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FOLIATIONS WITH GOOD GEOMETRY

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1. Introduction

The goal of this article is to show that there is a large class of closed hyperbolic 3-manifolds admitting codimension one foliations with good large scale geometric properties. We obtain results in two directions. First there is an internal result: A possibly singular foliation in a manifold is quasi-isometric if, when lifted to the universal cover, distance along leaves is efficient up to a bounded multiplicative distortion in measuring distance in the universal cover. This means that leaves reflect very well the geometry in the large of the universal cover and are geometrically tight - this is the best geometric behavior. We previously proved that nonsingular codimension one foliations in closed hyperbolic 3-manifolds can never be quasi-isometric. In this article we produce a large class of singular quasi-isometric, codimension one foliations in closed hyperbolic 3-manifolds. The foliations are stable and unstable foliations of pseudo-Anosov flows. Our second result is an external result, relating (nonsingular) foliations in hyperbolic 3-manifolds with their limit sets in the universal cover, that is, showing that leaves in the universal cover have good asymptotic behavior. Let $\mathcal{G}$ be a Reebless, finite depth foliation in a closed hyperbolic 3-manifold. Then $\mathcal{G}$ is not quasi-isometric, but suppose that $\mathcal{G}$ is transverse to a quasigeodesic pseudo-Anosov flow with quasi-isometric stable and unstable foliations - which are given by the internal result. We then show that the lifts of leaves of $\mathcal{G}$ to the universal cover extend continuously to the sphere at infinity and we also produce infinitely many examples satisfying the hypothesis. The main tools used to prove these results are first a link between geometric properties of stable/unstable foliations of pseudo-Anosov flows and the topology of these foliations in the universal cover, and second a topological theory of the joint structure of the pseudo-Anosov foliations in the universal cover.

Reebless codimension one foliations are extremely useful for understanding the topology of 3-manifolds. For instance they imply that the manifold is irreducible [Ro], its universal cover is homeomorphic to $\mathbf{R}^3$ [Pa], leaves are π_1-injective [No] and transversals are never null homotopic [No]. Hence they reflect topological properties of the manifold. As for which manifolds have Reebless foliations, Gabai [Ga1, Ga2, Ga3] proved that any compact, oriented, irreducible 3-manifold with nonzero first Betti number has many Reebless finite depth foliations. Roughly, a

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finite depth foliation is one that reflects the topology of the manifold extremely well in the sense that it is associated to a hierarchy (in the sutured manifold world) of the manifold. They can also be thought of as generalizations of fibrations. These results of Gabai had many important consequences for the study of 3-manifolds.

From a geometric point of view the geometrization conjecture [Th4] states that closed, irreducible, atoroidal 3-manifolds are either Seifert fibered or hyperbolic. An important question then is to understand how a Reebless foliation interacts with such structures. In Seifert fibered spaces, Reebless foliations are relatively well understood and interact very well with the Seifert fibration [Th1, Br]. There are also some structure results in graph manifolds [Ba, BNR]. Our interest is in understanding foliations in hyperbolic 3-manifolds. In that case it is essential to relate the foliation to the geometry (in the large) of the universal cover and to its ideal compactification with a sphere at infinity. This study of the geometry has some important antecedents. Consider a π_1-injective surface in a closed hyperbolic 3-manifold. Then deep work of Marden [Ma], Thurston [Th2] and Bonahon [Bo] implies a fundamental dichotomy: either the surface is geometrically very good (geometrically finite) or it is a virtual fiber (geometrically infinite), which is geometrically very bad. This is used in the proof of the hyperbolization theorem in the Haken case [Th2, Mor]. One important motivation to understand the possible geometric behaviors of Reebless foliations and also essential laminations in hyperbolic 3-manifolds, is that it may shed some light in trying to prove the geometrization conjecture for foliated or laminar manifolds.

In general what are good geometric properties for a foliation? The best property is that leaves are totally geodesic. These occur for instance in the 3-torus, in manifolds supporting suspension Anosov flows (which have solv geometry [Sc]) and in fact in any torus or sphere bundle over the circle. On the other hand Zeghib [Ze] showed that closed hyperbolic 3-manifolds do not admit geodesic foliations. One can relax this geodesic condition, introducing a notion which is extremely useful for hyperbolic 3-manifolds, namely quasi-isometric. A foliation is said to be *quasi-isometric* if, when lifted to the universal cover, the path distance along its leaves measures distance in the ambient manifold up to a bounded multiplicative distortion. The quasiproperties turn out to be almost as effective as the exact properties in hyperbolic manifolds. The lift of a quasi-isometric leaf has excellent properties: its limit set is a Jordan curve in the sphere at infinity and the leaf itself is a bounded distance from the hyperbolic convex hull of its limit set - it is geometrically tight [Ma, Th1].

We previously proved that there are no quasi-isometric (nonsingular) foliations in closed hyperbolic 3-manifolds [Fe2]. This basically depends on the leaf space of the foliation in the universal cover. If the leaf space is not Hausdorff there are points x, y in distinct leaves being approximated by x_i, y_i in the same leaf (for each i), contradicting quasi-isometric behavior. If the leaf space is Hausdorff, then every leaf has limit set the whole sphere and no leaf can be quasi-isometric. Hence the topology of the foliation in the universal cover strongly influences geometric properties.

Given this result, a natural question is: are there "generalized" foliations for which the quasi-isometric question makes sense? In this article we will consider singular foliations, where the singular set is a union of simple closed curves and near the singular set the foliation is a product of an interval with a singular foliation in the plane having a prong singularity. Examples are the stable and unstable foliations of a pseudo-Anosov flow in a 3-manifold [Mo1]. Roughly a pseudo-Anosov flow is one that is locally like a suspension of a pseudo-Anosov homeomorphism of a closed surface [Bl-Ca]. The first examples of quasi-isometric singular foliations in closed, hyperbolic 3-manifolds were given by Cannon and Thurston [Ca-Th] who proved that the suspension of a pseudo-Anosov homeomorphism of a closed surface has stable/unstable singular foliations which are quasi-isometric. The quasi-isometric property was the main tool used to produce examples of sphere filling curves [Ca-Th]. This highlights the usefulness of finding quasi-isometric objects in the foliation setting.

Therefore one asks: how common are quasi-isometric singular foliations? The first goal of this article is to produce a large class of quasi-isometric singular foliations in hyperbolic 3-manifolds by looking at pseudo-Anosov flows and their stable/unstable foliations. It is easy to prove that if the stable foliation of such a flow is quasi-isometric that implies that the flow lines themselves are *quasigeodesics* [Fe3], that is, when lifted to the universal cover they are uniformly efficient in measuring distance. Consequently we look for quasigeodesic pseudo-Anosov flows. Recently the author and Lee Mosher showed that if M^3 is closed, oriented, hyperbolic and with nonzero first Betti number, then M has a quasigeodesic pseudo-Anosov flow [Fe-Mo]. The reader may notice this is the same class that has (nonsingular) finite depth foliations in closed hyperbolic 3-manifolds, and this is not a coincidence since the flows are obtained from the foliations and are "almost" transverse to these foliations. This duality will be further explored in the second part of this article. Given that the pseudo-Anosov flow is quasigeodesic, how far is it from proving that the stable/unstable foliations are quasi-isometric? Not very far! The relationship is explained in our first result:

Theorem A. *Let Φ be a quasigeodesic pseudo-Anosov flow in M^3 closed, hyperbolic. Let $\mathcal{F}^s$ be the stable foliation of Φ and $\widetilde{\mathcal{F}}^s$ its lift to the universal cover. Then $\mathcal{F}^s$ is a quasi-isometric singular foliation if and only if $\widetilde{\mathcal{F}}^s$ has Hausdorff leaf space.*

This result highlights the fundamental role that the topology of the leaf space plays in understanding the geometry. One direction in Theorem A is very easy to prove as was discussed previously. The importance of this result is that it exchanges the verification of a geometric condition - which is usually extremely hard - and replaces it with understanding a topological condition which is very simple and in many cases possible to check, as explained below.

When does $\widetilde{\mathcal{F}}^s$ have Hausdorff leaf space? One is led to understand the topological theory of the stable/unstable foliations in the universal cover. Here the important result is the same as for Anosov flows [Fe6]. We stress that in the next result we do not assume that Φ is quasigeodesic or that M is hyperbolic.

Theorem B. *Let Φ be a pseudo-Anosov flow in M^3 closed. Let $F, L \in \widetilde{\mathcal{F}}^s$ which are not separated from each other in the leaf space of $\widetilde{\mathcal{F}}^s$. Then F, L are both left invariant by a nontrivial covering translation g of $\widetilde{M}$. This produces a nontrivial free homotopy between closed orbits of Φ in M.*

In fact the proof gives more information concerning the local structure of $\widetilde{\mathcal{F}}^s$ near a non-Hausdorff point. Regardless of geometric applications, Theorem B has independent interest for the theory of pseudo-Anosov flows, for instance see another application of Theorem B in [Fe7]. It follows from Theorem B that one way to obtain quasi-isometric singular foliations is to rule out freely homotopic closed orbits. In this article we prove this is true for infinitely many examples:

Theorem C. *There is a large class of examples of quasi-isometric singular foliations in closed hyperbolic* 3*-manifolds, which are not obtained from suspensions of pseudo-Anosov homeomorphisms.*

The examples of Theorem C were obtained from pseudo-Anosov flows transverse to depth one foliations, where it is easily checked that there are no freely homotopic closed orbits and in addition they are quasigeodesic by the main result of [Fe-Mo].

We now explain the duality between finite depth foliations and pseudo-Anosov flows. Given any transversely orientable foliation in a 3-manifold, a choice of a continuous transverse vector field produces a transverse flow. In the case of Reebless finite depth foliations one can find a dynamically "tight" representative in the class of flows transverse to the foliation and is led to a pseudo-Anosov flow as constructed by Mosher [Mo4]. Sometimes one has to consider flows which are "almost" transverse to the foliation [Mo4]. This is a technical condition which roughly means that, after blowing up a collection of singular orbits of the pseudo-Anosov flow into a collection of annuli, one obtains a flow transverse to the foliation. It is the property of the pseudo-Anosov flow being almost transverse to a finite depth foliation which makes it possible to analyse its geometric behavior and prove that it is quasigeodesic [Fe-Mo].

It is quite possible that all closed, oriented hyperbolic 3-manifolds with nonzero first homology may admit pseudo-Anosov flows which satisfy the conditions of Theorem A, and hence that any such manifold has singular quasi-isometric foliations. Checking the condition of Theorem A depends on a very careful analysis of the inductive step in Mosher's construction [Mo4] of pseudo-Anosov flows using sutured manifold hierarchies. We will analyse this in a future project.

If a (singular) foliation is quasi-isometric, then, in the universal cover, leaves are boundedly near their tightest position. The relative position of an object with respect to the leaves then tells us roughly where the object is in $\mathbf{H}^3$. This can give precise information about asymptotic behavior of the object in $\mathbf{H}^3$ as explained below.

The second goal of this article is to study the asymptotic behavior of (nonsingular) Reebless finite depth foliations in hyperbolic 3-manifolds. This will use the results obtained in the first part. Let $\mathcal{G}$ be such a foliation. Then $\mathcal{G}$ cannot be quasi-isometric. On the other hand, the foliation $\mathcal{G}$ reflects the topology of M extremely well because it is directly associated to a hierarchy of M. Therefore it is of interest to understand exactly how good its geometry can be, as related to that of $\widetilde{M}$. One is led to analyse the continuous extension property: Each leaf of $\mathcal{G}$ admits a hyperbolic metric quasiconformal with the Riemannian metric induced from the manifold. Let F be a lift of such a leaf to $\widetilde{M} = \mathbf{H}^3$. Then the intrinsic metric in F is isometric to the hyperbolic plane $\mathbf{H}^2$ and has a canonical compactification with a circle at infinity $\partial_\infty F$. The *continuous extension* question is the following: does the inclusion map $\varphi : F \to \mathbf{H}^3$ extend to a continuous map $\overline{\varphi} : F \cup \partial_\infty F \to \mathbf{H}^3 \cup S^2_\infty$ for every $F \in \widetilde{\mathcal{G}}$? Notice that the continuous extension property is weaker than the quasi-isometric property [Th2, Gr].

The continuous extension property was proved by Cannon and Thurston [Ca-Th] for fibrations (= depth 0 foliations) and by the author for a large class of depth one foliations in closed, hyperbolic 3-manifolds [Fe1]. The main idea was the following:

Cannon and Thurston introduced a pseudometric in M which is quasi-isometric to the hyperbolic metric when lifted to the universal cover. This means they are boundedly related up to a multiplicative constant. In this pseudometric the stable and unstable leaves (in the universal cover) are totally geodesic which implies they are quasi-isometric in the original hyperbolic metric. This gives information about asymptotic behavior of leaves of $\widetilde{\mathcal{G}}$. In [Fe1] we analysed the depth one case and produced a pseudometric in the complement of the compact leaves so that leaves of generalized stable/unstable laminations were totally geodesic and hence quasi-isometric in the hyperbolic metric. The depth one case was much more involved than the fibration case. The problem with this approach to study higher depth foliations (depth ≥ 2) is that it strongly depends on the topological structure of the foliation, which gets more and more complicated as the depth grows. In addition the ideas above only apply to the components of top depth leaves (which fiber over the circle with infinite genus fiber) and another prerequisite for this program to work is that the lower depth leaves are quasi-isometric. For depth one foliation this is true as lower depth leaves are compact and assumed not to be fibers of a fibration over the circle. However it is easy to construct examples of higher depth foliations where this requirement does not hold, presenting a further difficulty to the above program.

We bypass all these problems by using Theorem C which gives us quasi-isometric singular foliations not just in the top depth regions but in the whole manifold. We then prove the continuous extension property for a large class of finite depth foliations:

Theorem D. *Let $\mathcal{G}$ be a Reebless finite depth foliation in a closed hyperbolic 3-manifold M. Suppose that $\mathcal{G}$ is transverse to a quasigeodesic pseudo-Anosov flow Φ and that the stable and unstable foliations of Φ are quasi-isometric singular foliations. Let E be a leaf of $\mathcal{G}$ with hyperbolic metric quasiconformal with the induced Riemannian metric from M. Let F be a lift to $\widetilde{M}$ and $\overline{\varphi} : F \to \widetilde{M}$ be the inclusion map. Then φ extends to a continuous map $\overline{\varphi} : F \cup \partial_\infty F \to \widetilde{M} \cup S^2_\infty$ and $\overline{\varphi}_{|\partial_\infty F}$ gives a continuous parametrization of the limit set $\Lambda_F = \overline{\varphi}(\partial_\infty F)$ of F. In addition there is a large class of foliations satisfying the hypothesis of this theorem.*

Here are the basic ideas in the proof of this theorem. We consider the unit ball model for $\mathbf{H}^3 \cup S^2_\infty$ with the Euclidean metric induced from $\mathbf{R}^3$. Since $\widetilde{\mathcal{F}}^s$ and $\widetilde{\mathcal{F}}^u$ are transverse to F they induce singular one-dimensional foliations $\widetilde{\mathcal{F}}^s_F, \widetilde{\mathcal{F}}^u_F$ in F. We first prove that each regular leaf of $\widetilde{\mathcal{F}}^s_F$ is a bounded distance from a geodesic of F. A similar statement holds for singular leaves. Let p be a point in $\partial_\infty F$ and suppose first that p is not an ideal point of any leaf of $\widetilde{\mathcal{F}}^s_F, \widetilde{\mathcal{F}}^u_F$. One can then choose l_i leaves of $\widetilde{\mathcal{F}}^s_F$ so that l_i union its ideal points in $\partial_\infty F$ define a basis for the system of neighborhoods of p in $F \cup \partial_\infty F$. Let L_i be the leaf of $\widetilde{\mathcal{F}}^s$ with l_i contained in L_i. If the L_i escape compact sets in $\mathbf{H}^3$, then, because the L_i are uniformly quasi-isometric, it follows that the diameter of L_i in the Euclidean metric of $\mathbf{H}^3 \cup S^2_\infty$ (which is homeomorphic to a closed 3-ball) converges to 0 and hence it will define a single ideal point which is $\overline{\varphi}(p)$. This is what happens when $\mathcal{G}$ is a fibration or in the depth one cases analysed in [Fe1]. However, and this is a fundamental point, in the general case it may be that the L_i do not escape compact sets in $\mathbf{H}^3$ and the argument above is inconclusive. One is led to understand what else can happen "beyond" F in the leaf spaces of $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$. More specifically since in general

F does not intersect all leaves of $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$, what happens in the boundary of the set of leaves of $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$ intersected by F? Understanding the topological structure of the pseudo-Anosov foliations $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$ plays a fundamental role here. The other two cases to consider are that p is an ideal point of a leaf of $\widetilde{\mathcal{F}}^s_F, \widetilde{\mathcal{F}}^u_F$ or maybe of both. This last option could not occur for depth 0 and depth 1 foliations. Such is not the case here and it makes the analysis here more complex.

The nontrivial assumption in Theorem D is that Φ is transverse to $\mathcal{G}$ and not just almost transverse. We stress that there are many examples where the pseudo-Anosov flow Φ cannot be made transverse to $\mathcal{G}$ [Mo4]. However this is not a fatal difficulty for we expect that Theorem D will still hold under the weaker almost transverse hypothesis, even though the proof will be more complicated. This is also left for a future project.

Finally we describe a more conjectural project. The quasi-isometric property of singular foliations may be used to study the geometrical finiteness question for π_1-injective immersed surfaces as follows: Let Φ be a quasigeodesic pseudo-Anosov flow in M^3 closed hyperbolic and let R be an immersed, π_1-injective surface in M. Then R is a virtual fiber if and only if given a lift of R to $\widetilde{M}$ its limit set is S^2_∞ and R is geometrically finite if and only if the limit set is a Jordan curve [Bo, Ma, Th2]. Put R in general position with respect to $\widetilde{\mathcal{F}}^s$. It is possible that the induced singular foliations in R may give information enough to decide whether R is geometrically finite or not. This has been successfully done when R is in fact transverse to a pseudo-Anosov flow by work of Cooper, Long and Reid [CLR1, CLR2] and also the author [Fe7].

This article is organized as follows: In section 2 we review background material on pseudo-Anosov flows. In section 3 we prove Theorem A. In the following section we develop the topological theory of stable/unstable foliations of pseudo-Anosov flows. In sections 5 and 6 we consider a finite depth foliation transverse to a pseudo-Anosov flow and analyse topological and geometric properties of the singular 1-dim foliations induced by $\mathcal{F}^u, \mathcal{F}^s$ in leaves of $\mathcal{G}$. This is used in section 7 to prove Theorem D. Finally in section 8 we produce the examples mentioned in Theorems C and D.

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2. Pseudo-Anosov flows

Pseudo-Anosov flows are a generalization of suspension flows of pseudo-Anosov surface homeomorphisms. These flows behave much like Anosov flows, but they may have finitely many singular orbits which are periodic and have a prescribed behavior. In order to define pseudo-Anosov flows, first we recall singularities of pseudo-Anosov surface homeomorphisms.

Given $n \geq 2$, the quadratic differential $z^{n-2}dz^2$ on the complex plane $\mathbf{C}$ (see [St] for quadratic differentials) has a horizontal singular foliation f^u with transverse measure μ^u, and a vertical singular foliation f^s with transverse measure μ^s. These foliations have n-pronged singularities at the origin, and are regular and transverse to each other at every other point of $\mathbf{C}$. Given $\lambda > 1$, there is a homeomorphism $\psi : \mathbf{C} \to \mathbf{C}$ which takes f^u and f^s to themselves, preserving the singular leaves, stretching the leaves of f^u and compressing the leaves of f^s by the factor λ. Let R_θ

be the homeomorphism $z \to e^{2\pi\theta} z$ of $\mathbf{C}$. If $0 \leq k < n$, the map $R_{k/n} \circ \psi$ has a unique fixed point at the origin, and this defines the local model for a *pseudohyperbolic fixed point*, with n-prongs and rotation k. This map is everywhere smooth except at the origin. Let $d_{\mathbf{E}}$ be the singular Euclidean metric on $\mathbf{C}$ associated to the quadratic differential $z^{n-2} dz^2$, given by

$$d_{\mathbf{E}}^2 = \mu_u^2 + \mu_s^2.$$

Note that

$$(R_{k/n} \circ \psi)^* d_{\mathbf{E}}^2 = \lambda^{-2} \mu_u^2 + \lambda^2 \mu_s^2.$$

The mapping torus $N = \mathbf{C} \times \mathbf{R}/(z, r+1) \sim (R_{k/n} \circ \psi(z), r)$ has a suspension flow Ψ arising from the flow in the $\mathbf{R}$ direction on $\mathbf{C} \times \mathbf{R}$. The suspension of the origin defines a periodic orbit γ in N, and we say that (N, γ) is the local model for a *pseudohyperbolic periodic orbit*, with n prongs and with rotation k. The suspension of the foliations f^s, f^u define 2-dimensional foliations on N, singular along γ, called the *local weak stable and unstable foliations*.

Note that there is a singular Riemannian metric ds on $\mathbf{C} \times \mathbf{R}$ that is preserved by the gluing homeomorphism $(z, r+1) \sim (R_{k/n} \circ \psi(z), r)$, given by the formula

$$ds^2 = \lambda^{-2t} \mu_u^2 + \lambda^{2t} \mu_s^2 + dt^2.$$

The metric ds descends to a metric on N denoted ds_N.

Let Φ be a flow on a closed, oriented 3-manifold M. We say that Φ is a *pseudo-Anosov flow* if the following are satisfied:

- For each $x \in M$, the flow line $t \to \Phi(x,t)$ is C^1, it is not a single point, and the tangent vector bundle $D_t \Phi$ is C^0.

- There is a finite number of periodic orbits $\{\gamma_i\}$, called *singular orbits*, such that the flow is smooth off of the singular orbits.

- Each singular orbit γ_i is locally modelled on a pseudohyperbolic periodic orbit. More precisely, there exist n, k with $n \geq 3$ and $0 \leq k < n$, such that if (N, γ) is the local model for a pseudohyperbolic periodic orbit with n prongs and with rotation k, then there are neighborhoods U of γ in N and U_i of γ_i in M, and a diffeomorphism $f \colon U \to U_i$, such that f takes orbits of the semiflow $R_{k/n} \circ \psi \bigm| U$ to orbits of $\Phi \bigm| U_i$.

- There exists a path metric d_M on M, such that d_M is a smooth Riemannian metric off of the singular orbits, and for a neighborhood U_i of a singular orbit γ_i as above, the derivative of the map $f \colon (U - \gamma) \to (U_i - \gamma_i)$ has bounded norm, where the norm is measured using the metrics ds_N on U and d_M on U_i.

- On $M - \bigcup \gamma_i$, there is a continuous splitting of the tangent bundle into three 1-dimensional line bundles $E^u \oplus E^s \oplus T\Phi$, each invariant under $D\Phi$, such that $T\Phi$ is tangent to flow lines, and for some constants $\nu > 1, \theta > 1$ we have

1. if $v \in E^u$, then $|D\Phi_t(v)| \leq \theta \nu^t |v|$ for $t < 0$,
2. if $v \in E^s$, then $|D\Phi_t(v)| \leq \theta \nu^{-t} |v|$ for $t > 0$,

where norms of tangent vectors are measured using the metric d_M.

- In a neighborhood U_i of a singular orbit γ_i as above, $Df(E^s)$ is tangent to the local weak stable foliation and similarly for $Df(E^u)$.

With this definition, pseudo-Anosov flows are a generalization of Anosov flows in 3-manifolds [An, An-Si]. The entire theory of Anosov flows can be mimicked for pseudo-Anosov flows [Mo4]. In particular, a pseudo-Anosov flow Φ has a singular 2-dimensional weak unstable foliation $\mathcal{F}^u$ which is tangent to $E^u \oplus T\Phi$ away from

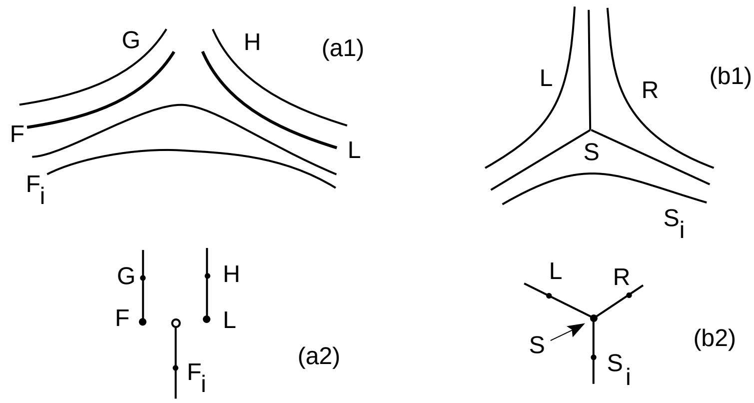

FIGURE 1. (a) Non-Hausdorff behavior in the leaf space of $\widetilde{\mathcal{F}}^s$: (a1) F, L nonseparated from each other, as seen in $\widetilde{M}$, (a2) the corresponding picture in the leaf space $\mathcal{H}(\widetilde{\mathcal{F}}^s)$. (b) A singular leaf of $\widetilde{\mathcal{F}}^s$: (b1) as seen in $\widetilde{M}$, (b2) as seen in $\mathcal{H}(\widetilde{\mathcal{F}}^s)$.

the singular orbits. A complete leaf of this foliation is called a *regular* leaf of $\mathcal{F}^u$. A noncomplete leaf can be completed by adding a singular orbit α. The union of α and the noncomplete leaves abutting α forms a *singular* leaf of $\mathcal{F}^u$ containing α. Similarly there is a 2-dimensional weak stable foliation $\mathcal{F}^s$ tangent to $E^s \oplus T\Phi$. These foliations are singular along the singular orbits of Φ, and regular everywhere else. In the neighborhood U_i of an n-pronged singular orbit γ_i, the images of $\mathcal{F}^s$ and $\mathcal{F}^u$ in the model manifold N are identical with the local weak stable and unstable foliations.

The pseudo-Anosov flow also has singular 1-dimensional strong foliations $\mathcal{F}^{ss}$, $\mathcal{F}^{uu}$. Outside the singular orbits, leaves of $\mathcal{F}^{ss}$ are obtained by integrating E^s. If $x \in \alpha$ and α is a singular orbit of Φ, then, in the local model $N = \mathbf{C} \times \mathbf{R}/\sim$, the point x corresponds to (O, t), where O is the origin in $\mathbf{C}$. Then $W^{ss}_{loc}(x)$ is $\zeta \times \{t\}$, where ζ is the singular leaf of f^s (which contains O). The $\{W^{ss}_{loc}(x)\}$, x in singular orbit, glue up with the leaves of $\mathcal{F}^{ss}$ outside singular orbits to form a singular foliation $\mathcal{F}^{ss}$. The foliation $\mathcal{F}^{ss}$ is flow invariant, that is, for any leaf ζ_1 of $\mathcal{F}^{ss}$ and any real t, $\Phi_t(\zeta_1)$ is a leaf of $\mathcal{F}^{ss}$. Furthermore for $t > 0$ Φ_t exponentially contracts distances along leaves of $\mathcal{F}^{ss}$. Similarly for $\mathcal{F}^{uu}$.

Notation/definition. The discussion above applies equally well to the lifted singular foliations $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u, \widetilde{\mathcal{F}}^{ss}, \widetilde{\mathcal{F}}^{uu}$ in $\widetilde{M}$. If $x \in M$ let $W^s(x)$ denote the leaf of $\mathcal{F}^s$ containing x. Similarly one defines $W^u(x), W^{ss}(x), W^{uu}(x)$ and in the universal cover $\widetilde{W}^s(x), \widetilde{W}^u(x), \widetilde{W}^{ss}(x), \widetilde{W}^{uu}(x)$. Similarly if α is an orbit of Φ define $W^s(\alpha)$, etc. Also let $\widetilde{\Phi}$ be the lifted flow to $\widetilde{M}$.

In Figure 1 we highlight the difference between non-Hausdorff behavior in the leaf space of $\widetilde{\mathcal{F}}^s$ and the splitting (or branching) of leaves associated to singular orbits of $\widetilde{\Phi}$. In part (a) the leaves F, L of $\widetilde{\mathcal{F}}^s$ are not separated from each other in the leaf space of $\widetilde{\mathcal{F}}^s$. Notice that the sequence F_i converges to F <u>and</u> L. In Figure 1 part (b) we sketch a singular leaf S with 3 prongs. Even though S separates $\widetilde{M}$ into 3 or more regions, non-Hausdorffness is not involved. The leaves S_i converge only to S. In this article, unless otherwise specified, all pictures of leaves of $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$ will describe them as subsets of $\widetilde{M}$, rather than in the leaf space of $\widetilde{\mathcal{F}}^s$.

We need to understand the intrinsic geometry of a leaf L of $\widetilde{\mathcal{F}}^s$. All results apply equally well to $\widetilde{\mathcal{F}}^u$. We start by reparametrizing the flow to have constant speed 1. Using the local model near singularities we may assume this was already true near the singular orbits. The resulting flow is still a pseudo-Anosov flow with the same weak stable and unstable foliations [An-Si], so we may assume it is the original flow.

There is a stable lamination associated to the flow Φ. Each singular leaf of $\mathcal{F}^s$ can be split up into a union of finitely many nonsingular leaves by blowing up the singularity, much like blowing air to inflate the leaves. If α is a p-prong singular orbit of Φ, then one sees p local leaves of $W^s(\alpha) - \alpha$ abutting α. The blow up operation will turn this into p local stable leaves. This produces a stable lamination Λ^s. There is an induced flow tangent to leaves of Λ^s. Let $\widetilde{\Lambda}^s$ be the lift of Λ^s to $\widetilde{M}$.

The leaves of $\widetilde{\Lambda}^s$ are negatively curved in the large, as defined by Gromov [Gr] who used the term hyperbolic. Notice first that there is no holonomy invariant transverse measure to Λ^s - because near the periodic orbits of Φ there is expanding holonomy. Using this Plante [Pl] proved that all leaves of Λ^s have exponential growth and Sullivan [Su] showed that leaves of $\widetilde{\Lambda}^s$ are uniformly hyperbolic in the Gromov sense. This immediately implies that the same is true for regular leaves of $\widetilde{\mathcal{F}}^s$ and we will prove the same for singular leaves. First we define quasi-isometries and standard leaves.

Definition 2.1. A quasi-isometric embedding is a map $\phi : (M, d) \to (M', d')$ between metric spaces for which there is $k \geq 1$ so that, for any $x, y \in M$,

$$\frac{1}{k} d(x,y) - k \ \leq \ d(\phi(x), \phi(y)) \ \leq \ k d(x,y) + k.$$

In addition ϕ is a quasi-isometry if every $y \in M'$ is a bounded distance from $\phi(M)$. Two metrics (M, d), (M, d') in M are quasi-isometric if $id : (M, d) \to (M, d')$ is a quasi-isometry.

Definition 2.2. Let $L \in \widetilde{\mathcal{F}}^s$. A standard leaf of L determined by orbit β of $\widetilde{\Phi}$ in L is $S = S' \cup \beta$, where S' is a component of $L - \beta$. If L is singular we only consider β to be the singular orbit in L. Similarly for $G \in \widetilde{\mathcal{F}}^u$.

Let L be a leaf of $\widetilde{\mathcal{F}}^s$. The Riemannian metric in $\widetilde{M}$ induces Riemannian metrics ds in each of the standard leaves of L. These glue together to produce a singular Riemannian metric ds in L. Let d be the metric which is its path integral: if $x, y \in L$, then $d(x,y) = \inf\{l(\gamma) \mid \gamma\}$, where the infimum is taken over all rectifiable paths γ from x to y in L. When L is regular ds is just the induced Riemannian metric from $\widetilde{M}$.

In any standard leaf S in a leaf L of $\widetilde{\mathcal{F}}^s$ there is a foliation (vertical) by flow lines and another (horizontal) by leaves of the strong stable foliation. Let dw (dt) be the length along the strong stable (flow) direction. Define the infinitesimal metric $ds' = dw + dy$ in L. Notice that ds' is only a Finsler metric. However lengths of paths can still be computed. Glueing this along all standard leaves of L produces a singular Finsler metric ds' in S. Let d' be its path integral.

With this background, then exactly as in [Fe3] it follows that: There is $k_1 > 0$ so that for any leaf L in $\widetilde{\mathcal{F}}^s$ (or for any standard leaf of $\widetilde{\mathcal{F}}^s$ or any leaf of $\widetilde{\Lambda}^s$), then d and d' are k_1-quasi-isometric metrics in L. To study the large scale geometry of L we will use the metric d' which has more obvious properties. When the dependence upon L is important the metrics will be denoted by d_L, d'_L.

A fundamental fact is that for $L \in \widetilde{\mathcal{F}}^s$ (or line leaf of $\widetilde{\mathcal{F}}^s$ or leaf in $\widetilde{\Lambda}^s$), then any orbit β of $\widetilde{\Phi}$ in L is a minimal geodesic in the d' metric: if $x, y \in \beta$ are a flow distance $|\Delta t|$, then any path from x to y in L has to cover at least that much $|\Delta t|$. This is because L has a global product structure by leaves of $\widetilde{\mathcal{F}}^{ss}$ and flow lines of $\widetilde{\Phi}$, and the measure dt is holonomy invariant under leaves of $\widetilde{\mathcal{F}}^{ss}$ in L.

Standard leaves in $\widetilde{\mathcal{F}}^s$ correspond to standard leaves of $\widetilde{\Lambda}^s$. Using this and the fact that any orbit β is a minimal geodesic in the d' metric, it follows that leaves of $\widetilde{\mathcal{F}}^s$ are also negatively curved in the large and have an associated ideal boundary.

We can now directly apply the analysis of [Fe3], section 5, to obtain the following:

Lemma 2.3. *If R is a standard leaf of $\widetilde{\mathcal{F}}^s$ bounded by a flow line α of $\widetilde{\Phi}$, then its ideal boundary $\partial_\infty R$ is a closed segment. The ideal points correspond to the common forward limit point of all flow lines and in addition a negative ideal limit point for each flow line. The boundary points of the segment $\partial_\infty R$ are the forward and backwards ideal point of α. Finally if $\alpha_i, i \in \mathbf{N}$, are orbits of $\widetilde{\Phi}$ in R which escape every compact set in R, then their negative ideal points converge to the positive ideal point of R.*

Now we can understand the intrinsic geometry of full leaves of $\widetilde{\mathcal{F}}^s$. If $L \in \widetilde{\mathcal{F}}^s$ is regular, then it is obtained by identifying two standard leaves of $\widetilde{\mathcal{F}}^s$ along their common boundary. Intuitively the geometric model for L is the hyperbolic plane $\mathbf{H}^2$, where the flow lines in L correspond to all oriented geodesics of $\mathbf{H}^2$ having the same positive limit point in the ideal boundary $\partial \mathbf{H}^2$. Then $\partial_\infty L$ is homeomorphic to a circle. On the other hand if L has a p-prong singular orbit β, then L is the union of p standard leaves of $\widetilde{\mathcal{F}}^s$ along β. Then $\partial_\infty L$ is a union of p closed segments which have the forward ideal points of β in the different standard leaves identified to a single point, and the same for the backward ideal points of β. Equivalently $\partial_\infty L$ is homotopic to a bouquet of $p-1$ circles.

3. Quasi-isometric singular foliations in hyperbolic 3-manifolds

The purpose of this section is to establish a strong relationship between the geometry and the topology of the stable foliation $\widetilde{\mathcal{F}}^s$ of a quasigeodesic pseudo-Anosov flow in M^3 hyperbolic. Our goal is to show that $\mathcal{F}^s$ is quasi-isometric if and only if the leaf space of $\widetilde{\mathcal{F}}^s$ is Hausdorff. We first define piece leaves in leaves of $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$.

Definition 3.1. A piece leaf of a leaf $Z \in \widetilde{\mathcal{F}}^s$ is $L = L_1 \cup \alpha \cup L_2$, where L_1, L_2 are distinct standard leaves of Z glued along α. If L is regular, then L itself is its only piece leaf. If L has p-prongs, then it has $p(p-1)/2$ piece leaves.

When α is regular, there are no singular orbits in the piece leaf L, so there is a stable bundle E^s induced in L. If on the other hand α is a singular orbit, then because of the local model in a neighborhood of α, there are strong stable directions in both of the sides of α in L and the directions define the same slope in L along

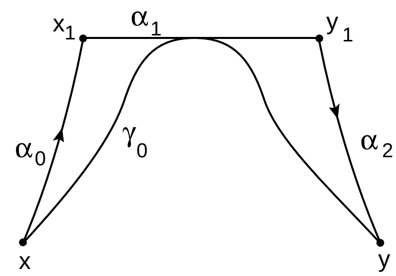

FIGURE 2. Paths "flow, strong stable and flow" which efficiently measure distance in L.

α. Hence for any piece leaf L of $\widetilde{\mathcal{F}}^s$ there is an induced continuous bundle E^s in the entire piece leaf L and for any $v \in E^s$ (in L) then $|D\widetilde{\Phi}_t(v)| \leq \theta\nu^{-t}|v|$ for any $t > 0$.

First we need a lemma. Let L be a piece leaf of $\widetilde{\mathcal{F}}^s$. Then $\widetilde{\Phi}$ and $\widetilde{\mathcal{F}}^{ss}$ induce foliations in L. Identify L to $\mathbf{R}^2$ as follows: fix $z \in L$, which will be the origin of $\mathbf{R}^2$. Flow lines are vertical lines in $\mathbf{R}^2$. Leaves of $\widetilde{\mathcal{F}}^{ss}$ are horizontal lines, where we identify $\widetilde{W}^{ss}(z)$ with its induced path metric to $\mathbf{R}$. Any point u in L is uniquely described as $(w,t) \in \mathbf{R}^2$, where $(w,0) \in \mathbf{R}^2$ is identified to $y \in \widetilde{W}^{ss}(z)$ and $u = \widetilde{\Phi}_t(y)$. Recall the infinitesimal metrics ds and ds' in L as well as their path integrals d and d'. Let l' be the length of a path in the ds' metric.

We denote by d the ambient metric in M or $\widetilde{M}$.

Lemma 3.2. *There are constants $a_0, a_1 > 0$ so that: For any piece leaf L in a leaf of $\widetilde{\mathcal{F}}^s$ and any $x, y \in L$, there is a path γ in L from x to y with $\gamma = \alpha_0 * \alpha_1 * \alpha_2$ and*

- α_0 *starts at* x, α_2 *ends in* y,
- $\alpha_0 \subset \widetilde{\Phi}_{[0,+\infty)}(x)$, $\alpha_2 \subset \widetilde{\Phi}_{[0,+\infty)}(y)$,
- α_1 *is contained in a leaf of* $\widetilde{\mathcal{F}}^{ss}$,
- $l'(\alpha_1) \leq a_1$ *and*
- $l'(\gamma) < a_0 d'(x,y)$.

This means that there are paths of type flow segment forwards, strong stable and then flow segment backwards which are uniformly efficient in measuring length in L.

Proof. Let γ_0 be a minimal geodesic from x to y in the d' metric. As $\widetilde{\Phi}_{\mathbf{R}}(x), \widetilde{\Phi}_{\mathbf{R}}(y)$ are minimal geodesics in the d' metric of L, it follows that γ_0 is contained in the region of L bounded by these flow lines; see Figure 2.

Under the identification $L \cong \mathbf{R}^2$ we have

$$\gamma_0 : \; [0,b] \;\rightarrow\; \mathbf{R}^2, \quad \gamma(\tau) \;=\; (w(\tau), t(\tau)).$$

Let $\tau' \in [0,b]$ so that $t(\tau') = \max\{t(\tau) \mid \tau \in [0,b]\}$. Let

- α_0 be the segment in $\widetilde{\Phi}_{\mathbf{R}}(x)$ from $x = (w(0), t(0))$ to $x_1 = (w(0), t(\tau'))$,
- α_1 the segment in $\widetilde{W}^{ss}(x_1)$ from x_1 to $y_1 = (w(b), t(\tau'))$ and
- α_2 the segment in $\widetilde{\Phi}_{\mathbf{R}}(y)$ from y_1 to $y = (w(b), t(b))$ - notice that α_2 moves in the direction opposite to the flow $\widetilde{\Phi}$.

Finally let $\gamma = \alpha_0 * \alpha_1 * \alpha_2$. Recall that

$$ds' \;=\; dw + dt \quad \text{ and } \quad l'(\gamma_0) \;=\; \int_{\gamma_0} (dw + dt).$$

Since γ_0 goes from height $t(0)$ (at x) to height $t(\tau')$ and then back to height $t(b)$ (at y), then

$$\int_{\gamma_0} dt \;\geq\; (t(\tau') - t(0)) \;+\; (t(\tau') - t(b)) \;=\; l'(\alpha_0) + l'(\alpha_2).$$

Since $t \leq t(\tau')$ in γ_0, push γ_0 forward along flow lines of $\widetilde{\Phi}$ to height $t(\tau')$. This destroys the dt component of ds' and since the change in t is always positive, it expands the dw part by at most θ, where θ was in the definition of the pseudo-Anosov flow. But the push forward of γ_0 contains α_1 so we get $l'(\alpha_1) \leq \theta \int_{\gamma_0} dw$. Therefore

$$l'(\gamma) \;=\; l'(\alpha_0) + l'(\alpha_1) + l'(\alpha_2) \;\leq\; \int_{\gamma_0} dt \;+\; \theta \int_{\gamma_0} dw \;\leq\; \theta \int_{\gamma_0} ds' \;=\; \theta d'(x, y).$$

The proof will be finished by showing that $l'(\alpha_1)$ is bounded. In the same way as above we can push forward α_1 by Δt along flow lines of $\widetilde{\Phi}$ and obtain a path α_3 which is homotopic to α_1 and consists of a flow segment followed by a segment in a leaf of $\widetilde{\mathcal{F}}^{ss}$ and another flow segment. By a similar argument as above:

$$l'(\alpha_3) \;\leq\; 2\Delta t + \theta \nu^{-\Delta t} l'(\alpha_1) \;\leq\; 2\Delta t + \theta^2 \nu^{-\Delta t} \int_{\gamma_0} dw.$$

By minimality of γ_0 it follows that

$$\int_{\gamma_0} ds' \;\leq\; l'(\alpha_0) + l'(\alpha_3) + l'(\alpha_2) \;\leq\; \int_{\gamma_0} dt + l'(\alpha_3).$$

Therefore

$$\int_{\gamma_0} dw \;\leq\; l'(\alpha_3) \;\leq\; 2\Delta t + \theta^2 \nu^{-\Delta t} \int_{\gamma_0} dw, \quad \forall\, \Delta t \geq 0.$$

Hence $\int_{\gamma_0} dw$ is bounded above and consequently $l'(\alpha_1)$ is also bounded above. This finishes the proof of the lemma. ☐

At this point we need to define the quasigeodesic property:

Definition 3.3. A quasigeodesic curve in a metric space (M, d) is the projection to M of a quasi-isometry $\phi : I \to \widetilde{M}$, where I is an interval in $\mathbf{R}$ which may be bounded, infinite or bi-infinite. The metric in I is the induced Euclidean metric. If M is compact, being quasigeodesic is independent of the choice of smooth metric in M. Once a metric is fixed, we say that α is a k-quasigeodesic if it is the image of a k-quasi-isometry. A flow for which all the flow lines are quasigeodesics is a quasigeodesic flow.

For the rest of this section we restrict to M^3 closed, with negatively curved fundamental group [Gr, Gh-Ha, CDP]. This was also defined by Gromov who again used the term hyperbolic [Gr]. Examples are manifolds of negative sectional curvature, therefore including all hyperbolic 3-manifolds. This property implies that $\mathbf{Z} \oplus \mathbf{Z}$ does not inject in $\pi_1(M)$ [Gr], [Gh-Ha], therefore M is atoroidal. Gromov [Gr] showed that $\widetilde{M}$ is compactified with an ideal boundary $\partial \widetilde{M}$. Bestvina and Mess [Be-Me] proved that $\partial \widetilde{M}$ is homeomorphic to a 2-dimensional sphere, which will be denoted by S^2_∞. They also showed that $\widetilde{M} \cup S^2_\infty$ is homeomorphic to the closed 3-ball $\mathbf{B}^3$. Given a k-quasigeodesic in M, any lift to $\widetilde{M}$ is a bounded distance from a minimal geodesic [Gr, Gh-Ha, CDP]. The bound depends only on k and how much $\pi_1(M)$ is negatively curved. It follows that any quasigeodesic in $\widetilde{M}$ has 2 well-defined distinct ideal points in S^2_∞ [Gr].

Now let Φ be a quasigeodesic pseudo-Anosov flow in M^3 with negatively curved $\pi_1(M)$. Given $x \in \widetilde{M}$, $\widetilde{\Phi}_{\mathbf{R}}(x)$ is a quasigeodesic in $\widetilde{M}$. This defines functions

$$\eta_+, \eta_- : \widetilde{M} \to S^2_\infty, \quad \eta_+(x) = \lim_{t \to +\infty} \widetilde{\Phi}_t(x), \quad \eta_-(x) = \lim_{t \to -\infty} \widetilde{\Phi}_t(x).$$

Any pseudo-Anosov flow in an atoroidal 3-manifold is transitive [Mo2]. Consequently orbits of Φ are uniformly quasigeodesic (that is, the same k can be used for all flow lines). This implies that η_+, η_- are continuous functions [Fe3]. In fact the next result says that leaves of $\widetilde{\mathcal{F}}^s$ extend continuously to the sphere at infinity.

Theorem 3.4. *Suppose Φ is a quasigeodesic pseudo-Anosov flow in a 3-manifold M with $\pi_1(M)$ negatively curved. Let L be a leaf of $\widetilde{\mathcal{F}}^s$. Then the embedding $\xi : L \to \widetilde{M}$ extends continuously to $\overline{\xi} : L \cup \partial_\infty L \to \widetilde{M} \cup S^2_\infty$.*

Proof. If β is an orbit of $\widetilde{\Phi}$ in L, then β is a minimal geodesic in the d' metric of L. Therefore it suffices to consider standard leaves of L to prove this result. With this in mind the proof is exactly the same as in Theorem 5.8 of [Fe3]. $\square$

Now we define a quasi-isometric singular foliation.

Definition 3.5. We say that the foliation $\mathcal{F}^s$ in M is quasi-isometric if there is $k > 0$ so that

$$\forall L \in \widetilde{\mathcal{F}}^s, \quad \forall x, y \in L, \quad d_L(x,y) \; < \; k d_{\widetilde{M}}(x,y) + k.$$

Similarly for $\mathcal{F}^u$. The property of M being hyperbolic is not needed for this definition.

In order to prove Theorem A of the introduction we start with a quasigeodesic pseudo-Anosov flow in M^3 closed, hyperbolic and first relate the quasi-isometric property for $\mathcal{F}^s$ with the continuous extension property for leaves of $\widetilde{\mathcal{F}}^s$. For simplicity we consider M hyperbolic. Analogous proofs work when $\pi_1(M)$ is negatively curved in the large.

Theorem 3.6. *Let Φ be a quasigeodesic pseudo-Anosov flow in M^3 closed, hyperbolic. Let $\mathcal{F}^s$ be the singular stable foliation of Φ. Then $\mathcal{F}^s$ is a quasi-isometric foliation if and only if, for each leaf $L \in \widetilde{\mathcal{F}}^s$, the extension $L \cup \partial_\infty L \to \mathbf{H}^3 \cup S^2_\infty$ is injective, that is, the map between ideal boundaries $\partial_\infty L \to S^2_\infty$ is a homeomorphism onto its image.*

Proof. Suppose first that $\mathcal{F}^s$ is quasi-isometric and let L be a leaf of the foliation $\widetilde{\mathcal{F}}^s$. Let $\xi : L \to M$ be the inclusion and $\overline{\xi} : L \cup \partial_\infty L \to \widetilde{M} \cup S^2_\infty$ the extension to the ideal compactifications given by Theorem 3.4. Since L and $\widetilde{M}$ are negatively curved in the large as defined by Gromov and $\xi : L \to \widetilde{M}$ is a quasi-isometric embedding, then Theorem 7.2 of [Gr] implies that $\overline{\xi}$ is injective.

Suppose now that $\overline{\xi} : L \cup \partial_\infty L \to \widetilde{M} \cup S^2_\infty$ is injective for all $L \in \widetilde{\mathcal{F}}^s$. Fix k_0 so that all flow lines of Φ are k_0-quasigeodesics. Recall that d_L, d'_L are uniformly quasi-isometric to each other in leaves or piece leaves L of $\widetilde{\mathcal{F}}^s$. It is simpler to use

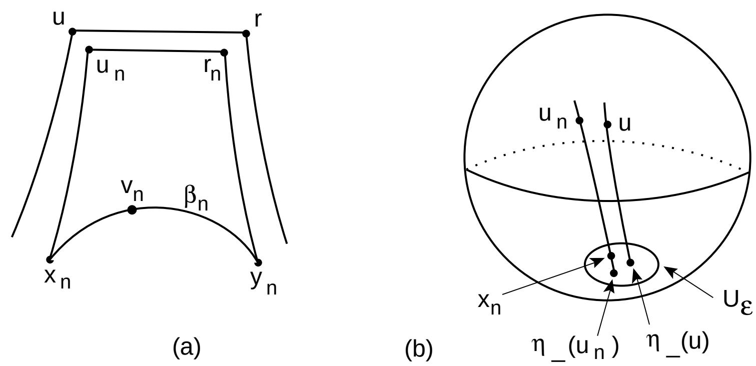

FIGURE 3. (a) Convergence of good paths in leaves. (b) Convergence in the sphere at infinity.

the d'_L metric. Suppose that $\widetilde{\mathcal{F}}^s$ is not quasi-isometric. Then there are leaves V_n of $\widetilde{\mathcal{F}}^s$ and $x_n, y_n \in V_n$ so that

$$(*) \qquad d'_{V_n}(x_n, y_n) > nd(x_n, y_n) + n, \quad \forall n \in \mathbf{N}.$$

Let L_n be piece leaves of V_n with $x_n, y_n \in L_n$. The previous lemma produces paths

$$\gamma_n \subset L_n, \quad \gamma_n = \alpha^0_n * \alpha^1_n * \alpha^2_n, \quad \text{with} \quad \alpha^0_n \subset \widetilde{\Phi}_{(-\infty,0]}(x_n), \quad \alpha^2_n \subset \widetilde{\Phi}_{(-\infty,0]}(y_n)$$

and α^1_n contained in a leaf of $\widetilde{\mathcal{F}}^{ss}$. In addition

$$l'(\alpha^1_n) \ \leq \ a_1 \quad \text{and} \quad l'(\gamma_n) \ \leq \ a_0 d'_{L_n}(x_n, y_n).$$

Up to subsequence assume that $l'(\alpha^0_n)$ or $l'(\alpha^2_n)$ is bounded, say the second option. The arc α^0_n is in a flow line of $\widetilde{\Phi}$, hence is k_0-quasigeodesic. Since γ_n is the union of $\alpha^1_n * \alpha^2_n$ of bounded length and α^0_n, it follows that γ_n is uniformly quasigeodesic (with a different constant) and $d'_{L_n}(x_n, y_n)$ is bounded above by a constant times $d(x_n, y_n)$. This contradicts the hypothesis.

We may now assume that $l'(\alpha^0_n), l'(\alpha^2_n) \to +\infty$ as $n \to \infty$. Given this we can also assume that $l'(\alpha^1_n)$ is bounded below, for otherwise we may save a lot of l' length by flowing α^1_n backwards. Since $l'(\alpha^1_n)$ is bounded above and below we may assume up to taking a subsequence that $\pi(\alpha^1_n)$ converges to a segment of positive length in a (possibly singular) leaf of $\mathcal{F}^{ss}$. Up to taking covering translations of $\widetilde{M}$, we may assume that the α^1_n themselves converge to a segment $[u, r]$ of positive length in a leaf of $\widetilde{\mathcal{F}}^{ss}$. Then α^0_n converges to $\widetilde{\Phi}_{(-\infty,0]}(u)$ and α^2_n converges to $\widetilde{\Phi}_{(-\infty,0]}(r)$.

Let u_n be the endpoint of α^0_n, r_n the starting point of α^2_n, so $u_n \to u, r_n \to r$. Let β_n be the geodesic arc of $\mathbf{H}^3$ connecting x_n, y_n, and let $v_n \in \beta_n$ be the closest point to u_n; see Figure 3 (a).

Suppose first that $d(u_n, v_n)$ does not converge to $+\infty$. Therefore up to subsequence assume that $d(u_n, v_n) \leq a_2$. Notice that $d(x_n, y_n) = d(x_n, v_n) + d(v_n, y_n)$. Since flow lines of $\widetilde{\Phi}$ are k_0 quasigeodesics, then

$$l'(\alpha^2_n) \ \leq \ k_0 d(r_n, y_n) + k_0, \quad l'(\alpha^0_n) \ \leq \ k_0 d(x_n, u_n) + k_0.$$

In addition

$$d(u_n, r_n) \ \leq \ l'(\alpha^1_n) \ \leq \ a_1, \quad d(r_n, y_n) \leq d(r_n, u_n) + d(u_n, y_n).$$

Therefore

$$l'(\alpha_n^1 * \alpha_n^2) \leq a_1 + k_0 d(r_n, y_n) + k_0 \leq a_1 + k_0 d(u_n, y_n) + k_0 a_1 + k_0.$$

Consequently

$$d'_{L_n}(x_n, y_n) \leq l'(\gamma_n) = l'(\alpha_n^0) + l'(\alpha_n^1 * \alpha_n^2)$$

$$\leq k_0 d(x_n, u_n) + k_0 + k_0 d(u_n, y_n) + k_0 a_1 + k_0 + a_1$$

$$= k_0(d(x_n, u_n) + d(u_n, y_n)) + k_0(2 + a_1) + a_1$$

$$\leq k_0 d(x_n, y_n) + 2k_0 d(u_n, v_n) + k_0(2 + a_1) + a_1$$

$$\leq k_0 d(x_n, y_n) + k_0(2a_2 + 2 + a_1) + a_1.$$

This contradicts the hypothesis (*). We conclude that $d(u_n, v_n) \to +\infty$.

Claim. $x_n \to \eta_-(u), y_n \to \eta_-(r)$ in $\widetilde{M} \cup S^2_\infty$ as $n \to \infty$.

We prove this for x_n. Consider the unit ball model for $\mathbf{H}^3$ with compactification $\widetilde{M} \cup S^2_\infty$ homeomorphic to a closed ball in Euclidean space. Let d_e be the Euclidean metric in this closed ball. Let ζ_n be the geodesic of $\mathbf{H}^3$ with endpoints $\eta_+(x_n), \eta_-(x_n)$, and let ζ be the geodesic of $\mathbf{H}^3$ with endpoints $\eta_+(u), \eta_-(u)$. Since $x_n = \widetilde{\Phi}_{t_n}(u_n)$, $u_n \to u$ as $n \to +\infty$ and η_-, η_+ are continuous, it follows that $\zeta_n \to \zeta$.

Fix $\epsilon > 0$. Let U_ϵ be the d_e-neighborhood of $\eta_-(u)$ in $\widetilde{M} \cup S^2_\infty$ with radius ϵ. There is n_0 so that for $n > n_0$ then $\eta_-(x_n) \in U_\epsilon$, so ζ_n has a ray contained in U_ϵ; see Figure 3 (b). Since $\widetilde{\Phi}_{\mathbf{R}}(x_n)$ is a uniformly bounded distance from ζ_n, then $\widetilde{\Phi}_{\mathbf{R}}(x_n)$ also has a ray contained in U_ϵ. In fact more is true: because flow lines of Φ are uniform k_0-quasigeodesics, then orthogonal projection of a flow line of $\widetilde{\Phi}$ to the geodesic with same ideal points is a k_1-quasi-isometry, where the k_1 depends only on k_0 [Th2, Gr]. Also distance between points and their images is uniformly bounded. Using this and the fact that $u_n \to u$, it follows that there is

$$n_0 \in \mathbf{N},\ t_0 < 0, \quad \text{so that} \quad \forall n > n_0,\ t < t_0, \quad \text{then} \quad \widetilde{\Phi}_t(u_n) \ \in \ U_\epsilon;$$

see Figure 3 (b). Since $x_n = \widetilde{\Phi}_{t_n}(u_n)$ and $t_n \to -\infty$, it follows that $x_n \in U_\epsilon$, for n big enough; see Figure 3 (b). This proves the claim.

Consequently β_n are geodesic arcs whose endpoints x_n, y_n converge to $\eta_-(u)$, $\eta_-(r)$. If $\eta_-(u) \neq \eta_-(r)$, then they define a geodesic β of $\mathbf{H}^3$ and $\beta_n \to \beta$, contradicting $d(u_n, \beta_n) \to +\infty$. Hence $\eta_-(u) = \eta_-(r)$. As $[u, r]$ has positive length, then u, r are not in the same orbit of $\widetilde{\Phi}$ in $C = \widetilde{W^s}(u)$. Let $u_-, r_- \in \partial_\infty C$ be the negative ideal points of $\widetilde{\Phi}_{\mathbf{R}}(u), \widetilde{\Phi}_{\mathbf{R}}(r)$ as seen in $C \cup \partial_\infty C$. Since u and r are not in the same flow line of $\widetilde{\Phi}$, then $u_- \neq r_-$. But

$$\overline{\xi}(u_-) \ = \ \eta_-(u) \ = \ \eta_-(r) \ = \ \overline{\xi}(r_-),$$

contradicting the fact that $\overline{\xi} : C \cup \partial_\infty C \to \widetilde{M} \cup S^2_\infty$ is injective. This finishes the proof of Theorem 3.6. $\square$

We now define sectors and line leaves, which will be used throughout the article.

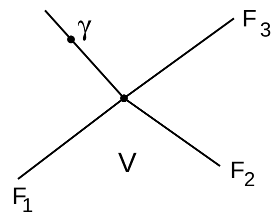

FIGURE 4. The sets associated to a leaf F of $\widetilde{\mathcal{F}}^s$: F_1, F_2, F_3 are standard leaves, V is a sector of F. Then $F_1 \cup F_2 = \partial V$ is a line leaf of F and so is $F_2 \cup F_3$. Also $F_1 \cup F_3$ is a piece leaf of F which is not a line leaf since it has prongs of F on each side. Each component of $F - \gamma$ is a half leaf of F.

Definition 3.7. The sectors defined by $L \in \widetilde{\mathcal{F}}^s$ are the components of $\widetilde{M} - L$. A line leaf of $L \in \widetilde{\mathcal{F}}^s$ is the boundary of a sector of L. Notice that it is a piece leaf of L, but if L is a singular leaf with 4 or more standard leaves, then there are many piece leaves of L which are not line leaves. If L has a p-prong singular orbit, then L has p line leaves.

Figure 4 illustrates all these concepts in a leaf F of $\widetilde{\mathcal{F}}^s$. We stress in the case of a regular leaf that the following simplifications occur: line leaves are the same as piece leaves and also half leaves are the same as standard leaves (modulo the boundary orbit). Half leaves are formally defined in the beginning of the next section.

Recall the stable lamination Λ^s. Its complementary regions are solid tori or solid Klein bottles, hence Λ^s is an essential lamination [Ga-Oe]. It follows that any piece leaf of $\widetilde{\mathcal{F}}^s$ separates in $\widetilde{M}$.

Given $B \subset \mathbf{H}^3$, its limit set is $\mathcal{I}_B = \overline{B} \cap S^2_\infty$, the closure taken in $\widetilde{M} \cup S^2_\infty$.

We will now prove Theorem A of the introduction.

Theorem 3.8. *Let Φ be a quasigeodesic pseudo-Anosov flow in M^3 closed, hyperbolic. Let $\mathcal{F}^s$ be the singular stable foliation of Φ. Then $\mathcal{F}^s$ is a quasi-isometric singular foliation if and only if $\widetilde{\mathcal{F}}^s$ has Hausdorff leaf space.*

Proof. One direction is very simple and the other uses the previous theorem.

First suppose that the leaf space of $\widetilde{\mathcal{F}}^s$ is not Hausdorff. There are $F \neq L \in \widetilde{\mathcal{F}}^s$ and $F_n \in \widetilde{\mathcal{F}}^s$ with $F_n \to F \cup L$ in the leaf space of $\widetilde{\mathcal{F}}^s$. Let $c \in F$ and $e \in L$ and let $c_n, e_n \in F_n$ with $c_n \to c$ and $e_n \to e$. Then $d(c_n, e_n) \to d(c, e)$ so $d(c_n, e_n)$ is bounded. If $d_{F_n}(c_n, e_n)$ has a bounded subsequence we may assume that it is bounded. Hence there is $a_4 > 0$ so that for all n, $e_n \in B^{F_n}_{a_4}(c_n) =$ the ball of radius a_4 in the d_{F_n} metric of F_n.

Here we use the fact that if $c_n \to c$, with $c_n \in F_n \in \widetilde{\mathcal{F}}^s$ and $c \in F \in \widetilde{\mathcal{F}}^s$, then

$$B^{F_n}_{a_4}(c_n) \ \to \ Z \text{ as } n \to \infty, \quad \text{where} \quad Z \ \subset \ B^F_{a_4}(c)$$

in the Gromov-Hausdorff topology of closed sets of $\widetilde{M}$. The reason is the following: if F is not singular, then $B^F_{a_4}(c)$ has a small product foliated neighborhood and $B^{F_n}_{a_4}(c_n)$ actually converges to $B^F_{a_4}(c)$. On the other hand if F is singular, then up to taking a subsequence of the c_n it follows that either all c_n are in F, in which case $W = B^F_{a_4}(c)$ again, or the c_n are all in one sector V of F. In this last case $B^{F_n}_{a_4}(c_n) \to B^{F'}_{a_4}(c)$, where F' is the line leaf of F which is the boundary of V.

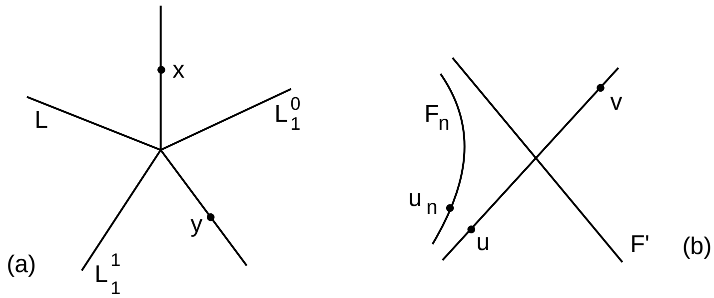

FIGURE 5. (a) Producing points on line leaves of L with same negative ideal points. (b) Impossible configuration of convergence.

As a consequence of this fact, since $e_n \in B^{F_n}_{a_4}(c_n)$, we conclude that $e \in B^F_{a_4}(c) \subset F$, contradicting $F \neq L$. This proves one implication of the theorem.

Assume now that $\mathcal{F}^s$ is not a quasi-isometric singular foliation. By the previous theorem there is $L \in \widetilde{\mathcal{F}}^s$ and $x, y \in L$, not in the same orbit of $\widetilde{\Phi}$ in L and so that $\eta_-(x) = \eta_-(y)$. First we claim that we can assume that x, y are in a line leaf of L. Suppose then that x and y are not contained in a line leaf of L; see Figure 5 (a). This implies that L is a singular leaf with 4 or more standard leaves. There is a piece leaf L_1 of L which separates x from y, that is, separates $\widetilde{\Phi}_{\mathbf{R}}(x)$ from $\widetilde{\Phi}_{\mathbf{R}}(y)$ in $\widetilde{M}$. In addition we can choose L_1 so that the two standard leaves L^0_1, L^1_1 of L_1 are part of line leaves of L which also contain y; see Figure 5 (a). As $\eta_-(x) = \eta_-(y)$ and L_1 separates $\widetilde{\Phi}_{\mathbf{R}}(x)$ from $\widetilde{\Phi}_{\mathbf{R}}(y)$ in $\widetilde{M}$, it follows that $\eta_-(x)$ is in $\mathcal{I}_{L_1}$, the limit set of L_1.

Let $\overline{\xi} : L_1 \cup \partial_\infty L_1 \to \widetilde{M} \cup S^2_\infty$ be the extension of $\xi : L_1 \to \widetilde{M}$. By Theorem 3.4, $\mathcal{I}_{L_1} = \overline{\xi}(\partial_\infty L_1)$. Let $c_0 \in \partial_\infty L_1$ with $\overline{\xi}(c_0) = \eta_-(y)$. By Lemma 2.3, we can choose $z \in L_1$ with either

$$\eta_-(z) = \overline{\xi}(c_0) = \eta_-(y) \quad \text{or} \quad \eta_+(z) = \overline{\xi}(c_0) = \eta_-(y).$$

Suppose first that $\eta_+(z) = \eta_-(y)$. As $z, y \in L \in \widetilde{\mathcal{F}}^s$, then $\eta_+(z) = \eta_+(y)$. But then $\eta_+(y) = \eta_-(y)$, a contradiction to $\widetilde{\Phi}_{\mathbf{R}}(y)$ being quasigeodesic. Therefore $\eta_-(z) = \eta_-(y)$. Suppose that $z \in \overline{L}^0_1$. Then z, y are in a line leaf L_2 of L. L_2 is the union of $\overline{L}^0_1$ and the standard leaf of L containing y. This proves the claim.

By moving z, y slightly along their unstable leaves into the sector of L defined by L_2 we find $z_1 \in \widetilde{W}^u(z), y_1 \in \widetilde{W}^u(y) \cap \widetilde{W}^s(z_1)$ and $\widetilde{W}^s(z_1)$ regular. Notice that

$$\eta_-(z_1) \; = \; \eta_-(z) \; = \; \eta_-(y) \; = \; \eta_-(y_1)$$

and z_1, y_1 are not in the same orbit in $\widetilde{W}^s(z_1)$. We may further assume that $y_1 \in \widetilde{W}^{ss}(z_1)$.

Let $v_n = \widetilde{\Phi}_{t_n}(z_1)$ with $t_n \to -\infty$, hence $v_n \to \eta_-(z_1)$. As $\eta_-(z_1) = \eta_-(y_1)$, then $\widetilde{\Phi}_{\mathbf{R}}(z_1), \widetilde{\Phi}_{\mathbf{R}}(y_1)$ are quasigeodesics of $\mathbf{H}^3$ with same ideal points, and there are $s_n \to -\infty$ so that $u_n = \widetilde{\Phi}_{s_n}(y_1)$ satisfies $d(v_n, u_n)$ is bounded.

Assume up to taking a subsequence that $\pi(v_n)$ converges to $b_0 \in M$. There are covering translations $g_n \in \pi_1(M)$ with $g_n(v_n) \to v$ and $\pi(v) = b_0$. As $d(g_n(v_n), g_n(u_n))$ is bounded, assume up to taking a further subsequence, that

$g_n(u_n)$ also converges to $u \in \widetilde{M}$. Let $F_n = g_n(\widetilde{W}^s(z_1))$. Notice that $d(v_n, u_n)$ is bounded below by a positive constant, because $t_n, s_n \rightarrow -\infty$, so in F_n, $d_{F_n}(u_n, v_n) \rightarrow +\infty$. Therefore $u \neq v$.

Suppose that u and v are in the same stable leaf F. We first claim that u and v cannot be separated by a piece leaf F' of F; see Figure 5 (b). Otherwise F' would separate u_n from v_n for n big, a contradiction to $u_n, v_n \in F_n$ and $u_n \rightarrow u$, $v_n \rightarrow v$. Let γ be a path in F from u to v. Since $F_n \rightarrow F$ and no piece leaf of F separates u from v, it follows that there are paths γ_n in F_n from $g_n(u_n)$ to $g_n(v_n)$ which are converging to γ. This implies that $l(\gamma_n)$ is bounded above and consequently $d_{F_n}(v_n, u_n)$ is bounded, a contradiction to the above. Hence u and v are not in the same leaf of $\widetilde{\mathcal{F}}^s$ and $\widetilde{\mathcal{F}}^s$ does not have Hausdorff leaf space. This finishes the proof of Theorem 3.8. $\square$

Remark. The property $\mathcal{F}^s$ quasi-isometric implies that Φ is quasigeodesic: Let β be a flow line of $\widetilde{\Phi}$ and $\beta \subset L \in \widetilde{\mathcal{F}}^s$. Then β is a minimal geodesic in the d'_L metric of L, that is, length along β measures length in L. The quasi-isometric property for L then immediately implies that β is a quasigeodesic in $\widetilde{M}$.

4. Topological theory of pseudo-Anosov flows

We showed in the previous section that if Φ is a quasigeodesic pseudo-Anosov flow in M^3 hyperbolic, then important geometric questions for $\widetilde{\mathcal{F}}^s$ fundamentally depend on the topology of the leaf space of $\widetilde{\mathcal{F}}^s$. When we want to think of this leaf space as a topological space we use the notation $\mathcal{H}(\widetilde{\mathcal{F}}^s)$. Then $\mathcal{H}(\widetilde{\mathcal{F}}^s)$ is a 1-dimensional object which is not Hausdorff when there are leaves of $\widetilde{\mathcal{F}}^s$ nonseparated from other leaves (in $\mathcal{H}(\widetilde{\mathcal{F}}^s)$). Also $\mathcal{H}(\widetilde{\mathcal{F}}^s)$ is not a manifold if Φ is really singular pseudo-Anosov (as opposed to being an Anosov flow). Then each p-prong leaf of $\widetilde{\mathcal{F}}^s$ gives rise to a p-prong in $\mathcal{H}(\widetilde{\mathcal{F}}^s)$. A leaf F of $\widetilde{\mathcal{F}}^s$ can be thought of as a subset of $\widetilde{M}$, that is, the union of the points of $\widetilde{M}$ which are in the leaf F; or as a point in $\mathcal{H}(\widetilde{\mathcal{F}}^s)$.

In this section we study the structure of the non-Hausdorff points in $\mathcal{H}(\widetilde{\mathcal{F}}^s)$. For starters we stress that the analysis will be done in a completely general setting: we will not assume that Φ is quasigeodesic or that M is hyperbolic. The study of non-Hausdorff points in $\mathcal{H}(\widetilde{\mathcal{F}}^s)$ was done in [Fe6] for general Anosov flows. Much of the analysis is very similar and we will refer to [Fe4, Fe5, Fe6] whenever possible. The main difficulty is to show that even though $\widetilde{\mathcal{F}}^s$ is a singular foliation, its singularities do not interfere with the analysis of the non-Hausdorff points of $\mathcal{H}(\widetilde{\mathcal{F}}^s)$. Regardless of geometric applications, the results of this section are of independent interest for the theory of pseudo-Anosov flows.

In [Fe6] a non-Hausdorff point of $\mathcal{H}(\widetilde{\mathcal{F}}^s)$ was called a branching point, and $F, L \in \widetilde{\mathcal{F}}^s$ was called a branching pair of $\widetilde{\mathcal{F}}^s$ if F is not separated from L as seen in $\mathcal{H}(\widetilde{\mathcal{F}}^s)$. In the case of pseudo-Anosov flows, a singular leaf $L \in \widetilde{\mathcal{F}}^s$ also produces some sort of branching in the leaf space $\mathcal{H}(\widetilde{\mathcal{F}}^s)$ - this is not associated to non-Hausdorff behavior in $\mathcal{H}(\widetilde{\mathcal{F}}^s)$. For this reason we do not find it appropriate to use the term branching leaves for the non-Hausdorff points of $\mathcal{H}(\widetilde{\mathcal{F}}^s)$ in the case of pseudo-Anosov flows.

The following facts and definitions will be needed here and in later sections. We proved in [Fe-Mo] that the orbit space of $\widetilde{\Phi}$ in $\widetilde{M}$ is homeomorphic to the plane $\mathbf{R}^2$. This orbit space is denoted by $\mathcal{O} \cong \widetilde{M}/\widetilde{\Phi}$. Let $\Theta : \widetilde{M} \rightarrow \mathcal{O} \cong \mathbf{R}^2$ be the projection map. As the foliations $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$ are invariant under $\widetilde{\Phi}$, they induce singular, transverse 1-dim foliations $\widetilde{\mathcal{F}}^s_{\mathcal{O}}, \widetilde{\mathcal{F}}^u_{\mathcal{O}}$ in $\mathcal{O}$. The singular points of $\widetilde{\mathcal{F}}^u_{\mathcal{O}}$ are the same as those of $\widetilde{\mathcal{F}}^s_{\mathcal{O}}$. If L is a leaf of $\widetilde{\mathcal{F}}^s$ or $\widetilde{\mathcal{F}}^u$, then $\Theta(L) \subset \mathcal{O}$ is a tree which is either homeomorphic to $\mathbf{R}$ if L is regular, or is a union of p-rays all with the same starting point if L has a singular p-prong orbit. In particular every orbit in L disconnects L.

Definition 4.1. Let L be a leaf of $\widetilde{\mathcal{F}}^s$ or a line leaf or a piece leaf of a leaf of $\widetilde{\mathcal{F}}^s$. Then a half leaf of L is a connected component A of $L - \gamma$, where γ is any full orbit in L. The closed half leaf is $\overline{A} = A \cup \gamma$ and its boundary is $\partial A = \gamma$. If ζ is an open, relatively compact, connected subset of $\Theta(L)$, then it defines a *flow band* L_1 of L by $L_1 = \Theta^{-1}(\zeta)$. Let $\overline{L}_1$ be the closure of L_1 in $\widetilde{M}$. If ζ is an open segment in $\Theta(L)$, then $\Theta^{-1}(\zeta)$ is called a segment flow band of L.

One difference from the case of Anosov flows is that even in the universal cover, the stable and unstable foliations may not be transversely orientable. In fact the foliations in the universal cover will be transversely orientable if and only if all singular orbits have an even number of prongs. For Anosov flows, much of the analysis in [Fe5, Fe6] was coached using the transversal orientations to $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$ and given $L \in \widetilde{\mathcal{F}}^s$, referring to the positive/negative sides of L in $\widetilde{M}$. Here this will not be possible. Nevertheless any leaf L in $\widetilde{\mathcal{F}}^s$ or $\widetilde{\mathcal{F}}^u$ separates $\widetilde{M}$ and we will use the connected components of $\widetilde{M} - L$, which are the sectors of L.

Here are two easy but important facts. First if $F \in \widetilde{\mathcal{F}}^s$ and $G \in \widetilde{\mathcal{F}}^u$, then F and G intersect in at most one orbit. This is a consequence of index computations for foliations in the plane as follows. Project to $\mathcal{O}$ and suppose that there are $a \neq b$ in $\Theta(F) \cap \Theta(G)$. Since $\Theta(F)$ is a tree, there is a unique segment l from a to b in $\Theta(F)$. By the local product structure of $\widetilde{\mathcal{F}}^s_{\mathcal{O}}$ and the fact that $\widetilde{\mathcal{F}}^s_{\mathcal{O}}, \widetilde{\mathcal{F}}^u_{\mathcal{O}}$ are transverse, it follows that intersections of l with $\Theta(G)$ are discrete in l and we may assume that $l \cap \Theta(G) = \{a, b\}$. Let l' be the segment in $\Theta(G)$ connecting a and b. Then $l \cap l' = \{a, b\}$. Let D be the disk of $\mathcal{O}$ bounded by $l \cup l'$. Consider the singular foliation in D induced by $\widetilde{\mathcal{F}}^s_{\mathcal{O}}$ and count the indices at each singularity of $\widetilde{\mathcal{F}}^s_{\mathcal{O}}$.

- Corner singularities. If there are n_i interior prongs of $\widetilde{\mathcal{F}}^s_{\mathcal{O}}$ from a corner (we do not count l here), then the index is $1/4 - n_i/2$.

- Transverse singularities in l'. Let p_i be a singularity of $\widetilde{\mathcal{F}}^s_{\mathcal{O}}$ in the interior of l' so that there are m_i prongs of $\widetilde{\mathcal{F}}^s_{\mathcal{O}}$ entering $int(D)$. The index is $(1 - m_i)/2$. Here $m_i = 1$ corresponds to l' being regular on the side facing $int(D)$.

- Interior singularities. Let q_i be an interior singularity with u_i prongs, $u_i \geq 3$. Then the index is $1 - u_i/2$.

The sum of the indices must be 1, the Euler characteristic of D. There are 2 corners, each with index $\leq 1/4$, with sum $\leq 1/2$. The other two index contributions are negative so this is impossible. Hence F and G intersect at most once.

A second important consequence of index computations is the following: Suppose that a leaf $F \in \widetilde{\mathcal{F}}^s$ intersects two leaves $G, H \in \widetilde{\mathcal{F}}^u$ and so does $L \in \widetilde{\mathcal{F}}^s$. Let $a = \Theta(F \cap G)$, a single point, and likewise $b = \Theta(F \cap H)$, $c = \Theta(H \cap L)$ and $d = \Theta(L \cap G)$; see Figure 6 (a). Let l_1 be the closed segment in $\Theta(F)$ with endpoints a, b and similarly $l_2 \subset \Theta(H)$ with endpoints b, c; $l_3 \subset \Theta(L)$ with endpoints c, d; and $l_4 \subset \Theta(G)$ with endpoints d, a. Then $(l_1 \cup l_2 \cup l_3 \cup l_4)$ bounds a disk D in $\mathcal{O}$.

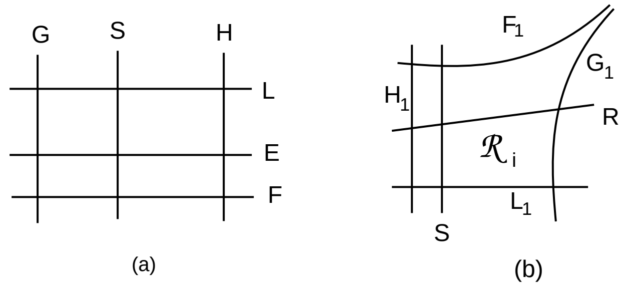

FIGURE 6. (a) Rectangles. (b) Perfect fits in the universal cover.

Consider the singular foliation induced by $\widetilde{\mathcal{F}}^s_{\mathcal{O}}$ in D. It is tangent to l_1 and l_3 and transverse to l_2 and l_4. Computing the index at singularities as above: the index at each corner is $\leq 1/4$ and all other indices are negative. Since $\chi(D) = 1$, the only possibility is that there are no interior prongs from the boundary or corners and no interior singularities. It follows that there is a product structure in D: a leaf of $\widetilde{\mathcal{F}}^s_{\mathcal{O}}$ intersects l_2 if and only if it intersects l_4, and a leaf in $\widetilde{\mathcal{F}}^u_{\mathcal{O}}$ intersects l_1 if and only if it intersects l_3. We therefore call the region D (or $\Theta^{-1}(D)$) a *rectangle*.

Recall that a line leaf L' of $L \in \widetilde{\mathcal{F}}^s$ is the boundary of a component of $M - L$. Then L' separates $\widetilde{M}$ into two components and we say that L' is *regular* on the side which is a sector of L. Notice that L is regular if and only if both sides of L' are regular. In the same way a line leaf of $\Theta(L) \in \widetilde{\mathcal{F}}^s_{\mathcal{O}}$ is $\Theta(B)$ where B is a line leaf of L.

Definition 4.2. Perfect fits - Two leaves $F \in \widetilde{\mathcal{F}}^s$ and $G \in \widetilde{\mathcal{F}}^u$ form a perfect fit if $F \cap G = \emptyset$ and there are line leaves F_0, G_0 of F, G respectively and half leaves F_1 of F_0 and G_1 of G_0 and also segment flow bands $L_1 \subset L \in \widetilde{\mathcal{F}}^s$ and $H_1 \subset H \in \widetilde{\mathcal{F}}^u$, so that F_0 is regular on the side containing L, G_0 is regular on the side containing H and:

$$\overline{L}_1 \cap \overline{G}_1 = \partial L_1 \cap \partial G_1, \quad \overline{L}_1 \cap \overline{H}_1 = \partial L_1 \cap \partial H_1, \quad \overline{H}_1 \cap \overline{F}_1 = \partial H_1 \cap \partial F_1,$$

$$\text{with} \quad \overline{L}_1 \cap \overline{G}_1 \neq \emptyset, \ \ \overline{L}_1 \cap \overline{H}_1 \neq \emptyset \ \text{ and } \ \overline{H}_1 \cap \overline{F}_1 \neq \emptyset.$$

Furthermore

$$\forall\ S \in \widetilde{\mathcal{F}}^u, \quad S \cap L_1 \neq \emptyset \ \Rightarrow S \cap F_1 \neq \emptyset \tag{1}$$

and

$$\forall\ E \in \widetilde{\mathcal{F}}^s, \quad E \cap H_1 \neq \emptyset \ \Rightarrow E \cap G_1 \neq \emptyset. \tag{2}$$

We refer to Figure 6 (b) for perfect fits. We claim that implications (1), (2) imply equivalences (that is, $S \cap L_1 \neq \emptyset \ \Leftrightarrow S \cap F_1 \neq \emptyset$ and the same for (2)). To see this let $S \in \widetilde{\mathcal{F}}^u$ with $S \cap F_1 \neq \emptyset$. Suppose that L_1 contains a singular orbit δ. Extend L_1 to a piece leaf L_2 of L. We first show that L_2 must be regular on the side containing F. Otherwise there is a component V of $L_1 - \delta$ which is separated from F by a line leaf of L. But then any unstable leaf U intersecting V cannot intersect this line leaf of L and hence cannot intersect F, a contradiction to condition (1). Using the fact that F_0 is regular on the side containing G, one finds $R \in \widetilde{\mathcal{F}}^s$ near enough F, so that $R \cap H_1 \neq \emptyset$ and $R \cap S \neq \emptyset$. By (2), $R \cap G_1 \neq \emptyset$. Under these conditions

$$R \cap G \neq \emptyset, \quad R \cap H \neq \emptyset, \quad L \cap G \neq \emptyset \quad \text{and} \quad L \cap H \neq \emptyset.$$

Therefore L, R, G and H form a rectangle. By the properties of rectangles it follows that S intersects L_1. This proves the claim.

The set $\overline{F}_1 \cup \overline{H}_1 \cup \overline{L}_1 \cup \overline{G}_1$ separates $\widetilde{M}$. Let A be the complementary region which does not contain $F - F_1$ in its closure. Let $p_i \in H_1$ with $p_i \to p \in F$ and so that $\widetilde{\Phi}_{\mathbf{R}}(p_i)$ separates $\widetilde{\Phi}_{\mathbf{R}}(p_{i-1})$ from $\widetilde{\Phi}_{\mathbf{R}}(p_{i+1})$ in H_1. Let $R_i = \widetilde{W}^s(p_i)$. Then $R_i \cap G \neq \emptyset$, so R_i, G, L, H form a rectangle $\mathcal{R}_i$ (see Figure 6 (b)) and there are no singularities in the interior of $\mathcal{R}_i$. The condition on the p_i's implies that $\mathcal{R}_i \subset \mathcal{R}_{i+1}$ for any i. Let

$$B \;=\; \bigcup_{i \in \mathbf{N}} \mathcal{R}_i.$$

Clearly $B \subset A$. We claim that $B = A$. Otherwise let $z \in A$ with $z \in \partial B$. Since $\mathcal{R}_i$ are increasing with i and only the stable boundary component of $\mathcal{R}_i$ (contained in R_i) is changing, it follows that $\widetilde{W}^u(z) \cap R_i \neq \emptyset$ for i big enough. Because R_i, H, L, G form a rectangle this forces $\widetilde{W}^u(z) \cap L_1 \neq \emptyset$. The definition of perfect fit implies that $\widetilde{W}^u(z) \cap F_1 \neq \emptyset$. The construction of the $\mathcal{R}_i$ implies that $z \in F_1$, which in turn implies that $z \in \partial A$, a contradiction. We conclude that $A = B$. This is important:

Conclusion. There are <u>no singularities</u> of $\widetilde{\Phi}$ in A.

Therefore perfect fits produce "ideal" rectangles, in the sense that even though F and G do not intersect, there is a product structure (of $\widetilde{\mathcal{F}}^s$ and $\widetilde{\mathcal{F}}^u$) in the interior of A. Notice that the flow bands L_1, H_1 (or the leaves L, H) are not uniquely determined by the perfect fit (F, G).

There is at most one leaf $G \in \widetilde{\mathcal{F}}^u$ making a perfect fit with a given half leaf of $F \in \widetilde{\mathcal{F}}^s$ and in a given side of F - the proof is the same as that for Anosov flows [Fe5]. Therefore if (F, G) forms a perfect fit and g is a covering translation preserving the half leaf F_1 and also the components of $M - F$, then $g(G) = G$. This follows from uniqueness of perfect fits and the fact that g takes perfect fits to perfect fits, because it acts by homeomorphisms in the leaf spaces.

Definition 4.3. Given $p \in \widetilde{M}$ (or $p \in \mathcal{O}$), and a half leaf H of $\widetilde{W}^u(p)$ defined by $\widetilde{\Phi}_{\mathbf{R}}(p)$, let

$$\mathcal{J}^u(H) \;=\; \{F \in \mathcal{H}(\widetilde{\mathcal{F}}^s) \mid F \cap H \neq \emptyset\} \;\subset\; \mathcal{H}(\widetilde{\mathcal{F}}^s).$$

Notice that $\widetilde{W}^s(p) \notin \mathcal{J}^u(H)$. Also let

$$\mathcal{L}^u(H) \;=\; \bigcup \{\; p \;\in \widetilde{M} \;\;|\;\; p \;\in\; F \;\in\; \mathcal{J}^u(H) \;\} \;\;\subset\;\; \widetilde{M}.$$

Then $\mathcal{L}^u(H) \subset \widetilde{M}$ and $\widetilde{W}^s(p) \subset \partial \mathcal{L}^u(H)$. Similarly define $\mathcal{J}^s(L), \mathcal{L}^s(L)$ for a stable half leaf L.

The notation $\mathcal{J}^u(H)$ is chosen to match the previous definition for Anosov flows [Fe5, Fe6]. In those articles the data was $p \in \widetilde{M}$, a symbol u or s (corresponding to a half leaf of $\widetilde{W}^u(p)$ or $\widetilde{W}^s(p)$) and a symbol $+$ or $-$ specifying that the half leaf is on the positive or negative side of the leaf through p of the dual foliation. For

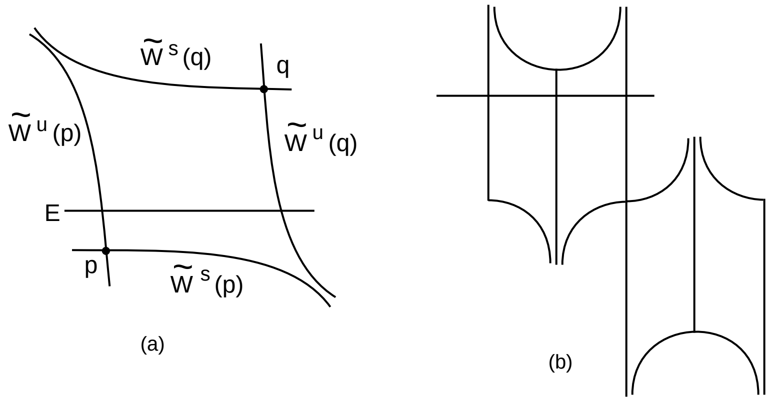

FIGURE 7. (a) A lozenge. (b) A chain of lozenges.

instance one such set was $\mathcal{J}^u_+(p)$. This entails transversal orientations which we do not have here. The data here is already a half leaf in $\widetilde{W}^u(p)$ or $\widetilde{W}^s(p)$.

Definition 4.4. Lozenges - Let $p, q \in \widetilde{M}$ and let there be half leaves L_p, H_p of $\widetilde{W}^s(p), \widetilde{W}^u(p)$ defined by $\widetilde{\Phi}_{\mathbf{R}}(p)$, and half leaves L_q, H_q of $\widetilde{W}^s(q), \widetilde{W}^u(q)$ defined by $\widetilde{\Phi}_{\mathbf{R}}(q)$ so that:

$$\mathcal{L}^u(L_p) \ \cap \ \mathcal{L}^s(H_q) \quad = \quad \mathcal{L}^u(L_q) \ \cap \ \mathcal{L}^s(H_p) \quad \subset \ \widetilde{M}.$$

Then this intersection is called a lozenge $\mathcal{A}$ in $\widetilde{M}$. The corners of the lozenge are $\widetilde{\Phi}_{\mathbf{R}}(p)$ and $\widetilde{\Phi}_{\mathbf{R}}(q)$ and $\mathcal{A}$ is a subset of $\widetilde{M}$. The sides of $\mathcal{A}$ are L_p, H_p, L_q, H_q. The sides are not contained in the lozenge, but are in the boundary of the lozenge.

Sometimes we also refer to p and q as corners of the lozenge.

An argument as done previously shows that there are no singularities in the lozenges. This in fact shows that $\mathcal{A}$ is an open region in $\widetilde{M}$. However there may be singular orbits on the sides of the lozenge and the corner orbits also may be singular. The definition of a lozenge implies that L_p, H_q form a perfect fit and so do L_q, H_p - this is an equivalent way to define a lozenge with corners $\widetilde{\Phi}_{\mathbf{R}}(p), \widetilde{\Phi}_{\mathbf{R}}(q)$. Given any four leaves there is at most one lozenge with sides in them, so we may refer to the full leaves as the sides of the lozenge.

Two lozenges are *adjacent* if they share a corner and there is a stable or unstable leaf intersecting both of them; see Figure 7 (b). Therefore they share a side. A *chain of lozenges* is a collection $\{\mathcal{A}_i\}, i \in I$, where I is an interval (finite or not) in $\mathbf{Z}$; so that if $i, i+1 \in I$, then $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$ share a corner; see Figure 7 (b). Consecutive lozenges may be adjacent or not. The chain is finite if I is finite.

Definition 4.5. Suppose $\zeta \subset E \in \widetilde{\mathcal{F}}^s$ is a (possibly infinite) strong stable segment so that for each $p \in \zeta$ there is a half leaf H_p of $\widetilde{W}^u(p)$ defined by $\widetilde{\Phi}_{\mathbf{R}}(p)$ so that

$$\forall \ p, q \in \zeta, \quad \mathcal{J}^u(H_p) = \mathcal{J}^u(H_q).$$

In that case let $\mathcal{P} = \bigcup_{p \in \zeta} H_p$. Then $\mathcal{P} \subset \widetilde{M}$ is called an unstable product region with base segment ζ. The base segment is not uniquely determined by $\mathcal{P}$. Similarly define stable product regions.

The main property of product regions is the following: for any $F \in \widetilde{\mathcal{F}}^s$, $G \in \widetilde{\mathcal{F}}^u$ so that

$$(i)\ F \cap \mathcal{P}\ \neq\ \emptyset \ \text{ and } \ (ii)\ G \cap \mathcal{P}\ \neq\ \emptyset, \quad \text{then} \quad F \cap G\ \neq\ \emptyset.$$

Suppose for instance that $\mathcal{P}$ is an unstable product region. First notice that (ii) implies that $\emptyset \neq G \cap \zeta = p$, so $H_p \subset G$. By (i) let $q \in \zeta$ with $F \cap H_q \neq \emptyset$. Then $F \in \mathcal{J}^u(H_q)$, hence $F \in \mathcal{J}^u(H_p)$, that is, $F \cap G \neq \emptyset$. As before it follows that there are no singular orbits of $\widetilde{\Phi}$ in $\mathcal{P}$.

We will also denote by rectangles, perfect fits, lozenges and product regions the projection of these regions to $\mathcal{O}$. One good way to visualize these objects in $\mathcal{O}$ is as follows. Consider proper embeddings $\xi : U \to \mathcal{O}$ of sets $U \subset \mathbf{R}^2$ into $\mathcal{O}$, sending the horizontal and vertical foliations induced in U to the stable and unstable foliations in $\xi(U) \subset \mathcal{O}$. Then a proper embedding is associated to a rectangle $\xi(U)$ if $U = [0,1] \times [0,1]$. A proper embedding is associated to a perfect fit if U is a rectangle without a corner, that is, $U = [0,1] \times [0,1] - \{(1,1)\}$. A lozenge is associated to the image of a rectangle without two opposite corners $U = [0,1] \times [0,1] - \{(1,1),(0,0)\}$ (the lozenge is the interior of $\xi(U)$). A stable product region is associated to the image of $U = [a,b] \times [0,\infty)$ (or $U = \mathbf{R} \times [0,\infty)$ when the base segment is infinite) and similarly for an unstable product region. The important fact here is that there are no singular orbits in any of these regions.

Definition 4.6. Convergence in the leaf space $\mathcal{H}(\widetilde{\mathcal{F}}^s)$ of $\widetilde{\mathcal{F}}^s$. We say that $L_i \in \widetilde{\mathcal{F}}^s$ converges to $L \in \widetilde{\mathcal{F}}^s$ if there is $x \in L$ and $x_i \in L_i$ with $x_i \to x$.

For a nonsingular foliation this is the same as requiring that for any $x \in L$ there are $x_i \in L_i$ with $x_i \to x$. The introduction of singularities changes this: the L_i usually only converge to a line leaf in L. Here's why: The trivial case is $L_i = L$ for all but finitely many i, in which case all points in L satisfy the requirement. So we may assume all L_i are distinct from each other and from L. Up to taking a subsequence we may assume all L_i are in the same sector V_0 of L. Let E be the line leaf ∂V_0. If $y \in E$, let B be a small segment in $\widetilde{W}^{uu}(y)$ with one endpoint in y and contained in V_0. Then since there are no components of $L - E$ contained in V_0 there is a local product structure of $\widetilde{\mathcal{F}}^s$ in this side of E, therefore for i big enough $L_i \cap B = y_i$ and clearly $y_i \to y$. Finally if $y' \in L - E$, then E separates y' from all L_i so no sequence in L_i can converge to y. This shows that taking a subsequence of the L_i is unnecessary. We conclude that the points $y \in L$ which are obtained as limits of $y_i \in L_i$ are exactly those in the line leaf E.

A leaf L of $\widetilde{\mathcal{F}}^s$ or $\widetilde{\mathcal{F}}^u$ is called *periodic* if there is a nontrivial covering translation g of $\widetilde{M}$ with $g(L) = L$. This is equivalent to $\pi(L)$ containing a periodic orbit of Φ. In the same way an orbit γ of $\widetilde{\Phi}$ is *periodic* if $\pi(\gamma)$ is a periodic orbit of Φ.

The following is a fundamental result:

Theorem 4.7. *Let Φ be a pseudo-Anosov flow in M^3. Let F be a non-Hausdorff point in the leaf space of $\widetilde{\mathcal{F}}^s$. Then F is periodic.*

Proof. We will make extensive use of the detailed analysis in [Fe6], particularly Theorem 3.5, which is the analog result for Anosov flows. The primary goal is to show that singularities do not affect this situation. Let then $L \in \widetilde{\mathcal{F}}^s$, $L \neq F$, so that F, L are not separated from each other in the leaf space of $\widetilde{\mathcal{F}}^s$. Let $L_i \in \widetilde{\mathcal{F}}^s$ with $L_i \to L \cup F$. Since $L \neq F$, only finitely many L_i can be either F or L. As

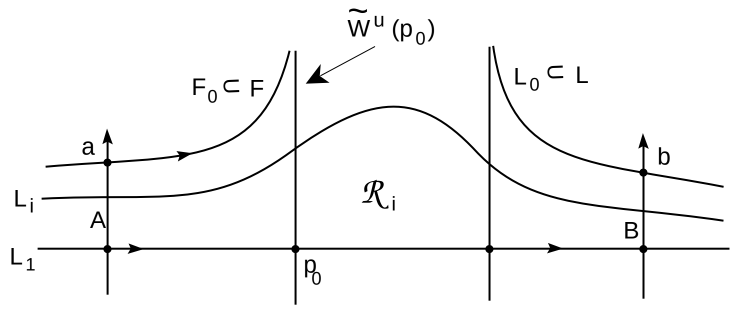

FIGURE 8. Non-Hausdorff points in $\widetilde{\mathcal{F}}^s$.

seen above there are line leaves F_0, L_0 of F, L respectively with $L_i \to L_0 \cup F_0$. Let W_0 be the sector of F bounded by F_0 and containing L and similarly define U_0 as the sector of L defined by L_0.

Fix $a \in F_0, b \in L_0$, with $\widetilde{W}^u(a), \widetilde{W}^u(b)$ not containing singular orbits of $\widetilde{\Phi}$. Fix compact segments

$$A \subset \widetilde{W}^{uu}(a), \quad B \subset \widetilde{W}^{uu}(b), \quad \text{with} \quad A - \{a\} \subset W_0, \quad B - \{b\} \subset U_0$$

and a, b are in the boundary of A, B. For i big enough $L_i \cap A \neq \emptyset$ and $L_i \cap B \neq \emptyset$, so we may assume this is true for all i. For simplicity assume that L_i separates L_{i-1} from L_{i+1} for all $i \in \mathbf{N}$. Now L_1 intersects $\widetilde{W}^u(a), \widetilde{W}^u(b)$ and so does L_i. Therefore $L_1, L_i, \widetilde{W}^u(a), \widetilde{W}^u(b)$ forms a rectangle $\mathcal{R}_i$. In particular there are no singularities in the interior of $\mathcal{R}_i$. Notice that $\mathcal{R}_i \subset \mathcal{R}_{i+1}$ and $\mathcal{R}_i \neq \mathcal{R}_{i+1}$ for all i. By omitting L_1 we may assume that $\mathcal{R} = \bigcup_{i \in \mathbf{N}} \mathcal{R}_i$ does not contain any singular orbits. There is a half leaf of F_0 contained in the closure of $\mathcal{R}$ and similarly for L_0.

Given the fact that $\mathcal{R}$ has no singularities we can now apply the proof contained in [Fe6] for Anosov flows. We sketch the steps. As in [Fe6] choose $p_0 \in L_1$ so that $\widetilde{W}^u(p_0)$ does not intersect either of L or F, but separates one from the other and so that: for any q in the flow band of L_1 bounded by $\widetilde{\Phi}_{\mathbf{R}}(\widetilde{W}^u(a) \cap L_1)$ and $\widetilde{\Phi}_{\mathbf{R}}(p_0)$, then $\widetilde{W}^u(q) \cap F_0 \neq \emptyset$; see Figure 8. The orbit $\widetilde{\Phi}_{\mathbf{R}}(p_0)$ is uniquely determined by these conditions. Then as shown in the claim of Theorem 3.5 of [Fe6], F_0 and $\widetilde{W}^u(p_0)$ form a perfect fit. Hence F is periodic if and only if $\widetilde{W}^u(p_0)$ is. We concentrate on $\widetilde{W}^u(p_0)$. If it is not periodic consider $C = \pi(\widetilde{W}^u(p_0))$ an unstable leaf in M. All orbits in C are backward asymptotic. If an orbit in C only limits (backwards) in a periodic (singular or not) orbit of Φ, then $\widetilde{W}^u(p_0)$ is in fact periodic as we wanted to prove. Otherwise let $c \in M$ be a nonsingular limit point of this backward orbit. Lifting to the universal cover produces covering translates of $\widetilde{W}^u(p_0)$ arbitrarily near c', where $\pi(c') = c$, that is, the picture in $\widetilde{M}$ can be perturbed in an arbitrarily small way. This produces a contradiction to F_0, L_0 not separated from each other, as proved in Theorem 3.5 of [Fe6]. Beware that instead of using the positive/negative half leaves as in [Fe6], one should use the appropriate half leaves in stable/unstable leaves. If $\widetilde{W}^u(p_0)$ is periodic, then all orbits in C only limit in the periodic orbit in C and there is no small perturbation. This is a topological rigidity of the stable/unstable foliations which also works in our situation. $\square$

We say that two orbits γ, α of $\widetilde{\Phi}$ (or the leaves $\widetilde{W}^s(\gamma), \widetilde{W}^s(\alpha)$) are connected by a chain of lozenges $\{\mathcal{A}_i\}, 1 \le i \le n$, if γ is a corner of $\mathcal{A}_1$ and α is a corner of $\mathcal{A}_n$.

Theorem 4.8. *Let Φ be a pseudo-Anosov flow in M^3 closed and let $F_0 \neq F_1 \in \widetilde{\mathcal{F}}^s$. Suppose that there is a nontrivial covering translation g with $g(F_i) = F_i, i = 0, 1$. Let $\alpha_i, i = 0, 1$, be the periodic orbits of $\widetilde{\Phi}$ in F_i so that $g(\alpha_i) = \alpha_i$. Then α_0 and α_1 are connected by a finite chain of lozenges $\{\mathcal{A}_i\}, 1 \le i \le n$, and g leaves invariant each lozenge $\mathcal{A}_i$ as well as their corners.*

Proof. This is exactly as was done for Anosov flows in Theorems 3.3 and 3.5 of [Fe4]. We sketch the main ideas here. There is a sector V_0 of F_0 which contains F_1. Let H_0 be the component of $(\widetilde{W}^u(\alpha_0) - \alpha_0)$ contained in V_0. Then $F_1 \cap H_0 = \emptyset$, or else g would leave invariant two orbits in F_1. So $F_1 \cap \mathcal{L}^u(H_0) = \emptyset$. The latter set is a union of leaves of $\widetilde{\mathcal{F}}^s$. Hence there is a unique leaf $L_1 \in \widetilde{\mathcal{F}}^s$ with a line leaf contained in $\partial \mathcal{L}^u(H_0)$ so that L_1 is either F_1 or the line leaf separates $\mathcal{L}^u(H_0)$ from F_1. These conditions imply that $g(L_1) = L_1$. If $L_1 = F_1$ stop. Otherwise reapply the argument to L_1, F_1 and eventually produce $L_2, L_3, \ldots$. As shown in [Fe4] this eventually stops, that is, there is $L_j = F_1$. We only need to consider $L_0 = F_0$ and L_1, both g-invariant. Let α^* be the periodic orbit of $\widetilde{\Phi}$ in L_1 and let H_1 be the component of $\widetilde{W}^u(\alpha^*) - \alpha^*$ intersecting $\mathcal{L}^u(H_0)$. Then $g(H_1) = H_1$ and H_0, H_1 intersect the same set of stable leaves. The last part of the proof is to show that α_0, α^* are connected by a finite chain of lozenges, all contained in $\mathcal{L}^u(H_0)$. Consecutive lozenges are adjacent. ◻

The main result concerning non-Hausdorff behavior in the leaf spaces of $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$ is the following:

Theorem 4.9. *Let Φ be a pseudo-Anosov flow in M^3. Suppose that $F \neq L$ are not separated in the leaf space of $\widetilde{\mathcal{F}}^s$. Let F_0, L_0 be the line leaves of F, L which are not separated from each other. Let V_0 be the sector of F bounded by F_0 and containing L. Let α be the periodic orbit in F_0 and H_0 be the component of $(\widetilde{W}^u(\alpha) - \alpha)$ contained in V_0. Let g be a nontrivial covering translation with $g(F_0) = F_0$, $g(H_0) = H_0$ and g leaves invariant the components of $(F_0 - \alpha)$. Then $g(L_0) = L_0$. This produces closed orbits of Φ which are freely homotopic in M. Theorem 4.8 then implies that F_0 and L_0 are connected by a finite chain of lozenges $\{\mathcal{A}_i\}, 1 \le i \le n$, all contained in $\mathcal{L}^u(H_0)$. Consecutive lozenges are adjacent. There is an even number of lozenges in the chain; see Figure 9. In addition let $\mathcal{B}_{F,L}$ be the set of leaves nonseparated from F and L. Put an order in $\mathcal{B}_{F,L}$ as follows: Let $C \in \widetilde{\mathcal{F}}^s$ not singular so that*

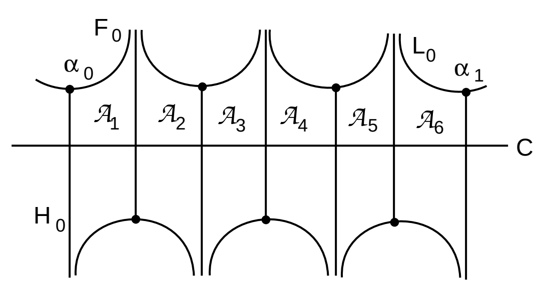

FIGURE 9. The correct picture between nonseparated leaves of $\widetilde{\mathcal{F}}^s$.

$C \cap H_0 \neq \emptyset$. Put an orientation in $\zeta_1 = \widetilde{W}^{ss}(a)$ where $a \in C$. If $R_1, R_2 \in \mathcal{B}_{F,L}$ let α_1, α_2 be the respective periodic orbits in R_1, R_2. Then $\widetilde{W}^u(\alpha_i) \cap C \neq \emptyset$ and let $a_i = \widetilde{W}^u(\alpha_i) \cap \zeta_1$. We define $R_1 < R_2$ in $\mathcal{B}_{F,L}$ if a_1 precedes a_2 in the orientation of ζ_1. Then $\mathcal{B}_{F,L}$ is either order isomorphic to $\{1, ..., n\}$ for some $n \in \mathbf{N}$; or $\mathcal{B}_{F,L}$ is order isomorphic to the integers $\mathbf{Z}$. In addition if there are $Z, S \in \widetilde{\mathcal{F}}^s$ so that $\mathcal{B}_{Z,S}$ is infinite, then there is an incompressible torus in M transverse to Φ. In particular M cannot be atoroidal. Finally up to covering translations, there are only finitely many non-Hausdorff points in the leaf space of $\widetilde{\mathcal{F}}^s$.

Given that the regions associated to non-Hausdorff F, L are free of singular orbits, it follows that the proof of this result is entirely analogous to the corresponding results in [Fe6]. See the proofs of Theorems 4.3, 4.9 and Corollaries 4.4, 4.5, 4.6 and 4.7 of [Fe6]. Furthermore if $\mathcal{B}_{F,L}$ is infinite there is a region in $\widetilde{M}$ associated to this, which is called a *scalloped* region. For its description see Theorem 5.3 of [Fe6].

Consider the chain $\mathcal{A}_i, 1 \leq i \leq n$, of lozenges connecting F_0 and L_0 produced by Theorem 4.9. Let α, β be the corner orbits of $\mathcal{A}_1$. Then $g(\mathcal{A}_1) = \mathcal{A}_1$, $g(\alpha) = \alpha$, $g(\beta) = \beta$. Let α_1, β_1 be the quotients of α, β by g. Then α_1, β_1 are periodic orbits of Φ - which may not be indivisible. Still their invariance by g implies that α_1, β_1 are freely homotopic. In fact suppose that g acts as a contraction on the set of orbits of $\widetilde{W}^s(\alpha)$. Then analysing the action of g on the lozenge $\mathcal{A}_1$, we see that g acts as an expansion in the set of orbits of $\widetilde{W}^s(\beta)$. This implies that α_1 is freely homotopic to the inverse of β_1. This is fundamental for us:

Conclusion. If $\widetilde{\mathcal{F}}^s$ is not Hausdorff, then there are 2 closed orbits γ, ζ of Φ which are freely homotopic to the inverse of each other, $\gamma \cong \zeta^{-1}$.

Theorem 4.10. *Let Φ be a pseudo-Anosov flow in M^3. If there is a product region in $\widetilde{M}$, then Φ is topologically conjugate to a suspension Anosov flow.*

This follows because product regions in $\widetilde{M}$ cannot contain singular orbits of $\widetilde{\Phi}$. The proof of Theorem 5.1 of [Fe6] then shows that in fact there are no singular orbits in $\widetilde{M}$, that is, Φ is an Anosov flow. The result follows as in Theorem 5.1 of [Fe6].

In section 8 we will produce infinitely many examples of pseudo-Anosov flows in closed hyperbolic 3-manifolds which have quasi-isometric stable/unstable foliations as follows: we will check that such flows do not have any freely homotopic closed orbits. We then use Theorem 4.9 to conclude that $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$ have Hausdorff leaf space and finally use 3.8 to conclude that $\mathcal{F}^s, \mathcal{F}^u$ are quasi-isometric.

5. Topology of the singular foliations in the leaves of $\widetilde{\mathcal{G}}$

The goal of the next three sections is to study the asymptotic behavior of Reebless foliations in closed hyperbolic 3-manifolds. Let $\mathcal{G}$ be a Reebless, finite depth foliation in M^3. Suppose that $\mathcal{G}$ is transverse to a pseudo-Anosov flow Φ. Let $\widetilde{\mathcal{G}}$ be the lift of $\mathcal{G}$ to $\widetilde{M}$. Given $F \in \widetilde{\mathcal{G}}$, then $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$ induce singular foliations $\widetilde{\mathcal{F}}^s_F, \widetilde{\mathcal{F}}^u_F$ in F. In this section we study topological properties of $\widetilde{\mathcal{F}}^s_F$ and $\widetilde{\mathcal{F}}^u_F$. This will be used in a fundamental way in section 7 to analyse asymptotic behavior of leaves of $\widetilde{\mathcal{G}}$ in $\widetilde{M}$. However, in this and the following section we will not assume that Φ is quasigeodesic or that M is hyperbolic.

Recall that $\Theta : \widetilde{M} \to \mathcal{O}$ is the projection into the orbit space of $\widetilde{\Phi}$ and that $\mathcal{O}$ is homeomorphic to the plane $\mathbf{R}^2$. For our analysis we will need to understand what the boundary $\partial\Theta(F) \subset \mathcal{O}$ is when F does not intersect every orbit of $\widetilde{\Phi}$. Since $\mathcal{G}$ is Reebless, then F is properly embedded in $\widetilde{M}$ [No] and therefore separates $\widetilde{M}$. Since $\widetilde{\Phi}$ is transverse to $\widetilde{\mathcal{G}}$ it follows that every orbit of $\widetilde{\Phi}$ can intersect F at most once. Therefore $\Theta : F \to \mathcal{O}$ is injective. In analogy with the previous section, a *sector* of a leaf l of $\widetilde{\mathcal{F}}^s_F$ is the closure (in F) of a component of $F - l$. A leaf is regular if and only if it produces exactly two sectors. Also a *line leaf* of a leaf l of $\widetilde{\mathcal{F}}^s_F$ is $\partial U = l'$ where U is a sector of l. The components of $l - l'$ are all in one side of l' and l' is regular in the other side. In the same way a *piece leaf* is a bi-infinite arc properly embedded in a leaf of $\widetilde{\mathcal{F}}^s_F$.

Proposition 5.1. *$\partial\Theta(F)$ is a disjoint union of line leaves of $\widetilde{\mathcal{F}}^s_{\mathcal{O}}, \widetilde{\mathcal{F}}^u_{\mathcal{O}}$, all regular on the side containing $\Theta(F)$. Also if $L \in \widetilde{\mathcal{F}}^s$ (or $\widetilde{\mathcal{F}}^u$), then $F \cap L$ is connected.*

Proof. Because Φ is transverse to $\mathcal{G}$ and M is compact, then for $\epsilon > 0$ sufficiently small there is $\eta(\epsilon)$ with $\eta(\epsilon) \to 0$, when $\epsilon \to 0$ so that: any orbit of $\widetilde{\Phi}$ which comes within ϵ of a point $z \in F \in \widetilde{\mathcal{G}}$ will in fact intersect F within $\eta(\epsilon)$ of z. Let $p \in \partial\Theta(F)$ and let $p_i \in \Theta(F)$ with $p_i \to p$. Let $z \in \widetilde{M}$ with $\Theta(z) = p$. Consider $D \subset \widetilde{M}$ a small embedded disk, transverse to $\widetilde{\Phi}$ with $z \in int(D)$ and Θ injective in D. Let $w_i \in F$ with $\Theta(w_i) = p_i$. By truncating finitely many terms if necessary, there are unique $z_i \in D$ and $t_i \in \mathbf{R}$ so that $w_i = \widetilde{\Phi}_{t_i}(z_i)$. If $|t_i| \not\to +\infty$, assume up to subsequence that $t_i \to t_0$. It follows that $w_i \to \widetilde{\Phi}_{t_0}(z)$. But since F is closed in $\widetilde{M}$, then $\widetilde{\Phi}_{t_0}(z) \in F$, contradicting $p \notin \Theta(F)$.

Assume then that there is a subsequence $t_i \to +\infty$. Suppose that the corresponding p_i are all in the closure of the same sector defined by $m = \Theta(\widetilde{W}^s(z))$ at p. Let l be the line leaf of m which bounds this sector.

Claim. $l \subset \partial\Theta(F)$.

Let $v \in \widetilde{M}$ with $\Theta(v) \in l$. For i big enough let $q_i = \widetilde{W}^{uu}(v) \cap \widetilde{W}^s(w_i)$. The intersection is nonvoid because l is regular on the side containing p_i. There are

$$s_i \in \mathbf{R}, \quad s_i \to +\infty \quad \text{so that} \quad \widetilde{\Phi}_{s_i}(q_i) \in \widetilde{W}^{ss}(w_i) \quad \text{and} \quad d(\widetilde{\Phi}_{s_i}(q_i), w_i) \to 0.$$

This implies that there are $\epsilon_i \to 0$ with $\widetilde{\Phi}_{s_i+\epsilon_i}(q_i) \in F$. Hence for i big enough $\Theta(q_i) \in \Theta(F)$. In fact this shows that the segment from $\widetilde{\Phi}_{s_i}(q_i)$ to w_i in $\widetilde{W}^{ss}(w_i)$ projects into $\Theta(F)$. One concludes that

$$\Theta(v) \in \Theta(F) \cup \partial\Theta(F).$$

If $\Theta(v) \in \Theta(F)$ let E be a small disk contained in F with v in the interior. Hence for i big enough there are bounded r_i with $\widetilde{\Phi}_{r_i}(q_i) \in E$. But the argument above shows that there are $\widetilde{\Phi}_{s_i+\epsilon_i}(q_i) \in F$ with $s_i + \epsilon_i \to +\infty$ as $i \to +\infty$. This would produce two points $\widetilde{\Phi}_{r_i}(q_i)$ and $\widetilde{\Phi}_{s_i+\epsilon_i}(q_i)$ of F in $\widetilde{\Phi}_{\mathbf{R}}(q_i)$, a contradiction.

We conclude that the stable line leaf $l \subset \partial\Theta(F)$, showing the claim.

Similarly given the t_i defined above, if there is some subsequence $t_{i_n} \to -\infty$, then we have an unstable line leaf I through p with $I \subset \partial\Theta(F)$. Since $\Theta(F)$ is connected this would force $I \cap \Theta(F) \neq \emptyset$ which is impossible. In addition l is regular on the side containing F. This proves the first statement.

Now let $F \in \widetilde{\mathcal{G}}$ and $L \in \widetilde{\mathcal{F}}^s$. Suppose that $F \cap L$ has two components u_1, u_2. Then $\Theta(u_1) \cap \Theta(u_2) = \emptyset$ and there is $x \in L$ with $\Theta(x)$ separating $\Theta(u_1)$ from $\Theta(u_2)$ in $\Theta(L)$ and $\Theta(x) \in \partial\Theta(u_1)$. By the proof above $\Theta(x) \in \partial\Theta(F)$ and there is a line leaf

$$m \subset \partial\Theta(F) \quad \text{with either} \quad m \subset \Theta(\widetilde{W}^s(x)) \quad \text{or} \quad m \subset \Theta(\widetilde{W}^u(x)).$$

Since $\Theta(u_1) \subset \Theta(F)$, $\Theta(u_1) \subset \Theta(\widetilde{W}^s(x))$ and m is regular on the side containing $\Theta(F)$, then $m \subset \Theta(\widetilde{W}^s(x))$ cannot happen. Now $\Theta(u_1)$ and $\Theta(u_2)$ must be in the same component of $\mathcal{O} - m$ and $\Theta(x)$ separates $\Theta(u_1)$ from $\Theta(u_2)$ in $\Theta(L)$. Hence there is a component of $\Theta(\widetilde{W}^u(x)) - \Theta(x)$ contained in $\Theta(F)$ and separating $\Theta(u_1)$ from $\Theta(u_2)$ in $\Theta(F)$. This contradicts the fact that m is a line leaf of $\Theta(\widetilde{W}^u(x))$ which is regular on the side containing $\Theta(F)$. This finishes the proof of the second statement. ☐

The following fact will be useful in the future. Since F separates $\widetilde{M}$ and $\widetilde{\Phi}$ is transverse to F, there are positive and negative flow sides of F in $\widetilde{M} - F$. By the above proof, if $l \subset \partial\Theta(F)$ and $l \subset \Theta(L)$ with $L \in \widetilde{\mathcal{F}}^s$, then L is on the negative flow side of F, because the $t_i \to +\infty$. Similarly if $l \subset \partial\Theta(F)$ with $l \subset \Theta(U)$ and $U \in \widetilde{\mathcal{F}}^u$, then U is on the positive flow side of F. It follows that F separates L from U.

The following proposition will be fundamental for the results in this article. We will assume that the lifted foliations $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$ to $\widetilde{M}$ have Hausdorff leaf spaces.

Proposition 5.2. *Suppose that $\widetilde{\mathcal{F}}^s$ has Hausdorff leaf space. Let $F \in \widetilde{\mathcal{G}}$, and let $\widetilde{\mathcal{F}}^s_F$ be the induced singular stable foliation in F. Then $\widetilde{\mathcal{F}}^s_F$ has Hausdorff leaf space. Similarly for $\widetilde{\mathcal{F}}^u_F$.*

Proof. Suppose there are leaves $l_i \in \widetilde{\mathcal{F}}^s_F$ converging to l and $l' \in \widetilde{\mathcal{F}}^s_F$. Let $L_i, L, L' \in \widetilde{\mathcal{F}}^s$ so that $l_i \subset L_i, l \subset L$ and $l' \subset L'$. Then clearly $L_i \to L$ and $L_i \to L'$ in $\mathcal{H}(\widetilde{\mathcal{F}}^s)$. By hypothesis $\widetilde{\mathcal{F}}^s$ has Hausdorff leaf space, therefore $L = L'$. The previous proposition shows that $F \cap L$ is connected, and it follows that $l = l'$. We conclude that $\widetilde{\mathcal{F}}^s_F$ has Hausdorff leaf space. ☐

6. Geometry of the singular foliations in the leaves of $\widetilde{\mathcal{G}}$

Let $\mathcal{G}$ be a finite depth foliation transverse to a pseudo-Anosov flow Φ, so that the lifted stable and unstable foliations $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$ have Hausdorff leaf space. In this section we do not assume that Φ is quasigeodesic or that M is hyperbolic. We prove some geometric properties of the leaves of $\widetilde{\mathcal{F}}^s_F, \widetilde{\mathcal{F}}^u_F$ (for $F \in \widetilde{\mathcal{G}}$), which will be needed in section 7 to analyse the asymptotic behavior of $\mathcal{G}$ when M is hyperbolic. This is a technical section - on a first reading the reader may just check properties (A), (B) and (C) for leaves of $\widetilde{\mathcal{F}}^s_F$ and then proceed to section 7.

Put a hyperbolic metric in the leaves of $\mathcal{G}$, so that it varies continuously in the transversal direction [Ca-Co, Can]. The metrics can be chosen so that projection to lower depth leaves is a local isometry, but that is not necessary for the proofs here. Given $F \in \widetilde{\mathcal{G}}$ we say that F has depth k if $\pi(F)$ has depth k in $\mathcal{G}$. Since F is isometric to $\mathbf{H}^2$, it has a canonical compactification with a circle at infinity which will be denoted by $\partial_\infty F$. Fix the depth k and consider the following properties:

Property (A). This has 3 parts: For any $F \in \widetilde{\mathcal{G}}$ of depth $\leq k$ then:

(i) If $\widetilde{\alpha}$ is a ray of $\widetilde{\mathcal{F}}^s_F$, then $\widetilde{\alpha}$ accumulates in a unique point of $\partial_\infty F$.

(ii) If $\widetilde{\alpha}$ is a bi-infinite arc in a leaf of $\widetilde{\mathcal{F}}^s_F$, then the endpoints corresponding to the two rays of $\widetilde{\alpha}$ are distinct points in $\partial_\infty F$.

(iii) If $\widetilde{\alpha}_i$ is a sequence of rays of $\widetilde{\mathcal{F}}^s_F$ converging to the ray $\widetilde{\alpha}$ of $\widetilde{\mathcal{F}}^s_F$, then the limit points of $\widetilde{\alpha}_i$ converge to the limit point of $\widetilde{\alpha}$ in $\partial_\infty F$.

Property (B). Given k there is $\mu_0 = \mu_0(k)$ so that: For any $F \in \widetilde{\mathcal{G}}$ of depth $\leq k$, (i) and (ii) above hold. In addition if γ is a line leaf of a leaf of $\widetilde{\mathcal{F}}^s_F$, x is any point in γ and γ^* is the geodesic with same ideal points as γ, then $d_F(x, \gamma^*) < \mu_0$.

Property (C). There is $\mu_1 > 0$ so that: Given any $F \in \widetilde{\mathcal{G}}$ of depth $\leq k$, then the line leaves of $\widetilde{\mathcal{F}}^s_F$ are uniform μ_1-quasigeodesics in F.

Property (C) implies the other two [Gr, Gh-Ha, CDP] and states that leaves of $\widetilde{\mathcal{F}}^s_F$ have excellent geometric properties. The strategy is to use induction and for each k, first prove the topological properties (A) and use (A) and induction to upgrade this to the geometric information (B) and (C). The Hausdorff property for the leaf spaces will be fundamental throughout this section. The following two lemmas show that (i) and (iii) of (A) hold under very general conditions:

Lemma 6.1. *Let $\mathcal{V}$ be a singular foliation in $\mathbf{H}^2$ with only p-prong singularities, $p \geq 3$. Suppose that each half plane $U \subset \mathbf{H}^2$ contains a curve γ which is a bounded distance from a geodesic in $\mathbf{H}^2$ and γ is either a leaf of $\mathcal{V}$ or is transverse to $\mathcal{V}$. Then the following happens: If β is a ray in a leaf $\mathcal{V}$, then β converges to a unique ideal point of S^1_∞.*

Proof. Suppose not. First notice that index computations imply that rays of leaves of $\mathcal{V}$ are properly embedded in $\mathbf{H}^2$. Let $p \neq q \in \mathbf{H}^2$ with β accumulating on p and q. Choose $x_i, y_i \in \beta$ with $x_i \to p, y_i \to q$. Let β_i be the segment of β between x_i and y_i. Since β does not accumulate on $\mathbf{H}^2$ assume (up to subsequence) that every β_i has a subset which is very near a component Z of $S^1_\infty - \{p, q\}$. Let Z' be a closed segment $\subset Z$, so that Z' bounds the half plane $U \subset \mathbf{H}^2$. By hypothesis there is $\gamma \subset U$ with γ a bounded distance from a geodesic and either γ a leaf of $\mathcal{V}$ or transverse to $\mathcal{V}$. If γ is a leaf of $\mathcal{V}$, then clearly β cannot limit in the interior of Z', a contradiction to the above argument. If γ is transverse to $\mathcal{V}$, then β cannot intersect γ more than once (index computations), hence β cannot limit in Z' and also in p, q. Hence β limits in only one point of S^1_∞. Notice it is fundamental here that there are no center singularities of $\mathcal{V}$! ☐

Lemma 6.2. *Let $\mathcal{V}$ be a singular foliation in $\mathbf{H}^2$ with only p-prong singularities, $p \geq 3$. Suppose that each ray in a leaf of $\mathcal{V}$ limits in a unique ideal point of S^1_∞. Also assume that the set of ideal points of leaves of $\mathcal{V}$ is dense in S^1_∞ and $\mathcal{V}$ has Hausdorff leaf space. Then if β_i is a collection of rays converging to a ray β, it follows that the ideal points of β_i converge to the ideal point of β.*

Proof. Let β_i converge monotonically to β. Let $x_i \in \beta_i$ converge to $x \in \beta$, with x nonsingular, all in a foliated box Q of $\mathcal{V}$; see Figure 10. Let p_i be the ideal point of β_i and p that of β; see Figure 10. Then $\{p_i\}$ is monotone with i in S^1_∞. If $p_i \not\to p$, then $p_i \to q \neq p$. Since ideal points of leaves of $\mathcal{V}$ are dense in S^1_∞, there is a leaf l with ideal point in q' between p and q. Therefore the sequence $\{\beta_i\}$ also has to limit

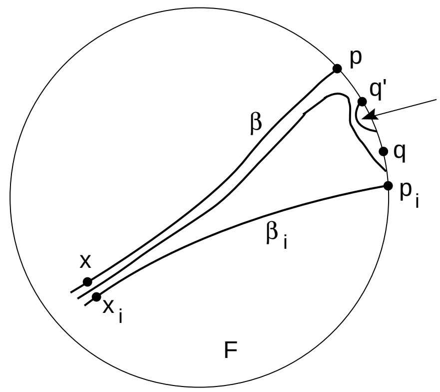

FIGURE 10. Proving continuity of ideal points.

on another leaf besides β and the leaf space of $\mathcal{V}$ is not Hausdorff, a contradiction. This finishes the proof. $\square$

We now analyse geometric properties of leaves of $\widetilde{\mathcal{F}}^s_F, F \in \widetilde{\mathcal{G}}$. Proposition 6.3 proves (A), Proposition 6.6 proves (B) and Proposition 6.7 proves (C).

Proposition 6.3. *Suppose that (A), (B) and (C) hold for any depth $< k$ (empty assumption for $k = 0$). Then (A) holds for depth k.*

Proof. For $k = 0$ (i) and (ii) were proved by Levitt in [Le]. Let $F \in \widetilde{\mathcal{G}}$ have depth 0. Since $\pi(F)$ is compact, the orbit of any point in $\partial_\infty F$ under covering translations is dense in $\partial_\infty F$. Part (iii) of (A) now follows from Lemma 6.2.

Using (A) for $k = 0$, we can prove Lemmas 6.4 and 6.5 which will be needed for higher depths. Notice that any leaf l of $\widetilde{\mathcal{F}}^s_F$ has at most one singularity. This is because $l = L \cap F$, with $L \in \widetilde{\mathcal{F}}^s$ and L has at most one singular orbit and any orbit of $\widetilde{\Phi}$ intersects F at most once. We can define a geodesic lamination $\mathcal{V}_F$ by pulling leaves of $\widetilde{\mathcal{F}}^s_F$ tight: a regular leaf of $\widetilde{\mathcal{F}}^s_F$ produces a geodesic with same ideal points and a p-prong leaf splits into p geodesics forming an ideal polygon with p sides. These polygons are the complementary regions of $\mathcal{V}_F$.

A convention in this section is: suppose γ is either an embedded segment in a leaf of $\widetilde{\mathcal{G}}$ or a properly embedded infinite ray with unique accumulation point in the corresponding circle at infinity or a properly embedded bi-infinite ray with well-defined (distinct) ideal points in both directions; then we denote by γ^* the associated geodesic segment, ray or bi-infinite geodesic having the same endpoints (ideal or not) as γ.

Lemma 6.4. *Any closed curve ζ in a compact leaf G can be chosen to be either contained in a leaf of $\mathcal{F}^s_G$ or to be be transverse to $\mathcal{F}^s_G$.*

Proof. We allow the curve to pass through a singularity of $\mathcal{F}^s_G$, if there are at least two prongs of the singularity in each side of the curve. Let $F \in \widetilde{\mathcal{G}}$ with $\pi(F) = G$. Let $\widetilde{\zeta}$ be a lift of ζ to F, and let a, b be the ideal points of $\widetilde{\zeta}$ in $\partial_\infty F$. There are two options:

Suppose first that a, b are not ideal points of leaves of $\widetilde{\mathcal{F}}^s_F$. As the complementary regions of $\mathcal{V}_F$ are finite sided ideal polygons, then there is a leaf m^* of $\mathcal{V}_F$ so that its ideal points separate a, b. Let g be the covering translation of F associated to ζ. Connect $y \in m^*$ to a $y' \in g(m^*)$ by a geodesic arc α. Collapsing α to the foliation setting produces a curve transverse to $\mathcal{F}^s_G$. Adjusting endpoints yields a closed curve isotopic to ζ. This transversal can pass through prongs with index ≥ 4, corresponding to a geodesic crossing an ideal polygon region with ≥ 4 sides.

If a is an ideal point of a line leaf l of $\widetilde{\mathcal{F}}^s_F$ with other ideal point $a' \neq a$, let h be the covering translation of F having a, b as fixed points, a the repelling one. Then $h^n(l^*), n \geq 0$, converges to $l^*_1 \in \mathcal{V}_F$ with $h(l^*_1) = l^*_1$. Let $l_1 \in \widetilde{\mathcal{F}}^s_F$, generating l^*_1. Then l_1 has ideal points a, b and considering $h^n(l_1)$ we obtain a limit l_2 with $h(l_2) = l_2$. Then $\pi(l_2)$ is a closed leaf of $\mathcal{F}^s_G$ isotopic to ζ. This proves the lemma. ☐

Lemma 6.5. *Let $F \in \widetilde{\mathcal{G}}$ (of any depth). The union of the accumulation points of leaves of $\widetilde{\mathcal{F}}^s_F$ is dense in $\partial_\infty F$.*

Proof. Let T be an interval in $\partial_\infty F$ and W the half plane in F with T as limit set.

We claim that the closure of $\pi(W)$ (in M) contains a depth 0 leaf. The hyperbolic metric in leaves of $\widetilde{\mathcal{G}}$ are quasiconformal with the induced Riemannian metrics from $\widetilde{M}$. Take $x_i \in W$ with $d_F(x_i, F - W) \to +\infty$ (distance in F). Obtain balls of arbitrarily large radius in W and conclude that the closure of $\pi(W)$ in M contains a leaf of $\mathcal{G}$. But any leaf of $\mathcal{G}$ contains compact leaves in its closure so the claim follows.

Hence $\pi(W)$ limits on a compact leaf R. The compact leaf has a *juncture* δ on that side which defines the spiralling of depth one leaves towards R - for details on juncture and its properties see [Ga1, Ga3, Ca-Co]. By the above we can assume that δ is either transverse to or contained in a leaf of $\mathcal{F}^s_R$. The transversal flow Φ lifts δ to a closed curve in a depth one leaf and also lifts δ completely to the higher depth leaf $\pi(F)$, to curves which are a bounded distance from a geodesic. Consider all such lifts to the piece of $\pi(F)$ contained in a small neighborhood of that side of R. In the intrinsic topology of $\pi(F)$, this collection is properly embedded, that is, all leaves are isolated and so the same occurs to the lift to F. By the claim above there are infinitely many lifts intersecting W. Given any two such lifts α_0, $\alpha_1 \subset F$, if neither of them is contained in W, then they intersect the geodesic α_2 of $F = \mathbf{H}^2$ which is the boundary of W. By the properness property there are only finitely many lifts intersecting the segment of α_2 between α_0 and α_1. It now follows that there is a lift α_3 entirely contained in W. Notice that α_3 is either a leaf of $\widetilde{\mathcal{F}}^s_F$ or transverse to $\widetilde{\mathcal{F}}^s_F$. In the first case the ideal points of α_3 are limit points of leaves of $\widetilde{\mathcal{F}}^s_F$. In the second case any ray of $\widetilde{\mathcal{F}}^s_F$ entering the region of F bounded by α_3 cannot double back, hence only accumulates in T. This finishes the proof. ☐

We proceed with the proof of the induction step in Proposition 6.3. (B) for depth 0 is proved in Proposition 6.6 and (C) for depth 0 is proved by the argument in Proposition 6.7. Let $k \geq 0$ and assume that (A), (B) and (C) hold for all depths $\leq k$. We prove (A) for $k+1$. Let $F \in \widetilde{\mathcal{G}}$ of depth $(k+1)$, $E = \pi(F)$.

Since E has depth $k+1$, it can only limit in lower depth leaves. Therefore for any $\epsilon > 0$ there is a compact core $\mathcal{C}(E)$ so that $E - \mathcal{C}(E)$ is ϵ-close to lower depth leaves. Let $\mathcal{C}(F) = \pi^{-1}(\mathcal{C}(E))$. Each component Z of $E - \mathcal{C}(E)$ projects entirely to a lower depth leaf G along flow segments of Φ of length $\leq \epsilon$. Then ∂Z projects to a closed curve in G. Let $\widetilde{Z}$ be a lift of Z to F and pr the projection to a lower depth leaf N with $\pi(N) = G$. If ϵ is sufficiently small, then the projection

$pr : \widetilde{Z} \to pr(\widetilde{Z}) \subset N$ is a quasiconformal homeomorphism with quasiconformal constant very near 1. Therefore pr is a quasi-isometry.

Now consider a ray $\widetilde{\alpha}$ of $\widetilde{\mathcal{F}}^s_F$. The argument in the proof of Lemma 6.5 shows that the hypothesis of Lemma 6.1 is satisfied and then Lemma 6.1 implies that $\widetilde{\alpha}$ converges to a unique ideal point in $\partial_\infty F$. This proves (i).

We now prove (ii). Properties (A), (B) and (C) for lower depths and $\partial\mathcal{C}(E)$ projecting to a lower depth leaf along the transversal flow Φ imply, as in Lemma 6.4 (applied to the projection of $\partial\mathcal{C}(E)$ to a lower depth leaf and pulled back to E by the transversal flow), that $\partial\mathcal{C}(E)$ can be chosen to be either in leaves of $\mathcal{F}^s_E$ or transverse to $\mathcal{F}^s_E$. Notice that $\partial\mathcal{C}(E)$ is a finite union of closed curves. Let $\widetilde{\alpha}$ be a bi-infinite arc in a leaf of $\widetilde{\mathcal{F}}^s_F$ and $\alpha = \pi(\widetilde{\alpha}) \subset E$.

The first option is that $\alpha \cap \partial\mathcal{C}(E) = \emptyset$. If $\alpha \subset E - \mathcal{C}(E)$, let Z be the component of $E - \mathcal{C}(E)$ containing α. Then α projects under some projection pr to a lower depth leaf G and $pr(\alpha)$ is a leaf of $\mathcal{F}^s_G$. By the induction hypothesis, $pr(\alpha)$ is a quasigeodesic in $pr(Z)$. Since Z and $pr(Z)$ are quasiconvex subsets in hyperbolic surfaces they are negatively curved. As pr is a quasi-isometry, then α itself is a quasigeodesic, hence (ii) follows. There are only finitely many components of $E - \mathcal{C}(E)$, hence α is a uniform quasigeodesic. The other possibility here is that $\alpha \subset \mathcal{C}(E)$ and the result follows from Lemma 6.8.

A second option is that $\widetilde{\alpha}$ intersects $\partial\mathcal{C}(F)$ only once in $\widetilde{\beta} \subset \partial\mathcal{C}(F)$. Let $\widetilde{\alpha}_1, \widetilde{\alpha}_2$ be the components of $\widetilde{\alpha} - \partial\mathcal{C}(F)$. If ideal points of $\widetilde{\alpha}_1, \widetilde{\alpha}_2$ are the same, then because $\widetilde{\beta}$ separates $\widetilde{\alpha}_1$ from $\widetilde{\alpha}_2$, these ideal points are an ideal point of $\widetilde{\beta}$. Suppose that $\widetilde{\alpha}_1 \subset (F - \mathcal{C}(F))$. As above it follows that $\widetilde{\alpha}_1$ is a quasigeodesic and hence that $\widetilde{\alpha}_1$ is a bounded distance from $\widetilde{\beta}$. As $\widetilde{\alpha}_2 \subset \mathcal{C}(F)$, Lemma 6.8 implies that $\widetilde{\alpha}_2$ is a bounded distance from $\widetilde{\beta}$. Hence α is a bounded distance from $\pi(\widetilde{\beta})$. Let $g \in \pi_1(E)$ so that $g(\widetilde{\beta}) = \widetilde{\beta}$ and the ideal point of $\widetilde{\alpha}$ is the repelling fixed point of g. Then all $g^n(\widetilde{\alpha})$ are a bounded distance from $\widetilde{\beta}$ and intersect it only once. It follows that $g^n(\widetilde{\alpha})$ converges (as $n \to +\infty$) to two distinct leaves of $\widetilde{\mathcal{F}}^s_F$. This contradicts $\widetilde{\mathcal{F}}^s_F$ having Hausdorff leaf space.

Finally, the third option is that α intersects $\partial\mathcal{C}(E)$ at least twice. Let $\widetilde{\beta}_1 \neq \widetilde{\beta}_2 \subset \partial\mathcal{C}(F)$ which are intersected by $\widetilde{\alpha}$. As $\widetilde{\beta}_1, \widetilde{\beta}_2$ are lifts of closed curves in E, they do not share an ideal point in $\partial_\infty F$. It follows that $\widetilde{\alpha}$ has distinct ideal points in this case too. This proves (ii).

Lemma 6.2 now proves (iii). This finishes the proof of Proposition 6.3. ∎

Proposition 6.6. *Property (B) holds for $k = 0$. In addition suppose that Properties (A), (B) and (C) hold for all depths $\leq k$ and that Property (A) holds for depth $k+1$. Then Property (B) holds for $(k+1)$.*

Proof. We first show (B) for $k = 0$. Let $F \in \widetilde{\mathcal{F}}^s$ of depth 0 and $E = \pi(F)$. If the result is not true for F there are $x_i \in F$ with x_i in line leaves γ_i of $\widetilde{\mathcal{F}}^s_F$ and geodesics γ_i^* with same ideal points as γ_i (because (A) holds for F), so that $d_F(x_i, \gamma_i^*) \to +\infty$. As $\pi(F)$ is compact, then up to subsequence and covering translations of F assume that $x_i \to x$. Then $\gamma_i \to \gamma$, where γ is a line leaf of $\widetilde{\mathcal{F}}^s_F$ containing x and regular on the side γ_i are limiting to. Let γ^* be the corresponding geodesic. The key fact here is that property (iii) of (A) shows that $\gamma_i^* \to \gamma^*$. As $d_F(x, \gamma^*) = a < \infty$, then for i big $d_F(x_i, \gamma_i^*) < a + 1$, a contradiction. Since there are only finitely many compact leaves there is a uniform bound on the distance $d_F(x, \gamma^*)$. This shows Property

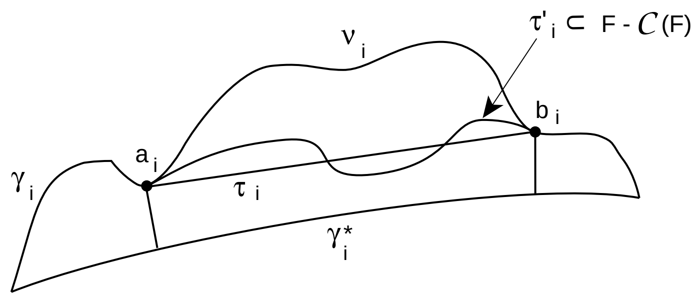

FIGURE 11. Producing bounded distance from the minimal geodesic.

(B) for $k = 0$. Compactness is the key here and we will extract some compactness from the higher depth leaves - the fact that the core $\mathcal{C}(E)$ is always compact.

We now prove the second assertion of the proposition. Suppose that result is true up to k and let $F \in \widetilde{\mathcal{G}}$ of depth $k+1$ and $E = \pi(F)$. If the result is not true for F, then as above find x_i and γ_i with $d_F(x_i, \gamma_i^*) \to +\infty$. If there is a subsequence of x_i such that $d_F(x_i, \mathcal{C}(F))$ is bounded, then the argument above finishes the proof.

Hence $d_F(x_i, \mathcal{C}(F)) \to +\infty$. Let $\beta_i = \pi(\gamma_i)$. There are two cases:

Case 1. β_i intersects $\partial\mathcal{C}(E)$.

Up to subsequence assume that β_i always intersect the same component ρ of $\partial\mathcal{C}(E)$. From $\pi(x_i)$ choose one side and the first point $u_i \in \beta_i \cap \rho$. Let $a_i \in \gamma_i$ corresponding to this first intersection so that $\pi(a_i) = u_i$ ($a_i \in \partial\mathcal{C}(F)$); see Figure 11. Since ρ is compact the proof for depth 0 shows that $d_F(a_i, \gamma_i^*)$ is bounded. The bound only depends on $k + 1$.

If the other side of γ_i from a_i intersects $\partial\mathcal{C}(F)$, let b_i be the first intersection. As above $d_F(b_i, \gamma_i^*)$ is bounded. Let τ_i be the hyperbolic geodesic segment in F between a_i and b_i; see Figure 11. The segments ν_i of γ_i between a_i and b_i are contained in a fixed component U of $(F - \mathcal{C}(F))$. The boundary of U is a union of uniform quasigeodesics and we can homotope τ_i by a bounded homotopy to $\tau_i' \subset U$, so that τ_i' are uniform quasigeodesics. Project U to a lower depth leaf $pr : U \to pr(U) \subset N$, N of lower depth. Then $pr(\tau_i')$ is also a uniform quasigeodesic in N. Also Property (C) for k implies that $pr(\nu_i)$ is a bounded distance from the corresponding geodesic arc in N [Gr, Gh-Ha, CDP], hence $d_N(pr(\nu_i), pr(\tau_i'))$ is bounded. The quasi-isometry pr implies that $d_F(\nu_i, \tau_i')$ is bounded and finally $d_F(\nu_i, \tau_i)$ is also bounded. Since $d_F(a_i, \gamma_i^*), d_F(b_i, \gamma_i^*)$ are also bounded, then τ_i is in a bounded neighborhood of γ_i^*. Hence $d(x_i, \gamma_i^*)$ is uniformly bounded.

The second option in Case 1 is that the other side of γ_i does not intersect $\partial\mathcal{C}(F)$. Then the corresponding ray of γ_i is contained in $(F - \mathcal{C}(F))$. A proof as above using Property (C) for rays in lower depth leaves implies the same result.

Case 2. β_i does not intersect $\partial\mathcal{C}(E)$.

If β_i is contained in $E - \mathcal{C}(E)$, then as seen in the proof of the induction step in Proposition 6.3 (first option), β_i is a uniform quasigeodesic and the result follows. Otherwise β_i is contained in $\mathcal{C}(E)$ and the result then follows from Lemma 6.8.

There are only finitely many isotopy classes of leaves at each new depth, except for the fibering regions. A fibering region minus a leaf has a product foliation

$F \times [0,1]$. An analysis as above in $F \times [0,1]$ yields the same result. Hence the bound μ_0 in Property (B) can be chosen to work for all leaves of depth $\leq k+1$. This finishes the proof of Proposition 6.6. $\square$

Proposition 6.7. *Suppose that Properties (A), (B) and (C) hold for all depths $\leq k$. In addition suppose that Properties (A) and (B) hold for depth $(k+1)$ also. Then Property (C) holds for $(k+1)$ also.*

Proof. The same proof works for $k=0$ or higher. First we claim that for any $a > 0$ there is $b(a) > 0$ so that if x, y are in the same leaf γ of $\widetilde{\mathcal{F}}^s_F$ (with $F \in \widetilde{\mathcal{F}}^s$), then

$$d_N(x,y) \; > \; a \quad \Rightarrow \quad l([x,y]) \; > \; b(a),$$

where $[x,y]$ is the segment of γ from x to y and l denotes length. Otherwise find $a > 0$ and x_i, y_i with $d_{N_i}(x_i, y_i) < a$, but $l([x_i, y_i]) > i$. Up to subsequence and covering translations assume that $x_i \to x$ and $y_i \to y$. As $d_{N_i}(x_i,y_i) < a$, the local product structure of $\widetilde{\mathcal{G}}$ implies that x, y are in the same leaf F of $\widetilde{\mathcal{G}}$. If they are in the same leaf γ of $\widetilde{\mathcal{F}}^s_F$, then the local pictures of $\widetilde{\mathcal{F}}^s$ and $\widetilde{\mathcal{G}}$ along γ show that $l([x_i,y_i])$ is bounded, a contradiction (notice that $\widetilde{\mathcal{F}}^s_F$ is regular along γ in the side the x_i, y_i are converging to). Since the leaf space of $\widetilde{\mathcal{F}}^s$ is Hausdorff, then x, y are in the same leaf L of $\widetilde{\mathcal{F}}^s$. But then $L \cap N$ is not connected, contradicting Proposition 5.1. This proves the claim.

Let γ be a line leaf of $\widetilde{\mathcal{F}}^s_F$ with $\text{depth}(F) \leq (k+1)$. By (B) there is $a > 0$ so that $\gamma \subset B_a(\gamma^*)$. Let $b' = b(2a+1)$. Given $x, y \in \gamma$, let $x', y' \in \gamma^*$ with $d_F(x,x'), d_F(y,y') < a$. Split the segment of γ^* from x' to y' into $\leq (d_F(x',y')+1)$ subsegments of length 1, except for the last with length ≤ 1. Let z'_i be the starting points of these subsegments and $z_i \in [x,y]$ with $d_F(z'_i, z_i) < a$. Then

$$d_F(z_i, z_{i+1}) \; < \; 2a+1, \qquad \text{so} \qquad l([z_i, z_{i+1}]) \; < \; b'.$$

Consequently

$$l([x,y]) \quad \leq \quad (d_F(x,y)+1)b'.$$

This shows the quasi-isometric behavior of line leaves of $\widetilde{\mathcal{F}}^s_F$ and finishes the proof. $\square$

Remark. Notice that Property (B) for line leaves easily implies the same property for piece leaves of $\widetilde{\mathcal{F}}^s_F$ (probably with a different constant) and the same is true for Property (C).

Lemma 6.8. *Let S be a compact surface with a hyperbolic metric and $\partial S \neq \emptyset$. Let $\mathcal{D}$ be a singular foliation in S with only p-prong singularities ($p \geq 3$ in the interior, $p \geq 2$ in the boundary); and in general position with respect to ∂S. Let $\widetilde{\mathcal{D}}$ be the lifted foliation to $\widetilde{S}$. $\widetilde{S}$ has a compactification with an ideal boundary $\partial \widetilde{S}$ which is a Cantor set. Suppose that $\widetilde{\mathcal{D}}$ has Hausdorff leaf space. Then Properties (A), (B) and (C) hold for leaves of $\widetilde{\mathcal{D}}$.*

Proof. If γ is a leaf of $\widetilde{\mathcal{D}}$, then a sector V of γ is a component of $\widetilde{S} - \gamma$ and a line leaf of γ is the closure (in $\widetilde{S}$) of $\partial V - \partial \widetilde{S}$. We stress that as $\widetilde{S}$ has boundary, line leaves of $\widetilde{\mathcal{D}}$ may be rays or even finite segments. Up to a bounded distortion in the metric assume that ∂S is geodesic. The proof of (i), (ii) of (A) is as in [Le]. As in Lemma 6.2, (iii) of Property (A) follows from the fact that $\widetilde{\mathcal{D}}$ has Hausdorff leaf space. The bounded distance from geodesics (Property (B)) is proved as in Proposition 6.6, for $k = 0$. Property (C) is proved as in Proposition 6.7. $\square$

7. Continuous extension of leaves

In this section we prove Theorem D of the introduction:

Theorem 7.1. *Let $\mathcal{G}$ be a Reebless, finite depth foliation in a closed hyperbolic 3-manifold M. Suppose that $\mathcal{G}$ is transverse to a quasigeodesic pseudo-Anosov flow Φ and that the stable and unstable foliations of Φ are quasi-isometric singular foliations. Let E be a leaf of $\mathcal{G}$ with hyperbolic metric quasiconformal to the induced Riemannian metric from M. Let F be a lift of E to $\widetilde{M}$ and $\varphi : F \to \widetilde{M}$ the inclusion map. Then φ extends to a continuous map $\overline{\varphi} : F \cup \partial_\infty F \to \widetilde{M} \cup S^2_\infty$. The ideal boundary map $\overline{\varphi}_{|\partial_\infty F} : \partial_\infty F \to S^2_\infty$ gives a parametrization of the limit set of F as the image of a continuous (closed) curve.*

Proof. Fix once and for all a hyperbolic metric in E, quasiconformal to the induced Riemannian metric from M. This may be the metric used in the last section.

We will use the unit ball model for $\mathbf{H}^3$ and identify $\widetilde{M} \cup S^2_\infty$ to a closed ball in Euclidean $\mathbf{R}^3$. Let d_e be the induced Euclidean metric in $\widetilde{M} \cup S^2_\infty$ and let $diam_e(B)$ be the Euclidean diameter of the set $B \subset \widetilde{M} \cup S^2_\infty$. Similarly there is a unit disk model for $F \cong \mathbf{H}^2$ and a Euclidean metric and diameter in $F \cup \partial_\infty F$, which are also denoted by d_e and $diam_e$, respectively.

Fix a base point $a_0 \in \mathbf{H}^3$. Let $B \subset \mathbf{H}^3$ be a set which is K-quasi-isometrically embedded in $\mathbf{H}^3$. Then if $d_{\mathbf{H}^3}(B, a_0)$ is very big, it follows that $diam_e(B)$ is very small [Th2]. Otherwise find $b_1, b_2 \in B$ not d_e close. Then the hyperbolic geodesic segment connecting them has big d_e diameter and therefore is $d_{\mathbf{H}^3}$ boundedly near a_0. By the quasi-isometric hypothesis, $d_{\mathbf{H}^3}(a_0, B)$ is also bounded, a contradiction. The same is true for quasigeodesics in $\mathbf{H}^3$ [Th2]. This fact applied to leaves of $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$ is fundamental for our result and will be used throughout the proof. Let $K > 0$ so that all leaves of $\mathcal{F}^u, \mathcal{F}^s$ are K-quasi-isometrically embedded and all flow lines of Φ are K-quasigeodesics.

Fix $F \in \widetilde{\mathcal{G}}$. As is the case for depth 0 leaves, the results of the previous section show that each leaf $\gamma \in \widetilde{\mathcal{F}}^s_F$ is quasi-isometrically embedded in F and hence can be pulled tight: If γ is regular it produces a geodesic γ^*. If γ contains a p-prong singularity it produces an ideal polygon with p-sides; each side is obtained by pulling tight a line leaf of l. Let $\mathcal{V}^s_F$ be the union of the geodesics thus produced. Clearly for any l, l' distinct leaves in $\mathcal{V}^s_F$, then $l \cap l' = \emptyset$ because leaves of $\widetilde{\mathcal{F}}^s_F$ do not intersect. Suppose that $l_i \in \mathcal{V}^s_F$ and $l_i \to l$ (l a geodesic of F). Then there are line leaves s_i of $\widetilde{\mathcal{F}}^s_F$ with $s_i^* = l_i$. By Property (B), s_i does not escape compact sets in F and thus $s_i \to s$ (up to subsequence). By (iii) of (A) it follows that $s^* = l$. Therefore $\mathcal{V}^s_F$ is closed and is a geodesic lamination in F.

Consider a complementary region C of $\mathcal{V}^s_F$. Take a leaf l of ∂C and consider the line leaf $s \in \widetilde{\mathcal{F}}^s_F$ so that $s^* = l$ and so that s separates all other line leaves of $\widetilde{\mathcal{F}}^s_F$ with same ideal points as s, from the remaining ideal points of C in $\partial_\infty F$. Notice s is unique. If the leaf α of $\widetilde{\mathcal{F}}^s_F$ containing s is not singular on the corresponding C side, then there are s_i line leaves of $\widetilde{\mathcal{F}}^s_F$, with $s_i \to s$ on that side. But then $s_i^* \to s^*$ on that side of s and by construction $s_i^* \neq s^*$. This implies that for i big enough s_i^* intersects the interior of C, a contradiction. Hence α is singular on the

C side. This shows that any complementary region C of $\mathcal{V}^s_F$ comes from splitting a singular leaf of $\widetilde{\mathcal{F}}^s_F$ and is an ideal polygon with a bounded number of sides.

We cannot stress enough the importance of the hypothesis that $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$ have Hausdorff leaf space for the arguments in this section. As the reader will see, this property is used in various strategic situations along the proof of Theorem 7.1.

Notation. Since there will be so many cases to consider, then in order to simplify the exposition we do the following: the notation l, l', l_i will almost always be reserved for leaves of $\widetilde{\mathcal{F}}^s_F$ and L, L_i will denote the leaves of $\widetilde{\mathcal{F}}^s$ containing these. In the same way g, g', g_i will be leaves of $\widetilde{\mathcal{F}}^u_F$ and G, G_i leaves of $\widetilde{\mathcal{F}}^u$.

Construction of extension of φ*.* We now start the proof of Theorem 7.1. We may assume that no depth 0 leaf of $\mathcal{G}$ is a virtual fiber of M over $\mathbf{S}^1$, for otherwise $\mathcal{G}$ is essentially a fibration [Fe-Mo] and the result was proved by Cannon and Thurston [Ca-Th]. We will use Properties (A), (B) and (C) of the previous section. Fix $F \in \widetilde{\mathcal{G}}$ and $\varphi : F \to \widetilde{M}$. First we show there is a natural extension of φ to S^1_∞. Let $q \in \partial_\infty F$.

Case 1. q is not an ideal point of a leaf of $\widetilde{\mathcal{F}}^s_F$ or $\widetilde{\mathcal{F}}^u_F$.

Let r be a geodesic ray in F with ideal point q. The goal is to show that $\varphi(r)$ has a unique ideal point in S^2_∞ and let this be $\overline{\varphi}(q)$. Since any other ray r' with ideal point q is asymptotic to r, then $\varphi(r), \varphi(r')$ are a bounded distance from each other in $\widetilde{M}$, so this is well defined.

The complementary components of $\mathcal{V}^s_F$ are <u>finite</u> sided ideal polygons in F. Since q is not an ideal point of leaves of $\mathcal{V}^s_F$, there are s_i leaves of $\mathcal{V}^s_F$, so that $\{\overline{s}_i\}, i \in \mathbf{N}$ (closure in $F \cup \partial_\infty F$), defines a neighborhood system for q in $F \cup S^1_\infty$. Let l_i be leaves of $\widetilde{\mathcal{F}}^s_F$ with $s_i = l^*_i$. As there are only countably many singular leaves in $\widetilde{\mathcal{F}}^s_F$, we may assume that l_i are regular and that l_i separates l_{i-1} from l_{i+1} for all $i \in \mathbf{N}$. Then r intersects each s_i once and, for each i, r is eventually on the q side of $\overline{l}_i$ in $F \cup S^1_\infty$. Property (B) implies that $d_F(l_i, s_i)$ is bounded, consequently $\{\overline{l_i}\}, i \in \mathbf{N}$, also defines a neighborhood system of q in $F \cup \partial_\infty F$. Let $L_i \in \widetilde{\mathcal{F}}^s$ with $l_i \subset L_i$. Because l_i separates l_{i-1} from l_{i+1} in F, it follows that L_i separates L_{i-1} from L_{i+1} in $\widetilde{M}$. Here is the key fact: the intersection $r \cap l_i$ is bounded in F and so r is eventually contained in the component W_i of $F - l_i$ which has q in its closure (in $F \cup \partial_\infty F$); see Figure 12 (a). This is the two-dimensional picture. Then from the point of view of $\widetilde{M}$, $\varphi(r)$ is eventually contained in the component of $\widetilde{M} - L_i$ containing $\varphi(W_i)$. Similarly let g_i be leaves of $\widetilde{\mathcal{F}}^u_F$ defining a system of neighborhoods of q and $G_i \in \widetilde{\mathcal{F}}^u$ with $g_i \subset G_i$. Assume that L_i, G_i are regular.

Case 1.1. One of the sequences L_i or G_i escapes in $\widetilde{M}$.

Suppose that (say) L_i escapes in $\widetilde{M}$. Fix a basepoint $a_0 \in \mathbf{H}^3$. Let U_i be the component of $(\mathbf{H}^3 - L_i)$ always containing a subray of $\varphi(r)$. Since L_i escapes $\widetilde{M}$, then $d_{\mathbf{H}^3}(L_i, a_0) \to +\infty$. As L_i is K-quasi-isometrically embedded, where K is fixed, then $diam_e(L_i) \to 0$ and consequently $diam_e(U_i) \to 0$. In addition the family $\{U_i\}, i \in \mathbf{N}$, is nested. Let $\overline{U}_i$ be the closure of U_i in $\widetilde{M} \cup S^2_\infty$. Then $diam_e(\overline{U}_i) \to 0$ also. Therefore

$$\bigcap_{i \in N} \overline{U}_i \quad = \quad \{z\},$$

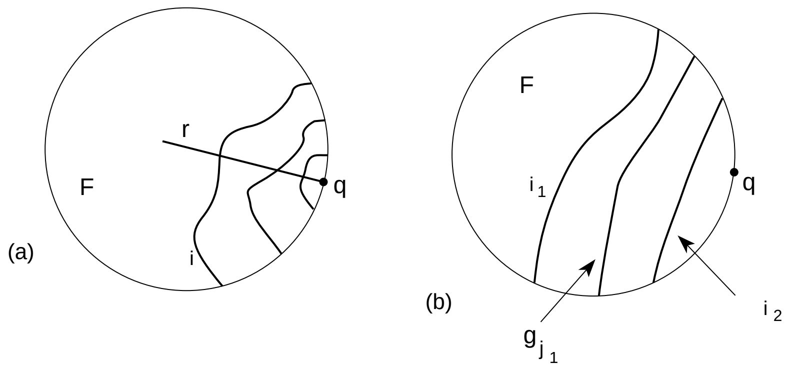

FIGURE 12. (a) System of neighborhoods of an ideal point of F.
(b) Separation properties of leaves.

a single point in S^2_∞. Since $\varphi(r)$ is eventually contained in any U_i and hence in $\overline{U}_i$, it follows that z is the unique limit point of $\varphi(r)$. Let this be $\overline{\varphi}(q)$.

Remark. In the case of fibration then: (1) No point $q \in \partial_\infty F$ is an ideal point of leaves of both $\widetilde{\mathcal{F}}^s_F$ and $\widetilde{\mathcal{F}}^u_F$, (2) If q is not an ideal point of leaves of $\widetilde{\mathcal{F}}^s_F$, then the leaves L_i defined above escape $\widetilde{M}$. Case 1.1 shows there is a natural extension of φ to $\overline{\varphi} : \partial_\infty F \to S^1_\infty$. In addition the sets L_i (or G_i) show that the extension is continuous (see proof of continuity later in this section). The difficulty in the higher depth case is that in general we cannot prove (and it may not be true) that (1) or (2) holds. This makes the analysis much more complicated.

Case 1.2. Neither L_i nor G_i escapes in $\widetilde{M}$.

Then the $\{L_i\}, i \in \mathbf{N}$, have to accumulate in $\widetilde{M}$ and since they are nested, it follows that $L_i \to L$ as $i \to +\infty$, in fact $L_i \to L'$ where L' is a line leaf in L. The leaf L is unique because $\mathcal{H}(\widetilde{\mathcal{F}}^s)$ is Hausdorff.

Let $i_1 = 1$. Since $\{\overline{g}_j\}, j \in \mathbf{N}$, defines a neighborhood system of q in $F \cup S^1_\infty$, choose j_1 so that g_{j_1} is in the interior of the neighborhood bounded by $\overline{l}_{i_1}$ and containing q; see Figure 12 (b). Inductively choose i_m so that l_{i_m} is in the neighborhood in $F \cup S^1_\infty$ bounded by $\overline{g}_{j_{m-1}}$ and containing q and choose j_m so that g_{j_m} is in the neighborhood bounded by $\overline{l}_{j_m}$ and containing q. For simplicity we assume these are the original sequences l_i, g_i. Then l_i separates g_{i-1} from g_i in F and g_i separates l_i from l_{i+1} for all i.

Lemma 7.2. *$L_i \cap G_j = \emptyset$ for all i, j.*

Proof. Suppose there are i, j so that $L_i \cap G_j \neq \emptyset$. Then there is $u \in \widetilde{M}$ with

$$\Theta(u) \ \in \ \partial\Theta(l_i) \ \subset \ \Theta(L_i)$$

and either

$$\Theta(u) \ \in \ \Theta(L_i \cap G_j) \quad \text{or} \quad \Theta(u) \ \text{ separates } \ \Theta(l_i) \ \text{ from } \ \Theta(L_i \cap G_j).$$

It follows that $\Theta(u) \in \partial\Theta(F)$ and by Proposition 5.1 there is either a line leaf v of $\Theta(\widetilde{W}^s(u))$ contained in $\partial\Theta(F)$ or a line leaf v of $\Theta(\widetilde{W}^u(u))$ contained in $\partial\Theta(F)$, with v regular on the side containing $\Theta(F)$. Since v is regular on the $\Theta(F)$ side and

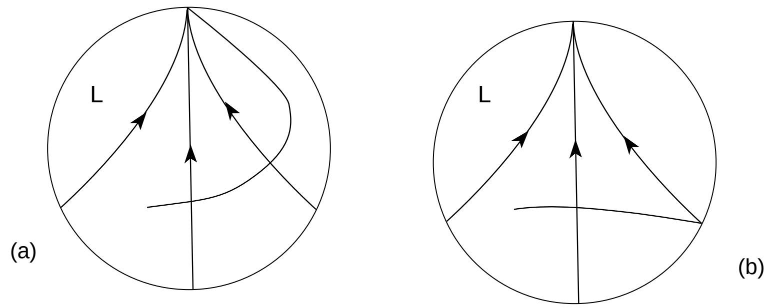

FIGURE 13. (a) The case $\Theta(l)$ is a ray in a leaf of $\widetilde{\mathcal{F}}^s_{\mathcal{O}}$. (b) The case $\Theta(l)$ is a bounded segment.

$L_i = \widetilde{W}^s(u)$ intersects F, then v cannot be contained in $\widetilde{W}^s(u)$. In the second case $v \cap \Theta(g_i) = \emptyset$ and therefore v separates $\Theta(F)$ from $\Theta(g_i)$, a contradiction. This proves the lemma. □

Since g_i separates l_i from l_{i+1} in F for all i and $L_i \cap G_j = \emptyset$, then G_i separates L_i from L_{i+1}. Choose $x \in L'$ nonsingular and a segment η in $\widetilde{W}^u(x)$ transverse to $\widetilde{\Phi}$ with one endpoint in x and in the side from which L_i is limiting to L. Then $L_i \cap \eta \neq \emptyset$ for i big enough. Since G_i separates L_i from L_{i+1}, then $G_i \cap \eta \neq \emptyset$ for i big. As $\eta \subset \widetilde{W}^u(x)$, it follows that $G_i = \widetilde{W}^u(x)$. But then clearly $G_i \cap L_i \neq \emptyset$, a contradiction.

We conclude that Case 1.2 cannot happen.

Case 2. q is an ideal point of a line leaf of $\widetilde{\mathcal{F}}^s_F$ but not of $\widetilde{\mathcal{F}}^u_F$ (or vice versa).

Let l be a ray in a line leaf of $\widetilde{\mathcal{F}}^s_F$ with q as ideal point. Let $L \in \widetilde{\mathcal{F}}^s$ with $l \subset L$. In L the ray l is transverse to flow lines of $\widetilde{\Phi}$. Parametrize l by arclength as $u_t, t \in [0, \infty)$. Let α_t be the flow line of $\widetilde{\Phi}$ with $u_t \in \alpha_t$.

Suppose first that $\Theta(l)$ is an infinite arc in $\Theta(L)$, see Figure 13 (a), which shows $l \subset L$. In that case α_t escapes in L as $t \to \infty$ and since L is properly embedded in $\widetilde{M}$, then the α_t also escape in $\widetilde{M}$. As the α_t are K-quasigeodesics in $\mathbf{H}^3$, then $diam_e(\alpha_t) \to 0$. All α_t have the same positive ideal point, because they are forward asymptotic. We denote this by L_+. Then $\varphi(u_t) \to L_+$.

The other option is that $\Theta(l)$ is a bounded segment in $\Theta(L)$, see Figure 13 (b), which shows $l \subset L$. In that case $\alpha_t \to \alpha$ as $t \to \infty$ where α is a flow line in L. Fix a disk D transverse to $\widetilde{\Phi}$ and with center in $x_0 \in \alpha$. For each t let

$$a_t \in \mathbf{R} \quad \text{with} \quad u_t = \widetilde{\Phi}_{a_t}(\widetilde{\Phi}_{\mathbf{R}}(u_t) \cap D).$$

Notice that u_t escapes in L as $t \to \infty$, otherwise F would not be properly embedded in $\widetilde{M}$. Therefore $|a_t| \to +\infty$. If there is a subsequence $a_{t_i} \to +\infty$, then since the α_t are all asymptotic to α in forward time it would follow that $l \cap \alpha \neq \emptyset$, a contradiction. Therefore $a_t \to -\infty$ as $t \to \infty$. Let $\alpha_- \in \partial_\infty L$ be the negative ideal point associated to α. Then $u_t \to \alpha_-$ in $L \cup \partial_\infty L$. Since the embedding $L \to \widetilde{M}$ extends continuously to $\partial_\infty L$, it follows that $\overline{\varphi}(u_t) \to \eta_-(\alpha)$.

If l' is another ray of $\widetilde{\mathcal{F}}^s_F$ with q as an ideal point, then Property (B) applied to F implies that, up to taking subrays, l and l' are a bounded distance apart in F

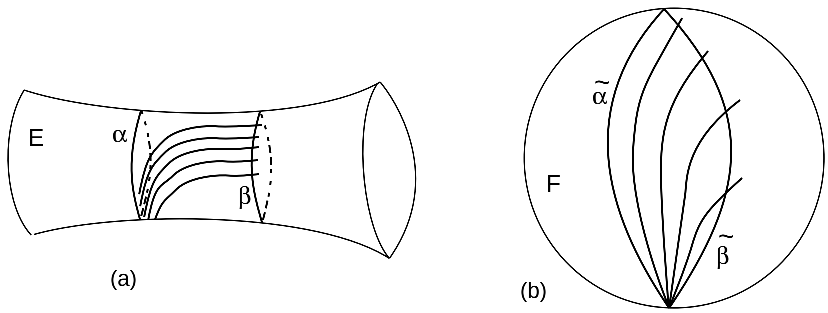

FIGURE 14. (a) Isotopic leaves of $\mathcal{F}^s_E, \mathcal{F}^u_E$. (b) Lifting to the universal cover.

[Th2, Gr, CDP]. Therefore $\varphi(l')$ is also a bounded distance apart from $\varphi(l)$ in $\widetilde{M}$ and $\varphi(l')$ converges to the same ideal point in S^2_∞ as $\varphi(l')$. Let $\overline{\varphi}(q)$ be this ideal point.

Case 3. q is an ideal point of leaves of both $\widetilde{\mathcal{F}}^s_F$ and $\widetilde{\mathcal{F}}^u_F$.

If l is a ray of $\widetilde{\mathcal{F}}^s_F$ with ideal point q, then as in Case 2, $\varphi(l)$ converges to a unique ideal point in S^2_∞. By Property (B) applied to F, any other ray of $\widetilde{\mathcal{F}}^s_F$ or $\widetilde{\mathcal{F}}^u_F$ with q as ideal point has a subray a bounded distance from l in F, so will have the same ideal point when seen in $\widetilde{M}$. Let $\overline{\varphi}(q)$ be this common ideal point. In Cases 2 and 3 notice that if ζ is any geodesic ray in F with ideal point q in $\partial_\infty F$, then $\varphi(\zeta)$ converges to a unique ideal point in S^2_∞, which is exactly $\overline{\varphi}(q)$. This was also true in Case 1.

Conclusion. This finishes the construction of a natural extension map $\overline{\varphi} : F \cup \partial_\infty F \to \widetilde{M} \cup S^2_\infty$.

Remark. Case 3 does not occur for suspension pseudo-Anosov flows transverse to fibrations and one might think at first they do not occur in general. However such is not the case! For instance suppose that $\mathcal{G}$ has a leaf E so that $\mathcal{F}^s_E, \mathcal{F}^u_E$ have closed leaves α and β which are isotopic, see Figure 14 (a), which shows α, β and $\mathcal{F}^s_E$ restricted to the annulus bounded by $\alpha \cup \beta$. This situation is quite possible; see the construction of pseudo-Anosov flows in [Mo4]. It also occurs in the case of intransitive Anosov flows as constructed by Franks and Williams [Fr-Wi]. Let F be a lift of E to $\widetilde{M}$ and lift α, β coherently to $\widetilde{\alpha}, \widetilde{\beta}$ respectively in F. Then $\widetilde{\alpha} \in \widetilde{\mathcal{F}}^s_F, \widetilde{\beta} \in \widetilde{\mathcal{F}}^u_F$ have the same ideal points in $\partial_\infty F$ providing an example where Case 3 happens.

Proof of continuity of $\overline{\varphi}$. We will now prove that the extension map $\overline{\varphi} : F \cup \partial_\infty F \to \widetilde{M} \cup S^2_\infty$ is continuous. Clearly we only need to check continuity of $\overline{\varphi}$ in $\partial_\infty F$. Let $q \in \partial_\infty F$. The proof of continuity is a bit tricky and long, with many cases to be checked. In some cases we prove continuity by considering neighborhoods of q in $F \cup \partial_\infty F$ bounded by $\overline{r}$, where r is a line leaf of $\widetilde{\mathcal{F}}^s_F$ or $\widetilde{\mathcal{F}}^u_F$ and $\overline{r}$ its closure in $F \cup \partial_\infty F$. Sometimes the neighborhood is obtained by a curve in F which is a concatenation of 2 or 3 segments or rays in different leaves of $\widetilde{\mathcal{F}}^s_F, \widetilde{\mathcal{F}}^u_F$. Finally in other times we fix a ray r of $\widetilde{\mathcal{F}}^s_F$ or $\widetilde{\mathcal{F}}^u_F$ with ideal point q and then consider continuity at q on each side of $\overline{r}$ in $F \cup \partial_\infty F$. Again the side neighborhoods may be bounded by one leaf or a union of pieces of various leaves.

Case 1. q is not an ideal point of a line leaf in $\widetilde{\mathcal{F}}^s_F$ or $\widetilde{\mathcal{F}}^u_F$.

We refer to the proof of the extension. Recall that only Case 1.1 can happen. Fix $\epsilon > 0$. Then one of L_i or G_i escapes in $\widetilde{M}$, so we assume that L_i escapes in $\widetilde{M}$ as $i \to \infty$. Recall that r is a geodesic ray in F with ideal point q and U_i is the component of $\widetilde{M} - L_i$ containing a subray of $\varphi(r)$. Since L_i escapes in $\widetilde{M}$, choose i big enough so that $diam_e(\overline{U}_i) < \epsilon$. Let A_i be the component of $F - l_i$ accumulating on q. Then $\{\overline{A}_i\}, i \in \mathbf{N}$, forms a neighborhood system of q in $F \cup \partial_\infty F$. By definition $\varphi(A_i) \subset U_i$.

Lemma 7.3. $\overline{\varphi}(\overline{A}_i) \subset \overline{U}_i$.

Proof. Let $w \in \overline{A}_i \cap \partial_\infty F$ (a closed segment in $\partial_\infty F$), but w not a boundary point of this segment. In all cases (1), (2) and (3) of the definition of $\overline{\varphi}(w)$ we considered a ray (geodesic or not) λ in F which has w as ideal point; then we showed that $\varphi(\lambda)$ has a single accumulation point in S^2_∞, and finally let $\overline{\varphi}(w)$ be this accumulation point. In case (1) λ was a geodesic ray and in cases (2) and (3) λ was a ray in a leaf of $\widetilde{\mathcal{F}}^s_F$ or $\widetilde{\mathcal{F}}^u_F$. The previous section shows that these rays are always uniform quasigeodesics, hence have subrays which are a bounded distance (in F) from geodesic rays with w as ideal point [Th2, Gr, CDP]. So up to taking a subray, we may assume that λ is a ray entirely contained in A_i. Since $\varphi(\lambda) \subset \varphi(A_i) \subset U_i$, we conclude that $\overline{\varphi}(w) \in \overline{U}_i$. Consequently $\overline{\varphi}(\overline{A}_i) \subset \overline{U}_i$. $\square$

Since we can make $diam_e(\overline{U}_i)$ as small as we want, we conclude that $\overline{\varphi}$ is continuous at q. This finishes the proof in Case 1.

Case 2. q is an ideal point of a line leaf of $\widetilde{\mathcal{F}}^s_F$ but not of $\widetilde{\mathcal{F}}^u_F$.

Let r be a ray of $\widetilde{\mathcal{F}}^s_F$ with q as an ideal point, so that r does not contain a singularity of $\widetilde{\mathcal{F}}^s_F$. Let l be a line leaf of $\widetilde{\mathcal{F}}^s_F$ with $r \subset l$ and let $L \in \widetilde{\mathcal{F}}^s$ with $l \subset L$. Choose

$$u_j \in r \quad \text{with} \quad u_j \to q, \quad \text{as} \quad j \to \infty.$$

Let g_j be the unstable leaf of $\widetilde{\mathcal{F}}^u_F$ through u_j; see Figure 15 (a). If g_j does not escape in F as $j \to \infty$, then $g_j \to g$, which is a leaf of $\widetilde{\mathcal{F}}^u_F$ with $g \cap l = \emptyset$. By continuity of ideal points of line leaves of $\widetilde{\mathcal{F}}^u_F$ it follows that one of the ideal points of g is q, a contradiction to the hypothesis. Therefore g_j escapes in F. In the same way all the endpoints of g_j converge to q as $j \to \infty$. Otherwise $\partial_\infty F$ has an interval free of ideal points of line leaves of $\widetilde{\mathcal{F}}^u_F$, a contradiction to Lemma 6.5.

Hence the $\overline{g}_j$ define a system of neighborhoods of q in $F \cup \partial_\infty F$. Let $G_j \in \widetilde{\mathcal{F}}^u$ with $g_j \subset G_j$; we may assume G_j is nonsingular. If G_j escapes $\widetilde{M}$, then an argument as in Case 1 shows that $\overline{\varphi}$ is continuous at q.

Suppose then that $G_j \to G$. We consider continuity at q in each side of $\overline{l}$ in $F \cup \partial_\infty F$. Consider all line leaves of $\widetilde{\mathcal{F}}^s_F$ in a side of l in F having q as one ideal point. If l_i is a sequence of such line leaves, let $q_i \neq q$ be the other ideal point of l_i in $\partial_\infty F$; see Figure 15 (a) (notice that $r \subset l$ in this figure).

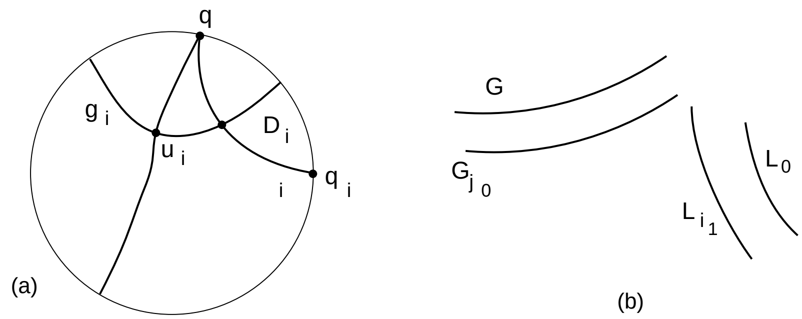

FIGURE 15. (a) Escaping l_i in the leaf F. (b) The picture in $\widetilde{M}$.

Case 2.1. There is a sequence l_i as above with $q_i \to q$; see Figure 15 (a).

This is equivalent to saying that l_i escapes in F when $i \to \infty$. Assume that the l_i are nested in F. Let $l_i \subset L_i \in \widetilde{\mathcal{F}}^s$. There are two subcases:

Case 2.1.1. The L_i escape in $\widetilde{M}$.

Let $\epsilon > 0$. Each l_i bounds a unique region $D_i \subset F$ with $l \cap D_i = \emptyset$; see Figure 15 (a). Let U_i be the component of $\widetilde{M} - L_i$ with $D_i \subset U_i$. The hypothesis implies that the U_i escape in $\widetilde{M}$. Choose i big enough so that $diam_e(\overline{U}_i) < \epsilon$.

Let N_1 be the wedge region bounded by l_i, l and a segment connecting their starting points. Since all line leaves of $\widetilde{\mathcal{F}}^s_F$ are uniform quasigeodesics, there is a fixed $\eta > 0$ satisfying: as l_i, l converge to the same ideal point q, then there are subrays r_i, r' of l_i, l which are in an η-neighborhood of each other in F [Th2, CDP, Gh-Ha]. In addition, because the hyperbolic metrics in leaves of $\mathcal{G}$ are uniformly quasiconformal to the induced Riemannian metrics from M, it follows that there is η' depending only on η so that $\varphi(r_i)$ is in an η'-neighborhood of $\varphi(r')$ in $\widetilde{M}$ and the same holds for $\varphi(N_1)$. This bounded thickness argument for wedges will be used a few times throughout the proof.

Hence there is a small neighborhood N_2 of q in $F \cup \partial_\infty F$, so that $diam_e(\varphi(N)) < \epsilon$, where $N = N_1 \cap N_2$. The set $N \cup \overline{D}_i$ is a neighborhood of q in that side of l in $F \cup S^1_\infty$. Since ϵ is arbitrary, this implies that $\overline{\varphi}$ is continuous at q in that side of l.

Case 2.1.2. The L_i converge to L_0 in $\widetilde{\mathcal{F}}^s$.

The goal is to show that this case cannot happen. As l_i escapes in F and $l_i \subset L_i$ with $L_i \to L_0$, then L_0 has a line leaf L' with $\Theta(L') \subset \partial\Theta(F)$.

We first claim that $G_j \cap L_0 = \emptyset$ for any j. Since $q_i \to q$ and l_i escapes in F, then there is

$$i_0 \quad \text{so that} \quad \forall i \geq i_0, \qquad l_i \cap g_j \ = \ \emptyset.$$

The argument of Lemma 7.2 shows that $L_i \cap G_j = \emptyset$ for any $i \geq i_0$. On the other hand if $G_j \cap L_0 \neq \emptyset$, then $G_j \cap L_i \neq \emptyset$ for i sufficiently big, contradicting the previous fact. This proves the claim.

Now as $G_j \to G$ as $j \to \infty$ and this sequence has no other limit ($\widetilde{\mathcal{F}}^u$ has Hausdorff leaf space), it follows that there is j_0 so that for $j \geq j_0$, G_j separates L_0 from G; see Figure 15 (b). In addition L_i converges to L_0 as $i \to \infty$, so there is an i_1 big enough so that L_{i_1} separates G_{j_0} from L_0; see Figure 15 (b). Therefore

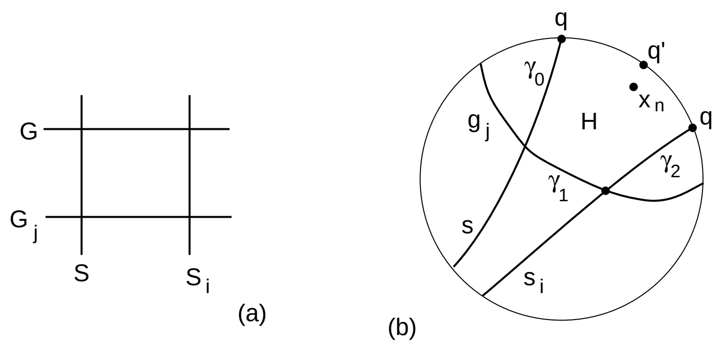

FIGURE 16. (a) Producing a rectangle in $\widetilde{M}$. (b) A region in F without singularities.

$G_j \cap L_{i_1} = \emptyset$ for any $j \geq j_0$. But this leads to a contradiction as follows: In F there is $j > j_0$ with

$$g_j \cap l_{i_1} \neq \emptyset.$$

Hence

$$G_j \cap L_{i_1} \neq \emptyset.$$

We conclude that Case 2.1.2 cannot happen.

Case 2.2. There is no sequence l_i as above with $q_i \to q$.

Let s be the outermost line leaf of $\widetilde{\mathcal{F}}^s_F$ on that side of l and with one ideal point q. Up to taking a subsequence of $\{g_j\}, j \in \mathbf{N}$, we may assume that $g_j \cap s \neq \emptyset$ for all $j \in \mathbf{N}$. Let $s \subset S \in \widetilde{\mathcal{F}}^s$. There are two subcases:

Case 2.2.1. $S \cap G \neq \emptyset$.

We will show that this case cannot happen. Choose $s_i \in \widetilde{\mathcal{F}}^s_F$, with $s_i \to s$ from the side opposite to l. Notice that s is regular on the s_i side. Choose s_i regular and $\{s_i\}, i \in \mathbf{N}$, nested. Then s_i has an ideal point q_i with $q_i \to q$ when i converges to ∞, but $q_i \neq q$ for any i. In addition assume that $q_i \neq q_m$ if $i \neq m$.

Let $S_i \in \widetilde{\mathcal{F}}^s$ with $s_i \subset S_i$. Let S' be the line leaf of S with $s \subset S'$ and $S_i \to S$. Choose i big enough so that $S_i \cap G \neq \emptyset$. Fix j so that $g_j \cap s_i \neq \emptyset$, $G_j \cap S \neq \emptyset$. Then $G_j \cap S_i \neq \emptyset$ and G, G_j, S, S_i form a rectangle $\mathcal{R}$; see Figure 16 (a). There is no singular orbit of $\widetilde{\Phi}$ in the interior of $\mathcal{R}$. Notice also that since s separates l from s_i (or $s = l$), then $g_j \cap s \neq \emptyset$.

Let γ_0 be the component of $s - g_j$ which is a ray with ideal point q. Let γ_2 be the component of $s_i - g_j$ which is a ray with ideal point q_i. Let γ_1 be the closure of the bounded component of $g_j - (s \cup s_i)$; see Figure 16 (b). Then γ_1 is a compact segment with endpoints in the starting points of γ_1, γ_2.

Let H be the component of $F - (\gamma_0 \cup \gamma_1 \cup \gamma_2)$ which intersects $s_{i'}$ for all $i' > i$. Since $F \cap G = \emptyset$ it follows that $H \subset \mathcal{R}$. In particular there are no singularities of $\widetilde{\mathcal{F}}^s_F$ in H. Choose points $x_n \in H$ with $x_n \to q' \in \partial_\infty F$, so that $q' \neq q$, $q' \neq q_i$. Because line leaves from $\widetilde{\mathcal{F}}^s_F$ are a bounded distance from geodesics, then there are balls in F, $B^F_{a_n}(x_n) \subset H$ and $a_n \to +\infty$ as $n \to \infty$. This is disallowed by the following lemma. We conclude that Case 2.2.1 cannot happen.

Lemma 7.4. *There is $a_0 > 0$ so that for any $F' \in \widetilde{\mathcal{G}}$ and any $x \in F'$, then the ball $B^{F'}_{a_0}(x)$ in F' has a singularity of $\widetilde{\mathcal{F}}^s_{F'}$.*

Proof. Otherwise find $F_n \in \widetilde{\mathcal{G}}$, $x_n \in F_n$, $a_n \to \infty$, with $B^{F_n}_{a_n}(x_n)$ without singularities. Therefore $\pi(B^{F_n}_{a_n}(x_n))$ also has no singularities. Assume that $\pi(x_n) \to y$ in M. The local product structure of foliations then implies that if C_1 is the leaf of $\mathcal{G}$ containing y, then C_1 does not intersect any singular orbit of Φ. But $\overline{C}_1$ contains a compact leaf, hence the same would be true for a compact leaf. Then $\mathcal{F}^s$ would induce a nonsingular foliation in this compact leaf, a contradiction to the compact leaf having genus > 1. $\square$

Case 2.2.2. $S \cap G = \emptyset$.

Let V_1 be the component of $\widetilde{M} - S'$ containing all S_i, and let V_2 be the other component of $\widetilde{M} - S'$.

Case 2.2.2.1. $G \subset V_1$; see Figure 17 (a).

We will show that this case does not happen. Let U_i be the component of $M - S_i$ not containing S. Since $S_i \to S$ when $i \to \infty$, then $\bigcup_{i \in \mathbf{N}}(U_i) = V_1$. As $G \subset V_1$ there is $i_0 \in \mathbf{N}$ so that $G \cap U_i \neq \emptyset$, for all $i \geq i_0$.

Suppose that

$$S_m \cap G \neq \emptyset, \quad S_n \cap G \neq \emptyset, \quad \text{for} \quad m \neq n, \quad \text{and} \quad m, n > i_0.$$

Let $j \in \mathbf{N}$ so that $G_j \cap S_m \neq \emptyset$, $G_j \cap S_n \neq \emptyset$. Then S_m, S_n, G, G_j form a rectangle in $\widetilde{M}$. Since $q_m \neq q_n$, Lemma 7.4 shows that this is impossible. Therefore there is at most one $i \geq i_0$ so that $S_i \cap G \neq \emptyset$. It now follows that there is $i_1 \geq i_0$ so that $G \subset U_i$ for all $i \geq i_1$.

Let $Y_0 \in \widetilde{\mathcal{F}}^s$ be a nonsingular leaf with $Y_0 \cap G \neq \emptyset$. Since $G_j \to G$ as $j \to \infty$, there is j_0 so that

$$\forall j \geq j_0, \quad G_j \cap Y_0 \neq \emptyset.$$

Notice that $Y_0 \subset U_i$ for any $i \geq i_1$. In addition since $s \cap g_j \neq \emptyset$ in F, then $S \cap G_j \neq \emptyset$ in $\widetilde{M}$. Since S_i separates Y_0 from S, then $G_j \cap S_i \neq \emptyset, \forall i \geq i_1, \forall j \geq j_0$.

Recall that G_j is regular. Fix $i > i_1$. Then $S \cap G_{j_0} \neq \emptyset$ and $S_i \cap G_{j_0} \neq \emptyset$. Let P be the segment flow band in G_{j_0} defined by these two orbits in G_{j_0} and let σ be a defining segment for this flow band. Our goal is to show that σ is a base segment for a stable product region in $\widetilde{M}$ and hence derive a contradiction. Let C_0 be the component of $\widetilde{M} - G_{j_0}$ containing G_j for $j > j_0$. For any $z \in \sigma$, let R_z be the component of $\widetilde{W}^s(z) - \widetilde{\Phi}_{\mathbf{R}}(z)$ contained in C_0. Fix $z \in \sigma$. Then $\widetilde{W}^s(z)$ separates S from S_i. Since

$$\forall j > j_0, \quad G_j \cap S \neq \emptyset, \quad G_j \cap S_i \neq \emptyset,$$

then

$$G_j \cap \widetilde{W}^s(z) \neq \emptyset, \quad \text{in fact} \quad G_j \cap R_z \neq \emptyset.$$

Let $\beta^j_z = G_j \cap R_z$. If β^j_z does not escape in R_z as $j \to \infty$, then $\beta^j_z \to \beta_z$ when $j \to \infty$, where β_z is an orbit of $\widetilde{\Phi}$ in R_z. Since

$$\beta^j_z \subset G_j \quad \text{and} \quad G_j \to G \quad \text{as } j \to \infty,$$

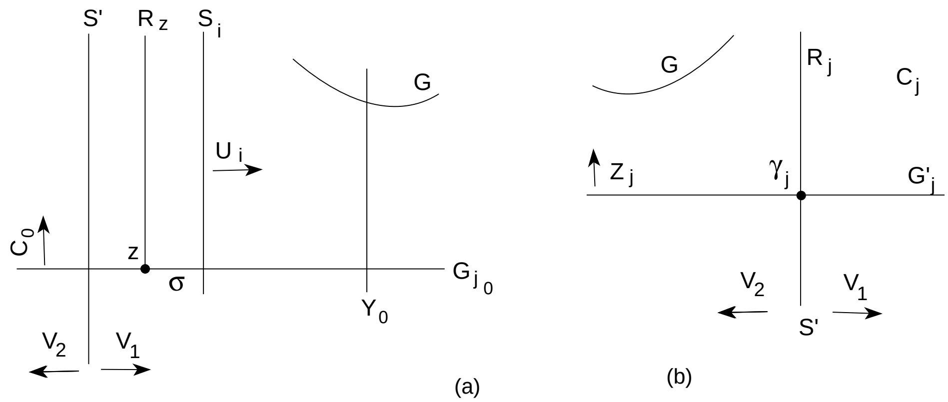

FIGURE 17. (a) Case $G \subset V_1$. (b) Case $G \subset V_2$.

then $\widetilde{W}^u(\beta_z)$ and G are not separated in the leaf space of $\widetilde{\mathcal{F}}^u$. But this leaf space is Hausdorff, hence $G = \widetilde{W}^u(\beta_z)$, that is, $G \cap R_z \neq \emptyset$. This contradicts $G \subset U_i$ and $R_z \cap U_i = \emptyset$. Therefore β_z^j escapes in R_z. This also implies that R_z has no singularities. Consequently

$$\forall z, w \in \sigma, \quad \mathcal{J}^s(R_z) = \mathcal{J}^s(R_w),$$

that is, σ is the defining segment of a stable product region in $\widetilde{M}$. Theorem 4.10 then implies that Φ is a suspension Anosov flow and in particular has no singular orbits, a contradiction. We conclude that Case 2.2.2.1 cannot happen.

Case 2.2.2.2. $G \subset V_2$; see Figure 17 (b).

Let $G'_j = G_j \cap V_1$. Recall that V_1 is the component of $\widetilde{M} - S'$ containing S_i. Let Z_j be the component of $M - G_j$ containing G and let $R_j = S' \cap Z_j$. Let $\gamma_j = S \cap G_j$. We may assume that R_j has no singularity. Let

$$D_j \ = \ R_j \ \cup \ \gamma_j \ \cup \ G'_j.$$

Then D_j separates $\widetilde{M}$. Let C_j be the component of $\widetilde{M} - D_j$ not containing G.

The γ_j escape in S' as $j \to \infty$. Otherwise $\gamma_j \to \gamma$ and as seen in Case 2.2.2.1 (Hausdorff leaf space of $\widetilde{\mathcal{F}}^u$), $\gamma \subset G$, so $S \cap G \neq \emptyset$, a contradiction. Therefore $R_j \cup \gamma_j$ escapes in S' and so escapes in $\widetilde{M}$ also (S' is properly embedded in $\widetilde{M}$). The leaves of $\widetilde{\mathcal{F}}^s$ are uniformly quasi-isometrically embedded in $\widetilde{M}$, hence this implies that $diam_e(R_j \cup \gamma_j) \to 0$ as $j \to \infty$. Suppose that $diam_e(G'_j) \not\to 0$. Then there is

$$a_2 > 0 \quad \text{and} \quad j_m \to \infty \quad \text{with} \quad diam_e(G'_{j_m}) \ > \ 2a_2.$$

Since all orbits in G'_{j_m} share the same negative ideal point in S^2_∞, it follows that there are orbits

$$\alpha_m \ \subset \ G'_{j_m} \quad \text{of} \quad \widetilde{\Phi} \quad \text{with} \quad diam_e(\alpha_m) \ > \ a_2.$$

As the α_m are uniform K-quasigeodesics, it follows that the α_m intersect a fixed compact set in $\widetilde{M}$. Up to subsequence assume that $\alpha_m \to \alpha$. Since $G_{j_m} \to G$, it now follows that $\alpha \subset G$. But

$$\alpha_m \ \subset \ G'_{j_m} \ \subset \ V_1, \quad \text{therefore} \quad \alpha \ \subset \ V_1 \cup \partial V_1 \ = \ V_1 \cup S'.$$

This contradicts $G \subset V_2$. We conclude that $diam_e(G'_j) \to 0$. Therefore $diam_e(D_j) \to 0$ and since G intersects $\widetilde{M} - C_j$ for any j, it follows that $diam_e(\overline{C}_j) \to 0$.

Let H_j be the component of $F - g_j$ so that $q \in \overline{H}_j$ (closure in $F \cup \partial_\infty F$). Let $I_j = H_j \cap V_1$. Then $\overline{I}_j$ is a neighborhood of q in that side of $\overline{s}$ in $F \cup \partial_\infty F$ (this is the side not containing l if $l \neq s$). These definitions imply that $\varphi(I_j) \subset C_j$ in $\widetilde{M}$. By Lemma 7.3, $\overline{\varphi}(\overline{I}_j) \subset \overline{C}_j$. Since $diam_e(\overline{C}_j) \to 0$ when $j \to \infty$, it follows that $\overline{\varphi}$ is continuous at q in that side of $\overline{s}$ in $F \cup \partial_\infty F$. The wedge between l and s has bounded thickness so continuity in the wedge follows as in Case 2.1.1.

This finishes the proof of continuity in Case 2.

Case 3. q is an ideal point of line leaves of both $\widetilde{\mathcal{F}}^s_F$ and $\widetilde{\mathcal{F}}^u_F$.

Case 3.1. For every pair of leaves $l \in \widetilde{\mathcal{F}}^s_F$ and $g \in \widetilde{\mathcal{F}}^u_F$ with both l and g having q as ideal point, then $l \cap g = \emptyset$.

Let $l' \in \widetilde{\mathcal{F}}^s_F$, $g' \in \widetilde{\mathcal{F}}^u_F$ with ideal point q. Assume for simplicity that l', g' do not have singularities of $\widetilde{\mathcal{F}}^s_F, \widetilde{\mathcal{F}}^u_F$. Choose u_i nested in l' and converging to q as $i \to \infty$. Let $g_i \in \widetilde{\mathcal{F}}^u_F$ be the unstable leaf of $\widetilde{\mathcal{F}}^u_F$ through u_i, assumed to be regular. By the hypothesis of Case 3.1, q is not an ideal point of g_i for any i. The g_i are nested in F getting closer to g' and never cross to the other side of g' in F. Therefore the sequence g_i does not escape in F as $i \to \infty$ and since they are nested, they converge to a line leaf g of $\widetilde{\mathcal{F}}^u_F$. By construction g separates g' from l'; see Figure 18 (a). As $u_i \to q$ and g_i are uniform quasigeodesics in F, it follows that g_i has at least one of the ideal points near q [Th2, CDP]. Therefore at least one of the ideal points of g_i converges to q. Continuity of ideal points in rays of $\widetilde{\mathcal{F}}^u_F$ implies that g has one ideal point q. In the same way choose v_i nested in g' with $v_i \to q$ as $i \to \infty$ and let l_i be the stable leaf of $\widetilde{\mathcal{F}}^s_F$ through v_i. Then $l_i \to l$, where l is a line leaf of $\widetilde{\mathcal{F}}^s_F$ with ideal point q. As l_i intersects g', then no l_i has q as ideal point.

The line leaves l and l' share a common ideal point q and since they are uniform quasigeodesics, there is a wedge W of bounded thickness between them [Th2].

We claim that g does not intersect the wedge W. If l and l' share the other ideal point q_1 too, then any leaf α of $\widetilde{\mathcal{F}}^s_F$ between them can only have q, q_1 as ideal points. Since distinct prongs of α have to limit in different ideal points, α can have only two prongs (α is nonsingular) and hence has one ideal point q. If g intersects the wedge W in p, the stable leaf through p will have ideal point q, contradicting the hypothesis of Case 3.1. On the other hand suppose l, l' have other ideal points y, y' respectively with $y \neq y'$. If a leaf γ of $\widetilde{\mathcal{F}}^s_F$ intersecting W does not have q as an ideal point, then all of its ideal points are in the segment of $\partial_\infty F$ from y to y' and not containing q. By the uniform quasigeodesics property, it follows that γ cannot have points close to q in the d_e metric of $F \cup \partial_\infty F$. Hence there is a subwedge W_1 of W so that any stable leaf intersecting W_1 will have q as ideal point. By hypothesis this implies that g does not intersect W_1. Since g does not intersect either l or l', but has an ideal point q, it now follows that g does not intersect W either. This proves the claim. Notice that a priori it might be g and l share both ideal points; see Figure 14 (b).

The claim shows that g', g, l, l' are nested in this order in F as shown in Figure 18 (a). This implies that as $l_i \to l$, then there is i_0 so that $l_i \cap g \neq \emptyset, \forall i > i_0$. Also since $g_i \to g$, there is

$$i_1 > 0 \quad \text{so that} \quad \forall\ i\ >\ i_1, \ \ g_i \cap l_{i_0}\ \neq\ \emptyset.$$

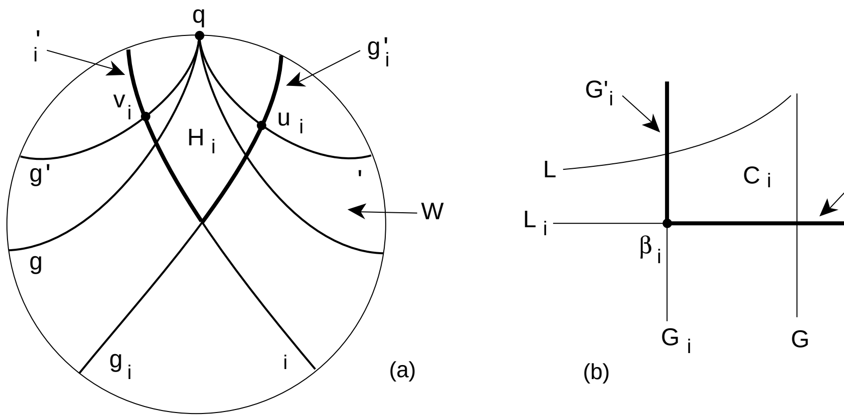

FIGURE 18. (a) Disjointness condition of the rays, producing a neighborhood of q in $F \cup \partial_\infty F$ bounded by $l'_i \cup (l_i \cap g_i) \cup g'_i$. (b) Escaping regions in $\widetilde{M}$.

As the l_i, g_i are nested it is easy to see that for any $i, j \geq i_2 = \max\{i_0, i_1\}$, then $l_i \cap g_j \neq \emptyset$.

Given $i > i_2$ let g'_i be the component of $(g_i - l_i)$ intersecting l. Let also l'_i be the component of $(l_i - g_i)$ intersecting g; see Figure 18 (a). Let

$$\alpha_i \;\; = \;\; l'_i \;\cup\; (l_i \cap g_i) \;\cup\; g'_i.$$

Then α_i separates F. Let H_i be the component of $F - \alpha_i$ so that $q \in \overline{H}_i$. The H_i are nested.

Claim. $\{\overline{H}_i\}, i \geq i_2$, defines a neighborhood system of q in $F \cup \partial_\infty F$.

In $\partial_\infty F$, q is an interior point of $\overline{H}_i \cap \partial_\infty F$, because none of the ideal points of g_i, l_i is q. Let

$$Z \;=\; \bigcap_{i \geq i_2} \; \overline{H}_i.$$

The ideal points of l'_i converge to q and the ideal points of g'_i converge to q. Since for any i, $\overline{H}_i \cap \partial_\infty F$ is an interval with boundary in these ideal points it follows that $Z \cap \partial_\infty F = \{q\}$.

Suppose now that $Z \neq \{q\}$. Then as $\{\overline{H}_i\}, i \geq 0$, is nested in $F \cup \partial_\infty F$, find $z_i \in \alpha_i$, $z_i \rightarrow z \in F$ and $i \rightarrow \infty$. Up to subsequence assume that (say)

$$z_i \;\in\; g'_i \;\cup\; (g_i \cap l_i) \;\subset\; g_i.$$

Since $g_i \rightarrow g$, it follows that $z \in g$. Since $l_i \cap g \rightarrow q$ as $i \rightarrow \infty$, choose i_3 big enough so that $z \notin \overline{H}_{i_3}$. Then $z \notin Z$, a contradiction. This proves the claim.

We now analyse the situation in the ambient 3-manifold $\widetilde{M}$, which in fact will be very similar to the setting in F. Let $l_i \subset L_i \in \widetilde{\mathcal{F}}^s$, $l \subset L \in \widetilde{\mathcal{F}}^s$, $g_i \subset G_i \in \widetilde{\mathcal{F}}^u$ and $g \subset G \in \widetilde{\mathcal{F}}^u$. Then $L_i \rightarrow L$, $G_i \rightarrow G$ as $i \rightarrow \infty$.

Since $l \cap g = \emptyset$, the argument of Lemma 7.2 shows that $L \cap G = \emptyset$.

We may assume that G_i and L_i are nonsingular for all i. Let U_i be the component of $\widetilde{M} - G_i$ containing G, and let Y_i be the component of $\widetilde{M} - L_i$ containing L. Let

$$L'_i = L_i \cap U_i, \; G'_i = G_i \cap Y_i \;\text{ and }\; \beta_i = L_i \cap G_i.$$

Then $D_i = L'_i \cup \beta_i \cup G'_i$ separates $\widetilde{M}$. Let C_i be the component of $\widetilde{M} - D_i$ containing the ray of g which has ideal point q in $F \cup \partial_\infty F$; see Figure 18 (b). The argument of Case 2.2.2.2 shows that $G_i - (G'_i \cup \beta_i)$ is the part of G_i which is converging to G and similarly $L_i - (L'_i \cup \beta_i) \to L$. This implies that D_i escapes compact sets in $\widetilde{M}$. Since orbits of $\widetilde{\Phi}$ in D_i are K-quasigeodesics, $diam_e(D_i) \to 0$. Consequently $diam(\overline{C}_i) \to 0$. The definitions of H_i and C_i imply that $\varphi(H_i) \subset C_i$ and Lemma 7.3 shows that $\overline{\varphi}(\overline{H}_i) \subset \overline{C}_i$. Since $\{H_i\}, i \geq i_2$, forms a neighborhood system of q in $F \cup \partial_\infty F$, this proves continuity of $\overline{\varphi}$ at q. This finishes Case 3.1.

Case 3.2. There are line leaves $l \in \widetilde{\mathcal{F}}^s_F$, $g \in \widetilde{\mathcal{F}}^u_F$, with l and g having q as ideal point in $\partial_\infty F$ and $l \cap g \neq \emptyset$.

This is the final case to be considered. In this case, particularly in subcase 3.2.2, we need to consider a continuous family of leaves, rather than just a countable set of leaves. Therefore parametrize l by arclength as

$$l: \; \{u_t\}, \; t \geq 0, \;\text{ with }\; u_t \to q \;\text{ when }\; t \to +\infty, \;\text{ so }\; l(t) \;=\; u_t.$$

In the same way let $g : \{v_t\}, t \geq 0$, with $v_t \to q$ when $t \to +\infty$. Suppose that $u_0 = v_0$. Here we only parametrize the rays of l and g starting in u_0 and having q as ideal point. Let

$$g_t \;\in\; \widetilde{\mathcal{F}}^u_F \;\text{ with }\; g_t \,\cap\, l \;=\; u_t \;\text{ and }\; l_t \;\in\; \widetilde{\mathcal{F}}^s_F \;\text{ with }\; l_t \cap g \;=\; v_t.$$

Let $g_t \subset G_t$ be leaves of $\widetilde{\mathcal{F}}^u$ and $l_t \subset L_t$ leaves of $\widetilde{\mathcal{F}}^s$. Let $\mathcal{A}'$ be the region of F bounded by $l \cup g$ and having q as its only ideal point; see Figure 19 (a).

We claim there is no singularity in the interior of $\mathcal{A}'$. From the point of view of $\widetilde{\mathcal{F}}^s_F$, the region $\mathcal{A}'$ has boundary consisting of a ray in the leaf l and a ray transverse to $\widetilde{\mathcal{F}}^s_F$ contained in g. Any ray of any leaf of $\widetilde{\mathcal{F}}^s_F$ or $\widetilde{\mathcal{F}}^u_F$ entirely contained in $\mathcal{A}'$ can only limit in q, hence has q as ideal point. Hence no leaf of $\widetilde{\mathcal{F}}^s_F, \widetilde{\mathcal{F}}^u_F$ can have two rays contained in $\mathcal{A}'$. If there is a singularity p in the interior of $\mathcal{A}'$, then at most one prong in its stable leaf (of $\widetilde{\mathcal{F}}^u_F$) can be contained in $\mathcal{A}'$. Therefore at least two prongs have to exit $\mathcal{A}'$, and they can only exit through the ray in g. This contradicts the fact that the stable leaf through p and the leaf g can intersect at most once. This proves the claim.

Given that $\mathcal{A}'$ has no singularity in the interior, then by taking a smaller $\mathcal{A}'$ we can assume it has no singularity in the boundary either and also that L_0 and G_0 are regular leaves.

For any $t > 0$, the leaf $(g_t - l)$ has a unique component g'_t contained in $\mathcal{A}'$. The rays g'_t have ideal point q for any t and do not have any singularity. In the same way $(l_t - g)$ has a nonsingular ray contained in $\mathcal{A}'$ with ideal point q. We will consider continuity on the side of l_0 containing $l_t, t > 0$.

Suppose first that L_t escapes in $\widetilde{M}$ when $t \to \infty$. Fix $\epsilon > 0$. For t sufficiently big $diam_e(\overline{L}_t) < \epsilon$. Also the wedge in F between l_t and l_0 has bounded thickness. Then continuity at q follows as in Case 2.1.1. Suppose from now on that $L_t \to L$ as $t \to \infty$.

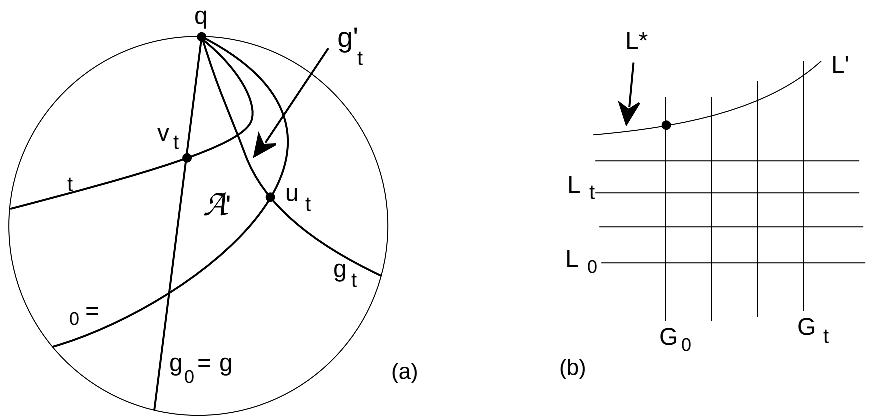

FIGURE 19. (a) Intersecting leaves of $\widetilde{\mathcal{F}}^s_F, \widetilde{\mathcal{F}}^u_F$ with same ideal point q in $F \cup \partial_\infty F$. (b) The case $G_0 \cap L \neq \emptyset$.

Case 3.2.1. $G_0 \cap L \neq \emptyset$; see Figure 19 (b).

In this subcase we will only use the countable set of leaves $L_i, G_i, i \in \mathbf{N}$ (that is, only those L_t for which t is a natural number). Let $w_0 \in G_0 \cap L$. Notice that

$$(G_0 \cap L_i) \cap F \;\;=\;\; (G_0 \cap F) \cap (L_i \cap F) \;\;=\;\; g_0 \cap l_i \;\;=\;\; v_i.$$

As

$$G_0 \;\cap\; L_i \;\rightarrow\; G_0 \;\cap\; L, \;\;\text{then}\;\; \Theta(v_i) \;\rightarrow\; \Theta(w_0).$$

Since Θ is injective when restricted to F and v_i escapes in F as $i \rightarrow \infty$, it follows that $\Theta(w_0) \in \partial\Theta(F)$. In addition $g_0 \subset \widetilde{W}^u(w_0) \cap F$, hence by Proposition 5.1, $\widetilde{W}^s(w_0) = L$ has a line leaf L' with $\Theta(L') \subset \partial\Theta(F)$. Notice that $L_i \rightarrow L'$. Therefore $L \cap F = \emptyset$ and l_i escapes in F as $i \rightarrow \infty$. Let L^* be the component of $L' - (L' \cap G_0)$ which does not intersect any $G_i, i > 0$; see Figure 20 (b). Choose

$$w_j \in \widetilde{W}^{ss}(w_0) \cap L^*, \quad w_j \;\;\text{nested}\;\;\text{and}\;\;\text{escaping}\;\;\text{in}\;\; \widetilde{W}^{ss}(w_0) \cap L^*.$$

Let $S_j = \widetilde{W}^u(w_j)$. As $\Theta(L') \subset \partial\Theta(F)$, then for any $j \in \mathbf{N}$, it follows that $S_j \cap F \neq \emptyset$. The intersection is a leaf $s_j = S_j \cap F$ of $\widetilde{\mathcal{F}}^u_F$; see Figure 20 (a). For $j = 0$, $s_0 = g_0$ and $S_0 = G_0$.

We claim that any $s_j \subset F$ has ideal point q in $\partial_\infty F$. Fix j and find $i(j)$ such that $L_{i(j)} \cap S_j \neq \emptyset$; see Figure 20 (b). Then

$$L_{i(j)}, \; L, \; S_0, \; S_j \quad \text{form a rectangle} \quad \mathcal{R} \;\subset\; \widetilde{M}$$

and there is no singularity in the interior of $\mathcal{R}$. The rectangle projects in F to a region bounded by a ray in s_j, a ray in $s_0 = g_0$ and a segment in $l_{i(j)}$, because $L \cap F = \emptyset$. This region has no singularity, so by Lemma 7.4, it follows that the ideal points of s_j and s_0 are the same, proving the claim.

Rather than checking whether S_j escapes in $\widetilde{M}$ or not we do a proof which deals with both cases. We may assume that S_j is regular for all j. For each j, consider

$$i(j) > j \quad \text{with} \quad L_{i(j)} \;\cap\; F \;\cap\; S_j \;\neq\; \emptyset$$

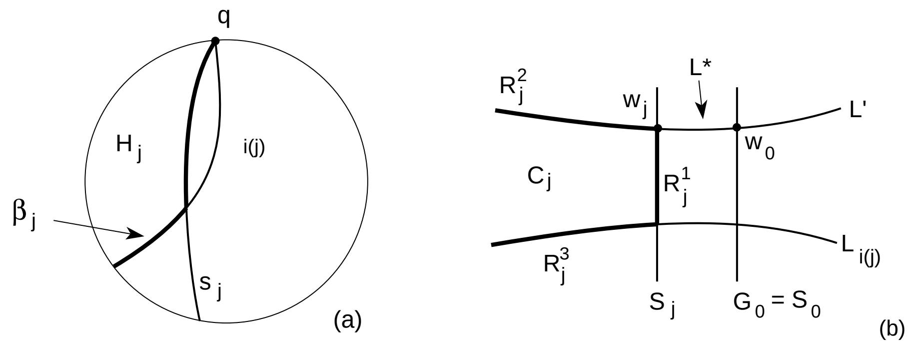

FIGURE 20. (a) Neighborhood of q in $F \cup \partial_\infty F$. (b) Neighborhood in $\widetilde{M}$.

and suppose for simplicity that $L_{i(j)}$ is regular. This implies that s_j intersects $l_{i(j)}$ in a single point; see Figure 20 (a). Consider the curve $\beta_j \subset F$ consisting of the union of:

(1) the ray of $s_j - l_{i(j)}$ with q as ideal point,
(2) $s_j \cap l_{i(j)}$ and
(3) the ray of $l_{i(j)} - s_j$ with ideal point $\neq q$; see Figure 20 (a).

Then β_j separates F. Let H_j be the component of $F - \beta_j$ not containing l_0. In addition let

$$R_j^1 \quad \text{be the component of} \quad S_j - (L' \cup L_{i(j)}) \quad \text{between} \quad L' \text{ and } L_{i(j)}.$$

Let R_j^2 (respectively R_j^3) be the component of $L' - S_j$ (respectively the component of $L_{i(j)} - S_j$) not intersecting S_0; see Figure 20 (b). Let

$$D_j \;=\; R_j^2 \,\cup\, (S_j \cap L') \,\cup\, R_j^1 \,\cup\, (S_j \cap L_{i(j)}) \,\cup\, R_j^3;$$

see Figure 20 (b). Then D_j separates $\widetilde{M}$. Let C_j be the component of $\widetilde{M} - D_j$ not containing G_0; see Figure 20 (b). We can choose $i(j)$ carefully so that the $\{C_j \cup D_j\}, j \in \mathbf{N}$, form a nested family of subsets in $\widetilde{M}$, that is, they decrease as j increases. These definitions imply that $D_j \cap F = \beta_j$ - notice that $L' \cap F = \emptyset$. In addition $\varphi(H_j) \subset C_j$.

We claim that C_j escapes in $\widetilde{M}$ as $j \to \infty$. Since $\widetilde{\Phi}_{\mathbf{R}}(w_j)$ escapes in L' as $j \to \infty$ and L' is properly embedded in $\widetilde{M}$, it follows that R_j^2 escapes in $\widetilde{M}$. Since $L_{i(j)} \to L'$ as $j \to \infty$, then if R_j^3 does not escape in $\widetilde{M}$ as $j \to \infty$, it has to limit in some $z \in L'$. But for j big enough S_j separates z from R_j^3, a contradiction. Finally we consider R_j^1. Let U_j' be the component of $\widetilde{M} - (L' \cup L_{i(j)})$ containing R_j^1 and let

$$U_j \;=\; U_j' \,\cup\, L' \,\cup\, L_{i(j)} \;=\; \text{closure of } U_j'.$$

Fix $j_0 \in \mathbf{N}$. Then $R_j^1 \subset U_{j_0}$ for any $j > j_0$, hence any limit point of $\{R_j^1\}, j \in \mathbf{N}$, has to be in U_{j_0}. Since this is true for any $j_0 \in \mathbf{N}$, then the limit points of $\{R_j^1\}, j \in \mathbf{N}$, have to be in

$$\bigcap_{j \in \mathbf{N}} U_j \;=\; L'.$$

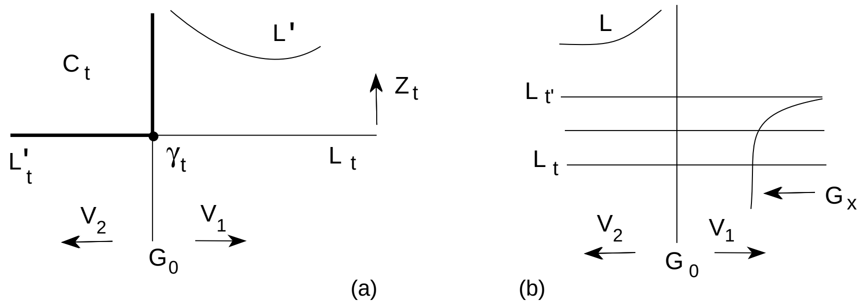

FIGURE 21. (a) The case $L \subset V_1$. (b) The impossible case $L \subset V_2$.

The last equality follows from $L_{i(j)} \to L'$. But as before any $z \in L'$ is separated from R^1_j by $S_{j'}$ for any $j' < j$, so again this cannot happen. We conclude that D_j escapes in $\widetilde{M}$. It follows that $diam_e(\overline{C}_j) \to 0$ as $j \to \infty$. In addition the wedge between $l_{i(j)}$ and l_0 in F has bounded thickness, hence the wedge between $\varphi(l_{i(j)})$ and $\varphi(l_0)$ in $\varphi(F)$ does also. A proof as in Case 2.1.1 shows continuity of $\overline{\varphi}$ in q in that side of l_0.

Case 3.2.2. $G_0 \cap L = \emptyset$.

Let V_1 be the component of $\widetilde{M} - G_0$ containing $G_t, t > 0$, and let V_2 be the other component of $\widetilde{M} - G_0$.

Case 3.2.2.1. $L \subset V_1$.

We refer to Figure 21 (a). Recall that L' is the line leaf of L with $L_t \to L'$ when $t \to +\infty$. Consider only those t for which L_t is regular. Let Z_t be the component of $\widetilde{M} - L_t$ containing L'. Let

$$L'_t = L_t \cap V_2, \quad \gamma_t = L_t \cap G_0, \quad R_t = G_0 \cap Z_t \quad \text{and} \quad D_t = L'_t \cup \gamma_t \cup R_t.$$

Then D_t separates $\widetilde{M}$. Since $L'_t \subset V_2$, no part of L'_t can converge to points in L. Hence L'_t escapes in $\widetilde{M}$. The γ_t also escape in G_0 as $t \to \infty$, hence the D_t also escape in $\widetilde{M}$. Let C_t be the component of $\widetilde{M} - D_t$ not containing L' and let $H_t = C_t \cap F$. The proof of continuity in this case is analogous to that of Case 3.2.1.

Case 3.2.2.2. $L \subset V_2$.

We will show this case cannot happen. Recall the continuous maps $\eta_-, \eta_+ : \widetilde{M} \to S^2_\infty$:

$$\eta_-(w) = \lim_{t \to -\infty} \widetilde{\Phi}_t(w) \ \in S^2_\infty, \qquad \eta_+(w) = \lim_{t \to +\infty} \widetilde{\Phi}_t(w) \ \in S^2_\infty.$$

Also if γ is an orbit of $\widetilde{\Phi}$ we define $\eta_-(\gamma), \eta_+(\gamma)$; if $Y \in \widetilde{\mathcal{F}}^s$, there is $\eta_+(Y)$ and if $Y \in \widetilde{\mathcal{F}}^u$, there is $\eta_-(Y)$.

Notice first that $G_0 \cap L_t \neq \emptyset$ for any $t \geq 0$, because $g_0 \cap l_t = u_t$. In addition $G_0 \cap L_t$ escapes in G_0 as $t \to +\infty$, for otherwise this intersection has to limit in an orbit in L', so $G_0 \cap L' \neq \emptyset$, a contradiction.

We refer to Figure 21 (b). Fix $x > 0$. Then

$$G_x \cap L_0 \cap F \ = \ g_x \cap l_0 \ = \ u_x,$$

hence G_x intersects L_t for any $t > 0$ sufficiently small. We are using here the fact that l_0 is regular.

The first option is that there is $t > 0$ so that $G_x \cap L_t = \emptyset$. Let $t' \geq 0$ be the smallest so that $G_x \cap L_{t'} = \emptyset$ - this is where we use the continuous family of leaves L_t. The orbits $G_x \cap L_t$ escape in G_x as $t \to t'$, $t < t'$, for otherwise $G_x \cap L_{t'} \neq \emptyset$, a contradiction. Therefore

$$diam_e(G_x \cap L_t) \to 0 \quad \text{and} \quad (G_x \cap L_t) \to \eta_-(G_x) \quad \text{in } \widetilde{M} \cup S^2_\infty \text{ as } t \to t', \quad t < t',$$

because all orbits in G_x are backward asymptotic. In particular

$$\eta_+(G_x \cap L_t) \ \to \ \eta_-(G_x), \quad \text{as} \quad t \to t' \ \text{ and } \ t < t'.$$

As $G_x \cap F = g_x$ and

$$g_x(t) = (G_x \cap F) \cap l(t) = (G_x \cap F) \cap (L_t \cap F) = (G_x \cap L_t) \cap F \in G_x \cap L_t,$$

it follows that

$$\varphi(g_x(t)) \ \to \ \eta_-(G_x) \ \text{ in } \ \widetilde{M} \cup S^2_\infty \ \text{ as } \ t \to t'$$

and consequently the ray $\varphi(g_x)$ limits to $\eta_-(G_x)$ in $\widetilde{M} \cup S^2_\infty$. But

$$\eta_+(G_x \cap L_t) \ = \ \eta_+(L_t) \ = \ \eta_+(G_0 \cap L_t) \ \to \ \eta_+(G_0 \cap L_{t'}), \text{ as } t \to t'.$$

Since $\eta_+(G_x \cap L_t) \to \eta_-(G_x)$ also, we conclude that $\eta_-(G_x) = \eta_+(G_0 \cap L_{t'})$. So $\varphi(g_x)$ limits to $\eta_+(G_0 \cap L_{t'})$.

Notice that $G_0 \cap F = g_0$ and the orbits $(G_0 \cap L_t)$ escape in G_0 as $t \to \infty$. Therefore $\varphi(g_0(t))$ converges to $\eta_-(G_0)$. In other words the ray $\varphi(g_0)$ limits to $\eta_-(G_0)$ in $\widetilde{M} \cup S^2_\infty$. Since g_0, g_x have rays which are a bounded distance from each other in F, then $\varphi(g_0), \varphi(g_x)$ have the same ideal points in S^2_∞. Putting this all together we conclude that

$$\eta_+(G_0 \cap L_{t'}) \ = \ \eta_-(G_0), \quad \text{which implies that} \quad \eta_+(G_0 \cap L_{t'}) \ = \ \eta_-(G_0 \cap L_{t'}).$$

This contradicts the fact that $G_0 \cap L_{t'}$ is a quasigeodesic in $\widetilde{M}$. Therefore the first option cannot occur.

The second option is that $G_x \cap L_t \neq \emptyset$ for any $t \geq 0$. Recall that Z_0 is the component of $\widetilde{M} - L_0$ containing L. If $G_x \cap L_t$ does not escape in G_x as $t \to \infty$, then $G_x \cap L' \neq \emptyset$, a contradiction to $L' \subset V_2$. Hence the half leaf $T_x = G_x \cap Z_0$ intersects the same set of stable leaves as $T_0 = G_0 \cap Z_0$. The same is true for any half leaf $T_{x'} = G_{x'} \cap Z_0, 0 \leq x' \leq x$, thus producing a product region. This contradicts Theorem 4.10, so this option cannot happen either. We conclude that Case 3.2.2.2 cannot happen. This finishes the analysis of Case 3.2.

This completes the proof of continuity and hence of Theorem 7.1. □

8. The examples

In this section we describe a class of examples of quasi-isometric singular foliations in hyperbolic 3-manifolds, as well as examples of finite depth foliations in closed hyperbolic manifolds with good asymptotic properties in the universal cover. The examples come from Mosher's construction of pseudo-Anosov flows transverse to finite depth foliations [Mo4].

First we need to describe round handles, which were created by Asimov [As]. A round handle is a standard neighborhood of a hyperbolic periodic orbit. We consider a modified round handle as defined by Mosher [Mo3]: Let O be an octagon in the plane with sides $O_1, ..., O_8$. Let $T = O \times \mathbf{S}^1$ which has a structure of sutured

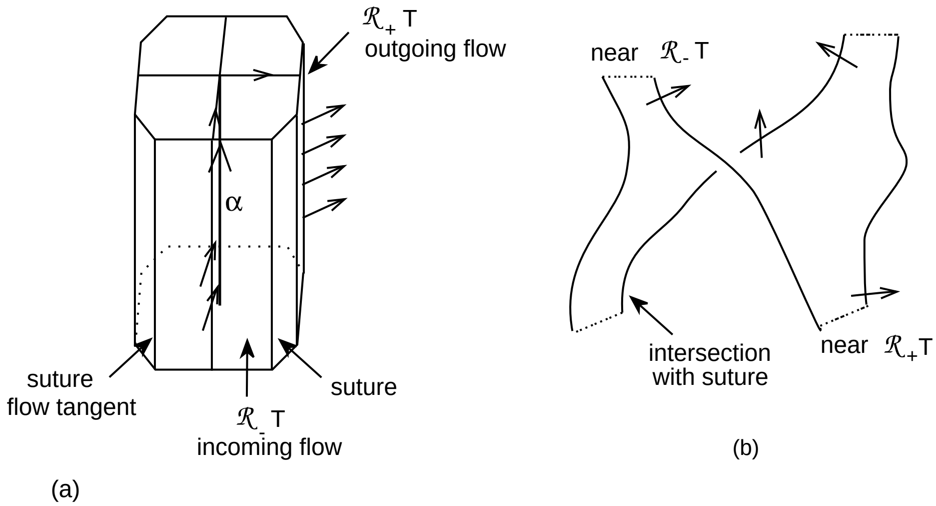

FIGURE 22. (a) The flow in the round handle. (b) The leaves of the foliation $\mathcal{G}_T$.

manifold (see [Ga1, Ga2, Ga3] for sutured manifolds) where the plus boundary is $\mathcal{R}_+T = (O_1 \times \mathbf{S}^1) \cup (O_5 \times \mathbf{S}^1)$, the minus boundary is $\mathcal{R}_-T = (O_3 \times \mathbf{S}^1) \cup (O_7 \times \mathbf{S}^1)$ and the sutures $\Delta = (O_2 \times \mathbf{S}^1) \cup (O_4 \times \mathbf{S}^1) \cup (O_6 \times \mathbf{S}^1) \cup (O_8 \times \mathbf{S}^1)$. There is a semiflow Φ_1 in T having a unique orbit which stays for all time, forwards and backwards, in T. This orbit α is periodic and hyperbolic. The sets $W^s(\alpha), W^u(\alpha)$ are properly embedded annuli in T with boundary components in $\mathcal{R}_-T, \mathcal{R}_+T$, respectively. The flow is incoming along $\mathcal{R}_-T$, outgoing along $\mathcal{R}_+T$ and tangent to Δ; see Figure 22 (a).

In T consider a foliation $\mathcal{G}_T$ so that components of $\mathcal{R}_-T, \mathcal{R}_+T$ are annuli leaves and all other leaves of $\mathcal{G}_T$ are homeomorphic to ideal quadrilaterals, which intersect ∂T transversely and only in the sutures ($= \Delta T$), and so that $\mathcal{G}_T$ is a fibration in $T - (\mathcal{R}_-T \cup \mathcal{R}_+T)$. See Figure 22 (b), where for simplicity we consider the picture in the universal cover of T. The interior leaves are noncompact and spiral towards the boundary leaves. Then $\mathcal{G}_T$ is a depth one foliation [Ga1]. The induced foliation in $O_2 \times \mathbf{S}^1$ has only the boundary components as compact leaves and the leaves in the interior spiral towards the boundary leaves in different directions, that is, this is not a Reeb foliated annulus; similarly for $(O_4 \times \mathbf{S}^1), (O_6 \times \mathbf{S}^1)$ and $(O_8 \times \mathbf{S}^1)$. The construction is done so that $\mathcal{G}_T$ is transverse to Φ_1.

Now consider S a compact surface with 4 boundary components. Consider the product flow in $S \times I$, also denoted by Φ_1. Choose two disjoint embedded arcs β_1, β_2 connecting the 4 boundary components of S two by two. Put a depth one foliation in $S \times I$ as follows: first put a product foliation in $(S - N(\beta_1 \cup \beta_2)) \times I$. Let $h : I \to I$ be a homeomorphism with $h(0) = 0, h(1) = 1$ and $h(t) < t$ for every $t \in (0, 1)$. Let $\partial N(\beta_1) = \zeta \cup \zeta'$. Put a foliation in $N(\beta_1) \times I$ so that all leaves are compact disks from $\zeta \times I$ to $\zeta' \times I$ with one boundary component $\zeta \times \{t\}$ and the other component $\zeta' \times \{h(t)\}$. This is a depth one foliation. Similarly for $N(\beta_2) \times I$. Glue these together to produce a depth one foliation $\mathcal{G}_{S\times I}$ in $S \times I$. This foliation is a fibration over the circle in $S \times (0, 1)$ and the only compact leaves are $(S \times \{0\}), (S \times \{1\})$. Choose $\mathcal{G}_{S\times I}$ so that it is transverse to Φ_1. Now glue the components of the suture in T to the components of $\partial S \times I$ so as to produce a manifold M_1 with two boundary components $\partial_- M_1$ and $\partial_+ M_1$. The glueing can be done so that foliation and flow match along the glueing set inducing a semiflow Φ_1 in M_1 and a depth one foliation $\mathcal{G}_{M_1}$ in M_1 so that: $\partial_- M_1, \partial_+ M_1$ are the only compact leaves of $\mathcal{G}_{M_1}$, the foliation $\mathcal{G}_{M_1}$ is transverse to Φ_1 and Φ_1 is outgoing along $\partial_+ M_1$ and incoming along $\partial_- M_1$.

The annuli $W^s(\alpha), W^u(\alpha)$ in T become properly embedded annuli in M_1 which are respectively the stable B^s_0 and unstable B^u_0 branched surfaces in M_1. B^s_0, B^u_0 induce laminations λ_-, λ_+ in ∂M_1. Each of λ_-, λ_+ is a union of two disjoint simple closed curves in ∂M_1.

Now glue $\partial_+ M_1$ to $\partial_- M_1$ by a homeomorphism g to produce a closed manifold M. Clearly M has a depth one foliation $\mathcal{G}$ with a unique compact leaf denoted by R and Φ_1 in M_1 induces a flow in M which is transverse to $\mathcal{G}$. This flow in M is also denoted by Φ_1. Choose g so that $g(\lambda_+)$ and λ_- intersect efficiently and bind $\partial_- M_1$. Under these conditions it follows that M is atoroidal [Mo3] and consequently hyperbolic [Th2, Th4, Mor]. In [Mo3] it is shown that R is not a fiber of a fibration of M over the circle. In particular $\mathcal{G}$ is not a perturbation of a fibration and Φ_1 is not a suspension flow. This class of examples was created by Mosher and is described in detail in [Mo2, Mo3].

We now make extensive use of the construction of pseudo-Anosov flow transverse or almost transverse to $\mathcal{G}$ which was carried out in detail in [Mo4]. The key fact here is that in M all orbits of Φ_1 intersect R transversely, with the exception of α. Mosher's construction of the pseudo-Anosov flow [Mo4] first produces a pair of good stable and unstable branched surfaces in M. In this case the procedure first cuts M along R and then decomposes M_1 into a product sutured manifold (which is $S \times I$) and the round handle T. The pair of branched surfaces B^s_0, B^u_0 intersect transversely and form a "dynamic pair" in T; see [Mo4], section 4.5. Once the glueing $\partial_+ M_1 \to \partial_- M_1$ is done, then B^u_0 is a branched surface with boundary in the interior of M and one can flow B^u_0 forwards along the Φ_1 flow in M to extend this branched surface - eventually pieces which are very near other pieces of the extended B^u_0 are tangentially collapsed together to produce a compact unstable branched surface B^u, which is properly embedded in M. The flow Φ_1 is slightly adjusted to be tangent to B^u. This produces a generalized flow Φ_2: it is not uniquely integrable. For instance in the branch set of B^u, the flow moves forward from the two sheeted side to the one sheeted side. Then orbits of Φ_2 in B^u are uniquely defined for all forward time, but have many divergent paths in backward time; see section 2.4 of [Mo4]. In the same way B^s is constructed. Orbits of Φ_2 in B^s have unique backward orbits, but forward orbits are not uniquely integrable along the branch set of B^s. In any case <u>all</u> orbits of Φ_2 are transverse to $\mathcal{G}$. The surfaces B^s, B^u are transverse to each other (see Figure 23 (a)) and their intersection τ is a graph. The flows in B^s, B^u induce a nonintegrable flow in τ - that is, τ is an orientable graph; see Figure 23 (b).

The surfaces B^s, B^u form a dynamic pair in the closed manifold M and Φ_2 is tangent to both of them. In particular this implies that $M - (B^s \cup B^u)$ is a disjoint union of pinched tetrahedra and solid tori; see section 2.4 of [Mo4]. Each corner orbit of a torus piece is a periodic orbit of Φ_2. Since B^s, B^u form a dynamic pair in M, then in section 3.3 of [Mo4], Mosher constructs a uniquely integrable flow Φ_3 associated to B^s, B^u. This flow is called a pA flow [Mo4] and has stable and unstable laminations which are contained in neighborhoods of B^s and B^u.

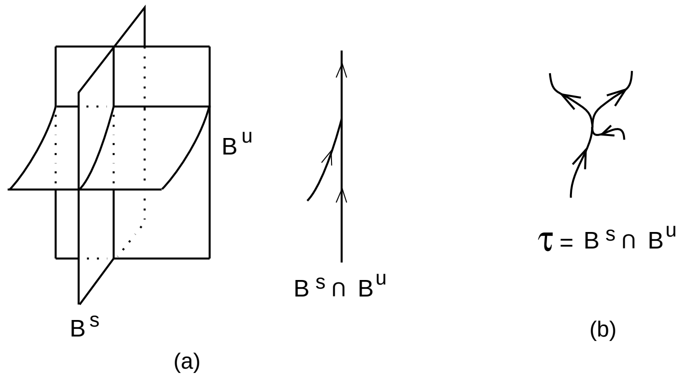

Figure 23. (a) Intersection of stable/unstable branched surfaces. (b) The oriented flow in $\tau = B^s \cap B^u$.

Roughly Φ_3 is constructed as follows: $\tau = B^s \cap B^u$ is an oriented graph and Φ_2 is a nonuniquely integrable flow in τ; see Figure 23 (b). The flow Φ_3 is constructed by essentially transforming the nonuniquely integrable dynamics of Φ_2 in τ into a uniquely integrable flow. The flow Φ_3 has nonwandering set which is the union of finitely many pseudohyperbolic orbits all contained in torus pieces, finitely many attracting and repelling periodic orbits which are contained in the boundary of the torus pieces and finally a hyperbolic set Z which is contained in a neighborhood $N(\tau)$ of the graph τ. The orbits of Φ_3 in Z are in one-to-one correspondence with oriented paths in τ. The flow Φ_3 is still transverse to $\mathcal{G}$.

Finally the pseudo-Anosov flow Φ_4 is obtained from Φ_3 by collapsing the complementary regions of $N(B^s \cup B^u)$; see section 3.4 of [Mo4]: a pinched tetrahedron is collapsed to a segment and a torus piece is collapsed into a pseudohyperbolic closed orbit. This collapsing preserves flow lines and produces a pseudo-Anosov flow Φ_4 in M. The collapsing operation is the inverse operation of the double DA blow up operation (the blow up operation is described in detail in [Mo1]). In the blow up operation the singular leaves of the stable and unstable foliations $\mathcal{F}^s, \mathcal{F}^u$ of Φ_4 split up to produce nonsingular laminations Λ^s, Λ^u of Φ_3. Notice that the dynamics of Φ_4 is essentially encoded by that of Φ_3 in $N(\tau)$. It is in the blow down operation $\Phi_3 \to \Phi_4$ that the flow may lose the property of being transverse to $\mathcal{G}$; see sections 3.5, 3.6 and 3.7 of [Mo4]. It may be that Φ_4 is only "almost transverse" to $\mathcal{G}$: this means that one needs to blow up a finite collection $\gamma_1, ..., \gamma_n$ of pseudohyperbolic orbits of Φ_4 into a collection of annuli and adjust the flow accordingly to produce a new flow transverse to $\mathcal{G}$. See a detailed definition in section 3.5 of [Mo4]. We will analyse the almost transversality of Φ_4 more carefully below.

It will be important to understand the oriented orbits of Φ_2 in $\tau = B^s \cap B^u$. First notice that since $\alpha = B^s_0 \cap B^u_0$, then α is a closed orbit of Φ_2 in τ. Let γ be an orbit of Φ_2 in τ. If γ enters T, then it is either eventually contained in α or, by the hyperbolic structure of Φ_2 in T, it follows that γ eventually has to leave T and γ intersects R transversely in an outgoing branch of τ. If a point in γ is in $S \times I$, then because Φ_1 was a product flow in $S \times I$ and Φ_2 is obtained by tangential collapsing of Φ_1, it follows that γ also intersects R transversely. The same is true for the negative direction. Putting all of this together, it follows that either γ is contained in α or γ will intersect R transversely.

Before concluding what this implies for orbits of Φ_4, we have to analyse whether Φ_4 is transverse to $\mathcal{G}$ or not. In order to do that we first consider torus pieces. Since $g(\lambda_+)$ intersects each component of λ_-, it follows that B^u_0 is extended by the flow Φ_1 to intersect B^s_0. Flowing forwards along Φ_1 will bring this unstable branched surface very near α and the original B^u_0 and eventually collapse pieces of it with the original B^u_0 (here Φ_2 will collapse forward orbits). The branch set created near α is transverse to α. This shows that α (as a full orbit) cannot be a corner orbit of a torus piece of $M - (B^s \cup B^u)$. As α is the only orbit of Φ_2 in τ which does not intersect R, it follows that all orbits of Φ_2 in τ which are corner orbits of torus pieces must transversely intersect R. The same is now true for the flow Φ_3 in $N(\tau)$.

The question as to whether Φ_4 is transverse to $\mathcal{G}$ or only almost transverse to $\mathcal{G}$ depends on the structure of the foliation $\mathcal{G}$ induced in the torus pieces of $M - N(B^s \cup B^u)$; see sections 3.5 and 3.6 of [Mo4]. Let V be such a torus piece. Then there are compact leaves of $\mathcal{G}$ induced on V and there are two options: either the compact leaves are meridian disks in V or they are peripheral annuli in V, which do not intersect the corner orbits. In the first case Φ_4 will be transverse to $\mathcal{G}$ and in the second case Φ_4 will not be transverse to $\mathcal{G}$ and will only be almost transverse to $\mathcal{G}$. In our situation the corner orbits (of Φ_3) in V all intersect R transversely. Since $R \cap V$ is compact, it follows that the compact leaf R cannot intersect V in a union of boundary parallel annuli (which would not intersect corner orbits). Therefore R has to intersect V in a union of meridian disks. Sections 3.5 and 3.6 of [Mo4] then imply that the collapsing of pinched tetrahedra and torus pieces can be done always transverse to the flow Φ_3. The conclusion is fundamental for us:

Fact 1. The pseudo-Anosov flow Φ_4 produced by Mosher's construction is transverse to $\mathcal{G}$ (and not just almost transverse).

Once we know this, we can now study orbits of Φ_4. Every orbit of Φ_2 in τ except for α will intersect R transversely. Therefore, except for α, every orbit of Φ_3 which is entirely contained in $N(\tau)$ has to intersect R transversely. Since orbits of Φ_4 are obtained by collapsing those of Φ_3 together and the dynamics of Φ_4 is entirely encoded by the nonwandering set of Φ_3 in $N(\tau)$ it now follows that:

Fact 2. Except for α, every orbit of Φ_4 intersects R transversely.

Let $\Phi = \Phi_4$ be the pseudo-Anosov flow thus constructed. Notice that α is a closed orbit which does not intersect R. Since R is not a fiber of a fibration of M over the circle, then Φ is not obtained as the suspension flow of a pseudo-Anosov homeomorphism.

Theorem 8.1. *Let M be as above, $\mathcal{G}$ a depth one foliation constructed as above and Φ a pseudo-Anosov flow transverse to $\mathcal{G}$ constructed as above. Then the stable and unstable foliations $\mathcal{F}^s, \mathcal{F}^u$ of Φ are quasi-isometric singular foliations in M. In addition leaves of $\widetilde{\mathcal{G}} \subset \widetilde{M} = \mathbf{H}^3$ extend continuously to the sphere at infinity S^2_∞.*

Proof. As seen before M is hyperbolic. Since Φ is a pseudo-Anosov flow transverse to a Reebless finite depth foliation $\mathcal{G}$ in a closed hyperbolic 3-manifold, then we proved in [Fe-Mo] that Φ is a quasigeodesic flow. Theorem 3.8 then shows that $\mathcal{F}^s$ is a quasi-isometric singular foliation if and only if $\widetilde{\mathcal{F}}^s$ has Hausdorff leaf space. Theorems 4.9 and 4.8 then show that if $\widetilde{\mathcal{F}}^s$ does not have Hausdorff leaf space, then there are closed orbits γ_1, γ_2 of Φ (which may not be indivisible) so that γ_1 is freely homotopic to the inverse of γ_2; see the conclusion statement after these theorems. The discussion following Theorem 4.9 shows that γ_1, γ_2 can be chosen so that there are lifts $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ to $\widetilde{M}$, satisfying: $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ are the corner orbits of a lozenge $\mathcal{B}$. Let us check whether this is possible.

First suppose that γ_1 intersects R. Then its intersection number with R is positive and since $\gamma_2 \cong (\gamma_1)^{-1}$ it follows that the intersection number of γ_2 and R is negative. This is impossible since R is transverse to Φ. Hence γ_1, γ_2 do not intersect R and by fact 2 above, it follows that $\gamma_1 = \alpha^n$, $\gamma_2 = \alpha^m$, where $nm < 0$. The free homotopy is realized by an element $h \in \pi_1(M)$ so that $h[\alpha]^n h^{-1} = [\alpha]^m$. Since M is Haken this implies that $|n| = |m| = 1$ [Ja-Sh]. Then $h^2, [\alpha]$ generate an abelian subgroup of $\pi_1(M)$.

We can choose h so that $h(\widetilde{\gamma}_1) = \widetilde{\gamma}_2$. If $h(\mathcal{B}) \cap \mathcal{B} = \emptyset$, then it is easy to see that for any $i \neq j \in \mathbf{Z}$, $h^i(\mathcal{B}) \cap h^j(\mathcal{B}) = \emptyset$. Since $[\alpha](\mathcal{B}) = \mathcal{B}$, it follows that $h^2, [\alpha]$ generate a $\mathbf{Z} \oplus \mathbf{Z}$ subgroup of $\pi_1(M)$, a contradiction to M being hyperbolic. The other option is that $h(\mathcal{B}) = \mathcal{B}$. This produces an orbit $\delta \subset \mathcal{B}$ of $\widetilde{\Phi}$ so that $h(\delta) = \delta$. Then $\widetilde{W}^s(\delta) \cap \widetilde{W}^u(\widetilde{\gamma}_1) \neq \emptyset$ and is equal to an orbit δ'. But then $h^2(\widetilde{\gamma}_1) = \widetilde{\gamma}_1$ and $h^2(\delta) = \delta$ imply $h^2(\delta') = \delta'$. This produces two periodic orbits in $W^u(\alpha)$, also a contradiction.

We conclude that $\widetilde{\mathcal{F}}^s$ has Hausdorff leaf space and consequently that $\mathcal{F}^s$ is a quasi-isometric singular foliation; similarly for $\mathcal{F}^u$. This proves the first assertion of the theorem. Given that $\mathcal{F}^s, \mathcal{F}^u$ are quasi-isometric and Φ is transverse to $\mathcal{G}$, then Theorem 7.1 proves that leaves of $\widetilde{\mathcal{G}}$ extend continuously to S^2_∞. This finishes the proof. $\square$

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