F}_l[\Gamma]}(\mathrm{Ind}^{\Gamma}_{\langle\xi\rangle}\mathbb{F}_l, M^{\vee}) \simeq \mathrm{Hom}_{\mathbb{F}_l[\langle\xi\rangle]}(\mathbb{F}_l, M^{\vee}) \simeq (M^{\vee})^{\langle\xi\rangle}.$$

By (5.4.11)(iv) $dim(M^{\vee})^{\langle\xi\rangle} = dim(M^{\langle\xi\rangle})$. An analogous calculation shows that P_0 occurs with multiplicity 1.

(v) The cokernel of the inclusion

$$H^2_{\rho^{-1}(x)}(C_{\bar{k}},\mu_l)_0 \to H^2_{\rho^{-1}(x)}(C_{\bar{k}},\mu_l)$$

is isomorphic to the trivial module, $\mathbb{F}_l$. Since $\rho^{-1}(x)$ supports a G_k-invariant divisor of degree prime to l, part (v) follows from (iv).

(vi) This was shown in the proof of (4.2.1). □

7. Proof of Theorem (3.4)

7.1. By (3.3.6) there is an element $\xi\in\Gamma$ of order e prime to l for which $M^{\langle\xi\rangle}\simeq\mathbb{F}_l$. Write $\tau: C\to\langle\xi\rangle\backslash C =: \bar{C}$ for the canonical quotient morphism. Choose points $x_0\in X_k$ and $c_0\in C$ with residue fields k as in (3.2.3–4). Define $\bar{c}_0=\tau(c_0)$. Define a curve $\tilde{C}$ by the Cartesian diagram

$$\begin{array}{ccc} \tilde{C} & \longrightarrow & Pic^0(\bar{C}) \\ \big\downarrow & & \big\downarrow{\scriptstyle m_l} \\ \bar{C} & \xrightarrow{\bar{i}_0} & Pic^0(\bar{C}), \end{array}$$

where $\bar{i}_0(\bar{c}) = \mathcal{O}_{\bar{C}}(\bar{c}-deg(\bar{c})\bar{c}_0)$ and m_l is multiplication by l. Since $gcd(e,l)=1$, the map

$$\tau^*: H^1(\bar{C}_{\bar{k}},\mu_l)\to(1+\xi+...+\xi^{e-1})_*H^1(C_{\bar{k}},\mu_l) \tag{7.1.1}$$

is an isomorphism. By (3.2.5) and (1.3.3)(iii) there are canonical isomorphisms

$$H^1(G_k,H^1(\bar{C}_{\bar{k}},\mu_l))\simeq H^1(\bar{C}_{\bar{k}},\mu_l)\simeq Gal(k(\tilde{C})/k(\bar{C})). \tag{7.1.2}$$

Now Theorem (3.4) is an immediate consequence of the following more precise result:

(7.2) Proposition. *Let $\varepsilon\in H^1(G_k,H^1(X_{\bar{k}},\mathcal{M}))$ be non-zero. Assume that the hypotheses* (3.2.1–6) *and* (3.3.1–6) *hold. Then*

(i) *there is a canonical isomorphism $H^1(G_k,H^1(X_{\bar{k}},\mathcal{M}))\simeq H^1(X_{\bar{k}},\mathcal{M})$;*

(ii) *there is an element $p\in(1+\xi+...+\xi^{e-1})_*H^1(C_{\bar{k}},\mathbb{Z}/l)$ such that the submodule $\mathbb{F}_l[\Gamma]p\subset H^1(C_{\bar{k}},\mathbb{Z}/l)$ is isomorphic to $P_M^{\vee}$ and the image of $\mathbb{F}_l\varepsilon$ under the map $\beta: H^1(X_{\bar{k}},\mathcal{M})\to\Lambda(H^1(C_{\bar{k}},\mathbb{Z}/l))$ of* (4.1) *coincides with the image of the induced map $\Lambda(P_M^{\vee})\to\Lambda(H^1(C_{\bar{k}},\mathbb{Z}/l))$;*

(iii) *there is an element $f\in Gal(k(\check{C})/k(\bar{C}))$ such that the image of f in $Gal(k(C)/k(\bar{C}))$ is ξ and the image of f in $Gal(k(\tilde{C})/k(\bar{C}))$ may be identified via* (7.1.1–2) *with an element of $(\mathbb{Z}/l)^*p\subset(1+\xi+...+\xi^{e-1})_*H^1(C_{\bar{k}},\mu_l)$;*

(iv) *there is a degree one point $x\in\ddot{X}\cap(X-\Sigma)$ with the property that f is contained in the Frobenius conjugacy class $Frob_x\subset Gal(k(\check{C})/k(X))$. For such an x, $\delta_{3.1.1}$ gives an isomorphism, $H^2_x(X_{\bar{k}},\mathcal{M})^{G_k}\to\mathbb{F}_l\varepsilon$.*

Proof. (i) The map $\beta : H^1(X_{\bar{k}}, \mathcal{M}) \to \Lambda(H^1(C_{\bar{k}}, \mathbb{Z}/l))$ is injective by (5.2.1). G_k acts trivially on the image by (3.2.5) and (3.2.2). Thus

$$H^1(G_k, H^1(X_{\bar{k}}, \mathcal{M})) \simeq Hom(G_k, H^1(X_{\bar{k}}, \mathcal{M})) \simeq H^1(X_{\bar{k}}, \mathcal{M}),$$

where the second isomorphism is obtained by evaluating a homomorphism at the Frobenius element.

(ii) Let $h := dim(H^1(X_{\bar{k}}, \mathcal{M}))$. By (5.2.1) there is an injective homomorphism $\iota : (P_M^h)^\vee \to H^1(C_{\bar{k}}, \mathbb{Z}/l)$ which gives rise to an isomorphism $im(\beta) \simeq \Lambda((P_M^h)^\vee)$. Applying β to $\mathbb{F}_l\varepsilon$ gives a one-dimensional subspace of $\Lambda((P_M^h)^\vee)$. Any such subspace may be obtained as the image of $\Lambda(\epsilon) : \Lambda(P_M^\vee) \to \Lambda((P_M^h)^\vee)$, where $\epsilon : P_M^\vee \to (P_M^h)^\vee$ is an appropriate injective $\mathbb{F}_l[\Gamma]$-module homomorphism.

Since $P_{M^\vee} \simeq P_M^\vee$, it remains only to show that $P_{M^\vee}$ has a generator contained in $(1+\xi+...+\xi^{e-1})P_{M^\vee}$. We need only check that $(1+\xi+...+\xi^{e-1})P_{M^\vee} \not\subset rad(P_{M^\vee})$. For this note that $M^{\langle\xi\rangle} \neq 0$ implies $(M^\vee)^{\langle\xi\rangle} \neq 0$, since M is a semi-simple $\mathbb{F}_l[\langle\xi\rangle]$-module. Now $gcd(e, l) = 1$ implies that the second map in the composition

$$(1 + \xi + ... + \xi^{e-1})P_{M^\vee} \to (1 + \xi + ... + \xi^{e-1})(P_{M^\vee}/rad(P_{M^\vee})) \to (M^\vee)^{\langle\xi\rangle}$$

is an isomorphism, so the composition is surjective.

For the remainder of the proof we fix an identification $\mathbb{Z}/l \simeq \mu_l$. This is permissable by (3.2.2).

(iii) Define $\hat{C} := C \times_{\bar{C}} \tilde{C}$. Now $\hat{C}$, being a quotient of $\breve{C}$, is geometrically irreducible. Clearly there is an isomorphism

$$Gal(k(\hat{C})/k(\bar{C})) \simeq Gal(k(C)/k(\bar{C})) \times Gal(k(\tilde{C})/k(\bar{C})). \tag{7.2.1}$$

Now (iii) follows from (7.1.1–2).

(iv) By (3.2.6) there is a point $\breve{c} \in \breve{C}$ whose image $x \in X_k$ lies in $\ddot{X}_k \cap (X_k - \Sigma)$ and has residue field k and such that $f \in Gal(k(\breve{C})/k(X))$ is the Frobenius at $\breve{c}$. The Frobenius at the image $c \in C$ of $\breve{c}$ is ξ. Since $dim(M^{\langle\xi\rangle}) = 1$, (6.1)(iv) implies that $\Lambda(H^2_{\rho^{-1}(x)}(C_{\bar{k}}, \mu_l)^{G_k}) \simeq \mathbb{F}_l$. The isomorphism $\beta_\bullet$ of (4.2.1) shows that $H^2_x(X_{\bar{k}}, \mathcal{M})^{G_k}$ has dimension one.

Set $\bar{c} = \tau(c)$. Since $\bar{c}$ has degree one, the zero cycle $\bar{c} - \bar{c}_0$ has degree zero. Let $\delta_{7.2.2}$ be the first coboundary map associated to the short exact sequence of G_k-modules

$$0 \to H^1(\bar{C}, \mu_l) \to H^1((\bar{C} - \{\bar{c}, \bar{c}_0\})_{\bar{k}}, \mu_l) \to H^2_{\{\bar{c},\bar{c}_0\}}(\bar{C}_{\bar{k}}, \mu_l)_0 \to 0. \tag{7.2.2}$$

By (iii) and (1.3.4)

(7.2.3)
$$\delta_{6.1.3}(\tau^*(\bar{c} - \bar{c}_0)) = \tau^*(\delta_{7.2.2}(\bar{c} - \bar{c}_0)) \in (\mathbb{Z}/l)^* p \subset (1 + \xi + ... + \xi^{e-1})H^1(C_{\bar{k}}, \mu_l).$$

Recall from (6.1)(iii) that $\delta_{6.1.3}$ induces an $\mathbb{F}_l[\Gamma]$-linear map

$$\delta_{6.1.2} : H^2_{\rho^{-1}(x)}(C_{\bar{k}}, \mu_l)^{G_k} \to H^1(G_k, H^1(C_{\bar{k}}, \mu_l)) \simeq H^1(C_{\bar{k}}, \mu_l).$$

By (6.1)(i) $H^2_{\rho^{-1}(x)}(C_{\bar{k}}, \mu_l)^{G_k}$ is a cyclic $\mathbb{F}_l[\Gamma]$-module generated by the image of $\tau^*(\bar{c} - \bar{c}_0)$ under (6.1.1). Thus (7.2.3) implies that the image of $\delta_{6.1.2}$ is $\mathbb{F}_l[\Gamma]p$. Now (ii) identifies the image of

$$\Lambda(\delta_{6.1.2}) : \Lambda(H^2_{\rho^{-1}(x)}(C_{\bar{k}}, \mu_l)^{G_k}) \to \Lambda(H^1(C_{\bar{k}}, \mu_l))$$

with $\beta(\mathbb{F}_l\varepsilon)$. By (6.1)(iii) and (vi) the images of $\Lambda(\delta_{6.1.2})$ and $\Lambda(\delta_{3.4.4})$ coincide. Since $\Lambda(\delta_{3.4.4})$ may be identified with $\delta_{3.1.1}$ by (4.3.1)(iii), the proposition follows. □

8. Tate classes

In this section we construct Tate classes in the fibers of certain proper, flat morphisms from odd dimensional varieties to curves.

8.1. Preliminaries on orthogonal groups. Let $\mathbf{E}$ be a vector space of odd dimension $2s+1$ over a field K of characteristic different from 2. Let $(\ ,\ ) : \mathbf{E} \otimes \mathbf{E} \to K$ be a non-degenerate symmetric bilinear pairing. Write $\mathbf{O}(\mathbf{E})$ (respectively $\mathbf{SO}(\mathbf{E})$) for the orthogonal (respectively special orthogonal) group of $(\ ,\ )$.

(8.1.1) Lemma. *Let $F \in \mathbf{O}(\mathbf{E})$. The eigenvalues $\lambda_0, \lambda_1, ..., \lambda_{2s} \in \bar{K}$ of F may be indexed so that*

$$\lambda_0 \in \{\pm 1\} \qquad \text{and} \qquad \lambda_i \lambda_{s+i} = 1 \quad \text{for} \quad 1 \leq i \leq s. \tag{8.1.2}$$

In particular if $F \in \mathbf{SO}(\mathbf{E})$, then $\lambda_0 = 1$.

Proof. Extend scalars to the algebraic closure and let $F^{s.s.}$ denote the semi-simple part in the Jordan decomposition of F [Hu, 15.2–3]. Let $\{v_i\}_{0 \leq i \leq 2s}$ be an eigenvector basis for $F^{s.s.}$ with corresponding eigenvalues $\{\lambda_i\}_{0 \leq i \leq 2s}$. Since $(\ ,\ )$ is non-degenerate, for each index $i \in \{0, 1, ..., 2s\}$ there exists an index i' such that $(v_i, v_{i'}) \neq 0$. Now

$$\lambda_i \lambda_{i'} (v_i, v_{i'}) = (F^{s.s.} v_i, F^{s.s.} v_{i'}) = (v_i, v_{i'}) \Rightarrow \lambda_i \lambda_{i'} = 1.$$

If for every i we have $i = i'$, then $\lambda_i^2 = 1$ for all i and the assertion follows. This applies in particular when $s = 0$. If there is some index i with $(v_i, v_i) = 0$, then $i' \neq i$ and

$$\mathbf{E} \simeq Span\{v_i, v_{i'}\} \oplus Span\{v_i, v_{i'}\}^{\perp}.$$

The lemma follows by induction on s. □

Let l be an odd prime and let E be a free $\mathbb{Z}_l$-module of odd rank $2s+1$ endowed with a symmetric bilinear pairing

$$\langle\ ,\ \rangle : E \otimes E \to \mathbb{Z}_l,$$

which induces an isomorphism $E \to \mathrm{Hom}(E, \mathbb{Z}_l)$. The orthogonal (respectively special orthogonal group) of $\langle\ ,\ \rangle$ will be denoted $\mathbf{O}(E)$ (respectively $\mathbf{SO}(E)$). Write

$$(\ ,\ ) : E/l \otimes E/l \to \mathbb{Z}/l$$

for the induced non-degenerate pairing.

(8.1.3) Lemma. *Let $F \in \mathbf{O}(E)$ and write $\xi \in GL(E/l)$ for the image of F under reduction mod l. If the order of ξ is prime to l and $dim_{\mathbb{F}_l}(E/l)^{\langle \xi \rangle} = 1$, then $F \in \mathbf{SO}(E)$, $E^{\langle F \rangle} \simeq \mathbb{Z}_l$ and the natural map $E^{\langle F \rangle}/l \to (E/l)^{\langle \xi \rangle}$ is an isomorphism.*

Proof. Write $Q(T) \in \mathbb{Z}_l[T]$ for the characteristic polynomial of F. Its roots are integral over $\mathbb{Z}_l$ and satisfy (8.1.2). Reduction mod l gives $\bar{Q}(T) \in \mathbb{Z}/l[T]$ with roots

$$\bar{\lambda}_0 \in \{\pm 1\} \qquad \text{and} \qquad \bar{\lambda}_i \bar{\lambda}_{s+i} = 1 \quad \text{for} \quad 1 \leq i \leq s.$$

These are the eigenvalues of ξ. The hypotheses on ξ imply that it is semi-simple and that 1 is an eigenvalue with multiplicity one. Thus $\bar{\lambda}_0 = 1$ and $\bar{\lambda}_j \neq 1$ for $j \geq 1$. Consequently, $\lambda_0 = 1$ and $\lambda_j \neq 1$ for $j \geq 1$. Clearly $F \in \mathbf{SO}(E)$ and $E^{\langle F \rangle} \simeq \mathbb{Z}_l$ is a saturated subgroup of E. Thus $E^{\langle F \rangle}/l \to (E/l)^{\langle \xi \rangle}$ is injective which proves the lemma. □

8.2. Existence of Tate classes. Recall the geometric situation (2.1.1). Suppose E is a torsion free, Galois invariant, orthogonal direct factor of $H^{2m}(f^{-1}(\bar{\eta}), \mathbb{Z}_l(m))$ with respect to the cup product pairing. Let $\langle \, , \, \rangle$ denote $(-1)^{m+1}$ times this pairing. By Poincaré duality it induces an isomorphism $E \to \mathrm{Hom}(E, \mathbb{Z}_l)$. The image of the corresponding Galois representation lies in the orthogonal group:

$$\tilde{\kappa} : \pi_1(\dot{X}, \bar{\eta}) \to \mathbf{O}(E). \tag{8.2.1}$$

Associated to this Galois module is an l-adic sheaf on the generic point η whose direct image under $g : \eta \to X$ will be denoted $\mathcal{E} = \{\mathcal{E}_r\}_{r \in \mathbb{N}}$. In particular, $\mathcal{E}_1$ is the direct image of the sheaf on η corresponding to the Galois module E/l. Suppose we are given $x \in \dot{X}(k)$ and a geometric point $\bar{x}$ above x. The Galois group $G_k \simeq \pi_1(x, \bar{x})$ acts on the stalks $\mathcal{E}_{r\bar{x}}, r \in \mathbb{N}$, and hence on $\mathcal{E}_{\bar{x}} := \varprojlim_r \mathcal{E}_{r\bar{x}}$. Suppose that E has odd rank and that $dim_{\mathbb{F}_l} H^2_x(X_{\bar{k}}, \mathcal{E}_1(1))^{G_k} = 1$.

(8.2.2) Lemma. *$\mathcal{E}_{\bar{x}}^{G_k} \simeq \mathbb{Z}_l$ and $\mathcal{E}_{\bar{x}}^{G_k}/l \simeq H^2_x(X_{\bar{k}}, \mathcal{E}_1(1))^{G_k}$.*

Proof. Since $\mathcal{E}_{1\bar{x}}^{G_k} \simeq H^2_x(X_{\bar{k}}, \mathcal{E}_1(1))^{G_k}$ and G_k is topologically generated by a single element, this follows from (8.1.3). □

Set $V = f^{-1}(x)$. The Tate conjecture implies that there is an algebraic cycle in $Z^m(V)$ whose cohomology class is a non-zero element of $\mathcal{E}_{\bar{x}}^{G_k}$. We would actually like to have a cycle whose cohomology class generates $\mathcal{E}_{\bar{x}}^{G_k}$. When this happens, reduction mod l gives a generator $\bar{z} \in H^2_x(X_{\bar{k}}, \mathcal{E}_1(1))^{G_k}$ which plays a crucial role in our analysis of the l-adic Abel-Jacobi map (cf. §2.4). The following lemma, which I learned from S. Bloch, shows that the Tate conjecture implies the existence of an algebraic cycle with the properties we want when $m = 1$.

(8.2.3) Lemma. *Let V/k be a smooth complete variety over a finite field. The cokernel of the cycle class map $CH^1(V) \otimes \mathbb{Z}_l \to H^2(V_{\bar{k}}, \mathbb{Z}_l(1))^{G_k}$ is torsion free.*

Proof. The cokernel is the Tate module of the Brauer group which is always torsion free [Ta, 5.10]. □

It is not clear if the cokernel of the cycle class map should be torsion free in higher codimension. (See [Sch6] for the case of cycles of dimension 1.)

9. Lefschetz pencils

The purpose of this section is to show that the machinery developed so far can be applied when the geometric situation (2.1.1) arises from a Lefschetz pencil of hyperplane sections. The vanishing cohomology will play the role of the Galois module E of §2.3. The first task is to describe under what conditions the rank of E is odd, since it is only in this case that the methods of §8 produce Tate cycles in the stalks of the associated sheaf $\mathcal{E}$. This is easy and is done in §9.2. The next task is to show that the monodromy representation on $M := E/l$ satisfies the hypotheses (3.3.1–6) so that Theorem (3.4) may be used to show the surjectivity of a piece of the mod l Abel-Jacobi map. For this it is necessary to have a rather precise description of the monodromy group $\Gamma \subset \mathrm{Aut}(M)$. In §9.4 we give an exact description of Γ when there is a lifting to characteristic zero using the theory of (symmetric) vanishing lattices developed in [Eb] and [Be]. The verification of the hypotheses (3.3.1–6) is carried out in §9.5.

9.1. Definition and existence of Lefschetz pencils. Let Y be a smooth projective variety of dimension $2m+1 \geq 3$ over a field K of characteristic different from 2. Let L be a very ample invertible sheaf on Y. Fix a basis for $H^0(Y, L)$. This specifies an embedding $Y \subset \mathbb{P}(H^0(Y, L)^\vee)$. A one-dimensional linear subspace $X \subset \mathbb{P}H^0(Y, L)$ determines a family of hyperplanes $\{H_x\}_{x \in X}$ in $\mathbb{P}(H^0(Y, L)^\vee)$ whose intersection is a codimension 2 linear subspace $\mathbf{B}$. In this paper we say that X is a Lefschetz pencil if the following five conditions are met:

(i) $\mathbf{B}$ meets Y transversely.

(ii) The locus $\dot{X} \subset X$ of points x such that H_x meets Y transversely is non-empty.

(iii) $\Sigma := X - \dot{X}$ consists of K-rational points $\{x_i\}$ and the scheme-theoretic intersection $H_{x_i} \cap Y$ is non-singular except for a single ordinary double point [SGA 7, XVII.1].

(iv) $\dot{X}(K) \neq \emptyset$.

(v) The dual variety $Y^\vee \subset \mathbb{P}(H^0(Y, L))$ of Y is a hypersurface.

Associated to a Lefschetz pencil there is a diagram

$$\begin{tikzcd}
 & & P \arrow[r] \arrow[d] & B \arrow[d] \\
V \arrow[r, "i_V"] \arrow[d] & \dot{W} \arrow[r] \arrow[d, "\dot{f}"'] & W \arrow[r, "\sigma"] \arrow[d, "f"] & Y \\
x \arrow[r, "i_x"] & \dot{X} \arrow[r, "j"] & X & \eta, \arrow[l, "g"']
\end{tikzcd} \tag{9.1.1}$$

where B is the base locus of the pencil, σ is the blow-up of Y along B, $\dot{f}$ is smooth, i_x is a K-rational point, g is the inclusion of the generic point, and all squares are Cartesian.

(9.1.2) Proposition. *Let Y and K be as above and suppose that L_0 is a very ample invertible sheaf on Y. Then there is an integer n_0 such that for any $n \geq n_0$ there exist a finite extension $K \subset K_n$ and a Lefschetz pencil $X \subset \mathbb{P}H^0(Y_{K_n}, L_0^{\otimes n})$.*

Proof. Most of this is in [SGA 7, XVII.3]. The fact that $Y^\vee$ will be a hypersurface for n sufficiently large is a consequence of the degree formula in [Fu, 3.2.21]. □

9.2. The parity of the rank of the vanishing cohomology. We keep the previous notation and assume that the Lefschetz pencil is defined over the base field K. In particular $V \subset Y$ is a non-singular divisor in the linear system $|L| = |L_0^{\otimes n}|$. Write $s_V : V \to Y$ for the inclusion, $\Theta_{V/K}$ for the tangent sheaf to V and $E \subset H^{2m}(V_{\bar{K}}, \mathbb{Z}_l(m))$ for the vanishing cohomology [De2, 4.2.4].

(9.2.1) Lemma. (i) *If n is even and $h^{2m}(Y_{\bar{K}})$ is odd, then rank(E) is odd.*

(ii) *If $h^{2m}(Y'_{\bar{K}})$ is even and $Y \to Y'$ is the blow up at a point of odd degree, then $h^{2m}(Y_{\bar{K}})$ is odd.*

(iii) *Every smooth projective variety over a finite field has a point of odd degree.*

Proof. (i) By Poincaré duality the topological Euler characteristic, $e(V)$, is congruent to $h^{2m}(V_{\bar{K}})$ modulo 2. A standard Chern class computation yields

$$e(V) = deg(s_{V*}c_{2m}(\Theta_{V/K})) = deg(s_{V*}s_V^*(\mathfrak{z})),$$

where $\mathfrak{z}$ is the codimension $2m$ part of the product of total Chern classes $c_\bullet(\Theta_{Y/K}) \cdot c_\bullet(L^{-1})$.

Assume $n \equiv 0 \mod 2$. Since

$$s_{V*}s_V^*(\mathfrak{z}) = nc_1(L_0) \cdot \mathfrak{z},$$

we find that $e(V)$ and hence $h^{2m}(V_{\bar{K}})$ is even. The hard Lefschetz theorem gives an isomorphism

$$H^{2m}(V_{\bar{K}}, \mathbb{Q}_l(m)) \simeq E \otimes \mathbb{Q}_l \quad \oplus \quad H^{2m}(Y_{\bar{K}}, \mathbb{Q}_l(m)).$$

Clearly E has odd rank.

(ii) This is standard.

(iii) It suffices to prove this for curves, in which case it follows from the Riemann hypothesis for the zeta function of the curve. □

9.3. Lifting to characteristic zero. In this subsection we discuss lifting Lefschetz pencils to characteristic 0 and we compare the monodromy associated to the original Lefschetz pencil with that of its lift.

(9.3.1). Let k_0 be a finite field of characteristic $\neq 2$ and let Y/k_0 be a smooth, projective variety with an ample invertible sheaf L_0. Write R for the Witt vectors of k_0, $S := Spec(R)$, and $\epsilon \in S$ for the generic point. Assume that there is a projective, flat morphism $h : \mathcal{Y} \to S$ whose special fiber is isomorphic to Y. Assume in addition that there is an invertible sheaf $\mathcal{L}_0$ on $\mathcal{Y}$ whose restriction to Y is isomorphic to L_0. We refer to $(\mathcal{Y}, \mathcal{L}_0)$ as a lifting of (Y, L_0).

(9.3.2) Lemma. *There is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$*

(i) *$L_0^{\otimes n}$ is very ample,*

(ii) *$H^i(Y, L_0^{\otimes n}) = 0$ for $i > 0$, and*

(iii) *after replacing k_0 by a finite extension if necessary, there is a Lefschetz pencil $X \subset \mathbb{P}H^0(Y, L_0^{\otimes n})$.*

Proof. [Ha, II.Ex7.5e, III.5.3]. □

Fix an n as in (9.3.2) and define $L := L_0^{\otimes n}$ and $\mathcal{L} := \mathcal{L}_0^{\otimes n}$.

(9.3.3) Lemma. (i) *$R^i h_* \mathcal{L} = 0$ for $i > 0$.*

(ii) *$h_*\mathcal{L}$ is the sheaf associated to a finitely generated, free R-module $\mathbb{L}$.*

(iii) *The natural map $\mathbb{L} \otimes_R \mathcal{O}_{\mathcal{Y}} \to \mathcal{L}$ is surjective and gives rise to an embedding of S-schemes $\mathcal{Y} \to \mathbb{P}\mathbb{L}^\vee$.*

(iv) *Any lift $\mathcal{X} \subset \mathbb{P}\mathbb{L}$ of $X \subset \mathbb{P}H^0(Y, L)$ has the property that the generic fiber $\mathcal{X}_\epsilon$ gives a Lefschetz pencil on $\mathcal{Y}_\epsilon$ in the sense of* §9.1. *Furthermore the locus $\Sigma_S \subset \mathcal{X}$ where $\mathcal{X}$ meets the dual of $\mathcal{Y}$ is étale over S.*

Proof. (i) follows by upper semi-continuity [Ha, III.12.8-9]. (ii) follows from [Ha, 12.8-9] and the constancy of Euler characteristics in a flat family [Ha, III.9.9]. (iii) The surjectivity is a consequence of L being generated by global sections and Nakayama's lemma. The resulting morphism of S-schemes is an embedding because it is so on the special fiber. (iv) Most of this is proved in [SGA 7, XVII.6]. To verify condition (9.1)(iv) use Hensel's Lemma. To show that the dual of $\mathcal{Y}_\epsilon$ is a hypersurface recall that the dual of $\mathcal{Y}$ is obtained by projecting the projectivized conormal bundle $\mathbb{P}_{\mathcal{Y}}(\mathcal{N}^{\vee}_{\mathcal{Y}/\mathbb{PL}^{\vee}}) \subset \mathbb{PL} \times \mathbb{PL}^{\vee}$ to the second factor. Since $Y^{\vee}$ is a hypersurface, the irreducibility of $\mathbb{P}_{\mathcal{Y}}(\mathcal{N}^{\vee}_{\mathcal{Y}/\mathbb{PL}^{\vee}})$ implies that the dual variety of $\mathcal{Y}_{\epsilon}$ is also a hypersurface. □

Let $\mathcal{B} \subset \mathbb{PL}^{\vee}$ be the linear space dual to $\mathcal{X}$. The intersection $\mathcal{B} \cap \mathcal{Y}$ is transverse since it is transverse along the special fiber. Blowing up $\mathcal{Y}$ along $\mathcal{B} \cap \mathcal{Y}$ gives rise to a regular $\mathcal{X}$-scheme $F : \mathcal{W} \to \mathcal{X}$. Set $\dot{\mathcal{X}} := \mathcal{X} - \Sigma_S$. Consider the diagram

$$\begin{array}{ccccc} \dot{W} & \longrightarrow & \dot{\mathcal{W}} & \longrightarrow & \mathcal{W} \\ {\scriptstyle \dot{f}}\downarrow & & {\scriptstyle \dot{F}}\downarrow & & {\scriptstyle F}\downarrow \\ \dot{X} & \longrightarrow & \dot{\mathcal{X}} & \longrightarrow & \mathcal{X}, \end{array}$$

in which the squares are Cartesian. Both $\dot{F}$ and $\dot{f}$ are smooth. Let $\bar{\eta}_S$ be a geometric generic point of $\dot{\mathcal{X}}$. Let $\bar{R}$ be the Witt vectors for $\bar{k}$, the algebraic closure of k_0, and let $\bar{\epsilon}$ denote the generic point of $\bar{S} := Spec(\bar{R})$.

(9.3.4) Lemma. *There is a (non-canonical) identification of the images of the two representations*

$$\pi_1(\dot{\mathcal{X}}_{\bar{\epsilon}}, \bar{\eta}_S) \to \operatorname{Aut}(\{R^{2m}\dot{F}_*\mathbb{Z}/l^r(m)_{|\bar{\eta}_S}\}_{r\in\mathbb{N}})$$

and

$$\pi_1(\dot{X}_{\bar{k}}, \bar{\eta}) \to \operatorname{Aut}(\{R^{2m}\dot{f}_*\mathbb{Z}/l^r(m)_{|\bar{\eta}}\}_{r\in\mathbb{N}}),$$

where the brackets $\{\ \}_{r\in\mathbb{N}}$ indicate an inverse system.

Proof. Since $\dot{F}$ is smooth, the first representation factors through $\pi_1(\dot{\mathcal{X}}_{\bar{S}}, \bar{\eta}_S)$. In fact there are group homomorphisms

$$\pi_1(\dot{\mathcal{X}}_{\bar{\epsilon}}, \bar{\eta}_S) \twoheadrightarrow \pi_1(\dot{\mathcal{X}}_{\bar{S}}, \bar{\eta}_S) \simeq \pi_1(\dot{\mathcal{X}}_{\bar{S}}, \bar{\eta}) \xleftarrow{\ i_*\ } \pi_1(\dot{X}_{\bar{k}}, \bar{\eta})$$

and an isomorphism of inverse systems

$$\{R^{2m}\dot{F}_*\mathbb{Z}/l^r(m)_{|\bar{\eta}}\}_{r\in\mathbb{N}} \to \{R^{2m}\dot{F}_*\mathbb{Z}/l^r(m)_{|\bar{\eta}_S}\}_{r\in\mathbb{N}}$$

such that the action of $\pi_1(\dot{\mathcal{X}}_{\bar{S}}, \bar{\eta})$ on the left corresponds to the action of $\pi_1(\dot{\mathcal{X}}_{\bar{S}}, \bar{\eta}_S)$ on the right. The integral closure of $\dot{\mathcal{X}}_{\bar{S}}$ in the fixed field of the kernel of

$$\varsigma_r : \pi_1(\dot{\mathcal{X}}_{\bar{S}}, \bar{\eta}) \to \operatorname{Aut}(R^{2m}\dot{F}_*\mathbb{Z}/l^r(m)_{|\bar{\eta}})$$

is an irreducible étale $\dot{\mathcal{X}}_{\bar{S}}$ -scheme, ${}_r\dot{\mathcal{C}}_{\bar{S}}$. Since $\mathcal{X}_{\bar{S}} - \dot{\mathcal{X}}_{\bar{S}}$ is étale over $\bar{S}$, a theorem of Fulton [Fu2, 3.3] implies that the closed fiber of ${}_r\dot{\mathcal{C}}_{\bar{S}}$ is irreducible. In other words, ς_r and $\varsigma_r \circ i_*$ have the same image. An application of the base change isomorphism,

$$R^{2m}\dot{F}_*\mathbb{Z}/l^r(m)_{|\bar{\eta}} \simeq R^{2m}\dot{f}_*\mathbb{Z}/l^r(m)_{|\bar{\eta}},$$

completes the proof. □

(9.3.5) Corollary. *Suppose that there is a direct sum decomposition of inverse systems of representations of $\pi_1(\dot{\mathcal{X}}_{\bar{\epsilon}}, \bar{\eta}_S)$,*

$$\{R^{2m}\dot{F}_*\mathbb{Z}/l^r(m)_{|\bar{\eta}_S}\}_{r\in\mathbb{N}} \simeq \{E_r\}_{r\in\mathbb{N}} \oplus \{\breve{E}_r\}_{r\in\mathbb{N}}.$$

Then there is a corresponding decomposition of $\{R^{2m}\dot{f}_*\mathbb{Z}/l^r(m)_{|\bar{\eta}}\}_{r\in\mathbb{N}}$ as a representation of $\pi_1(\dot{X}_{\bar{k}}, \bar{\eta})$.

9.4. Vanishing lattices and a description of the monodromy group. In this subsection we temporarily leave the category of schemes and work with complex analytic spaces. The notion of Lefschetz pencil makes sense for closed complex submanifolds of projective space. We use the same notation (X, Σ, V, etc.) introduced in §9.1 for Lefschetz pencils of varieties to signify the corresponding objects in the category of complex analytic spaces.

For each homotopy class of paths $\gamma : [0,1] \to X$ with $\gamma(0) = x$, $\gamma(1) \in \Sigma$ and $\gamma((0,1)) \subset \dot{X}$ there are associated vanishing cycles $\pm\delta_\gamma \in H_{2m}(V, \mathbb{Z})$. The subgroup generated by all vanishing cycles, E, will be viewed as a subgroup of $H^{2m}(V, \mathbb{Z})$ by Poincaré duality. Fix one vanishing cycle $\delta_{\gamma_0} \in E$ and let $\Delta \subset H^{2m}(V, \mathbb{Z})$ denote the orbit of δ_{γ_0} under the action of $\pi_1(\dot{X}, x)$. The following is well known.

(9.4.1) Proposition. (i) *For each homotopy class of paths γ_1 joining x and a point in Σ there is $g \in \pi_1(\dot{X}, x)$ such that $g(\delta_{\gamma_0}) = \pm\delta_{\gamma_1}$.*

(ii) *The set Δ consists of vanishing cycles and generates E.*

(iii) *For each $\delta \in \Delta$, $\delta \cdot \delta = (-1)^m \cdot 2$.*

(iv) *There is an element $g_\gamma \in \pi_1(\dot{X}, x)$ whose action on $H^{2m}(V, \mathbb{Z})$ is described by*

$$g_\gamma(\xi) = \xi + (-1)^{m+1}(\xi \cdot \delta_\gamma)\delta_\gamma.$$

Proof. (i) [Lam, 7.3.5]. (ii) follows from (i). (iii) [Lam, 6.3.2]. (iv) [Lam, 6.3.3]. □

(9.4.2) Definition ([Be, §2], [Eb, §2])**.** We call a pair (E', Δ') a vanishing lattice if

(i) E' is a free, finite rank $\mathbb{Z}$-module with a symmetric bilinear form $\langle\ ,\ \rangle$.

(ii) $\Delta' \subset E'$ is a generating set consisting of elements δ with $\langle \delta, \delta \rangle = -2$.

(iii) The subgroup $\Gamma_{\Delta'} \subset GL(E')$ generated by the reflections

$$g_\delta(\xi) = \xi + \langle \xi, \delta \rangle \delta, \qquad \delta \in \Delta',$$

acts transitively on Δ'.

(9.4.3) Example. (i) Let E' denote E modulo its torsion subgroup, let Δ' be the image of Δ and define $\langle \xi_1, \xi_2 \rangle = (-1)^{m+1}\xi_1 \cdot \xi_2$. Then (E', Δ') is a vanishing lattice and $\langle\ ,\ \rangle$ is non-degenerate.

(ii) The middle dimensional homology of the Milnor fiber of an isolated singularity on a $2m$-dimensional complex analytic space gives rise to a vanishing lattice. Here again there is a notion of vanishing cycle and intersection product [Be, §1], [Eb]. We modify the latter by multiplying by $(-1)^{m+1}$. In the particular case that the isolated singular point is given by the equation

$$z_1^3 + z_2^3 + z_3^4 + \sum_{i=4}^{2m+1} z_i^2 = 0, \qquad m \geq 1, \tag{9.4.4}$$

we write $(E'_{(m)}, \Delta'_m)$ for the corresponding vanishing lattice.

For any vanishing lattice it is clear that the group $\Gamma_{\Delta'}$ is contained in the orthogonal group $\mathbf{O}(E')$. Assume that $\langle\ ,\ \rangle$ is non-degenerate. Define a homomorphism $\sigma : \mathbf{O}(E') \to \{\pm 1\}$ by first embedding E' in $E' \otimes \mathbb{R}$ and then mapping to $\mathbb{R}^*/(\mathbb{R}^*)^2 \simeq \{\pm 1\}$ by the product of the determinant and the real spinor norm. The cokernel of the map $E' \to (E')^\vee$ which sends $e \mapsto \langle e,\ \rangle$ is a finite Abelian group which we denote $D(E')$. Write $\tau : \mathbf{O}(E') \to \mathrm{Aut}(D(E'))$ for the induced homomorphism. Define $\mathbf{O}^*(E') := \mathrm{Ker}(\sigma) \cap \mathrm{Ker}(\tau)$. Since each $f \in (E')^\vee$ satisfies

$$f \circ g_\delta^{-1}(\xi) = f(\xi) + f(\delta)\langle \delta, \xi \rangle \qquad \forall\ \delta \in \Delta',$$

one sees easily that $\Gamma_{\Delta'} \subset \mathbf{O}^*(E')$.

(9.4.5) Proposition. *If (E', Δ') contains a vanishing sublattice isomorphic to $(E'_{(m)}, \Delta'_m)$ for some $m \in \mathbb{N}$, then $\Gamma_{\Delta'} = \mathbf{O}^*(E')$.*

Proof. [Be, §2], [Eb, §2]. □

Write r (respectively d) for the rank (respectively the discriminant) of E'. Fix an odd prime l and write $(\ ,\ )$ for the induced pairing on E'/l, $\bar{\Delta}' \subset E'/l$ for the image of Δ' and $\nu : \mathbf{O}(E'/l) \to \mathbb{F}_l^*/(\mathbb{F}_l^*)^2 \simeq \{\pm 1\}$ for the composition of the spinor norm with the obvious isomorphism. Define $\nu' := \nu$ when $-2 \in (\mathbb{F}_l^*)^2$ and $\nu' := \nu \cdot det$ otherwise.

(9.4.6) Proposition. *Suppose that (E', Δ') contains a vanishing sublattice isomorphic to $(E'_{(m)}, \Delta'_m)$ for some $m \in \mathbb{N}$. If $r \geq 5$, $l \nmid 2 \cdot d$ and the extension of $\langle\ ,\ \rangle$ to $E' \otimes \mathbb{R}$ is indefinite, then the image $\Gamma_{\bar{\Delta}'} \subset \mathbf{O}(E'/l)$ of $\Gamma_{\Delta'}$ is equal to* $\mathrm{Ker}(\nu')$.

Proof. Since $l \nmid d$, the induced pairing $(\ ,\ )$ on E'/l is non-degenerate. Let $R_- \subset \mathbf{O}(E')$ (respectively $\bar{R}_- \subset \mathbf{O}(E'/l)$) denote the subgroup generated by reflections in all elements $e \in E'$ satisfying $\langle e, e\rangle = -2$ (respectively elements $\bar{e} \in E'/l$ satisfying $(\bar{e}, \bar{e}) = -2 \in (\mathbb{Z}/l)^*$). The fact that (E', Δ') contains a vanishing sublattice $(E'_{(m)}, \Delta'_m)$ for some $m \in \mathbb{N}$ implies that it is complete in the sense of Ebeling [Be, §2]. Now an argument of Ebeling [Eb, Theorem 2.3] shows that $\Gamma_{\Delta'} = R_-$. The affine variety $(\bar{e}, \bar{e}) = -2$ is non-singular and has an $\mathbb{F}_l$-rational point. Any such point lifts to a solution $\langle e, e\rangle = -2$ in $E' \otimes \mathbb{Z}_l$ by Hensel's Lemma. Since $\langle\ ,\ \rangle$ is indefinite, [Ca, Theorem 1.5, p. 131] implies that each $\bar{e} \in E/l$ with $(\bar{e}, \bar{e}) = -2$ lifts to some $e \in E'$ with $\langle e, e\rangle = -2$. Thus R_- maps surjectively to $\bar{R}_-$ and $\Gamma_{\bar{\Delta}'} = \bar{R}_-$. Now $\bar{R}_- \subset \mathbf{O}(E'/l)$ is a non-trivial, normal subgroup which is not contained in the kernel of the determinant since it is generated by reflections. On the other hand $\bar{R}_- \subset \mathrm{Ker}(\nu')$. Since $r \geq 5$, the only normal subgroup of the commutator subgroup of $\mathbf{O}(E'/l)$ is $\{\pm Id\}$ and this occurs only if $r = 0 \mod 2$ [Di, II.6.2, II.6.5, II.9C]. Furthermore the commutator subgroup is the kernel of the map

$$\mathbf{O}(E'/l) \to \{\pm 1\}\ \times\ \{\pm 1\}, \qquad \vartheta \mapsto (\nu(\vartheta), det(\vartheta))$$

[Di, II.8]. The proposition follows easily. □

9.5. The structure of M as an $\mathbb{F}_l[\Gamma]$-module. In this subsection we are concerned with the following

(9.5.1) Geometric situation. Y is a smooth projective variety of dimension $2m+1$ over a finite field k_0, whose algebraic closure we denote by $\bar{k}$. Assume that the Betti number $h^{2m}(Y_{\bar{k}})$ is odd. Let L_0 be an ample invertible sheaf on Y. Assume that the pair (Y, L_0) lifts to a pair $(\mathcal{Y}, \mathcal{L}_0)$ over the Witt ring R of k_0. Fix a complex embedding of the fraction field of R. Then $\mathcal{Y}$ gives rise to a complex projective manifold Y^{an} and $\mathcal{L}_0$ gives rise to an invertible sheaf L_0^{an} on Y^{an}.

(9.5.2) Lemma. *Let (Y, L_0) be as in* (9.5.1). *After replacing k_0 by a finite extension if necessary there is an $n \in \mathbb{N}$ so that all of the following conditions hold:*

(i) (9.3.2) *holds and the Lefschetz pencil* $X \subset \mathbb{P}H^0(Y, L_0^{\otimes n})$ *is actually defined over* k_0*.*

(ii) $n = 0 \mod 2$.

(iii) *There is a hypersurface on* Y^{an} *belonging to the linear system* $|(L_0^{an})^{\otimes n}|$ *which has an isolated singular point given by the local equation* (9.4.4) *and no other singularities.*

(iv) *The rank of the vanishing cohomology* E *is at least 5.*

(v) *The signature* (a, b) *of the intersection product restricted to* E *satisfies* $a > 1$ *and* $b > 1$*.*

Proof. Only (iii), (iv) and (v) require comment. Since some power of L_0 is very ample, it is not difficult to produce for any large n a section of $(L_0^{an})^{\otimes n}$ with only one singular point such that the local equation agrees with (9.4.4) modulo a high power of the maximal ideal. It follows that the singularity is locally analytically isomorphic to (9.4.4) [Ar, Corollary 1.6], [Sa, Lemma 2].

To establish (iv) and (v) we appeal to Hodge theory. For V a smooth divisor belonging to $|L_0^{\otimes n}|$ we have

$$\chi(O_V) - \chi(O_Y) = -\chi(L_0^{-n}) = \chi(\Omega_Y^{2m+1} \otimes L_0^{\otimes n}),$$

where the right-hand side is a polynomial in n of degree $2m+1$ with positive leading coefficient. In the decomposition of rational Hodge structures,

$$H^\bullet(V^{an}, \mathbb{Q}) \simeq E \otimes \mathbb{Q} \quad \oplus \quad H^\bullet,$$

the second term is independent of n by the Lefschetz theorems and Poincaré duality. Thus $dim(E_{\mathbb{C}}^{0,2m}) = \chi(O_V) - \chi(O_Y)$ increases without bound as $n \to \infty$ and the same holds for the rank of E.

Since the vanishing cohomology is contained in the primitive cohomology, the signature, (a, b), of the intersection pairing on $E \otimes \mathbb{R}$ is determined by the Hodge numbers of E [We, V.5]. To verify that both a and b approach infinity as n does, we need only show that the dimensions of $E_{\mathbb{C}}^{0,2m}$ and $E_{\mathbb{C}}^{1,2m-1}$ increase without bound. This follows from the above and

$$-dim(E_{\mathbb{C}}^{1,2m-1}) = \chi(\Omega_V) - \chi(\Omega_Y) = -\chi(\Omega_Y \otimes L_0^{-n}) + \chi(L_0^{-2n}) - \chi(L_0^{-n}),$$

which is a polynomial in n of degree $2m+1$ with negative leading coefficient. □

Let n be as in (9.5.2). Set $L = L_0^{\otimes n}$ and let $V^{an} \subset Y^{an}$ be a non-singular divisor which corresponds to a point $x \in \dot{X}^{an}$. Let $\mathfrak{t} \subset H^{2m}(V^{an}, \mathbb{Z})$ denote the torsion subgroup. Write $\langle\ ,\ \rangle$ for $(-1)^{m+1}$ times the intersection form on $E/(\mathfrak{t} \cap E)$ and let d denote the discriminant of $\langle\ ,\ \rangle$. Observe that the torsion subgroup of $H^{2m+1}(V^{an}, \mathbb{Z})$ is isomorphic to $\mathfrak{t}$ by the universal coefficient theorem and Poincaré duality. Let l be a prime such that $l \nmid 2 \cdot p \cdot d \cdot |\mathfrak{t}|$. There are isomorphisms of inverse systems,

(9.5.3)
$$\{H^{2m}(V^{an}, \mathbb{Z}/l^r)\}_{r\in\mathbb{N}} \simeq \{H^{2m}(V^{an}, \mathbb{Z})/l^r\}_{r\in\mathbb{N}} \simeq \{E/l^r\}_{r\in\mathbb{N}} \ \oplus\ \{(E/l^r)^\perp\}_{r\in\mathbb{N}},$$

which give rise to a direct sum decomposition of the local systems $\{R^{2m}\dot{F}_*^{an}\mathbb{Z}/l^r\}_{r\in\mathbb{N}}$. After fixing a compatible system of roots of unity and applying "Riemann's existence theorem" [Mi, III.3.14] we obtain a decomposition of $\{R^{2m}\dot{F}_*\mathbb{Z}/l^r(m)_{|\dot{\mathcal{X}}_{\bar{S}}}\}_r$. By (9.3.5) there is a corresponding decomposition of $\pi_1(\dot{X}_{\bar{k}}, \bar{\eta})$-modules:

$$\{R^{2m}\dot{f}_*\mathbb{Z}/l^r(m)_{|\bar{\eta}}\}_{r\in\mathbb{N}} \simeq \{E_r\}_{r\in\mathbb{N}} \ \oplus\ \{(E_r)^\perp\}_{r\in\mathbb{N}}. \tag{9.5.4}$$

The fundamental group acts trivially on the second factor in the sum. Using the purely algebraic notion of vanishing cohomology one can show that the isomorphism (9.5.4) respects the action of the arithmetic monodromy group $\pi_1(\dot{X}_{k_0}, \bar{\eta})$ [Fr-Ki, III.7.7]. By ([De2, 4.3])

$$\{R^{2m} f_* \mathbb{Z}/l^r(m)\}_{r\in\mathbb{N}} \simeq \{g_*(R^{2m} f_* \mathbb{Z}/l^r(m)_{|\eta})\}_{r\in\mathbb{N}}.$$

By the choice of l and (9.5.3) this is an l-adic sheaf in the sense of [Mi, p. 163] and $R^{2m} f_* \mathbb{Z}/l^r(m)$ is $\mathbb{Z}/l^r$-flat for each r. Set $\mathcal{E}_r = g_* E_r$ and $M = E_1(1)$. Let $k_0 \subset k$ be a finite field extension such that (3.2.1) and (3.2.2) hold. The image, Γ, of $\pi_1(\dot{X}_k, \bar{\eta}) \to \mathrm{Aut}_{\mathbb{F}_l}(M)$ may be identified with the image of $\pi_1(\dot{X}^{an}, x) \to \mathbf{O}(E/l)$ by (3.2.1). By (9.4.6) we have $\Gamma = \mathrm{Ker}(\nu')$.

With this precise description of the monodromy group in hand we can now verify that the hypotheses (3.3.1–3) and (3.3.6) hold:

(9.5.5) Lemma. (i) *$M \simeq M^\vee$.*

(ii) *$M^\Gamma = 0$ and M is an absolutely irreducible $\mathbb{F}_l[\Gamma]$-module.*

(iii) *There exists $\xi \in \Gamma$ of order prime to l such that $M^{\langle\xi\rangle} \simeq \mathbb{F}_l$.*

(iv) *The inertia subgroup of Γ corresponding to a point above any $x \in \Sigma$ is isomorphic to $\mathbb{Z}/2$.*

Proof. (i) Since both $R^{2m} \dot{f}_* \mathbb{Z}/l^r(m)_{|\bar{\eta}}$ and the trivial representation are self-dual, this follows from (9.5.4).

(ii) Let $m \neq 0$ be contained in an $\bar{\mathbb{F}}_l[\Gamma]$-submodule $N \subset M \otimes \bar{\mathbb{F}}_l$. Since the bilinear form $(\ ,\ )$ induced from $\langle\ ,\ \rangle$ is non-degenerate and M is generated by the image $\bar{\Delta}$ of $\Delta \subset E$, there exists $\delta \in \bar{\Delta}$ such that $(m, \delta) \neq 0$. Since $g_\delta(m) = m + (m, \delta)\delta$, we must have $\delta \in N$. Since Γ acts transitively on $\bar{\Delta}$, we have $\bar{\Delta} \subset N$. But $\bar{\Delta}$ generates M, so $N = M \otimes \bar{\mathbb{F}}_l$.

(iii) By (9.5.1–2) and (9.2.1)(i) $dim_{\mathbb{F}_l}(M) \equiv 1 \mod 2$. The intersection pairing gives rise to a non-degenerate symmetric bilinear pairing on M. The only invariant of a non-degenerate symmetric bilinear form of fixed rank over a finite field of odd characteristic is the discriminant [Se2, IV.5]. Thus we may regard M as the orthogonal direct sum of hyperbolic planes and a single one-dimensional non-degenerate quadratic space. When $l > 3$, take for ξ a block diagonal matrix with one block $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda \in (\mathbb{F}_l^*)^2 - \{1\}$, for each hyperbolic plane and with a 1 in the remaining 1×1 block. When $l = 3$, Γ is the kernel of the spinor norm and we may take $\xi = -Id \cdot g_m$, where g_m is the reflection in the hyperplane $m^\perp$ for some $m \in M$ where (m, m) and the discriminant d have the same class in $\mathbb{F}_l^*/(\mathbb{F}_l^*)^2$. This works because the spinor norm of $-Id$ may be expressed as $\prod(m_i, m_i) \in \mathbb{F}_l^*/(\mathbb{F}_l^*)^2$, where the product is over an orthogonal basis. This is also the discriminant.

(iv) This follows from the Picard-Lefschetz formula for the local monodromy [SGA 7, XV.3.4]. ∎

The next result shows that the hypotheses (3.3.4) and (3.3.5), which are equivalent by (9.5.5)(i), are also satisfied, at least if we avoid a finite set of primes l. To emphasize that the group Γ depends on the choice of prime l we denote it by $\Gamma_{(l)}$ in the remainder of this section.

(9.5.6) Proposition. *$H^1(\Gamma_{(l)}, E/l) = 0$ for all except finitely many l.*

The proposition will be deduced from rather general facts about the cohomology of arithmetic groups.

Let $\mathbf{G}$ be a linear algebraic group over $\mathbb{Q}$ and let $E_{\mathbb{Q}}$ be a finite-dimensional $\mathbb{Q}$-vector space. Consider a representation $\varpi : \mathbf{G} \to GL(E_{\mathbb{Q}})$. Fix a full lattice $E \subset E_{\mathbb{Q}}$ and let $\mathbf{\Gamma} \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup which stabilizes E. For each prime l the image of $\mathbf{\Gamma}$ in $GL(E/lE)$ will be denoted Γ_l. The set of primes l for which $H^1(\Gamma_l, E/lE) \neq 0$ is denoted $\mathfrak{P}$.

(9.5.7) Proposition. *$\mathfrak{P}$ is a finite set when the following hypotheses hold:*

(i) *$\mathbf{G}$ is absolutely simple, simply connected and isotropic.*

(ii) *The Lie algebra of $\mathbf{G}_{\mathbb{R}}$ is not isomorphic to $\mathfrak{so}(m,1)$ for any m and the Lie algebra of $\mathbf{G}_{\mathbb{C}}$ is not isomorphic to $\mathfrak{sl}(n,\mathbb{C})$ for any n.*

(iii) *The representation $E_{\mathbb{Q}}$ of $\mathbf{G}$ is non-trivial and absolutely irreducible.*

Proof. The hypotheses imply that $H^1(\mathbf{\Gamma}, E_{\mathbb{Q}}) = 0$ [Ra1, §3, Corollary 1] and [Ra2, Theorem 3]. Thus $H^1(\mathbf{\Gamma}, E)$ is a torsion group. Since $\mathbf{\Gamma}$ is arithmetic, it has a torsion free subgroup of finite index. (In fact the intersection with the principal congruence subgroup of level 3 in $GL(E)$ is torsion free [Min, §1].) We may now apply [Br, VIII.9.5 and 5.1] to conclude that the $\mathbb{Z}[\mathbf{\Gamma}]$-module $\mathbb{Z}$ has a resolution by finitely generated projective $\mathbb{Z}[\mathbf{\Gamma}]$-modules. Thus $H^i(\mathbf{\Gamma}, E)$ is a finitely generated Abelian group for each i. It follows from the long exact cohomology sequence associated to the sequence of $\mathbf{\Gamma}$-modules

$$0 \to E \xrightarrow{l} E \to E/lE \to 0$$

that $H^1(\mathbf{\Gamma}, E/lE) = 0$ for all but finitely many primes l. Now (9.5.7) follows from the injectivity of the restriction map

$$H^1(\Gamma_l, E/lE) \to H^1(\mathbf{\Gamma}, E/lE).$$

□

Proof of Proposition (9.5.6). We take for E the image of the vanishing cohomology in $H^{2m}(V^{an}, \mathbb{Z})/\mathfrak{t}$, for $\mathbf{G}$ the universal cover of $\mathbf{SO}(E_{\mathbb{Q}})$, and for $\mathbf{\Gamma}$ the inverse image of $\mathbf{O}^*(E) \cap \mathbf{SO}(E_{\mathbb{Q}})$ in $\mathbf{G}(\mathbb{Q})$. $\mathbf{G}$ is isotropic because $\operatorname{rank}(E) \geq 5$ and the intersection form is indefinite by (9.5.2)(v) [Se4, IV.3, Corollary 2]. The Lie algebra of $\mathbf{G}$ satisfies (9.5.7)(ii) by (9.5.2)(v).

We may assume $l \nmid 2 \cdot p \cdot d \cdot |\mathfrak{t}|$. By (9.4.6) there is a short exact sequence of groups,

$$1 \to \Gamma_l \to \Gamma_{(l)} \xrightarrow{det} \{\pm 1\} \to 1,$$

which yields an exact inflation-restriction sequence,

$$0 \to H^1(\{\pm 1\}, (E/l)^{\Gamma_l}) \to H^1(\Gamma_{(l)}, E/l) \to H^1(\Gamma_l, E/l)^{\{\pm 1\}}.$$

The first term is zero because l is odd and the last term is zero for almost all l by (9.5.7). Proposition (9.5.6) follows. □

9.6. Integral Tate classes in the fibers of a Lefschetz pencil. As mentioned in the introduction, the analog of (0.1) for varieties of dimension > 4 does not follow from the Tate conjecture. It will however follow from

(9.6.1) Hypothesis H. Let $\dot{f} : \dot{W} \to \dot{X}$ be as in (9.1.1) and let $\mathcal{E}$ be as in §9.5. There exists a number $h(f) \in \mathbb{N}$ such that if $l \nmid h(f)$. Then the map introduced in (2.3.4),

$$q_{\#} : Z_f^{m+1}(W_{\bar{k}}) \to Z(\mathcal{E}),$$

is surjective.

(9.6.2) *Remarks.* (i) In case $m = 1$, hypothesis **H** follows from the Tate conjecture; cf. (8.2.3).

(ii) Hypothesis **H** is a weak version of an integral analog of the Tate conjecture. There are known counterexamples to the integral Hodge conjecture and hence to certain integral analogs of the Tate conjecture. However these counterexamples do not seem to be helpful in evaluating the plausibility of **H**.

(iii) The discussion of an integral version of the Tate conjecture in [Sch6] only concerns one-dimensional cycles and thus has no bearing on **H** when $m \geq 2$.

(iv) It appears to the author that a serious investigation of the plausibility of hypothesis **H** will have to wait until more is known about the Tate conjecture.

10. The proof of Theorems (0.1) and (0.2)

10.1. Let Y (respectively Y') denote a smooth, projective variety of dimension n (respectively n') over a finite field k_0.

(10.1.1) Lemma. (i) *If* $\mathfrak{Z} \in Z^{n'+r-s}(Y' \times Y)$ *induces a surjection*

$$\mathfrak{Z}_* : H^{2s-1}(Y'_{\bar{k}}, \mathbb{Q}_l(s)) \to H^{2r-1}(Y_{\bar{k}}, \mathbb{Q}_l(r)),$$

then $\mathbf{a}^s_{Y'}$ *surjective implies* $\mathbf{a}^r_Y$ *surjective.*

(ii) *If* $n = n'$, $\sigma : Y' \to Y$ *is a dominant morphism and* $\mathbf{a}^r_{Y'} : CH^r_{hom}(Y'_{\bar{k}}) \to J^r_l(Y')$ *is surjective, then* $\mathbf{a}^r_Y : CH^r_{hom}(Y_{\bar{k}}) \to J^r_l(Y)$ *is also surjective.*

(iii) $\mathbf{a}^1_Y : CH^1_{hom}(Y_{\bar{k}}) \to J^1_l(Y)$ *is surjective.*

(iv) $\mathbf{a}^n_Y : CH^n_{hom}(Y_{\bar{k}}) \to J^n_l(Y)$ *is surjective.*

Proof. (i) This follows from (1.1.5)(i) and the compatibility of the l-adic Abel-Jacobi map with correspondences [Sch, 1.10].

(ii) Take $\mathfrak{Z}$ to be the graph of σ and apply (i).

(iii) By replacing Y by a curve which generates the Picard variety of Y one is reduced to the case that Y is a curve. This case follows from (1.3.4) and the variant of the Tchebotarev density theorem described in [Sch, 9.9] and proven in [Lan]. Alternately, one can prove the assertion by relating $\mathbf{a}^1_Y$ to the inverse of the map $H^1(Y_{\bar{k}}, \mathbb{Q}_l/\mathbb{Z}_l(1)) \to \varinjlim_r H^1(Y_{\bar{k}}, \mathbb{G}_m)[l^r]$ in Kummer theory [Bu-Sch-Top, 1.12].

(iv) By replacing Y by a curve mapping to Y which generates $Alb(Y)$ one is reduced to the case that Y is a curve. Now the argument in (iii) applies. $\square$

10.2. Proof of (0.1) and (0.2). The proof is by induction on the dimension n of Y. By (10.1.1)(iii) and (iv) the theorem is true when $n \leq 2$. We assume now that Y is given of dimension $n \geq 3$ and that (0.1) and (0.2) hold for all smooth projective varieties of dimension $\leq n-1$. Fix $r, 0 < r \leq \frac{n}{2}$, and let $T \subset Y$ be a subvariety of dimension $2r-1$ which is the transverse intersection of non-singular ample divisors. The Lefschetz hyperplane theorem says that the Gysin map

$$s_{T*} : H^{2r-1}(T_{\bar{k}}, \mathbb{Q}_l(r)) \to H^{2n-2r+1}(Y_{\bar{k}}, \mathbb{Q}_l(n-r+1))$$

is surjective. Since $\mathbf{a}^r_{T,l}$ is surjective for all except finitely many primes l by the induction hypothesis, it follows from (10.1.1)(i) that $\mathbf{a}^{n-r+1}_{Y,l}$ is surjective for all except finitely many primes l.

The Tate conjecture for $Y \times Y$ (or the somewhat weaker Lefschetz standard conjecture for Y [Kl, §4]) implies the existence of a cycle $\mathfrak{z} \in Z^{2r-1}(Y \times Y)$ such that

$$\mathfrak{z}_* : H^{2n-2r+1}(Y_{\bar{k}}, \mathbb{Q}_l(n-r+1)) \to H^{2r-1}(Y_{\bar{k}}, \mathbb{Q}_l(r))$$

is an isomorphism. By (10.1.1)(i) the surjectivity of $\mathbf{a}_{Y,l}^{n-r+1}$ implies the surjectivity of $\mathbf{a}_{Y,l}^{r}$. If n is even, this completes the inductive step in the proof of Theorems (0.1) and (0.2).

Suppose that $n = 2m+1 \geq 3$. To complete the inductive step in this case we need only check that $\mathbf{a}_{Y,l}^{m+1}$ is surjective for almost all l. At this point we would like to assume that Y together with a Lefschetz pencil lifts to characteristic zero. Unfortunately, this will not always be the case. So we fix a prime l_0 distinct from the characteristic of k_0 and use the Tate conjecture and the assumption that Frobenius acts semi-simply on $H^{2m+1}(Y_{\bar{k}}, \mathbb{Q}_{l_0})$ to construct curves $T_1,...,T_{2m+1}$ defined over k_0 and a cycle $\mathfrak{z} \in Z^{2m+1}((\prod_{i=1}^{2m+1} T_i) \times Y)$ such that the induced map

$$H^{2m+1}(\mathfrak{z}, \mathbb{Q}_{l_0}) : H^{2m+1}((\prod_{i=1}^{2m+1} T_i)_{\bar{k}}, \mathbb{Q}_{l_0}) \to H^{2m+1}(Y_{\bar{k}}, \mathbb{Q}_{l_0})$$

is surjective [Sch3, 7.1.2]. The next lemma addresses the issue of "independence of l".

(10.2.1) Lemma. *If the Tate conjecture holds, then the cycle $\mathfrak{z}$ induces a surjective map, $H^{2m+1}(\mathfrak{z}, \mathbb{Q}_l)$, for each prime l distinct from the characteristic of k_0.*

Proof. Katz and Messing [Ka-Me, Theorems 1 and 2] construct motives

$$H^{2m+1}(\prod_{i=1}^{2m+1} T_i) \quad \text{and} \quad H^{2m+1}(Y)$$

with respect to any adequate equivalence relation which is coarser than $\mathbb{Q}_l$-cohomological equivalence for some prime $l \neq char(k_0)$ by taking certain explicit $\mathbb{Q}$-linear combinations of graphs of powers of Frobenius as projectors. The cycle $\mathfrak{z}$ induces a morphism

$$\mathfrak{z}_{\#} : H^{2m+1}(\textstyle\prod_{i=1}^{2m+1} T_i) \to H^{2m+1}(Y).$$

We work with motives for numerical equivalence, so that we may identify the cokernel of $\mathfrak{z}_{\#}$ with a submotive (Y, pr) of (Y, Δ_Y) [Ja2]. Write $Z_{num}(Y \times Y)_{\mathbb{Q}}$ (respectively $Z_{hom_l}(Y \times Y)_{\mathbb{Q}}$) for the subspace of $Z(Y \times Y) \otimes \mathbb{Q}$ consisting of cycles which are numerically (respectively $\mathbb{Q}_l$-cohomologically) equivalent to zero. Certainly $Z_{hom_l}(Y \times Y)_{\mathbb{Q}} \subset Z_{num}(Y \times Y)_{\mathbb{Q}}$ and the Tate conjecture implies the opposite inclusion. Thus we may speak of the $\mathbb{Q}_l$-cohomology of (Y, pr) and compute its dimension using the Lefschetz fixed point formula [Kl2, 1.3.6]:

$$\text{(10.2.2)} \quad dim_{\mathbb{Q}_l} pr_* H^{2m+1}(Y_{\bar{k}}, \mathbb{Q}_l) = -\sum_i (-1)^i Tr(pr \circ pr|_{H^i(Y_{\bar{k}}, \mathbb{Q}_l)}) = -pr \cdot^t pr,$$

where the right-hand side, being the intersection number of two cycles in $Y \times Y$, is independent of l. Since the left-hand side of (10.2.2) may be identified with the dimension of the cokernel of $H^{2m+1}(\mathfrak{z}, \mathbb{Q}_l)$, the lemma follows. $\square$

Now (10.1.1)(i) allows us to forget about the original variety and to concentrate on establishing the surjectivity of $\mathbf{a}_l^{m+1}$ for the product of curves $\prod_{i=1}^{2m+1} T_i$. We rename this product Y if the $2m$-th Betti number is odd. If the $2m$-th Betti number is even, we blow up a point of odd degree and denote the resulting variety by Y.

(10.2.3) Lemma. *There exist an ample invertible sheaf L_0 on Y and a lifting of the pair (Y, L_0) to a pair $(\mathcal{Y}, \mathcal{L}_0)$ over the Witt vectors of k_0.*

Proof. Let $\mathfrak{d}_i$ be a divisor of positive degree on T_i. Since T_i is a smooth curve,

$$\mathrm{Ext}^2_{\mathcal{O}_{T_i}}(\Omega_{T_i}, \mathcal{O}_{T_i}) \simeq H^2(T_i, Hom(\Omega_{T_i}, \mathcal{O}_{T_i})) = 0$$

and

$$\mathrm{Ext}^2_{\mathcal{O}_{T_i}}(\mathcal{O}_{T_i}, \mathcal{O}_{T_i}) \simeq H^2(T_i, \mathcal{O}_{T_i}) = 0.$$

Now deformation theory implies that the pair $(T_i, \mathcal{O}_{T_i}(\mathfrak{d}_i))$ lifts to the Witt vectors [SGA 1, III.7]. By forming the fiber product over S we obtain a lifting of the pair $(\prod_{i=1}^{2m+1} T_i, \bigotimes_{i=1}^{2m+1} pr_i^* \mathcal{O}_{T_i}(\mathfrak{d}_i))$. If the $2m$-th Betti number of the product of curves is odd, we set $L_0 = \bigotimes_{i=1}^{2m+1} pr_i^* \mathcal{O}_{T_i}(\mathfrak{d}_i)$ and are done. If the $2m$-th Betti number is even, we denote by q the point of odd degree which gets blown up to produce Y. By Hensel's lemma q lifts to an étale multisection of the fiber product over the Witt ring. Let $\mathcal{Y}$ denote the blow up of the fiber product along this multisection. Let $D \subset Y$ denote the special fiber of the exceptional divisor. The pull back of some high multiple of $\bigotimes_{i=1}^{2m+1} pr_i^* \mathcal{O}_{T_i}(\mathfrak{d}_i)$ tensored with $\mathcal{O}_Y(-D)$ is an ample invertible sheaf [Ha, II.Ex.7.5(b)]. This sheaf, which we denote by L_0, clearly lifts to $\mathcal{Y}$. $\square$

Fix n as in (9.5.2). If necessary replace k_0 by a finite extension to be sure that a Lefschetz pencil $X \subset \mathbb{P}H^0(Y, L_0^{\otimes n})$ exists over k_0. Set $L = L_0^{\otimes n}$ and let $f: W \to X$ be as in (9.1.1). By (10.1.1)(ii) $\mathbf{a}^{m+1}_{Y,l}$ will be surjective if $\mathbf{a}^{m+1}_{W,l}$ is. The Leray spectral sequence for f gives rise to a filtration on $H^{2m+1}(W_{\bar{k}}, \mathbb{Q}_l(m+1))$. Now (2.2.3) allows us to verify the surjectivity of $\mathbf{a}^{m+1}_{W,l}$ by checking surjectivity for the three simpler Abel-Jacobi maps (2.2.3)(i), (ii), and (iii). Of these, (2.2.3)(i) is already surjective by the induction hypothesis.

(10.2.4) Lemma. *With W as above the map* (2.2.3)(iii) *is surjective.*

Proof. Let $L^\bullet H^{2m+1}(W_{\bar{k}}, \mathbb{Q}_l)$ denote the filtration which comes from the Leray spectral sequence for f. Define $s_V = \sigma \circ i_V : V \to Y$ as in §9.2. We claim that the composition

$$H^{2m-1}(V_{\bar{k}}, \mathbb{Q}_l) \xrightarrow{s_{V*}} H^{2m+1}(Y_{\bar{k}}, \mathbb{Q}_l) \xrightarrow{\sigma^*} H^{2m+1}(W_{\bar{k}}, \mathbb{Q}_l) \xrightarrow{c} \frac{H^{2m+1}(W_{\bar{k}}, \mathbb{Q}_l)}{L^1 H^{2m+1}(W_{\bar{k}}, \mathbb{Q}_l)},$$

in which c is the tautological map, is surjective. For this we need only note that there is a commutative diagram

$$\begin{array}{ccc} H^{2m+1}(W_{\bar{k}}, \mathbb{Q}_l)/L^1 H^{2m+1}(W_{\bar{k}}, \mathbb{Q}_l) & \longrightarrow & H^{2m+1}(V_{\bar{k}}, \mathbb{Q}_l) \\ {\scriptstyle c\circ\sigma^*}\Big\uparrow & & \Big\uparrow{\scriptstyle s_V^* \circ s_{V*}} \\ H^{2m+1}(Y_{\bar{k}}, \mathbb{Q}_l) & \xleftarrow{s_{V*}} & H^{2m-1}(V_{\bar{k}}, \mathbb{Q}_l), \end{array}$$

in which the top arrow comes from the Leray spectral sequence and is injective while the right-hand arrow is an isomorphism by the hard Lefschetz theorem [De2]. Thus the composition $J^m_l(V) \overset{\sigma^* \circ s_{V*}}{\to} J^{m+1}_l(W) \xrightarrow{c} J(L^0/L^1)$ is surjective by (1.1.5)(i). Since $\mathbf{a}^m_{V,l}$ is surjective by the induction hypothesis, the lemma follows. $\square$

According to (2.2.3) the surjectivity of $\mathbf{a}^{m+1}_{W,l}$ will follow if the map $\mathbf{a}_f : Z^{m+1}_f(W_{\bar{k}}) \to J(H^1(X_{\bar{k}}, R^{2m} f_* \mathbb{Z}_l(m+1)))$ is surjective. Recall from §9.5 that the intersection product on the vanishing cohomology $E \subset (R^{2m} F^{an}_* \mathbb{Z})_x$ has a non-zero discriminant

d. Since we are allowed to ignore finitely many primes l, we assume $l \nmid 2 \cdot d \cdot p \cdot |\mathfrak{t}|$, where $\mathfrak{t} \subset H^{2m}(V^{an}, \mathbb{Z})$ is the torsion subgroup. As in §9.5 the vanishing cohomology gives rise to an orthogonal direct sum decomposition of l-adic sheaves on X,

$$\{R^{2m} f_* \mathbb{Z}/l^r(m+1)\}_{r \in \mathbb{N}} \simeq \{\mathcal{E}_r\}_{r \in \mathbb{N}} \oplus \{\mathcal{E}_r^{\perp}\}_{r \in \mathbb{N}},$$

where the geometric monodromy acts trivially on $\mathcal{E}_r^{\perp}$. Since $X \simeq \mathbb{P}^1_{k_0}$, $H^1(X_{\bar{k}}, \mathcal{E}^{\perp}) = 0$, $J(H^1(X_{\bar{k}}, R^{2m} f_* \mathbb{Z}_l(m+1))) \simeq J(H^1(X_{\bar{k}}, \mathcal{E}(1)))$. Recall the definitions of $Z_f(W_{\bar{k}})$ and $Z(\mathcal{E})$ from §2.2 and §2.3. If we assume the Tate conjecture and either that $m = 1$ or that hypothesis **H** holds and $l \nmid h(f)$, then the cycle class map of (2.3.4), $q_\# : Z_f^{m+1}(W_{\bar{k}}) \to Z(\mathcal{E})$, is surjective by §9.6. Thus the surjectivity of $\mathbf{a}_f$ will follow if

$$\mathbf{a}_{\mathcal{E}} : Z(\mathcal{E}) \to J(H^1(X_{\bar{k}}, \mathcal{E}(1)))$$

is surjective. To verify this we use (2.4.3). To check that the hypotheses of (2.4.3) hold we recall from §9.5 that $\mathcal{E}_r$ is flat over $\mathbb{Z}/l^r$, $j^*\mathcal{E}_r$ is locally constant, and $\mathcal{E}_1(1) = g_* \underline{M}$, where $\underline{M}$ is a sheaf of $\mathbb{F}_l$-vector spaces on the generic point of X. By (9.5.5) the associated Galois module, M, satisfies $M = M^{\vee}$ and $M^{\Gamma} = 0$. Thus (2.4.3)(i) is satisfied. To verify (2.4.3)(ii) we apply Theorem (3.4). By (9.5.5)(iv) the Galois representation on the vanishing cohomology is tamely ramified. We may thus fix a finite extension field $k_0 \subset k_1$ such that (3.2.1–6) hold for any base field k which is a finite extension of k_1. In §9.5 we checked that hypotheses (3.3.1–6) hold for all except finitely many choices of l. Thus Theorem (3.4) tells us that hypothesis (2.4.4)(ii) holds for almost all choices of l, so $\mathbf{a}_{\mathcal{E}}$ and hence $\mathbf{a}^{m+1}_{W,l}$ are surjective for almost all primes l. This completes the inductive step in the case that n is odd.

The proof of Theorems (0.1) and (0.2) is complete.

11. The surjectivity of the l-adic Abel-Jacobi map for desingularized self-fiber products of elliptic surfaces

11.1. Statement of the theorem. Let k_0 be a finite field of characteristic p and let X be a smooth projective curve over k_0. Let $\pi : Y \to X$ be a non-isotrivial, relatively minimal, semi-stable, elliptic surface with a section, $s : X \to Y$. The inclusion of the non-empty open affine subset over which π is smooth is denoted $j : \dot{X} \to X$. Base change with respect to j is denoted by adding a dot, $\dot{}$, to the notation. For instance $\dot{\pi} : \dot{Y} \to \dot{X}$ is an Abelian scheme.

The fiber product $Y \times_X Y$ is non-singular away from finitely many ordinary double points. A point $(v_1, v_2) \in Y \times_X Y$ is singular when π fails to be smooth at both v_1 and v_2. The blow-up of the ideal sheaf of the reduced singular locus will be written $\sigma : W \to Y \times_X Y$. W is non-singular and has a tautological morphism $f : W \to X$.

(11.1.1) Definition. Let m_π denote the least common multiple of all n such that π has a singular fiber of Kodaira type I_n.

The main result of this section is

(11.1.2) Theorem. *For $l \nmid 2 \cdot 5 \cdot p \cdot m_\pi$ the l-adic Abel-Jacobi map $\mathbf{a}^2_{W,l} : Z^2_{hom}(W_{\bar{k}}) \to J^2_l(W)$ is surjective.*

We prove the theorem by evaluating $\mathbf{a}^2_{W,l}$ at complex multiplication cycles whose definition and properties we now recall.

11.2. Complex multiplication cycles. Let $k_0 \subset k'$ be an algebraic field extension.

(11.2.1) Definition. A point $x \in \dot{X}(k')$ is called a complex multiplication (CM) point if the geometric fiber $\pi^{-1}(x)_{\bar{k}}$ is not a supersingular elliptic curve.

For $x \in \dot{X}(k')$ there is a canonical isomorphism $f^{-1}(x) \simeq \pi^{-1}(x) \times \pi^{-1}(x)$. If $[k' : k_0] < \infty$, then the graph of the Frobenius relative to k' and the diagonal give two divisors Δ_x and $F_x \subset f^{-1}(x)$.

(11.2.2) Lemma. (i) *The complement $\ddot{X} \subset \dot{X}$ of the points where the geometric fiber of π is supersingular is a non-empty affine open subset.*

(ii) *If $x \in \dot{X}(k')$ is a CM point, then $N.S.(f^{-1}(x)) = N.S.(f^{-1}(x)_{\bar{k}}) \simeq \mathbb{Z}^4$. If in addition $[k' : k_0] < \infty$, then the subset $\{\pi^{-1}(x) \times s(x), s(x) \times \pi^{-1}(x), \Delta_x, F_x\} \subset N.S.(f^{-1}(x))$ is linearly independent.*

Proof. (i) [Sil, V.3.1(iii)]. (ii) is straightforward. □

(11.2.3) Definition. Let $x \in \dot{X}(k')$ be a CM point. An element $z \in Z^1(f^{-1}(x))$ is called a CM cycle if its image in $N.S.(f^{-1}(x))$ is a generator of the free rank one $\mathbb{Z}$-module

$$Span\{\pi^{-1}(x) \times s(x), s(x) \times \pi^{-1}(x), \Delta_x\}^{\perp} \subset N.S.(f^{-1}(x)).$$

The image of the CM cycle z under the natural map $Z^1(f^{-1}(x)) \to Z^2(W_{k'})$ gives rise to an element of $CH^2(W_{k'})$. The subgroup of $CH^2(W_{k'})$ generated by all CM cycles in fibers $f^{-1}(x)$, where $x \in \ddot{X}(k')$ ranges over all CM points, is denoted $CH^2_{CM}(W_{k'})$.

(11.2.4) Lemma. *Assume $l \nmid 2 \cdot p$.*

(i) *The cohomology class of a CM cycle lies in the subgroup*

$$Sym^2 H^1(\pi^{-1}(x)_{\bar{k}}, \mathbb{Z}_l)(1) \subset H^2(f^{-1}(x)_{\bar{k}}, \mathbb{Z}_l(1)).$$

(ii) *The element of $CH^2_{CM}(W_{\bar{k}})$ determined by a CM-cycle $z \in Z^1(f^{-1}(x))$ depends only on the cohomology class of z in $H^2(f^{-1}(x)_{\bar{k}}, \mathbb{Z}_l(1))$.*

Proof. (i) This is straightforward. (ii) [Sch, 5.18]. □

(11.2.5) Lemma. *There is a cycle $P \in Z^3(W \times W) \otimes \mathbb{Z}[\frac{1}{2}]$ with the following properties:*

(i) *The class of P in $CH^3(W \times W) \otimes \mathbb{Z}[\frac{1}{2}]$ is an idempotent.*

(ii) *P gives rise to a relative correspondence in $CH^2(\dot{W} \times_{\dot{X}} \dot{W}) \otimes \mathbb{Z}[\frac{1}{2}]$ which is an idempotent.*

(iii) *For primes $l \nmid 2 \cdot p$, $P_* R^2 \dot{f}_* \mathbb{Z}_l(2) \simeq (Sym^2 R^1 \dot{\pi}_* \mathbb{Z}_l)(2)$.*

(iv) *If $l \nmid 2 \cdot p \cdot m_\pi$, then $H^0(X_{\bar{k}}, j_* Sym^2 R^1 \dot{\pi}_* \mathbb{Z}/l^n) = 0 \ \ \forall\ n$.*

(v) *If $l \nmid 2 \cdot p \cdot m_\pi$, $P_* H^3(W_{\bar{k}}, \mathbb{Z}_l(2)) \simeq H^1(X_{\bar{k}}, j_*(Sym^2 R^1 \dot{\pi}_* \mathbb{Z}_l)(2))$ and this group is torsion free.*

(vi) *P acts as the identity on $CH^2_{CM}(W_{\bar{k}})$.*

(vii) *For $l \nmid 2 \cdot p$ the l-adic Abel-Jacobi map induces a surjection*

$$(Id - P)_* CH^2_{alg}(W_{\bar{k}}) \to (Id - P)_* J^2_l(W).$$

Proof. (i)–(iii) [Sch, 5.8]. (iv) This is a consequence of [Sch, 5.6]. (v) [Sch, 5.13, 5.14]. (vi) [Sch, 5.8]. (vii) [Sch, 10.8]. □

Define $\mathcal{M} := j_*(Sym^2 R^1\dot{\pi}_*\mathbb{Z}/l)(2)$. Let $k_0 \subset k$ be a finite extension and let $z \in Z^1(f^{-1}(x))$ be a CM cycle for some CM point $x \in \dot{X}(k)$. The cohomology class of z in

$$Sym^2 H^1(\pi^{-1}(x)_{\bar{k}}, \mathbb{Z}/l)(1) \simeq H^2_x(X_{\bar{k}}, \mathcal{M})$$

will be denoted $[z]$. For the next lemma we regard z as an element of $Z^2(W_k)$.

(11.2.6) Lemma. *Assume* $l \nmid 2 \cdot p \cdot m_\pi$.
(i) $z \in Z^2_{hom}(W_k)$.
(ii) $[z] \neq 0$.
(iii) *The monodromy group* Γ *associated to* $\mathcal{M}$ *is isomorphic to* $SL(2, \mathbb{F}_l)/\pm Id$.
(iv) *The Galois representation associated to* $\mathcal{M}$ *is tamely ramified. In particular,* (3.3.1) *holds.*

Proof. (i) [Sch, 5.4]. (ii) [Sch, 5.3]. (iii) [Sch, 5.6]. (iv) Since π is semi-stable, the Galois representation associated to $R^1\dot{\pi}_*\mathbb{Z}/l$ is tamely ramified [Ogg, §II] and (iv) follows. □

Let $k_0 \subset k_1$ be a finite extension with the property that for any finite extension $k_1 \subset k$ the hypotheses (3.2.1–6) hold. The existence of such a k_1 was proved at the end of §3.2.

(11.2.7) Lemma. *Assume* $l \nmid 2 \cdot 5 \cdot p \cdot m_\pi$. *Then hypotheses* (3.3.2–6) *hold.*

Proof. As a representation of $\Gamma \simeq SL(2, \mathbb{F}_l)/\pm Id$, M is isomorphic to the second symmetric power of the tautological representation of $SL(2, \mathbb{F}_l)$. We leave the verification of (3.3.2–3) to the reader. For (3.3.6) we may take $\xi \in \Gamma$ to be the image of any semi-simple element of $SL(2, \mathbb{F}_l)$ other than $\pm Id$. Finally (3.3.4) and (3.3.5) are equivalent since $M \simeq M^\vee$. We will view (3.3.4) as a statement about extensions in the category of $\mathbb{F}_l[\Gamma]$-modules. When $\Gamma \simeq SL(2, \mathbb{F}_l)/\pm Id$, this category is well understood. There are only finitely many isomorphism classes of indecomposable modules and the structure of each indecomposable as an extension of simple modules is known [Al, p. 49]. For our purposes it suffices to know that there is a non-split exact sequence

$$0 \to \mathbb{F}_l \to rad(P_0) \to S \to 0, \tag{11.2.8}$$

where S is isomorphic to the $l-3$-rd symmetric power of the tautological representation of $SL(2, \mathbb{F}_l)$. S is absolutely simple and self-dual. There are isomorphisms

$$H^1(\Gamma, M) \simeq \mathrm{Ext}^1_{\mathbb{F}_l[\Gamma]}(\mathbb{F}_l, M) \simeq \mathrm{Hom}_{\mathbb{F}_l[\Gamma]}(rad(P_0), M) \simeq \mathrm{Hom}_{\mathbb{F}_l[\Gamma]}(S, M) = 0,$$

where the second map comes from $\mathrm{Hom}_{\mathbb{F}_l[\Gamma]}(\ \ , M)$ applied to

$$0 \to rad(P_0) \to P_0 \to \mathbb{F}_l \to 0$$

and the final equality is a consequence of the assumption that the odd prime $l \neq 5$. □

11.3. Proof of Theorem (11.1.2). Set $\mathcal{E} := j_*(Sym^2 R^1\dot{\pi}_*\mathbb{Z}_l)(1)$. By (11.2.5)(v) $P_* J^2_l(W) \simeq J(H^1(X_{\bar{k}}, \mathcal{E}(1)))$. Recall the map

$$\mathbf{a}_{\mathcal{E}} : Z(\mathcal{E}) \to J(H^1(X_{\bar{k}}, \mathcal{E}(1)))$$

from (2.3.2). Let k_1 be as below (11.2.6) and let $k_1 \subset k$ be a finite extension. Fix an element $\epsilon \in H^1(G_k, H^1(X_{\bar{k}}, \mathcal{M}))$. By Theorem (3.4) there is a CM point $x \in \ddot{X}(k)$

such that

(i) $H^2_x(X_{\bar{k}}, \mathcal{M})^{G_k} \simeq \mathbb{F}_l$.

(ii) The image of $\delta_{3.1.1} : H^2_x(X_{\bar{k}}, \mathcal{M})^{G_k} \to H^1(G_k, H^1(X_{\bar{k}}, \mathcal{M}))$ is $\mathbb{F}_l\epsilon$.

Let $z_x \in Z^1(f^{-1}(x))$ be a CM cycle. By (11.2.6)(ii) the cohomology class $[z_x] \in H^2_x(X_{\bar{k}}, \mathcal{M})^{G_k}$ is not zero. Thus for any finite extension $k_1 \subset k$ and any $\epsilon \in H^1(G_k, H^1(X_{\bar{k}}, \mathcal{M}))$ there is a CM cycle z_x such that $\delta_{3.1.1}([z_x])$ is a non-zero multiple of ϵ. By (2.4.3) this implies that $\mathbf{a}_{\mathcal{E}}$ is surjective. Since we have only made use of Tate classes in $Z(\mathcal{E})$ which correspond to CM cycles, we may use (11.2.5)(v) to conclude that $\mathbf{a}^2_{W,l}$ induces a surjective map

$$CH_{CM}(W_{\bar{k}}) \to J(P_* H^3(W_{\bar{k}}, \mathbb{Z}_l(2))) \simeq P_* J^2_l(W). \tag{11.3.1}$$

Now (11.2.5)(vii) shows that

$$\mathbf{a}^2_{W,l} : Z^2_{hom}(W_{\bar{k}}) \to J^2_l(W)$$

is surjective.

(11.3.2) Corollary. *For $l \nmid 2 \cdot 5 \cdot p \cdot m_\pi$ the restriction of the l-adic Abel-Jacobi map to the subgroup $CH^2_{alg}(W_{\bar{k}}) + CH_{CM}(W_{\bar{k}})$ of $CH^2_{hom}(W_{\bar{k}})$ is a surjection to $J^2_l(W)$.*

11.4. The Chow group of the threefold product of the Fermat cubic curve. We now prove Theorem (0.4) of the introduction. Fix a prime $p \neq 3$ and a second prime $l \nmid 3 \cdot p$. Let $E \subset \mathbb{P}^2_{\mathbb{F}_p}$ be the Fermat cubic curve. When $p \equiv -1$ mod 3, $E^3_{\bar{\mathbb{F}}_p}$ is supersingular and the proof of [Sch, 14.4(1)] shows that

$$CH^2_{alg}(E^3_{\bar{\mathbb{F}}_p}) \otimes \mathbb{Z}_l = CH^2_{hom}(E^3_{\bar{\mathbb{F}}_p}) \otimes \mathbb{Z}_l \xrightarrow{\mathbf{a}^2_{E^3,l}} J^2_l(E^3)$$

is an isomorphism. When $p \equiv 1$ mod 3 and $l \equiv -1$ mod 3, the proof of [Sch, 14.4(3)] shows that

$$\mathbf{a}^2_{E^3,l} : CH^2_{hom}(E^3_{\bar{\mathbb{F}}_p}) \otimes \mathbb{Z}_l \to J^2_l(E^3) \tag{11.4.1}$$

is an isomorphism. Now (11.3.2) shows that the map [Sch, 14.3] is surjective even when $l \equiv 1$ mod 3. The proof of [Sch, 14.4] now gives immediately that (11.4.1) is an isomorphism for $l \equiv 1$ mod 3.

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