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ON THE DISTRIBUTION OF THE LENGTH OF THE LONGEST INCREASING SUBSEQUENCE OF RANDOM PERMUTATIONS

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1. Introduction

Let S_N be the group of permutations of $1, 2, \dots, N$. If $\pi \in S_N$, we say that $\pi(i_1), \dots, \pi(i_k)$ is an increasing subsequence in π if $i_1 < i_2 < \cdots < i_k$ and $\pi(i_1) < \pi(i_2) < \cdots < \pi(i_k)$. Let $l_N(\pi)$ be the length of the longest increasing subsequence. For example, if $N = 5$ and π is the permutation 5 1 3 2 4 (in one-line notation: thus $\pi(1) = 5$, $\pi(2) = 1, \dots$), then the longest increasing subsequences are 1 2 4 and 1 3 4, and $l_N(\pi) = 3$. Equip S_N with uniform distribution,

$$q_{n,N} = Prob(l_N \le n) = \frac{f_{N,n}}{N!},$$

where $f_{N,n} = \#(\text{permutations } \pi \text{ in } S_N \text{ with } l_N \le n)$. The goal of this paper is to determine the asymptotics of $q_{n,N}$ as $N \to \infty$. This problem was raised by Ulam in the early 60's [Ul], and on the basis of Monte Carlo simulations, he conjectured that the limit

$$c \equiv \lim_{N\to\infty} \frac{1}{\sqrt{N}} \mathbb{E}_N(l_N) \tag{1.1}$$

exists. (Here $\mathbb{E}_N(\cdot)$ denotes the expectation value with respect to the distribution function $q_{n,N}$.) The problem of proving the existence of this limit and the computation of c has become known as "Ulam's problem". An argument of Erdös and Szekeres [ES] shows that $\mathbb{E}_N(l_N) \ge \frac{1}{2}\sqrt{N-1}$, so that if the limit exists, then $c \ge \frac{1}{2}$. Subsequent numerical work by Baer and Brock [BB] in the late 60's suggested that value of c is 2. The existence of the limit was rigorously established by Hammersley [Ha] in 1972. In [LS], Logan and Shepp proved that $c \ge 2$ and simultaneously Vershik and Kerov [VK1] (see also [VK2]) showed that $c = 2$, thus settling Ulam's problem. Alternative proofs of Ulam's problem are due to Aldous and Diaconis [AD], Seppäläinen [Se1] and Johansson [Jo1]. Over the years, various

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conjectures have been made concerning the variance $Var(l_N)$ of l_N, and Monte Carlo simulations of Odlyzko and Rains beginning in 1993 indicated that

$$\lim_{N\to\infty} \frac{1}{N^{1/3}} Var(l_N) = c_0 \tag{1.2}$$

for some numerical constant $c_0 \sim 0.819$. Also Odlyzko and Rains computed $\mathbb{E}(l_N)$ to higher order and found

$$\lim_{N\to\infty} \frac{\mathbb{E}(l_N) - 2\sqrt{N}}{N^{1/6}} = c_1, \tag{1.3}$$

where $c_1 \sim -1.758$. Further historical information on Ulam's problem, together with some discussions of the methods used by various authors, can be found in [AD] and [OR]

Before stating our results, we need to define the Tracy-Widom distribution [TW1] (see below). Let $u(x)$ be the solution of the Painlevé II (PII) equation,

$$u_{xx} = 2u^3 + xu, \quad \text{and} \quad u \sim -Ai(x) \quad \text{as} \quad x \to \infty, \tag{1.4}$$

where Ai is the Airy function. The (global) existence and uniqueness of this solution was first established in [HM]: the asymptotics as $x \to \pm\infty$ are

$$\begin{aligned} u(x) &= -Ai(x) + O\Big(\frac{e^{-(4/3)x^{3/2}}}{x^{1/4}}\Big) \quad \text{as} \quad x \to \infty, \\ u(x) &= -\sqrt{\frac{-x}{2}}\Big(1 + O\big(\frac{1}{x^2}\big)\Big) \qquad \text{as} \quad x \to -\infty \end{aligned} \tag{1.5}$$

(see, for example, [HM], [IN], [DZ2]). Recall [AS] that $Ai(x) \sim \frac{e^{-(2/3)x^{3/2}}}{2\sqrt{\pi}x^{1/4}}$ as $x \to \infty$. Define the Tracy-Widom distribution

$$F(t) = \exp\Big(-\int_t^\infty (x-t)u^2(x)dx\Big). \tag{1.6}$$

From (1.5) and (1.6), $F'(t) > 0$, $F(t) \to 1$ as $t \to +\infty$ and $F(t) \to 0$ as $t \to -\infty$, so that F is indeed a distribution function. Our first result concerns the convergence of l_N in distribution after appropriate centering and scaling.

Theorem 1.1. *Let S_N be the group of all permutations of N numbers with uniform distribution and let $l_N(\pi)$ be the length of the longest increasing subsequence of $\pi \in S_N$. Let χ be a random variable whose distribution function is F. Then, as $N \to \infty$,*

$$\chi_N \equiv \frac{l_N - 2\sqrt{N}}{N^{1/6}} \to \chi \qquad \textit{in distribution,}$$

i.e.

$$\lim_{N\to\infty} Prob\Big(\chi_N \equiv \frac{l_N - 2\sqrt{N}}{N^{1/6}} \le t\Big) = F(t) \quad \textit{for all } t \in \mathbb{R}.$$

In order to show that the moments of χ_N converge to the corresponding moments of χ we need estimates for the distribution function $F_N(t)$ of χ_N for large $|t|$. From the large deviation formulas for l_N (see below), we expect that $F_N(t)$ (resp., $1 - F_N(t)$) should go to zero rapidly as $t \to -\infty$ (resp., $t \to +\infty$). In fact, we will prove that, for $M > 0$ sufficiently large, there are positive constants c and $C(M)$ such that

$$F_N(t) \leq C(M)e^{ct^3} \tag{1.7}$$

if $-2N^{1/3} \leq t \leq -M$, and

$$1 - F_N(t) \leq C(M)e^{-ct^{3/5}} \tag{1.8}$$

if $M \leq t \leq N^{5/6} - 2N^{1/3}$. Together with Theorem 1.1 these estimates yield

Theorem 1.2. *For any $m = 1, 2, 3, \ldots$, we have*

$$\lim_{N\to\infty} \mathbb{E}_N(\chi_N^m) = \mathbb{E}(\chi^m),$$

where $\mathbb{E}(\cdot)$ denotes expectation with respect to the distribution function F. In particular,

$$\lim_{N\to\infty} \frac{Var(l_N)}{N^{1/3}} = \int_{-\infty}^{\infty} t^2 dF(t) - \left(\int_{-\infty}^{\infty} t dF(t)\right)^2 \tag{1.9}$$

and

$$\lim_{N\to\infty} \frac{\mathbb{E}_N(l_N) - 2\sqrt{N}}{N^{1/6}} = \int_{-\infty}^{\infty} t dF(t). \tag{1.10}$$

If one solves the Painlevé II equation (1.4) numerically (see [TW1]), and then computes the integrals on the RHS of the formulae of (1.9) and (1.10), one obtains the values 0.8132 and -1.7711 which agree with c_0 and c_1 in (1.2) and (1.3) respectively, up to two decimal places.

The distribution function $F(t)$ in Theorems 1.1 and 1.2 first arose in the work of Tracy and Widom on the Gaussian Unitary Ensemble (GUE) of random matrix theory. In this theory (see, e.g., [Me]), one considers the $N \times N$ hermitian matrix $M = (M_{ij})$ with probability density

$$Z_N^{-1} e^{-tr(M^2)} dM = Z_N^{-1} e^{-tr(M^2)} \left(\prod_{i=1}^N dM_{ii}\right) \prod_{i=1}^N d(ReM_{ij}) d(ImM_{ij}),$$

where Z_N is the normalization constant. In [TW1], Tracy and Widom showed that as the size of the hermitian matrices increases, the distribution of the (properly centered and scaled) largest eigenvalue of a random GUE matrix converges precisely to $F(t)$! In other words, properly centered and scaled, the length of the longest increasing subsequence for a permutation $\pi \in S_N$ behaves statistically for large N like the largest eigenvalue of a random GUE matrix (see the Appendix for an intuitive argument). In [TW1], the authors also computed the distribution functions of the second, third, ... largest eigenvalues of such random matrices, and the question arises whether such distribution functions describe the statistics of quantities identifiable in the random permutation context.

Recall the Robinson-Schensted correspondence (see, e.g., [Sa], and also Section 5.1.4 in [Kn]) which establishes a bijection $\pi \mapsto (P(\pi), Q(\pi))$ from S_N to pairs of Young tableaux with shape$(P(\pi)) = $ shape$(Q(\pi))$. Under this correspondence, the number of boxes in the first row of $P(\pi)$ (equivalently $Q(\pi)$) is precisely $l_N(\pi)$ (see [Sa], [Kn]). In other words, the results on l_N can be rephrased as results on the statistics of the number of boxes in the first row of Young tableaux. Monte Carlo simulations of Odlyzko and Rains [OR] indicate that $\widetilde{l}_N$, the number of boxes in the second row of $P(\pi)$ (equivalently $Q(\pi)$), behaves statistically for large N, like the second largest eigenvalue of a random GUE matrix. More precisely, their simulations indicate that

$$\lim_{N\to\infty} \frac{\mathbb{E}_N(\widetilde{l}_N) - 2\sqrt{N}}{N^{1/6}} = -3.618$$

and

$$\lim_{N\to\infty} \frac{Var(\widetilde{l}_N)}{N^{1/3}} = 0.545.$$

These values agree, once again, to two decimal places with the mean and variance of the suitably centered and scaled second largest eigenvalue of a GUE matrix, as computed in [TW1]. Presumably, the number of boxes in the third row of $P(\pi)$ should behave statistically like the third largest eigenvalue of a GUE matrix as $N \to \infty$, etc. In recent work [BDJ], the authors have shown that this conjecture is indeed true for the second row. Also, beautiful results of Okounkov [Ok], using arguments from combinatorial topology, have now provided an elegant basis for understanding the relationship between the statistics of Young tableaux and the eigenvalues of random matrices. Over the last year, many other intriguing results have been obtained on a variety of problems arising in mathematics and mathematical physics, which are closely related to, or motivated by, the longest increasing subsequence problem. We refer the reader to [TW2], [Bo], [Jo2], [Jo3] and [BR].

As in [Jo1], we consider the Poissonization $\phi_n(\lambda)$ of $q_{n,N}$,

$$\phi_n(\lambda) \equiv \sum_{N=0}^{\infty} \frac{e^{-\lambda}\lambda^N}{N!} q_{n,N}. \tag{1.11}$$

The function $\phi_n(\lambda)$ is a distribution function (in n) of a random variable $L(\lambda)$ coming from a superadditive process introduced by Hammersley in [Ha], and used by him to show that the limit (1.1) exists. The random variable $L(\lambda)$ is defined as follows. Consider a homogeneous rate one Poisson process in the plane and let $L(\lambda)$ denote the maximum number of points in an up-right (increasing) path through the points starting at $(0, 0)$ and ending at $(\sqrt{\lambda}, \sqrt{\lambda})$. For more details see [AD] and [Se2], and for a generalization to the non-homogeneous case see [DeZe1]. Theorems 1.1 and 1.2 hold for the random variable $L(\lambda)$ as $\lambda \to \infty$. Referring to the "de-Poissonization" Lemmas 8.2 and 8.3 below, we see that it is easy to recover the asymptotics of $q_{n,N}$ as $N \to \infty$ from the knowledge of $\phi_n(\lambda)$ for $\lambda \sim N$. In other words, in order to compute the asymptotics of l_N, we must investigate the double scaling limit of $\phi_n(\lambda)$ when $\lambda \to \infty$ and $1 \le n \le N \sim \lambda$, and this is the technical thrust of the paper.

To this end we use the following representation for $\phi_n(\lambda)$,

$$\phi_n(\lambda) = e^{-\lambda} D_{n-1}(\exp(2\sqrt{\lambda}\cos\theta)), \tag{1.12}$$

where D_{n-1} denotes the $n \times n$ Toeplitz determinant with weight function $f(e^{i\theta}) = \exp(2\sqrt{\lambda}\cos\theta)$ on the unit circle (see, e.g., [Sz1]). The above formula follows from work of Gessel in [Ge] using well known results about Toeplitz determinants. As noted in [Jo1], the formula can also be proved using the following representation for $q_{n,N}$, $1 \le n \le N$, discovered by [OPWW],

$$q_{n,N} = \frac{2^{2N} N!}{(2N)!} \int_{[-\pi,\pi]^n} \Big(\sum_{j=1}^{n} \cos\theta_j\Big)^{2N} \prod_{1 \le j < k \le n} |e^{i\theta_j} - e^{i\theta_k}|^2 \frac{d^n\theta}{(2\pi)^n n!}. \tag{1.13}$$

In addition, an earlier result of Diaconis and Shahshahani ([DS]) shows that the above formula (1.13) is true also in the case $n > N$ when $q_{n,N} \equiv 1$. Inserting (1.13) into (1.11), we obtain

$$\phi_n(\lambda) = e^{-\lambda} \frac{1}{(2\pi)^n n!} \int_{[-\pi,\pi]^n} \exp(2\sqrt{\lambda} \sum_{j=1}^{n} \cos\theta_j) \prod_{1 \le j < k \le n} |e^{i\theta_j} - e^{i\theta_k}|^2 d^n\theta, \tag{1.14}$$

which is precisely (1.12) by standard methods in the theory of Toeplitz determinants (see [Sz1]). An additional proof of (1.12) can be found in [GWW], and also an alternative derivation of formula (1.13) is given in [Ra]. For the convenience of the reader we provide (yet another) proof of (1.12) in the Appendix to this paper.

Using the integral representation (1.12), Johansson ([Jo1]) proved the following bound for $\phi(\lambda)$: for any given $\epsilon > 0$, there exist C and $\delta > 0$ such that

$$\begin{aligned} &0 \le \phi_n(\lambda) \le Ce^{-\delta\lambda} \quad \text{if} \quad (1+\epsilon)n < 2\sqrt{\lambda}, \\ &0 \le 1 - \phi_n(\lambda) \le \frac{C}{n} \quad \text{if} \quad (1-\epsilon)n > 2\sqrt{\lambda}. \end{aligned} \tag{1.15}$$

This information and the de-Poissonization Lemma 8.2 are enough to give a new proof ([Jo1]) that

$$\lim_{N\to\infty} L_N/2\sqrt{N} = 1. \tag{1.16}$$

The first estimate in (1.15) is a consequence of the following lower tail large deviation formula for $\phi_n(\lambda)$,

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \phi_{[x\sqrt{\lambda}]}(\lambda) = -1 + 2x - \frac{3}{4}x^2 - \frac{x^2}{2} \log\frac{2}{x} \equiv -U(x), \tag{1.17}$$

if $x < 2$. For the upper tail Seppäläinen in [Se2] used the interacting particle system implicitly introduced by Hammersley in [Ha] to show that

$$\lim_{\lambda\to\infty} \frac{1}{\sqrt{\lambda}} \log\big(1 - \phi_{[x\sqrt{\lambda}]}(\lambda)\big) = -2x\cosh^{-1}(x/2) + 2\sqrt{x^2-4} \equiv -I(x) \tag{1.18}$$

if $x > 2$. We note that Hammersley's interacting particle system was also used earlier by Aldous and Diaconis in [AD]. The super-additivity of the process described above implies that we actually have (see [Se2] and also [Ki])

$$1 - \phi_{[xM]}(M^2) \le e^{-MI(x)} \tag{1.19}$$

if M is a positive integer and $x \ge 2$. This estimate can be used to show (1.8), but in this paper we will give an independent proof of (1.8). The large deviation formula (1.18) implies, via a de-Poissonization argument, that for $x > 2$,

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \log Prob\big(l_N > x\sqrt{N}\big) = -I(x). \tag{1.20}$$

For the lower tail the large deviation formula for l_N is not the same as for $L(\lambda)$, the Poissonized case. Deuschel and Zeitouni in [DeZe2] use combinatorial and variational ideas from Logan and Shepp [LS] to prove that

$$\lim_{N\to\infty} \frac{1}{N} \log Prob\big(l_N < x\sqrt{N}\big) = -H(x) \tag{1.21}$$

if $0 < x < 2$, where

$$H(x) = -\frac{1}{2} + \frac{x^2}{8} + \log\frac{x}{2} - \big(1+\frac{x^2}{4}\big)\log\big(\frac{2x^2}{4+x^2}\big). \tag{1.22}$$

For the lower tail we have no analogue of (1.19). The rate functions U and H are related via a Legendre transform; see [Se2]. The above results show clearly that the distribution function for l_N is sharply concentrated in the region $\{(2-\epsilon)\sqrt{N} < l_N < (2+\epsilon)\sqrt{N}\}$ for any $\epsilon > 0$, and they can be used to see heuristically that the variance for l_N should be of order $N^{1/3}$; see [Ki].

As is well known (see [Sz1]) the Toeplitz determinant D_{n-1} in (1.12) is intimately connected with the polynomials $p_n(z;\lambda) = \kappa_n(\lambda)z^n + \cdots$, which are orthonormal with respect to the weight $f(e^{i\theta})\frac{d\theta}{2\pi} = \exp(\sqrt{\lambda}(z+z^{-1}))\frac{dz}{2\pi iz}$ on the unit circle,

$$\int_{-\pi}^{\pi} p_n(e^{i\theta})\overline{p_m(e^{i\theta})} f(e^{i\theta})\frac{d\theta}{2\pi} = \delta_{n,m} \quad \text{for } n,m\ge 0. \tag{1.23}$$

The leading coefficient $\kappa_n^2(\lambda)$ can be expressed in terms of Toeplitz determinants,

$$\kappa_n^2(\lambda) = \frac{D_{n-1}(\lambda)}{D_n(\lambda)}, \tag{1.24}$$

where $D_n(\lambda) = D_n(\exp(2\sqrt{\lambda}\cos\theta))$. But by Szegö's strong limit theorem ([Sz2]) for Toeplitz determinants, $\lim_{n\to\infty} D_n(\lambda) = e^{\lambda}$, and hence

$$\log\phi_n(\lambda) = \sum_{k=n}^{\infty} \log\kappa_k^2(\lambda). \tag{1.25}$$

Therefore, if one can control the large k,λ behavior of $\kappa_k^2(\lambda)$ for all $k\ge n$, one will control the large n,λ behavior of $\phi_n(\lambda)$.

The key point in our analysis is that $\kappa_k^2(\lambda)$ can be expressed in terms of the following Riemann-Hilbert Problem (RHP): Let Σ be the unit circle oriented counterclockwise. Let $Y(z;k+1,\lambda)$ be the 2×2 matrix-valued function satisfying

$$\begin{cases} Y(z;k+1,\lambda) \quad \text{is analytic in} \quad \mathbb{C}-\Sigma, \\ Y_+(z;k+1,\lambda) = Y_-(z;k+1,\lambda)\begin{pmatrix} 1 & \frac{1}{z^{k+1}}e^{\sqrt{\lambda}(z+z^{-1})} \\ 0 & 1\end{pmatrix} \quad \text{on} \quad \Sigma, \\ Y(z;k+1,\lambda)z^{-(k+1)\sigma_3} = I + O(\frac{1}{z}) \quad \text{as} \quad z\to\infty, \end{cases} \tag{1.26}$$

where Y_+ and Y_- denote the limit from inside and outside of the circle respectively, and $\sigma_3 = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, so that $z^{-(k+1)\sigma_3} = \left(\begin{smallmatrix} z^{-(k+1)} & 0 \\ 0 & z^{k+1}\end{smallmatrix}\right)$. Here I is the 2×2 identity matrix. This RHP has a unique solution (see (4.1) below), and the fact of the matter is that

$$\kappa_k^2(\lambda) = -Y_{21}(0;k+1,\lambda), \tag{1.27}$$

where $Y_{21}(0;k,\lambda)$ is the (21)-entry of the solution Y at $z=0$. In [DZ1] and [DZ2], Deift and Zhou introduced a steepest descent type method to compute the asymptotic behavior of RHP's containing large oscillatory and/or exponentially growing/decaying factors as in (1.26). This method was further extended in [DVZ1] and eventually placed in a very general form by Deift, Zhou and Venakides in

[DVZ2], making possible the analysis of the limiting behavior of a large variety of asymptotic problems in pure and applied mathematics (see, e.g., [DIZ]). As we will see, the application of this method to (1.26) makes it possible to control the large k, λ behavior of $\kappa_k^2(\lambda)$. The calculations in this paper have many similarities to the computations in [DKMVZ1], where the authors use the steepest descent method to obtain Plancherel-Rotach type asymptotics for polynomials orthogonal with respect to varying weights, $e^{-NV(x)}dx$ on the real line, and hence to prove universality for a class of random matrix models. The Riemann-Hilbert formulation of the theory of orthogonal polynomials on the line is due to Fokas, Its and Kitaev ([FIK]): the RHP (1.26) is an adaptation of the construction in [FIK] to the case of an orthogonal polynomial with respect to a weight on the unit circle.

This paper is arranged as follows. In Section 2, we discuss some of the basic theory of RHP's and also provide some information on the RHP associated with the PII equation. This information will be used in the construction of an approximate solution, i.e. a *parametrix*, for the RHP (1.26) in subsequent sections. The appearance of the PII equation in the limiting distribution $F(t)$ for χ_N originates in this construction of the parametrix. A connection of $\phi_n(\lambda)$ to Toda lattice and the Painlevé III equation is presented in Section 3. Section 4 is the starting point for the analysis of the RHP (1.26). In this section, (1.26) is transformed into an equivalent RHP via a so-called g-function. The role of the g-function, first introduced in [DZ2], and then analyzed in full generality in [DVZ2], is to replace exponentially growing terms in a RHP by oscillatory or exponentially decreasing terms. It turns out that in the case of (1.26), as in [DKMVZ1], the g-function can be constructed in terms of an associated equilibrium measure $d\mu(s)$ as follows,

$$g(z) \equiv \int_\Sigma \log(z-s)d\mu(s). \tag{1.28}$$

The measure $d\mu$ is the unique minimizer of the following variational problem:

$$E^V = \inf\{I^V(\widetilde{\mu}) : \widetilde{\mu} \text{ is a probability measure on the unit circle } \Sigma\}, \tag{1.29}$$

where

$$I^V(\widetilde{\mu}) = \iint_{\Sigma\times\Sigma} \log|s-w|^{-1} d\widetilde{\mu}(s)d\widetilde{\mu}(w) + \int_\Sigma V(s)d\widetilde{\mu}(s) \tag{1.30}$$

and $V(s) = -\sqrt{\lambda}(s+s^{-1})$. The variational problem (1.29) describes the equilibrium configuration of electrons, say, confined to the unit circle with Coulomb interactions, and acted on by an external field V. It turns out that the support of the equilibrium measure depends critically on the quantity

$$\gamma = \frac{2\sqrt{\lambda}}{k+1}. \tag{1.31}$$

We need to distinguish these two cases, $\gamma \le 1$ and $\gamma > 1$. As noted by Gross and Witten ([GW]), and also by Johansson ([Jo1]), the point $\frac{2\sqrt{\lambda}}{k+1} = 1$ corresponds to a (third order) phase transition for a statistical system with partition function (1.14). The first case, when $\gamma = \frac{2\sqrt{\lambda}}{k+1} \le 1$, is discussed in Section 5, and the second case, when $\gamma = \frac{2\sqrt{\lambda}}{k+1} > 1$, is discussed in Section 6. The principal results of the above two sections are summarized in Lemmas 5.1 and 6.3. We obtain full asymptotics of $\kappa_n^2(\lambda)$ for $n, \lambda > 0$ when $n, \lambda \to \infty$. In Section 7, by summing up $\kappa_k^2(\lambda)$ for all $k \ge n$, we obtain the asymptotics of $\phi_n(\lambda)$ in Lemma 7.1. The relation between

$\phi_n(N)$ and $q_{n,N}$ (de-Poissonization) is discussed in Section 8. Finally, the proofs of Theorems 1.1 and 1.2 are given in Section 9.

Notational remarks. The primary variables in this paper are n, N, and λ. The letters C, c denote general positive constants. Rather than introducing many such constants $C_1, C_2, \dots, c_1, c_2, \dots$, we always interpret C, c in a general way. For example, we write $|f(x)| \le 2C|g(x)| + e^c|h(x)| \le C(|g(x)| + |h(x)|)$, etc. We will also use certain auxiliary positive parameters $M, M_1, M_2, \dots, M_7$. If a constant depends on some of these parameters, we indicate this explicitly, for example, $C(M_2, M_4)$. In addition to the standard big O notation, we also use a notation O_M. Thus $f = O(\frac{1}{n^{1/3}})$ means $|f| \le \frac{C}{n^{1/3}}$, where C is independent of $M, M_1, \dots$. On the other hand, $f = O_M(\frac{1}{n^{1/3}})$ means $|f| \le \frac{C(M,M_1,\dots)}{n^{1/3}}$, where $C(M, M_1, \dots)$ depends on at least one of the parameters $M, M_1, \dots$.

In the estimates that follow we will often claim that an inequality is true "as $n \to \infty$". For example, in (7.3) below, we say that

$$|\log \phi_n(\lambda)| \le C \exp\biggl(-c(n+1)\bigl(1 - \frac{2\sqrt{\lambda}}{n+1}\bigr)^{3/2}\biggr),$$

as $n \to \infty$. This mean that there exists a number n_0, say, which may depend on *all* the other relevant constants in the problem, such that the inequality is true for $n \ge n_0$, etc. (For this particular inequality the only other parameter is M_5, but it turns out that the constants C, c can be chosen independent of M_5 (see below).)

2. Riemann-Hilbert theory

In this section, we first summarize some basic facts about RHP's in general, and then discuss the RHP for the PII equation in some detail. Basic references for RHP's are [CG], [GK], and the material on PII is taken from [DZ2].

Let Σ be an oriented curve in the plane (see, for example, Figure 1). By convention, the $(+)$-side (resp., $(-)$-side) of an arc in Σ lies to the left (resp., right) as one traverses the arc in the direction of the orientation. Thus, corresponding to Figure 1, we have Figure 2. Let $\Sigma_0 = \Sigma - \{\text{points of self-intersection}\}$, and let v be a smooth map from $\Sigma_0 \to Gl(n, \mathbb{C})$, for some n. If Σ is unbounded, we require that $v(z) \to I$ as $z \to \infty$ along Σ. The RHP (Σ, v) consists of the following (see, e.g., [CG]): establish the existence and uniqueness of an $n \times n$ matrix valued function

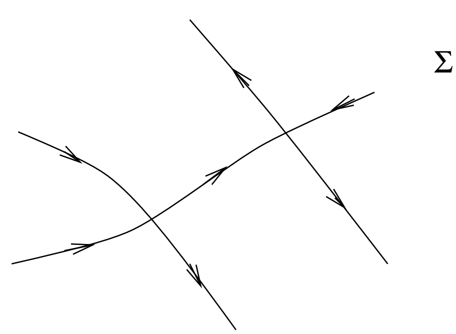

Figure 1

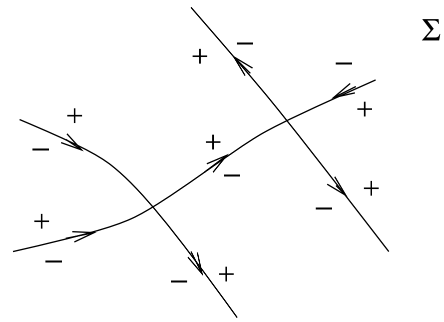

FIGURE 2

$Y(z)$ (the *solution* of the RHP (Σ, v)) such that

$$\begin{cases} Y(z) & \text{is analytic in } \mathbb{C} - \Sigma, \\ Y_+(z) = Y_-(z)v(z), & z \in \Sigma_0, \\ Y(z) \to I & \text{as } z \to \infty. \end{cases} \tag{2.1}$$

Here $Y_\pm(z) = \lim_{z' \to z} Y(z')$ where $z' \in (\pm)$-side of Σ. The precise sense in which these boundary values are attained, and also the precise sense in which $Y(z) \to I$ as $z \to \infty$, are technical matters that should be specified for any given RHP (Σ, v). In *this* paper, by a solution Y of a RHP (Σ, v), we always mean that

(2.2)
$Y(z)$ is analytic in $\mathbb{C} - \Sigma$ and continuous up to the boundary (including the points in $\Sigma - \Sigma_0$) in each component.
The jump relation $Y_+(z) = Y_-(z)v(z)$ is taken in the sense of continuous boundary values, and $Y(z) \to I$ as $z \to \infty$ means
$Y(z) = I + O\big(\frac{1}{|z|}\big)$ uniformly as $z \to \infty$ in $\mathbb{C} - \Sigma$.

Given (Σ, v), the existence of Y under appropriate technical assumptions on Σ and v is in general a subtle and difficult question. However, for the RHP (1.26), and hence for all RHP's obtained by deforming (1.26) (see, e.g., (4.9)), we will prove the existence of Y directly by construction (see Lemma 4.1): uniqueness, as we will see, is a simple matter.

The solution of a RHP (Σ, v) can be expressed in terms of the solution of an associated singular integral equation on Σ (see (2.7), (2.8) below) as follows. Let $C_\pm$ be the Cauchy operators

$$(C_\pm f)(z) = \lim_{z' \to z_\pm} \int_\Sigma \frac{f(s)}{s - z'} \frac{ds}{2\pi i}, \quad z \in \Sigma, \tag{2.3}$$

where $z' \to z_\pm$ denotes the non-tangential limit from the $\pm$-side of Σ respectively. A useful reference for Cauchy operators on curves which may have points of self-intersection is [GK]. Under mild assumptions on Σ, which will always be satisfied for the curves that arise in this paper, the non-tangential limits in (2.3) will exist pointwise a.e. on Σ. Furthermore, if $f \in L^p(\Sigma, |dz|)$, $1 < p < \infty$, then the boundary values (appropriately interpreted at the points $\Sigma - \Sigma_0$ of self-intersection) of $\int_\Sigma \frac{f(s)}{s-z} \frac{ds}{2\pi i}$ are also taken in the sense of L^p and $\|C_\pm f\|_{L^p(\Sigma, |dz|)} \le c_p \|f\|_{L^p(\Sigma, |dz|)}$.

A simple calculation shows that

$$C_+ - C_- = 1. \tag{2.4}$$

Let

$$v = b_-^{-1} b_+ \equiv (I - w_-)^{-1}(I + w_+) \tag{2.5}$$

be any factorization of v. We assume $b_\pm$, and hence $w_\pm$, are smooth on Σ_0, and if Σ is unbounded, we assume $b_\pm(z) \to I$ as $z \to \infty$ along Σ. Define the operator

$$C_w(f) \equiv C_+(fw_-) + C_-(fw_+). \tag{2.6}$$

By the above discussion, if $w_\pm \in L^\infty(\Sigma, |dz|)$, then C_w is bounded from $L^2(\Sigma, |dz|) \to L^2(\Sigma, |dz|)$. Suppose that the equation

$$(1 - C_w)\mu = I \quad \text{on } \Sigma \tag{2.7}$$

has a solution $\mu \in I + L^2(\Sigma)$, or more precisely, suppose $\mu - I \in L^2(\Sigma)$ solves

$$(1 - C_w)(\mu - I) = C_w I = C_+(w_-) + C_-(w_+), \tag{2.8}$$

which is a well-defined equation in $L^2(\Sigma)$ provided that $w_\pm \in L^\infty \cap L^2(\Sigma, |dz|)$. Then the solution of the RHP (2.1) is given by (see [CG], [BC])

$$Y(z) = I + \int_\Sigma \frac{\mu(s)(w_+(s) + w_-(s))}{s - z} \frac{ds}{2\pi i}, \quad z \notin \Sigma. \tag{2.9}$$

Indeed for a.e. $z \in \Sigma$, from (2.7) and (2.4),

$$\begin{aligned} Y_+(z) &= I + C_+(\mu(s)(w_+(s) + w_-(s))) \\ &= I + C_w(\mu) + (C_+ - C_-)(\mu w_+) \\ &= \mu + \mu w_+ \\ &= \mu(z) b_+(z), \end{aligned}$$

and similarly $Y_-(z) = \mu(z) b_-(z)$, so that $Y_+(z) = Y_-(z) b_-^{-1}(z) b_+(z) = Y_-(z) v(z)$ a.e. on Σ. Under the appropriate regularity assumptions on Σ and v, one then shows that $Y(z)$ solves the RHP (Σ, v) in the sense of (2.2).

As indicated, the above approach to the RHP goes through for any factorization $v = (I - w_-)^{-1}(I + w_+)$. Different factorizations may be used at different points in the analysis of any given problem (see, e.g., [DZ1]). However, in this paper we will *always* take $w_- = 0$, so that $v = (I + w_+)$. Thus C_w always denotes the operator $C_-\big(\cdot(v - I)\big)$.

In this paper we will not develop the general theory for the solution of RHP's, giving conditions under which (2.7) has a (unique) solution, etc. Rather, for the convenience of the reader who may not be familiar with Riemann-Hilbert theory, we will use the above calculations and computations as a guide, and verify all the steps directly as they arise.

We now consider the RHP for the PII equation ([FN], [JMU]: see also [IN], [FZ], [DZ2]). We will consider two equivalent versions of the RHP for PII. These two RHP's will be used in the later sections for the construction of parametrices for the solution of (1.26).

Let Σ^{PII} denote the oriented contour consisting of 6 rays in Figure 3. Thus $\Sigma^{PII} = \bigcup_{k=1}^6 \{\Sigma_k^{PII} = e^{i(k-1)\pi/3}\mathbb{R}_+\}$, with associated jump matrix $v^{PII} : \Sigma^{PII} \to$

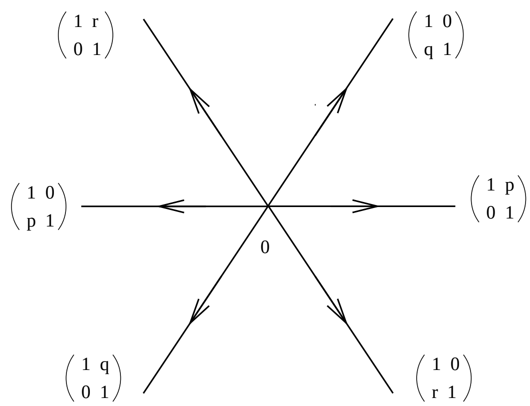

FIGURE 3. v^{PII} and Σ^{PII}

$M_2(\mathbb{C})$, where the monodromy data p, q and r are complex numbers satisfying the relation

$$p + q + r + pqr = 0. \tag{2.10}$$

For $x \in \mathbb{R}$ and $z \in \Sigma^{PII} - \{0\}$, set

$$\begin{aligned} v_x^{PII}(z) &= e^{-i\theta_{PII}\sigma_3} v^{PII} e^{i\theta_{PII}\sigma_3} \\ &\equiv e^{-i\theta_{PII} ad\sigma_3} v^{PII}, \end{aligned} \tag{2.11}$$

where

$$\theta_{PII} \equiv \frac{4z^3}{3} + xz. \tag{2.12}$$

The contour Σ^{PII} consists precisely of the set $Re(i4z^3/3) = 0$. This implies, in particular, that $v_x^{PII}(z) - I \notin L^2(\Sigma^{PII})$. For example, as $z \to +\infty$ along the real axis, $v_x^{PII}(z) - I$ is oscillatory (on the other rays, Σ_k^{PII}, $k = 1, 2, 4, 5$, $v_x^{PII}(z) - I$ could grow), and so we cannot expect that the RHP $(\Sigma^{PII}, v_x^{PII})$ has a solution in the sense of (2.2). However, if we rotate Σ^{PII} in the clockwise direction by any angle θ_0, $0 < \theta_0 < \pi/3$, $\Sigma^{PII} \to \Sigma_{\theta_0}^{PII} \equiv e^{-i\theta_0}\Sigma^{PII}$, then it is easy to see that $v_x^{PII}(z) - I \in L^2 \cap L^\infty(\Sigma_{\theta_0}^{PII})$, and we may expect that the RHP $(\Sigma_{\theta_0}^{PII}, v_x^{PII})$ has a solution in the sense of (2.2). Moreover, as $v_x^{PII}(z)$ is analytic, it is clear that if one can solve $(\Sigma_{\theta_0}^{PII}, v_x^{PII})$ for some $0 < \theta_0 < \pi/3$, then one can solve $(\Sigma_{\widetilde{\theta}_0}^{PII}, v_x^{PII})$ for any other $0 < \widetilde{\theta}_0 < \pi/3$, and the solution of the $\widetilde{\theta}_0$-problem can be obtained from the θ_0-problem by an analytic continuation, and vice versa. So suppose that for some fixed $0 < \theta_0 < \pi/3$, and for $x \in \mathbb{R}$, $m_{\theta_0}^{PII}(z; x)$ is a $(2 \times 2$ matrix) solution of the RHP $(\Sigma_{\theta_0}^{PII}, v_x^{PII})$,

$$\begin{cases} m_{\theta_0}^{PII}(z) \qquad \text{analytic in} \qquad \mathbb{C} - \Sigma_{\theta_0}^{PII}, \\ \left(m_{\theta_0}^{PII}\right)_+(z) = \left(m_{\theta_0}^{PII}\right)_-(z) v_x^{PII}(z), \qquad 0 \neq z \in \Sigma_{\theta_0}^{PII}, \\ m_{\theta_0}^{PII}(z) \to I \qquad \text{as} \qquad z \to \infty, \end{cases} \tag{2.13}$$

in the sense of (2.2). Let $m_1^{PII}(x)$ denote the residue at ∞ of $m_{\theta_0}^{PII}(z)$, given by

$$m_{\theta_0}^{PII}(z;x) = I + \frac{m_1^{PII}(x)}{z} + O(\frac{1}{z^2})$$

as $z \to \infty$. Then

$$u(x) \equiv 2im_{1,12}^{PII}(x) = -2im_{1,21}^{PII}(x) \tag{2.14}$$

solves PII (see [FN], [JMU]),

$$u_{xx} = 2u^3 + xu, \qquad x \in \mathbb{R},$$

where $m_{1,12}^{PII}(x)$ (resp., $m_{1,21}^{PII}(x)$) denotes the (12)-entry (resp, (21)-entry) of $m_1^{PII}(x)$. It is easy to see that $m_1^{PII}(x)$, and hence $u(x)$ in (2.14), is independent of the choice of $\theta_0 \in (0, \pi/3)$

A solution of the RHP $(\Sigma_{\theta_0}^{PII}, v_x^{PII})$ for some θ_0, hence for all $\theta_0 \in (0, \pi/3)$, may not exist for all p, q, r satisfying (2.10) and $x \in \mathbb{R}$. A sufficient condition (see [FZ]) for the RHP to have a unique solution (in the sense of (2.2)) for all $x \in \mathbb{R}$, is that

$$|q - \bar{p}| < 2 \qquad \text{and} \qquad r \in \mathbb{R}.$$

In this paper, we need the singular case

$$p = -q = 1 \qquad \text{and} \qquad r = 0. \tag{2.15}$$

The latter condition $r = 0$ implies that there is no jump across the rays $\pm e^{i(2\pi/3-\theta_0)}$, and we may replace $\Sigma_{\theta_0}^{PII}$ by $\Sigma_{\theta_0}^{PII,1}$ as in Figure 4 (note that the orientations across the rays $e^{-i\theta_0}, e^{i(2\pi/3-\theta_0)}$ have been reversed). As noted in [DZ2], a unique solution in the sense of (2.2) still exists in this singular case for all $x \in \mathbb{R}$: a proof of this fact is not given in [DZ2], but can be found in [DKMVZ3, nonregular case, Case II]. In addition, the solution has the property that

$$\begin{aligned} &m_{\theta_0}^{PII,1}(z;x) \text{ and its inverse are uniformly bounded} \\ &\text{for } (z,x) \in (\mathbb{C} - \Sigma_{\theta_0}^{PII,1}) \times [-M, M], \end{aligned} \tag{2.16}$$

for any fixed $M > 0$. As $m_{\theta_0}^{PII,1}(z;x)$ solves (2.13) in the sense of (2.2), we see in particular that (2.16) holds up to the boundary in each sector.

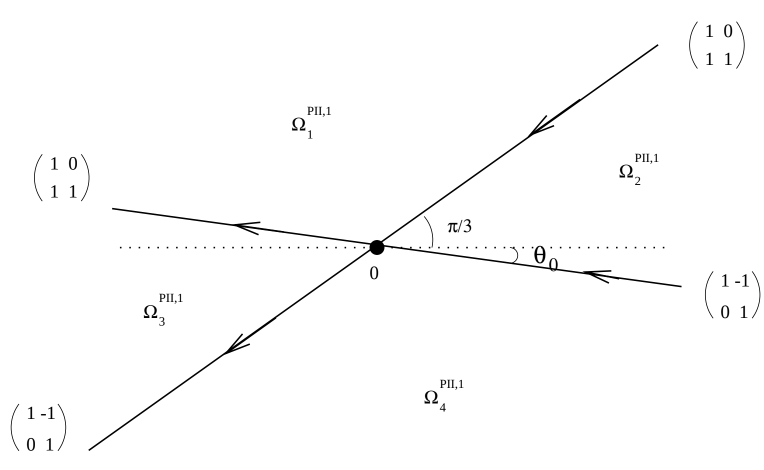

FIGURE 4. $p = -q = 1, r = 0$ case; $\Sigma_{\theta_0}^{PII,1}$ and $v^{PII,1}$

The asymptotics of $u(x) = 2im_{1,12}^{PII,1}(x)$ given in (1.5) is computed in [DZ2] via the above RHP and from the proof in [DZ2], one learns that

$$(2.17)\qquad \begin{aligned} m_{1,22}^{PII,1}(x) &= O\left(\frac{e^{-(4/3)x^{3/2}}}{x^{1/4}}\right) \quad \text{as} \quad x \to \infty, \\ m_{1,22}^{PII,1}(x) &\sim \frac{i}{8}x^2 \quad \text{as} \quad x \to -\infty, \end{aligned}$$

where $m_{1,22}^{PII,1}$ denotes the (22)-entry of m_1^{PII}. Also, using the methods in [DZ2], for example, one obtains the relation

$$(2.18)\qquad \frac{d}{dx} 2im_{1,22}^{PII,1}(x) = u^2(x),$$

and verifies directly that $2im_{1,22}^{PII,1}(x)$ is real-valued.

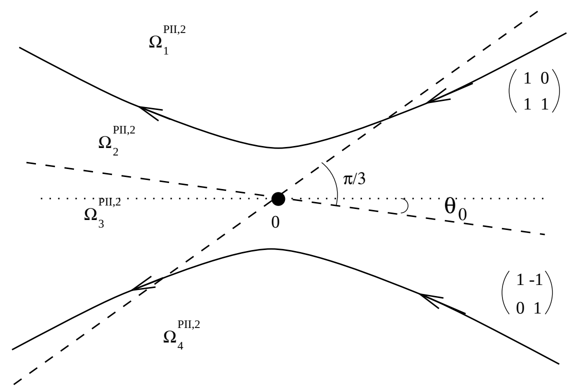

FIGURE 5. $\Sigma^{PII,2}$ and $v^{PII,2}$

For the first of the two equivalent RHP's advertised above, we consider Figure 5, which consists of the real axis (the dotted line), $\Sigma_{\theta_0}^{PII,1}$ for some fixed, small $\theta_0 > 0$ (the dashed lines), and a contour $\Sigma^{PII,2}$ consisting of a pair of curved solid lines. The contour $\Sigma^{PII,2}$ is of the general shape indicated in the figure, with one component in $\mathbb{C}_+$ and one component in $\mathbb{C}_-$, and we require that $\Sigma^{PII,2}$ is asymptotic to straight lines lying *strictly* within the region $\{|arg z| < \pi/3\} \cup \{2\pi/3 < arg z < 4\pi/3\}$. Together with the line $\{xe^{-i\theta_0} : x \in \mathbb{R}\}$, these contours divide the complex plane into 4 open regions, $\Omega_k^{PII,2}$, $k = 1, 2, 3, 4$, as shown in Figure 5. Let $v^{PII,2}$ be the jump matrix on $\Sigma^{PII,2}$ which is given by $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ in $\mathbb{C}_+$ and by $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$ in $\mathbb{C}_-$. We define

$$\begin{cases} m^{PII,2} = m_{\theta_0}^{PII,1} e^{-i(\theta_{PII}) ad\sigma_3} \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)^{-1} & \text{in} \quad \Omega_1^{PII,2} \cap \Omega_2^{PII,1}, \\ m^{PII,2} = m_{\theta_0}^{PII,1} e^{-i(\theta_{PII}) ad\sigma_3} \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) & \text{in} \quad \Omega_2^{PII,2} \cap \Omega_1^{PII,1}, \\ m^{PII,2} = m_{\theta_0}^{PII,1} e^{-i(\theta_{PII}) ad\sigma_3} \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)^{-1} & \text{in} \quad \Omega_3^{PII,2} \cap \Omega_4^{PII,1}, \\ m^{PII,2} = m_{\theta_0}^{PII,1} e^{-i(\theta_{PII}) ad\sigma_3} \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right) & \text{in} \quad \Omega_4^{PII,2} \cap \Omega_3^{PII,1}, \\ m^{PII,2} = m_{\theta_0}^{PII,1} \quad \text{otherwise}, \end{cases}$$

where the regions $\Omega_k^{PII,1}$, $k = 1, 2, 3, 4$, are defined in Figure 4. A straightforward calculation with the jump relations for $m_{\theta_0}^{PII,1}$ shows that $m^{PII,2}$ solves the new

RHP

$$\begin{cases} m^{PII,2} \quad \text{is analytic in} \quad \mathbb{C} - \Sigma^{PII,2}, \\ m_+^{PII,2} = m_-^{PII,2} v_x^{PII,2} \quad \text{on} \quad \Sigma^{PII,2}, \\ m^{PII,2} \to I \quad \text{as} \quad z \to \infty, \end{cases} \tag{2.19}$$

where $v_x^{PII,2} = e^{-i(\theta_{PII}) ad\sigma_3} v^{PII,2}$ and $v^{PII,2}$ is given in Figure 5. This deformed RHP is clearly equivalent to the original RHP for $m_{\theta_0}^{PII,1}$ in the sense that a solution of the one RHP yields a solution of the other RHP, and vice versa. Also we have

$$(m_{\theta_0}^{PII,1})_1 = m_1^{PII,2}, \tag{2.20}$$

for the residues of $m_{\theta_0}^{PII,1}$ (resp., $m^{PII,2}$) at ∞. From (2.16), we see that for any fixed $M > 0$,

$$\begin{aligned} &m^{PII,2}(z;x) \quad \text{and its inverse are uniformly bounded} \\ &\text{for} \quad (z,x) \in (\mathbb{C} - \Sigma^{PII,2}) \times [-M, M]. \end{aligned} \tag{2.21}$$

A particular choice of contour Σ^{PII} will be made in Section 5 (see below).

The second of the equivalent RHP's is restricted to the case $x < 0$, and we consider Figure 6, which consists of the real axis (the dotted line), $\Sigma_{\theta_0}^{PII,1}$ for some fixed small $\theta_0 > 0$ (the dashed lines) and a contour $\Sigma^{PII,3} = \bigcup_{k=1}^5 \Sigma_k^{PII,3}$ consisting of 5 straight lines, one finite and four infinite. The regions $\Omega_k^{PII,3}$, $1 \le k \le 4$, are the components of $\mathbb{C} - \Sigma^{PII,3}$.

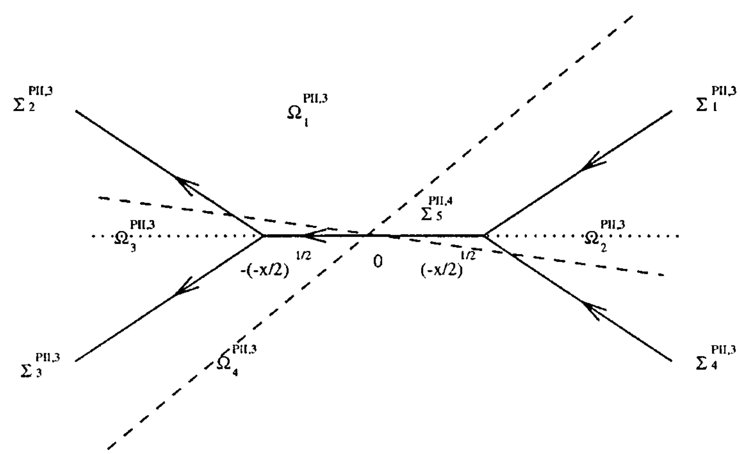

FIGURE 6. $\Sigma^{PII,3}$

The infinite lines make an angle strictly between 0 and $\pi/3$ with the real axis. Set

$$g^{PII}(z) = \frac{4}{3}\big(z^2 + \frac{x}{2}\big)^{3/2} \tag{2.22}$$

which is defined to be analytic in $\mathbb{C} - [-\sqrt{\frac{-x}{2}}, \sqrt{\frac{-x}{2}}]$, and behaves like $\frac{4}{3}z^3 + xz + \frac{x^2}{8z} + O(\frac{1}{z^3}) = \theta_{PII}(z) + O(\frac{1}{z})$ as $z \to \infty$. Therefore for any $M > 0$,

(2.23)
$$e^{i(g^{PII}(z)-\theta_{PII}(z))} \quad \text{is bounded for} \quad (z,x) \in (\mathbb{C} - [-\sqrt{\frac{-x}{2}}, \sqrt{\frac{-x}{2}}]) \times [-M, 0]$$

and

$$e^{i(g^{PII}(z)-\theta_{PII}(z))} \to 1 \quad \text{as } z \to \infty \text{ uniformly for } -M \le x \le 0. \tag{2.24}$$

We define $m^{PII,3}$ by

$$\begin{cases} m^{PII,1}_{\theta_0}[e^{-i(\theta_{PII})ad\sigma_3}\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)^{-1}]e^{i(g^{PII}(z)-\theta_{PII}(z))\sigma_3} & \text{in} \quad \Omega^{PII,3}_1 \cap (\Omega^{PII,1}_2 \cup \Omega^{PII,1}_3), \\ m^{PII,1}_{\theta_0}[e^{-i(\theta_{PII})ad\sigma_3}\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)]e^{i(g^{PII}(z)-\theta_{PII}(z))\sigma_3} & \text{in} \quad \Omega^{PII,3}_3 \cap \Omega^{PII,1}_1, \\ m^{PII,1}_{\theta_0}[e^{-i(\theta_{PII})ad\sigma_3}\left(\begin{smallmatrix}1&-1\\0&1\end{smallmatrix}\right)^{-1}]e^{i(g^{PII}(z)-\theta_{PII}(z))\sigma_3} & \text{in} \quad \Omega^{PII,3}_2 \cap \Omega^{PII,1}_4, \\ m^{PII,1}_{\theta_0}[e^{-i(\theta_{PII})ad\sigma_3}\left(\begin{smallmatrix}1&-1\\0&1\end{smallmatrix}\right)]e^{i(g^{PII}(z)-\theta_{PII}(z))\sigma_3} & \text{in} \quad \Omega^{PII,3}_4 \cap (\Omega^{PII,1}_2 \cup \Omega^{PII,1}_3), \\ m^{PII,1}_{\theta_0} \quad \text{otherwise.} \end{cases}$$

Then from the jump relations for $m^{PII,1}_{\theta_0}$, we see that $m^{PII,3}$ solves the new RHP $(\Sigma^{PII,3}, v^{PII,3}_x)$ in the sense of (2.2),

$$\begin{cases} m^{PII,3} \quad \text{is analytic in} \quad \mathbb{C} - \Sigma^{PII,3}, \\ m^{PII,3}_+ = m^{PII,3}_- v^{PII,3}_x \quad \text{on} \quad \Sigma^{PII,3}, \\ m^{PII,3} \to I \quad \text{as} \quad z \to \infty, \end{cases} \tag{2.25}$$

where $v^{PII,3}_x$ is given by

$$\begin{cases} \begin{pmatrix} 1 & 0 \\ e^{2ig^{PII}} & 1 \end{pmatrix} & \text{on} \quad \Sigma^{PII,3}_1, \Sigma^{PII,3}_2, \\ \begin{pmatrix} 1 & -e^{-2ig^{PII}} \\ 0 & 1 \end{pmatrix} & \text{on} \quad \Sigma^{PII,3}_3, \Sigma^{PII,3}_4, \\ \begin{pmatrix} e^{-2ig^{PII}_-} & -1 \\ 1 & 0 \end{pmatrix} & \text{on} \quad \Sigma^{PII,3}_5. \end{cases} \tag{2.26}$$

Also we have

$$m^{PII}_1 = m^{PII,3}_1 - (\frac{ix^2}{8})\sigma_3, \tag{2.27}$$

for the respective residues of $m^{PII,1}_{\theta_0}$ and $m^{PII,3}$ at ∞. Finally, from (2.16) and (2.23), we see that, for any fixed $M \in \mathbb{R}$

$$\begin{aligned} &m^{PII,3}(z;x) \quad \text{and its inverse are uniformly bounded} \\ &\text{for} \quad (z,x) \in (\mathbb{C} - \Sigma^{PII,3}) \times [-M, 0]. \end{aligned} \tag{2.28}$$

3. Connection to the Toda lattice and the Painlevé III equation

In this section, we discuss the connection of the RHP (1.26) for κ_k^2 and the RHP for the Toda lattice and the Painlevé III equation. In the RH context, the connection results from the specific form of the weight, $e^{\sqrt{\lambda}(z+z^{-1})}$. Connections can also be seen from the Toeplitz determinant/orthogonal polynomial point of view as in [PS], [Hi] and [Wi]. The purpose of this short section is purely to establish the various connections, but we do not use the results in the sequel.

Write $q = k + 1$ in (1.26). We define

$$
\text{(3.1)}\quad m^{TL}(z;q) = \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Y(z;q) \begin{pmatrix} z^{-q} e^{\sqrt{\lambda} z^{-1}} & 0 \\ 0 & z^{q} e^{-\sqrt{\lambda} z^{-1}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & |z| > 1, \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Y(z;q) \begin{pmatrix} e^{\sqrt{\lambda} z} & 0 \\ 0 & e^{-\sqrt{\lambda} z} \end{pmatrix}, & |z| < 1. \end{cases}
$$

A simple calculation shows that m^{TL} solves the following RHP:

$$
\text{(3.2)}\qquad \begin{cases} m^{TL}(z) \quad \text{is analytic in} \quad \mathbb{C} - \Sigma, \\ m_{+}^{TL}(z) = m_{-}^{TL}(z) \begin{pmatrix} 0 & -z^{-q} e^{-\sqrt{\lambda}(z - z^{-1})} \\ z^{q} e^{\sqrt{\lambda}(z - z^{-1})} & 1 \end{pmatrix}, \\ m^{TL}(z) \to I \quad \text{as} \quad z \to \infty. \end{cases}
$$

Once again, the RHP for Y is equivalent to the RHP for m^{TL} in the sense that a solution of one problem yields a solution of the other problem.

Recall that the RHP related to the Toda lattice problem, for $-\infty < m < \infty$,

$$
\text{(3.3)}\qquad \begin{aligned} \frac{da_m}{dt} &= 2(b_m^2 - b_{m-1}^2), \\ \frac{db_m}{dt} &= b_m(a_{m+1} - a_m), \end{aligned}
$$

under initial data decaying at infinity is the following (see, e.g., [Ka]). Suppose that there are no solitons and denote the reflection coefficient by $r(z)$, $z \in \Sigma$. Then we find $Q(z)$ such that

$$
\text{(3.4)}\qquad \begin{cases} Q(z) \qquad \text{is analytic in } \mathbb{C} - \Sigma, \\ Q_{+}(z) = Q_{-}(z) \begin{pmatrix} 1 - |r(z)|^2 & -\bar{r}(z) z^{2m} e^{-t(z - z^{-1})} \\ r(z) z^{-2m} e^{t(z - z^{-1})} & 1 \end{pmatrix}, \\ Q(z) \to I \qquad \text{as } z \to \infty. \end{cases}
$$

When q is even, if we set $\sqrt{\lambda} = t$ and $q = -2m$ in (3.2), then the RHP is identical with the above RHP with $r(z) \equiv 1$.

For the connection to the Painlevé III equation, define

$$
\text{(3.5)}\qquad m^{PIII}(z) = \begin{cases} (-1)^q m^{TL}(z), & |z| < 1, \\ m^{TL}(z), & |z| > 1. \end{cases}
$$

Note in (3.2),

$$
\text{(3.6)}\qquad \begin{aligned} &(-1)^q \begin{pmatrix} 0 & -z^{-q} e^{-\sqrt{\lambda}(z - z^{-1})} \\ z^{q} e^{\sqrt{\lambda}(z - z^{-1})} & 1 \end{pmatrix} \\ &\quad = (-1)^q e^{-\frac{\sqrt{\lambda}}{2}(z - z^{-1}) ad\sigma_3} z^{-\frac{q}{2} ad\sigma_3} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{-1}, \end{aligned}
$$

where $z^{\frac{q}{2}}$ is analytic in $\mathbb{C} - (-\infty, 0]$ and real-valued for real z. If we set $\sqrt{\lambda} = -ix$, then this is the same RHP for the particular Painlevé III equation (see [FMZ] for results and notations)

$$
u_{xx} = \frac{u_x^2}{u} - \frac{1}{x} u_x + \frac{1}{x}\big(-4qu^2 + 4(1 - q)\big) + 4u^3 + \frac{-4}{u}
$$

with monodromy data

$$\theta_\infty = -\theta_0 = q,$$
$$a_0 = b_0 = a_\infty = b_\infty = 0,$$
$$E = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right).$$

In the RHP (1.26), we are interested directly in the quantity $-Y_{21}(0; k+1, \lambda)$, or by (3.1), $m_{11}^{TL}(0;q)$. On the other hand, for the Toda lattice and the PIII equation, one is interested in quantities other than $m_{11}^{TL}(0;q)$ which are related to the respective RHP's. For example, the solution $u(x)$ of the PIII equation is given by $u(x) = -ix(m_1^{PIII})_{12}$ where $m^{PIII} = I + \frac{m_1^{PIII}}{z} + O(\frac{1}{z})$, which is clearly different from $(-1)^q m_{11}^{PIII}(0;q)$. However, the importance of the connection of (1.26) to the RHP's for the Toda lattice and the PIII equation lies precisely in the fact that (a_m, b_m) (resp., $u(x)$) solve differential-difference (resp., differential) equations which in turn imply that the coefficients of the generating function $\phi_n(\lambda)$, $2\sqrt{\lambda} = -ix$, must satisfy a certain class of identities. We plan to investigate these relations in a later publication.

Finally, note that for PIII, the interesting asymptotic question is to evaluate the limit $x = i\sqrt{\lambda} \to \infty$, with q fixed. In this paper, as in the Toda lattice, we are interested in the double limit when $\lambda \to \infty$ and q is allowed to vary (note that in [Ka], the singular case $r(z) \equiv 1$ is not considered). When $\lambda \to \infty$, $\frac{2\sqrt{\lambda}}{q} \sim 1$, we are in a region where the solution of the PIII equation degenerates to a solution of the PII equation, and this explains the appearance of PII in the parametrix for the solution of Y of the RHP (1.26).

4. Equilibrium measure and the g-function

In this section, the equilibrium measure is explicitly calculated for each $\gamma > 0$ (Lemma 4.3) and, using this equilibrium measure, the g-function (4.8) is introduced in order to convert the RHP (1.26) into a RHP which is normalized to be I at ∞.

Let Σ denote the unit circle oriented counterclockwise, and let $f(e^{i\theta}) = f(z)$ be a non-negative, periodic, smooth function on Σ. Let $p_q(z) = \kappa_q z^q + \cdots$ be the q-th normalized orthogonal polynomial with respect to the weight $f(e^{i\theta})\frac{d\theta}{2\pi}$ on the unit circle. Define the polynomial $p_q^*(z) \equiv z^q \bar{p}_q(1/z) = z^q \overline{p_q(1/\bar{z})}$ (see [Sz1]). We consider the following RHP: Let $Y(z)$ be the 2×2 matrix-valued function satisfying

$$\begin{cases} Y(z) \quad \text{is analytic in} \quad \mathbb{C} - \Sigma, \\ Y_+(z) = Y_-(z) \begin{pmatrix} 1 & \frac{1}{z^{k+1}} f(z) \\ 0 & 1 \end{pmatrix} \quad \text{on} \quad \Sigma, \\ Y(z) z^{-(k+1)\sigma_3} = I + O(\frac{1}{z}) \quad \text{as} \quad z \to \infty. \end{cases} \tag{4.1}$$

The following lemma is the starting point of our calculations.

Lemma 4.1 (cf. [FIK], [DKMVZ1]). *The RHP* (4.1) *has a unique solution*

$$Y(z) = \begin{pmatrix} \frac{1}{\kappa_{k+1}} p_{k+1}(z) & \frac{1}{\kappa_{k+1}} \int_\Sigma \frac{p_{k+1}(s)}{s-z} \frac{f(s)ds}{2\pi i s^{k+1}} \\ -\kappa_k p_k^*(z) & -\kappa_k \int_\Sigma \frac{p_k^*(s)}{s-z} \frac{f(s)ds}{2\pi i s^{k+1}} \end{pmatrix}.$$

Proof. Existence: Using the property of Cauchy operator $C_+ - C_- = I$, where $Ch(z) \equiv \int_\Sigma \frac{h(s)}{s-z} ds$, it is a straightforward calculation to show that the above expression for Y satisfies the jump condition. The asymptotics at ∞ codes in precisely the fact that the p_k's are the normalized orthogonal polynomials for the weight $f(e^{i\theta})d\theta$.

Uniqueness: Suppose that there is another solution $\tilde{Y}$ of RHP. Noting

$$\det\begin{pmatrix} 1 & \frac{1}{z^{k+1}}f(z) \\ 0 & 1 \end{pmatrix} = 1,$$

we have that $\det\tilde{Y}$ is entire, and $\to 1$ as $z\to\infty$. Therefore by Liouville's theorem, $\det\tilde{Y}\equiv 1$. In particular, $\tilde{Y}$ is invertible. Now set $Z = Y\tilde{Y}^{-1}$. Then it has no jump on Σ, hence is entire. Also, $Z\to I$ as $z\to\infty$, and therefore $Z\equiv I$. □

From this lemma, we have

$$\kappa_k^2 = -Y_{21}(0). \tag{4.2}$$

Therefore the RHP (1.26) has a unique solution and (1.27) is verified.

Again set $q = k+1$ and

$$\gamma = \frac{2\sqrt{\lambda}}{q}. \tag{4.3}$$

We are interested in the case when q and $2\sqrt{\lambda}$ are of the same order, or more precisely, $\gamma\to 1$. In this section, and also in Sections 5 and 6, we consider the RHP (1.26) with parameter γ and q,

$$\begin{cases} Y(z;q) \quad \text{analytic in} \quad \mathbb{C}-\Sigma, \\ Y_+(z;q) = Y_-(z;q)\begin{pmatrix} 1 & \frac{1}{z^q}e^{\frac{q\gamma}{2}(z+z^{-1})} \\ 0 & 1 \end{pmatrix} \quad \text{on} \quad \Sigma, \\ Y(z;q) = \big(I + O(\tfrac{1}{z})\big)z^{q\sigma_3} \quad \text{as} \quad z\to\infty, \end{cases} \tag{4.4}$$

rather than λ and q. With γ fixed, the RHP (4.4) is of the Plancherel-Rotach type with *varying* exponential weight $e^{\frac{q\gamma}{2}(z+z^{-1})}$ on the unit circle (see [Sz1], [DKMVZ1]). A similar problem on the real line is analyzed in [DKMVZ1] without double scaling limit (γ is kept fixed). Our goal is to find the large q behavior of $Y_{21}(0;q)$ for all $\gamma>0$.

Let $d\mu(s)$ be a probability measure on the unit circle. Define

$$g(z) \equiv \int_\Sigma \log(z-s)d\mu(s), \tag{4.5}$$

where for each θ, the branch is chosen such that $\log(z-e^{i\theta})$ is analytic in $\mathbb{C} - (-\infty,-1]\cup\{e^{it} : -\pi\le t\le\theta\}$ (see Figure 7) and $\log(z-e^{i\theta})\sim\log z$ for real $z\to\infty$. The following lemma is based on related calculations in [DKM].

Lemma 4.2. *Suppose $d\mu(z) = u(\theta)d\theta$ is an absolutely continuous probability measure on the unit circle and $u(\theta) = u(-\theta)$. Then $g(z)$ has the following properties:*

(i) *g is analytic in $\mathbb{C}-\Sigma\cup(-\infty,-1)$.*
(ii) *On $(-\infty,-1), g_+(z)-g_-(z) = 2\pi i$.*
(iii) *$g(z) = \log z + O(\frac{1}{z})$ as $z\to\infty$.*
(iv) *$e^{qg(z)}$ is analytic in $\mathbb{C}-\Sigma$.*
(v) *$e^{qg(z)} = z^q(1+O(\frac{1}{z}))$ as $z\to\infty$.*
(vi) *$g(0) = \pi i$.*
(vii) *$g_+(z)+g_-(z) = 2\int_{-\pi}^{\pi}\log|z-s|d\mu(s) + i(\phi+\pi)$ on $z\in\Sigma$ where $\phi = arg(z)$.*
(viii) *$g_+(z)-g_-(z) = 2\pi i\int_\phi^\pi d\mu(s)$ on $z\in\Sigma$.*

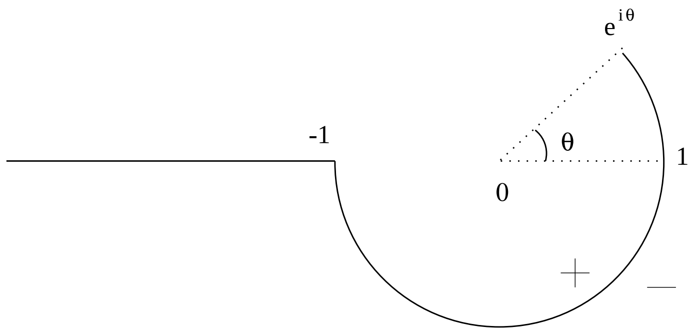

FIGURE 7. Branch cut of $\log(z - e^{i\theta})$

Proof. (i)–(v) are trivial. For (vi),

$$g(0) = \int_{-\pi}^{\pi} \log{(0 - e^{i\theta})} u(\theta) d\theta = \int_{-\pi}^{\pi} i(\theta + \pi) u(\theta) d\theta = \pi i$$

using the evenness of $u(\theta)$.

For (vii), fix $z = e^{i\phi} \in \Sigma$. Then $arg(z - e^{i\theta})$ is analytic if $-\pi < \theta < \phi$ and

$$g_+(z) = \int_{-\pi}^{\pi} \log|z - e^{i\theta}| d\mu(z) + i \int_{-\pi}^{\phi} arg(z - e^{i\theta}) d\mu(z) + i \int_{\phi}^{\pi} arg_+(z - e^{i\theta}) d\mu(z),$$
$$g_-(z) = \int_{-\pi}^{\pi} \log|z - e^{i\theta}| d\mu(z) + i \int_{-\pi}^{\phi} arg(z - e^{i\theta}) d\mu(z) + i \int_{\phi}^{\pi} arg_-(z - e^{i\theta}) d\mu(z).$$

Note that for $\phi < \theta < \pi$,

$$arg_+(e^{i\phi} - e^{i\theta}) - arg_-(e^{i\phi} - e^{i\theta}) = 2\pi.$$

This yields

$$g_+(z) + g_-(z)$$
$$= 2 \int_{-\pi}^{\pi} \log|z - e^{i\theta}| d\mu(z) + 2i \int_{-\pi}^{\pi} arg_+(e^{i\phi} - e^{i\theta}) d\mu(z) - i \int_{\phi}^{\pi} 2\pi d\mu(z).$$

Set

$$F(\phi) = 2 \int_{-\pi}^{\phi} arg(e^{i\phi} - e^{i\theta}) d\mu(z) + 2 \int_{\phi}^{\pi} arg_+(e^{i\phi} - e^{i\theta}) d\mu(z) - 2\pi \int_{\phi}^{\pi} d\mu(z) - \phi.$$

If we show $F(\phi) \equiv \pi$, then (vii) is proved. Note that (a) $arg(e^{i\phi} - e^{i\phi_-}) = \phi + \frac{\pi}{2}$, (b) $arg_+(e^{i\phi} - e^{i\phi_+}) = \phi + \frac{3\pi}{2}$ and (c) $\frac{d}{d\phi} arg(e^{i\phi} - e^{i\theta}) \equiv \frac{1}{2}$. This gives us $F'(\phi) = 2arg(e^{i\phi} - e^{i\phi_-})u(\phi) - 2arg_+(e^{i\phi} - e^{i\phi_+})u(\phi) + 2\pi u(\phi) \equiv 0$. But $F(\pi) = \pi$. Therefore $F(\phi) \equiv \pi$.

For (viii),

$$g_+(z) - g_-(z) = i \int_{\phi}^{\pi} [arg_+(e^{i\phi} - e^{i\theta}) - arg_-(e^{i\phi} - e^{i\theta})] d\mu(z)$$
$$= i \int_{\phi}^{\pi} 2\pi d\mu(z).$$

□

Let M be the set of probability measures on Σ. The equilibrium measure $d\mu_V(z)$ for potential $V(z) = -\frac{\gamma}{2}(z+z^{-1})$ on the unit circle is defined by the following minimization problem:

$$\inf_{\mu\in M} \iint_{\Sigma\times\Sigma} \log|z-w|^{-1} d\mu(z)d\mu(w) + \int_\Sigma V(z)d\mu(z). \tag{4.6}$$

The infimum is achieved uniquely (see, e.g., [ST]) at the equilibrium measure. Let J denote the support of $d\mu_V$. The equilibrium measure and its support are uniquely determined by the following Euler-Lagrange variational conditions:

$$\begin{aligned} &\text{there exists a real constant } l \text{ such that} \\ &2\int_\Sigma \log|z-s|d\mu_V(s) - V(z) + l = 0 \quad \text{for } z\in \bar{J}, \\ &2\int_\Sigma \log|z-s|d\mu_V(s) - V(z) + l \le 0 \quad \text{for } z\in \Sigma-\bar{J}. \end{aligned} \tag{4.7}$$

In Lemma 4.3 below, we find $d\mu_V$, its support and l explicitly from this variational condition with the aid of Lemma 4.2. Let

$$g(z) = g_V(z) \equiv \int_\Sigma \log(z-s)d\mu_V(s), \tag{4.8}$$

where $d\mu_V$ is the equilibrium measure. Following [DKMVZ1], we define

$$m^{(1)}(z) \equiv e^{\frac{ql}{2}\sigma_3} Y(z) e^{-qg(z)\sigma_3} e^{-\frac{ql}{2}\sigma_3}.$$

Then $m^{(1)}$ solves the following new RHP,

$$\begin{cases} m^{(1)}(z) & \text{is analytic in} \quad \mathbb{C}-\Sigma, \\ m^{(1)}_+(z) = m^{(1)}_-(z)v^{(1)} & \text{on} \quad \Sigma, \\ m^{(1)}(z) = I + O(\frac{1}{z}) & \text{as} \quad z\to\infty, \end{cases} \tag{4.9}$$

where $v^{(1)} = \begin{pmatrix} e^{q(g_- - g_+)} & \frac{1}{z^q}e^{q(g_+ + g_- - V + l)} \\ 0 & e^{q(g_+ - g_-)} \end{pmatrix}$, and

$$\kappa_{q-1}^2 = -Y_{21}(0;q) = -m^{(1)}_{21}(0)e^{ql}e^{qg(0)} = -(-1)^q m^{(1)}_{21}(0)e^{ql}, \tag{4.10}$$

from Lemma 4.2 (vi).

Once again we note that this RHP for $m^{(1)}$ is equivalent to the RHP for Y in the sense that a solution of one RHP yields a solution of the other RHP, and vice versa. Using Lemma 4.2, the jump matrix $v^{(1)}$ is given by

$$\begin{cases} \text{inside the support of } d\mu_V, \\ \qquad \begin{pmatrix} e^{-2q\pi i\int_\phi^\pi d\mu_V(\theta)} & (-1)^q \\ 0 & e^{2q\pi i \int_\phi^\pi d\mu_V(\theta)} \end{pmatrix}, \\ \text{outside the support of } d\mu_V, \\ \qquad \begin{pmatrix} e^{-2q\pi i\int_\phi^\pi d\mu_V(\theta)} & (-1)^q e^{q\left[2\int_{-\pi}^\pi \log|z-e^{i\theta}|d\mu_V(\theta) - V(z) + l\right]} \\ 0 & e^{2q\pi i\int_\phi^\pi d\mu_V(\theta)} \end{pmatrix}. \end{cases} \tag{4.11}$$

As indicated in the Introduction, the purpose of the g-function is to turn exponentially growing terms in the jump matrix for the RHP into oscillatory or exponentially decaying terms: this can be seen explicitly in (4.11), using (4.7).

We have explicit formulae for the equilibrium measure and l. For $0 < \gamma \leq 1$, the equilibrium measure has the whole circle as its support but for $\gamma > 1$, a gap opens up. See also [GW] and [Jo1].

Notation. $\chi_B(\theta)$ denotes the indicator function of the set $B \subset \Sigma$.

Lemma 4.3. *For the weight $V(z) = -\frac{\gamma}{2}(z + z^{-1})$, the equilibrium measure and l are given as follows:*

(i) *If $0 \leq \gamma \leq 1$, then*

$$d\mu_V(\theta) = \frac{1}{2\pi}(1 + \gamma\cos\theta)d\theta \tag{4.12}$$

and $l = 0$.

(ii) *If $\gamma > 1$, then*

$$d\mu_V(\theta) = \frac{\gamma}{\pi}\cos(\frac{\theta}{2})\sqrt{\frac{1}{\gamma} - \sin^2(\frac{\theta}{2})}\chi_{[-\theta_c,\theta_c]}(\theta)d\theta \tag{4.13}$$

and

$$l = -\gamma + \log\gamma + 1, \tag{4.14}$$

where $\sin^2\frac{\theta_c}{2} = \frac{1}{\gamma}, 0 < \theta_c < \pi$. In this case, the inequality in the variational condition (4.7) *is strict.*

Proof. (i) First, it is easy to check that $d\mu_V(\theta)$ defined above in (4.12) is a positive probability measure. We set

$$g(z) = \int_{-\pi}^{\pi} \log(z - e^{i\theta})\frac{1}{2\pi}(1 + \gamma\cos\theta)d\theta.$$

Then

$$g'(z) = \frac{1}{2\pi i}\int_{\Sigma} \frac{1}{z - s}(1 + \frac{\gamma}{2}(s + s^{-1}))\frac{ds}{s}.$$

Using a residue calculation with $g(z) = \log z + O(\frac{1}{z})$ as $z \to \infty$ and $g(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log e^{i(\theta+\pi)}(1 + \gamma\cos\theta)d\theta = \pi i$, we have

$$g(z) = \begin{cases} \log z - \frac{\gamma}{2z}, & |z| > 1, z \notin (-\infty, -1), \\ -\frac{\gamma}{2}z + \pi i, & |z| < 1. \end{cases} \tag{4.15}$$

Therefore we have

$$g_+(z) + g_-(z) = \log z - \frac{\gamma}{2}(z + z^{-1}) + \pi i.$$

From Lemma 4.2 (vii), we have

$$2\int_{-\pi}^{\pi} \log|z - e^{i\theta}|\frac{1}{2\pi}(1 + \gamma\cos\theta)d\theta + \frac{\gamma}{2}(z + z^{-1}) = 0$$

for any $z = e^{i\phi}$ with $l = 0$ as $\log z = i\phi$.

(ii) It is straightforward to check that the above measure (4.13) is a positive probability measure. For $g(z)$ defined as before, we have

$$\begin{aligned} g'(z) &= \int_{-\theta_c}^{\theta_c} \frac{1}{z - e^{i\theta}} \frac{\gamma}{\pi} \cos(\frac{\theta}{2}) \sqrt{\frac{1}{\gamma} - \sin^2 \frac{\theta}{2}} d\theta \\ &= \frac{\gamma}{4\pi i} \int_{-\theta_c}^{\theta_c} \frac{1}{z - s} \frac{s+1}{s^2} \sqrt{(s-\xi)(s-\bar{\xi})} ds, \end{aligned}$$

where $\xi = e^{i\theta_c}$, and the branch is chosen to be analytic in $\mathbb{C} - \{e^{i\theta} : \theta_c \le |\theta| \le \pi\}$ and $\sqrt{(s-\xi)(s-\bar{\xi})} > 0$ for real $s >> 0$. From a residue calculation, we obtain

$$g'(z) = \frac{1}{2z} - \frac{\gamma}{4}(1 - z^{-2}) + \frac{\gamma}{4} \frac{z+1}{z^2} \sqrt{(z-\xi)(z-\bar{\xi})}.$$

Integrating, we have for $|z| > 1$, $z \notin (-\infty, -1)$,

$$g(z) = \frac{1}{2} \log z - \frac{\gamma}{4}(z + z^{-1}) + \frac{\gamma}{2} + \frac{\gamma i}{4} \int_{1_{+0}}^{z} \frac{s+1}{s} \sqrt{(s-\xi)(s-\bar{\xi})} \frac{ds}{si} + g_-(1)$$

and for $|z| < 1$, $z \notin (-1, 0]$,

$$g(z) = \frac{1}{2} \log z - \frac{\gamma}{4}(z + z^{-1}) + \frac{\gamma}{2} + \frac{\gamma i}{4} \int_{1_{-0}}^{z} \frac{s+1}{s} \sqrt{(s-\xi)(s-\bar{\xi})} \frac{ds}{si} + g_+(1),$$

where g_+, g_- denote the limit from inside and outside each and 1_{+0}, 1_{-0} denote the outside and inside limits.

(a) For $|\phi| \le \theta_c$,

$$g_+(z) + g_-(z) = \log z - \frac{\gamma}{2}(z + z^{-1}) + \gamma + g_+(1) + g_-(1).$$

From Lemma 4.2 (vii), we obtain

$$\begin{aligned} g_+(1) + g_-(1) &= 2 \int_{-\theta_c}^{\theta_c} \log|1 - e^{i\theta}| \frac{\gamma}{\pi} \cos\frac{\theta}{2} \sqrt{\frac{1}{\gamma} - \sin^2 \frac{\theta}{2}} d\theta + i(0 + \pi) \\ &= \frac{4\gamma}{\pi} \int_0^{\theta_c} \log(2|\sin\frac{\theta}{2}|) \cos\frac{\theta}{2} \sqrt{\frac{1}{\gamma} - \sin^2\frac{\theta}{2}} d\theta + \pi i \\ &= 2\log 2 - \log\gamma + \frac{8}{\pi} \int_0^{\frac{\pi}{2}} \log(\sin\theta) \cos^2\theta d\theta + \pi i \\ &= 2\log 2 - \log\gamma - 1 + \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \log(\sin\theta) d\theta + \pi i, \end{aligned}$$

after a simple change of variables and integration by parts. But we have

$$\int_0^{\frac{\pi}{2}} \log(\sin\theta) d\theta = -\frac{\pi}{2} \log 2$$

from

$$\begin{aligned}2\int_0^{\frac{\pi}{2}}\log(\sin\theta)d\theta &= \int_0^{\frac{\pi}{2}}\log(\sin\theta)d\theta + \int_0^{\frac{\pi}{2}}\log(\cos\theta)d\theta = \int_0^{\frac{\pi}{2}}\log(\frac{1}{2}\sin 2\theta)d\theta \\ &= -\frac{\pi}{2}\log 2 + \int_0^{\frac{\pi}{4}}\log(\sin 2\theta)d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\log(\sin 2\theta)d\theta \\ &= -\frac{\pi}{2}\log 2 + \int_0^{\frac{\pi}{2}}\log(\sin\theta)d\theta.\end{aligned}$$

Therefore

$$g_+(z) + g_-(z) = \log z - \frac{\gamma}{2}(z + z^{-1}) + \gamma - \log\gamma - 1 + \pi i.$$

From Lemma 4.2 (vii), we obtain the desired result for $|\phi| \le \theta_c$, $z = e^{i\phi}$,

$$2\int_{-\pi}^{\pi}\log|z - e^{i\theta}|d\mu_V(\theta) + \frac{\gamma}{2}(z + z^{-1}) - \gamma + \log\gamma + 1 = 0.$$

(b) for $\theta_c < \phi < \pi$ (the $-\pi < \phi < -\theta_c$ case is similar),

$$\begin{aligned}g_+(z) + g_-(z) = \log z - \frac{\gamma}{2}(z + z^{-1}) + g_+(1) + g_-(1) \\ + \frac{\gamma}{2}\int_{\theta_c}^{\phi}\frac{s+1}{s^2}\sqrt{(s-\xi)(s-\bar{\xi})}ds.\end{aligned}$$

But

$$\frac{\gamma}{2}\int_{\theta_c}^{\phi}\frac{s+1}{s^2}\sqrt{(s-\xi)(s-\bar{\xi})}ds = -\frac{\gamma}{2}\int_{\theta_c}^{\phi}\cos\frac{\theta}{2}\sqrt{\sin^2\frac{\theta}{2} - \frac{1}{\gamma}}d\theta < 0.$$

Therefore, using Lemma 4.2 (vii) and calculations in (a), we obtain for $|\phi| > \theta_c$,

$$2\int_{-\pi}^{\pi}\log|z - e^{i\theta}|d\mu_V(\theta) + \frac{\gamma}{2}(z + z^{-1}) - \gamma + \log\gamma + 1 < 0.$$

$\square$

In the following sections, we distinguish the two cases $\gamma \le 1$ and $\gamma > 1$ due to the difference of the supports of their equilibrium measures.

5. $0 \le \gamma \le 1$

From (4.15), we have the explicit formula for the g-function:

$$g(z) = \begin{cases}\log z - \frac{\gamma}{2z}, & |z| > 1, z \notin (-\infty, -1), \\ -\frac{\gamma}{2}z + \pi i, & |z| < 1.\end{cases}$$

With this g, $l = 0$ from Lemma 4.3 (i), and our RHP (4.9), or equivalently (4.11), becomes

$$\text{(5.1)}\qquad \begin{cases} m^{(1)} \quad \text{analytic in } \ \mathbb{C} - \Sigma, \\ m^{(1)}_+ = m^{(1)}_- \begin{pmatrix} (-1)^q z^q e^{\frac{q\gamma}{2}(z - z^{-1})} & (-1)^q \\ 0 & (-1)^q z^{-q} e^{\frac{-q\gamma}{2}(z - z^{-1})} \end{pmatrix} \quad \text{on } \ \Sigma, \\ m^{(1)} = I + O(\frac{1}{z}) \quad \text{as } \ z \to \infty \end{cases}$$

and $\kappa_{q-1}^2 = -(-1)^q m^{(1)}_{21}(0)$ from (4.10).

We define $m^{(2)}$ in terms of $m^{(1)}$ as follows:

$$\text{for even } q,\quad \begin{cases} m^{(2)} \equiv m^{(1)}, & |z|>1,\\ m^{(2)} \equiv m^{(1)}\left(\begin{smallmatrix} 0 & -1\\ 1 & 0\end{smallmatrix}\right), & |z|<1.\end{cases}$$
$$\text{for odd } q,\quad \begin{cases} m^{(2)} \equiv \left(\begin{smallmatrix} 1 & 0\\ 0 & -1\end{smallmatrix}\right) m^{(1)}\left(\begin{smallmatrix} 1 & 0\\ 0 & -1\end{smallmatrix}\right), & |z|>1,\\ m^{(2)} \equiv \left(\begin{smallmatrix} 1 & 0\\ 0 & -1\end{smallmatrix}\right) m^{(1)}\left(\begin{smallmatrix} 0 & -1\\ -1 & 0\end{smallmatrix}\right), & |z|<1.\end{cases} \tag{5.2}$$

Then we have a new equivalent RHP

$$\begin{cases} m_+^{(2)} = m_-^{(2)} v^{(2)} & \text{on} \quad \Sigma,\\ m^{(2)} = I + O(\frac{1}{z}) & \text{as} \quad z\to\infty,\end{cases} \tag{5.3}$$

where $v^{(2)} = \begin{pmatrix} 1 & -(-1)^q z^q e^{\frac{q\gamma}{2}(z-z^{-1})}\\ (-1)^q z^{-q} e^{\frac{-q\gamma}{2}(z-z^{-1})} & 0\end{pmatrix}$ and $\kappa_{q-1}^2 = m_{22}^{(2)}(0)$.

Introduce $\Sigma^{(3)} = \Sigma^{(3)}(\gamma) = \Sigma_{in}^{(3)} \cup \Sigma_{out}^{(3)}$ (see Figure 8) as follows. For fixed $\pi/2 < |\theta| \le \pi$,

$$F(\rho) \equiv F(\rho,\theta) = Re\Big(\frac{\gamma}{2}(z-z^{-1}) + \log z\Big) = \frac{\gamma}{2}(\rho-\rho^{-1})\cos\theta + \log\rho, \tag{5.4}$$

where $z=\rho e^{i\theta}$, $0<\rho\le 1$, has the minimum at

$$\rho = \rho_\theta \equiv \frac{1-\sqrt{1-\gamma^2\cos^2\theta}}{-\gamma\cos\theta} \tag{5.5}$$

and $F(\rho_\theta)<0$. (Note that $\rho_\theta<0$ for $|\theta|<\pi/2$.) For $\frac{1}{2}\le\gamma\le 1$, we take

$$\begin{aligned} \Sigma_{in}^{(3)} &= \{\rho_\theta e^{i\theta} : 3\pi/4 \le |\theta| \le \pi\} \cup \{\rho_{3\pi/4} e^{i\theta} : |\theta| \le 3\pi/4\},\\ \Sigma_{out}^{(3)} &= \{\rho_\theta^{-1} e^{i\theta} : 3\pi/4 \le |\theta| \le \pi\} \cup \{\rho_{3\pi/4}^{-1} e^{i\theta} : |\theta| \le 3\pi/4\}.\end{aligned} \tag{5.6}$$

Orient $\Sigma^{(3)}$ as in Figure 8. And finally, for $0\le\gamma\le\frac{1}{2}$, set $\Sigma^{(3)}(\gamma) = \Sigma^{(3)}(\frac{1}{2})$.

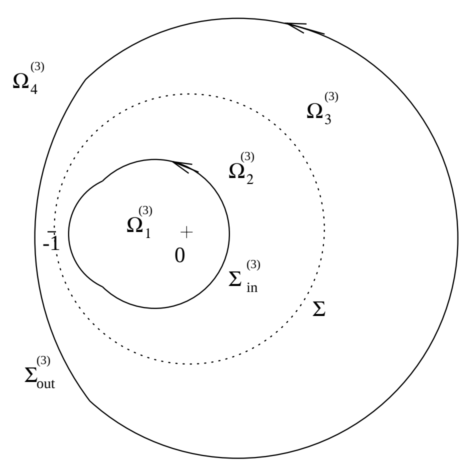

FIGURE 8. $\Sigma^{(3)}$ and $\Omega^{(3)}$

Of course, $\Sigma^{(3)}$ varies with $\gamma \in [0,1]$. However, using estimates from [GK], it is not difficult to show that the Cauchy operators $C_\pm$ on $L^2(\Sigma^{(3)})$ are uniformly bounded,

$$\|C_\pm\|_{L^2(\Sigma^{(3)})\to L^2(\Sigma^{(3)})} \le C < \infty \tag{5.7}$$

for all $0 \le \gamma \le 1$. Observe also that in the limit $\gamma \to 1$, $\Sigma^{(3)}$ takes the form of the cross

$$y = \pm|x+1| \quad \text{for } z = x+iy \text{ near } -1. \tag{5.8}$$

Apart from the neighborhood of $z=-1$, there is considerable freedom in the choice of $\Sigma^{(3)}$. For example, $3\pi/4$ could be replaced by any angle between $\pi/2$ and π. Also the form of the contour for $|\theta| < 3\pi/4$ is not critical, as long as it has the general shape drawn in Figure 8: all that we really need is that the jump matrix $v^{(3)}$ below has the property $\sup_{\{z\in\Sigma^{(3)}:|arg(z)|<3\pi/4\}} |v^{(3)} - I| \to 0$ exponentially as $q\to\infty$.

Using the factorization

$$v^{(2)} = \begin{pmatrix} 1 & 0 \\ (-1)^q z^{-q} e^{\frac{-q\gamma}{2}(z-z^{-1})} & 1 \end{pmatrix} \begin{pmatrix} 1 & -(-1)^q z^q e^{\frac{q\gamma}{2}(z-z^{-1})} \\ 0 & 1 \end{pmatrix} \equiv (b_-^{(2)})^{-1} b_+^{(2)},$$

we define

$$\begin{cases} m^{(3)} = m^{(2)}(b_+^{(2)})^{-1} & \text{in} \quad \Omega_2^{(3)}, \\ m^{(3)} = m^{(2)}(b_-^{(2)})^{-1} & \text{in} \quad \Omega_3^{(3)}, \\ m^{(3)} = m^{(2)} \quad \text{in} \quad \Omega_1^{(3)}, \Omega_4^{(3)}. \end{cases}$$

Then $m^{(3)}$ solves the RHP $(\Sigma^{(3)}, v^{(3)})$, where

$$\begin{cases} v^{(3)} = \begin{pmatrix} 1 & -(-1)^q z^q e^{\frac{q\gamma}{2}(z-z^{-1})} \\ 0 & 1 \end{pmatrix} & \text{on} \quad \Sigma_{in}^{(3)}, \\ v^{(3)} = \begin{pmatrix} 1 & 0 \\ (-1)^q z^{-q} e^{\frac{-q\gamma}{2}(z-z^{-1})} & 1 \end{pmatrix} & \text{on} \quad \Sigma_{out}^{(3)}, \end{cases} \tag{5.9}$$

and

$$\kappa_{q-1}^2 = m_{22}^{(3)}(0). \tag{5.10}$$

As $q\to\infty$, $v^{(3)}(z) \to I$. Set $\Sigma^\infty = \Sigma^{(3)}$. The RHP

$$\begin{cases} m_+^\infty = m_-^\infty \mathrm{I} \quad \text{on} \quad \Sigma^\infty, \\ m^\infty = I + O(\frac{1}{z}) \quad \text{as} \quad z\to\infty \end{cases} \tag{5.11}$$

has, of course, the unique solution $m^\infty(z) \equiv I$.

Let $0 \le \gamma \le 1-\delta_1$ for some $0<\delta_1<1$. From the choice of $\Sigma^{(3)}$,

$$\begin{aligned} \|v^{(3)} - I\|_{L^\infty(\Sigma^{(3)})} &= \sup_{3\pi/4\le\theta\le5\pi/4} |e^{F(\rho_\theta,\theta)}| \le \sup_{3\pi/4\le\theta\le5\pi/4} |e^{F(\rho_\pi,\theta)}| \\ &\le e^{F(\rho_\pi,\pi)} = e^{q(\sqrt{1-\gamma^2}+\log\frac{1-\sqrt{1-\gamma^2}}{\gamma})}. \end{aligned} \tag{5.12}$$

But, for $0<\gamma\le1$, a straightforward estimate shows that

$$\sqrt{1-\gamma^2} + \log(1-\sqrt{1-\gamma^2}) - \log\gamma \le -\frac{2\sqrt{2}}{3}(1-\gamma)^{3/2} \tag{5.13}$$

so that

$$\|w^{(3)}\|_\infty = \|v^{(3)} - I\|_\infty \le e^{-\frac{2\sqrt{2}}{3}\delta_1^{3/2} q} \to 0 \quad \text{as} \quad q \to \infty. \tag{5.14}$$

Since $\|C_{w^{(3)}}\|_{L^2\to L^2} \le C\|w^{(3)}\|_\infty$, for some constant C independent of γ (see (5.7)), $(I - C_{w^{(3)}})^{-1}$ is invertible for large q and the solution for the RHP $(\Sigma^{(3)}, v^{(3)})$ is given by (see (2.9))

$$m^{(3)}(z) = I + \int_{\Sigma^{(3)}} \frac{((I - C_{w^{(3)}})^{-1}I)(s)(v^{(3)}(s) - I)}{s - z} \frac{ds}{2\pi i}, \quad z \notin \Sigma^{(3)}, \tag{5.15}$$

and (see (5.10))

$$\kappa_{q-1}^2 = m_{22}^{(3)}(0) = 1 + \left(\int_{\Sigma^{(3)}} \frac{((I - C_{w^{(3)}})^{-1}I)(s)(v^{(3)}(s) - I)}{s} \frac{ds}{2\pi i} \right)_{22}. \tag{5.16}$$

Now from the fact that the length of $\Sigma^{(3)}$ is uniformly bounded and $dist(0, \Sigma) \ge c > 0$ for all $\gamma \in [0, 1]$, we obtain,

$$|\kappa_{q-1}^2 - 1| \le C\|v^{(3)} - I\|_\infty \le Ce^{-\frac{2\sqrt{2}}{3}\delta_1^{3/2} q}. \tag{5.17}$$

The above calculation also applies to the case when $\gamma \to 1$ slowly. Indeed, suppose $\frac{1}{2} \le \gamma \le 1 - \frac{M_1}{2^{1/3}q^{2/3}}$, where $M_1 > 0$ is a fixed, sufficiently large number. (The lower bound $\frac{1}{2}$ is chosen for convenience. Any fixed number between 0 and 1 would work.) From (5.13), (5.14), for some constant C which is independent of γ,

$$\|C_{w^{(3)}}\|_{L^2\to L^2} \le Ce^{-\frac{2\sqrt{2}}{3} q(1-\gamma)^{3/2}} \le Ce^{-\frac{2\sqrt{2}}{3} M_1^{3/2}} \le \frac{1}{2} < 1, \tag{5.18}$$

if M_1 is sufficiently large. For convenience, we only consider $M_1 \ge 1$. From (5.15),

$$\begin{aligned} m^{(3)} &= I + \frac{1}{2\pi i} \int_{\Sigma^{(3)}} \frac{[(I - C_{w^{(3)}})^{-1}I](s)w^{(3)}(s)}{s - z} \, ds \\ &= I + \frac{1}{2\pi i} \int_{\Sigma^{(3)}} \frac{w^{(3)}(s) + [(I - C_{w^{(3)}})^{-1}C_{w^{(3)}}I](s)w^{(3)}(s)}{s - z} \, ds \end{aligned}$$

and, as $diag(w^{(3)}) = 0$,

$$\kappa_{q-1}^2 = m_{22}^{(3)}(0) = 1 + \Big(\frac{1}{2\pi i} \int_{\Sigma^{(3)}} [(I - C_{w^{(3)}})^{-1}C_{w^{(3)}}I](s)w^{(3)}(s) \frac{ds}{s} \Big)_{22}. \tag{5.19}$$

Hence, we have

$$\begin{aligned} |\kappa_{q-1}^2 - 1| &\le \|(I - C_{w^{(3)}})^{-1}C_{w^{(3)}}I\|_{L^2} \|\frac{w^{(3)}(s)}{2\pi i s}\|_{L^2} \\ &\le \|(I - C_{w^{(3)}})^{-1}\|_{L^2\to L^2} \|C_{w^{(3)}}I\|_{L^2} \|\frac{w^{(3)}(s)}{2\pi i s}\|_{L^2} \\ &\le C\|w^{(3)}\|_{L^2}^2 \\ &\le C\|w^{(3)}\|_{L^\infty} \|w^{(3)}\|_{L^1} \\ &\le Ce^{-\frac{2\sqrt{2}}{3} q(1-\gamma)^{3/2}} \|w^{(3)}\|_{L^1}, \end{aligned} \tag{5.20}$$

where the (final) constant C is independent of γ, q and M_1 (sufficiently large), provided that $0 \le \gamma \le 1 - \frac{M_1}{2^{1/3}q^{2/3}}$.

Since the length of $\Sigma^{(3)}$ is bounded, we have $\|w^{(3)}\|_{L^1} \le Ce^{-\frac{2\sqrt{2}}{3} q(1-\gamma)^{3/2}}$, which is the same estimation (5.17) as in the case $\gamma < 1 - \delta_1$. But for future calculations (see (7.2) below), we need a sharper result. We estimate $\|w^{(3)}\|_{L^1}$ as follows: Focus on $\Sigma_{in}^{(3)}$. For $\Sigma_{out}^{(3)}$, similar computations apply. Only the 12-component of $w^{(3)}$ is non-zero. Set $\widetilde{\theta} = \frac{1}{q^{1/3}} \log q$.

$$\int_{\Sigma_{in}^{(3)}} |z^q e^{\frac{q\gamma}{2}(z-z^{-1})}||dz| = (1) + (2),$$

where (1) is an integration over $|\theta| \le \pi - \widetilde{\theta}$ and (2) covers the remainder. Note from (5.8) that $|dz| \le C d\theta$. Substituting ρ_θ (5.5) into $F(\rho, \theta)$ (5.4), we obtain on $\Sigma_{in}^{(3)}$,

$$|z^q e^{\frac{q\gamma}{2}(z-z^{-1})}| \le e^{q\left(\sqrt{1-\gamma^2\cos^2\theta} + \log(1-\sqrt{1-\gamma^2\cos^2\theta}) - \log(-\gamma\cos\theta)\right)}. \tag{5.21}$$

Setting $\gamma \to -\gamma\cos\theta$ in (5.13), we obtain for $z \in \Sigma_{in}^{(3)}$,

$$|z^q e^{\frac{q\gamma}{2}(z-z^{-1})}| \le e^{-\frac{2\sqrt{2}q}{3}(1+\gamma\cos\theta)^{3/2}} \le e^{-Cq(\pi-|\theta|)^3}. \tag{5.22}$$

Hence, adjusting the constants C if necessary, we have

$$(1) \le C e^{-Cq\widetilde{\theta}^3} \le \frac{C}{q^{1/3}}$$

and

$$(2) \le C \int_0^{\widetilde{\theta}} e^{-Cq\theta^3} d\theta \le C \int_0^{\log q} e^{-Ct^3} \frac{dt}{q^{1/3}} \le \frac{C}{q^{1/3}}.$$

Therefore,

$$\|w^{(3)}\|_{L^1} \le \frac{C}{q^{1/3}} \tag{5.23}$$

and we obtain

$$|\kappa_{q-1}^2 - 1| \le \frac{C}{q^{1/3}} e^{-\frac{2\sqrt{2}}{3}q(1-\gamma)^{3/2}}. \tag{5.24}$$

Let $M_2 > 0$ be a fixed number and consider $1 - \frac{M_2}{2^{1/3}q^{2/3}} \le \gamma \le 1$. For this case, as $q \to \infty$, $\gamma \to 1$ and $\rho_{\theta=\pi} \to 1$. We need to devote special attention to the neighborhood of $z = -1$, where we will introduce a parametrix for the RHP, which is related to the special solution of the Painlevé II (PII) equation (1.4) given in Section 2. For a discussion of parametrices in RHP's, see, e.g., [DZ2], [DKMVZ1]. Set $\gamma = 1 - \frac{t}{2^{1/3}q^{2/3}}$. The region above corresponds to $0 \le t \le M_2$. Let $\mathcal{O}$ be a small neighborhood of size ϵ around $z = -1$, where $\epsilon > 0$ is a fixed number which is small enough so that first,

(5.25) the map u defined below is a bijection from $\mathcal{O}$,

and second,

(5.26) the inequality (5.32) below is satisfied.

The goal is to solve the RHP for $m^{(3)}$ explicitly in this small region.

Let $u = \frac{1}{2}(z - z^{-1})$ in $\mathcal{O}$. As noted above, we choose and fix $\epsilon > 0$ sufficiently small (in fact, any number $0 < \epsilon < 1$ would do) so that $z \to u(z)$ is a bijection from $\mathcal{O}$ onto some open neighborhood of 0 in the u-plane: under the bijection, $\Sigma \cap \mathcal{O}$ becomes a part of the imaginary axis. Set

$$\lambda(z) = \frac{q^{1/3}u(z)}{i2^{4/3}} = \frac{q^{1/3}}{i2^{4/3}} \frac{1}{2}(z - z^{-1}).$$

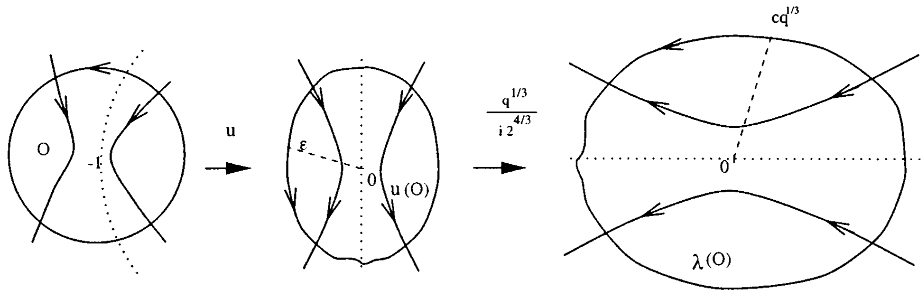

FIGURE 9. The map $z \to \lambda(z)$

Note that with ϵ fixed, there are constants $c_1, c_2 > 0$ such that

$$c_1 q^{1/3} \le |\lambda(z)| \le c_2 q^{1/3}, \tag{5.27}$$

for all $z \in \partial\mathcal{O}$. Under the map $z \to \lambda(z)$ (see Figure 9), $\Sigma \cap \mathcal{O}$ now becomes part of the real axis and

$$\lambda(\Sigma^{(3)} \cap \mathcal{O}) = \{x + iy : y^2 = \frac{\frac{q^{2/3}}{2^{8/3}}(1-\gamma^2) + x^2}{1 + \frac{2^{8/3}\gamma^2 x^2}{q^{2/3}}}, |x| \le cq^{1/3}\}, \tag{5.28}$$

where c is a fixed small number. As $\frac{q^{2/3}}{2^{8/3}}(1-\gamma^2) = \frac{t}{4}\big(1 - \frac{t}{2^{4/3}q^{2/3}}\big)$, $0 \le t \le M_2$, we see that the contour $\lambda(\Sigma^{(3)} \cap \mathcal{O})$ makes an angle $\le \pi/4$ and uniformly bounded away from zero as $q \to \infty$, hence has the general shape of the contour in Figure 5, Section 2, within the ball $\lambda(\mathcal{O})$. We *define*

$$\Sigma^{PII,2} \cap \lambda(\mathcal{O}) \equiv \lambda(\Sigma^{(3)} \cap \mathcal{O}) \tag{5.29}$$

and extend $\Sigma^{PII,2}$ smoothly outside $\lambda(\mathcal{O})$ in such a way that it is asymptotic to straight lines making angles between 0 and $\pi/3$ with the real axis. It is clear from the estimation in Section 2, and the preceding calculations, that for such a contour $\Sigma^{PII,2}$, the bound (2.21) for the solution $m^{PII,2}(z,t)$ of $(\Sigma^{PII,2}, v_t^{PII,2})$

$$\text{is uniform for } \gamma, q \text{ satisfying the relation } 1 - \frac{M_2}{2^{1/3}q^{2/3}} \le \gamma \le 1. \tag{5.30}$$

Introduce the parametrix around $z = -1$ as follows. Define

$$\begin{cases} m_p(z) = m^{PII,2}(\lambda(z), t) & \text{in} \quad \mathcal{O} - \Sigma^{(3)}, \\ m_p(z) = I & \text{in} \quad \bar{\mathcal{O}}^c - \Sigma^{(3)}. \end{cases}$$

As $q \to \infty$, $|\lambda(z)| \to \infty$ for $z \in \partial\mathcal{O}$, and we have for $v_p(z) \equiv v_t^{PII,2}(\lambda(z))$,

$$\begin{cases} m_p(z) \text{ is analytic in } \mathbb{C} - (\Sigma^{(3)} \cup \partial\mathcal{O}), \\ m_{p+}(z) = m_{p-}(z) v_p(z) \text{ on } \mathcal{O} \cap \Sigma^{(3)}, \\ m_{p+}(z) = m_{p-}(z) I \text{ on } \mathcal{O}^c \cap \Sigma^{(3)}, \\ m_{p+}(z) = I + \frac{m_1^{PII,2}(t)}{\lambda(z)} + O(\frac{1}{\lambda(z)^2}) \text{ on } \partial\mathcal{O} \text{ as } q \to \infty. \end{cases} \tag{5.31}$$

The key fact is that v_p is an approximation to $v^{(3)}$ with error of order $\frac{1}{q^{2/3}}$. We compare, for example, the 12-components of $v^{(3)}$ and v_p on $\Sigma^{(3)}$. We focus on

$\mathcal{O}\cap\Sigma_{in}^{(3)}$. Using the u variable, the 12-entries of $v^{(3)}$ and v_p are

$$-\exp(q[\gamma u+\log(\sqrt{1+u^2}-u)])$$

and

$$-\exp(q[\gamma u-u+\frac{1}{6}u^3])$$

respectively. By (5.21) and (5.22), we have for $z\in\mathcal{O}\cap\Sigma_{in}^{(3)}$,

$$|e^{q[\gamma u+\log(\sqrt{1+u^2}-u)]}|=|z^q e^{\frac{q\gamma}{2}(z-z^{-1})}|\le e^{-\frac{2\sqrt{2}q}{3}(1+\gamma\cos\theta)^{3/2}}.$$

From the Taylor expansion of the odd function $\log(\sqrt{1+u^2}-u)$,

$$\log(\sqrt{1+u^2}-u)=-u+\frac{1}{6}u^3+u^5r(u),$$

where $r(0)=-\frac{3}{40}$ and $r(u)$ is bounded for small u, say $|u|\le\frac{1}{2}$. Set $\hat{c}=\sup_{|u|\le\frac{1}{2}}|r(u)|$. Note that for $z=\rho_\theta e^{i\theta}\in\mathcal{O}\cap\Sigma_{in}^{(3)}$ (see (5.5)), as $q\to\infty$, we have $|u|\le\widetilde{c}(1+\gamma\cos\theta)^{1/2}\le\widetilde{c}(c'\epsilon)^{1/2}$. Therefore, if we have chosen $\epsilon>0$ small enough so that

$$-\frac{2\sqrt{2}}{3}+\hat{c}(\widetilde{c})^5c'\epsilon\le-\frac{1}{2}, \tag{5.32}$$

we obtain, as $q\to\infty$,

$$\begin{aligned}
&|e^{q[\gamma u+\log(\sqrt{1+u^2}-u)]}-e^{q[\gamma u-u+\frac{1}{6}u^3]}|\\
&\quad=|e^{q[\gamma u+\log(\sqrt{1+u^2}-u)]}||1-e^{-qu^5r(u)}|\\
&\quad\le e^{-\frac{2\sqrt{2}q}{3}(1+\gamma\cos\theta)^{3/2}}\times q|u|^5\|r\|_{L^\infty(\{|u|\le\frac{1}{2}\})}e^{q\hat{c}(\widetilde{c})^5(1+\gamma\cos\theta)^{5/2}}\\
&\quad\le Cq(1+\gamma\cos\theta)^{5/2}e^{q(-\frac{2\sqrt{2}}{3}+\hat{c}(\widetilde{c})^5c'\epsilon)(1+\gamma\cos\theta)^{3/2}}\\
&\quad\le Cq(1+\gamma\cos\theta)^{5/2}e^{-\frac{q}{2}(1+\gamma\cos\theta)^{3/2}}\\
&\quad\le\frac{C}{q^{2/3}},
\end{aligned}\tag{5.33}$$

where we have used the basic inequality $|1-e^z|\le|z|e^{|z|}$ and the fact that

$$\|x^{5/3}e^{-x^{3/2}}\|_{L^\infty_{(0,\infty)}}\le C.$$

Since

$$v^{(3)}v_p^{-1}=\begin{pmatrix}1 & -e^{q[\gamma u+\log(\sqrt{1+u^2}-u)]}+e^{q[\gamma u-u+\frac{1}{6}u^3]}\\ 0 & 1\end{pmatrix}\quad\text{on}\quad\mathcal{O}\cap\Sigma_{in}^{(3)},$$

we have

$$\|v^{(3)}v_p^{-1}-I\|_{L^\infty(\mathcal{O}\cap\Sigma_{in}^{(3)})}=O(\frac{1}{q^{2/3}}).$$

For $\mathcal{O}\cap\Sigma_{out}^{(3)}$, we have a similar estimation. On the other hand, for $\mathcal{O}^c$, the error is exponentially small; $\|v^{(3)}v_p^{-1}-I\|_{L^\infty(\mathcal{O}^c\cap\Sigma^{(3)})}=\|v^{(3)}-I\|_{L^\infty(\mathcal{O}^c\cap\Sigma^{(3)})}=O(e^{-cq})$.

Now define $R(z) = m^{(3)} m_p^{-1}$. The ratio is analytic in $\mathbb{C} - (\Sigma^{(3)} \cup \partial\mathcal{O})$ and the above calculations show that the jump matrix $v_R = m_{p-} v^{(3)} v_p^{-1} m_{p-}^{-1}$ satisfies

$$(5.34) \quad \begin{cases} \|v_R - I\|_\infty \le \frac{C(M_2)}{q^{2/3}} & \text{on } \mathcal{O} \cap \Sigma^{(3)}, \\ \|v_R - I\|_\infty \le Ce^{-cq} & \text{on } \mathcal{O}^c \cap \Sigma^{(3)}, \\ v_R = v_p^{-1} = m_{p+}^{-1} = I - \frac{m_1^{PII,2}(t)}{\lambda(z)} + O_{M_2}(\frac{1}{\lambda(z)^2}) & \text{on } \partial\mathcal{O}, \text{ as } q \to \infty. \end{cases}$$

In (5.34), we have used the fact that $m^{PII,2}(z,t)$, and hence m_p, is invertible and bounded for $(z,t) \in \mathbb{C} \times [0, M_2]$ (see (5.30)).

From (5.34) and (5.27), we see that $\|v_R - I\|_\infty = \|w_R\|_\infty \le \frac{C(M_2)}{q^{1/3}}$. In particular, $(I - C_{w_R})$ is invertible for large q and by (2.9), R is given by

$$R(z) = I + \frac{1}{2\pi i} \int_{\Sigma^{(3)} \cup \partial\mathcal{O}} \frac{\mu(s)(v_R - I)}{s - z}\, ds,$$

where μ solves $(I - C_{w_R})\mu = I$. As $\|v_R - I\|_\infty \le \frac{C(M_2)}{q^{1/3}}$, we have $\|\mu - I\|_{L^2} = O_{M_2}(\frac{1}{q^{1/3}})$, and also

$$R_{22}(0) = 1 + \frac{1}{2\pi i} \int_{\Sigma^{(3)} \cup \partial\mathcal{O}} (v_R - 1)_{22}(s) \frac{ds}{s} + O_{M_2}(\frac{1}{q^{2/3}}).$$

Thus, using $\|v_R - I\|_\infty \le \frac{C(M_2)}{q^{2/3}}$ in (5.34) for the second equality, and $m_1^{PII,2} = m_1^{PII,1}$ (see (2.20)) for the last equality, we obtain

$$(5.35) \quad \begin{aligned} \kappa_{q-1}^2 &= R_{22}(0) \\ &= 1 + \frac{1}{2\pi i} \int_{\partial\mathcal{O}} (v_R - 1)_{22}(z) \frac{dz}{z} + O_{M_2}(\frac{1}{q^{2/3}}) \\ &= 1 - \frac{1}{2\pi i} \int_{\partial\mathcal{O}} \frac{m_{1,22}^{PII,2}(t)}{\lambda(z)} \frac{dz}{z} + O_{M_2}(\frac{1}{q^{2/3}}) \\ &= 1 - \frac{m_{1,22}^{PII,2}(t)}{2\pi i} \int_{u(\partial\mathcal{O})} \frac{1}{-i\frac{q^{1/3}}{2^{4/3}} u} \frac{du}{(-\sqrt{u^2+1})} + O_{M_2}(\frac{1}{q^{2/3}}) \\ &= 1 + \frac{i 2^{4/3}}{q^{1/3}} m_{1,22}^{PII,2}(t) + O_{M_2}(\frac{1}{q^{2/3}}) \\ &= 1 + \frac{i 2^{4/3}}{q^{1/3}} m_{1,22}^{PII,1}(t) + O_{M_2}(\frac{1}{q^{2/3}}). \end{aligned}$$

Note that error in (5.35) is uniform for $0 \le t \le M_2$.

We summarize as follows.

Lemma 5.1. *Let $M_1 > 0$ be a fixed number which is sufficiently large so that* (5.18) *is satisfied. Also let $M_2 > 0$ and $0 < \delta_1 < 1$ be fixed numbers. As $q \to \infty$, we have the following results.*

(i) *If $0 \le \gamma \le 1 - \delta_1$, then, for some constants C, c which may depend on δ_1,*

$$|\kappa_{q-1}^2 - 1| \le Ce^{-cq}.$$

(ii) *If $\frac{1}{2} \le \gamma \le 1 - \frac{M_1}{2^{1/3} q^{2/3}}$, then, for some constant C which is independent of M_1 satisfying* (5.18),

$$|\kappa_{q-1}^2 - 1| \le \frac{C}{q^{1/3}} e^{-\frac{2\sqrt{2}}{3} q (1-\gamma)^{3/2}}.$$

(iii) *If* $1 - \frac{M_2}{2^{1/3}q^{2/3}} \le \gamma \le 1$,

$$\left|\kappa_{q-1}^2 - 1 - \frac{i2^{4/3}}{q^{1/3}} m_{1,22}^{PII}(t)\right| \le \frac{C(M_2)}{q^{2/3}},$$

where t is defined by $\gamma = 1 - \frac{t}{2^{1/3}q^{2/3}}$.

6. $\gamma > 1$

Let θ_c be as given in Lemma 4.3, $\sin^2\frac{\theta_c}{2} = \frac{1}{\gamma}$, $0 < \theta_c < \pi$. Decompose $\Sigma = \overline{C_1} \cup C_2$ where $C_1 = \{e^{i\theta} : \theta_c < |\theta| \le \pi\}$ and $C_2 = \Sigma - \overline{C_1}$. Note that on the support of the measure $d\mu_V$ in (4.13), $d\mu_V(\theta) = \frac{\gamma}{\pi}\cos(\frac{\theta}{2})\sqrt{\frac{1}{\gamma} - \sin^2(\frac{\theta}{2})}\chi_{[-\theta_c,\theta_c]}(\theta)d\theta = \frac{\gamma}{4\pi i}\frac{s+1}{s^2}\sqrt{(s-\xi)(s-\xi^{-1})}ds$ for $s = e^{i\theta}$.

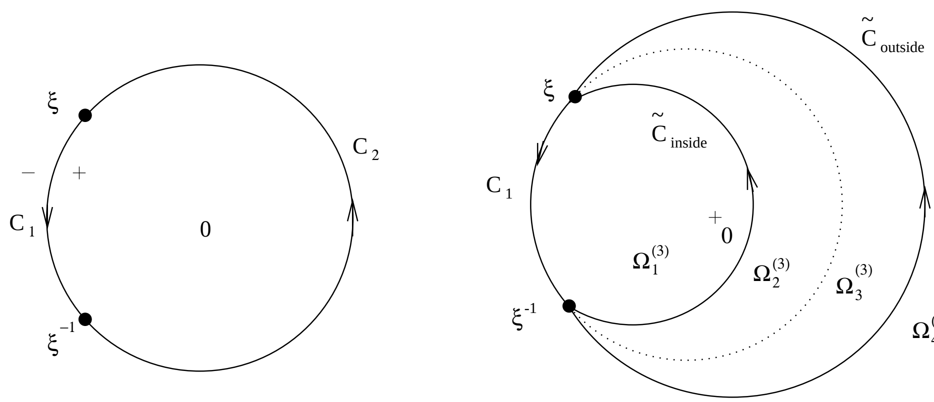

FIGURE 10. Σ and $\Sigma^{(3)}$

Lemma 6.1. *Define* $\alpha(z) = -\frac{\gamma}{4}\int_\xi^z \frac{s+1}{s^2}\sqrt{(s-\xi)(s-\xi^{-1})}ds$, *where* $\xi = e^{i\theta_c}$ *and the branch is chosen to be analytic in* $\mathbb{C} - \overline{C_1}$ *and* $\sqrt{(s-\xi)(s-\xi^{-1})} > 0$ *for real* $s > 0$. *Then*

(i) $e^{2\alpha}$ *is independent of the path in* $\mathbb{C} - (\bar{C}_1 \cup \{0\})$.

(ii) $\exp\bigl(-2\pi i\int_{\theta_c}^{\phi} d\mu_V(\theta)\bigr) = \exp\bigl(2\alpha(z)\bigr)$ *for* $z = e^{i\phi}, |\phi| < \theta_c$.

(iii) $\exp\bigl(2\int_{-\pi}^{\pi}\log|z-e^{i\theta}|d\mu_V(\theta) - V(z) + l\bigr) = \exp\bigl(-2\alpha_-(z)\bigr)$ *for* $z = e^{i\phi}, |\phi| > \theta_c$.

Proof. Property (i) follows from a standard residue calculation: the change in $\alpha(z)$ around the point at 0 is $-\pi i$, and the change in $\alpha(z)$ around C_1 is 0. Property (ii) follows from the definition of $\alpha(z)$. For (iii), set

$$F(\phi) = 2\int_{-\pi}^{\pi}\log|e^{i\phi} - e^{i\theta}|d\mu_V(\theta) + \gamma\cos\phi + l + 2\alpha_-(e^{i\phi})$$

for $z = e^{i\phi}$, $|\phi| > \theta_c$. From the variational condition (4.7), we have $F(\theta_c) = 0$. Differentiating,

$$\begin{aligned} F'(\phi) &= \int_{-\theta_c}^{\theta_c} i\big[\frac{2e^{i\phi}}{e^{i\phi}-e^{i\theta}} - 1\big] d\mu_V(\theta) - \frac{\gamma}{2i}\big(e^{i\phi} - e^{-i\phi}\big) \\ &\quad - \frac{i\gamma}{2}\frac{e^{i\phi}+1}{e^{i\phi}}\sqrt{(e^{i\phi}-\xi)(e^{i\phi}-\xi^{-1})}_- \\ &= \frac{\gamma}{2\pi}\int_{C_2} \frac{z}{z-s}\frac{s+1}{s^2}\sqrt{(s-\xi)(s-\xi^{-1})}ds \\ &\quad - i - \frac{\gamma}{2i}(z - z^{-1}) - \frac{i\gamma}{2}\frac{z+1}{z}\sqrt{(z-\xi)(z-\xi^{-1})}_- . \end{aligned}$$

A residue calculation similar to that in (i) now shows that $F'(\phi) = 0$. Therefore we have $F(\phi) \equiv 0$. □

Note that $e^{2q\pi i \int_\phi^\pi d\mu_V(\theta)} = 1$ for ϕ outside the support of $d\mu_V$, i.e. for $|\phi| > \theta_c$. By (4.11) and the above lemma, our RHP becomes

$$\begin{cases} m^{(1)}(z) \ \text{ is analytic in } \ \mathbb{C} - \Sigma, \\ m_+^{(1)} = m_-^{(1)} \begin{pmatrix} e^{-2q\alpha} & (-1)^q \\ 0 & e^{2q\alpha} \end{pmatrix} \ \text{ on } \ C_2, \\ m_+^{(1)} = m_-^{(1)} \begin{pmatrix} 1 & (-1)^q e^{-2q\alpha_-} \\ 0 & 1 \end{pmatrix} \ \text{ on } \ C_1, \\ m^{(1)} = I + O(\frac{1}{z}) \ \text{ as } \ z \to \infty \end{cases} \tag{6.1}$$

and $\kappa_{q-1}^2 = -(-1)^q m_{21}^{(1)}(0) e^{ql} = -(-1)^q e^{q(-\gamma+\log\gamma+1)} m_{21}^{(1)}(0)$ by (4.10) and (4.14).

We use the same conjugation (5.2) for $m^{(1)}$ as in the case $\gamma \le 1$. Then our new jump matrices for $m^{(2)}$ are

$$\begin{cases} v^{(2)} = \begin{pmatrix} 1 & -e^{-2q\alpha} \\ e^{2q\alpha} & 0 \end{pmatrix} \ \text{ on } \ C_2, \\ v^{(2)} = \begin{pmatrix} e^{-2q\alpha_-} & -1 \\ 1 & 0 \end{pmatrix} \ \text{ on } \ C_1 \end{cases} \tag{6.2}$$

and $\kappa_{q-1}^2 = e^{q(-\gamma+\log\gamma+1)} m_{22}^{(2)}(0)$.

Set $\Sigma^{(3)} = \overline{C_1} \cup \widetilde{C}_{inside} \cup \widetilde{C}_{outside}$ where $\widetilde{C}_{inside}$ and $\widetilde{C}_{outside}$ are open arcs as chosen below. Note the factorization $v^{(2)} = \begin{pmatrix} 1 & 0 \\ e^{2q\alpha} & 1 \end{pmatrix} \begin{pmatrix} 1 & -e^{-2q\alpha} \\ 0 & 1 \end{pmatrix}$ on C_2. Set $Re\alpha = R$, $Im\alpha = I$ so that $\alpha = R + iI$. Recall the Cauchy-Riemann equations in polar coordinates (r, θ),

$$r\frac{\partial R}{\partial r} = \frac{\partial I}{\partial \theta}, \quad r\frac{\partial I}{\partial r} = -\frac{\partial R}{\partial \theta}.$$

For $z = e^{i\theta} \in C_2$, $\alpha(z) = -\pi i \int_{\theta_c}^{\theta} d\mu_V(\theta')$ is purely imaginary and

$$\frac{\partial I}{\partial \theta} = \frac{\partial}{\partial \theta}(-i\alpha) = -\pi\frac{\gamma}{\pi}\cos(\frac{\theta}{2})\sqrt{\frac{1}{\gamma} - \sin^2(\frac{\theta}{2})} < 0.$$

Hence

$$R = 0 \quad \text{and} \quad \frac{\partial R}{\partial r} = \frac{\partial I}{\partial \theta} < 0 \quad \text{on} \quad C_2.$$

Therefore for fixed θ, $e^{i\theta} \in C_2$, there is $\epsilon_1 = \epsilon_1(\theta) > 0$ such that $R = Re\alpha > 0$ (resp. < 0) for $z = re^{i\theta}$ with $1 - \epsilon_1 < r < 1$ (resp. $1 < r < 1 + \epsilon_1$). We take $\widetilde{C}_{inside}$ (resp. $\widetilde{C}_{outside}$) such that $|e^{-2\alpha}| < 1$ (resp. $|e^{2\alpha}| < 1$) on $\widetilde{C}_{inside}$ (resp. $\widetilde{C}_{outside}$). Clearly there exist $0 < \rho_1, \rho_2 < 1$ such that $|e^{-2\alpha}| < \rho_1$ (resp. $|e^{2\alpha}| < \rho_2$), for all $z \in \widetilde{C}_{inside}$ (resp. $z \in \widetilde{C}_{outside}$), apart from a small neighborhood of the endpoints. Introduce the regions $\Omega_k^{(3)}$, $k = 1, 2, 3, 4$, as in Figure 10. Define $m^{(3)}$ as follows,

$$\begin{cases} m^{(3)} = m^{(2)} \begin{pmatrix} 1 & -e^{-2q\alpha} \\ 0 & 1 \end{pmatrix}^{-1} & \text{in} \quad \Omega_2^{(3)}, \\ m^{(3)} = m^{(2)} \begin{pmatrix} 1 & 0 \\ e^{2q\alpha} & 1 \end{pmatrix} & \text{in} \quad \Omega_3^{(3)}, \\ m^{(3)} = m^{(2)} & \text{in} \quad \Omega_1^{(3)}, \Omega_4^{(3)}. \end{cases}$$

Then $v^{(3)}$ is given by

$$\begin{cases} \begin{pmatrix} 1 & -e^{-2q\alpha} \\ 0 & 1 \end{pmatrix} & \text{on} \quad \widetilde{C}_{inside}, \\ \begin{pmatrix} 1 & 0 \\ e^{2q\alpha} & 1 \end{pmatrix} & \text{on} \quad \widetilde{C}_{outside}, \\ \begin{pmatrix} e^{-2q\alpha_-} & -1 \\ 1 & 0 \end{pmatrix} & \text{on} \quad C_1, \end{cases}$$

and

$$\kappa_{q-1}^2 = e^{q(-\gamma + \log\gamma + 1)} m_{22}^{(3)}(0). \tag{6.3}$$

From Lemma 6.1 (iii) and the second variational condition in (4.7), we have, for any $z \in C_1$,

$$e^{-2q\alpha_-} \to 0 \quad \text{as} \quad q \to \infty.$$

(Recall that the inequality in (4.7) is strict from Lemma 4.3 (ii).) Also from the choice of $\widetilde{C}_{inside}$ and $\widetilde{C}_{outside}$,

$$e^{-2q\alpha} \to 0, \quad e^{2q\alpha} \to 0 \quad \text{as} \quad q \to \infty$$

on $\widetilde{C}_{inside}$, $\widetilde{C}_{outside}$, respectively. Therefore $v^{(3)} \to v^{\infty}$ as $q \to \infty$,

$$\begin{cases} v^{\infty} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{on} \quad \widetilde{C}_{inside} \cup \widetilde{C}_{outside}, \\ v^{\infty} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \text{on} \quad C_1. \end{cases} \tag{6.4}$$

The following result can be verified by direct calculation.

Lemma 6.2. *RHP (C_1, v^∞) can be solved explicitly.*

$$m^\infty = \begin{pmatrix} \frac{1}{2}(\beta+\beta^{-1}) & \frac{1}{2i}(\beta-\beta^{-1}) \\ \\ -\frac{1}{2i}(\beta-\beta^{-1}) & \frac{1}{2}(\beta+\beta^{-1}) \end{pmatrix}, \tag{6.5}$$

where $\beta(z) \equiv (\frac{z-\xi}{z-\xi^{-1}})^{1/4}$, is analytic in $\mathbb{C} - \bar{C}_1$ such that $\beta \sim +1$ as $z \to \infty$.

Hence from (6.3) we expect $\kappa_{q-1}^2 \sim e^{q(-\gamma+\log\gamma+1)}\frac{1}{\sqrt{\gamma}}$, because $m_{22}^\infty(0) = \frac{1}{\sqrt{\gamma}}$.

Our goal now is to show that indeed $m^{(3)} \to m^\infty$ as $q \to \infty$. As in Section 5, we must control the behavior of the solution of the RHP for $m^{(3)}$ near the endpoints, where the rate of exponential convergence $v^{(3)} \to v^\infty$ becomes smaller and smaller.

Let $\delta_3, M_4 > 0$ be fixed numbers, let $0 < \delta_4 < 1$ be a fixed, sufficiently small number satisfying (6.35) below, and let $M_3 > 0$ be a fixed, sufficiently large number satisfying (6.32) and (6.39) below. We consider 3 cases for γ:

(i) $1 + \delta_3 \le \gamma$.
(ii) $1 + \frac{M_3}{2^{1/3}q^{2/3}} \le \gamma \le 1 + \delta_4$.
(iii) $1 \le \gamma \le 1 + \frac{M_4}{2^{1/3}q^{2/3}}$.

Calculations similar to those that are needed for the asymptotics of the orthogonal polynomial on the real line (see [DKMVZ1]) show that for $\gamma > 1 + \delta_3$,

$$\kappa_{q-1}^2 = e^{q(-\gamma+\log\gamma+1)}\frac{1}{\sqrt{\gamma}}(1 + O(\frac{1}{q})). \tag{6.6}$$

The error is uniform for $1 + \delta_3 \le \gamma \le L$ for any fixed $L < \infty$. However, we will not use this result, utilizing instead (stronger) estimates from [Jo1] (see the next section).

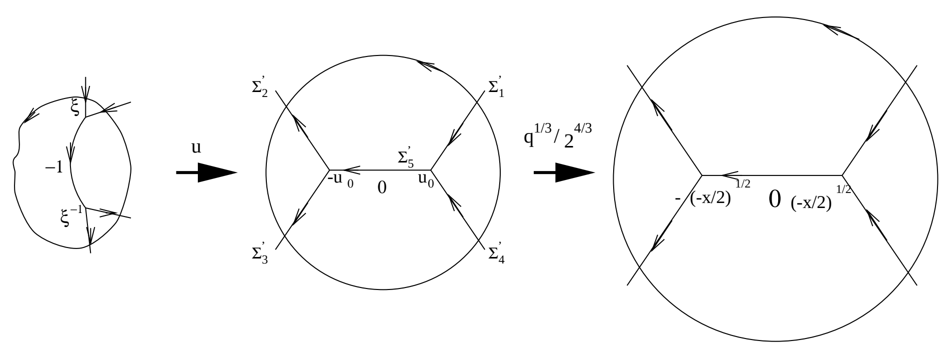

FIGURE 11. The map $z \to \frac{q^{1/3}}{2^{4/3}}u(z)$

We consider case (iii). Set $\gamma = 1 + \frac{t}{2^{1/3}q^{2/3}}$ with $0 \le t \le M_4$ and $u_0 = \sin\theta_c = \frac{2}{\gamma}\sqrt{\gamma - 1}$. In defining $\Sigma^{(3)}$ above, there is some freedom in the choice of $\widetilde{C}_{inside}$ and $\widetilde{C}_{outside}$. We make the following choice (see (6.11) below). Set $x = -\frac{q^{2/3}u_0^2}{2^{5/3}} = -t(1 + \frac{t}{2^{1/3}q^{2/3}})^{-2} \sim -t < 0$ as $q \to \infty$, and let $\Sigma^{PII,3}$ be the contour defined in Figure 6 for this specific x. Let $\Sigma' = \{u = \frac{2^{4/3}}{q^{1/3}}\lambda : \lambda \in \Sigma^{PII,3}\} = \bigcup_{k=1}^5 \Sigma'_k$, and let $\epsilon' > 0$ be small and fixed (see (6.7), (6.10) below). For definiteness, we can and do assume that the rays $\Sigma'_1, \ldots, \Sigma'_4$ make an angle of $\pi/6$ with the real axis (see

Figure 11). Consider $\mathcal{O}' = \{u : |u| < \epsilon'\}$. If q is large enough, then $u_0 \in \mathcal{O}'$. Set $u = u(z) = \frac{1}{2i}(z - z^{-1})$. We choose ϵ' such that (cf. (5.25))

(6.7) u is a bijection from an open neighborhood of $z = -1$ onto $\mathcal{O}'$.

Clearly there are constants $c_1, c_2 > 0$ such that $c_1 \le |z(u)| \le c_2$ for all $u \in \partial\mathcal{O}'$.

Under u^{-1}, the points $u_0, -u_0$ are mapped into $\xi, \bar{\xi}$ respectively, and $u^{-1}(\Sigma_5') = C_1$. Consider a point $z \in u^{-1}(\Sigma_4' \cap \mathcal{O}')$, the inverse image of a point $u \in \Sigma_4' \cap \mathcal{O}$. Changing variables twice, $v = \frac{1}{2i}(s - s^{-1})$ and $w = v^2$,

(6.8)

$$\begin{aligned}
-2\alpha(z) &= \frac{i\gamma}{2}\int_{\xi}^{z} \left(\sqrt{s} + \sqrt{s}^{-1}\right)\sqrt{(s+s^{-1}) - (\xi + \xi^{-1})}\,\frac{ds}{is} \\
&= -i\gamma \int_{u_0}^{u} \left(\sqrt{1-u_0^2} - \sqrt{1-v^2}\right)^{1/2} \frac{\sqrt{1-\sqrt{1-v^2}}}{\sqrt{1-v^2}}\,dv \\
&= \frac{-i\gamma}{2}\int_{u_0}^{u} \left(\frac{v^2 - u_0^2}{1 + k(u_0^2) + k(v^2)}\right)^{1/2} v\big(1 + h(v^2)\big)\,dv \\
&= -\frac{i\gamma}{4}\int_{u_0^2}^{u^2} (w - u_0^2)^{1/2}\big[1 + (h(w) - h(u_0^2)) + h(u_0^2)\big]\,dw \\
&\quad - \frac{i\gamma}{4}\int_{u_0^2}^{u^2} (w-u_0^2)^{1/2} \frac{\big[(k(u_0^2) - k(w)) - 2k(u_0^2)\big](1 + h(w))\,dw}{(1 + k(u_0^2) + k(w))^{1/2}\big[1 + (1 + k(u_0^2) + k(w))^{1/2}\big]} \\
&= \frac{-i\gamma}{6}(u^2 - u_0^2)^{3/2} + O\Big(|u^2 - u_0^2|^{5/2}\Big) + O\Big(|u^2 - u_0^2|^{3/2}u_0^2\Big),
\end{aligned}$$

where $\sqrt{u^2 - u_0^2}$ is defined to be analytic in $\mathbb{C} - [-u_0, u_0]$ and positive for real $u > u_0$; $h(w) = \frac{\sqrt{2}\sqrt{1-\sqrt{1-w}}}{\sqrt{w}\sqrt{1-w}} - 1$, which is analytic in $|w| \le \epsilon'$ and $h(0) = 0$; $k(w) = \frac{1}{2}(\sqrt{1-w} - 1)$, which is also analytic in $|w| \le \epsilon'$ and $k(0) = 0$. Since Σ_4' is a straight ray of angle $\frac{-\pi}{6}$ at u_0, $Re(-i(u^2 - u_0^2)^{3/2}) \le -\frac{1}{2}|u^2 - u_0^2|^{3/2}$, which yields

$$|\exp(-2\alpha(z))| \le \exp\big(-\frac{1}{24}|u^2 - u_0^2|^{3/2}\big) < 1, \tag{6.9}$$

provided ϵ' is sufficiently small so that

$$\begin{aligned}
O(|u^2 - u_0^2|^{5/2}) + O(|u^2 - u_0^2|^{3/2}u_0^2) &\le c|u^2 - u_0^2|^{3/2}(|u^2 - u_0^2| + u_0^2) \\
&\le \frac{1}{24}|u^2 - u_0^2|^{3/2}
\end{aligned} \tag{6.10}$$

for $u \in \Sigma_4' \cap \mathcal{O}'$, where the terms on the LHS are given in (6.8). The same choice of ϵ' gives rise to the same result for $z \in u^{-1}(\Sigma_3' \cap \mathcal{O}')$, and also $|\exp(2\alpha(z))| \le \exp(-\frac{1}{24}|u^2 - u_0^2|^{3/2}) < 1$ for $z \in u^{-1}((\Sigma_1' \cup \Sigma_2') \cap \mathcal{O}')$. We thus fix $\Sigma^{(3)}$ by choosing

$$\Sigma^{(3)} \equiv u^{-1}(\Sigma' \cap \mathcal{O}') \quad \text{inside } \mathcal{O}', \tag{6.11}$$

and extending it to a contour of the general shape $\bar{C}_1 \cup \widetilde{C}_{inside} \cup \widetilde{C}_{outside}$ as in Figure 10.

Define

$$\begin{cases} m_p(z) = m^{PII,3}\big(\frac{q^{1/3}}{2^{4/3}}u(z), x\big) & \text{in} \quad \mathcal{O} - \Sigma^{(3)}, \\ m_p(z) = I & \text{in} \quad \bar{\mathcal{O}}^c - \Sigma^{(3)}, \end{cases} \tag{6.12}$$

where $m^{PII,3}(z,x)$ solves the RHP of the Painlevé II equation given by (2.25) and (2.26). Then m_p solves the RHP on $\Sigma^{(3)} \cup \partial\mathcal{O}$ in which the jump matrix $v_p(z)$ is given by

$$\begin{cases} v^{PII,3}(\frac{q^{1/3}}{2^{4/3}}u(z)), & z \in \Sigma^{(3)} \cap \mathcal{O}, \\ I, & z \in \Sigma^{(3)} \cap \mathcal{O}^c, \\ m_{p+}(z), & z \in \partial\mathcal{O}, \end{cases} \tag{6.13}$$

where $v^{PII,3}$ is given in (2.26).

We compare $v^{(3)}$ and v_p. First, let $z \in \Sigma^{(3)} \cap \mathcal{O}$ such that $u(z) \in \Sigma_4' \cap \mathcal{O}'$. The 12-entries of $v^{(3)}$ and v_p are $-\exp(-2q\alpha(z))$ and $-\exp(-2ig^{PII}(\frac{q^{1/3}}{2^{4/3}}u(z))) = -\exp(-\frac{iq}{6}(u^2-u_0^2)^{3/2})$, respectively. With ϵ' chosen small as above, using

$$\begin{aligned} &Re(-i(u^2-u_0^2)^{3/2}) \le -\frac{1}{2}|u^2-u_0^2|^{3/2}, \\ &u_0^2 \le \frac{4M_4}{q^{2/3}}, \\ &\|x^{5/2}e^{-x^{3/2}}\|_{L^\infty[0,\infty)} \le C, \\ &\gamma - 1 \le \frac{M_4}{2^{1/3}q^{2/3}}, \end{aligned}$$

we obtain from (6.8),

$$\begin{aligned} &|e^{-2q\alpha(z)} - e^{-2ig^{PII}(\frac{q^{1/3}}{2^{4/3}}u(z))}| \\ &\quad \le e^{Re(-\frac{iq}{6}(u^2-u_0^2)^{3/2})}|e^{-2q\alpha(z)+2ig^{PII}(\frac{q^{1/3}}{2^{4/3}}u(z))} - 1| \\ &\quad \le Ce^{-\frac{q}{24}|u^2-u_0^2|^{3/2}}\left[q|u^2-u_0^2|^{3/2}(|u^2-u_0^2| + u_0^2 + (\gamma-1))\right] \\ &\quad \le \frac{C(M_4)}{q^{2/3}}. \end{aligned} \tag{6.14}$$

In a similar manner, for z such that $u(z) \in \Sigma_3' \cap \mathcal{O}'$, the same result holds and for $z \in \widetilde{C}_{outside} \cap \mathcal{O}$, the difference of the 21-entries of $v^{(3)}$ and v_p satisfies

$$|e^{2q\alpha(z)} - e^{2ig^{PII}(\frac{q^{1/3}}{2^{4/3}}u(z))}| \le \frac{C(M_4)}{q^{2/3}}.$$

For $z \in C_1$, $Re(-i(u^2-u_0^2)_-^{3/2}) = -|u^2-u_0^2|^{3/2}$. Again by (6.8), the difference of the 11-entries of $v^{(3)}$ and v_p satisfies

$$|e^{-2q\alpha_-(z)} - e^{-2ig_-^{PII}(\frac{q^{1/3}}{2^{4/3}}u(z))}| \le \frac{C(M_4)}{q^{2/3}}.$$

Therefore, we have

$$\|v^{(3)}v_p^{-1} - I\|_{L^\infty(\Sigma^{(3)}\cap\mathcal{O})} \le \frac{C(M_4)}{q^{2/3}}. \tag{6.15}$$

Secondly, for $z \in \Sigma^{(3)} \cap \mathcal{O}^c$, $|z-\xi|, |z-\xi^{-1}| \ge c > 0$ implies exponential decay for $e^{-2q\alpha(z)}$ and $e^{2q\alpha(z)}$ for $z \in \widetilde{C}_{inside} \cap \mathcal{O}^c$ and $z \in \widetilde{C}_{outside} \cap \mathcal{O}^c$, respectively. Therefore we have

$$\|v^{(3)}v_p^{-1} - I\|_{L^\infty(\Sigma^{(3)}\cap\mathcal{O}^c)} \le Ce^{-cq}. \tag{6.16}$$

Finally, for $z \in \partial\mathcal{O}$, $|u(z)| = \epsilon'$ and

$$m_{p+}(z) = m^{PII,3}\big(\frac{q^{1/3}}{2^{4/3}}u(z), x\big) = I + \frac{m_1^{PII,3}(x)}{\frac{q^{1/3}}{2^{4/3}}u(z)} + O_{M_4}(\frac{1}{q^{2/3}}), \tag{6.17}$$

by (2.25). Here the error is uniformly for $0 \le x \le 2M_4$.

Now as in the case $\gamma \le 1$, define $R(z) = m^{(3)}m_p^{-1}$. Then the jump matrix for R is given by $v_R = m_{p-}v^{(3)}v_p^{-1}m_{p-}^{-1}$. From (6.15), (6.16) and (6.17), and also (2.28), we have

$$\begin{cases} \|v_R - I\|_{L^\infty(\Sigma^{(3)}\cap\mathcal{O})} \le \frac{C(M_4)}{q^{2/3}} & \text{on } \mathcal{O}\cap\Sigma^{(3)}, \\ \|v_R - I\|_{L^\infty(\Sigma^{(3)}\cap\mathcal{O}^c)} \le Ce^{-cq} & \text{on } \mathcal{O}^c\cap\Sigma^{(3)}, \\ v_R = m_{p+}^{-1} = I - \frac{2^{4/3}m_1^{PII,3}(x)}{q^{1/3}u(z)} + O_{M_4}(\frac{1}{q^{2/3}}) & \text{on } \partial\mathcal{O}. \end{cases} \tag{6.18}$$

As in (5.35) for the case $\gamma \le 1$, using $m_{1,22}^{PII} = m_{1,22}^{PII,3} + (ix^2/8)$ from (2.27),

$$\begin{aligned} m_{22}^{(3)}(0) &= R_{22}(0) \\ &= 1 + \frac{i2^{4/3}}{q^{1/3}}m_{1,22}^{PII,3}(x) + O_{M_4}(\frac{1}{q^{2/3}}) \\ &= 1 + \frac{i2^{4/3}}{q^{1/3}}\Big[m_{1,22}^{PII}(x) - \frac{it^2}{8}\big(1 + \frac{t}{2^{1/3}q^{2/3}}\big)^{-4}\Big] + O_{M_4}(\frac{1}{q^{2/3}}) \\ &= 1 + \frac{i2^{4/3}}{q^{1/3}}m_{1,22}^{PII}(x) + \frac{t^2}{2^{5/3}q^{1/3}} + O_{M_4}(\frac{1}{q^{2/3}}). \end{aligned} \tag{6.19}$$

Therefore, from (6.3) using $x = -t(1 + \frac{t}{2^{1/3}q^{2/3}})^{-2} = -t + O_{M_4}(\frac{1}{q^{2/3}})$ and the fact that $\frac{d}{dt}m_{1,22}^{PII}(t)$ is bounded for $0 \le t \le M_4$ (this follows, for example, from (2.18) and the boundedness of $u(x) = 2im_{1,12}^{PII}$; alternatively statements like (2.28) are true also for all the x-derivatives of $m^{PII,3}(z;x)$, etc.),

(6.20)

$$\begin{aligned} \kappa_{q-1}^2 &= e^{q(-\gamma+\log\gamma+1)}m_{22}^{(3)}(0) \\ &= e^{q(-\gamma+\log\gamma+1)}\Big(1 + \frac{i2^{4/3}}{q^{1/3}}m_{1,22}^{PII}(x) + \frac{t^2}{2^{5/3}q^{1/3}} + O_{M_4}(\frac{1}{q^{2/3}})\Big) \\ &= \Big(1 - \frac{t^2}{2^{5/3}q^{1/3}} + O_{M_4}(\frac{1}{q})\Big)\Big(1 + \frac{i2^{4/3}}{q^{1/3}}m_{1,22}^{PII}(x) + \frac{t^2}{2^{5/3}q^{1/3}} + O_{M_4}(\frac{1}{q^{2/3}})\Big) \\ &= 1 + \frac{i2^{4/3}}{q^{1/3}}m_{1,22}^{PII}(x) + O_{M_4}(\frac{1}{q^{2/3}}) \\ &= 1 + \frac{i2^{4/3}}{q^{1/3}}m_{1,22}^{PII}(-t) + O_{M_4}(\frac{1}{q^{2/3}}). \end{aligned}$$

Finally we consider the case (ii), $1 + \frac{M_3}{2^{1/3}q^{2/3}} \le \gamma \le 1 + \delta_4$. We conjugate $m^{(2)}$ with jump matrix $v^{(2)}$ given by (6.2), as follows:

$$\begin{cases} m^{(4)} \equiv m^{(2)}, & |z| > 1, \\ m^{(4)} \equiv m^{(2)}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), & |z| < 1. \end{cases} \tag{6.21}$$

Define $\widetilde{\alpha}(z) = -\pi i\int_\xi^z \frac{\gamma}{4\pi i}\frac{s+1}{s^2}\sqrt{(s-\xi)(s-\xi^{-1})}ds$, where $\widetilde{\alpha}(z)$ is the same as $\alpha(z)$ in Lemma 6.1, but now we choose the branch so that $\sqrt{(s-\xi)(s-\xi^{-1})}$ is analytic

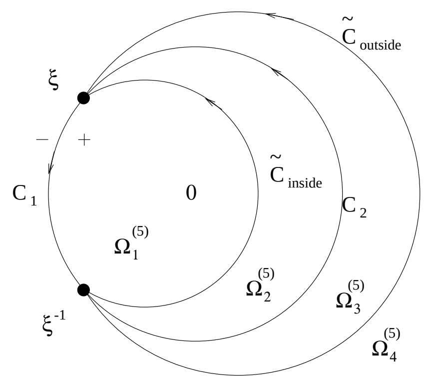

FIGURE 12

in $\mathbb{C} - \bar{C}_2$, and $\sqrt{(s-\xi)(s-\xi^{-1})} \sim +s$ as $s \to \infty$. Then the jump matrix $v^{(4)}$ for $m^{(4)}$ is given by

$$
\begin{cases} v^{(4)} = \begin{pmatrix} e^{2q\tilde{\alpha}_+} & 1 \\ 0 & e^{2q\tilde{\alpha}_-} \end{pmatrix} \quad \text{on} \quad C_2, \\ v^{(4)} = \begin{pmatrix} 1 & e^{-2q\tilde{\alpha}} \\ 0 & 1 \end{pmatrix} \quad \text{on} \quad C_1 \end{cases} \tag{6.22}
$$

and $\kappa_{q-1}^2 = -e^{q(-\gamma+\log\gamma+1)} m_{21}^{(4)}(0)$. Noting the factorization $v^{(4)} = \begin{pmatrix} 1 & 0 \\ e^{2q\tilde{\alpha}_-} & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{2q\tilde{\alpha}_+} & 1 \end{pmatrix}$ on C_2, we define (see Figure 12)

$$
\begin{cases} m^{(5)} = m^{(4)} \begin{pmatrix} 1 & 0 \\ e^{2q\tilde{\alpha}} & 1 \end{pmatrix}^{-1} & \text{in} \quad \Omega_2^{(5)}, \\ m^{(5)} = m^{(4)} \begin{pmatrix} 1 & 0 \\ e^{2q\tilde{\alpha}} & 1 \end{pmatrix} & \text{in} \quad \Omega_3^{(5)}, \\ m^{(5)} = m^{(4)} \quad \text{in} \quad \Omega_1^{(5)} \cup \Omega_4^{(5)}, \end{cases}
$$

so that

$$
\begin{cases} v^{(5)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{on} \quad C_2, \\ v^{(5)} = \begin{pmatrix} 1 & e^{-2q\tilde{\alpha}} \\ 0 & 1 \end{pmatrix} \quad \text{on} \quad C_1, \\ v^{(5)} = \begin{pmatrix} 1 & 0 \\ e^{2q\tilde{\alpha}} & 1 \end{pmatrix} \quad \text{on} \quad \widetilde{C}_{inside} \cup \widetilde{C}_{outside}. \end{cases}
$$

As in the case of α, we have $|e^{-\widetilde{\alpha}(z)}| < 1$ for $z \in C_1$ and $|e^{\widetilde{\alpha}(z)}| < 1$ for $z \in \widetilde{C}_{inside} \cup \widetilde{C}_{outside}$. Therefore taking $q \to \infty$, we have

$$\begin{cases} v^{(5,\infty)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{on } \ C_2, \\ v^{(5,\infty)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{on } \ C_1. \end{cases} \tag{6.23}$$

This RHP can be solved explicitly as in Lemma 6.2, and we find

$$m^{(5,\infty)} = \begin{pmatrix} \frac{1}{2}(\widetilde{\beta} + \widetilde{\beta}^{-1}) & \frac{1}{2i}(\widetilde{\beta} - \widetilde{\beta}^{-1}) \\ -\frac{1}{2i}(\widetilde{\beta} - \widetilde{\beta}^{-1}) & \frac{1}{2}(\widetilde{\beta} + \widetilde{\beta}^{-1}) \end{pmatrix}, \tag{6.24}$$

where $\widetilde{\beta}(z) \equiv (\frac{z-\xi}{z-\xi^{-1}})^{1/4}$ is now analytic in $\mathbb{C} - \bar{C}_2$ and $\widetilde{\beta} \sim +1$ as $z \to \infty$. From (6.24), we have $m_{21}^{(5,\infty)}(0) = -\frac{1}{\sqrt{\gamma}}$ and $\kappa_{q-1}^2 \sim e^{q(-\gamma+\log\gamma+1)}\frac{1}{\sqrt{\gamma}}$ as $q \to \infty$. Again, we need to construct parametrices around ξ and ξ^{-1} in order to prove that indeed $m^{(5)} \to m^{(5,\infty)}$. Note that $\det m^{(5,\infty)} = 1$: this follows either by direct calculation or by a general argument as $\det v^{(5,\infty)} = 1$.

Let Σ'' be the contour $\Sigma'' = \mathbb{R} \cup \mathbb{R}_+ e^{2\pi i/3} \cup \mathbb{R}_+ e^{4\pi i/3}$ shown in Figure 13. Let $\omega = e^{2\pi i/3}$ and set (see [DZ2])

$$\begin{cases} \Psi(s) = \begin{pmatrix} Ai(s) & Ai(\omega^2 s) \\ Ai'(s) & \omega^2 Ai'(\omega^2 s) \end{pmatrix} e^{-\frac{i\pi}{6}}\sigma_3, & 0 < args < \frac{2\pi}{3}, \\ \Psi(s) = \begin{pmatrix} Ai(s) & Ai(\omega^2 s) \\ Ai'(s) & \omega^2 Ai'(\omega^2 s) \end{pmatrix} e^{-\frac{i\pi}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & \frac{2\pi}{3} < args < \pi, \\ \Psi(s) = \begin{pmatrix} Ai(s) & -\omega^2 Ai(\omega s) \\ Ai'(s) & -Ai'(\omega s) \end{pmatrix} e^{-\frac{i\pi}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \pi < args < \frac{4\pi}{3}, \\ \Psi(s) = \begin{pmatrix} Ai(s) & -\omega^2 Ai(\omega s) \\ Ai'(s) & \omega Ai'(\omega s) \end{pmatrix} e^{-\frac{i\pi}{6}}\sigma_3, & \frac{4\pi}{3} < args < 2\pi, \end{cases} \tag{6.25}$$

where $Ai(s)$ is the Airy function. Then Ψ satisfies the jump conditions

$$\begin{cases} \Psi_+ = \Psi_- \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & z \in \mathbb{R}_+, \\ \Psi_+ = \Psi_- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in \mathbb{R}_-, \\ \Psi_+ = \Psi_- \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & z \in \mathbb{R}_+ e^{2\pi i/3}, \mathbb{R}_+ e^{4\pi i/3}. \end{cases} \tag{6.26}$$

Let $\mathcal{O}_\xi$ and $\mathcal{O}_{\bar{\xi}}$ be neighborhoods around ξ and $\bar{\xi}$ of size $\epsilon''\sqrt{\gamma - 1}$, respectively, where $\epsilon'' > 0$ is a small, fixed number chosen to satisfy (6.27), (6.29) below. Since $\frac{1}{2}|\xi - \bar{\xi}| = \frac{2}{\gamma}\sqrt{\gamma - 1} > \epsilon''\sqrt{\gamma - 1}$, $\mathcal{O}_\xi$ and $\mathcal{O}_{\bar{\xi}}$ have no intersection, provided

$$0 < \epsilon'' < 1. \tag{6.27}$$

For definiteness we assume that $\partial\mathcal{O}_\xi$ and $\partial\mathcal{O}_{\bar{\xi}}$ are oriented counterclockwise. In $\mathcal{O}_\xi$, a simple substitution shows that $\widetilde{\alpha}(z) = \frac{2}{3}(z - \xi)^{3/2} G(z)$, where G is analytic

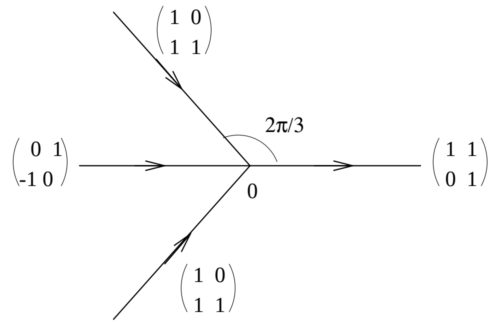

FIGURE 13. Σ'' and v_Ψ

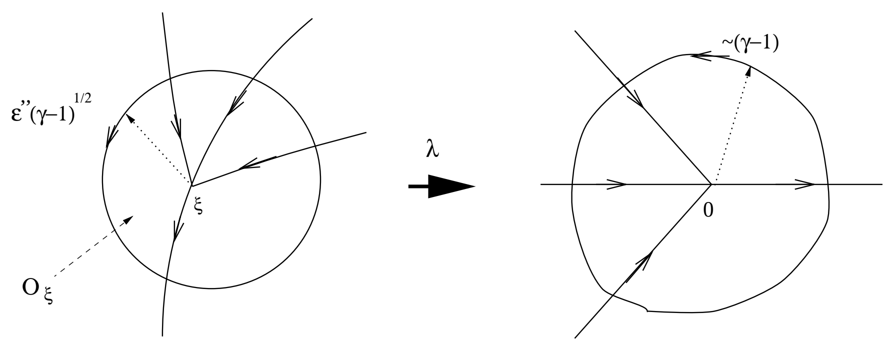

FIGURE 14. The map $z \to (z-\xi)(G(z))^{2/3}$

and $G(\xi) = (\gamma-1)^{3/4}e^{-i(\frac{3}{2}\theta_c+\frac{3}{4}\pi)}$. Here $(z-\xi)^{3/2} = |z-\xi|^{3/2}e^{\frac{3}{2}iarg(z-\xi)}$ and $\theta_c - \pi/2 < arg(z-\xi) < \theta_c + 3\pi/2$. Define (see Figure 14)

$$\lambda(z) \equiv (z-\xi)(G(z))^{2/3}, \tag{6.28}$$

where $(G(z))^{2/3}$ is analytic in $\mathcal{O}_\xi$ and $(G(z))^{2/3} \to (\gamma-1)^{1/2}e^{-i(\theta_c+\pi/2)}$ as $z \to \xi$. Of course, $\lambda^{3/2} = \frac{3}{2}\widetilde{\alpha}$. It is a simple calculus question to verify that we may choose ϵ'' sufficiently small so that

$$\begin{aligned}&z \to \lambda(z) \text{ is a bijection from } \mathcal{O}_\xi \text{ onto an open neighborhood}\\ &\text{of } 0 \text{ in the } \lambda\text{-plane, of radius } \sim (\gamma-1).\end{aligned} \tag{6.29}$$

Define $\Sigma^{(5)} \cap \mathcal{O}_\xi \equiv \{z \in \mathcal{O}_\xi : \lambda(z) \in \Sigma''\}$. As in the construction in [DZ2], set (cf. (4.34) in [DZ2])

$$E(z) = \begin{pmatrix}1 & -1\\ -1 & -1\end{pmatrix} \sqrt{\pi}e^{i\pi/6}q^{\frac{\sigma_3}{6}}\left((z-\bar{\xi})(G(z))^{2/3}\right)^{\frac{\sigma_3}{4}}, \tag{6.30}$$

and for $z \in \mathcal{O}_\xi - \Sigma^{(5)}$, define the parametrix for $m^{(5)}$ by

$$m_p(z) = E(z)\Psi(q^{2/3}\lambda(z))e^{q\widetilde{\alpha}(z)\sigma_3}. \tag{6.31}$$

Then m_p satisfies the same jump conditions on $\Sigma^{(5)} \cap \mathcal{O}_\xi$ as $m^{(5)}$; $m_{p+} = m_{p-}v^{(5)}$. And if q becomes large, then for $z \in \partial\mathcal{O}_\xi$, $|q^{2/3}\lambda(z)| \geq cq^{2/3}(\gamma-1) \geq cM_3/2^{1/3}$.

Therefore,

$$\text{if } M_3 \text{ is sufficiently large so that the leading terms dominate} \\ \text{in the asymptotics for the Airy functions in } \Psi(q^{2/3}\lambda(z)) \text{ (see, e.g., [AS]),} \tag{6.32}$$

then by the explicit choice of $E(z)$ in (6.30), we find for $z \in \partial\mathcal{O}_\xi$,

$$m_{p+} = m^{(5,\infty)}(z)\Big(I + O(\frac{1}{q\lambda^{3/2}(z)})\Big) = m^{(5,\infty)}(z)\Big(I + O(\frac{1}{q(\gamma-1)^{3/2}})\Big). \tag{6.33}$$

Noting the symmetry $m^{(5)} = \overline{m^{(5)}(\bar{z})}$, define $\Sigma^{(5)} \cap \mathcal{O}_{\bar{\xi}} \equiv \overline{\Sigma^{(5)} \cap \mathcal{O}_\xi}$, and for $z \in \mathcal{O}_{\bar{\xi}} - \Sigma^{(5)}$ set $m_p(z) = \overline{m_p(\bar{z})}$. We now extend $\Sigma^{(5)} \cap (\mathcal{O}_\xi \cup \mathcal{O}_{\bar{\xi}})$ to $\Sigma^{(5)}$ to have the same general shape as in Figure 12. Finally, for $z \in \mathbb{C} - (\mathcal{O}_\xi \cup \mathcal{O}_{\bar{\xi}} \cup \Sigma^{(5)})$, define $m_p(z) = m^{(5,\infty)}$.

Set $\mathcal{O} = \mathcal{O}_\xi \cup \mathcal{O}_{\bar{\xi}}$. Then $\widetilde{R} \equiv m^{(5)} m_p^{-1}$ solves a RHP on $\Sigma^{(5)} \cup \partial\mathcal{O}$ with the jump matrix $v_{\widetilde{R}} = m_{p-} v^{(5)} v_p^{-1} m_{p-}^{-1}$,

$$\begin{cases} v_{\widetilde{R}} = I & \text{on} \quad (\Sigma^{(5)} \cap \mathcal{O}) \cup C_2, \\ v_{\widetilde{R}} = m^{(5,\infty)} \begin{pmatrix} 1 & e^{-2q\tilde{\alpha}} \\ 0 & 1 \end{pmatrix} (m^{(5,\infty)})^{-1} & \text{on} \quad C_1 \cap \bar{\mathcal{O}}^c, \\ v_{\widetilde{R}} = m^{(5,\infty)} \begin{pmatrix} 1 & 0 \\ e^{2q\tilde{\alpha}} & 1 \end{pmatrix} (m^{(5,\infty)})^{-1} & \text{on} \quad (\widetilde{C}_{inside} \cup \widetilde{C}_{outside}) \cap \bar{\mathcal{O}}^c, \\ v_{\widetilde{R}} = I + O(\frac{1}{q(\gamma-1)^{3/2}}) & \text{on} \quad \partial\mathcal{O}. \end{cases} \tag{6.34}$$

Let $\epsilon_1 > 0$ be a fixed, small number: for example, we may take $\epsilon_1 = \epsilon'$ satisfying (6.7), (6.10) above. Choose δ_4 sufficiently small so that

$$\xi, \bar{\xi} \in \{z : |z+1| \le \epsilon_1\} \tag{6.35}$$

for $1 \le \gamma \le 1 + \delta_4$. By calculations similar to (6.8) and (6.9), $Re(2\tilde{\alpha}(z)) \le -c(\gamma-1)^{3/2}$ for $z \in (\widetilde{C}_{inside} \cup \widetilde{C}_{outside}) \cap \{|z+1| \le \epsilon_1\} \cap \bar{\mathcal{O}}^c$ (in fact the estimate is true on the full set $(\widetilde{C}_{inside} \cup \widetilde{C}_{outside}) \cap \{|z+1| \le \epsilon_1\}$) and also $|e^{2q\tilde{\alpha}(z)}| \le e^{-cq}$ for $z \in (\widetilde{C}_{inside} \cup \widetilde{C}_{outside}) \cap \{|z+1| > \epsilon_1\}$. Thus, $|e^{2q\tilde{\alpha}(z)}| \le e^{-cq(\gamma-1)^{3/2}}$ for $z \in (\widetilde{C}_{inside} \cup \widetilde{C}_{outside}) \cap \mathcal{O}^c$. Also, by calculations similar to (6.8) and (6.9) again, $Re(-2\tilde{\alpha}(z)) \le -c(\gamma-1)^{3/2}$ for $z \in C_1 \cap \mathcal{O}^c$. Therefore we have L^∞ estimation

$$\|v_{\widetilde{R}} - I\|_{L^\infty(\Sigma^{(5)} \cap \mathcal{O}^c)} \le Ce^{-cq(\gamma-1)^{3/2}}. \tag{6.36}$$

Furthermore, from calculations similar to (5.23), on $\widetilde{C}_{inside} \cap \mathcal{O}^c \cap \{Im(z) \ge 0\}$, using $|u + u_0| \ge |u - u_0|$ on the integration contour for the second inequality,

$$\begin{aligned} \int |e^{-2q\tilde{\alpha}(z)}||dz| &\le \int_{\{u=u_0+xe^{-i\pi/3}: x \ge c\sqrt{\gamma-1}\}} Ce^{-qc|u^2-u_0^2|^{3/2}} du + Ce^{-cq} \\ &\le \int_{c\sqrt{\gamma-1}}^\infty Ce^{-qcx^3} dx + Ce^{-cq} \\ &\le \frac{C}{q(\gamma-1)}. \end{aligned} \tag{6.37}$$

The same calculations apply to the other part of $\widetilde{C}_{inside} \cap \mathcal{O}^c$ and also to $\widetilde{C}_{outside} \cap \mathcal{O}^c$, so that $\|v_{\widetilde{R}} - I\|_{L^1((\widetilde{C}_{inside} \cup \widetilde{C}_{outside}) \cap \mathcal{O}^c)} \leq \frac{C}{q(\gamma-1)}$. On the other hand, $length(\partial\mathcal{O}) \leq C\sqrt{\gamma-1}$ and $length(C_1 \cap \mathcal{O}^c) \leq C\sqrt{\gamma-1}$, and hence, by the above L^∞ estimates, $\|v_{\widetilde{R}} - I\|_{L^1(\partial\mathcal{O})} \leq C/(q(\gamma-1))$ and $\|v_{\widetilde{R}} - I\|_{L^1(C_1 \cap \mathcal{O}^c)} \leq C/(q(\gamma-1))$. Thus

$$\|v_{\widetilde{R}} - I\|_{L^1(\Sigma^{(5)} \cup \partial\mathcal{O})} \leq \frac{C}{q(\gamma-1)}. \tag{6.38}$$

Using the choice of $\mathcal{O}_\xi$ and $\mathcal{O}_{\bar{\xi}}$, direct calculation shows that $m^{(5,\infty)}$, hence $(m^{(5,\infty)})^{-1}$ (as $\det m^{(5,\infty)} = 1$), are uniformly bounded for γ in the region $1 + \frac{M_3}{2^{1/3}q^{2/3}} \leq \gamma \leq 1 + \delta_4$, for $z \in \mathcal{O}^c - \Sigma^{(5)}$. On the other hand, even though the contour $\Sigma^{(5)} \cup \partial\mathcal{O}$ varies with γ and q, the length of $\Sigma^{(5)} \cup \partial\mathcal{O}$ is uniformly bounded for $1 + \frac{M_3}{2^{1/3}q^{2/3}} \leq \gamma \leq 1 + \delta_4$. Also a simple scaling argument shows that the Cauchy operators $C_\pm$ on $L^2(\Sigma^{(5)} \cup \partial\mathcal{O})$ are uniformly bounded for $1 + \frac{M_3}{2^{1/3}q^{2/3}} \leq \gamma \leq 1 + \delta_4$. Therefore,

$$\begin{aligned} \|C_{w_{\widetilde{R}}}\|_{L^2(\Sigma^{(5)} \cup \partial\mathcal{O}) \to L^2(\Sigma^{(5)} \cup \partial\mathcal{O})} &\leq C\|w_{\widetilde{R}}\|_{L^\infty(\Sigma^{(5)} \cup \partial\mathcal{O})} \\ &\leq \frac{C}{q(\gamma-1)^{3/2}} + Ce^{-Cq(\gamma-1)^{3/2}} \\ &\leq \frac{C}{M_3^{2/3}} + Ce^{-CM_3^{3/2}} \\ &\leq \frac{1}{2} < 1 \end{aligned} \tag{6.39}$$

provided that M_3 is sufficiently large.

From (2.9) and (6.38), we have

$$\begin{aligned} |\widetilde{R}_{22}(0) - 1| &= \left| \frac{1}{2\pi i} \int_{\Sigma^{(5)} \cup \partial\mathcal{O}} \left(\frac{w_{\widetilde{R}} - [(I - C_{w_{\widetilde{R}}})^{-1} C_{w_{\widetilde{R}}} I](z) w_{\widetilde{R}}(z)}{z} \, ds \right)_{22} \right| \\ &\leq C(\|w_{\widetilde{R}}\|_{L^1(\Sigma^{(5)} \cup \partial\mathcal{O})} + \|w_{\widetilde{R}}\|^2_{L^2(\Sigma^{(5)} \cup \partial\mathcal{O})}) \\ &\leq C\|w_{\widetilde{R}}\|_{L^1(\Sigma^{(5)} \cup \partial\mathcal{O})}, \qquad \text{as } \|w_{\widetilde{R}}\|_{L^\infty(\Sigma^{(5)} \cup \partial\mathcal{O})} \text{ is bounded,} \\ &\leq \frac{C}{q(\gamma-1)}, \end{aligned}$$

and

$$|\widetilde{R}_{21}(0)| \leq \frac{C}{q(\gamma-1)}.$$

Therefore, from $m_{21}^{(5)}(0) = \widetilde{R}_{22}(0) m_{21}^{(5,\infty)}(0) + \widetilde{R}_{21}(0) m_{11}^{(5,\infty)}(0)$, we obtain

$$\kappa_{q-1}^2 = -e^{q(-\gamma+\log\gamma+1)} m_{21}^{(5)}(0) = e^{q(-\gamma+\log\gamma+1)} \frac{1}{\sqrt{\gamma}} \left(1 + O\big(\frac{1}{q(\gamma-1)}\big) \right). \tag{6.40}$$

Note that this is consistent with the result (6.6) for case (i) where $\gamma - 1 \geq \delta_3$.

Summarizing, we have proven the following results.

Lemma 6.3. *Let $\delta_3, M_4 > 0$ be fixed numbers. Let $\delta_4 > 0$ be a fixed sufficiently small number satisfying (6.35), and let $M_3 > 0$ be a fixed, sufficiently large number satisfying (6.32) and (6.39). As $q \to \infty$, we have the following asymptotics.*

(i) *If* $1+\delta_3 \le \gamma$,

$$\kappa_{q-1}^2 = e^{q(-\gamma+\log\gamma+1)}\frac{1}{\sqrt{\gamma}}\Big(1+O\big(\frac{1}{q}\big)\Big),$$

where the error is uniform for $1+\delta_4 \le \gamma \le L$ *for any fixed* $L<\infty$.

(ii) *If* $1+\frac{M_3}{2^{1/3}q^{2/3}} \le \gamma \le 1+\delta_4$,

$$\kappa_{q-1}^2 = e^{q(-\gamma+\log\gamma+1)}\frac{1}{\sqrt{\gamma}}\Big(1+O\big(\frac{1}{q(\gamma-1)}\big)\Big),$$

where the error is uniform in the region.

(iii) *If* $1<\gamma\le 1+\frac{M_4}{2^{1/3}q^{2/3}}$,

$$\big|\kappa_{q-1}^2 - 1 - \frac{i2^{4/3}}{q^{1/3}}m_{1,22}^{PII}(-t)\big| \le \frac{C(M_4)}{q^{2/3}},$$

where t *is defined by* $\gamma = 1+\frac{t}{2^{1/3}q^{2/3}}$, $0\le t\le M_4$.

Note that, comparing Lemma 6.3 (iii) with Lemma 5.1 (iii), we have the same result everywhere in the region $1-\frac{M}{2^{1/3}q^{2/3}} \le \gamma \le 1+\frac{M}{2^{1/3}q^{2/3}}$,

$$\big|\kappa_{q-1}^2 - 1 - \frac{i2^{4/3}}{q^{1/3}}m_{1,22}^{PII}(t)\big| \le \frac{C(M)}{q^{2/3}}, \tag{6.41}$$

where t is defined by $\gamma = 1-\frac{t}{2^{1/3}q^{2/3}}$, and M is any fixed positive number.

Also note from Lemma 6.3 (ii), that as $q\to\infty$,

$$\log\kappa_{q-1}^2 \le q(-\gamma+\log\gamma+1)+\frac{C^{\#}}{q(\gamma-1)}, \tag{6.42}$$

where $C^{\#}$ is independent of M_3 and is fixed once δ_4 satisfying (6.35) is determined.

7. Asymptotics of $\phi_n(\lambda)$ as $n\to\infty$

In this section, using Lemmas 5.1 and 6.3, we obtain the large n behavior of $\phi_n(\lambda)$. In the following, $\delta_5,\delta_6,\delta_7$ are fixed numbers between 0 and 1, and M_5, M_6, M_7 are fixed and positive. These numbers are free apart from the following requirements:

(a) δ_6 satisfies (6.35),
(b) $M_5 \ge 1$ satisfies (5.18),
(c) $\dfrac{1}{2}M_6 \ge 1$ satisfies (5.18), and
(d) $M_7\ge 1$ satisfies (6.32), (6.39) and condition (7.8) below.

We consider the following five cases for $\lambda>0$ and n:

(i) $0\le \frac{2\sqrt{\lambda}}{n+1} \le 1-\delta_5$.
(ii) $\frac{1}{2} \le \frac{2\sqrt{\lambda}}{n+1} \le 1-\frac{M_5}{2^{1/3}(n+1)^{2/3}}$.
(iii) $1-\frac{M_6}{2^{1/3}(n+1)^{2/3}} \le \frac{2\sqrt{\lambda}}{n+1} \le 1+\frac{M_6}{2^{1/3}(n+1)^{2/3}}$.
(iv) $1+\frac{M_7}{2^{1/3}(n+1)^{2/3}} \le \frac{2\sqrt{\lambda}}{n+1} \le 1+\delta_6$.
(v) $1+\delta_7 \le \frac{2\sqrt{\lambda}}{n+1}$.

Consider case (i). For any $k \geq n$, $\frac{2\sqrt{\lambda}}{k+1} \leq 1 - \delta_5$. From Lemma 5.1 (i), we have as $n \to \infty$,

$$|\log \phi_n(\lambda)| = \Big|\sum_{k=n}^{\infty} \log \kappa_k^2(\lambda)\Big| \leq \sum_{k=n}^{\infty} Ce^{-ck} \leq Ce^{-cn}. \tag{7.1}$$

Consider case (ii). We split the sum into two pieces:

$$\begin{aligned}\log \phi_n(\lambda) &= \sum_{k=n}^{\infty} \log \kappa_k^2(\lambda) \\ &= \sum_{(1)} \log \kappa_k^2(\lambda) + \sum_{(2)} \log \kappa_k^2(\lambda),\end{aligned}$$

where (1) and (2) represent the regions

$$\begin{aligned}&(1)\ \ n+1 \leq k+1 \leq 4\sqrt{\lambda}, \\ &(2)\ \ 4\sqrt{\lambda} < k+1.\end{aligned}$$

For (1), $\frac{1}{2} \leq \frac{2\sqrt{\lambda}}{k+1} \leq 1 - \frac{M_5}{2^{1/3}(k+1)^{2/3}}$. From Lemma 5.1 (ii), for some constant C, independent of M_5 satisfying (5.18),

$$\big|\log \kappa_k^2(\lambda)\big| \leq C \frac{e^{-\frac{2\sqrt{2}}{3}(k+1)(1-\frac{2\sqrt{\lambda}}{k+1})^{3/2}}}{(k+1)^{1/3}}.$$

Using the fact that $f(x) = \frac{1}{x^{1/3}} e^{-\frac{2\sqrt{2}}{3}x(1-\frac{2\sqrt{\lambda}}{x})^{3/2}}$ is monotone decreasing in the second inequality below, we have, as $n \to \infty$,

(7.2)

$$\begin{aligned}\Big|\sum_{(1)} \log \kappa_k^2(\lambda)\Big| &\leq C \sum_{(1)} \frac{e^{-\frac{2\sqrt{2}}{3}(k+1)(1-\frac{2\sqrt{\lambda}}{k+1})^{3/2}}}{(k+1)^{1/3}} \\ &\leq C \int_{n+1}^{4\sqrt{\lambda}} e^{-\frac{2\sqrt{2}}{3}x(1-\frac{2\sqrt{\lambda}}{x})^{3/2}} \frac{dx}{x^{1/3}} + C\frac{e^{-\frac{2\sqrt{2}}{3}(n+1)(1-\frac{2\sqrt{\lambda}}{n+1})^{3/2}}}{(n+1)^{1/3}} \\ &\leq C(2\sqrt{\lambda})^{2/3} \int_{\frac{(n+1)}{2\sqrt{\lambda}}-1}^{1} e^{-\frac{4\sqrt{2\lambda}y^{3/2}}{3(1+y)^{1/2}}} \frac{dy}{(1+y)^{1/3}} + Ce^{-\frac{1}{2}(n+1)(1-\frac{2\sqrt{\lambda}}{n+1})^{3/2}} \\ &\leq C(2\sqrt{\lambda})^{2/3} \int_{\frac{(n+1)}{2\sqrt{\lambda}}-1}^{1} e^{-\sqrt{\lambda}y^{3/2}} dy + Ce^{-\frac{1}{2}(n+1)(1-\frac{2\sqrt{\lambda}}{n+1})^{3/2}} \\ &\leq C \int_{\sqrt{\lambda}(\frac{(n+1)}{2\sqrt{\lambda}}-1)^{3/2}}^{\infty} e^{-s} \frac{ds}{s^{1/3}} + Ce^{-\frac{1}{2}(n+1)(1-\frac{2\sqrt{\lambda}}{n+1})^{3/2}} \\ &\leq Ce^{-\sqrt{\lambda}(\frac{(n+1)}{2\sqrt{\lambda}}-1)^{3/2}} + Ce^{-\frac{1}{2}(n+1)(1-\frac{2\sqrt{\lambda}}{n+1})^{3/2}} \\ &\leq C \exp\Big(-\frac{1}{2}(n+1)\big(1-\frac{2\sqrt{\lambda}}{n+1}\big)^{3/2}\Big).\end{aligned}$$

We use the change of variable $y = \frac{x}{2\sqrt{\lambda}} - 1$ for the integral in the third line. The fifth inequality is obtained from the substitution $s = \sqrt{\lambda}y^{3/2}$, and at the end, we have used $\frac{2\sqrt{\lambda}}{n+1} \leq 1$.

For (2), $\frac{2\sqrt{\lambda}}{k+1} \leq \frac{1}{2}$. Therefore, from Lemma 5.1 (i), we have

$$\Big|\sum_{(2)} \log \kappa_k^2(\lambda)\Big| \leq \sum_{k+1=[4\sqrt{\lambda}]}^{\infty} Ce^{-ck} \leq Ce^{-cn}.$$

Summing up the above two calculations, we have, for case (ii),

$$|\log \phi_n(\lambda)| \leq C \exp\Big(-c(n+1)\big(1-\frac{2\sqrt{\lambda}}{n+1}\big)^{3/2}\Big), \tag{7.3}$$

as $n \to \infty$. Note again that the constants C, c can be taken independent of M_5.

Consider case (iii). Set

$$\frac{2\sqrt{\lambda}}{n+1} = 1 - \frac{t}{2^{1/3}(n+1)^{2/3}} \tag{7.4}$$

so that $-M_6 \leq t \leq M_6$. We divide the sum into three pieces:

$$\begin{aligned}\log \phi_n(\lambda) &= \sum_{k=n}^{\infty} \log \kappa_k^2(\lambda)\\ &= \sum_{(1)} \log \kappa_k^2(\lambda) + \sum_{(2)} \log \kappa_k^2(\lambda) + \sum_{(3)} \log \kappa_k^2(\lambda),\end{aligned}$$

where $(1), (2)$ and (3) indicate the following regions:

(1) $n+1 \leq k+1 \leq (n+1) + \dfrac{(M_6-t)}{2^{1/3}}(n+1)^{1/3}$,

(2) $(n+1) + \dfrac{(M_6-t)}{2^{1/3}}(n+1)^{1/3} < k+1 < \dfrac{3}{2}(n+1) - \dfrac{t}{2^{1/3}}(n+1)^{1/3}$,

(3) $\dfrac{3}{2}(n+1) - \dfrac{t}{2^{1/3}}(n+1)^{1/3} \leq k+1$.

For (1), as $n \to \infty$,

$$1 - \frac{6M_6}{2^{1/3}(k+1)^{2/3}} \leq \frac{2\sqrt{\lambda}}{k+1} \leq 1 + \frac{2M_6}{2^{1/3}(k+1)^{2/3}}.$$

Hence from (6.41), we have as $k \geq n \to \infty$,

$$\log \kappa_k^2(\lambda) = \frac{i2^{4/3}}{(k+1)^{1/3}} m_{1,22}^{PII}\Big(2^{\frac{1}{3}}(k+1)^{\frac{2}{3}}(1-\frac{2\sqrt{\lambda}}{k+1})\Big) + O_{M_6}\big(\frac{1}{k^{2/3}}\big).$$

This leads to

$$
\begin{aligned}
&\sum_{(1)} \log \kappa_k^2(\lambda) \\
&= \sum_{k+1=n+1}^{[(n+1)+\frac{(M_6-t)}{2^{1/3}}(n+1)^{1/3}]} \Big[\frac{i2^{4/3}}{(k+1)^{1/3}} m_{1,22}^{PII}(2^{\frac{1}{3}}(k+1)^{\frac{2}{3}}(1-\frac{2\sqrt{\lambda}}{k+1})) + O_{M_6}(\frac{1}{k^{2/3}}) \Big] \\
&= \int_{(n+1)}^{(n+1)+\frac{(M_6-t)}{2^{1/3}}(n+1)^{1/3}} \frac{i2^{4/3}}{x^{1/3}} m_{1,22}^{PII}(2^{1/3}x^{2/3}(1-\frac{2\sqrt{\lambda}}{x}))dx + O_{M_6}(\frac{1}{n^{1/3}}) \\
&= \int_0^{\frac{(M_6-t)}{2^{1/3}}} i2^{4/3} m_{1,22}^{PII}\Big(2^{1/3}(n+1)^{2/3}(1+\frac{2s}{3(n+1)^{2/3}}+\cdots) \\
&\qquad (1-\frac{(n+1)-\frac{t}{2^{1/3}}(n+1)^{1/3}}{(n+1)+s(n+1)^{1/3}}) \Big) \frac{ds}{(1+\frac{s}{3(n+1)^{2/3}}+\cdots)} + O_{M_6}(\frac{1}{n^{1/3}}) \\
&= \int_0^{\frac{(M_6-t)}{2^{1/3}}} i2^{4/3} m_{1,22}^{PII}\Big((t+2^{1/3}s)(1-\frac{s}{3(n+1)^{2/3}}+\cdots) \Big) \\
&\qquad (1-\frac{s}{3(n+1)^{2/3}}+\cdots)ds + O_{M_6}(\frac{1}{n^{1/3}}) \\
&= \int_0^{\frac{(M_6-t)}{2^{1/3}}} i2^{4/3} m_{1,22}^{PII}(t+2^{1/3}s)ds + O_{M_6}(\frac{1}{n^{1/3}}) \\
&= \int_t^{M_6} 2im_{1,22}^{PII}(y)dy + O_{M_6}(\frac{1}{n^{1/3}}).
\end{aligned}
$$

The fourth equation is obtained using the change of variable $x = (n+1) + s(n+1)^{1/3}$, and for the sixth equation, we use the fact that $\frac{d}{dt}m_{1,22}^{PII}(t)$ is uniformly bounded for $-M_6 \le t \le M_6$ (see the remark below (6.19)). To pass from the second to the third line, note that for integers $b > a$,

$$
\begin{aligned}
|\sum_{n=a}^{b-1} f(x) - \int_a^b f(x)| &\le \sum_{n=a}^{b-1} \sup\{|f(\alpha)-f(\beta)| : n \le \alpha, \beta \le n+1\} \\
&\le \|f'\|_{L^\infty(a,b)}(b-a).
\end{aligned}
\tag{7.5}
$$

For the case at hand, a simple calculation shows that

$$
\|f'\|_{L^\infty\left(n+1,[(n+1)+\frac{(M_6-t)}{2^{1/3}}(n+1)^{1/3}]\right)} \le C(M_6)/n^{2/3}.
$$

Also, the contribution to the integral from the interval $\left((n+1)+\frac{(M_6-t)}{2^{1/3}}(n+1)^{1/3}, [(n+1)+\frac{(M_6-t)}{2^{1/3}}(n+1)^{1/3}]+1\right)$ is $O_{M_6}(1/n^{1/3})$.

For (2),

$$
\frac{1}{2} \le \frac{2\sqrt{\lambda}}{k+1} \le 1 - \frac{\frac{1}{2}M_6}{2^{1/3}(k+1)^{2/3}} \quad \text{as} \quad k \ge n \to \infty.
$$

As $n \to \infty$, by a calculation similar to the case (ii), again using the monotonicity of $f(x) = \frac{1}{x^{1/3}} e^{-\frac{2\sqrt{2}}{3} x (1-\frac{2\sqrt{\lambda}}{x})^{3/2}}$ for the second inequality, we have

$$
\begin{aligned}
\Big|\sum_{(2)} \log \kappa_k^2(\lambda)\Big| &\le C \sum_{(2)} \frac{e^{-\frac{2\sqrt{2}}{3}(k+1)(1-\frac{2\sqrt{\lambda}}{k+1})^{3/2}}}{(k+1)^{1/3}} \\
&\le C \int_{(n+1)+\frac{(M_6-t)}{2^{1/3}}(n+1)^{1/3}}^{\frac{3}{2}(n+1)-\frac{t}{2^{1/3}}(n+1)^{1/3}} e^{-\frac{2\sqrt{2}}{3} x (1-\frac{2\sqrt{\lambda}}{x})^{3/2}} \frac{dx}{x^{1/3}} + Ce^{-(\frac{M_6}{2})^{3/2}} \\
&\le C(2\sqrt{\lambda})^{2/3} \int_{\frac{M_6(n+1)^{1/3}}{2^{4/3}\sqrt{\lambda}}}^{\frac{(n+1)}{4\sqrt{\lambda}}} e^{-\frac{4\sqrt{2\lambda}}{3}(\frac{y^3}{1+y})^{1/2}} \frac{dy}{(1+y)^{1/3}} + Ce^{-(\frac{M_6}{2})^{3/2}} \\
&\le C(2\sqrt{\lambda})^{2/3} \int_{\frac{M_6(n+1)^{1/3}}{2^{4/3}\sqrt{\lambda}}}^{\frac{(n+1)}{4\sqrt{\lambda}}} e^{-\sqrt{\lambda} y^{3/2}} dy + Ce^{-(\frac{M_6}{2})^{3/2}} \\
&\le C \int_{(\frac{n+1}{2\sqrt{\lambda}})^{\frac{1}{2}}(\frac{M_6}{2})^{\frac{3}{2}}}^{\infty} e^{-s} \frac{ds}{s^{1/3}} + Ce^{-(\frac{M_6}{2})^{3/2}} \\
&\le C \int_{\frac{1}{4}M_6^{3/2}}^{\infty} \frac{e^{-s}}{s^{1/3}} ds + Ce^{-(\frac{M_6}{2})^{3/2}} \\
&\le Ce^{-\frac{1}{4}M_6^{3/2}} + Ce^{-(\frac{M_6}{2})^{3/2}} \le Ce^{-\frac{1}{4}M_6^{3/2}}.
\end{aligned}
$$

The first inequality follows from Lemma 5.1 (ii) (note that, by assumption, $\frac{1}{2}M_6$ satisfies (5.18)). For the second line, in order to control the contribution to the integral from the interval $[[(n+1) + \frac{(M_6-t)}{2^{1/3}}(n+1)^{1/3}], (n+1) + \frac{(M_6-t)}{2^{1/3}}(n+1)^{1/3}]$, we use the inequality $1 + \frac{M_6-t}{2^{1/3}(n+1)^{2/3}} \le 1 + \frac{2^{2/3}M_6}{(n+1)^{2/3}} \le \frac{32}{9}$ for large enough n. For the third line, we use the change of variable $y = \frac{x}{2\sqrt{\lambda}} - 1$, and for the fourth line, we use the inequality $\frac{4\sqrt{\lambda}}{(n+1)\delta} \ge 1 - \frac{M_6}{2^{1/3}(n+1)^{2/3}} \ge \frac{9}{23}$ for sufficiently large n. The fifth equation is obtained from the substitution $s = \sqrt{\lambda} y^{3/2}$, and for the sixth line, we have used the inequality $\frac{2\sqrt{\lambda}}{n+1} \le 1 + \frac{M_6}{2^{1/3}(n+1)^{2/3}} \le 2$ for sufficiently large n.

For (3), as $n \to \infty$, $0 \le \frac{2\sqrt{\lambda}}{k+1} \le \frac{3}{4}$, which yields, from Lemma 5.1 (i),

$$
\Big|\sum_{(3)} \log \kappa_k^2(\lambda)\Big| \le Ce^{-cn}.
$$

Summing up all these calculations, for $\frac{2\sqrt{\lambda}}{n+1} = 1 - \frac{t}{2^{1/3}(n+1)^{2/3}}$ with $-M_6 \le t \le M_6$, we have, as $n \to \infty$,

$$
\Big|\log \phi_n(\lambda) - \int_t^{M_6} 2im_{1,22}^{PII}(y)dy\Big| \le \frac{C(M_6)}{n^{1/3}} + Ce^{-\frac{1}{4}M_6^{3/2}},
$$

for a constant $C(M_6)$ which depends on M_6, and for a constant C which is independent of M_6. Using the asymptotics of $m_{1,22}^{PII}(x)$ as $x \to +\infty$ (see (2.17)), we have (recall $\frac{1}{2}M_6 \ge 1$)

$$
\Big|\log \phi_n(\lambda) - \int_t^{\infty} 2im_{1,22}^{PII}(y)dy\Big| \le \frac{C(M_6)}{n^{1/3}} + Ce^{-\frac{1}{4}M_6^{3/2}}. \tag{7.6}
$$

Now we consider case (iv), $1+\frac{M_7}{2^{1/3}(n+1)^{2/3}} \le \frac{2\sqrt{\lambda}}{n+1} \le 1+\delta_6$. We write

$$\log \phi_n(\lambda) = \sum_{(1)} \log \kappa_k^2 + \sum_{(2)} \log \kappa_k^2,$$

where $(1),(2)$ indicate the following regions:

$$(1) \quad n+1 \le k+1 \le 2\sqrt{\lambda} - \frac{M_7}{2^{1/3}}(n+1)^{1/3},$$
$$(2) \quad 2\sqrt{\lambda} - \frac{M_7}{2^{1/3}}(n+1)^{1/3} \le k+1.$$

For (1), we have for n sufficiently large, $1+\frac{M_7}{2^{1/3}(k+1)^{2/3}} \le \frac{2\sqrt{\lambda}}{k+1} \le 1+\delta_6$. Therefore, using (6.42), we obtain

(7.7)

$$\begin{aligned}
&\sum_{(1)} \log \kappa_k^2(\lambda) \\
&\le \sum_{k+1=n+1}^{[2\sqrt{\lambda}-\frac{M_7}{2^{1/3}}(n+1)^{1/3}]} (k+1)\Big(-\frac{1}{4}(\frac{2\sqrt{\lambda}}{k+1}-1)^2\Big) + \frac{C^{\#}}{2\sqrt{\lambda}-(k+1)} \\
&\le -\frac{1}{4}\int_{n+1}^{2\sqrt{\lambda}-\frac{M_7}{2^{1/3}}(n+1)^{1/3}} x\big(\frac{2\sqrt{\lambda}}{x}-1\big)^2 dx + \int_{n+1}^{2\sqrt{\lambda}-\frac{M_7}{2^{1/3}}(n+1)^{1/3}} \frac{C^{\#}}{2\sqrt{\lambda}-x} dx + C(M_7) \\
&\le -\frac{1}{4}\int_{\frac{n+1}{2\sqrt{\lambda}}}^{1-\frac{M_7}{2^{1/3}(n+1)^{2/3}}\frac{n+1}{2\sqrt{\lambda}}} (2\sqrt{\lambda})^2 \frac{(1-y)^2}{y} dy + C^{\#}\log\Big(\frac{2\sqrt{\lambda}-(n+1)}{(n+1)^{1/3}}\Big) + C(M_7) \\
&\le -\frac{1}{4}\int_{\frac{n+1}{2\sqrt{\lambda}}}^{1-\frac{M_7}{2^{1/3}(n+1)^{2/3}}\frac{n+1}{2\sqrt{\lambda}}} (2\sqrt{\lambda})^2 (1-y)^2 dy + C^{\#}\frac{2\sqrt{\lambda}-(n+1)}{(n+1)^{1/3}} + C(M_7) \\
&\le \frac{1}{4}\int_{1-\frac{n+1}{2\sqrt{\lambda}}}^{\frac{M_7}{2^{1/3}(n+1)^{2/3}}\frac{n+1}{2\sqrt{\lambda}}} (2\sqrt{\lambda})^2 z^2 dz + C^{\#}(n+1)^{2/3}(\frac{2\sqrt{\lambda}}{n+1}-1) + C(M_7) \\
&\le \frac{1}{12}(2\sqrt{\lambda})^2\Big[(\frac{M_7}{2^{1/3}(n+1)^{2/3}}\frac{n+1}{2\sqrt{\lambda}})^3 - (1-\frac{n+1}{2\sqrt{\lambda}})^3\Big] \\
&\qquad + C^{\#}(n+1)^{2/3}(\frac{2\sqrt{\lambda}}{n+1}-1) + C(M_7) \\
&\le \frac{M_7^3}{24}(\frac{n+1}{2\sqrt{\lambda}}) - \frac{n+1}{24\sqrt{\lambda}}(n+1)^2(\frac{2\sqrt{\lambda}}{n+1}-1)^3 + C^{\#}(n+1)^{2/3}(\frac{2\sqrt{\lambda}}{n+1}-1) + C(M_7) \\
&\le -\frac{1}{48}\Big[2^{1/3}(n+1)^{2/3}(\frac{2\sqrt{\lambda}}{n+1}-1)\Big]^3 + C^{\#}(n+1)^{2/3}(\frac{2\sqrt{\lambda}}{n+1}-1) + C(M_7) \\
&\le -\frac{1}{96}\Big[2^{1/3}(n+1)^{2/3}(\frac{2\sqrt{\lambda}}{n+1}-1)\Big]^3 + C(M_7).
\end{aligned}$$

The first line follows from the inequality $-\gamma+\log\gamma+1 \le -\frac{(\gamma-1)^2}{4}$ for $1 \le \gamma \le 2$ (note from (6.42) that $C^{\#}$ is independent of M_7). In the second line, we use the monotonicity of $x(\frac{2\sqrt{\lambda}}{x}-1)^2$ and of $(2\sqrt{\lambda}-x)^{-1}$ in the region $n+1 \le x \le 2\sqrt{\lambda}-\frac{M_7}{2^{1/3}}(n+1)^{1/3}$. In the succeeding lines, we have used the changes of variables

$x = 2\sqrt{\lambda}y$, $1 - y = z$ and $2\sqrt{\lambda}z^{3/2} = s$. For the last line, note that

$$2^{1/3}(n+1)^{2/3}(\frac{2\sqrt{\lambda}}{n+1} - 1) \geq M_7,$$

and we require

$$M_7 \geq \sqrt{96C^{\#}}. \tag{7.8}$$

Remark. In estimating the sum in the second line of (7.7) by an integral, the monotonicity of the integrand plays a crucial role: we cannot, for example, use an estimate of the form (7.5), as the derivative is not sufficiently small.

For (2), we have $\frac{2\sqrt{\lambda}}{k+1} \leq 1 + \frac{2M_7}{2^{1/3}(k+1)^{2/3}}$. Calculations similar to the previous cases (i), (ii) and (iii) show that

$$\sum_{(2)} \log \kappa_k^2(\lambda) \leq \Big| \int_{-2M_7}^{\infty} 2im_{1,22}^{PII}(y)dy \Big| + \frac{C(M_7)}{n^{1/3}} + Ce^{-M_7} \leq C(M_7).$$

The result follows by splitting the sum $\sum_{(2)}$ into the following regions: $2\sqrt{\lambda} - \frac{M_7}{2^{1/3}}(n+1)^{1/3} \leq k+1 \leq 2\sqrt{\lambda} + \frac{M_7}{2^{1/3}}(n+1)^{1/3}$, $2\sqrt{\lambda} + \frac{M_7}{2^{1/3}}(n+1)^{1/3} < k+1 < 3\sqrt{\lambda}$ and $3\sqrt{\lambda} \leq k+1$: we leave the details to the reader.

Therefore, for $1 + \frac{M_7}{2^{1/3}(n+1)^{2/3}} \leq \frac{2\sqrt{\lambda}}{n+1} \leq 1 + \delta_6$, we have

$$\log \phi_n(\lambda) \leq -\frac{1}{96}\left[2^{1/3}(n+1)^{2/3}(\frac{2\sqrt{\lambda}}{n+1} - 1)\right]^3 + C(M_7). \tag{7.9}$$

For case (v), we use the estimation of [Jo1] given in Lemma 7.1 (v) below.

Summarizing, we have

Lemma 7.1. *Let $0 < \delta_5, \delta_6, \delta_7 < 1$ and $M_5, M_6, M_7 > 0$ be fixed numbers. Suppose that δ_6, M_5, M_6 and M_7 satisfy conditions* (a), (b), (c) *and* (d) *given at the beginning of this section, respectively. Set*

$$t = 2^{1/3}(n+1)^{2/3}\big(1 - \frac{2\sqrt{\lambda}}{n+1}\big) \quad \text{so that} \quad \frac{2\sqrt{\lambda}}{n+1} = 1 - \frac{t}{2^{1/3}(n+1)^{2/3}}. \tag{7.10}$$

We have the following estimates for the large n behavior of $\phi_n(\lambda)$:

(i) *If $0 \leq \frac{2\sqrt{\lambda}}{n+1} \leq 1 - \delta_5$,*

$$\big|\log \phi_n(\lambda)\big| \leq C\exp(-cn),$$

for some constants C, c which may depend on δ_5.

(ii) *If $\frac{1}{2} \leq \frac{2\sqrt{\lambda}}{n+1} \leq 1 - \frac{M_5}{2^{1/3}(n+1)^{2/3}}$,*

$$\big|\log \phi_n(\lambda)\big| \leq C\exp(-ct^{3/2}),$$

for constants C,c independent of M_5.

(iii) *If $1 - \frac{M_6}{2^{1/3}(n+1)^{2/3}} \leq \frac{2\sqrt{\lambda}}{n+1} \leq 1 + \frac{M_6}{2^{1/3}(n+1)^{2/3}}$, so that $-M_6 \leq t \leq M_6$, there is a constant $C(M_6)$ which depends on M_6, and a constant C which is independent of M_6, such that*

$$\Big|\log \phi_n(\lambda) - \int_t^{\infty} 2im_{1,22}^{PII}(y)dy\Big| \leq \frac{C(M_6)}{n^{1/3}} + Ce^{-\frac{1}{4}M_6^{3/2}}.$$

(iv) *If* $1+\frac{M_7}{2^{1/3}(n+1)^{2/3}} \le \frac{2\sqrt{\lambda}}{n+1} \le 1+\delta_6$,

$$\log \phi_n(\lambda) \le \frac{1}{96}t^3 + C(M_7),$$

for a constant $C(M_7)$.

(v) [Jo1] *If* $1+\delta_7 \le \frac{2\sqrt{\lambda}}{n+1}$,

$$\phi_n(\lambda) \le Ce^{-C\lambda} \le Ce^{-cn^2}.$$

The really new results in this lemma are (iii) and (iv). Indeed, (i) and (ii) can also be obtained from (1.19), and as indicated, (v) is given in [Jo1].

8. De-Poissonization lemmas

In this section, we present two lemmas which show that $\phi_n(N)$ is a good approximation of $q_{n,N} = f_{N,n}/N!$.

We need a lemma showing the monotonicity of $q_{n,N}$ in N. The statement and proof can be found in [Jo1].

Lemma 8.1. *For all* $n, N \ge 1$,

$$q_{n,N+1} \le q_{n,N}.$$

Using this monotonicity result, the following Tauberian-like "de-Poissonization" lemma can be proved. This is a modification of Lemma 2.5 in [Jo1] and the proof is the same.

Lemma 8.2. *Let* $m > 0$ *be a fixed real number. Set*

$$\mu_N^{(m)} = N + (2\sqrt{m+1}+1)\sqrt{N\log N}$$

and

$$\nu_N^{(m)} = N - (2\sqrt{m+1}+1)\sqrt{N\log N}.$$

Then there are constants $C = C(m)$ *and* $N_0 = N_0(m)$ *such that*

$$\phi_n(\mu_N^{(m)}) - \frac{C}{N^m} \le q_{n,N} \le \phi_n(\nu_N^{(m)}) + \frac{C}{N^m}$$

for $N \ge N_0$, $0 \le n \le N$.

The reader will observe that the above lemma is actually enough for all of our future calculations. Nevertheless, for convenience and the purpose of illustration, we use the following lemma for the convergence of moments.

Lemma 8.3. *There exists* $C > 0$ *such that*

$$q_{n,N} \le C\phi_n(N-\sqrt{N}), \quad 1 - q_{n,N} \le C\big(1-\phi_n(N+\sqrt{N})\big)$$

for all sufficiently large N, $0 \le n \le N$.

Proof. Note that $q_{n,N} \geq 0$. Using Lemma 8.1 and Stirling's formula for sufficiently large N, we have from (1.11),

$$\begin{aligned}
\phi_n(N-\sqrt{N}) &= \sum_{N'=0}^{\infty} \frac{e^{-(N-\sqrt{N})}(N-\sqrt{N})^{N'}}{(N')!} q_{n,N'} \\
&\geq \sum_{N' \geq N-\sqrt{N}}^{N} \frac{e^{-(N-\sqrt{N})}(N-\sqrt{N})^{N'}}{(N')!} q_{n,N'} \\
&\geq q_{n,N} \sum_{N' \geq N-\sqrt{N}}^{N} \frac{e^{-(N-\sqrt{N})}(N-\sqrt{N})^{N'}}{(N')!} \\
&\geq C q_{n,N} \sum_{N' \geq N-\sqrt{N}}^{N} \frac{e^{-(N-\sqrt{N})}(N-\sqrt{N})^{N'}}{(N')^{N'+1/2} e^{-N'}} = C q_{n,N} \sum_{N' \geq N-\sqrt{N}}^{N} e^{f(N')},
\end{aligned}$$

where $f(x) = -(N-\sqrt{N}) + x \log(N-\sqrt{N}) + x - (x+\frac{1}{2}) \log x$. One can easily check that $f(x)$ is a decreasing function for $x \geq (N-\sqrt{N})$. Thus

$$\phi_n(N-\sqrt{N}) \geq C q_{n,N} \sqrt{N} e^{f(N)} = C q_{n,N} e^{\sqrt{N}+N\log(1-1/\sqrt{N})} \geq C q_{n,N},$$

for sufficiently large N, $0 \leq n \leq N$.

For the second inequality, note that $q_{n,N} \leq 1$ by definition. Again, using Lemma 8.1 and Stirling's formula for sufficiently large N,

$$\begin{aligned}
1-\phi_n(N+\sqrt{N}) &= \sum_{N'=0}^{\infty} \frac{e^{-(N+\sqrt{N})}(N+\sqrt{N})^{N'}}{(N')!} (1-q_{n,N'}) \\
&\geq C(1-q_{n,N}) \sum_{N'=N}^{N+\sqrt{N}} e^{g(N')},
\end{aligned}$$

where $g(x) = -(N+\sqrt{N}) + x\log(N+\sqrt{N}) + x - (x+\frac{1}{2})\log x$. One can check that for $N \leq x \leq N+\sqrt{N}$, $g''(x) < 0$ so that $\min g(x) = \min\big(g(N), g(N+\sqrt{N})\big)$. If N is sufficiently large, $\min\big(g(N), g(N+\sqrt{N})\big) = g(N+\sqrt{N}) = -\frac{1}{2}\log(N+\sqrt{N})$. Therefore

$$1-\phi_n(N+\sqrt{N}) \geq C(1-q_{n,N})\sqrt{N} e^{g(N)} \geq C(1-q_{n,N}),$$

for sufficiently large N, $0 \leq n \leq N$. □

9. Proofs of the main theorems

In this section, we prove the main theorems.

Proof of Theorem 1.1. Assume for definiteness that $t < 0$. For $t \geq 0$, the calculation is similar. From the definition of $q_{n,N} \equiv \frac{f_{N,n}}{N!}$,

$$F_N(t) = Prob\Big(\frac{l_N - 2\sqrt{N}}{N^{1/6}} \leq t\Big) = q_{[2\sqrt{N}+tN^{1/6}],N}. \tag{9.1}$$

Set

$$n = [2\sqrt{N} + tN^{1/6}].$$

As t is fixed, observe that $0 \le n \le N$, as $N \to \infty$. Using Lemma 8.2 with any fixed value of $m > 0$, we have

$$\phi_n(\mu_N^{(m)}) - \frac{C}{N^m} \le F_N(t) \le \phi_n(\nu_N^{(m)}) + \frac{C}{N^m}.$$

Set

$$t_N = 2^{1/3}(n+1)^{2/3}\big(1 - \frac{2\sqrt{\mu_N^{(m)}}}{n+1}\big)$$

(cf. the definition of t in (7.10)). Then, for all large N,

$$2t \le t_N \le \frac{1}{2}t \quad \text{and} \quad \lim_{N\to\infty} t_N = t.$$

Let $M_6 \ge 2|t|$ be any sufficiently large, fixed number satisfying condition (c) in Lemma 7.1. Using Lemma 7.1 (iii), we have, for some constant $C(M_6)$ which depends on M_6, and a constant C which is independent of M_6,

$$\phi_n(\mu_N^{(m)}) = \exp\bigg(\int_{t_N}^{\infty} 2im_{1,22}^{PII}(y)dy\bigg)\big(1 + O_{M_6}(\frac{1}{n^{1/3}}) + O(e^{-\frac{1}{4}M_6^{3/2}})\big).$$

Taking $N \to \infty$, and then taking $M_6 \to \infty$, we obtain

$$\lim_{N\to\infty} \phi_n(\mu_N^{(m)}) = \exp\bigg(\int_t^{\infty} 2im_{1,22}^{PII}(y)dy\bigg).$$

For $\phi_n(\nu_N^{(m)})$, we obtain the same limit by a similar calculation,

$$\lim_{N\to\infty} \phi_n(\nu_N^{(m)}) = \exp\bigg(\int_t^{\infty} 2im_{1,22}^{PII}(y)dy\bigg).$$

Thus, recalling $\frac{d}{dx}2i(m_1^{PII})_{22}(x) = u^2(x)$ in (2.18), integration by parts yields

$$\lim_{N\to\infty} Prob\bigg(\frac{l_N - 2\sqrt{n}}{N^{1/6}} \le t\bigg) = \exp\bigg(\int_t^{\infty} 2im_{1,22}^{PII}(y)dy\bigg) = F(t).$$

□

Proof of Theorem 1.2. Integrating by parts,

$$\mathbb{E}_N(\chi_N^m) = \int_{-\infty}^{\infty} t^m dF_N(t) = -\int_{-\infty}^{0} mt^{m-1}F_N(t)dt + \int_0^{\infty} mt^{m-1}(1 - F_N(t))dt,$$

where $F_N(t) \equiv Prob\bigg(\frac{l_N - 2\sqrt{N}}{N^{1/6}} \le t\bigg)$ as in Theorem 1.1. From Theorem 1.1, we have pointwise convergence of $F_N(t)$ to $F(t)$. We need uniform control of F_N for large N. Let $M > 0$ be a sufficiently large, fixed number, and let $0 < \delta < \frac{1}{4}$ be a fixed, sufficiently small number.

Set $n = [2\sqrt{N} + tN^{1/6}]$. First consider the case when $t \le -M$. If $t < -2N^{1/3}$, then $F_N(t) = Prob(l_N \le 2\sqrt{N} + tN^{1/6}) \le Prob(l_N < 0) = 0$. For $-2N^{1/3} \le t \le -M$, (9.1) and Lemma 8.3 yield

$$F_N(t) = q_{n,N} \le C\phi_n(N - \sqrt{N}). \tag{9.2}$$

If $-2N^{1/3} \le t \le -2\delta N^{1/3}$, when N is sufficiently large,

$$\frac{2\sqrt{N-\sqrt{N}}}{n+1} \ge \frac{2\sqrt{N}(1 - \frac{1}{\sqrt{N}})^{1/2}}{2\sqrt{N} + tN^{1/6} + 1} \ge \frac{2\sqrt{N}(1 - \frac{\delta}{4})}{2(1-\delta)\sqrt{N} + 1} \ge 1 + \frac{\delta}{2}.$$

Thus, using Lemma 7.1 (v), for large N,

$$\phi_n(N-\sqrt{N}) \le Ce^{-cN} \le Ce^{ct^3}. \tag{9.3}$$

If $-2\delta N^{1/3} \le t \le -M$,

$$\frac{2\sqrt{N-\sqrt{N}}}{n+1} \le \frac{2\sqrt{N}}{2\sqrt{N}-2\delta\sqrt{N}} \le 1+2\delta$$

and, using the monotonicity of $(2\sqrt{\lambda}-x)/x^{1/3}$ as a function of $x \le 2\sqrt{\lambda}$,

$$\begin{aligned}
&2^{1/3}\Big(\frac{2\sqrt{N-\sqrt{N}}-(n+1)}{(n+1)^{1/3}}\Big)\\
&\qquad \ge 2^{1/3}\Big(\frac{2\sqrt{N}(1-\frac{1}{\sqrt{N}})^{1/2}-(2\sqrt{N}-MN^{1/6}+1)}{(2\sqrt{N}-MN^{1/6}+1)^{1/3}}\Big) \ge \frac{M}{2}
\end{aligned}$$

as $N \to \infty$. Thus, we have for $-2\delta N^{1/3} \le t \le -M$,

$$1+\frac{\frac{1}{2}M}{2^{1/3}(n+1)^{2/3}} \le \frac{2\sqrt{N-\sqrt{N}}}{n+1} \le 1+2\delta.$$

Therefore, from Lemma 7.1 (iv), provided $\frac{M}{2}$ satisfies condition (d) and 2δ satisfies condition (a),

$$\phi_n(N-\sqrt{N}) \le C_M \exp\Big(-\frac{1}{96}\big(2^{1/3}(n+1)^{2/3}(\frac{2\sqrt{N-\sqrt{N}}}{n+1}-1)\big)^3\Big). \tag{9.4}$$

On the other hand, using the monotonicity of $(2\sqrt{\lambda}-x)/x^{1/3}$ as a function of $x \le 2\sqrt{\lambda}$,

$$\begin{aligned}
&2^{1/3}\Big(\frac{2\sqrt{N-\sqrt{N}}-(n+1)}{(n+1)^{1/3}}\Big)\\
&\qquad \ge 2^{1/3}\Big(\frac{2\sqrt{N}(1-\frac{1}{\sqrt{N}})^{1/2}-(2\sqrt{N}+tN^{1/6}+1)}{(2\sqrt{N}+tN^{1/6}+1)^{1/3}}\Big) \ge -\frac{t}{2}
\end{aligned}$$

for all $-2\delta N^{1/3} \le t \le -M$, as $N \to \infty$. Therefore (9.4) gives us

$$\phi_n(N-\sqrt{N}) \le C(M)\exp(\frac{1}{768}t^3) \tag{9.5}$$

for $-2\delta N^{1/3} \le t \le -M$.

Inserting the above estimates (9.3) and (9.5) into (9.2), we obtain

$$F_N(t) \le C(M)e^{ct^3} \qquad \text{for } -2N^{1/3} \le t \le -M, \tag{9.6}$$

and as $F_N(t)=0$ for $t<-2N^{1/3}$, it follows by the dominated convergence theorem that

$$\lim_{N\to\infty}\int_{-\infty}^0 mt^{m-1}F_N(t)dt = \int_{-\infty}^0 mt^{m-1}F(t)dt. \tag{9.7}$$

Now consider the case when $t \ge M$. If $t > N^{5/6}-2N^{1/3}$, then $1-F_N(t) = 1-Prob(l_N \le 2\sqrt{N}+tN^{1/6}) \le 1-Prob(l_N > N) = 0$. For $M \le t \le N^{5/6}-2N^{1/3}$, again, (9.1) and Lemma 8.3 yield

$$1-F_N(t) = 1-q_{n,N} \le C(1-\phi_n(N+\sqrt{N})). \tag{9.8}$$

If $2\delta N^{1/3} \le t \le N^{5/6} - 2N^{1/3}$, when N is sufficiently large,

$$\frac{2\sqrt{N+\sqrt{N}}}{n+1} \le \frac{2\sqrt{N}(1+\frac{\delta}{4})}{2\sqrt{N}+2\delta\sqrt{N}} \le 1 - \frac{\delta}{2}.$$

Thus, using Lemma 7.1 (i),

$$1 - \phi_n(N+\sqrt{N}) \le Ce^{-cn} \le Ce^{-c\sqrt{N}} \le Ce^{-ct^{3/5}}. \tag{9.9}$$

If $M \le t \le 2\delta N^{1/3}$, similar calculations to the case $-2\delta N^{1/3} \le t \le -M$ yield

$$\frac{1}{2} \le 1 - 2\delta \le \frac{2\sqrt{N+\sqrt{N}}}{n+1} \le 1 - \frac{\frac{1}{2}M}{2^{1/3}(n+1)^{2/3}}.$$

Therefore, from Lemma 7.1 (ii), as $N \to \infty$,

$$1 - \phi_n(N-\sqrt{N}) \le \exp\left(-c\big(2^{1/3}(n+1)^{2/3}(1 - \frac{2\sqrt{N+\sqrt{N}}}{n+1})\big)^{3/2}\right), \tag{9.10}$$

provided $\frac{1}{2}M$ satisfies condition (d). However, as in the case $-2\delta N^{1/3} \le t \le -M$, we have

$$2^{1/3}\left(\frac{(n+1) - 2\sqrt{N+\sqrt{N}}}{(n+1)^{1/3}}\right) \ge 2^{1/3}\left(\frac{(2\sqrt{N}+tN^{1/6}) - 2\sqrt{N}(1+\frac{1}{\sqrt{N}})^{1/2}}{(2\sqrt{N}+tN^{1/6})^{1/3}}\right) \ge \frac{t}{2}$$

for all $M \le t \le 2\delta N^{1/3}$, as $N \to \infty$. Therefore (9.10) gives us

$$1 - \phi_n(N-\sqrt{N}) \le C\exp(-ct^{3/2}) \tag{9.11}$$

for $M \le t \le 2\delta N^{1/3}$.

Inserting the above estimates (9.9) and (9.11) into (9.8), we obtain for $M \le t \le N^{5/6} - 2N^{1/3}$

$$1 - F_N(t) \le Ce^{-ct^{3/5}} \tag{9.12}$$

as $N \to \infty$. Once again, as $1 - F_N(t) = 0$ for $t > N^{5/6} - 2N^{1/3}$, it follows by the dominated convergence theorem that

$$\lim_{N\to\infty} \int_0^\infty mt^{m-1}(1-F_N(t))dt = \int_0^\infty mt^{m-1}(1-F(t))dt. \tag{9.13}$$

□

Appendix A.

As advertised in the Introduction, in this Appendix we give a new derivation of the formula

$$\sum_{N=0}^{\infty} \frac{\lambda^N F_N(n)}{N!} = \det(d_{j-k})_{0\le j,k\le n-1}, \tag{A.1}$$

where $d_j = (2\pi)^{-1}\int_0^{2\pi} \exp(2\sqrt{\lambda}\cos\theta - ij\theta)d\theta$, and $F_N(n)$ is the distribution function for the length, $\ell_N(\pi)$, of the longest increasing subsequence in the random permutation π from S_N. We set $F_0(0) = 1$.

Let $\mu = (\mu_1, \mu_2, \dots, \mu_r, 0, 0, \dots)$, $\mu_1 \ge \mu_2 \ge \cdots$, be a partition of N, i.e. μ_j, $1 \le j \le r$, are positive integers and $N = \mu_1 + \cdots + \mu_r$; we write $\mu \vdash N$. With μ we can associate a Young diagram, also denoted by μ, in the standard way; see for example [Sa]. In the Young diagram there are μ_j boxes in the j-th row. If we insert the numbers $1, \dots, N$ in the boxes in such a way that the numbers in every row and column are increasing we get a (standard) Young tableau t; t has shape μ, $s(t)=\mu$. Let $r(\mu)$ denote the number of rows in μ.

Schensted [Sc] has constructed a certain bijection, the Schensted correspondence, between the permutation group S_N and pairs of Young tableaux (t,t') with the same shape $s(t)=s(t')=\mu$, where $\mu\vdash N$. This correspondence has the property that if $S_N\ni\pi\to(t,t')$, $\mu=s(t)$, then $\ell_N(\pi)$ equals the length, μ_1, of the first row in μ, and the length, $\ell'_N(\pi)$, of the longest *decreasing* subsequence in π equals $r(\mu)$, the number of rows in μ. For details see [Sa].

If we put the uniform probability distribution on S_N, then clearly the random variables ℓ_N and ℓ'_N have the same distribution (just reverse the permutation). Let $f(\mu)$ denote the number of Young tableaux with shape μ. Then, by the Schensted correspondence,

$$F_N(n)=\frac{1}{N!}\sum_{\substack{\mu\vdash N\\ r(\mu)\le n}} f(\mu)^2. \tag{A.2}$$

If we set $h_j=\mu_j+r-j$, $r=r(\mu)$, we have the following formula, due to Frobenius and Young:

$$f(\mu)=N!\prod_{1\le i<j\le r}(h_i-h_j)\prod_{i=1}^{r}\frac{1}{h_i!} \tag{A.3}$$

(see for example [Si]). Note that $N=\sum_{j=1}^r\mu_j=\sum_{j=1}^r h_j-r(r-1)/2$ and $h_{j-1}-h_j=\mu_{j-1}-\mu_j+1\ge1$. Combining the formulas (A.2) and (A.3) we get

$$F_N(n)=N!\sum_{r=1}^{n}\frac{1}{r!}\sum_{(*)}\Delta(h)^2\prod_{j=1}^{r}\frac{1}{(h_j!)^2}, \tag{A.4}$$

where the $(*)$ means that we sum over all different integers $h_i\ge1$ such that $\sum h_j=N+r(r-1)/2$, and $\Delta(h)=\prod_{i<j}(h_j-h_i)$ is the Vandermonde determinant. That we can remove the ordering of the h_j 's in (A.4) follows from symmetry under permutation of $h_1,\dots,h_r$. The constraint $\sum h_j=N+r(r-1)/2$ is removed by the Poissonization

$$\phi_n(\lambda)=e^{-\lambda}\sum_{N=0}^{\infty}\frac{\lambda^N}{N!}F_N(n)=e^{-\lambda}[1+\sum_{r=1}^{n}\lambda^{-r(r-1)/2}H_r(\lambda)], \tag{A.5}$$

where

$$H_r(\lambda)=\frac{1}{r!}\sum_{h\in\mathbb{Z}_+^r}\Delta(h)^2\prod_{j=1}^{r}\frac{\lambda^{h_j}}{(h_j!)^2}.$$

We have used the fact that $\sum h_j\ge1+\dots+r=r(r-1)/2+r$ and $N\ge r$, since the h_j 's are different integers. The condition that the h_j 's are different can then be removed since otherwise $\Delta(h)=0$. Observe that $H_r(\lambda)$ is a Hankel determinant with respect to the discrete measure

$$\nu(\{m\})=\frac{\lambda^m}{(m!)^2},\quad m\in\mathbb{Z}_+$$

(see [Sz1]), i.e.

$$H_r(\lambda)=\det(\sum_{m=1}^{\infty}m^{j+k}\frac{\lambda^m}{(m!)^2})_{0\le j,k\le r-1}.$$

If q_j, $j \geq 0$, are any polynomials with $\deg q_j = j$ and leading coefficient 1, row and column operations on the determinant give

$$H_r(\lambda) = \det(\sum_{m=1}^{\infty} q_j(m) q_k(m) \frac{\lambda^m}{(m!)^2})_{0 \leq j,k \leq r-1}. \tag{A.6}$$

We now make a particular choice of q_j, $q_j(x) = x(x-1)\cdots(x-(j-1))$, if $j \geq 1$ and $q_0(x) = 1$, so that

$$a^j \frac{d^j}{da^j} a^m = q_j(m) a^m, \quad m, j \geq 0. \tag{A.7}$$

The elements in the Hankel determinants can then be written

$$\sum_{m=1}^{\infty} q_j(m) q_k(m) \frac{\lambda^m}{(m!)^2} = a^j b^k \frac{d^j}{da^j} \frac{d^k}{db^k} \sum_{m=0}^{\infty} \frac{a^m b^m}{(m!)^2} \bigg|_{a=b=\sqrt{\lambda}} - \delta_{j0}\delta_{k0}. \tag{A.8}$$

Now,

$$\sum_{m=0}^{\infty} \frac{a^m b^m}{(m!)^2} = \frac{1}{2\pi} \int_0^{2\pi} e^{ae^{i\theta} + be^{-i\theta}} d\theta$$

and hence we can perform the differentiations in (A.8) and get

$$\sum_{m=1}^{\infty} q_j(m) q_k(m) \frac{\lambda^m}{(m!)^2} = \lambda^{(j+k)/2} d_{j-k} - \delta_{j0}\delta_{k0},$$

where $d_{j-k} = (2\pi)^{-1} \int_0^{2\pi} \exp(2\sqrt{\lambda}\cos\theta - i(j-k)\theta) d\theta$. Inserting this identity into the formula (A.6) yields

$$H_r(\lambda) = \lambda^{r(r-1)/2}(D_r - D_{r-1}), \quad r \geq 1, \tag{A.9}$$

where D_r is the Toeplitz determinant $\det(d_{j-k})_{0 \leq j,k \leq r-1}$ and $D_0 = 1$. Hence, using the formula (A.5), we get $\phi_n(\lambda) = e^{-\lambda} D_n$, which is what we wanted to prove.

In the remaining part of this Appendix we will give a heuristic argument showing why we can expect the random variable $\ell_N(\pi)$ to behave like the largest eigenvalue of a random hermitian matrix. From our considerations above we see that

$$F_N(n) = \frac{1}{N!} \sum_{\substack{\mu \vdash N \\ \mu_1 \leq n}} f(\mu)^2.$$

By the same computations as above this leads to

$$\phi_n(\lambda) = e^{-\lambda}[1 + \sum_{r=1}^{\infty} \lambda^{-r(r-1)/2} H_r(\lambda; n)], \tag{A.10}$$

where

$$H_r(\lambda; n) = \frac{1}{r!} \sum_{h \in \{1,\dots,n+r-1\}^r} \Delta(h)^2 \prod_{j=1}^{r} \frac{\lambda^{h_j}}{(h_j!)^2}.$$

Note that $H_r(\lambda; n) \nearrow H_r(\lambda)$ as $n \to \infty$. We can think of

(A.11)

$$\frac{1}{r! H_r(\lambda)} \Delta(h)^2 \prod_{j=1}^{r} \frac{\lambda^{h_j}}{(h_j!)^2} = \frac{1}{r! H_r(\lambda)} e^{-2\sum_{i<j} \log|h_i - h_j|^{-1} + \sum_j [(\log\lambda) h_j + 2\log(h_j!)]}$$

as the probability of the configuration $h \in \mathbb{Z}_+^r$. This probability has the form of a *discrete* Coulomb gas on $\mathbb{Z}_+$ at inverse temperature $\beta = 2$ confined by an external potential. An $N \times N$ random hermitian matrix with a probability density of the form $Z_N^{-1} \exp(-\operatorname{Tr} V(M))$ has an eigenvalue density

$$\frac{1}{Z_N} e^{-2\sum_{i<j} \log |x_i - x_j|^{-1} + \sum_j V(x_j)},$$

with $x \in \mathbb{R}^N$; $x_1, \dots, x_N$ are the eigenvalues of M. Thus we can think of the h_j 's as some kind of "eigenvalues".

Let

$$P_r(\lambda; n) = H_r(\lambda; n)/H_r(\lambda),$$

i.e. $P_r(\lambda; n)$ is the probability that the largest "eigenvalue" is $\leq n + r - 1$. Then, by (A.9) and (A.10),

(A.12)

$$\phi_n(\lambda) = e^{-\lambda}[1 + \sum_{r=1}^{\infty} P_r(\lambda; n)(D_r - D_{r-1})] = e^{-\lambda} + \sum_{r=1}^{\infty} P_r(\lambda; n)(\phi_r(\lambda) - \phi_{r-1}(\lambda)).$$

Now, the essential contribution to the right-hand side of (A.12) comes from r around $2\sqrt{\lambda}$ since otherwise $\phi_r(\lambda) - \phi_{r-1}(\lambda)$ is very small. Thus

$$\phi_n(\lambda) \approx P_{2\sqrt{\lambda}}(\lambda; n),$$

i.e. $\phi_n(\lambda)$ is like the probability that the largest "eigenvalue" in the discrete Coulomb gas (A.11) is $\leq n + 2\sqrt{\lambda}$.

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